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ASYMPTOTIC PROPERTIES OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN THE SUBLINEAR REGIME

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In this paper, we investigate stochastic heat equation with sublinear diffusion coefficients. By assuming certain concavity of the diffusion coefficient, we establish non-trivial moment upper bounds and almost sure spatial asymptotic properties for solutions. These results shed light on the *smoothing intermittency effect* under *weak diffusion* (i.e., sublinear growth) previously observed by Zeldovich *et al.* [ZMRS87]. The sample-path spatial asymptotics obtained in this paper partially bridge a gap in earlier works of Conus *et al.* [CJKS13, CJK13], which focused on two extreme scenarios: a linear diffusion coefficient and a bounded diffusion coefficient. Our approach is highly robust and applicable to a variety of stochastic partial differential equations, including the one-dimensional stochastic wave equation.

CONTENTS

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1. Introduction. In this paper, we investigate the *stochastic partial differential equations* (SPDEs) in the sublinear regime. More precisely, we focus on the asymptotic behavior of the following *stochastic heat equation* (SHE)

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \dfrac{1}{2}\Delta u(t,x) + \sigma(u(t,x))\dot{W}(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ u(0,\cdot)=\mu, \end{cases} \tag{1.1}$$

where the diffusion coefficient $\sigma(\cdot)$ is assumed to be locally bounded and exhibits sublinear growth at infinity. Previous studies have extensively examined the case when $\sigma(\cdot)$ has linear growth at infinity, which results in an intermittent solution. In particular, a solution is said to be *intermittent* if the *moment Lyapunov exponents* $\overline{\lambda}_p$ and $\underline{\lambda}_p$ of this solution, defined by

$$\overline{\lambda}_p \coloneqq \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[|u(t,x)|^p\right] \quad \text{and} \quad \underline{\lambda}_p \coloneqq \liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[|u(t,x)|^p\right] \quad (p\geq 2),$$

satisfy the property that $\underline{\lambda}_2>0$. The literature on this topic is extensive, and interested readers may consult [CM94, FK09, CD15b, Che15a, KKX17] and references therein. Zeldovich *et al.* [ZRS90, Chapter 9] have observed that intermittency is a universal phenomenon. It occurs irrespective of the underlying properties of the instability in a random medium, as long as the random field is of multiplicative type. However, they have also noted in [ZMRS87] (see also [ZRS90, Section 8.9]) that "*smoothing intermittency*", where the high maxima of solutions have smaller growth, should be expected in the presence of "*weak diffusion*"—when $\sigma(\cdot)$ exhibits sublinear growth at infinity. The authors substantiated their statement by highlighting the power growth of the moments when $\sigma(\cdot)$ is bounded, such as

$$\sigma(u) = \frac{u}{1+u}, \tag{1.2}$$

and when the noise is white in time and space-independent. Inspired by these previous works, this paper aims to analyze the smoothing intermittency property in broader weak diffusion cases and demonstrate the influence of $\sigma(\cdot)$ on the moment growth rate of solutions.

Note that this paper focuses on studying the intermittency property, which examines the distribution and propagation of peaks and valleys across the entire spatial domain $\mathbb{R}^d$, as seen in property (1.3) below. While this setup is ideal for our analysis, it's worth noting that SHEs on bounded domains have also been extensively studied; see, e.g., [Sow92].

Examining SPDEs in the sublinear regime is also driven by the need for more realistic biological population models. As noted by König in the Appendix of [Kön16], the *parabolic Anderson model* (PAM) (i.e., $\sigma(u)=\lambda u$) used in population dynamics results in unrealistic branching and killing rates, due to the absence of birth or death control. Sublinear growth for $\sigma(\cdot)$, on the other hand, offers a potential solution, allowing for models that better represent biological population dynamics.

Both the PAM and the SHE with additive noise, or simply the *additive SHE*, (i.e., $\sigma\equiv 1$, or more generally, the case when σ is bounded) have been extensively studied in the literature. They represent two extreme cases where rich properties have been previously derived. For instance, when $d=1$, the noise is space-time white noise, and the initial condition μ is constant, Conus *et al.* [CJKS13] showed that

$$\begin{cases} \displaystyle\sup_{|x|\leq R} u(t,x) \asymp [\log R]^{1/2} & \text{SHE with bounded } \sigma \text{ (Theorem 1.2 \textit{ibid.});} \\ \displaystyle\log\sup_{|x|\leq R} u(t,x) \asymp [\log R]^{2/3} & \text{PAM (Theorem 1.3 \textit{ibid.}).} \end{cases} \tag{1.3}$$

Here and in this paper, we use the notation $f(x) \lesssim g(x)$ to denote that there exists a non-random constant $C > 0$ such that $\limsup_{x\to\infty} f(x)/g(x) \le C$; the notation $f(x) \gtrsim g(x)$ is defined analogously. We also write $f(x) \asymp g(x)$ if both $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$. The distinct behaviors exhibited in (1.3) naturally raise questions regarding the dynamics when σ is neither bounded nor linear, but rather demonstrates sublinear growth, thereby providing a potential interpolation between these two extreme scenarios. Addressing sublinear growth requires a novel approach, which is the primary contribution of this paper.

Sublinear examples of $\sigma(u)$ (for $u > 0$) typically include

$$\sigma(u) = \frac{u}{(r+u)^{1-\alpha}}, \quad \alpha \in [0,1) \text{ and } r \ge 0; \tag{1.4}$$

$$\sigma(u) = u^\alpha \left[\log\left(e+u^2\right)\right]^{-\beta}, \quad \text{with } \begin{cases} \alpha = 0 \text{ and } \beta < 0 & \text{Case (i)}, \\ \alpha \in (0,1) \text{ and } \beta \in \mathbb{R} & \text{Case (ii)}, \\ \alpha = 1 \text{ and } \beta > 0 & \text{Case (iii)}; \end{cases} \tag{1.5}$$

and

$$\sigma(u) = u\exp\left(-\beta\left(\log\left(\log\left(e+u^2\right)\right)\right)^\kappa\right) \quad \text{with } \kappa > 0 \text{ and } \beta > 0. \tag{1.6}$$

Here, e is the *Euler constant*. In particular, letting $r = 1$ and $\alpha = 0$ in (1.4), we reduce to the case of bounded σ in (1.2). The next important example is a special case of (1.4) when $r = 0$:

$$\sigma(u) = u^\alpha, \quad \alpha \in (0,1). \tag{1.7}$$

It should be noted that SHEs with the diffusion coefficient given in (1.7) are closely related to superprocesses; see, e.g., [Daw93, Eth00, Per02]. Among the three cases in (1.5), cases (i) and (iii) are also interesting, as they are logarithmic perturbations of the additive SHE and the PAM, respectively. When $\kappa = 1$, σ in (1.6) reduces to case (iii) of (1.5). One would expect properties, such as the moment growth rates and spatial asymptotics, to transition from those of the additive SHE to those of the PAM. For example, for (1.4), one would expect a polynomial growth in t of moments, while the case (1.6) should lead to some exponential growth, but with a sublinear dependence on t in the exponent. Note that examples in (1.4)–(1.6) are all for the case when $u \ge 0$. For signed solutions, one needs to replace u by $|u|$.

Let us proceed to set up the problem. The noise $\dot{W}$ in SHE (1.1) is a centered Gaussian noise white in time and homogeneously colored in space. Its covariance structure is given by

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \delta_0(t-s)f(x-y). \tag{1.8}$$

Here, δ_0 denotes the Dirac delta measure at 0 and f is the correlation (generalized) function on $\mathbb{R}^d$, which satisfies the following hypothesis:

HYPOTHESIS 1.1. The correlation f is a tempered nonnegative definite measure on $\mathbb{R}^d$ that is not identically zero, such that the following *Dalang's condition* [Dal99] is satisfied:

$$\int_{\mathbb{R}^d} \frac{\widehat{f}(\mathrm{d}\xi)}{1+|\xi|^2} < \infty, \tag{1.9}$$

where $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(\mathrm{d}x)\, e^{-ix\cdot\xi}$ is the Fourier transform of f.

By using the Fourier transform and the Plancherel theorem, it is easy to verify that Dalang's condition (1.9) is equivalent to the next condition:

$$h(t) := \int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y')\, p_s(y)p_s(y-y') = \frac{1}{2}\int_0^{2t} \mathrm{d}s \int_{\mathbb{R}^d} f(\mathrm{d}y')\, p_s(y') < \infty, \tag{1.10}$$

for all $t > 0$, where $p_t(x) \coloneqq (2\pi)^{-d/2} \exp\left(-|x|^2/(2t)\right)$ refers to the heat kernel. The function h plays an essential role in our main result—Theorem 1.6 below, while the nonnegativity of f in Hypothesis 1.1 ensures that $h(\cdot)$ is an increasing function on $\mathbb{R}_+$.

To facilitate the analysis, we introduce the following hypothesis on σ, which covers all examples given in (1.4)–(1.6) with all u replaced $|u|$:

HYPOTHESIS 1.2. The diffusion coefficient $\sigma : \mathbb{R} \to \mathbb{R}$ satisfies the following properties:

(i) σ is a locally bounded function.

(ii) $\displaystyle\lim_{x\to\pm\infty} \frac{\sigma(x)}{x} = 0$;

(iii) There exists $M_0 \geq 0$, such that $|\sigma|$ is concave separately on $(-\infty, -M_0]$ and $[M_0, \infty)$.

REMARK 1.3. In Hypothesis 1.2, we do not impose any regularity conditions on σ beyond boundedness on $[-M_0, M_0]$. The absolutely continuity of σ outside this interval is due to its concavity (see Lemma 3.1). Thus, existence and uniqueness of solutions to (1.1) are not always guaranteed under Hypothesis 1.2. For a globally Lipschitz σ, such as in (1.4)–(1.6), the existence and uniqueness of solutions with rough initial conditions are well-established (see [CD15c, CK19]). In contrast, without the Lipschitz condition, as in (1.7), our main result, Theorem 1.6 below, provides *a priori* moment estimates that enable potential solutions using an approximation method. This method involves approximating the non-Lipschitz σ with a sequence of globally Lipschitz functions to construct random field solutions. Under mild conditions, these solutions exhibit jointly Hölder continuity, and thus has a convergent subsequence yielding a solution to (1.1) via the Kolmogorov-Chentsov criterion (see [Kal02, Corollary 16.9]). The approximating approach has been successfully applied in [MPS06, Appendix] for constructing solutions to SHEs with non-Lipschitz coefficients. However, establishing well-posedness for SHEs with non-Lipschitz coefficients remains challenging, with limited results; see e.g., [Myt98, MPS06, MP11] for the sublinear case; [DKZ19, Sal22, CH23] for the superlinear case; [MMP14, Che15b] on non-uniqueness. As our focus is not on existence and uniqueness, we do not explore this further.

The initial condition μ in (1.1) also plays an active role in shaping the properties of any solution, for which we make the following assumption:

HYPOTHESIS 1.4. μ is a signed Borel measure[1] on $\mathbb{R}^d$ such that

(i) For any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, it holds that, in the sense of Lebesgue integral,

$$\begin{aligned} \mathcal{J}_0(t,x) &\coloneqq \int_{\mathbb{R}^d} \mu(\mathrm{d}y)\, p_t(x-y) \in (-\infty, \infty), \quad \text{or equivalently,} \\ \mathcal{J}_+(t,x) &\coloneqq \int_{\mathbb{R}^d} |\mu|(\mathrm{d}y)\, p_t(x-y) < \infty, \end{aligned} \tag{1.11}$$

where $\mu = \mu_+ - \mu_-$ is the Hahn decomposition of μ and $|\mu| = \mu_+ + \mu_-$;

(ii) Moreover, if $d \geq 2$, then for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\mathcal{J}_1(t,x) \coloneqq \int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y')\, p_{t-s}(x-y) p_{t-s}(x-y+y') \mathcal{J}_0^2(s,y) < \infty. \tag{1.12}$$

[1] We follow the convention that when μ is absolutely continuous with the Lebesgue measure, it is identified as its Lebesgue density.

As usual, any solution to (1.1) is understood as a mild solution to the corresponding integral equation:

$$u(t,x)=\mathcal{J}_0(t,x)+\int_0^t\int_{\mathbb{R}^d}p_{t-s}(x-y)\sigma\left(u(s,y)\right)W(\mathrm{d}s,\mathrm{d}y),\quad t>0,\,x\in\mathbb{R}^d, \tag{1.13}$$

where the stochastic integral is understood in the sense of Dalang–Walsh [Wal86, Dal99].

1.1. *Main results.* The following notation is used in our main results.

DEFINITION 1.5. Suppose that σ fulfills Hypothesis 1.2. Let $\sigma_2:\mathbb{R}_+\mapsto\mathbb{R}_+$ be given by

$$\sigma_2(x)\coloneqq\sigma_2^+(x)+\sigma_2^-(x)\quad\text{with}\quad\sigma_2^\pm(x)\coloneqq\left|\sigma\left(\pm\sqrt{x}\right)\right|^2. \tag{1.14}$$

Define $F:[M^2,\infty)\to\mathbb{R}_+\cup\{\infty\}$ as

$$F(x)\coloneqq\frac{x}{4\sigma_2(x)},\quad x\ge M^2, \tag{1.15}$$

with the conventions that

$$\frac{x}{0}\coloneqq\infty\quad\text{if }x>0;\quad\text{and}\quad F(0)\coloneqq\lim_{x\downarrow 0}\frac{x}{4\sigma_2(x)}\in\mathbb{R}_+\cup\{\infty\},\quad\text{if }M=0\text{ and }\sigma_2(0)=0,$$

where $M>M_0$ such that σ_2 is concave on $[M,\infty)$ (see as in part (iii) of Lemma 3.3). We use F^{-1} to denote the (right) inverse of F restricted on $[2M^2,\infty)$ as follows,

$$F^{-1}(x)\coloneqq\inf\left\{y\in[2M^2,+\infty):\,F(y)\ge x\right\}. \tag{1.16}$$

Now we are ready to state our main result as follows:

THEOREM 1.6. *Under Hypotheses 1.1, 1.2, and 1.4, let $u(t,x)$ be a solution to SHE (1.1). Then, for all $(t,x,p)\in\mathbb{R}_+\times\mathbb{R}^d\times[2,\infty)$, it holds that*

$$\|u(t,x)\|_p^2\le 2\mathcal{J}_0^2(t,x)+2(2\pi)^d\left(h(t)^{-1}\mathcal{J}_1(t,x)+4K_M^2p\,h(t)+F^{-1}(2p\,h(t))\right), \tag{1.17}$$

where the function $F(\cdot)$ and its inverse $F^{-1}(\cdot)$, both determined by σ, are defined in (1.15) and (1.16) above, respectively. In (1.17), $\mathcal{J}_0(\cdot,\circ)$, $\mathcal{J}_1(\cdot,\circ)$ and $h(\cdot)$ are defined in (1.11), (1.12) and (1.10), respectively; and

$$K_M\coloneqq\sup_{x\in(-M,M)}|\sigma(x)|, \tag{1.18}$$

with M the same as in (1.16). In particular, the following statements hold:

(i) *If $d=1$, one can replace $4h(t)^{-1}\mathcal{J}_1(t,x)$ in (1.17) by $2^{7/2}\pi\mathcal{J}_+^2(t/2,x)$ (see (1.11));*
(ii) *If $|\sigma(\cdot)|$ is concave separately on $\mathbb{R}_+$ and $\mathbb{R}_-$, then taking $M=K_M=0$ in (1.17),*

$$\|u(t,x)\|_p^2\le 2\,\mathcal{J}_0^2(t,x)+2(2\pi)^d\left(h(t)^{-1}\mathcal{J}_1(t,x)+F^{-1}(2p\,h(t))\right); \tag{1.19}$$

(iii) *If $\sigma(\cdot)$ is not identically 0 on $(-\infty,-M_0]\cup[M_0,\infty)$, then there exists a constant $C>0$ such that for all $(t,x,p)\in[1,\infty)\times\mathbb{R}^d\times[2,\infty)$,*[2]

$$\|u(t,x)\|_p^2\le 2\,\mathcal{J}_0^2(t,x)+C\left(h(t)^{-1}\mathcal{J}_1(t,x)+F^{-1}\left(Cp\,h(t)\right)\right). \tag{1.20}$$

Moreover, if the initial condition is bounded, i.e., $\mu(\mathrm{d}x)=u_0(x)\mathrm{d}x$ with $u_0\in L^\infty(\mathbb{R}^d)$, the moment bound in (1.20) can be simplified as follows:

$$\|u(t,x)\|_p^2\le C_*F^{-1}\left(C_*p\,h(t)\right)\quad\textit{for some }C_*>0. \tag{1.21}$$

[2] Here t can be relaxed to $t>0$, but the constant C in (1.20) will depend on t when t is close to 0

The general moment bounds in the above theorem demonstrate how different components of the SPDE affect the moment growth of the solution, which is a concrete manifestation of the statement in Zeldovich *et al* [ZMRS87] (see also [ZRS90, Section 8.9]) that *the behavior of nonlinear solutions depends radically on the time behavior of the potential and on the form of the nonlinearity.* Theorem 1.6 is proved in Section 3.2.

