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High-Level Moving Excursions for Spatiotemporal Gaussian Random Fields with Long Range Dependence

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Abstract
The asymptotic behavior of an extended family of integral geometric random functionals, including spatiotemporal Minkowski functionals under moving levels, is analyzed in this paper. Specifically, sojourn measures of spatiotemporal long-range dependence (LRD) Gaussian random fields are considered in this analysis. The limit results derived provide general reduction principles under increasing domain asymptotics in space and time. The case of time-varying thresholds is also studied. Thus, the family of morphological measures considered allows the statistical and geometrical analysis of random physical systems displaying structural changes over time. Motivated by cosmological applications, the derived results are applied to the context of sojourn measures of spatiotemporal spherical Gaussian random fields. The results are illustrated for some families of spatiotemporal Gaussian random fields displaying complex spatiotemporal dependence structures.

Keywords Central limit theorem · Gaussian subordinated random fields · LRD in physics · Moving levels · Reduction theorems · Spatiotemporal increasing domain asymptotics

1 Introduction

1.1 Connections with Statistical Physics

Geometric integral functionals defining sojourn measures in the context of random fields play a crucial role in Statistical Physics. These morphological descriptors arising in integral geometry allow the characterization of connectivity, content and shape of stochastic spatial structures in the analysis of physical systems. In particular, they complement the spatial statistical and physical analysis of systems, whose stochastic structures are characterized by

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the Boolean model. Several physical phenomena involved in different issues of Statistical Physics, such as complex fluids, porous media, and pattern formation in dissipative systems, can be suitably analyzed in terms of these morphological measures. Particularly, Minkowski functionals, as additive functionals of spatial patterns, provide the geometric description of physical phenomena from integral instead of differential expressions. Special attention has been paid to Minkowski formalism in the three dimensional Euclidean space, where this family of functionals consists of the volume, the surface area of the pattern, its integral mean curvature, and the Euler characteristic or integral of the Gaussian curvature.

The additivity property of Minkowski functionals is well-suited and applicable in several areas of statistical mechanics where systems have structures or properties that can be decomposed and analyzed in a linear manner. Some scenarios where additivity of Minkowski functionals is suitable, and hence, effectively applied are homogeneous and isotropic systems. Additivity is used in porous media or composite materials to quantify geometric properties like volume, surface area, and connectivity. For well-separated pores or grains, the functional is evaluated over the entire structure by summing of the functional values for each component. This approach is useful in estimating properties such as permeability, mechanical strength, and conductivity by decomposing the overall geometry into additive components. In percolation theory, additivity is applicable in the analysis of disconnected clusters. For systems below the percolation threshold, where clusters are typically separated and finite, the additivity property allows easy computation of geometric measures by summing the contributions of individual clusters. In materials science, Minkowski functionals are used to describe the geometric features of grains. If the grains are mostly non-overlapping or exhibit simple intersections, the additivity property is useful in characterizing grain growth, phase boundaries, and interfaces in such materials, simplifying the calculation of volume, surface area, and curvature. Summarizing, in systems where interactions are weak or the geometry approximates simple shapes (like spheres or cubes), Minkowski functionals can be applied. Assuming limited or negligible overlap, in density fields or point distributions, like Poisson point processes or random tessellations, the Minkowski functionals for each region can be summed to describe the global geometry.

Minkowski formalism offers robust morphological measures contributing to the introduction of order parameters characterizing pattern transitions in dissipative systems, dynamical quantities characterizing spinodal decomposition, generalized molecular distribution functions for the characterization of the atomic structure of simple fluids (see, e.g., Sect. 3 in Mecke [37] and the references therein). Related statistical physical problems arising in these fields can be solved applying mathematical properties of Minkowski functionals (e.g., computation of virial coefficients and definition of accurate density functionals, prediction of percolation thresholds, and formulation of general morphological models for complex fluids based on the completeness of these additive functionals). Some drawbacks still must be solved opening new research lines related to tractability of third and higher virial coefficients in a cluster expansion; the improvement of the accuracy of threshold estimates based on Minkowski functionals; the computation of analytic expressions of Minkowski functionals beyond mean values of additive measures, and the definition of these functionals on lattices and their associated second-order moments (see Sects. 4 and 5 in Mecke [37]).

The additivity property of Minkowski functionals becomes a limitation in systems exhibiting non-linear interactions, non-linear phase behavior (e.g., phase separation in binary fluids or emulsions), fractal or self-similar structures (non-integer dimensions and scale-dependent structures), complex topologies (the presence of higher-order holes, handles, or cavities), overlapping particles in granular and disordered systems, systems with non-additive energy contributions (e.g., curvature effects in membranes and vesicles involve non-linear energy terms), and complex percolation structures, where near the critical threshold, clusters exhibit scale-free and non-linear growth. In these cases, additional tools, such as fractal analysis, non-additive geometric measures, or topological data analysis (TDA), may complement or replace the use of Minkowski functionals to better describe the system's behavior.

The present paper considers a more general family of morphological measures allowing the dynamical analysis of spatial random structures in physical systems, beyond the purely spatial analysis previously developed in the literature, addressing the case where thresholds can change over time. The asymptotic probabilistic properties of this extended family of geometric functionals allow, in particular, the characterization and prediction of future spatial configurations in random physical systems, providing basic quantities, like asymptotic means and variances, that play a crucial role in morphological analysis. In the context of percolation theory and Boolean model, our approach introduces a more flexible stochastic modelling of spatial patterns incorporating structural changes over time induced by time-varying thresholds.

It is worthwhile noticing the interest in Cosmology of the extended asymptotic spatiotemporal analysis of Minkowski functionals addressed in this paper. Particularly, this analysis has special significance in Cosmic Microwave Background (CMB) Radiation Variation studies over time (see, e.g., Carones et al. [9], Duque et al. [12], Marinucci [34], among others). The CMB encodes information from the Early Universe in the intensity and polarisation of the light. The characterization of the statistical distribution of CMB plays a crucial role in these studies. The analysis of morphological properties of CMB spherical maps is based on completeness theorem, the invariance assumption under translations and rotations, and on the additivity property of Minkowski functionals. Departure from Gaussianity or deviations from isotropy assumption must be detected to validate statistical inference procedures, and to discriminate between competitive scenarios for Big Bang dynamics. The central role of Minkowski functionals in non-Gaussianity tests is well-known (see, e.g., Marinucci [34]). In addition, Minkowski functionals allow to address computational burden, the ease of masking or weighting data, and the analysis of deviations from different thresholds. To go beyond the limited computation of Minkowski functionals to CMB temperature and weak lensing, in Duque et al. [12], Minkowski functionals are applied to CMB polarisation data, introducing a new formalism that incorporates spin effects. CMB polarisation field is decomposed into two rotationally invariant fields, usually analyzed through their angular power spectra (see Sect. 7). The angular power spectrum is not sensitive to the possible deviations from Gaussianity or departure from statistical isotropy. This fact motivates the extended application of Minkowski functionals to CMB polarisation data, exploiting their more efficient discrimination of Gaussian or isotropic deviations, with respect to the bispectrum, based on three point correlation function, and trispectrum, based on four point correlation function (see also Carones et al. [9] for further insights in the analysis of CMB polarisation intensity maps from Minkowski functionals). The asymptotic probability distribution results derived in the present paper, in particular, for spatiotemporal Minkowski functionals, provide new tools for the derivation of a wider family of non-Gaussianity tests detecting changes or deviations through time (see Sect. 7).

From a statistical point of view, the implementation of non-Gaussianity tests, in the spatiotemporal context addressed in this paper, also requires the choice of a suitable parametric scenario for the family of spatiotemporal covariance functions, characterizing the two point correlation structure of the underlying spatiotemporal isotropic random field. Note that variances of Minkowski functionals also depend on the parameters of these covariance functions. In the context of spatiotemporal Gaussian random fields (STGRFs), the Gneiting class of spatiotemporal covariance functions offers a flexible nonseparable modelling framework (see

Gneiting [14]). An extended formulation of Matérn covariance function family to the spatiotemporal context can be achieved within this Gneiting class (see Bevilacqua et al. [6] for an unified framework in the purely spatial case). See also Sect. 4.1, where additional examples of Gneiting class of nonseparable spatiotemporal covariance functions are analyzed in the illustration of Theorem 2, providing the asymptotic Gaussian distribution of spatiotemporal Minkowski functionals.

1.2 State of the Art

Sojourn functionals were extensively analyzed since the nineties in the context of weak-dependent random fields (see, e.g., Bulinski [7] and Ivanov and Leonenko [19]). Special attention has been paid to the long-range dependent random field case (see Berman [5], Leonenko [24], Leonenko and Olenko [26], Makogin and Spodarev [33], Marinucci et al. [35], among others). Limit theorems for level functionals of stationary Gaussian processes and fields constitute a major topic in this literature (see, e.g., Auffinger and Ben Arous [2], Azäis and Wschebor [3], Cabaña [8], Estrade and León [13], Iribarren [18], Kratz and León [21], Kratz and León [22], Marinucci and Vadlamani [36], Slud [43]). These papers contribute to the characterization of topological and geometrical properties of random fields, from the analysis of morphological descriptors like the Euler characteristic of an excursion set, the number of up-crossings at level u on a bounded closed cube of $\mathbb{R}^d$, the probability of the maximum to be greater than a given threshold u, or the analysis of anisotropy based on the line integral with respect to the level curve at any threshold u (see also Müller [39], Kratz and Vadlamani [23] and Pham [40]).

In the characterization of the asymptotic distribution of geometric functionals under LRD, the limit results derived are based on reduction principle. This principle was first discovered by Taqqu [44] (see also Dobrushin and Major [11], Leonenko et al. [29, 30], Taqqu [45]). Theorem 1 in Leonenko and Ruiz-Medina [28] provides a general reduction principle, under increasing domain asymptotics in time, leading to the limiting distributions of properly normalised integral functionals, when the underlying spatiotemporal Gaussian random field displays LRD in time. The restriction of this Gaussian field to the sphere allows the application of these results in the context of CMB analysis. In this spherical context, interesting asymptotic results for sojourn functionals, under increasing domain asymptotics in time, have been derived in Marinucci et al. [35], covering the cases where the underlying spatiotemporal random field displays Short-Range Dependence (SRD) and LRD (see also Marinucci and Vadlamani [36]). The special case of Hermite rank equal to two deserves special attention, since this case can be connected with Chi-squared statistics, usually arising in non-Gaussianity tests in CMB analysis (see, e.g., Marinucci [34]). See also Leonenko et al. [29]. The results of this paper extend the asymptotic analysis achieved in the above cited references to the case of increasing domain asymptotics in time and space, incorporating moving thresholds (see Sect. 1.3 for more technical details).

1.3 Our Contribution

Up to our knowledge, the problem of determining, via reduction theorems, the asymptotic probability distribution of integral functionals of nonlinear transformations of LRD STGRFs, under increasing domain asymptotics in space and time, has not been addressed in the current literature. The extended analysis to the case of time-varying subordination, via a nonlinear transformation depending on the size of the temporal domain constitutes the main contribution of the present paper. First Minkowski functionals involving moving levels subordinated to STGRFs arise as interesting particular case of this extended analysis. Namely, we restrict our attention to the case where time domain is a temporal continuous interval $[0, T]$, and $\Lambda(T) = T^{\gamma}$, $\gamma \geq 0$, defines the scale factor of the homothetic transformation of a convex compact set $K \subset \mathbb{R}^d$, with center at point $\mathbf{0} \in K$. The case $\gamma = 0$ was addressed in Leonenko and Ruiz-Medina [28] (see also Marinucci and Vadlamani [36], where alternative conditions were formulated in a Hilbert space framework, under a parametric modelling of spherical scale dependent LRD). When $\gamma = 1$, our Theorem 1 extends Theorem 1 in Leonenko et al. [29] to the spatiotemporal context (see also Leonenko et al. [30]).