REMARK 1.7. If the diffusion coefficient σ does not satisfy Hypothesis 1.2, the moment bounds in the above Theorem 1.6 can still be applied to derive nontrivial moment upper bounds under the following conditions:

1. the initial condition μ is nonnegative;
2. there exists an even dominating diffusion coefficient σ' such that $\sigma'(0)=0$, σ' satisfies Hypothesis 1.2, and the following cone condition holds: $0 \le \sigma(x) \le \sigma'(x)$ for all $x \ge 0$.

In this case, thanks to the stochastic (moment) comparison principle (see, e.g., [CK20] and [JKM17]), the moment bounds in (1.17) holds, with F^{-1} being obtained via σ'. This is true also for Theorems 1.8, 1.10, and 1.11 below.

1.2. *Applications: tail probabilities, Hölder regularity and spatial asymptotics.* The moment bounds can be applied to derive the tail probability, sample-path Hölder regularity, and sample-path asymptotics in the spatial variable of solutions to (1.1). Specifically, the sample-path spatial asymptotics have been previously studied by Conus *et al.* [CJKS13, CJK13] in two extreme cases, when σ is linear and bounded. Our result in Theorem 1.11 below serves as an initial attempt to bridge the gap between these two extreme cases by allowing σ to have sublinear growth. Here, we have to admit that only the asymptotic upper bounds have been obtained, while the more challenging lower bounds will be left for future investigation.

THEOREM 1.8 (Tail probability). *Assume that both Hypotheses 1.1 and 1.2 hold. Let $u(t,x)$ be a solution to SHE (1.1) with the initial condition $\mu(\mathrm{d}x)=u_0(x)\mathrm{d}x$ with $u_0 \in L^\infty(\mathbb{R}^d)$. Then for all $t \ge 1$, $x \in \mathbb{R}^d$ and $z \ge L_t$,*

$$\mathbb{P}\left(|u(t,x)| \ge z\right) \le \exp\left(-(C_* h(t))^{-1} F\left(z^2/\left(C_* e^2\right)\right)\right), \tag{1.22}$$

where $C_*>0$ is the same as in (1.21), $F(\cdot)$ and $F^{-1}(\cdot)$ are defined Definition 1.5,

$$L_t := e\sqrt{2C_* M^2 + C_* F^{-1}\left(2C_* h(t)\right)}\,, \tag{1.23}$$

and the constant M in (1.23) is the same as those in Theorem 1.6.

Theorem 1.8 is proved in Section 3.3. To state the next two results, we need the following *Dalang-Sanz-Solé-Sarrà condition* ([Dal99, SSS00]) for the spatial correlation function f:

HYPOTHESIS 1.9. The correlation function $f:\mathbb{R}^d \to \mathbb{R}$ is a nonnegative-definite tempered measure that is not identically zero such that,

$$\int_{\mathbb{R}^d} \frac{\widehat{f}(\mathrm{d}\xi)}{(1+|\xi|^2)^{1-\eta}} < \infty, \quad \text{for some } \eta \in (0,1). \tag{1.24}$$

THEOREM 1.10 (Hölder regularity). *Under parts (i) and (ii) of Hypothesis 1.2, part (i) of Hypothesis 1.4, and Hypothesis 1.9, let $u(t,x)$ be a solution to (1.1). Then u has a version which is a.s. η_1–Hölder continuous in time and η_2–Hölder continuous in space on $(0,\infty)\times\mathbb{R}^d$ for all $\eta_1 \in (0,\eta/2)$ and $\eta_2 \in (0,\eta)$, where η is given in (1.24).*

Theorem 1.10 is proved in Section 3.4.

THEOREM 1.11 (Spatial asymptotics). *Assume that both Hypotheses 1.2 and 1.9 hold. Suppose that $\sigma(\cdot)$ is not identically zero on $(-\infty,-M_0]\cup[M_0,\infty)$. Let $u(t,x)$ be a solution to SHE (1.1) with the constant initial condition $\mu(\mathrm{d}x)=u_0(x)\mathrm{d}x$ with $u_0\in L^\infty\left(\mathbb{R}^d\right)$. Then, there exists a positive constant C such that for all $t>1$,*

$$\sup_{|x|\le R}|u(t,x)|\lesssim\sqrt{F^{-1}\Big(C\,h(t)\log R\Big)}\,,\quad \text{a.s., as } R\to\infty,$$

where $F^{-1}(\cdot)$ is defined in (1.16).

Theorem 1.11 is proved in Section 3.5.

1.3. *One-dimensional stochastic wave equation.* The method for proving Theorem 1.6 for SHE is quite robust and it can can be easily adapted to other SPDEs. In this part, we will make this extension for the one-dimensional *stochastic wave equation* (SWE) from the linear growth regime as studied in [DM09, CD15a, BC16, HW21, CGS22] to the sublinear growth regime. Consider the following SWE

$$\begin{cases}\dfrac{\partial^2}{\partial t^2}u(t,x)=\Delta u(t,x)+\sigma(u(t,x))\dot{W}(t,x), & (t,x)\in\mathbb{R}_+\times\mathbb{R},\\ u(0,\cdot)=\mu_0, \quad \dfrac{\partial}{\partial t}u(t,x)\Big|_{t=0}=\mu_1, \end{cases}\tag{1.25}$$

where $\dot{W}$ is a centered Gaussian nosie with covariance given by (1.8). Similar to the SHE, solutions to (1.25) are formulated in the following mild form:

$$u(t,x)=\mathcal{J}_0(t,x)+\int_0^t\int_{\mathbb{R}}G(t-s,x-y)\sigma(u(s,y))W(\mathrm{d}s,\mathrm{d}y),$$

where $G(t,x):=\frac{1}{2}\mathrm{1\!I}_{[-t,t]}(x)$ refers to the wave kernel on $\mathbb{R}$ and

$$\mathcal{J}_0(t,x):=\frac{1}{2}\left[\mu_0(x+t)+\mu_0(x-t)\right]+\int_{\mathbb{R}}\mu_1(\mathrm{d}y)G(t,x-y).\tag{1.26}$$

THEOREM 1.12. *Let $u(t,x)$ be a solution to the SWE (1.25) with initial position $\mu_0\in L^2_{loc}(\mathbb{R})$ and initial velocity μ_1 which is a locally finite Borel measure on $\mathbb{R}$. Under Hypothesis 1.1 (with $d=1$) and Hypothesis 1.2, it holds for all $(t,x,p)\in\mathbb{R}_+\times\mathbb{R}\times[2,\infty)$, that*

$$\|u(t,x)\|_p^2\le 2\mathcal{J}_0^2(t,x)+K_1\left(h(t)^{-1}\mathcal{J}_1(t,x)+K_2p\,h(t)+F^{-1}(2p\,h(t))\right),\tag{1.27}$$

where $\mathcal{J}_0$ and F^{-1} are given in (1.26) and (1.16), respectively,

$$h(t):=\int_0^t\mathrm{d}s\iint_{\mathbb{R}^2}\mathrm{d}y\,f(\mathrm{d}y')\,G(s,y)G(s,y-y')\quad\text{and}\tag{1.28}$$

$$\mathcal{J}_1(t,x):=\int_0^t\mathrm{d}s\iint_{\mathbb{R}^2}\mathrm{d}y\,f(\mathrm{d}y')\,G\left(t-s,x-y\right)G\left(t-s,x-y+y'\right)\mathcal{J}_0^2(s,y).\tag{1.29}$$

In (1.27), K_1 and K_2 are some positive constants not depending on t and p. In particular,

(i) *if $|\sigma(\cdot)|$ is concave separately on $\mathbb{R}_+$ and $\mathbb{R}_-$, then one can take $K_2=0$ in (1.27);*
(ii) *there exist some constants $C>0$ such that the moment bound (1.20) holds, where $\mathcal{J}_0$, $\mathcal{J}_1$, and $h(t)$ are given in (1.26), (1.29), and (1.28), respectively. Moreover, if μ_0 is bounded and $|\mu_1|(\mathbb{R})<\infty$, then in (1.20) can be simplified to (1.21) with $h(t)$ given in (1.28);*

(iii) *assuming Hypothesis 1.9, $\mu_0 \equiv 1$ and $\mu_1 \equiv 0$, any solution $u(t,x)$ to (1.25) has a version which is a.s. η_1–Hölder continuous in time and η_2–Hölder continuous in space on $(0,\infty)\times\mathbb{R}$ for all $\eta_1,\eta_2 \in (0,\eta)$, where η is given in (1.24);*

(iv) *assuming that $\mu_0 \equiv 1$ and $|\mu_1| \equiv 0$, with the same C_* as in part (ii) and L_t given in (1.23) but with $h(t)$ replaced by (1.28), then for all $(t,x) \in [T,\infty)\times\mathbb{R}^d$ and $z \ge L_t$,*

$$\mathbb{P}\left(|u(t,x)| \ge z\right) \le \exp\left(-(C_* h(t))^{-1} F\left(z^2 / \left(C_* e^2\right)\right)\right),$$

and under Hypothesis 1.9, with some universal constant $C' > 0$,

$$\sup_{|x|\le R} u(t,x) \lesssim \sqrt{F^{-1}\Big(C'\, h(t) \log R\Big)}, \quad \textit{a.s., as } R \to \infty.$$

Theorem 1.12 is proved in Section 3.6.

Outline The paper is organized as follows. Section 2 expands on our main results, with their proofs provided in Section 3. Section 3.1 introduces several technical lemmas, with proofs in Section 3.7. Theorems 1.6, 1.8, 1.10, 1.11, and 1.12 are proved in Sections 3.2, 3.3, 3.4, 3.5, and 3.6, respectively. Finally, Section 4 presents examples illustrating our main results.

2. Remarks and proof strategy. In the following, we will first highlight, in Section 2.1, our mathematical innovation—solving a generalized *Bihari–LaSalle inequality*. Then from Sections 2.2–2.5, we will make some comments/discussions on our assumptions and results.

2.1. *Solving a generalized form of the Bihari–LaSalle integral inequality.* The core contribution of this paper's methodology lies in solving the following integral inequality for $\|u(t,x)\|_p$ ($p \ge 2$) with σ having sublinear growth rate at infinity:

$$\begin{aligned} \|u(t,x)\|_p^2 \le \mathcal{J}_0^2(t,x) + \int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y')\, G(t-s,x-y) G(t-s,x-y+y') \\ \times \|\sigma(u(s,y))\|_p \left\|\sigma\left(u(s,y-y')\right)\right\|_p. \end{aligned} \tag{2.1}$$

When σ is of linear growth, (2.1) can be reduced to

$$\begin{aligned} \|u(t,x)\|_p^2 \le & \mathcal{J}_0^2(t,x) + C\int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y')\, G(t-s,x-y) \\ & \times G(t-s,x-y+y') \Big(1 + \|u(s,y)\|_p\Big) \Big(1 + \left\|u(s,y-y')\right\|_p\Big). \end{aligned}$$

This is a generalized version of the renewal inequality and thus one should expect some Grönwall-type estimates. Dalang [Dal99] has provided a solution for the case when the initial condition is bounded. In this situation, by taking the supremum in the spatial variable on both sides, one obtains the following integral inequality of single variable:

$$Y(t) \le K + \int_0^t (K' + Y(s)) g(t-s)\mathrm{d}s, \quad t \ge 0, \quad \text{where } Y(t) := \sup_{x\in\mathbb{R}^d} \|u(t,x)\|_p^2. \tag{2.2}$$

It is worthy noting that the above integral inequality differs from the standard Grönwall inequality where the time variable takes a convolution form. However, this approach is not suitable for studying the equation with a rough initial condition (see Section 2.2 below) since the supremum norm in (2.2) does not always exist. One has to keep the spatial integral in (2.1) and a genuine PDF-type inequality has to be solved. This difficulty has been recently resolved by the first author and Huang [CH19] with an additional assumption $\sigma(0) = 0$. They obtained

$$\|u(t,x)\|_p \le \mathcal{J}_0(t,x) H_{f,d}(t), \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d.$$

Here, the function $H_{f,d}(t)$ depends on the correlation function f and the spatial dimension d. In most cases, $H_{f,d}(t)$ exhibits an exponential growth in t, as is typically expected, though this is not always true; see [CK19] for more details. More recently, the above moment bounds have been extended to the case when $\sigma(0) \neq 0$ together with a Lipschitz drift term in the equation; see [CFHS23, part (1) of Theorem 2.5].

The previously mentioned methods capture the correct moment growth in t in case $\sigma(x) \asymp |x|$. In contrast, when σ satisfies a sublinear growth condition, as examined in this paper, these approaches fail to produce the moment upper bounds with the right growth rate in t. To achieve more precise moment upper bounds, a robust and novel approach to solve the integral inequality (2.1) is needed. Indeed, when there is only one time variable, there is the well-known *Bihari–LaSalle inequality* [Bih56, LaS49], which has been applied for nonlinear stochastic differential equations (SDEs), such as in [FZ05].

LEMMA 2.1. Let $Y(\cdot)$ and $F(\cdot)$ be positive continuous functions on $[a,b]$. Let $k \ge 0$, $M \ge 0$, and let $\omega(\cdot) \ge 0$ be a nondescending continuous function on $\mathbb{R}_+$. Then the inequality

$$Y(x) \le k + M \int_a^x F(t)\omega(Y(t))\mathrm{d}t, \quad \text{for all } x \in [a,b], \tag{2.3}$$

implies the inequality

$$Y(x) \le \Omega^{-1}\left(\Omega(k) + M\int_a^x F(t)\mathrm{d}t\right), \quad \text{for all } x \in [a,b], \tag{2.4}$$

where $\Omega(u) := \displaystyle\int_{u_0}^u \frac{\mathrm{d}t}{\omega(t)}$ with $u_0 > 0$ and $u \ge 0$.