In the second part of the paper, Theorem 3 provides a general reduction principle under T-varying nonlinear transformations of Gaussian random fields, with, as before, T denoting the size of the time interval. Additionally to the assumed conditions on LRD in the first part of the paper, in the derivation of Theorem 4, the divergence rate of the moving threshold parameter must be controlled at the logarithm scale by the increasing of the size T of the temporal interval (see **Condition 6**). In that sense, the methodological approach adopted in the proof of this result is not standard. This reduction principle characterizes the limit Gaussian distribution of Minkowski functionals involving moving levels. The application of the above results to the context of sojourn measures of spatiotemporal Gaussian random fields restricted to the sphere is then contemplated, motivated by the current literature on CMB radiation variation analysis.

The outline of the paper is now introduced. Some preliminary results on geometrical probabilities are first reviewed in Sect. 2. Section 3 derives the conditions for a general reduction principle under increasing domain asymptotics in time and in space (see Theorem 1). In Sect. 4, Theorem 2 derives the asymptotic Gaussian distribution of spatiotemporal Minkowski functionals by applying reduction principle provided in Theorem 1. Some examples are also analyzed, where subordination to STGRFs with nonseparable covariance function is considered. The case where geometric integral functionals are computed from a T-dependent nonlinear transformations of a Gaussian random field is studied in Sect. 5 (see Theorem 3). In Sect. 6, the asymptotic distribution of suitable normalized first Minkowski functionals involving moving levels, under increasing domain asymptotics in space–time, is derived in Theorem 4. The obtained results are applied in Sect. 7 to the case of $\gamma = 0$, when the underlying Gaussian spatiotemporal random field is the restriction to the sphere of a stationary in time, and homogeneous and isotropic in space, STGRF.

2 Preliminaries

Under spatial isotropy, variance components in the chaotic expansion of spatiotemporal geometric integral functionals can be computed from geometrical probabilities (see, e.g., Lord [31], Ivanov and Leonenko [19]). Some extended results can also be found in Aharonyan and Khalatyan [1], and the references therein. We now summarize them and introduce the corresponding notation.

Let $\nu_d(\cdot)$ be the Lebesgue measure in $\mathbb{R}^d$, $d \geq 2$, and K be a convex body in $\mathbb{R}^d$, i.e., a compact convex set with non-empty interior with center at the point $0 \in K$. Let $\mathcal{D}(K) = \max\{\|x - y\|,\ x, y \in K\}$ be the diameter of K. Let also $\nu_d(K) = |K|$ be the volume of K, and $\nu_{d-1}(\delta K) = U_{d-1}(K)$ be the surface area of K, where δK is the boundary of K. Note that for $K = \mathcal{B}(1) = \{x \in \mathbb{R}^d;\ \|x\| \leq 1\}$ is the unit ball, and $\delta K = S_{d-1}(1) = \{x \in$

$\mathbb{R}^d;\ \|x\| = 1\}$ is the unit sphere, then $\mathcal{D}(\mathcal{B}(1)) = 2$, and $|\mathcal{B}(1)| = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$, $U_{d-1}(\mathcal{B}(1)) = |S_{d-1}| = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$.

Let $\Lambda(T)K$ be a homothetic transformation of body K with center $0 \in K$, and coefficient or scaling factor $\Lambda(T) > 0$. We assume that $\Lambda(T) = T^{\gamma}$, $\gamma \geq 0$. Then, $|\Lambda(T)K| = |K|T^{\gamma d}$. Following the approach presented in Lord [31] (see also Ivanov and Leonenko [19]), the probability density $\psi_{\Lambda(T),\mathcal{B}(1)}(z)$ of the random variable $Z = \rho_{\mathcal{B}(1)} = \|P_1 - P_2\|$, with P_1 and P_2 being two independent random points with uniform distribution in $\mathcal{B}(1)$, is given by, for $0 \leq z \leq 2T^{\gamma}$,

$$\psi_{\Lambda(T),\mathcal{B}(1)}(z) = \frac{d}{[\Lambda(T)]^d} z^{d-1} I_{1-\left(\frac{z}{2\Lambda(T)}\right)^2}\left(\frac{d+1}{2}, \frac{1}{2}\right), \tag{1}$$

in terms of the incomplete beta-function

$$I_{\mu}(p,q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^{\mu} t^{p-1}(1-t)^{q-1} dt, \quad \mu \in (0,1]. \tag{2}$$

In formula (2.6) in Aharonyan and Khalatyan [1], an extended version of the probability density of the distance between two independent uniformly distributed points in a convex body K in $\mathbb{R}^d$ is derived, applying an alternative methodology to Lord [31] for the case of hyperspheres. Specifically, let $\mathcal{J}$ be the space of all straight lines in $\mathbb{R}^d$, and $d\gamma$ is an element of a locally finite measure in the space $\mathcal{J}$, which is invariant, with respect to the group M of all Euclidean motions in the space $\mathbb{R}^d$ (the uniform measure on $\mathcal{J}$). Let also $F_K(v)$ be the chord length distribution function of body K, defined as

$$F_K(v) = \frac{2(d-1)}{|S_{d-2}|} \int_{|\chi(\gamma)| \leq v} d\gamma,$$

where $\chi(\gamma) = \gamma \cap K$ is a chord in K. Then, for $0 \leq z \leq \mathcal{D}(K)$,

$$\psi_{\rho_K}(z) = \frac{1}{|K|^2}\left[z^{d-1}|S_{d-1}||K| - z^{d-1}|S_{d-2}|\frac{U_{d-1}(K)}{d-1}\int_0^z (1 - F_K(v))\, dv\right]. \tag{3}$$

From (3), we also have, for $0 \leq z \leq \mathcal{D}(\Lambda(T)K)$,

$$\begin{aligned}\psi_{\Lambda(T),K}(z) = \psi_{\rho_{\Lambda(T)K}}(z) = \frac{1}{[(|K|[\Lambda(T)]^d)]^2}\Big[\left(|K|[\Lambda(T)]^d\right) z^{d-1}|S_{d-1}| \\ -|S_{d-2}|z^{d-1}\frac{U_{d-1}(\Lambda(T)K)}{d-1}\int_0^z \left(1 - F_{\Lambda(T)K}(v)\right) dv\Big].\end{aligned} \tag{4}$$

3 Reduction Theorems for Spatiotemporal Random Fields with LRD

We consider the spatiotemporal random field $Z : \left(\Omega \times \mathbb{R}^d \times \mathbb{R}\right) \longrightarrow \mathbb{R}$, with $(\Omega, \mathcal{A}, P)$ denoting the basic probability space.

Condition 1. Assume that Z is a measurable mean-square continuous homogeneous and isotropic in space and stationary in time Gaussian random field with

$$\mathbb{E}[Z(x,t)] = 0, \quad \mathbb{E}[Z^2(x,t)] = 1$$
$$\widetilde{C}(\|x-y\|, |t-s|) = \mathbb{E}[Z(x,t)Z(y,s)] \geq 0.$$

Under **Condition 1**, one can write

$$C(z,\tau)=\widetilde{C}(\|x-y\|,|t-s|),\quad z=\|x-y\|\geq 0,\quad \tau=|t-s|\geq 0,$$

where C denotes the covariance function as a function of the arguments z and τ, respectively representing the norm of the spatial argument, and the absolute value of the time argument. While $\widetilde{C}$ means that we are considering the values of the covariance function depending on the input arguments $x-y\in\mathbb{R}^d$ and $t-s\in\mathbb{R}$.

Remark 1 The non-negativeness condition of the covariance function has been usually assumed in the literature of LRD stationary Gaussian processes, and isotropic random fields during nineties, as standard assumption in reduction theorems, leading to central and non-central limit results for integral functionals of nonlinear transformations of Gaussian random fields. For an alternative approach in the derivation of limit results for weighted nonlinear transformations of LRD Gaussian stationary processes we refer the reader to Ivanov et al. [20]. See also Mainia and Nourdin [32], where the application of the Malliavin-Stein method and Fourier analysis techniques in the derivation of spectral limit theorems is considered. The general setting of product of spatial domains of different dimensions requiring different scaling factors is also analyzed in Leonenko et al. [25].

Let $Z\sim\mathcal{N}(0,1)$ be a standard Gaussian random variable with density

$$\phi(w)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{w^2}{2}\right),\ w\in\mathbb{R},\ \Phi(u)=\int_{-\infty}^{u}\phi(w)dw.$$

Let $G(z)$ be a real-valued Borel function satisfying

Condition 2. $\mathbb{E}G^2(Z(x,t))<\infty$.

Under **Condition 2**, G admits a chaotic expansion in terms of the normalized Hermite polynomials in the Hilbert space $L_2(\mathbb{R},\phi(u)du)$ of square integrable functions with respect to the standard Gaussian measure. This expansion is given by

$$G(z)=\sum_{n=0}^{\infty}\frac{\mathcal{J}_n}{n!}H_n(z),\ \mathcal{J}_n=\int_{\mathbb{R}}G(z)H_n(z)\phi(z)dz,$$

where the Hermite polynomial of order n, denoted by $H_n(z)$, is defined by the equation $\frac{d^n}{dz^n}\phi(z)=(-1)^nH_n(z)\phi(z)$. Note that $H_0=1$, $H_1(z)=z$, and $H_2(z)=z^2-1,\ldots$

Following Taqqu [44], we will introduce the following condition:

Condition 3. The Hermite rank of the function G is $m\geq 1$. Hence, for $m=1$, $\mathcal{J}_1\neq 0$, or for $m\geq 2$, $\mathcal{J}_1=\cdots=\mathcal{J}_{m-1}=0$, $\mathcal{J}_m\neq 0$ (see also Taqqu [45]).

Under **Condition 3**, the following spatiotemporal integral functional of $Z(x,t)$ is defined:

$$\begin{aligned}
A_T &= \int_0^T\int_{\Lambda(T)K}G(Z(x,t))dxdt=\mathcal{J}_0+\sum_{n\geq m}\frac{\mathcal{J}_n}{n!}\xi_{n,T}\\
E[A_T] &= \int_0^T\int_{\Lambda(T)K}\mathcal{J}_0dxdt=\mathcal{J}_0|T||K|[\Lambda(T)]^d\\
\xi_{n,T} &= \int_0^T\int_{\Lambda(T)K}H_n(Z(x,t))dxdt\\
\mathbb{E}[\xi_{n,T}] &= 0,\ \mathbb{E}[\xi_{n,T}\xi_{l,T}]=0,\ n\neq l,\ n,l\geq m,
\end{aligned}\tag{5}$$

where, as given in **Condition 3**, m denotes the Hermite rank of function G, and the integrals are interpreted in the mean square sense.