Comparing (2.3) and (2.1), one needs to generalize the above lemma involving a space-time convolution. Directly transplanting the arguments in [Dal99, CH19, CFHS23] from the Grönwall to the Bihari–LaSalle framework seems ineffective due to the intricate nonlinear nature of ω in (2.3). Instead, we solving (2.1) by, first, deriving inequality (3.19) of the form

$$X(t,x) \le k\sigma_2(X(t,x)) + b, \quad \text{with } X(t,x) = \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}y\, p_{t-s}^2(x-y)\,\|u(s,y)\|_p^2, \tag{2.5}$$

where $\sigma_2(\cdot)$ is a sublinear function on $\mathbb{R}_+$, k and b are some constants depending on p and t. Since σ_2 is sublinear, any nonnegative solution to inequality (2.5) has to lie in a compact interval, e.g., $[0, x_0]$. In other words, as X satisfies (2.5), we get $X \le x_0$; see Lemma 3.7.

To illustrate the idea of solving inequality (2.5), consider the case when $\sigma(u) = u^a$, $u \ge 0$, with $a \in [0,1)$ fixed. In this case, $\sigma_2(u) = \sigma(u)$ and the inequality in (2.5) becomes

$$x \le kx^a + b, \quad \text{for } x \ge 0 \text{ with } a \in (0,1),\ b > 0 \text{ and } k > 0 \text{ being fixed.} \tag{2.6}$$

By the concavity of the function x^a, the corresponding equation $x = kx^a + b$ has a unique positive solution, which is denoted by x^*. Hence, inequality (2.6) holds provided that $x \in [0, x^*]$, i.e., x^* is an upper bound estimate for x. Moreover, one can apply Newton's method for one step, properly started, to obtain an upper bound for x^*.

$$x^* \le k^{1/(1-a)} + b/(1-a). \tag{2.7}$$

To handle more general cases, a more meticulous approach is required, which yields a bound akin to (2.7) for solutions to (2.5). Overall, this constitutes the general strategy behind the proof of Theorem 1.6, and our moment bounds (1.19) should be compared to (2.4).

2.2. *Assumptions on initial conditions.* Here are some comments on the assumptions made for the initial conditions in Hypothesis 1.4: **(i)** Following [CD15b], we call the initial condition satisfying inequality (1.11) the *rough initial condition* for SHE (1.1). In particular, the Dirac delta measure is a special case; see also [CJKS14]. Note that the Dirac delta measure plays an important role in studying the asymptotic properties of the PAM on $\mathbb{R}$; see [ACQ11, Cor12]. As an easy exercise, condition (1.11) is equivalent to

$$-\infty < \int_{\mathbb{R}^d} e^{-a|x|^2} \mu(\mathrm{d}x) < \infty, \quad \text{for all } a > 0.$$

(ii) Condition (1.12) is only a technical assumption. We believe that this assumption can be removed. This will be left for future investigation. If $\mathcal{J}_0(s,y)^2$ in (1.12) is replaced by $\mathcal{J}_0(s,y)\mathcal{J}_0(s,y')$, then due to [CK19, Lemma 2.7], the integral is finite under Dalang's condition (1.9) for all rough initial conditions. This extra condition comes from the application of the inequality $\mathcal{J}_0(s,y)\mathcal{J}_0(s,y') \leq \frac{1}{2}\left(\mathcal{J}_0^2(s,y) + \mathcal{J}_0^2(s,y')\right)$ in the proof of Theorem 1.6. **(iii)** In case $d=1$, condition (1.12) is automatically satisfied by Lemma 3.8 below. On the other hand, for $d \geq 2$, condition (1.12) excludes the Dirac delta initial condition. This can be seen by setting $f(\cdot) \equiv 1$ (the space-independent noise), then the integral in (1.12) with delta initial condition reduces to $\int_0^t s^{-d/2} \mathrm{d}s = \infty$. Instead, condition (1.12) holds when $\mu(\mathrm{d}x) = |x|^{-\ell}\mathrm{d}x$ for any $\ell \in (0,1)$. This is due to the bound in (3.10) and the fact that for such initial data, $\mathcal{J}_0(t,x) \lesssim t^{-\ell/2}$; see [CE24]. Roughly speaking, the Dirac delta measure corresponds to the case when $\ell = d$. Removing part (ii) of Hypothesis 1.4 will be left for future investigation.

2.3. *Generality versus sharpness.* The moment bounds obtained in Theorem 1.6 strike a balance between their level of generality and their sharpness. Obtaining sharp moment asymptotics in general can be extremely challenging and is typically only possible in some specific settings. For instance, in the case when $d=1$, $\dot{W}$ is the space-time white noise, and $\sigma(u) = \sqrt{u}$ (the super-Brownian motion) with $u(0,x) \equiv 1$, the second author and his collaborators [HWXZ23] recently found the following exact moment asymptotics:

$$\mathbb{E}\left[u(t,x)^p\right] \asymp K^p p! \left(1 + t^{(p-1)/2}\right), \quad \text{as } t \to \infty,$$

for any positive integer p and $x \in \mathbb{R}$; while Theorem 1.6 suggests:

$$\mathbb{E}\left[u(t,x)^p\right] \leq K^p p! \left(1 + t^{p/2}\right).$$

The difference between above two inequality highlights that Theorem 1.6 is not sharp. While this appears to be the only known case of non-sharpness, we believe it is likely more general. Nonetheless, the method used in the proof of Theorem 1.6 are robust and can be easily extended to more general settings and a broad class of SPDEs, including one-dimensional SWEs. Moreover, despite being sub-optimal, Theorem 1.6 is sufficient to describe in a quantitative way for the "smoothing intermittency" phenomenon from Zeldovich *et al.* [ZRS90].

2.4. *Spatial asymptotics and tail estimates.* Theorem 1.11 provides an almost sure asymptotic upper bound for solutions to (1.1) in space. The proof of this theorem closely follows the approach presented in [CJKS13], which relies on tail estimates (Theorem 1.8) and the Borel–Cantelli lemma. While the moment bounds may not be particularly sharp, we find that the tail estimates are indeed sharp, at least in the case of super Brownian motion; see Proposition 4.4 with $\alpha = 1/2$ and [HWXZ23, Proposition 1.4]. As a result, we believe Theorem 1.11 provides a sharp bound for super-Brownian motion, specifically, fix $t > 0$,

$$\sup_{|x| \leq R} u(t,x) \asymp \sqrt{t}\log(R), \quad \text{almost surely, as } R \to \infty.$$

2.5. *Interaction with initial conditions.* Due to the multiplicative nature of the diffusion term, the initial condition interacts with the noise, actively shaping solutions to (1.1). For the PAM, this interaction leads to the following moment bound (see [CH19, Theorem 1.7]):

$$\|u(t,x)\|_p^2 \le C\mathcal{J}_0^2(t,x)\,\Upsilon(t), \quad \text{for all } t>0,\, x\in\mathbb{R}^d, \text{ and } p\ge 2, \tag{2.8}$$

where $\Upsilon(\cdot)$ represents the contribution of the driving noise. The *multiplicative interaction* of the initial conditions and the noise in (2.8) naturally gives rise to the propagation of tall peaks in some space-time cone $\{|x|\le \kappa t\}$, which was earlier observed in physical contexts, cf. [ZRS90, Section 8.10] and later rigorously studied by Conus and Khoshnevisan [CK12]. Since then, additional researches (cf. [CD15b, CK19, HLN17]) has expanded upon this cone property. In essence, the cone property asserts the pretense of a space-time cone sized κ within which the moments of the PAM grow exponentially fast, contrasting with rapid exponentially decay outside this cone. The size κ is termed the *intermittency front*.

However, in this paper, we obtain an *additive interaction* of the initial data and the noise as in Theorem 1.6. This additive interaction arises from the way we solve the inequality (2.6), or from the application of Lemma 3.7 in general, where the coefficient b corresponds to the initial condition. By sending b to zero in (2.6), the linear equation $x = kx^\alpha + b$ shifts from having a single nonnegative solution to having two nonnegative solutions, one of which is zero. Accordingly, assuming $b=0$, the inequality $x\le kx^\alpha$ can only assures that $x\in[0,k^{1/(1-\alpha)}]$, but additional information is needed to determine the exact value of x. In the context of SPDEs, such additional information may be related to the (non-)uniqueness of solutions. In fact, when $\sigma(u)=|u|^\alpha$ with $\alpha\in(0,3/4)$ and $\dot W$ is the one-dimensional space-time white noise, Mueller *et al* [MMP14] constructed nontrivial solutions starting from zero initial data. Hence, the propagation of high peaks will be more subtle and will be left for future research.

If σ is globally Lipschitz, the moment comparison theorem may be applied, and thus the moment bounds for the "dominant PAM" can serve as an upper bound for the propagation of solutions to (1.1). When σ is not globally Lipschitz, solutions to (1.1), assuming $\sigma(u)=u^\alpha$ with $\alpha\in(0,1)$, is compactly supported if the initial data is a finite measure (cf. [DIP89, MP92]). Additionally, given that precise analysis on front propagation (a related but distinct property) for Fisher-KPP equations has been presented in [MMQ11, MMR21], it should be possible to study propagation of tall peaks, even for non-Lipschitz cases with necessary restrictions on initial datum. We hope that this question can be resolved in the future.

3. Proof of the theorems.

3.1. *Technical lemmas.* In this subsection, we provide some technical lemmas that will be used in the proof of Theorem 1.6. The proofs of these lemmas are postponed to Section 3.7.

LEMMA 3.1. Let σ be a function satisfying Hypothesis 1.2. Then there exist two functions $g^+:[M_0,\infty)\to\mathbb{R}$ and $g^-:(-\infty,-M_0]\to\mathbb{R}$ that satisfy the following properties:

(i) g^+ is nonnegative, non-increasing, right-continuous, and it satisfies that

$$\lim_{x\to\infty} g^+(x)=0 \quad \text{and} \quad |\sigma(x)|-|\sigma(M_0)| = \int_{M_0}^x g^+(y)\mathrm{d}y \quad \text{for all } x\ge M_0;$$

(ii) Similarly, g^- is nonnegative, non-decreasing, left-continuous, and it satisfies that

$$\lim_{x\to-\infty} g^-(x)=0 \quad \text{and} \quad |\sigma(x)|-|\sigma(-M_0)| = \int_x^{-M_0} g^-(y)\mathrm{d}y \quad \text{for all } x\le -M_0.$$

Let $\theta_p : \mathbb{R} \to \mathbb{R}_+$ denote the power function $\theta_p(x) = |x|^p$ for $p \in \mathbb{R}$. When $\sigma(u) = |u|^\alpha$ with $u \ge 0$ and $\alpha \in (0,1]$, we claim that

$$\|\sigma(u)\|_p^2 \le \sigma\left(\|u\|_p^2\right) \quad \text{for all } p > 0.$$

Indeed, it is clear that

$$\begin{aligned}\|\sigma(u)\|_p^2 &= \left(\theta_{\frac{2}{p}} \circ \mathbb{E} \circ \theta_p \circ |\sigma|\right)(u) = \left(\theta_{\frac{2}{p}} \circ \mathbb{E} \circ \theta_p \circ \theta_\alpha\right)(u) = \left(\theta_{\frac{2}{p}} \circ \mathbb{E} \circ \theta_\alpha \circ \theta_p\right)(u) \\ &\le \left(\theta_{\frac{2}{p}} \circ \theta_\alpha \circ \mathbb{E} \circ \theta_p\right)(u) = \left(\theta_\alpha \circ \theta_{\frac{2}{p}} \circ \mathbb{E} \circ \theta_p\right)(u) = \sigma\left(\|u\|_p^2\right),\end{aligned}$$

where the inequality is due to the concavity of $\sigma = \theta_\alpha$ and we used twice the commutative property: $\sigma \circ \theta_p = \theta_\alpha \circ \theta_p = \theta_p \circ \theta_\alpha = \theta_p \circ \sigma$. Yet for a general σ fulfilling Hypothesis 1.2, we need to introduce σ_p for this purpose, that generalizes σ_2 in (1.14) to arbitrary $p > 0$.

DEFINITION 3.2. For any function $\sigma : \mathbb{R} \to \mathbb{R}$ and any positive number p, let the functions σ_p^+, σ_p^- and $\sigma_p : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as follows: for all $x \in \mathbb{R}_+$,

$$\sigma_p(x) := \sigma_p^+(x) + \sigma_p^-(x) \quad \text{with} \quad \sigma_p^\pm(x) := \left|\sigma\left(\pm\, x^{1/p}\right)\right|^p. \tag{3.1}$$

The next lemma shows that $\sigma_p(\cdot)$ and $\sigma_p^\pm(\cdot)$ inherit the properties from $\sigma(\cdot)$.

LEMMA 3.3. Suppose that $\sigma(\cdot)$ is a function satisfying Hypothesis 1.2. For any $p > 0$, let $\sigma_p(\cdot)$, $\sigma_p^+(\cdot)$ and $\sigma_p^-(\cdot)$ be given in Definition 3.2. Then, the following properties hold:

(i) For any $x \ge M_0^p$, with $g^\pm(\cdot)$ given in Lemma 3.1

$$\rho_p^+(x) - \rho_p^+(M_0^p) = \int_{M_0^p}^x g_p^+(y)\mathrm{d}y, \quad \text{and} \quad \rho_p^-(x) - \rho_p^-(M_0^p) = \int_{M_0^p}^x g_p^-(y)\mathrm{d}y,$$

where

$$g_p^+(x) := \left|\rho\left(+x^{1/p}\right)\right|^{p-1} g^+\left(+x^{1/p}\right) x^{-(p-1)/p}, \quad \text{and} \tag{3.2}$$

$$g_p^-(x) := \left|\rho\left(-x^{1/p}\right)\right|^{p-1} g^-\left(-x^{1/p}\right) x^{-(p-1)/p}. \tag{3.3}$$

(ii) Both σ_p^+ and σ_p^- are non-decreasing on $[M_0^p, \infty)$;
(iii) There exists $M \ge M_0$, independent of p, such that all functions σ_p, σ_p^+, and σ_p^- are concave on $[M^p, \infty)$;
(iv) If $M_0 = 0$ in part (iii) of Hypothesis 1.2, then σ_p^+, σ_p^- and σ_p are all concave on $\mathbb{R}_+$.

LEMMA 3.4. For any $U \in L^p(\Omega)$ and $p > 0$, it holds that

$$\mathbb{E}\left[\sigma_p^+(|U|^p)\mathbb{1}_{[+M,+\infty)}(U)\right] \le \sigma_p^+\left(M^p + \|U\|_p^p\right) \quad \text{and} \tag{3.4}$$

$$\mathbb{E}\left[\sigma_p^-(|U|^p)\mathbb{1}_{(-\infty,-M]}(U)\right] \le \sigma_p^-\left(M^p + \|U\|_p^p\right), \tag{3.5}$$

where the constant M is given in part (iii) of Lemma 3.3.

LEMMA 3.5. Suppose that σ satisfies Hypotheses 1.2. Let M be the associated constant given in part (iii) of Lemma 3.3. Then for all $p \ge 2$ and any $U \in L^p(\Omega)$, it holds that

$$\|\sigma(U)\|_p^2 \le K_M^2 + \sigma_2\left(M^2 + \|U\|_p^2\right), \tag{3.6}$$

where K_M is defined in (1.18). In particular, if $M_0 = 0$, one can take $M = K_M = 0$ in (3.6).