Thus, for $n \geq m$,

$$\begin{aligned}
\sigma^2_{n,K}(T) &= \mathrm{Var}(\xi_{n,T}) \\
&= 2n!T \int_0^T \left(1-\frac{\tau}{T}\right) \int_{K\Lambda(T)\times K\Lambda(T)} \widetilde{C}^n(\|x-y\|,\tau)d\tau dx dy \\
&= 2n!T|K\Lambda(T)|^2 \int_0^T \left(1-\frac{\tau}{T}\right) \mathbb{E}\left[\widetilde{C}^n\left(\|P_1-P_2\|,\tau\right)\right] d\tau \\
&= 2n!T|K|^2 T^{2\gamma d} \int_0^T \left(1-\frac{\tau}{T}\right) \int_0^{\mathcal{D}(K\Lambda(T))} \psi_{\Lambda(T),K}(z) C^n(z,\tau) dz d\tau,
\end{aligned} \tag{6}$$

where $\psi_{\Lambda(T),K}(z)$, denotes, as before, the probability density of the random variable $\rho_{\Lambda(T)K} = \|P_1 - P_2\|$, where P_1 and P_2 are two independent random points with uniform distribution in $\Lambda(T)K$. In particular, from Eq. (1), for $K = \mathcal{B}(1)$, for $0 \leq z \leq 2T^\gamma$,

$$\psi_{T^\gamma,\mathcal{B}(1)}(z) = \frac{d}{T^{\gamma d}} z^{d-1} I_{1-\left(\frac{z}{2T^\gamma}\right)^2}\left(\frac{d+1}{2},\frac{1}{2}\right), \tag{7}$$

in terms of the incomplete beta function (2). Thus,

$$\begin{aligned}
\sigma^2_{m,\mathcal{B}(1)}(T) &= \mathrm{Var}\left(\int_0^T \int_{\Lambda(T)\mathcal{B}(1)} H_m\left(Z(x,t)\right)\right) dx dt \\
&= 2m!T|\mathcal{B}(1)|^2[\Lambda(T)]^{2d} \int_0^T \left(1-\frac{\tau}{T}\right) \int_0^{\mathcal{D}(K\Lambda(T))} C^m(z,\tau) \\
&\quad \times \left[\frac{d}{[\Lambda(T)]^d} z^{d-1} I_{1-\left(\frac{z}{2\Lambda(T)}\right)^2}\left(\frac{d+1}{2},\frac{1}{2}\right)\right] dz d\tau \\
&= 2m!|\mathcal{B}(1)|^2 d T^{\gamma d+1} \int_0^T \left(1-\frac{\tau}{T}\right) \\
&\quad \times \int_0^{2T^\gamma} z^{d-1} C^m(z,\tau) I_{1-\left(\frac{z}{2T^\gamma}\right)^2}\left(\frac{d+1}{2},\frac{1}{2}\right) dz d\tau \\
&= 8m! \frac{\pi^d}{d\Gamma^2\left(\frac{d}{2}\right)} T^{\gamma d+1} \int_0^T \left(1-\frac{\tau}{T}\right) \int_0^{2T^\gamma} z^{d-1} C^m(z,\tau) \\
&\quad \times I_{1-\left(\frac{z}{2T^\gamma}\right)^2}\left(\frac{d+1}{2},\frac{1}{2}\right) dz d\tau.
\end{aligned} \tag{8}$$

Remark 2 In the following, we will apply Lord (1954) results on the derivation of the probability density of the distance between two independent uniform points in hyperspheres. Consider the positive constants F_1 and F_2 respectively defining the supremum and infimum of the radius of the balls such that the following inclusions hold

$$F_1\mathcal{B}(1) \subseteq K \subseteq F_2\mathcal{B}(1). \tag{9}$$

From Eqs. (4)–(6) in Lord (1954),

$$C_1\psi_{T^\gamma,\mathcal{B}(S_1)} \leq \psi_{T^\gamma,K} \leq C_2\psi_{T^\gamma,\mathcal{B}(S_2)}, \ 0 \leq C_1 \leq C_2, \tag{10}$$

where $\psi_{T^{\gamma},K}(z)$ denotes the probability density of the random variable $\rho_{KT^{\gamma}} = \|P_1 - P_2\|$, with P_1 and P_2 being two independent random points with uniform distribution in KT^{γ}. As before, K denotes a compact convex set with non-empty interior, and with center at the point $0 \in K$. In particular, $\mathcal{B}(S_i)$ denotes the ball with center $0 \in \mathcal{B}(S_i)$, and radius S_i, $i = 1, 2$, according to constants F_i, $i = 1, 2$, in Eq. (9).

The following additional condition will be assumed.

Condition 4.

(i) $C(z, \tau) \to 0$, if $\max\{z, \tau\} \to \infty$.

(ii) For some fixed $m \in \{1, 2, 3 \dots\}$, there exist $\delta_1 \in (0, 1)$ and $\delta_2 \in (0, 1)$ such that for $\Lambda(T) = T^{\gamma}$, for certain $\gamma \geq 0$,

$$\lim_{T\to\infty} \frac{\sigma^2_{m,K}(T)}{T^{1+\delta_1} T^{\gamma d(1+\delta_2)}} = \infty.$$

From Remark 2 (see also Eqs. (6), (8) and (10)), it is straightforward that **Condition 4(ii)** holds if there exist $\delta_1 \in (0, 1)$ and $\delta_2 \in (0, 1)$ such that as $T \to \infty$,

$$\frac{1}{T^{\delta_1+\gamma d\delta_2}} \int_0^T \left(1 - \frac{\tau}{T}\right) \int_0^{2T^{\gamma}} z^{d-1} C^m(z, \tau) I_{1-\left(\frac{z}{2T^{\gamma}}\right)^2}\left(\frac{d+1}{2}, \frac{1}{2}\right) dz d\tau \to \infty. \tag{11}$$

In the subsequent development, **Condition 4(ii)** will be verified in terms of Eq. (11).

Remark 3 **Condition 4** and Eq. (11) mean that the spatiotemporal Gaussian random field Z displays LRD in space and time. The general reduction principle provided in Theorem 1 essentially follows from this condition, for any function $G \in L_2(\mathbb{R}, \phi(u)du)$, under the Gaussian scenario introduced in **Condition 1** (see also Eq. (28) in the proof of this result). This fact is illustrated in Sects. 3.1 and 4.1. Specifically, the case of separable covariance functions in space and time is considered in Sect. 3.1. In this case, **Condition 4(ii)** holds for $0 < A < 1/m$, and $0 < \widetilde{\alpha} < d/m$ (see Eqs. (14) and (16)), which corresponds to the case of LRD in time and space. The nonseparable covariance function case is illustrated in Sect. 4.1 within the Gneiting covariance function class (see Eqs. (30)–(34)). The sufficient conditions considered, $0 < \delta_2 < 1 - \frac{\alpha\beta}{\gamma} < 1$, $0 < \delta_1 < 1 - 2\widetilde{\gamma}(\gamma - 2\alpha\beta) < 1$, in the first example, and $0 < \delta_2 < 1 - \frac{\alpha\beta}{\gamma} < 1$, $0 < \delta_1 < 1 - 2\nu\widetilde{\gamma}(\gamma - 2\alpha\beta) < 1$, in the second example, reflect LRD in time and space, involving space–time interaction in the restrictions on the LRD parameters α, β (time), $\widetilde{\gamma}, \nu$ (space), as well as on the shape parameter γ characterizing the scaling factor of the homothetic transformation of $K \subset \mathbb{R}^d$. Note that, in both cases separable and nonseparable covariance function cases, the LRD parameter are involved in the variance of Minkowski functionals.

3.1 Separable Covariance Functions

For the case of separable spatiotemporal covariance functions in the unit ball,

$$C(z, \tau) = C_{Space}(z) C_{Time}(\tau),$$

we have

$$\sigma^2_{m,\mathcal{B}(1)}(T) = m! b_{1m}(T) b_{2m}(T), \tag{12}$$

where

$$b_{1m}(T) = 2T \int_0^T \left(1 - \frac{\tau}{T}\right) C_{Time}^m(\tau) d\tau$$
$$b_{2m}(T) = |\mathcal{B}(1)|^2 d T^{\gamma d} \int_0^{2T^\gamma} z^{d-1} C_{Space}^m(z) I_{1-\left(\frac{z}{2T^\gamma}\right)^2}\left(\frac{d+1}{2}, \frac{1}{2}\right) dz.$$

Then, in the weak dependent case in time, i.e., when the temporal covariance function is absolutely integrable, as T $\to \infty$, we obtain

$$b_{1m}(T) = 2L_1 T(1 + o(1))$$
$$L_1 = \int_0^\infty C_{Time}^m(\tau) d\tau < \infty, \quad \int_0^\infty C_{Time}^m(\tau) d\tau \neq 0. \tag{13}$$

Suppose that for some bounded slowly varying functions at infinity $\mathcal{L}_1(\tau)$:

$$C_{Time}(\tau) = \frac{\mathcal{L}_1(\tau)}{\tau^A}, \quad A > 0. \tag{14}$$

Then,

$$b_{1m}(T) = 2T \int_0^T \left(1 - \frac{\tau}{T}\right) \left[\frac{\mathcal{L}_1(\tau)}{\tau^A}\right]^m d\tau.$$

Under covariance model (14), Eq. (13) holds for $A > 1/m$, corresponding to the weak-dependent case in time. For $A = \frac{1}{m}$, $b_{1\,m}(T) = 2T \log(T) \mathcal{L}_1^m(T)(1 + o(1))$. While for $0 < A < \frac{1}{m}$, applying the change of variable $x = \frac{\tau}{T}$,

$$\begin{aligned} b_{1m}(T) &= 2T^{2-mA} \mathcal{L}_1^m(T) \int_0^1 \frac{\mathcal{L}_1(xT)}{\mathcal{L}_1^m(T)} \frac{1}{x^{Am}} (1-x) dx \\ &= 2L_2 T^{2-mA} \mathcal{L}_1^m(T)(1 + o(1)) \end{aligned}$$
$$L_2 = \left[\int_0^1 (1-\tau)\tau^{-Am} d\tau\right] = \left[(1-mA)(2-Am)\right]^{-1}. \tag{15}$$

Similarly, as $T \to \infty$, under weak-dependence in space, i.e., when the spatial covariance function is absolutely integrable over the spatial domain,

$$b_{2\,m}(T) = L_3 T^{d\gamma}(1 + o(1)),$$

where

$$L_3 = |\mathcal{B}(1)|^2 d \int_0^\infty z^{d-1} C_{Space}^m(z) dz < \infty \int_0^\infty z^{d-1} C_{Space}^m(z) dz \neq 0.$$