REMARK 3.6. Here are some remarks on the functions F and F^{-1}:

(i) As stated in Definition 1.5, we allow $F(x)=\infty$ in case $x>0$ and $\sigma_2(x)=0$, see e.g., $\sigma(x)=1$ for $|x|<1$ and $\sigma(x)=|x-1|^\alpha$ for $|x|\geq 1$. If $\sigma\equiv 0$ on $[M_0,\infty)$, then $F\equiv\infty$ on $[M_0,\infty)$ as well. This implies that $F^{-1}\equiv 2M^2$ on $\mathbb{R}_+$. As a result, $F^{-1}(2ph(t))$ as in (1.17) is uniformly bounded in t, this coincides with the SHE with additive noise.
(ii) Under part (ii) of Hypothesis 1.2, the set in (1.16) is nonempty for any $x>0$, and thus $F^{-1}(\cdot)$ is a real-valued function.
(iii) From the definitions, and noticing that F is continuous on $[M^2,\infty)$, it is easy to see that

$$F\circ F^{-1}(x)\geq x,\quad x\geq 0,\quad \text{and} \tag{3.7}$$

$$F^{-1}\circ F(x)\leq x,\quad x\geq 2M^2. \tag{3.8}$$

LEMMA 3.7. Suppose that the function $\sigma(\cdot)$ satisfies Hypothesis 1.2. Let $\sigma_2(\cdot)$ be defined as in (3.1) with $p=2$. Then, thanks to part (iii) of Lemma 3.3, σ_2 is concave on $[M^p,\infty)$ with some $M>M_0$. For any $k,\ b>0$, suppose $x\geq 0$ such that $x\leq k\sigma_2(x)+b$. Then,

$$x\leq 2F^{-1}(k)+2b<\infty. \tag{3.9}$$

LEMMA 3.8. Under Hypothesis 1.1 and part (i) of Hypothesis 1.4, for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, $\mathcal{J}_1(t,x)$ defined in (1.12) satisfies that

$$\mathcal{J}_1(t,x)\leq\int_0^t \mathrm{d}s\, k(t-s)\int_{\mathbb{R}^d}\mathrm{d}y\, p_{t-s}(x-y)\mathcal{J}_0^2(s,y),\quad\text{where} \tag{3.10}$$

$$k(t):=\int_{\mathbb{R}^d} f(\mathrm{d}z)\, p_t(z)=h'(t/2)<\infty. \tag{3.11}$$

In particular, when $d=1$, it holds that

$$\mathcal{J}_1(t,x)\leq 2^{3/2}\pi\, h(t)\, \mathcal{J}_+(t/2,x)^2<\infty. \tag{3.12}$$

3.2. *Proof of moment growth formulas—Theorem 1.6.* The proof consists of four steps:

Step 1. In this step, we derive the following inequality thanks to Hypothesis 1.2:

$$\begin{aligned}\|u(t,x)\|_p^2\leq 2\mathcal{J}_0^2(t,x)+8K_M^2 p\, h(t)+8p\int_0^t\mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}y f(\mathrm{d}y')\, p_{t-s}(x-y)\\ \times p_{t-s}(x-y+y')\sigma_2\left(M^2+\|u(s,y)\|_p^2\right).\end{aligned} \tag{3.13}$$

Indeed, by the Burkholder-Davis-Gundy and Minkowski's inequalities to the mild form (1.13),

$$\begin{aligned}\|u(t,x)\|_p^2\leq& 2\mathcal{J}_0^2(t,x)+8p\int_0^t\mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}y f(\mathrm{d}y')\, p_{t-s}(x-y)p_{t-s}(x-y+y')\\ &\times\left\|\sigma(u(s,y))\sigma(u(s,y-y'))\right\|_{p/2}.\end{aligned}$$

Then an application of the Cauchy-Schwarz inequality for the $L^{p/2}(\Omega)$–norm yields that

$$\begin{aligned}\|u(t,x)\|_p^2\leq 2\mathcal{J}_0^2(t,x)+8p\int_0^t\mathrm{d}s\iint_{\mathbb{R}^{2d}}\mathrm{d}y f(\mathrm{d}y')\, p_{t-s}(x-y)\,\|\sigma(u(s,y))\|_p\\ \times p_{t-s}(x-y+y')\left\|\sigma(u(s,y-y'))\right\|_p.\end{aligned} \tag{3.14}$$

Taking account of the fact that $ab \le \frac{1}{2}(a^2+b^2)$ for all $a, b \in \mathbb{R}$, we can further deduce that

$$\|u(t,x)\|_p^2 \le 2\mathcal{J}_0^2(t,x) + 8p\int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y')\, p_{t-s}(x-y)p_{t-s}(x-y-y') \\ \times \|\sigma(u(s,y))\|_p^2 .$$

Next, applying Lemma 3.5, we get

$$\|\sigma(u(s,y))\|_p^2 \le K_M^2 + \sigma_2\left(M^2 + \|u(s,y)\|_p^2\right). \tag{3.15}$$

Plugging (3.15) into the previous inequality proves (3.13).

Step 2. In this step, we will solve the nonlinear integral inequality (3.13). Fix $t > 0$ and $x \in \mathbb{R}^d$. Using the function $h(t)$ in (1.10) as a normalization constant and thanks to the concavity of σ_2, we can apply Jensen's inequality to the triple integrals in (3.13) to write that

$$\|u(t,x)\|_p^2 \le 2\mathcal{J}_0^2(t,x) + 8K_M^2 p\, h(t) + 8ph(t)\sigma_2\left(X\right), \tag{3.16}$$

where $X := M^2 + h(t)^{-1}Y$ with

$$Y := \int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y')\, p_{t-s}(x-y)p_{t-s}(x-y+y')\, \|u(s,y)\|_p^2 .$$

It reduces to find an upper bound for X. By using (3.13), we deduce that $Y \le 2\mathcal{J}_1(t,x) + 8K_M^2 p\, h(t)^2 + \mathcal{I}$ with

$$\mathcal{I} := 8p\int_0^t \mathrm{d}r \iint_{\mathbb{R}^{2d}} \mathrm{d}z f(\mathrm{d}z')\, \sigma_2\big(M^2 + \|u(r,z)\|_p^2\big) \int_r^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} dy f(\mathrm{d}y')\, p_{t-s}(x-y) \\ \times p_{t-s}(x-y+y')p_{s-r}(y-z)p_{s-r}(y-y'-z+z').$$

Using the following formula,

$$p_{t-s}(a)p_s(b) = p_{s(t-s)/t}\left(b - \frac{s}{t}(a+b)\right)p_t(a+b), \quad \text{for all } 0 < s < t \text{ and } a, b \in \mathbb{R}, \tag{3.17}$$

and [CK19, Lemma 2.6 and inequalities (2.21)–(2.23)], it is not hard to deduce that

$$\int_r^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y') p_{t-s}(x-y)p_{t-s}(x-y+y')p_{s-r}(y-z) \\ \times p_{s-r}(y-y'-z-z') \le (2\pi)^{-d}\, p_{t-r}(x-z)p_{t-r}(x-z+z')h(t). \tag{3.18}$$

Thus,

$$\begin{aligned}\mathcal{I} \le & \frac{8ph(t)}{(2\pi)^d}\int_0^t \mathrm{d}r \iint_{\mathbb{R}^{2d}} \mathrm{d}z f(\mathrm{d}z')\, p_{t-r}(x-z)p_{t-r}(x-z+z')\sigma_2\left(M^2 + \|u(r,z)\|_p^2\right)\\ \le & 8\,(2\pi)^{-d} ph^2(t)\sigma_2(X),\end{aligned}$$

due to the concavity of $\sigma_2(\cdot)$ (see Lemma 3.3), (1.10), and Jensen's inequality. Therefore,

$$Y \le 2\mathcal{J}_1(t,x) + 8K_M^2 p\, h(t)^2 + 8\,(2\pi)^{-d} ph^2(t)\sigma_2(X),$$

or equivalently,

$$X \le \underbrace{M^2 + 2h(t)^{-1}\mathcal{J}_1(t,x) + 8K_M^2 p\, h(t)}_{:= b(p,t,x)} + \underbrace{8\,(2\pi)^{-d}\, p\, h(t)}_{:= k(p,t)}\sigma_2(X). \tag{3.19}$$

By Lemma 3.7, we have that $X \leq 2F^{-1}(k(p,t)) + 2b(p,t,x)$. Finally, thanks to the monotonicity of $\sigma_2(\cdot)$ when $x \geq M^2$; see Lemma 3.3, plugging the above moment bounds back to (3.16) proves the moment following bounds

$$\|u(t,x)\|_p^2 \leq 2\mathcal{J}_0^2(t,x) + 8K_M^2 ph(t) + 8ph(t)\sigma_2\left(2b(p,t,x) + 2F^{-1}\big(k(p,t)\big)\right). \tag{3.20}$$

Step 3. In this step, we will simplify the expression in (3.20) and prove inequality (1.17). Recall the definition of F^{-1} in (1.16), one can show that for any $x \geq 2F^{-1}(k)$,

$$\frac{\sigma_2(x)}{x} = \frac{\sigma_2(x) - \sigma_2(2F^{-1}(k)) + \sigma_2(2F^{-1}(k))}{x - 2F^{-1}(k) + 2F^{-1}(k)} \leq \frac{1}{2k},$$

because

$$\frac{\sigma_2\left(2F^{-1}(k)\right)}{2F^{-1}(k)} \leq \frac{1}{4k} \leq \frac{1}{2k} \quad \text{and} \quad \frac{\sigma_2(x) - \sigma_2\left(2F^{-1}(k)\right)}{x - 2F^{-1}(k)} \leq g_2\left(2F^{-1}(k)\right) \leq \frac{1}{2k},$$

where the last inequality is proved in (3.34) below. As a result, concerning the fact that $F^{-1}(k) \geq 2M^2$ for all $k > 0$, and $8(2\pi)^{-d} \leq 2$ for all $d \geq 1$, we can write

$$\begin{aligned} 8p\,h(t)\,\sigma_2\Big(2b(p,t,x) + 2F^{-1}\big(k(p,t)\big)\Big) &\leq \frac{4p\,h(t)}{k(p,t)}\Big(b(p,t,x) + F^{-1}\big(k(p,t)\big)\Big) \\ &\leq 2(2\pi)^d\Big(h(t)^{-1}\mathcal{J}_1(t,x) + 4K_M^2 p\,h(t) + F^{-1}(2p\,h(t))\Big). \end{aligned}$$

Plugging the above upper bound back to (3.20) proves (1.17).

Step 4. The special case when $d = 1$ is an application of Lemma 3.8 and the case when $\sigma_2(\cdot)$ is concave separately on $\mathbb{R}_+$ and $\mathbb{R}_-$ is due to Lemma 3.5. This proves both parts (i) and (ii). As for part (iii), if $M = 0$, inequality (1.20) follows from part (ii). Otherwise, if $M > 0$, then (1.20) holds provided that one can verify that there exists $C > 0$ such that for all $t \geq 1$,

$$4K_M^2 p\,h(t) + F^{-1}\left(2p\,h(t)\right) \leq CF^{-1}(Cph(t)). \tag{3.21}$$

Indeed, the assumption of part (iii) ensures that $\sigma_2(x) \geq c > 0$ with some uniform constant $c > 0$ if $x > M$ is large enough, which implies that $F^{-1}(k) \geq ck$ for large k (see (1.15)). Hence, inequality (3.21) holds by noticing that h is a non-decreasing function such that $h(t) > 0$ for all $t > 0$ under Hypothesis 1.1. This proves (1.20).

Finally, if $\mu(\cdot)$ is bounded, then $\mathcal{J}_0(t,x)$ is bounded on $\mathbb{R}_+ \times \mathbb{R}^d$, and the same is true for $h(t)^{-1}\mathcal{J}_1(t,x)$ (see Lemma 3.8). Then, inequality (1.21) is a consequence of the fact that $F^{-1}(k)$ is bounded below by a positive constant for all k large enough because σ is not identically zero on $(-\infty, -M_0] \cup [M_0, \infty)$. This completes the proof of Theorem 1.6. $\square$

3.3. *Proof of tail probability—Theorem 1.8.* We first prove a lemma that extends [CJKS13, Lemma 3.4], generalizing the exponent $\alpha(\cdot)$ from a power function to a more general form.

LEMMA 3.9. Let X be a random variable such that for some function $\alpha\colon [2,\infty) \to \mathbb{R}$,

$$\mathbb{E}\left[|X|^p\right] \leq \exp(\alpha(p)) < \infty, \quad \text{for all } p \geq 2. \tag{3.22}$$

Then, for all $z > 0$, it holds that

$$\mathbb{P}(|X| \geq z) \leq \exp\big(-\alpha^*(\log(z))\big), \tag{3.23}$$

where $\alpha^*(\cdot)$ is the Legendre-type transform of $\alpha(\cdot)$ on $[2,\infty)$, namely,

$$\alpha^*(x) := \sup\Big\{xp - \alpha(p)\colon p \in [2,\infty)\Big\} \in \mathbb{R} \cup \{\infty\}, \qquad \text{for all } x \in \mathbb{R}. \tag{3.24}$$

PROOF. The lemma follows from Chebyshev's inequality: for any $z > 0$ and $p \geq 2$,

$$\mathbb{P}\left(|X| \geq z\right) = \mathbb{P}\left(|X|^p \geq z^p\right) \leq z^{-p} \exp\left(\alpha(p)\right) = \exp\left(-\left[p \log(z) - \alpha(p)\right]\right).$$

Then, (3.23) follows from the minimization of the above expression with respect to p. $\square$

PROOF OF THEOREM 1.8. For all $t \geq 1$ and $p \geq 2$, we apply Lemma 3.9 with the moment bounds given in (1.21) of Theorem 1.6 to see that

$$\mathbb{P}(u(t,x) \geq z) \leq \exp\left(-H^*(\log(z))\right), \tag{3.25}$$

where $H^*\colon \mathbb{R} \to \mathbb{R}$ is the Legendre-type transform (see (3.24)) of $H\colon [2,\infty) \to \mathbb{R}$ given by

$$H(p) := \frac{p}{2} \log\left(C_* F^{-1}(C_* p h(t))\right), \qquad \text{for all } p \geq 2, \tag{3.26}$$

with the constant C_* given in (1.21). Next, notice that if

$$y \geq \max\left\{2^{-1} \log\left(2C_* M^2\right) + 1, 2^{-1} \log\left(C_* F^{-1}\left(2C_* h(t)\right)\right) + 1\right\}, \tag{3.27}$$

then, we have $e^{2(y-1)}/C_* \geq 2M^2$, and (thanks to (3.7))

$$p^*(y) := \left(C_* h(t)\right)^{-1} F\left(e^{2(y-1)}/C_*\right) \geq 2.$$

Therefore,

$$\begin{aligned} H(p^*(y)) &= \left(2C_* h(t)\right)^{-1} F\left(e^{2(y-1)}/C_*\right) \log\left(C_* F^{-1} \circ F\left(e^{2(y-1)}/C_*\right)\right) \\ &\leq \left(2C_* h(t)\right)^{-1} F\left(e^{2(y-1)}/C_*\right) \log\left(e^{2(y-1)}\right) = p^*(y)(y-1), \end{aligned}$$

where we have used (3.8) for the inequality. This yields that

$$H^*(y) \geq y p^*(y) - H(p^*(y)) = y p^*(y) - p^*(y)(y-1) = \frac{F\left(e^{2(y-1)}/C_*\right)}{(C_* h(t))}. \tag{3.28}$$

Therefore, (1.22) is justified by plugging (3.28) in (3.25) with y replaced by $\log z$. Similarly, the expression of L_t in (1.23) can be obtained by replacing y by $\log z$ in (3.27). $\square$

3.4. *Proof of Hölder regularity—Theorem 1.10.* The proof follows the same arguments as those in [CH19, Theorem 1.8] with the moment bounds obtained in Lemma 3.10 below. $\square$

LEMMA 3.10. Assume Hypothesis 1.1, parts (i) and (ii) of Hypothesis 1.2, and part (i) of Hypothesis 1.4. Let $u(t,x)$ be a solution SHE (1.1). Then, for all $(t,x,p) \in \mathbb{R}_+ \times \mathbb{R}^d \times [2,\infty)$

$$\|u(t,x)\|_p \leq \left(\mu' * p_t\right)(x) \ \left[H\left(t\,;\,32pK^2\right)\right]^{1/2}, \tag{3.29}$$

where $\mu' := \sqrt{2} + 2|\mu|$ and $H(t;\lambda)$ is non-decreasing in t with a parameter $\lambda > 0$.

Referring to the precise definition of $H(t;\lambda)$, one can consult [CH19, Formula (2.4)]. It should be noted that $H(0;\lambda) > 0$, which means the function $H(t;\lambda)$ does not introduce any singularity at $t = 0$. In general, $H(t;\lambda)$ exhibits an exponential growth rate in t; see [CK19, Lemma 2.5]. Certainly, the moment bounds (3.29) are considerably less accurate compared to those as in (1.17), especially for large p or t. This is a worthy trade-off for removing part (ii) of Hypothesis 1.4 in Lemma 3.10, which is sufficient for deducing the Hölder continuity of solutions in Theorem 1.10. To achieve more precise tail estimates in Theorem 1.8, additional improved estimates for the moment increments, as in Lemma 3.11 below, are required.

PROOF OF LEMMA 3.10. Parts (i) and (ii) of Hypothesis 1.2 imply that $|\sigma(x)| \le K_\sigma(1+|x|)$ with a universal constant $K_\sigma > 0$ for all $x \in \mathbb{R}$. Hence, from (3.14), we see that for all $(t,x,p) \in \mathbb{R}_+ \times \mathbb{R}^d \times [2,\infty)$,

$$\|u(t,x)\|_p^2 \le 2\mathcal{J}_0^2(t,x) + 8pK_\sigma^2 \int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y')\; p_{t-s}(x-y)\big(1+\|u(s,y)\|_p\big) \times p_{t-s}(x-y+y')\big(1+\|u(s,y-y')\|_p\big)\,.$$

By setting $g(t,x) := 1 + \|u(t,x)\|_p$ and denoting the above triple integral by $I(t,x)$, we have

$$g(t,x)^2 \le 2 + 2\|u(t,x)\|_p^2 \le \left(\sqrt{2} + 2\mathcal{J}_+(t,x)\right)^2 + 16pK_\sigma^2 I(t,x).$$

Therefore, for $\mu' = \sqrt{2}K_\sigma + |\mu|$, $g(t,x)$ satisfies the following integral inequality:

$$g(t,x)^2 \le \left[\left(\mu' * p_t\right)(x)\right]^2 + 16pK_\sigma^2 \int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}y f(\mathrm{d}y')\, p_{t-s}(x-y) g(s,y) \times p_{t-s}(x-y+y') g(s,y-y')\,.$$

It is easy to see that μ' also satisifes part (i) of Hypothesis 1.4. Then, an application of [CH19, Lemma 2.2] with the above μ' and $\lambda = 16pK_\sigma^2$ implies the moment bound (3.29). The property of $H(t;\lambda)$ can be found in [CK19, Lemma 2.5]. □

3.5. *Proof of spatial asymptotics—Theorem 1.11.* The proof of Theorem 1.11 is based on a tail estimate for solutions to SHE (1.1) given in Theorem 1.8. We also need moment increments in the space variable, in case of the constant initial condition, with a sharper constant than the one implicitly appearing in Theorem 1.10.

LEMMA 3.11. Assume Hypotheses 1.2 and 1.9. Let $u(t,x)$ be a solution to SHE (1.1) with a bounded initial condition (i.e., $\mu(\mathrm{d}x) = u_0(x)\mathrm{d}x$ with $u_0 \in L^\infty(\mathbb{R}^d)$). Let $C_* > 0$ be the constant in (1.21). Then, the following statements are satisfied.

(i) For all $x, y \in \mathbb{R}^d$, $t \ge 1$, and $p \ge 2$, there exists a constant $C > 0$ such that
$$\|u(t,x) - u(t,y)\|_p^2 \le CF^{-1}\left(C_* p h(t)\right)|x-y|^{2\eta};$$

(ii) For any $x \in \mathbb{R}^d$, with B_x denoting the unit ball centered at x, it holds that
$$\left\| \sup_{y_1,y_2 \in B_x} |u(t,y_1) - u(t,y_2)| \right\|_p^2 \le C'F^{-1}\left(C_* p h(t)\right),$$

where $C' > 0$ is another universal constant.

PROOF. Following the same lines as in the proof of [CH19, Theorem 1.8], one has

$$\|u(t,x) - u(t,y)\|_p^2 \le 4p \int_0^t \iint_{\mathbb{R}^{2d}} \mathrm{d}z f(\mathrm{d}z')\; \|\sigma(u(s,z))\|_p \|\sigma(u(s,z))\|_p \times |p_{t-s}(x-z) - p_{t-s}(y-z)||p_{t-s}(x-z+z') - p_{t-s}(y-z+z')|.$$

Thanks to (3.6) and part (iii) of Theorem 1.6, one can write

$$\sup_{s\in[0,t]} \sup_{z\in\mathbb{R}^d} \|\sigma(u(s,z))\|_p^2 \le K_M^2 + \sigma_2\left(M^2 + C_* F^{-1}(C_* p h(t))\right),$$

for all $t \geq 1$, where $C_* > 0$ is the same as in (1.21). If $F^{-1}(C_*ph(t)) < 2M^2$, one can apply part (ii) of Lemma 3.3 to see that

$$I \coloneqq \sigma_2\left(M^2 + C_*F^{-1}(C_*ph(t))\right) \leq \sigma_2\left((1+2C_*)M^2\right).$$

Otherwise, with $g_2(\cdot) \coloneqq g_2^+(\cdot) + g_2^-(\cdot)$ defined as in (3.2) and (3.3), respectively, we have,

$$\begin{aligned}
I &= \sigma_2(M^2) + \int_{M^2}^{M^2+C_*F^{-1}(C_*ph(t))} g_2(x)\mathrm{d}x \\
&\leq \sigma_2(M^2) + \frac{C_*F^{-1}(C_*ph(t))}{F^{-1}(C_*ph(t)) - M^2}\int_{M^2}^{F^{-1}(C_*ph(t))} g_2(x)\mathrm{d}x \\
&\leq (2C_*+1)\sigma_2\left(F^{-1}(C_*ph(t))\right).
\end{aligned}$$

Combining the above two cases, we have that

$$I \leq K_1 + C_1\sigma_2\left(F^{-1}(C_*ph(t))\right) = K_1 + C_1\frac{F^{-1}(C_*ph(t))}{F\circ F^{-1}(Cph(t))} \leq K_1 + C_1\frac{F^{-1}(C_*ph(t))}{C_*ph(t)}.$$

with some universal positive constants C_1 and K_1. Additionally, following the same idea as in the proof of (3.21), and noting $h(\cdot)$ is a non-decreasing function with $h(t) > 0$ for $t > 0$, we can further simplify above inequality as $I \leq C_2 p^{-1}F^{-1}\left(C_*ph(t)\right)$, where $C_2 > 0$ is a universal constant. Hence,

$$\begin{aligned}
\|u(t,x) - u(t,y)\|_p^2 \leq 4C_2F^{-1}\big(C_*ph(t)\big)\int_0^t \mathrm{d}s \iint_{\mathbb{R}^{2d}} \mathrm{d}z f(\mathrm{d}z')\ |p_{t-s}(x-z) - p_{t-s}(y-z)| \\
\times \left|p_{t-s}(x-z+z') - p_{t-s}(y-z+z')\right|.
\end{aligned}$$

The rest proof of part (i) is the same as Step 1 in the proof of [CH19, Theorem 1.8]. Part (ii) follows from a classical argument for Kolmogorov's continuity criterion. Thus, we omit it and refer interested readers to, e.g., [DKM+09, Theorem 4.3 on page 10] for more references. □

PROOF OF THEOREM 1.11. Let C_1 and C_2 be two positive constants, the values of which will be determined later. Set $Q(R) \coloneqq C_1\sqrt{F^{-1}\Big(C_2h(t)\log(R)\Big)}$, $\quad R > 0$. It is clear that for any $R > 0$ fixed, $Q(R)$ is increasing in both C_1 and C_2. Following the idea as in [CJKS13], to apply the Borel–Cantelli lemma, we need to estimate

$$\mathcal{T}_1(R) \coloneqq \mathbb{P}\left\{\max_{x\in\{y\in\mathbb{Z}^d:\ |y|\leq R\}} |u(t,x)| \geq Q(R)\right\} \quad \text{and}$$

$$\mathcal{T}_2(R) \coloneqq \mathbb{P}\left\{\max_{x\in\{y\in\mathbb{Z}^d:\ |y|\leq R\}} \sup_{y\in B_x} |u(t,y) - u(t,x)| \geq Q(R)\right\},$$

which come from the following inequality:

$$\mathbb{P}\left\{\sup_{|x|\leq R} |u(t,x)| \geq 2Q(R)\right\} \leq \mathcal{T}_1(R) + \mathcal{T}_2(R),$$

for all positive integer R such that $Q(R) \geq L_t$; see Theorem 1.8.

Let C_* be the constant in (1.21). Set $C_1 = \sqrt{C_*}e$ and $C_2 = (2+2d)C_*$. By Theorem 1.8,

$$\begin{aligned}
\mathcal{T}_1(R) \leq& \sum_{x\in\{y\in\mathbb{Z}^d:\ |y|\leq R\}} \mathbb{P}\left\{|u(t,x)| \geq Q(R)\right\} \\
\leq& (2R)^d \exp\left(-(C_*h(t))^{-1}F\left(Q(R)^2/\left(C_*e^2\right)\right)\right) = R^{-2}.
\end{aligned}$$

The estimate for $\mathcal{T}_2$ is quite similar. By using the same argument as in Theorem 1.8 and taking account of part (ii) of Lemma 3.11, one can show that with some $L_t'>0$ and the same constants C_* and C' as in part (ii) of Lemma 3.11, for all $z\ge L_t'$,

$$\mathbb{P}\left(\sup_{y_1,y_2\in B_x}|u(t,y_1)-u(t,y_2)|\ge z\right)\le \exp\left(-(C_*h(t))^{-1}F\left(z^2/(C'e^2)\right)\right).$$

Then, with $C_1=\sqrt{C'}e$ and $C_2=(2+2d)C_*$, we have $\mathcal{T}_1(R)\le R^{-2}$ as well. Therefore, with appropriate $C_1,C_2>0$ and $L_t>0$ the same as in (1.23), the next inequality holds,

$$\sum_{R=1}^{\infty}\mathbb{P}\left\{\sup_{|x|\le R}|u(t,x)|\ge 2Q(R)\right\}\le \sum_{R=1}^{\lfloor L_t\vee L_t'\rfloor}\mathbb{P}\left\{\sup_{|x|\le R}|u(t,x)|\ge 2Q(R)\right\} + \sum_{R=\lfloor L_t\vee L_t'\rfloor+1}^{\infty}(\mathcal{T}_1(R)+\mathcal{T}_2(R))<\infty.$$

An application of the first Borel–Cantelli lemma completes the proof of Theorem 1.11. □

3.6. *Proof of the one-dimensional SWE—Theorem 1.12.* The validity of Theorem 1.12 relies on the following lemmas.

LEMMA 3.12. The wave kernel function $G(\cdot,\circ)$ satisfies the following properties:

(i) For all $t,s\in\mathbb{R}_+$ and $x,y\in\mathbb{R}$,

$$G(t,x-y)G(s,y)=2G(t,x-y)G(s,y)G(t+s,x)\le G(s,y)G(t+s,x); \tag{3.30}$$

(ii) For all $t>0$ and $x\in\mathbb{R}$,

$$tG(t,x)\le G(2t,x)(2t-|x|)=2\left(G(t,\cdot)*G(t,\cdot)\right)(x)\le 2tG(2t,x); \tag{3.31}$$

(iii) If the correlation function $f:\mathbb{R}\to\mathbb{R}_+$ satisfies Hypothesis 1.1, then for all $t>0$,

$$\sup_{(s,x)\in[0,t]\times\mathbb{R}}\int_{\mathbb{R}}f(\mathrm{d}z)\,G\left(s,x+z\right)\le 2k(2t)<\infty, \tag{3.32}$$

where $k(t)$ is defined as in (3.11) with $p_t(x)$ replaced by $G(t,x)$;

LEMMA 3.13. Under the same setting as Theorem 1.12, $\mathcal{J}_1(t,x)$ defined in (1.29) is finite for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$. In particular, for every $(t,x)\in\mathbb{R}_+\times\mathbb{R}$,

$$\mathcal{J}_1(t,x)\le 4tk(2t)\left(t\int_{\mathbb{R}}G(t,x-y)\mu_0^2(y)\mathrm{d}y+2\left(t\int_{\mathbb{R}}|\mu_1|(\mathrm{d}y)G(t,x-y)\right)^2\right)<\infty,$$

where $k(t)$ is defined in (3.11) with $p_t(x)$ replaced by $G(t,x)$.

PROOF OF THEOREM 1.12. Having Lemmas 3.12 and 3.13, the poof of Theorem 1.12 is straightforward. Here, we only provide the proof for part (i), which follows a similar approach to that of Theorem 1.6, with one notable difference arising from the absence of the corresponding formula (3.17) for the heat equation case. However, we can leverage (3.30) to overcome this obstacle and conclude

$$\begin{aligned}
&\int_r^t\mathrm{d}s\iint_{\mathbb{R}^2}\mathrm{d}yf(\mathrm{d}y')\,G(t-s,x-y)G(t-s,x'-y+y')G(s-r,y)G(s-r,y-y')\\
&\le G(t-r,x)G(t-r,x')\int_0^{t-r}\mathrm{d}s\iint_{\mathbb{R}^2}\mathrm{d}yf(\mathrm{d}y')\,G(s,y)G(s,y-y')\\
&=G(t-r,x)G(t-r,x')h(t-r)\le G(t-r,x)G(t-r,x')h(t).
\end{aligned}$$

The last sequence of inequalities play the same role as those in (3.18). Following the same idea as in the proof of Theorem 1.6, we can show that

$$X \le M^2 + 2h(t)^{-1}\mathcal{J}_1(t,x) + 8K_M^2 p\, h(t) + 8ph(t)\sigma_2(X),$$

where K_M is given in (1.18) and

$$X := M^2 + h(t)^{-1}\int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} \mathrm{d}y f(\mathrm{d}y')G(t-s,x-y)G(t-s,x-y+y')\,\|u(s,y)\|_p^2 .$$

The remaining proof follows the same line of reasoning as presented in Theorem 1.6, 1.8 and 1.11, if one can show the Hölder continuity for solution(s) to (1.25) under the improved Dalang condition. In the following, we only outline the proof for spatial continuity, and the time continuity can be verified similarly. Assuming $\mu_0 \equiv 1$ and $\mu_1 \equiv 0$, we can write

$$\begin{aligned}
\|u(t,x)-u(t,y)\|_p^2 \le & \, C\int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} \mathrm{d}z f(\mathrm{d}z')|G(t-s,x-z)-G(t-s,y-z)| \\
& \times |G(t-s,x-z+z')-G(t-s,y-z+z')|\,\|\sigma(u(s,z))\|_p \left\|\sigma(u(s,z-z'))\right\|_p \\
\le & \, C\int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} \mathrm{d}z f(\mathrm{d}z')|G(t-s,x-z)-G(t-s,y-z)| \\
& \times |G(t-s,x-z+z')-G(t-s,y-z+z')|\left[1+F^{-1}\big(C_* ph(s)\big)\right]^2 .
\end{aligned}$$

Then, by using the same arguments as in [DSS05, Theorem 5], one finds that u is Hölder continuous in space with and exponent $\alpha < \eta$ and constant of the form $CF^{-1}(C_* ph(t))$. $\square$

Now, it remains to prove Lemmas 3.12 and 3.13.