Furthermore, if for some bounded slowly varying functions at infinity $\mathcal{L}(z)$:

$$C_{Space}(z) = \frac{\mathcal{L}(z)}{z^{\widetilde{\alpha}}}, \quad \widetilde{\alpha} > 0, \tag{16}$$

then, for $\widetilde{\alpha} = \frac{d}{m}$,

$$\begin{aligned} b_{2m}(T) &= |\mathcal{B}(1)|^2 d T^{\gamma d} \int_0^{2T^\gamma} z^{d-1} \left[\frac{\mathcal{L}(z)}{z^{\widetilde{\alpha}}}\right]^m I_{1-\left(\frac{z}{2T^\gamma}\right)^2}\left(\frac{d+1}{2}, \frac{1}{2}\right) dz \\ &= L_4 T^{\gamma d} \log(T^\gamma) \mathcal{L}^m(T^\gamma)(1 + o(1)), \end{aligned}$$

with $L_4 = \frac{4\pi^d}{d\Gamma^2(d/2)}$. For $0 < \widetilde{\alpha} < \frac{d}{m}$, considering the change of variable $u = \frac{z}{T^\gamma}$

$$
\begin{aligned}
b_{2m}(T) &= |\mathcal{B}(1)|^2 dT^{\gamma d} \int_0^{2T^\gamma} z^{d-1} \left[\frac{\mathcal{L}(z)}{z^{\widetilde{\alpha}}}\right]^m I_{1-\left(\frac{z}{2T^\gamma}\right)^2}\left(\frac{d+1}{2}, \frac{1}{2}\right) dz \\
&= |\mathcal{B}(1)|^2 dT^{\gamma(2d-m\widetilde{\alpha})} \mathcal{L}^m(T^\gamma) \int_0^2 u^{d-1} \left(\frac{\mathcal{L}^m(T^\gamma u)}{\mathcal{L}^m(T^\gamma)} - 1 + 1\right) I_{1-(\frac{u}{2})^2}\left(\frac{d+1}{2}, \frac{1}{2}\right) du \\
&= |\mathcal{B}(1)|^2 dT^{\gamma(2d-m\widetilde{\alpha})} \mathcal{L}^m(T^\gamma) \left[\int_0^2 u^{d-1} I_{1-(\frac{u}{2})^2}\left(\frac{d+1}{2}, \frac{1}{2}\right) du\right. \\
&\quad \left. + \int_0^2 u^{d-1} \left(\frac{\mathcal{L}^m(T^\gamma u)}{\mathcal{L}^m(T^\gamma)} - 1\right) I_{1-(\frac{u}{2})^2}\left(\frac{d+1}{2}, \frac{1}{2}\right) du\right] \\
&= L_5 T^{\gamma(2d-m\widetilde{\alpha})} \mathcal{L}^m(T^\gamma)(1+o(1)),
\end{aligned}
\tag{17}
$$

where

$$
L_5 = \frac{2^{d-m\widetilde{\alpha}+1} \pi^{d-\frac{1}{2}} \Gamma\left(\frac{d-m\widetilde{\alpha}+1}{2}\right)}{(d-m\widetilde{\alpha})\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{2d-m\widetilde{\alpha}+2}{2}\right)},
$$

since

$$
\begin{aligned}
\int_0^2 u^{d-1} I_{1-(\frac{u}{2})^2}\left(\frac{d+1}{2}, \frac{1}{2}\right) du &= 2^d \int_0^1 w^{d-1} I_{1-w^2}\left(\frac{d+1}{2}, \frac{1}{2}\right) dw \\
&= \frac{2^d}{B\left(\frac{d+1}{2}, \frac{1}{2}\right)} \int_0^1 \int_0^{1-w^2} w^{d-1} t^{\frac{d-1}{2}} (1-t)^{-1/2} dt dw \\
&= \frac{2^d}{B\left(\frac{d+1}{2}, \frac{1}{2}\right)} \int_0^1 t^{\frac{d-1}{2}} (1-t)^{-1/2} \left[\int_0^{\sqrt{1-t}} w^{d-1} dw\right] dt \\
&= \frac{2^d}{dB\left(\frac{d+1}{2}, \frac{1}{2}\right)} \int_0^1 (1-t)^{(d-1)/2} t^{\frac{d-1}{2}} dt \\
&= \frac{2^d B\left(\frac{d+1}{2}, \frac{d+1}{2}\right)}{dB\left(\frac{d+1}{2}, \frac{1}{2}\right)},
\end{aligned}
$$

and

$$
\lim_{T\to\infty} \int_0^2 u^{d-1} \left(\frac{\mathcal{L}^m(T^\gamma u)}{\mathcal{L}^m(T^\gamma)} - 1\right) I_{1-(\frac{u}{2})^2}\left(\frac{d+1}{2}, \frac{1}{2}\right) du = 0 \tag{18}
$$

(see also Lemma 2.1.3 of Ivanov and Leonenko [19]).

Summarizing, for the introduced separable covariance function family, when $K = \mathcal{B}(1)$, **Condition 4(ii)** holds in the following cases:

From Eq. (12), when $\gamma = 0$, and $0 < A < \frac{1}{m}$,

$$
\sigma^2_{m,\mathcal{B}(1)}(T) = m! L_2 T^{2-mA} \mathcal{L}_1^m(T)(1+o(1)), \tag{19}
$$

hence,

$$
\lim_{T\to\infty} \frac{\sigma^2_{m,\mathcal{B}(1)}(T)}{T^{1+\delta_1}} = \infty, \tag{20}
$$

for $0 < \delta_1 < 1 - mA < 1$, in the case of LRD in time. This case has been considered by Leonenko and Ruiz-Medina [28]. While for $\gamma > 0$, $0 < A < \frac{1}{m}$, and $0 < \widetilde{\alpha} < \frac{d}{m}$,

$$
\sigma^2_{m,\mathcal{B}(1)}(T) = m! L_2 L_5 T^{2-mA} \mathcal{L}_1^m(T) T^{\gamma(2d-m\widetilde{\alpha})} \mathcal{L}^m(T^\gamma)(1+o(1)), \tag{21}
$$

and

$$\lim_{T\to\infty} \frac{\sigma^2_{m,\mathcal{B}(1)}(T)}{T^{1+\delta_1}T^{\gamma d(1+\delta_2)}} = \infty,$$

for $0 < \delta_1 < 1 - mA < 1$, and $0 < \delta_2 < 1 - (m\widetilde{\alpha})/d < 1$, in the case of LRD in time and space. For the remaining cases,

$$\lim_{T\to\infty} \frac{\sigma^2_{m,\mathcal{B}(1)}(T)}{T^{1+\delta_1}T^{\gamma d(1+\delta_2)}} = 0, \tag{22}$$

for any $\delta_i > 0$, $i = 1, 2$, and **Condition 4(ii)** does not hold.

Theorem 1 *Under **Conditions 1,2,3 and 4**, the random variables*

$$Y_T = \frac{[A_T - \mathbb{E}[A_T]]}{|\mathcal{J}_m|\sigma_{m,K}(T)/m!} \tag{23}$$

and

$$Y_{m,T} = \frac{sgn(\mathcal{J}_m)\int_0^T \int_{T^\gamma K} H_m(Z(x,t))dxdt}{\sigma_{m,K}(T)} \tag{24}$$

have the same limiting distributions as $T \to \infty$ (if one of them exists).

Proof We consider the decomposition $A_T - \mathbb{E}[A_T] = S_{1,T} + S_{2,T}$, where

$$S_{1,T} = \frac{\mathcal{J}_m}{m!}\xi_{m,T}, \quad S_{2,T} = \sum_{n=m+1}^{\infty} \frac{\mathcal{J}_n}{n!}\xi_{n,T}, \quad \sum_{n=m}^{\infty} \frac{\mathcal{J}_n^2}{n!} < \infty. \tag{25}$$

From (5),

$$\mathrm{Var}(A_T) = \mathrm{Var}[S_{1,T}] + \mathrm{Var}[S_{2,T}], \tag{26}$$

and we will prove that $\mathrm{Var}[S_{2,T}]/\sigma^2_{m,K\Lambda(T)}(T) \to 0$, $T \to \infty$.

From **Condition 4(i)**,

$$\sup_{(z,\tau)\in B_T^{\delta_1,\delta_2}} C(z,\tau) \to 0, \quad T \to \infty$$

$$B_T^{\delta_1,\delta_2} = \{(z,\tau);\ \tau \geq T^{\delta_1} \text{ or } z \geq T^{\gamma\delta_2}\}. \tag{27}$$

Furthermore, for the set

$$\overline{B}_T^{\delta_1,\delta_2} = \left\{(z,\tau);\ 0 \leq \tau \leq T^{\delta_1},\ 0 \leq z \leq T^{\gamma\delta_2}\right\},$$

one can use the estimate $C^{m+1}(z,\tau) \leq 1$. Let us consider $\delta_1 \in (0,1)$, $\delta_2 \in (0,1)$ as in **Condition 4**. Then, from Eq. (6),

$$\begin{aligned}\mathrm{Var}(S_{2,T}) &= \sum_{n=m+1}^{\infty} \frac{\mathcal{J}_n^2}{(n!)^2}\sigma^2_{n,K}(T) \\ &\leq k_1 \left\{T^{1+\gamma 2d}\left[\int_{\overline{B}_T^{\delta_1,\delta_2}} + \int_{B_T^{\delta_1,\delta_2}}\right]\left(1 - \frac{\tau}{T}\right)C^{m+1}(z,\tau)\psi_{T^\gamma,K}(z)dzd\tau\right\},\end{aligned}$$

for some positive constant $k_1 > 0$, where we have used $\sum_{n=m}^{\infty} \frac{\mathcal{J}_n^2}{n!} < \infty$, and $C^{m+l} \leq C^{m+1}$, for $l \geq 2$, since $C(z,\tau) \leq 1$, under **Condition 1**. We then obtain for some positive constants k_2, k_3, k_4, keeping in mind equation (6), and Remark 2

$$\mathrm{Var}(S_{2,T}) \leq k_2 \left\{ k_3 T^{1+\delta_1}[T^{\gamma}]^{d(1+\delta_2)} \right.$$

$$\left. + k_4 T^{1+\gamma 2d} \sup_{(z,\tau)\in B_T^{\delta_1,\delta_2}} \{C(z,\tau)\} \int_{B_T^{\delta_1,\delta_2}} \left(1 - \frac{\tau}{T}\right) C^m(z,\tau)\psi_{T^{\gamma},K}(z)dzd\tau \right\}.$$

Hence, as $T \to \infty$,

$$\frac{\mathrm{Var}(S_{2,T})}{\sigma^2_{m,K}(T)} \leq k_5 \left\{ \frac{1}{\frac{\sigma^2_{m,K}(T)}{T^{1+\delta_1}T^{\gamma d(1+\delta_2)}}} + k_6 \sup_{(z,\tau)\in B_T^{\delta_1,\delta_2}} \{C(z,\tau)\} \right.$$
$$\left. \times \frac{\int_{B_T^{\delta_1,\delta_2}} \left(1 - \frac{\tau}{T}\right) C^m(z,\tau)\psi_{T^{\gamma},K}(z)dzd\tau}{\int_{B_T^{\delta_1,\delta_2}\cup \overline{B}_T^{\delta_1,\delta_2}} \left(1 - \frac{\tau}{T}\right) C^m(z,\tau)\psi_{T^{\gamma},K}(z)dzd\tau} \right\} \to 0, \tag{28}$$

where we have applied that from **Condition 1**, for all $T > 0$,

$$\frac{\int_{B_T^{\delta_1,\delta_2}} \left(1 - \frac{\tau}{T}\right) C^m(z,\tau)\psi_{T^{\gamma},K}(z)dzd\tau}{\int_{B_T^{\delta_1,\delta_2}\cup \overline{B}_T^{\delta_1,\delta_2}} \left(1 - \frac{\tau}{T}\right) C^m(z,\tau)\psi_{T^{\gamma},K}(z)dzd\tau} \leq 1,$$

and that from **Condition 4(i)**, $\sup_{(z,\tau)\in B_T^{\delta_1,\delta_2}} \{C(z,\tau)\} \to 0$, $T \to \infty$, as well as from **Condition 4(ii)** (see also Remark 2 and Eq. (11))

$$\frac{\sigma^2_{m,K}(T)}{T^{1+\delta_1}T^{\gamma d(1+\delta_2)}} \to \infty$$

as $T \to \infty$. From Eqs. (25), (26) and (28),

$$\mathrm{Var}\left(Y_T - Y_{m,T}\right) \to 0, \quad T \to \infty, \tag{29}$$

as we wanted to prove. □

Remark 4 From Eq. (29), if the limit exists, Y_T and $Y_{m,T}$ have the same limit in probability, and hence, in distribution.

4 Sojourn Functionals for Spatiotemporal Fields

We consider the first Minkowski functional

$$M_1(T) = \left|\left\{0 \leq t \leq T,\ x \in T^{\gamma}K;\ Z(x,t) \geq u\right\}\right|$$
$$= \int_0^T \int_{T^{\gamma}K} G_u(Z(x,t))dxdt, \quad u \geq 0,\ \gamma \geq 0,$$

where $G_u(z) = \mathbb{I}_{z\geq u}$, that is, $G_u(Z(x,t))$ is the indicator function of the set

$$\left\{0 \leq t \leq T,\ x \in T^{\gamma}K;\ Z(x,t) \geq u\right\}.$$

Then,

$$
\begin{aligned}
E[M_1(T)] &= (1-\Phi(u))T|K|T^{\gamma d} \\
\mathcal{J}_q(u) &= \phi(u)H_{q-1}(u), \quad q \geq 1,
\end{aligned}
$$

and from Theorem 1 with $m=1$ we arrive to the following result.

Theorem 2 *Under **Conditions 1–4** with $m=1$, the random variable*

$$
\left\{M_1(T)-(1-\Phi(u))|K|T^{1+\gamma d}\right\}
$$

$$
\times\left(\phi(u)\left[2|K|^2T^{1+2\gamma d}\int_0^T\left(1-\frac{\tau}{T}\right)\int_0^{\mathcal{D}(T^{\gamma}K)}C(z,\tau)\psi_{T^{\gamma},K}(z)dzd\tau\right]^{1/2}\right)^{-1}
$$

converges to a standard normal distribution as $T\to\infty$.

Note that the case $\gamma=0$ was proved in Theorem 1 of Leonenko and Ruiz-Medina [28]. In particular, for the ball $K=\mathcal{B}(1)\subset\mathbb{R}^d$, the random variable

$$
\left\{M_1(T)-(1-\Phi(u))\pi^{d/2}T^{1+\gamma d}\left[\Gamma\left(\frac{d}{2}+1\right)\right]^{-1}\right\}\left[\phi(u)\left[(8\pi^d\Gamma^{-2}(d/2)(1/d))\right.\right.
$$

$$
\left.\left.\times T^{1+\gamma d}\int_0^T\left(1-\frac{\tau}{T}\right)\int_0^{2T^{\gamma}}z^{d-1}C(z,\tau)I_{1-\left(\frac{z}{2T^{\gamma}}\right)^2}\left(\frac{d+1}{2},\frac{1}{2}\right)dzd\tau\right]^{1/2}\right]^{-1}
$$

is asymptotically distributed as a standard normal random variable, as $T\to\infty$.

4.1 Examples

Let us analyze **Condition 4(ii)** with $m=1$ for nonseparable covariance functions in the Gneiting class (see Gneiting [14]). This family of covariance functions is given by

$$
\widetilde{C}(\|x\|,\tau)=\frac{\sigma^2}{[\psi(\tau^2)]^{d/2}}\varphi\left(\frac{\|x\|^2}{\psi(\tau^2)}\right),\quad \sigma^2\geq 0,\quad (x,\tau)\in\mathbb{R}^d\times\mathbb{R}, \tag{30}
$$

in terms of a completely monotone function φ and a positive function $\psi(u)$, $u\geq 0$, with a completely monotone derivative. We will analyze two special cases of functions φ and ψ in (30) in the following two examples.

4.1.1 Example 1

Let us consider the one-parameter Mittag–Leffler function E_ν, for $0<\nu\leq 1$, which is a completely monotone function (see, e.g., Gorenflo et al. [15], Haubold et al. [17]), given by

$$
E_\nu(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\beta+1)},\quad z\in\mathbb{C},\quad 0<\beta<1.
$$

Let function φ in Eq. (30) be defined as

$$
\varphi_\nu(z)=E_\nu(-z^{\widetilde{\gamma}}),\quad 0<\nu\leq 1,\quad 0<\widetilde{\gamma}<1, \tag{31}
$$

for nonnegative argument $z \geq 0$. This function is a complete monotone function, that is,

$$(-1)^r \frac{d^r}{dz^r} \varphi_\nu(z) \geq 0,$$

for all $r = 0, 1, 2, \ldots,$ and $0 < \nu \leq 1$ (see Barndoff-Nielsen and Leonenko [4]). From Theorem 4 in Simon [42], for $z \in \mathbb{R}_+$, that is, for $z \in \mathbb{R}$ and $z \geq 0$, and $\nu \in (0, 1)$,

$$\frac{1}{1+\Gamma(1-\nu)z} \leq E_\nu(-z) \leq \frac{1}{1+[\Gamma(1+\nu)]^{-1}z}. \tag{32}$$

The function $\psi(\tau) = (1 + a\tau^\alpha)^\beta$, $a > 0$, $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $\tau \geq 0$, has completely monotone derivatives (see Gneiting [14]).

From Eqs. (7) and (32),

$$\begin{aligned}
&\int_0^T \left(1-\frac{\tau}{T}\right)\int_0^{2T^\gamma} z^{d-1}C(z,\tau)I_{1-\left(\frac{z}{2T^\gamma}\right)^2}\left(\frac{d+1}{2},\frac{1}{2}\right)dzd\tau \\
&\geq \int_0^T \left(1-\frac{\tau}{T}\right)\int_0^{2T^\gamma} \frac{z^{d-1}\sigma^2}{(a\tau^{2\alpha}+1)^{\beta d/2}[1+\Gamma(1-\nu)\left(z^{2\widetilde{\gamma}}/((a\tau^{2\alpha}+1)^{\beta\widetilde{\gamma}})\right)]} \\
&\quad \times I_{1-\left(\frac{z}{2T^\gamma}\right)^2}\left(\frac{d+1}{2},\frac{1}{2}\right)dzd\tau \\
&= T^{1+\gamma d-\alpha\beta d-2\widetilde{\gamma}\gamma+2\alpha\beta\widetilde{\gamma}}\int_0^1 (1-u)\int_0^2 \frac{x^{d-1}\sigma^2}{(au^{2\alpha}+T^{-2\alpha})^{\beta d/2}} \\
&\quad \times \frac{I_{1-(\frac{x}{2})^2}\left(\frac{d+1}{2},\frac{1}{2}\right)}{T^{2\alpha\beta\widetilde{\gamma}-2\widetilde{\gamma}\gamma}+\Gamma(1-\nu)\left[x^{2\widetilde{\gamma}}/(au^{2\alpha}+T^{-2\alpha})^{\beta\widetilde{\gamma}}\right]}dudx,
\end{aligned} \tag{33}$$

where the last equality in Eq. (33) has been obtained by applying the change of variables $\tau = Tu$ and $z = T^\gamma x$. From (33), **Condition 4(ii)** holds if $1+\gamma d-\alpha\beta d-2\gamma\widetilde{\gamma}+2\alpha\beta\widetilde{\gamma} > \gamma d\delta_2+\delta_1$ for some $\delta_1, \delta_2 \in (0, 1)$. In particular, a sufficient condition for **Condition 4(ii)** to hold is $\gamma > \alpha\beta$, and $\widetilde{\gamma} < \frac{1}{2(\gamma-\alpha\beta)}$. Under this condition, one can consider, for instance, $\gamma d - \alpha\beta d > \gamma d\delta_2$, and $1 - 2\gamma\widetilde{\gamma} + 2\alpha\beta\widetilde{\gamma} > \delta_1$, and hence, $0 < \delta_2 < 1 - \frac{\alpha\beta}{\gamma} < 1$, $0 < \delta_1 < 1 - 2\widetilde{\gamma}(\gamma - 2\alpha\beta) < 1$.

4.2 Example 2

Consider now, in Eq. (30),

$$\begin{aligned}
\varphi(z) &= \frac{1}{(1+\widetilde{c}z^{\widetilde{\gamma}})^\nu},\ z > 0,\ \widetilde{c} > 0,\ 0 < \widetilde{\gamma} \leq 1,\ \nu > 0 \\
\psi(\tau) &= (1+a\tau^\alpha)^\beta,\ a > 0,\ 0 < \alpha \leq 1,\ 0 < \beta \leq 1,\ \tau \geq 0.
\end{aligned} \tag{34}$$

It is known that the function $\varphi(z)$ is completely monotone while, as before, function $\psi(\tau)$ has completely monotone derivatives. We restrict our attention to the ball $\mathcal{B}(1)$. In a similar way to Eq. (33), one can obtain

$$
\begin{aligned}
&\int_0^T \left(1-\frac{\tau}{T}\right)\int_0^{2T^{\gamma}} z^{d-1}C(z,\tau)I_{1-\left(\frac{z}{2T^{\gamma}}\right)^2}\left(\frac{d+1}{2},\frac{1}{2}\right)dzd\tau \\
&= T^{1+\gamma d-\alpha\beta d-2\widetilde{\gamma}\gamma\nu+2\alpha\beta\widetilde{\gamma}\nu}\int_0^1 (1-u)\int_0^2 \frac{x^{d-1}\sigma^2}{(au^{2\alpha}+T^{-2\alpha})^{\beta d/2}} \\
&\quad\times\frac{I_{1-(\frac{x}{2})^2}\left(\frac{d+1}{2},\frac{1}{2}\right)}{T^{2\alpha\beta\widetilde{\gamma}\nu-2\widetilde{\gamma}\gamma\nu}+\widetilde{c}\left[x^{2\widetilde{\gamma}\nu}/(au^{2\alpha}+T^{-2\alpha})^{\beta\widetilde{\gamma}\nu}\right]}dudx.
\end{aligned}
\tag{35}
$$

Condition 4(ii) is then satisfied if $1+\gamma d-\alpha\beta d-2\gamma\widetilde{\gamma}\nu+2\alpha\beta\widetilde{\gamma}\nu > \gamma d\delta_2+\delta_1$, for some $\delta_1,\delta_2 \in (0,1)$. In particular, a sufficient condition for **Condition 4(ii)** to hold is $\gamma > \alpha\beta$, and $\widetilde{\gamma}\nu < \frac{1}{2(\gamma-\alpha\beta)}$. Under this condition, one can consider, for instance, $\gamma d-\alpha\beta d > \gamma d\delta_2$, and $1-2\gamma\widetilde{\gamma}\nu+2\alpha\beta\widetilde{\gamma}\nu > \delta_1$, leading to $0 < \delta_2 < 1-\frac{\alpha\beta}{\gamma} < 1$, $0 < \delta_1 < 1-2\nu\widetilde{\gamma}(\gamma-2\alpha\beta) < 1$.