PROOF OF LEMMA 3.12. Part (i) is due to the identity: for any $s,t\in\mathbb{R}_+$ and $x,y\in\mathbb{R}$,

$$1\!\!1_{[-t,t]}(x-y)1\!\!1_{[-s,s]}(y) = 1\!\!1_{[-t,t]}(x-y)1\!\!1_{[-s,s]}(y)1\!\!1_{[-(s+t),t+s]}(x).$$

Part (ii) can be obtained by direct computation. As for part (iii), by the nonnegativity of f and non-decreasing property of $t\to G(t,x)$ with $x\in\mathbb{R}$ fixed, we see that

$$\sup_{(s,x)\in[0,t]\times\mathbb{R}} \int_{\mathbb{R}} f(\mathrm{d}z)\, G\,(s,x+z) \le \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} f(\mathrm{d}z)\, G\,(t,x+z)$$

Next, thanks to the first inequality in (3.31), for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, we have that

$$\begin{aligned}
\int_{\mathbb{R}} f(\mathrm{d}z)\, G\,(t,x+z) \le & \, t^{-1}\int_{\mathbb{R}} f(\mathrm{d}z)\, G(2t,x+z)(2t-|x+z|) \\
= & \frac{2}{t}\iint_{\mathbb{R}^2} \mathrm{d}y f(\mathrm{d}z)\, G(t,x+y)G(t,y-z) = \frac{1}{\pi t}\int_{\mathbb{R}} e^{ix}\left|\frac{\sin(t\xi)}{\xi}\right|^2 \widehat{f}(\mathrm{d}\xi) \\
\le & \frac{1}{\pi t}\int_{\mathbb{R}} \left|\frac{\sin(t\xi)}{\xi}\right|^2 \widehat{f}(\mathrm{d}\xi) = \frac{2}{t}\iint_{\mathbb{R}^2} \mathrm{d}y f(\mathrm{d}y')\, G(t,y)G(t,y-y').
\end{aligned}$$

Then by the second inequality in (3.31), we see that $\int_{\mathbb{R}} f(\mathrm{d}z)\, G\,(t,x+z) \le 2k(2t)$. Notice that thanks to Dalang's condition (1.9), $\frac{1}{\pi t}\int_{\mathbb{R}}\left|\frac{\sin(t\xi)}{\xi}\right|^2 \widehat{f}(\mathrm{d}\xi) \le C_t \int_{\mathbb{R}} \frac{\widehat{f}(\mathrm{d}\xi)}{1+|\xi|^2} < \infty$ for all $t > 0$. This proves part (iii). The proof of Lemma 3.12 is complete. $\square$

PROOF OF LEMMA 3.13. We decompose $\mathcal{J}_0(t,x)$ in (1.26) into two parts:

$$\mathcal{J}_{0,1}(t,x) \coloneqq \frac{1}{2}[\mu_0(x+t)+\mu_0(x-t)] \quad \text{and} \quad \mathcal{J}_{0,2}(t,x) \coloneqq \int_{\mathbb{R}} \mu_1(\mathrm{d}y) G(t,x-y).$$

Then, using (3.32), we have

$$\begin{aligned}\mathcal{J}_{1,1}(t,x) \coloneqq& \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} \mathrm{d}y f(\mathrm{d}y')\, G(t-s,x-y)\, G\left(t-s,x-y+y'\right) \mathcal{J}_{0,1}(s,y)^2 \\ \leq& 2k(2t) \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, G(t-s,x-y)\, \mathcal{J}_{0,1}(s,y)^2 =: 2k(2t)\, \Theta_1(t).\end{aligned}$$

Notice that

$$\begin{aligned}\Theta_1(t) \leq& \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, G(t-s,x-y) \left(\mu_0^2(y+s)+\mu_0^2(y-s)\right) \\ =& \int_{\mathbb{R}} \mathrm{d}z\, \mu_0^2(z) \int_0^t \mathrm{d}s\, [G(t-s,x-z+s)+G(t-s,x-z-s)].\end{aligned}$$

Because $s-|x-z| \leq |x-z\pm s| \leq t-s$, we see that the above $\mathrm{d}s$-integral is bounded $t\mathbb{1}_{\{|x-z|\leq t\}} = 2tG(t,x-z)$. Hence, the condition $\mu_0 \in L^2_{\text{loc}}(\mathbb{R})$ implies that

$$\mathcal{J}_{1,1}(t,x) \leq 2tk(2t) \int_{x-t}^{x+t} \mu_0^2(z)\mathrm{d}z = 4t\, k(2t) \int_{\mathbb{R}} \mathrm{d}z\, G(t,x-z)\mu_0^2(z) < \infty.$$

It remains to show that

$$\mathcal{J}_{1,2}(t,x) \coloneqq \int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}y f(\mathrm{d}y')\, G(t-s,x-y)\, G\left(t-s,x-y+y'\right) \mathcal{J}_{0,2}(s,y)^2 < \infty.$$

Thanks to (3.32), we see that

$$\mathcal{J}_{1,2}(t,x) \leq 2k(2t) \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\, G(t-s,x-y)\, \mathcal{J}_{0,2}(s,y)^2 =: 2k(2t)\, \Theta_2(t).$$

By the Minkowski inequality with respect to the $\mathrm{d}y$-integral, we can write

$$\begin{aligned}\Theta_2(t) =& 2\int_{\mathbb{R}} [G(t-s,x-y)\mathcal{J}_{0,2}(s,y)]^2\, \mathrm{d}y \\ \leq& 2\left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} G(t-s,x-y)G(s,y-z)^2 \mathrm{d}y\right)^{1/2} |\mu_1|(\mathrm{d}z)\right]^2.\end{aligned}$$

Due to (3.30), we see that $\int_{\mathbb{R}} G(t-s,x-y)G(s,y-z)^2\mathrm{d}y \leq \frac{s}{4}\mathbb{1}_{[-t,t]}(x-z)$. Therefore, $\Theta_2(t) \leq \frac{s}{2}\left(\int_{x-t}^{x+t}|\mu_1|(\mathrm{d}z)\right)^2$ and

$$\mathcal{J}_{1,2}(t,x) \leq 2t^2\, k(2t) \left(\int_{x-t}^{x+t} |\mu_1|(\mathrm{d}z)\right)^2 = 8t^2\, k(2t) \left(\int_{\mathbb{R}} |\mu_1|(\mathrm{d}z) G(t,x-z)\right)^2 < \infty.$$

This completes the proof of Lemma 3.13. □

3.7. *Proofs of technical lemmas.*

PROOF OF LEMMA 3.1. Since $|\sigma|$ is concave on $[M_0,\infty)$, we can find a non-increasing and right-continuous function g^+ on $[M_0,\infty)$ such that $|\sigma(x)|-|\sigma(M_0)| = \int_{M_0}^x g^+(y)\mathrm{d}y$ holds for all $x\in[M_0,\infty)$; cf. [NP18, Theorems 1.4.2 and 1.5.2]. The properties of g^+ follows from simple exercises in calculus. The case for g^- can be proved similarly. □

PROOF OF LEMMA 3.3. The representations in both (3.2) and (3.3) are direct consequences of the definitions of $\sigma_p^{\pm}$ in (3.1) and Lemma 3.1. Part (ii) is an immediate consequence of part (i). Now we prove part (iii). It suffices to show the case for $\sigma_p^+(\cdot)$ since the case for $\sigma_p^-(\cdot)$ can be proved in the same way and the case for σ_p follows from those two cases. Let g^+ be given in (3.2). We need to show that g_p^+ is non-increasing on $[M^p,\infty)$ with some $M\geq M_0$. We know that g^+ is non-increasing on (M_0,∞), it suffices to show that $\varphi(x)=\frac{|\sigma(x)|}{x}$ is non-increasing for x large enough. To show this property, we write

$$|\sigma(x)| = \int_{M_0}^{x} g^+(y)dy + |\sigma(M_0)|.$$

Thus for almost every $x\in(M_0,\infty)$,

$$\begin{aligned}\varphi'(x) =& x^{-2}\left[g^+(x)\,(x-M_0) - \int_{M_0}^{x} g^+(y)\mathrm{d}y + g^+(x)M_0 - |\sigma(M_0)|\right]\\ \leq& g^+(x)M_0 - |\sigma(M_0)|,\end{aligned}$$

where the last inequality follows from the fact that g^+ is non-increasing on (M_0,∞). Since $\lim_{x\uparrow\infty} g^+(x)=0$ (see Lemma 3.1), we conclude that $g^+(x)M_0-|\sigma(M_0)|<0$ for x large enough. In other words, there exists $M\geq M_0$, such that φ is non-increasing on $[M,\infty)$. Therefore, σ_p^+ is concave on $[M^p,\infty)$. For the same reason, we can show that σ_p^-, and thus σ_p, are also concave on $[M^p,\infty)$ with a possibly different $M\geq M_0$. This proves part (ii). Finally, when $M_0=0$, then $g^+(x)M_0-|\sigma(M_0)|=-|\sigma(0)|\leq 0$ for all $x\geq 0$. Hence, part (iv) follows. This completes the proof of Lemma 3.3. □

PROOF OF LEMMA 3.4. In the following, we will prove (3.4) only. The proof of (3.5) is similar. Let $p>0$ and $U\in L^p(\Omega)$. Set $\alpha:=\mathbb{P}(U\geq M)$. It is clear that when $\alpha=0$, the inequality (3.4) is trivially true. So, we may assume that $\alpha>0$. Since $\sigma_p^+(\cdot)$ is concave on $[M^p,\infty)$ (see Lemma 3.3), we can apply Jensen's inequality to see that

$$\mathbb{E}\left[\sigma_p^+(|U|^p)\mathbb{1}_{[M,\infty)}(U)\right] \leq \alpha\sigma_p^+\left(\frac{1}{\alpha}\mathbb{E}\left[|U|^p\mathbb{1}_{[M,\infty)}(U)\right]\right) \leq \alpha\sigma_p^+\left(\frac{1}{\alpha}\mathbb{E}\left[|U|^p\right]\right),$$

where the last inequality is due to the monotonicity of $\sigma_p^+(\cdot)$; see Lemma 3.3. Denote $x=\mathbb{E}[|U|^p]$. Since $x/\alpha\geq M^p$, by the monotonicity and the concavity of $\sigma_p^+(\cdot)$,

$$\begin{aligned}\sigma_p^+(\alpha^{-1}x)\leq \sigma_p^+(M^p+\alpha^{-1}x) \leq& \sigma_p^+(M^p)+\alpha^{-1}\left(\sigma_p^+(M^p+x)-\sigma_p^+(M^p)\right)\\ \leq& \alpha^{-1}\sigma_p^+(M^p+x).\end{aligned}$$

This implies that

$$\mathbb{E}\left[\sigma_p^+(|U|^p)\mathbb{1}_{[M,\infty)}(U)\right] \leq \sigma_p^+\left(M^p+\mathbb{E}\left[|U|^p\right]\right) = \sigma_p^+\left(M^p+\|U\|_p^p\right),$$

which completes the proof of Lemma 3.4. □

PROOF OF LEMMA 3.5. Fix an arbitrary $p\geq 2$. By the subadditivity of $\theta_{2/p}(\cdot)$,

$$\begin{aligned}\|\sigma(U)\|_p^2 \leq& \left(\mathbb{E}\left[|\sigma(U)|^p 1_{\{|U|\leq M\}}\right]\right)^{2/p} + \left(\mathbb{E}\left[|\sigma(U)|^p 1_{\{u\geq M\}}\right]\right)^{2/p}\\ &+\left(\mathbb{E}\left[|\sigma(U)|^p 1_{\{u\leq -M\}}\right]\right)^{2/p}\\ \leq& K_M^2 + \left(\theta_{2/p}\circ\mathbb{E}_{\geq}\circ\theta_p\circ|\sigma|\right)(U) + \left(\theta_{2/p}\circ\mathbb{E}_{\leq}\circ\theta_p\circ|\sigma|\right)(U)\\ =& K_M^2 + \left(\theta_{2/p}\circ\mathbb{E}_{\geq}\circ\sigma_p^+\circ\theta_p\right)(U) + \left(\theta_{2/p}\circ\mathbb{E}_{\leq}\circ\sigma_p^-\circ\theta_p\right)(U),\end{aligned}$$

where $\mathbb{E}_{\geq}$ is the expectation given $\{U \geq M\}$, i.e., $\mathbb{E}_{\geq}(\phi(U)) = \mathbb{E}\left(\phi(U)1_{\{U\geq M\}}\right)$ for any measurable function ϕ; and $\mathbb{E}_{\leq}$ is defined similarly. Set $y := M^p + \|U\|_p^p$. By Lemma 3.4,

$$\begin{aligned}\|\sigma(U)\|_p^2 \leq & K_M^2 + \left(\theta_{2/p}\circ\sigma_p^+\right)(y) + \left(\theta_{2/p}\circ\sigma_p^-\right)(y)\\ = & K_M^2 + \left(\sigma_2^+\circ\theta_{2/p}\right)(y) + \left(\sigma_2^-\circ\theta_{2/p}\right)(y)\\ \leq & K_M^2 + \sigma_2^+\left(M^2 + \|U\|_p^2\right) + \sigma_2^-\left(M^2+\|U\|_p^2\right),\end{aligned}$$

where the last step is due to the subadditivity of $\theta_{2/p}(\cdot)$ and the monotonicity of $\sigma_2^{\pm}(\cdot)$. □

PROOF OF LEMMA 3.7. The finiteness of $F^{-1}(k) + b$ is a consequence of part (ii) of Hypothesis 1.2. Thus it suffices to show the first inequality in (3.9). To this end, we first prove it by verifying $x \leq \gamma_0(k) + 2b$, with

$$\gamma_0(k) := \inf\left\{x\in(M^2,\infty)\colon \frac{\sigma_2(x)}{x}\leq\frac{1}{k} \text{ and } g_2(x)\leq\frac{1}{2k}\right\}.$$