5 Reduction Theorem for Time Varying Subordination

For each fixed $T > 0$, let $G_T \in L_2(\mathbb{R},\phi(z)dz)$ such that

$$
G_T(z)=\sum_{q=0}^{\infty}\frac{\mathcal{J}_q(T)}{q!}H_q(z),\quad \mathcal{J}_q(T)=\int_{\mathbb{R}}G_T(z)H_q(z)\phi(z)dz.
$$

For example, one can consider the indicator function with moving threshold,

$$
G_T(Z(x,t))=\mathbb{I}_{Z(x,t)\geq u(T)},
$$

or, equivalently,

$$
G_T(z)=\mathbb{I}_{z\geq u(T)} \in L_2(\mathbb{R},\phi(z)dz),
$$

for each fixed $T > 0$. Here, $u:\mathbb{R}_+\to\mathbb{R}$ be such that $u(T)\to\infty$, as $T\to\infty$, and $G_T(z)$ admits the following orhogonal expansion in terms of Hermite polynomials:

$$
\begin{aligned}
\mathcal{J}_0(T) &= \int_{u(T)}^{\infty}\phi(\xi)d\xi = 1-\Phi(u(T)) \\
\mathcal{J}_q(T) &= \int_{u(T)}^{\infty}H_q(x)\phi(x)dx = H_{q-1}(u(T))\phi(u(T)),\ q\geq 1.
\end{aligned}
\tag{36}
$$

In the subsequent development, we consider the general case of a T-varying nonlinear transformation $G_T \in L_2(\mathbb{R},\phi(z)dz)$. We then analyze the asymptotic behavior of the functional

$$
A(T)=\int_0^T\int_{\Lambda(T)K}G_T(Z(x,t))dxdt=\sum_{n=0}^{\infty}\frac{\mathcal{J}_n(T)}{n!}\xi_n(T),
\tag{37}
$$

where, as before, $\Lambda(T)=T^{\gamma}$, $\gamma\geq 0$, and

$$
\xi_n(T)=\int_0^T\int_{T^{\gamma}K}H_n(Z(x,t))dxdt,\quad \forall n\in\mathbb{N},\ T>0.
$$

The mean and variance can be computed as follows:

$$E[A(T)] = T^{1+\gamma d}|K|\mathcal{J}_0(T).$$
$$\mathrm{Var}(A(T)) = \mathbb{E}[A(T) - \mathbb{E}[A(T)]]^2 = \sum_{q=m}^{\infty} \frac{\mathcal{J}_q^2(T)}{(q!)^2}\sigma^2_{q,K\Lambda(T)}(T), \tag{38}$$

where

$$\sigma^2_{q,\Lambda(T)K}(T) = 2q!T|K|^2T^{2d\gamma}\int_0^T\left(1-\frac{\tau}{T}\right)\int_0^{\mathcal{D}(T^\gamma K)} C^q(z,\tau)\psi_{T^\gamma K}(z)dzd\tau. \tag{39}$$

We assume that function G_T has Hermite rank $m \geq 1$, for every $T > 0$, i.e., **Condition 3** is satisfied. The derivation of Theorem 3 is obtained under the following additional condition.

Condition 5. For some $m \geq 1$,

$$\overline{\lim}_{T\to\infty} \frac{\mathrm{Var}(A(T))}{\frac{\mathcal{J}_m^2(T)}{(m!)^2}\sigma^2_{m,K\Lambda(T)}(T)} \leq 1. \tag{40}$$

The technical nature of **Condition 5** does not hinder its verification in practice, as follows from the proof of Theorem 4, where this condition is proved to hold for spatiotemporal Minkowski functionals with moving levels.

Theorem 3 *Under Eq. (40) in* ***Condition 5****,*

$$Y(T) = \frac{[A(T) - \mathbb{E}[A(T)]]}{|\mathcal{J}_m(T)|\sigma_{m,T^\gamma K}(T)/m!} \tag{41}$$

and

$$Y_m(T) = \frac{sgn(\mathcal{J}_m(T))\int_0^T\int_{T^\gamma K} H_m(Z(x,t))dxdt}{\sigma_{m,T^\gamma K}(T)} \tag{42}$$

have the same limiting distributions as $T \to \infty$ (if one of them exists).

Proof The proof is straightforward from **Condition 5**. Specifically, from Eq. (38),

$$\mathrm{Var}(A(T)) \geq \frac{\mathcal{J}_m^2(T)\sigma^2_{m,\Lambda(T)K}(T)}{(m!)^2}. \tag{43}$$

Considering $R(T) = Y(T) - Y_m(T)$, or, equivalently, $Y(T) = R(T) + Y_m(T)$, since $\mathrm{Var}(Y_m(T)) = 1$, from Eq. (43), under Eq. (40),

$$1 = \overline{\lim}_{T\to\infty} \frac{\mathrm{Var}(A(T))}{\frac{\mathcal{J}_m^2(T)}{(m!)^2}\sigma^2_{m,K\Lambda(T)}(T)} = 1 + \overline{\lim}_{T\to\infty}\mathrm{Var}(R(T)).$$

Thus, $\lim_{T\to\infty}\mathrm{Var}(R(T)) = 0$ (if the limit exists). □

6 First Minkowski Functional with Moving Level

We consider the geometric functional

$$\begin{aligned} M(T) &= \left|\left\{0 \le t \le T,\ x \in T^{\gamma} K;\ Z(x,t) \ge u(T)\right\}\right| \\ &= \int_0^T \int_{T^{\gamma} K} \mathbb{I}_{Z(x,t)\ge u(T)} dx dt \\ &= (1-\Phi(u(T)))\,|K|T^{\gamma d} + \sum_{q=1}^{\infty} \frac{\mathcal{J}_q(T)}{q!} \int_0^T \int_{T^{\gamma} K} H_q(Z(x,t)) dx dt, \end{aligned} \tag{44}$$

where

$$\mathcal{J}_q(T) = \phi(u(T)) H_{q-1}(u(T)),$$

and $u(T)$ is a continuous function such that $u(T) \to \infty$, as $T \to \infty$.

In the next result, the following condition is assumed:

Condition 6. Assume that **Condition 4** is satisfied, and $u(T)$ is such that $u^2(T) = o(log(T))$, and $u^2(T) \sup_{(z,\tau)\in B_T^{\beta_1,\beta_2}} C(z,\tau) \to 0$, as $T \to \infty$, where

$$B_T^{\beta_1,\beta_2} = \{(z,\tau);\ \tau \ge T^{\beta_1} \text{ or } z \ge T^{\gamma\beta_2}\}, \tag{45}$$

for some $\beta_1 \in (0,\delta_1)$, $\beta_2 \in (0,\delta_2)$.

Remark 5 Note that, for

$$\Lambda(T) = T^{\gamma},\quad \gamma \ge 0,\ u^2(T) = o(\log(T[\Lambda(T)]^d)) = o((\gamma d+1)\log(T)),\ T \to \infty,$$

if and only if $u^2(T) = o(log(T))$. In particular, for any $\varepsilon_i > 0$, $i = 1,2$, $u^2(T) = o(\log(T^{\varepsilon_1}[\Lambda(T)]^{d\varepsilon_2})) = o((\gamma d\varepsilon_2 + \varepsilon_1)\log(T)) = o(\log(T))$.

Remark 6 Note that the set $B_T^{\delta_1,\delta_2}$ introduced in Eq. (27) is not included in the set family $\left\{B_T^{\beta_1,\beta_2},\ \beta_1 \in (0,\delta_1),\ \beta_2 \in (0,\delta_2)\right\}$. This set family is considered in the proof of Theorem 4 (see Eqs. (52)–(53)), to apply **Condition 6** incorporating **Condition 4(ii)** (see also Remark 5).

Theorem 4 *Under **Conditions 1–3**, and **Condition 6**, as $T \to \infty$, the random variables*

$$\frac{M(T) - T^{1+\gamma d}|K|(1-\Phi(u(T)))}{\phi(u(T))\left[2T|K|^2 T^{2\gamma d}\int_0^T \left(1-\frac{\tau}{T}\right)\int_0^{\mathcal{D}(T^{\gamma}K)} C(z,\tau)\psi_{\rho_{T^{\gamma}K}}(z,\tau) dz d\tau\right]^{1/2}}$$

and

$$\frac{\int_0^T \int_{T^{\gamma}K} Z(x,t) dx dt}{\left[2T|K|^2 T^{2d\gamma}\int_0^T \left(1-\frac{\tau}{T}\right)\int_0^{\mathcal{D}(T^{\gamma}K)} C(z,\tau)\psi_{\rho_{T^{\gamma}K}}(z,\tau) dz d\tau\right]^{1/2}}$$

have the same asymptotic distribution, that is, a standard normal distribution, where $M(T)$ has been introduced in (44).

Proof It is known that for bivariate normal density (see, e.g.,Eq. 10.8.3 in Cramer and Leadbetter [10])

$$\phi(x, y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 + y^2 - 2\rho xy)\right), \tag{46}$$

the following identity holds:

$$\begin{aligned}
&\int_{u(T)}^{\infty}\int_{u(T)}^{\infty} \phi(x, y, \rho) dx dy \\
&= \left(\int_{u(T)}^{\infty} \phi(y) dy\right)^2 + \frac{1}{2\pi}\int_0^{\rho} \exp\left(-\frac{u^2(T)}{1+v}\right) \frac{dv}{\sqrt{1-v^2}}.
\end{aligned} \tag{47}$$

In our case, in Eq. (47), $\rho = \widetilde{C}(\|x - y\|, |t - s|)$, we then obtain

$$\begin{aligned}
E[M(T)] &= (1 - \Phi(u(T)))|K|T^{1+\gamma d} \\
E[M^2(T)] &= \int_{[0,T]\times[0,T]}\int_{T^\gamma K\times T^\gamma K} \mathbb{E}\left[\mathbb{I}_{Z(x,t)>u(T)}\mathbb{I}_{Z(y,s)>u(T)}\right] dx dy dt ds \\
&= \int_{[0,T]\times[0,T]}\int_{T^\gamma K\times T^\gamma K}\int_{u(T)}^{\infty}\int_{u(T)}^{\infty} \phi(u, w, \rho) du dw dx dy dt ds \\
&= T^{2+2d\gamma}|K|^2\left[\int_{u(T)}^{\infty}\phi(u)du\right]^2 \\
&\quad + \frac{1}{2\pi}\int_0^T\int_0^T\int_{T^\gamma K\times T^\gamma K}\int_0^{\widetilde{C}(\|x-y\|,|t-s|)} \exp\left(-\frac{u^2(T)}{1+v}\right)\frac{dv}{\sqrt{1-v^2}} dx dy dt ds \\
&= T^{2+2d\gamma}|K|^2[1 - \Phi(u(T))]^2 \\
&\quad + \frac{1}{2\pi}\int_0^T\int_0^T\int_{T^\gamma K\times T^\gamma K}\int_0^{\widetilde{C}(\|x-y\|,|t-s|)} \exp\left(-\frac{u^2(T)}{1+v}\right)\frac{dv}{\sqrt{1-v^2}} dx dy dt ds.
\end{aligned} \tag{48}$$

Thus, from Eq. (48), operating in a similar way to Eq. (6) in the integrals over time and space, we have

$$\begin{aligned}
\mathrm{Var}(M(T)) &= E\left[M^2(T)\right] - [E[M(T)]]^2 \\
&= 2T|K|^2T^{2\gamma d}\int_0^T\left(1 - \frac{\tau}{T}\right)\int_0^{\mathcal{D}(T^\gamma K)} z^{d-1}\widetilde{\psi}_{\rho_{T^\gamma K}}(z) \\
&\quad \times\frac{1}{2\pi}\int_0^{C(z,\tau)} \exp\left(-\frac{u^2(T)}{1+v}\right)\frac{dv}{\sqrt{1-v^2}} dz d\tau,
\end{aligned} \tag{49}$$

where $\widetilde{\psi}_{\rho_{T^\gamma K}}(z) = (z^{-(d-1)})\psi_{\rho_{T^\gamma K}}(z)$, with $\psi_{\rho_{T^\gamma K}}(z)$ being, as before, the probability density of the random variable $\rho_K = \|P_1 - P_2\|$, where P_1 and P_2 are two independent random points with uniform distribution in $T^\gamma K$ (see also Remark 2). The proof is based on verifying that **Condition 5** holds under the assumptions made, and hence, applying Theorem 3 for $m = 1$. Specifically, we prove that

$$\overline{\lim}_{T\to\infty}\frac{\mathrm{Var}(M(T))}{\mathcal{J}_1^2(T)\sigma^2_{1,KT^\gamma}(T)} \leq 1.$$

Note that $\mathcal{J}_1(T) = \phi(u(T)) = \frac{1}{\sqrt{2\pi}} \exp(-(u^2(T))/2) > 0$, and

$$\frac{\exp\left(-\frac{u^2(T)}{1+v}\right)}{\phi^2(u(T))} = 2\pi \exp\left(u^2(T)\left(\frac{v}{1+v}\right)\right). \tag{50}$$

Hence, from Eqs. (49) and (50),

$$\begin{aligned}
\frac{\mathrm{Var}(M(T))}{[\phi(u(T))]^2} &= 2T|K|^2 T^{\gamma 2d} \int_0^T \left(1-\frac{\tau}{T}\right) \int_0^{\mathcal{D}(T^\gamma K)} z^{d-1} \widetilde{\psi}_{\rho_{T^\gamma K}}(z) \\
&\quad \times \int_0^{C(z,\tau)} \exp\left(u^2(T)\left(\frac{v}{1+v}\right)\right) \frac{dv}{\sqrt{1-v^2}} dz d\tau \\
&= 2T|K|^2 T^{2\gamma d} \left[\int_{\overline{B}_T^{\beta_1,\beta_2}} + \int_{B_T^{\beta_1,\beta_2}}\right] \left(1-\frac{\tau}{T}\right) z^{d-1} \widetilde{\psi}_{\rho_{T^\gamma K}}(z) \\
&\quad \times \int_0^{C(z,\tau)} \exp\left(u^2(T)\left(\frac{v}{1+v}\right)\right) \frac{dv}{\sqrt{1-v^2}} dz d\tau \\
&= S_1(T) + S_2(T),
\end{aligned} \tag{51}$$

where the set $B_T^{\beta_1,\beta_2}$ has been introduced in Eq. (45), and the set $\overline{B}_T^{\beta_1,\beta_2}$ is defined as

$$\overline{B}_T^{\beta_1,\beta_2} = \left\{(z,\tau);\ 0 \le \tau \le T^{\beta_1},\ 0 \le z \le T^{\gamma\beta_2}\right\},$$

with $\beta_1 \in (0, \delta_1)$, $\beta_2 \in (0, \delta_2)$.

From **Condition 6**, applying Remark 5, $u^2(T) = o\left(\log((T)^{\varepsilon_1}([\Lambda(T)]^d)^{\varepsilon_2})\right)$, for any $\varepsilon_1, \varepsilon_2 > 0$. Keeping in mind that $\frac{v}{1+v} \le 1$, we obtain

$$\begin{aligned}
S_1(T) &\le k_8\ T^{1+2\gamma d} \int_{\overline{B}_T^{\beta_1,\beta_2}} \left(1-\frac{\tau}{T}\right) z^{d-1} \widetilde{\psi}_{\rho_{T^\gamma K}}(z) \\
&\quad \times \exp\left(u^2(T)\right) \int_0^{C(z,\tau)} \frac{dv}{\sqrt{1-v^2}} dz d\tau \\
&= k_8\ T^{1+2\gamma d} \exp\left(u^2(T)\right) \int_{\overline{B}_T^{\beta_1,\beta_2}} \left(1-\frac{\tau}{T}\right) z^{d-1} \widetilde{\psi}_{\rho_{T^\gamma K}}(z) \\
&\quad \times \arcsin C(z,\tau) dz d\tau \\
&\le \widetilde{k}_8\ T^{1+\varepsilon_1} T^{2d\gamma + d\gamma\varepsilon_2} \int_{\overline{B}_T^{\beta_1,\beta_2}} \left(1-\frac{\tau}{T}\right) z^{d-1} \widetilde{\psi}_{\rho_{T^\gamma K}}(z) dz d\tau \\
&\le k_9 T^{1+\beta_1+\varepsilon_1} [\Lambda(T)]^{d+d\beta_2+d\varepsilon_2},
\end{aligned} \tag{52}$$

for some positive constants k_8, and k_9. In particular, for $\varepsilon_i = \delta_i - \beta_i, i = 1, 2$, under **Condition 4(ii)**, as $T \to \infty$,

$$\frac{S_1(T)}{\sigma_{1,K}^2(T)} \le \frac{k_9}{\frac{1}{T^{1+\delta_1} T^{\gamma d(1+\delta_2)}} \sigma_{1,K}^2(T)} \to 0. \tag{53}$$

Let us now consider $S_2(T)$, under **Condition 6**,

$$\sup_{(z,\tau)\in B_T^{\beta_1,\beta_2}} C(z,\tau) \to 0, \quad T \to \infty$$

$$B_T^{\beta_1,\beta_2} = \{(z,\tau);\ \tau \ge T^{\beta_1} \text{ or } z \ge T^{\gamma\beta_2}\},$$

and, since $\frac{1}{1+v} \leq 1$ and $\frac{1}{\sqrt{1-v^2}} \to 1$, $0 \leq v \leq C(z,\tau)$, $C(z,\tau) \to 0$, $(z,\tau) \in B_T^{\beta_1,\beta_2}$, as $T \to \infty$, we obtain that $S_2(T)$ satisfies

$$
\begin{aligned}
S_2(T) &= 2T^{1+2\gamma d}|K|^2 \int_{B_T^{\beta_1,\beta_2}} \left(1-\frac{\tau}{T}\right)\psi_{\rho_{T^\gamma K}}(z) \\
&\quad \times \int_0^{C(z,\tau)} \exp\left(u^2(T)\frac{v}{1+v}\right)\frac{dv}{\sqrt{1-v^2}}dz d\tau \\
&\leq 2T^{1+2\gamma d}|K|^2 \int_{B_T^{\beta_1,\beta_2}} \left(1-\frac{\tau}{T}\right)\psi_{\rho_{T^\gamma K}}(z) \\
&\quad \times \int_0^{C(z,\tau)} \exp\left(u^2(T)v\right)\frac{dv}{\sqrt{1-v^2}}dz d\tau \\
&\leq k_{10} 2T^{1+2\gamma d}|K|^2 \exp\left(u^2(T)\sup_{(z,\tau)\in B_T^{\beta_1,\beta_2}} C(z,\tau)\right) \\
&\quad \times \int_{B_T^{\beta_1,\beta_2}} \left(1-\frac{\tau}{T}\right)\psi_{\rho_{T^\gamma K}}(z) C(z,\tau) dz d\tau,
\end{aligned}
$$

for some positive constant k_{10}. Hence, for T sufficiently large,

$$
\begin{aligned}
\frac{S_2(T)}{\sigma_{1,K}^2(T)} &\leq k_{11}\exp\left(u^2(T)\sup_{(z,\tau)\in B_T^{\beta_1,\beta_2}} C(z,\tau)\right) \\
&\quad \times \frac{\int_{B_T^{\beta_1,\beta_2}}\left(1-\frac{\tau}{T}\right)\psi_{\rho_{T^\gamma K}}(z)C(z,\tau)dz d\tau}{\int_{\overline{B}_T^{\beta_1,\beta_2}\cup B_T^{\beta_1,\beta_2}}\left(1-\frac{\tau}{T}\right)\psi_{\rho_{T^\gamma K}}(z)C(z,\tau)dz d\tau} \leq 1,
\end{aligned}
\tag{54}
$$

since under **Condition 6**,

$$
\exp\left(u^2(T)\sup_{(z,\tau)\in B_T^{\beta_1,\beta_2}} C(z,\tau)\right) \to 1, \quad T \to \infty,
$$

and $k_{11} \leq 1$, for sufficiently large T. Finally, from (53)–(54),

$$
\begin{aligned}
&\overline{\lim}_{T\to\infty}\frac{\mathrm{Var}(M(T))}{\phi^2(u(T))\sigma_{1,K}^2(T)} \\
&= \overline{\lim}_{T\to\infty}\left(\frac{S_1(T)}{\sigma_{1,K}^2(T)} + \frac{S_2(T)}{\sigma_{1,K}^2(T)}\right) \leq 1.
\end{aligned}
\tag{55}
$$

Thus, the desired result follows from Theorem 3, and the asymptotic normality of

$$
\frac{\int_0^T\int_{T^\gamma K} Z(x,t)dx dt}{\left[2T|K|^2T^{2d\gamma}\int_0^T\left(1-\frac{\tau}{T}\right)\int_0^{\mathcal{D}(T^\gamma K)} C(z,\tau)\psi_{\rho_{T^\gamma K}}(z,\tau)dz d\tau\right]^{1/2}}.
$$

□

7 Sojourn Functionals for a Class of Spherical Random Fields

This section derives a central limit result for sojourn functionals subordinated to STGRFs homogeneous and isotropic in space, and stationary in time, restricted to the unit sphere.

Let $\mathbb{S}_{d-1}(1) = \{x \in \mathbb{R}^d;\ \|x\| = 1\}$ be the unit sphere embedded into $\mathbb{R}^d$, for some $d \geq 2$, and denote by $d\nu_{d-1}(x)$ the normalized Riemannian measure on $\mathbb{S}_{d-1}(1)$. Denote also by $\theta = \arccos\left(\langle x, x'\rangle\right)$ the angle between two points $x, x' \in \mathbb{S}_{d-1}(1)$. For every $x, x' \in \mathbb{S}_{d-1}(1)$, $\|x - x'\| = 2\sin\left(\frac{\theta}{2}\right)$, with $\|\cdot\|$ being the Euclidean distance. Let us denote $S^{(d)}_{lm}(u)$, $u \in \mathbb{S}_{d-1}(1)$, $m = 1, 2, \ldots, h(l, d)$, $l \in \mathbb{N}_0$, the real spherical harmonics on $\mathbb{S}_{d-1}(1)$ (see Leonenko [24], Müller [38], and the references therein), with $h(l, d) = (2l + d - 2)\frac{(l+d-3)!}{(d-2)!l!}$ denoting the dimension of the eigenspace of the Laplace Beltrami operator generated by

$$\left\{S^{(d)}_{lm},\quad m = 1, 2, \ldots, h(l, d)\right\},\quad l \in \mathbb{N}_0.$$

Let $\left\{Z(x, t),\ x \in \mathbb{R}^d,\ t \in \mathbb{R}\right\}$ be a zero-mean, mean-square continuous, STGRF, homogeneous and isotropic in space, and stationary in time, with covariance function satisfying

$$\widetilde{C}(\|x\|, \tau) = 2^{\frac{d-2}{2}+1}\Gamma\left(\frac{d}{2}\right)\int_0^\infty \cos(\mu\tau)\int_0^\infty \frac{J_{\frac{d-2}{2}}(\lambda\|x\|)}{(\lambda\|x\|)^{\frac{d-2}{2}}}\mathcal{G}(d\lambda, d\mu), \tag{56}$$

where $\mathcal{G}$ is defined from the spectral measure F of Z arising in Bochner Theorem (see, e.g., Ivanov and Leonenko [19], Schoenberg [41]) satisfying

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} F(d\widetilde{\omega}, d\widetilde{\mu}) = \int_0^\infty\int_0^\infty \mathcal{G}(d\lambda, d\mu) < \infty,$$
$$\mathcal{G}(\lambda, \mu) = \int_{\|\widetilde{\omega}\|<\lambda}\int_{|\widetilde{\mu}|<\mu} F(d\widetilde{\omega}, d\widetilde{\mu}). \tag{57}$$

Let us consider $T_R(x, t) = Z(x, t)$, for every $x \in \mathbb{S}_{d-1}(1)$, and $t \in \mathbb{R}$, defining the restriction of $Z(x, t)$ to the unit sphere $\mathbb{S}_{d-1}(1)$. The following identities will be applied in the characterization of the second-order pure point spectral properties of $T_R(x, t)$ from the spectral representation (56) of the covariance function $\widetilde{C}$ of Z.