Here, $g_2(\cdot) := g_2^+(\cdot) + g_2^-(\cdot)$ with $g_2^+(\cdot)$ and $g_2^-(\cdot)$ defined in (3.2) and (3.3), respectively. Note that $\gamma_0(k) \geq M^2$ and $\sigma_2(\cdot)$ is concave on $[M^2,\infty)$. Thus, if $x \geq \gamma_0(k) + 2b$, we have

$$\begin{aligned}k\sigma_2(x) + b = & k\sigma_2(\gamma_0(k)) + k\int_{\gamma_0(k)}^x g_2(y)\mathrm{d}y + b \leq \gamma_0(k) + \frac{1}{2}\left(x-\gamma_0(k)\right) + b\\ \leq & \gamma_0(k) + \frac{1}{2}\left(x-\gamma_0(k)\right) + \frac{1}{2}(x-\gamma_0(k)) = x.\end{aligned}$$

In the next step, we can show that

$$\gamma_0(k) \leq 2F^{-1}(k), \qquad \text{for all } k>0. \tag{3.33}$$

First if $M=0$, then

$$\frac{2}{F(x)} = \frac{x}{2\sigma_2(x)} \leq \frac{x}{\sigma_2(x)} = x\left(\int_0^x \mathrm{d}y g_2(y)\right)^{-1} \leq \frac{1}{g_2(x)},$$

by the non-increasing property of $g_2(\cdot)$, and (3.33) follows immediately from the definition of F^{-1}; see (1.16). On the other hand, if $M>0$, for any $x \geq 2F^{-1}(k) \geq 3F^{-1}(k)/2$,

$$g_2(x) \leq \frac{\sigma_2(x) - \sigma_2(M^2)}{x - M^2} \leq \frac{\sigma_2(3F^{-1}(k)/2) - \sigma_2(M^2)}{3F^{-1}(k)/2 - M^2}.$$

Notice that $F^{-1}(k) \geq 2M^2$ implies $3F^{-1}(k)/2 - M^2 \geq F^{-1}(k)$ and

$$\sigma_2(3F^{-1}(k)/2) - \sigma_2(M^2) \leq \frac{3F^{-1}(k)/2 - M^2}{F^{-1}(k) - M^2}\int_{M^2}^{F^{-1}(k)} \mathrm{d}y g_2(y) \leq 2\sigma_2\left(F^{-1}(k)\right).$$

Combining above inequalities and (3.7), we get

$$g_2(x) \leq \frac{2\sigma_2\left(F^{-1}(k)\right)}{F^{-1}(k)} = \frac{1}{2F\circ F^{-1}(k)} \leq \frac{1}{2k}. \tag{3.34}$$

This implies that (3.33), and thus completes the proof of Lemma 3.7. □

PROOF OF LEMMA 3.8. Without loss generality, we assume that μ is nonnegative. Otherwise, one simply replaces μ by $|\mu|$. Since f is a nonnegative-definite tempered measure, and the heat kernel is rapidly decreasing at infinity, by the Plancherel theorem, we can write

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathrm{d}y)\, p_t(x-y) = \sup_{x\in\mathbb{R}^d}(2\pi)^{-d}\int_{\mathbb{R}^d}\mathrm{d}\xi\, e^{-ix\cdot\xi - \frac{t|\xi|^2}{2}}\widehat{f}(\xi) = k(t),$$

from which one proves (3.10). As for (3.12), we proceed with a general $d \geq 1$ and make the restriction to $d=1$ when some integrability issue comes up. By the Cauchy-Schwarz inequality with respect to the $\mathrm{d}y$ integral, we see that

$$\int_{\mathbb{R}^d} \mathrm{d}y\, p_{t-s}(x-y)\mathcal{J}_0^2(s,y) \leq \left[\int_{\mathbb{R}^d} |\mu|(\mathrm{d}z) \left(\int_{\mathbb{R}^d} \mathrm{d}y\, p_{t-s}(x-y)p_s(y-z)^2\right)^{1/2}\right]^2.$$

Because $p_t^2(x) = (4\pi t)^{-d/2}p_{t/2}(x) \leq (2\pi t)^{-d/2}p_t(x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we can write

$$(2\pi s)^{d/2}\int_{\mathbb{R}^d} \mathrm{d}y\, p_{t-s}(x-y)p_s(y-z)^2 \leq \int_{\mathbb{R}^d} \mathrm{d}y\, p_{t-s}(x-y)p_s(y-z) = p_t(x-z),$$

which implies that

$$\begin{aligned}\mathcal{J}_1(t,x) \leq& (2t)^{d/2}\int_0^t \mathrm{d}s\, s^{-d/2}k(t-s)\left(\int_{\mathbb{R}^d} |\mu|(\mathrm{d}z)p_{t/2}(x-z)\right)^2\\ =&(2t)^{d/2}\mathcal{J}_+\left(t/2,x\right)^2\int_0^t \mathrm{d}s\, s^{-d/2}\,k(t-s).\end{aligned}$$

Notice that the integral against the time argument in the last expression is finite for all $t>0$, if and only if $d=1$. Moreover, in case $d=1$, we can deduce that

$$\int_0^t \mathrm{d}s\, s^{-1/2}k(t-s) \leq (2\pi)^{-1/2}\int_0^t \mathrm{d}s\, s^{-1/2}(t-s)^{-1/2}\int_{\mathbb{R}} f(\mathrm{d}z)\, e^{-\frac{x^2}{2t}} = \pi t^{1/2}k(t).$$

Therefore, $\mathcal{J}_1(t,x) \leq \sqrt{2}\pi\, t\, k(t)\, \mathcal{J}_+\left(t/2,x\right)^2 < \infty$. Finally, notice that $k(t)$ is a nonincreasing function of t. From (1.10), $2h(t) \geq 2h(t/2) = \int_0^t \mathrm{d}s\, k(s) \geq \int_0^t \mathrm{d}s\, k(t) = tk(t)$. Plugging this inequality to the above inequality for $\mathcal{J}_1$ verifies (3.12). ☐

4. Examples. In this section, we present some examples for the moment bounds with various sublinear diffusion coefficient σ in examples (1.4)–(1.6). The proofs are direct applications of Theorems 1.6, 1.8, 1.11, and 1.12, and are omitted for conciseness.

4.1. *F and F^{-1} for various σ.* In this subsection, we present explicit expressions for $F(\cdot)$ and its inverse $F^{-1}(\cdot)$, both depending solely on σ, as summarized in Figure 4.1.

PROPOSITION 4.1. Let $\sigma(u) = \frac{|u|}{(r+|u|)^{1-\alpha}}$ for $u \in \mathbb{R}$, with $r \geq 0$ and $\alpha \in [0,1)$. Then, σ satisfies Hypothesis 1.2 with $M_0 = 0$; and F and F^{-1} take the following forms:

$$\begin{aligned}F(u) &= \frac{1}{8}\left(r+u^{1/2}\right)^{2(1-\alpha)}, \quad \text{for all } u \geq 0, \quad \text{and}\\ F^{-1}(x) &= \left((8x)^{1/(2(1-\alpha))} - r\right)^2 \mathbb{1}_{\{x \geq 8^{-1}r^{2(1-\alpha)}\}}.\end{aligned}$$

In particular, $F^{-1}(x) \leq (8x)^{1/(1-\alpha)}$ for all $x \geq 0$.

The diffusion term σ in the following examples does not exhibit the global concavity, which motivates us to propose the asymptotic concave condition in Hypothesis 1.2.

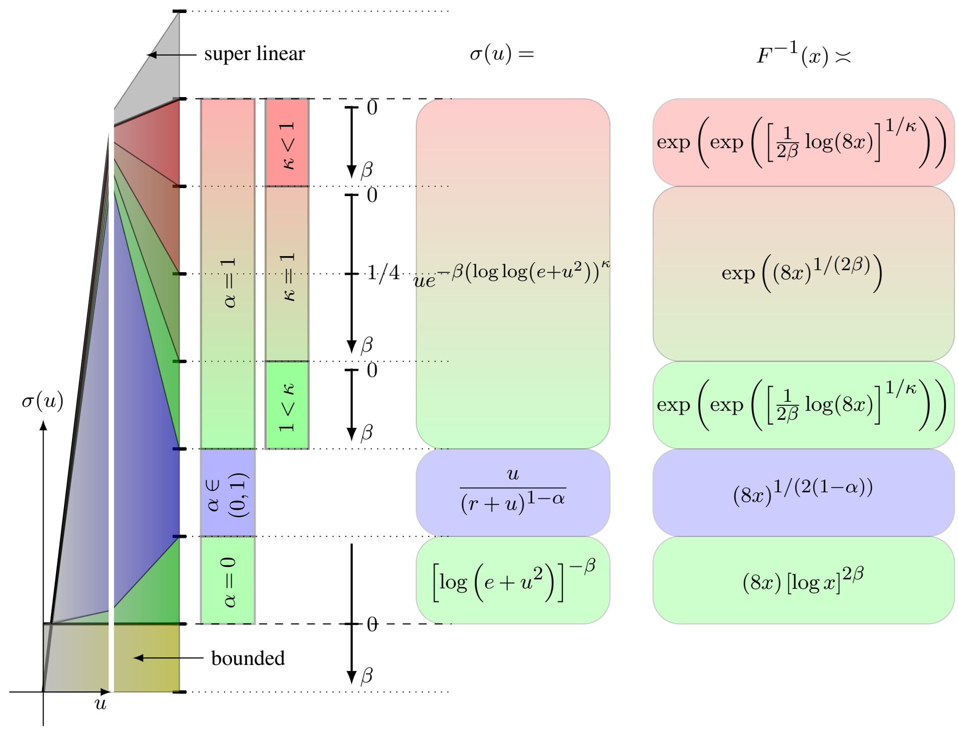

Fig 4.1: Summary of the asymptotics of F^{-1} for σ in Propositions 4.1, 4.2 and 4.3.

PROPOSITION 4.2. Suppose that $\sigma(u) = |u|^\alpha \left[\log\left(e+u^2\right)\right]^{-\beta}$, with α, β in one of the cases in (1.5). Then, σ satisfies Hypothesis 1.2 with $M_0 > 0$. Moreover,

$$F(u) = \frac{1}{8} u^{1-\alpha} \left[\log(e+u)\right]^{2\beta}, \quad u > 0; \quad \text{and}$$

$$\begin{cases} F^{-1}(x) \asymp (8x)^{1/(1-\alpha)} \left[\dfrac{\log(e+x)}{1-\alpha}\right]^{2\beta/(1-\alpha)}, & x \to \infty, \quad \text{Cases (i) and (ii)}, \\ F^{-1}(x) = \left(\exp\left(2^{3/(2\beta)} x^{1/(2\beta)}\right) - e\right) \mathbb{1}_{\{x > 8^{-1}\}}, & \text{Case (iii)}. \end{cases}$$

In particular, in Cases (i) and (ii), $F^{-1}(x) \lesssim \left[x\,(\log x)^{-2\beta}\right]^{1/(1-\alpha)}$ as $x \to \infty$.

PROPOSITION 4.3. The diffusion coefficient $\sigma(u) = |u| \exp\left(-\beta\Big(\log\log\left(e+u^2\right)\Big)^\kappa\right)$ with $\kappa > 0$ and $\beta > 0$ satisfies Hypothesis 1.2 with some $M_0 > 0$. Moreover, in this case,

$$F(u) = \frac{1}{8}\exp\left(2\beta\left(\log\log(e+u)\right)^\kappa\right), \quad \text{for all } u > 0, \quad \text{and}$$

$$F^{-1}(x) = \left(\exp\left\{\exp\left(\left((2\beta)^{-1}\log(8x)\right)^{1/\kappa}\right)\right\} - e\right) \mathbb{1}_{\{x > 1/8\}}.$$

4.2. *Moment growth, tail probability and spatial asymptotics for SHE.* In this part, we showcase the moment growth, tail probability, and spatial asymptotics for solutions for SHEs as applications of Theorems 1.6, 1.8, and 1.11. For conciseness, we focus the case when σ is defined in Proposition 4.1. The results for other scenarios are summarized in Figure 4.2.

PROPOSITION 4.4. Let σ be given in Proposition 4.1, and let u be a solution to SHE (1.1) under the same setting as Theorem 1.6, but with the diffusion coefficient σ given in Proposition 4.1. Then, for all $(p,t,x)\in[2,\infty)\times\mathbb{R}_+\times\mathbb{R}^d$, it holds that

$$\|u(t,x)\|_p^2\le C\left[\mathcal{J}_0^2(t,x)+ph(t)+(ph(t))^{1/(1-\alpha)}+h(t)^{-1}\mathcal{J}_1(t,x)\right].$$

Furthermore, we have the following three cases:

(i) If $d=1$ and $f=\delta$, then

$$\|u(t,x)\|_p^2\le C\left(\mathcal{J}_0^2(t,x)+p\sqrt{t}+\left(p\sqrt{t}\right)^{1/(1-\alpha)}+\mathcal{J}_+^2(t/2,x)\right). \tag{4.1}$$

Moreover, if $\mu(\mathrm{d}x)=u_0(x)\mathrm{d}x$ with $u_0\in L^\infty(\mathbb{R})$, then $\|u(t,x)\|_p^2\le C\left(p\sqrt{t}\right)^{1/(1-\alpha)}$, $t\ge 1$.

(ii) If $d=1$ and $f(x)=|x|^{-\beta}$ with $\beta\in(0,1)$, then

$$\|u(t,x)\|_p^2\le C\Big(\mathcal{J}_0^2(t,x)+p\,t^{1-\beta/2}+\left(p\,t^{1-\beta/2}\right)^{1/(1-\alpha)}+\mathcal{J}_+^2(t/2,x)\Big).$$

If $\mu(\mathrm{d}x)=u_0(x)\mathrm{d}x$ with $u_0\in L^\infty(\mathbb{R}^d)$, then $\|u(t,x)\|_p^2\le C\left(p\,t^{1-\beta/2}\right)^{1/(1-\alpha)}$, $t\ge 1$.

(iii) If $f=|x|^{-\beta}$ with $\beta\in(0,2\wedge d)$, then for all $d\ge 1$ it holds that

$$\|u(t,x)\|_p^2\le C\left(\mathcal{J}_0^2(t,x)+p\,t^{1-\beta/2}+\left(p\,t^{1-\beta/2}\right)^{1/(1-\alpha)}+t^{-1+\beta/2}\,\mathcal{J}_1(t,x)\right). \tag{4.2}$$

Moreover, (a) if $\mu(x)=|x|^{-\ell}$ with $\ell\in(0,2\wedge d)$, then we have

$$\|u(t,x)\|_p^2\le C\left(t^{-\ell}+\left(p\,t^{1-\beta/2}\right)^{1/(1-\alpha)}+p\,t^{1-\beta/2}\right); \tag{4.3}$$

(b) if $\mu(x)=e^{\ell|x|}$ with $\ell\in\mathbb{R}$, then with universal constants C_1 and $C_2>0$,

$$\|u(t,x)\|_p^2\le C\left(e^{C_1\ell^2t+C_2\ell|x|}+p\,t^{1-\beta/2}+\left(p\,t^{1-\beta/2}\right)^{1/(1-\alpha)}\right). \tag{4.4}$$

4.3. *Results for SWEs.* Let u be a solution to SWE (1.25) with space-time white noise. Assume the diffusion coefficient σ given in Proposition 4.1. Then Theorem 1.12 implies that

$$\|u(t,x)\|_p^2\le C\left(\mathcal{J}_0^2(t,x)+t^{-2}\mathcal{J}_1(t,x)+p\,t^2+\left(p\,t^2\right)^{1/(1-\alpha)}\right). \tag{4.5}$$

Furthermore, with the initial condition $\mu_0\equiv 1$ and $\mu_1\equiv 0$,

$$\log\mathbb{P}\left(|u(t,x)|\ge z\right)\lesssim -t^{-2}z^{2(1-\alpha)},\quad \text{as } z\to\infty \quad\text{and}$$

$$\sup_{|x|\le R}u(t,x)\lesssim\left(t^2\log R\right)^{1/(2(1-\alpha))},\quad \text{a.s., as } R\to\infty.$$

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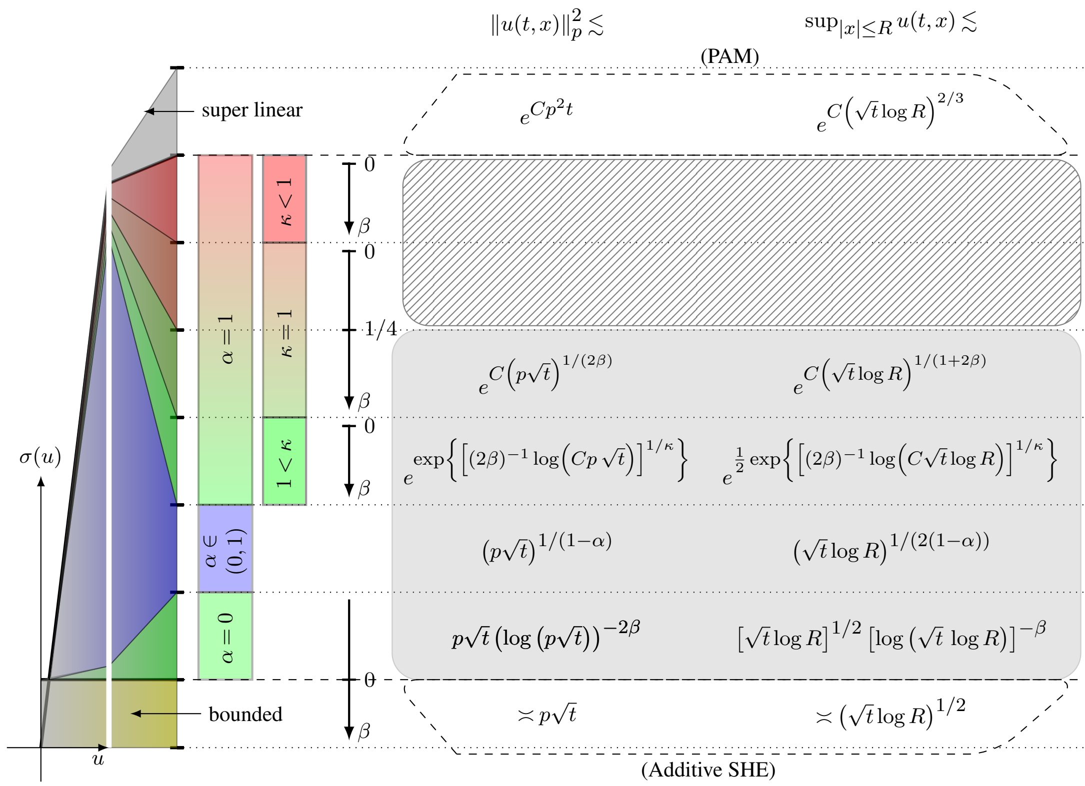

Fig 4.2: Summary of moments bounds and spatial asymptotics of various σ's given in Propositions 4.1–4.3 in case of $d = 1$, space-time white noise, and bounded initial conditions.

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REFERENCES

[ACQ11] Gideon Amir, Ivan Corwin, and Jeremy Quastel, *Probability distribution of the free energy of the continuum directed random polymer in* $1+1$ *dimensions*, Comm. Pure Appl. Math. **64** (2011), no. 4, 466–537. MR2796514

[BC16] Raluca M. Balan and Daniel Conus, *Intermittency for the wave and heat equations with fractional noise in time*, Ann. Probab. **44** (2016), no. 2, 1488–1534. MR3474476

[Bih56] I. Bihari, *A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations*, Acta Math. Acad. Sci. Hungar. **7** (1956), 81–94. MR79154

[CD15a] Le Chen and Robert C. Dalang, *Moment bounds and asymptotics for the stochastic wave equation*, Stochastic Process. Appl. **125** (2015), no. 4, 1605–1628. MR3310358

[CD15b] ———, *Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions*, Ann. Probab. **43** (2015), no. 6, 3006–3051. MR3433576

[CD15c] ———, *Moments, intermittency and growth indices for the nonlinear fractional stochastic heat equation*, Stoch. Partial Differ. Equ. Anal. Comput. **3** (2015), no. 3, 360–397. MR3383450

[CE24] Le Chen and Nicholas Eisenberg, *Invariant Measures for the Nonlinear Stochastic Heat Equation with No Drift Term*, J. Theoret. Probab. **37** (2024), no. 2, 1357–1396. MR4751295

[CFHS23] Le Chen, Mohammud Foondun, Jingyu Huang, and Michael Salins, *Global solution for superlinear stochastic heat equation on* $\mathbb{R}^d$ *under Osgood-type conditions*, preprint arXiv:2310.02153 (2023).

[CGS22] Le Chen, Yuhui Guo, and Jian Song, *Moments and asymptotics for a class of spdes with space-time white noise*, preprint arXiv:2206.10069, to appear in Trans. Amer. Math. Soc. (2022).

[CH19] Le Chen and Jingyu Huang, *Comparison principle for stochastic heat equation on* $\mathbb{R}^d$, Ann. Probab. **47** (2019), no. 2, 989–1035. MR3916940

[CH23] ______, *Superlinear stochastic heat equation on* $\mathbb{R}^d$, Proc. Amer. Math. Soc. **151** (2023), no. 9, 4063–4078. MR4607649

[Che15a] Xia Chen, *Precise intermittency for the parabolic Anderson equation with an* $(1+1)$*-dimensional time-space white noise*, Ann. Inst. Henri Poincaré Probab. Stat. **51** (2015), no. 4, 1486–1499. MR3414455

[Che15b] Yu-Ting Chen, *Pathwise nonuniqueness for the SPDEs of some super-Brownian motions with immigration*, Ann. Probab. **43** (2015), no. 6, 3359–3467. MR3433584

[CJK13] Daniel Conus, Mathew Joseph, and Davar Khoshnevisan, *On the chaotic character of the stochastic heat equation, before the onset of intermitttency*, Ann. Probab. **41** (2013), no. 3B, 2225–2260. MR3098071

[CJKS13] Daniel Conus, Mathew Joseph, Davar Khoshnevisan, and Shang-Yuan Shiu, *On the chaotic character of the stochastic heat equation, II*, Probab. Theory Related Fields **156** (2013), no. 3-4, 483–533. MR3078278

[CJKS14] ______, *Initial measures for the stochastic heat equation*, Ann. Inst. Henri Poincaré Probab. Stat. **50** (2014), no. 1, 136–153. MR3161526

[CK12] Daniel Conus and Davar Khoshnevisan, *On the existence and position of the farthest peaks of a family of stochastic heat and wave equations*, Probab. Theory Related Fields **152** (2012), no. 3-4, 681–701. MR2892959

[CK19] Le Chen and Kunwoo Kim, *Nonlinear stochastic heat equation driven by spatially colored noise: moments and intermittency*, Acta Math. Sci. Ser. B (Engl. Ed.) **39** (2019), no. 3, 645–668. MR4066498

[CK20] ______, *Stochastic comparisons for stochastic heat equation*, Electron. J. Probab. **25** (2020), Paper No. 140, 38. MR4186259

[CM94] René A. Carmona and S. A. Molchanov, *Parabolic Anderson problem and intermittency*, Mem. Amer. Math. Soc. **108** (1994), no. 518, viii+125. MR1185878

[Cor12] Ivan Corwin, *The Kardar-Parisi-Zhang equation and universality class*, Random Matrices Theory Appl. **1** (2012), no. 1, 1130001, 76. MR2930377

[Dal99] Robert C. Dalang, *Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s*, Electron. J. Probab. **4** (1999), no. 6, 29. MR1684157

[Daw93] Donald A. Dawson, *Measure-valued Markov processes*, École d'Été de Probabilités de Saint-Flour XXI—1991, Lecture Notes in Math., vol. 1541, Springer, Berlin, 1993, pp. 1–260. MR1242575

[DIP89] D. A. Dawson, I. Iscoe, and E. A. Perkins, *Super-Brownian motion: path properties and hitting probabilities*, Probab. Theory Related Fields **83** (1989), no. 1-2, 135–205. MR1012498

[DKM+09] Robert Dalang, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao, *A minicourse on stochastic partial differential equations*, Lecture Notes in Mathematics, vol. 1962, Springer-Verlag, Berlin, 2009, Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006, Edited by Khoshnevisan and Firas Rassoul-Agha. MR1500166

[DKZ19] Robert C. Dalang, Davar Khoshnevisan, and Tusheng Zhang, *Global solutions to stochastic reaction-diffusion equations with super-linear drift and multiplicative noise*, Ann. Probab. **47** (2019), no. 1, 519–559. MR3909975

[DM09] Robert C. Dalang and Carl Mueller, *Intermittency properties in a hyperbolic Anderson problem*, Ann. Inst. Henri Poincaré Probab. Stat. **45** (2009), no. 4, 1150–1164. MR2572169

[DSS05] Robert C. Dalang and Marta Sanz-Solé, *Regularity of the sample paths of a class of second-order spde's*, J. Funct. Anal. **227** (2005), no. 2, 304–337. MR2168077

[Eth00] Alison M. Etheridge, *An introduction to superprocesses*, University Lecture Series, vol. 20, American Mathematical Society, Providence, RI, 2000. MR1779100

[FK09] Mohammud Foondun and Davar Khoshnevisan, *Intermittence and nonlinear parabolic stochastic partial differential equations*, Electron. J. Probab. **14** (2009), no. 21, 548–568. MR2480553

[FZ05] Shizan Fang and Tusheng Zhang, *A study of a class of stochastic differential equations with non-Lipschitzian coefficients*, Probab. Theory Related Fields **132** (2005), no. 3, 356–390. MR2197106

[HLN17] Jingyu Huang, Khoa Lê, and David Nualart, *Large time asymptotics for the parabolic Anderson model driven by space and time correlated noise*, Stoch. Partial Differ. Equ. Anal. Comput. **5** (2017), no. 4, 614–651. MR3736656

[HW21] Yaozhong Hu and Xiong Wang, *Intermittency properties for a large class of stochastic pdes driven by fractional space-time noises*, preprint arXiv:2109.03473, to appear in Stoch. Partial Differ. Equ. Anal. Comput. (2021).

[HWXZ23] Yaozhong Hu, Xiong Wang, Panqiu Xia, and Jiayu Zheng, *Moment asymptotics for super-brownian motions*, preprint arXiv:2303.12994 (2023).

[JKM17] Mathew Joseph, Davar Khoshnevisan, and Carl Mueller, *Strong invariance and noise-comparison principles for some parabolic stochastic PDEs*, Ann. Probab. **45** (2017), no. 1, 377–403. MR3601652

[Kal02] Olav Kallenberg, *Foundations of modern probability*, second ed., Probability and its Applications (New York), Springer-Verlag, New York, 2002. MR1876169

[KKX17] Davar Khoshnevisan, Kunwoo Kim, and Yimin Xiao, *Intermittency and multifractality: a case study via parabolic stochastic PDEs*, Ann. Probab. **45** (2017), no. 6A, 3697–3751. MR3729613

[Kön16] Wolfgang König, *The parabolic Anderson model*, Pathways in Mathematics, Birkhäuser/Springer, [Cham], 2016, Random walk in random potential. MR3526112

[LaS49] J. LaSalle, *Uniqueness theorems and successive approximations*, Ann. of Math. (2) **50** (1949), 722–730. MR31165

[MMP14] Carl Mueller, Leonid Mytnik, and Edwin Perkins, *Nonuniqueness for a parabolic SPDE with $\frac{3}{4}-\epsilon$-Hölder diffusion coefficients*, Ann. Probab. **42** (2014), no. 5, 2032–2112. MR3262498

[MMQ11] Carl Mueller, Leonid Mytnik, and Jeremy Quastel, *Effect of noise on front propagation in reaction-diffusion equations of KPP type*, Invent. Math. **184** (2011), no. 2, 405–453. MR2793860

[MMR21] Carl Mueller, Leonid Mytnik, and Lenya Ryzhik, *The speed of a random front for stochastic reaction-diffusion equations with strong noise*, Comm. Math. Phys. **384** (2021), no. 2, 699–732. MR4259374

[MP92] Carl Mueller and Edwin A. Perkins, *The compact support property for solutions to the heat equation with noise*, Probab. Theory Related Fields **93** (1992), no. 3, 325–358. MR1180704

[MP11] Leonid Mytnik and Edwin Perkins, *Pathwise uniqueness for stochastic heat equations with Hölder continuous coefficients: the white noise case*, Probab. Theory Related Fields **149** (2011), no. 1-2, 1–96. MR2773025

[MPS06] Leonid Mytnik, Edwin Perkins, and Anja Sturm, *On pathwise uniqueness for stochastic heat equations with non-Lipschitz coefficients*, Ann. Probab. **34** (2006), no. 5, 1910–1959. MR2271487

[Myt98] Leonid Mytnik, *Weak uniqueness for the heat equation with noise*, Ann. Probab. **26** (1998), no. 3, 968–984. MR1634410

[NP18] Constantin P. Niculescu and Lars-Erik Persson, *Convex functions and their applications*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, Cham, 2018, A contemporary approach, Second edition of [MR2178902]. MR3821518

[Per02] Edwin Perkins, *Dawson-Watanabe superprocesses and measure-valued diffusions*, Lectures on probability theory and statistics (Saint-Flour, 1999), Lecture Notes in Math., vol. 1781, Springer, Berlin, 2002, pp. 125–324. MR1915445

[Sal22] Michael Salins, *Global solutions for the stochastic reaction-diffusion equation with super-linear multiplicative noise and strong dissipativity*, Electron. J. Probab. **27** (2022), Paper No. 12, 17. MR4372099

[Sow92] Richard B. Sowers, *Large deviations for a reaction-diffusion equation with non-Gaussian perturbations*, Ann. Probab. **20** (1992), no. 1, 504–537. MR1143433

[SSS00] Marta Sanz-Solé and Mònica Sarrà, *Path properties of a class of Gaussian processes with applications to spde's*, Stochastic processes, physics and geometry: new interplays, I (Leipzig, 1999), CMS Conf. Proc., vol. 28, Amer. Math. Soc., Providence, RI, 2000, pp. 303–316. MR1803395

[Wal86] John B. Walsh, *An introduction to stochastic partial differential equations*, École d'été de probabilités de Saint-Flour, XIV—1984, Lecture Notes in Math., vol. 1180, Springer, Berlin, 1986, pp. 265–439. MR876085

[ZMRS87] Ya. B. Zel′dovich, S. A. Molchanov, A. A. Ruzmauikin, and D. D. Sokoloff, *Self-excitation of a nonlinear scalar field in a random medium*, Proc. Nat. Acad. Sci. U.S.A. **84** (1987), no. 18, 6323–6325. MR907831

[ZRS90] Ya. B. Zel′dovich, A. A. Ruzmauikin, and D. D. Sokoloff, *The almighty chance*, World Scientific Lecture Notes in Physics, vol. 20, World Scientific Publishing Co., Inc., River Edge, NJ, 1990, Translated from the Russian by Anvar Shukurov. MR1141627