In the following, we denote $\mathbb{S}_{d-1}(u) = \left\{x \in \mathbb{R}^d;\ \|x\| = u\right\}$. Let us first consider the characteristic function of $\mathbb{S}_{d-1}(u)$

$$\frac{1}{|\mathbb{S}_{d-1}(u)|}\int_{\mathbb{S}_{d-1}(u)} \exp\left(i\,\langle\lambda, x\rangle\right) d\nu_{d-1}(\lambda) = Y_d(ux), \tag{58}$$

where Y_d denotes the spherical Bessel function, and $d\nu_{d-1}$ is the normalized Riemannian measure on $\mathbb{S}_{d-1}(1)$. Applying Eq. (1.2.13a) in Ivanov and Leonenko [19], the following relationship holds between the spherical Bessel function and the Bessel function of the first kind

$$Y_d(z) = 2^{(d-2)/2}\Gamma\left(\frac{d}{2}\right) J_{\frac{d-2}{2}}(z) z^{(2-d)/2},\quad z \geq 0, \tag{59}$$

where

$$J_\nu(z) = \sum_{m=0}^\infty (-1)^m \left(\frac{z}{2}\right)^{2m+\nu}[m!\Gamma(m+\nu+1)]^{-1}$$

is the Bessel function of the first kind of order $\nu > 1/2$.

Thus, from Eqs. (58) and (59),

$$\frac{1}{|\mathbb{S}_{d-1}(u)|}\int_{\mathbb{S}_{d-1}(u)} \exp\left(i\left\langle \lambda, x-x'\right\rangle\right) d\nu_{d-1}(\lambda)$$
$$= 2^{(d-2)/2}\Gamma\left(\frac{d}{2}\right)\frac{J_{\frac{d-2}{2}}(u\|x-x'\|)}{(u\|x-x'\|)^{\frac{d-2}{2}}}. \tag{60}$$

Applying now addition theorem of spherical Bessel function

$$Y_d(\lambda\rho) = c_1^2(d)\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,d)} S_{lm}^{(d)}(u)S_{lm}^{(d)}(v)\frac{J_{l+\frac{d-2}{2}}(\lambda r_1)}{(\lambda r_1)^{\frac{d-2}{2}}}\frac{J_{l+\frac{d-2}{2}}(\lambda r_2)}{(\lambda r_2)^{\frac{d-2}{2}}}, \tag{61}$$

where $c_1^2(d) = 2^{d-1}\Gamma\left(\frac{d}{2}\right)\pi^{d/2}$, and

$$\begin{aligned} &x = (r_1, u),\ r_1 \geq 0,\ u = x/\|x\| \in \mathbb{S}_{d-1}(1)\\ &y = (r_2, v),\ r_2 \geq 0,\ v = y/\|y\| \in \mathbb{S}_{d-1}(1)\\ &\rho = \|x-y\| = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\gamma)}, \quad \cos(\gamma) = \frac{\langle x, y\rangle}{\|x\|\|y\|},\ \lambda \geq 0. \end{aligned} \tag{62}$$

From Eqs. (58)–(62),

$$\begin{aligned} &\widetilde{C}(\|x-y\|,\tau)\\ &= 2[c_1(d)]^2\sum_{l=0}^{\infty}\left[\int_0^{\infty}\int_0^{\infty}\cos(\mu\tau)\frac{J_{l+\frac{d-2}{2}}(\lambda r_1)}{(\lambda r_1)^{\frac{d-2}{2}}}\frac{J_{l+\frac{d-2}{2}}(\lambda r_2)}{(\lambda r_2)^{\frac{d-2}{2}}}\mathcal{G}(d\lambda, d\mu)\right]\\ &\qquad\times\sum_{m=1}^{h(l,d)} S_{lm}^{(d)}(u)S_{lm}^{(d)}(v), \end{aligned} \tag{63}$$

where $(x, y), (u, v)$, and (r_1, r_2) are defined as in Eq. (62). For $r_1 = r_2 = 1$, that is, in the case of considering the covariance function $\widetilde{C}_R$ of the restricted random field $\{T_R(x,t),\ x \in \mathbb{S}_{d-1}(1),\ t \in \mathbb{R}\}$, we obtain

$$\begin{aligned} &\widetilde{C}_R(\|x-y\|,\tau)\\ &= 2[c_1(d)]^2\sum_{l=0}^{\infty}\left[\int_0^{\infty}\int_0^{\infty}\cos(\mu\tau)\left[\frac{J_{l+\frac{d-2}{2}}(\lambda)}{\lambda^{\frac{d-2}{2}}}\right]^2\mathcal{G}(d\lambda, d\mu)\right]\\ &\qquad\times\sum_{m=1}^{h(l,d)} S_{lm}^{(d)}(u)S_{lm}^{(d)}(v). \end{aligned} \tag{64}$$

Random field $\{T_R(x,t),\ x \in \mathbb{S}_{d-1}(1),\ t \in \mathbb{R}\}$ then has τ-varying angular power spectrum $\{A_l(\tau),\ \tau \geq 0,\ l \in \mathbb{N}_0\}$ given by

$$A_l(\tau) = 2\Gamma\left(\frac{d}{2}\right)\pi^{d/2}\int_0^{\infty}\int_0^{\infty}\cos(\mu\tau)\left[\frac{J_{l+\frac{d-2}{2}}(\lambda)}{\lambda^{\frac{d-2}{2}}}\right]^2\mathcal{G}(d\lambda, d\mu),\ \tau \geq 0,\ l \in \mathbb{N}_0. \tag{65}$$

Equation (65) allows the interpretation of the elements of the τ-varying angular spectrum $\{A_l(\tau),\ \tau \geq 0,\ l \in \mathbb{N}_0\}$ as the inverse Fourier transforms of the temporal spectral measures

$f(d\mu)=2\Gamma\left(\frac{d}{2}\right)\pi^{d/2}\int_0^\infty\left[\frac{J_{l+\frac{d-2}{2}}(\lambda)}{\lambda^{\frac{d-2}{2}}}\right]^2\mathcal{G}(d\lambda,d\mu)$. Equivalently,

$$\begin{aligned} A_l(\tau) &= \int_0^\infty \cos(\mu\tau) f(d\mu) \\ &= 2\Gamma\left(\frac{d}{2}\right)\pi^{d/2}\int_0^\infty \cos(\mu\tau)\int_0^\infty\left[\frac{J_{l+\frac{d-2}{2}}(\lambda)}{\lambda^{\frac{d-2}{2}}}\right]^2\mathcal{G}(d\lambda,d\mu). \end{aligned} \tag{66}$$

Note also that, from (56), applying trigonometric identity $\|x-x'\|=2\sin\left(\frac{\theta}{2}\right)$, for every $x,x'\in\mathbb{S}_{d-1}(1)$,

$$\begin{aligned} &\widetilde{C}_R(\|x-x'\|,\tau)=2^{\frac{d-2}{2}+1}\Gamma\left(\frac{d}{2}\right)\int_0^\infty\cos(\mu\tau)\int_0^\infty\frac{J_{\frac{d-2}{2}}(\lambda\|x\|)}{(\lambda\|x\|)^{\frac{d-2}{2}}}\mathcal{G}(d\lambda,d\mu) \\ &2^{\frac{d-2}{2}+1}\Gamma\left(\frac{d}{2}\right)\int_0^\infty\cos(\mu\tau)\int_0^\infty\frac{J_{\frac{d-2}{2}}\left(\lambda 2\sin\left(\frac{\theta}{2}\right)\right)}{\left(\lambda 2\sin\left(\frac{\theta}{2}\right)\right)^{\frac{d-2}{2}}}\mathcal{G}(d\lambda,d\mu) \\ &=C_R(\cos(\theta),\tau), \end{aligned} \tag{67}$$

where as before, θ denotes the angle between vectors x and x' in $\mathbb{S}_{d-1}(1)$. Thus, T_R has covariance function (67) (see, e.g., Leonenko and Ruiz-Medina [27]).

Condition K. Random field $\{T_R(x,t),\ x\in\mathbb{S}_{d-1}(1),\ t\in\mathbb{R}\}$ is defined on the sphere as the restriction of a zero-mean STGRF with covariance function (56).

Under *Condition K* (see Eqs. (64)–(66)), random field T_R admits the following orthogonal expansion, in the mean-square sense, for every fixed $t\in\mathbb{R}$, and $x\in\mathbb{S}_{d-1}(1)$,

$$T_R(x,t)=\sum_{l=0}^{\infty}\sum_{m=1}^{h(l,d)}a_{lm}(t)S_{lm}^{(d)}(x),$$

where $a_{lm}(t)$, $m=1,\dots,h(l,d)$, $l\in\mathbb{N}_0$, are independent zero-mean Gaussian stochastic processes such that $E[a_{lm}(t)]=0$, $E[a_{lm}(t)a_{l'm'}(t')]=\delta_{ll'}\delta_{mm'}A_l(|t-t'|)$, with $A_l(|t-t'|)=A_l(\tau)$ satisfying (66), and $\sum_{l=0}^{\infty}h(l,d)A_l(\tau)<\infty$, for every $\tau\in\mathbb{R}_+$.

Let now consider the first Minkowski functional subordinated to $T_R(x,t)$, given by

$$\begin{aligned} N_T &= \int_0^T\int_{\mathbb{S}_{d-1}(1)}\mathbb{I}_{T_R(x,t)\geq u}\,d\nu_{d-1}(x)dt \\ &= |\{0\leq t\leq T;\ T_R(x,t)\geq u,\ x\in\mathbb{S}_{d-1}(1)\}| \\ &= E[N_T]+\sum_{n\geq 1}\frac{\mathcal{J}_n}{n!}\eta_n(T), \end{aligned} \tag{68}$$

where $E[N_T]=(1-\Phi(u))\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$, and $\mathcal{J}_n(u)=\phi(u)H_{n-1}(u)$, $n\geq 1$, and

$$\eta_n(T)=\int_0^T\int_{\mathbb{S}_{d-1}(1)}H_n(T_R(x,t))\,d\nu_{d-1}(x)dt.$$

Thus, $E[\eta_n(T)] = 0$, $E[\eta_n(T)\eta_l(T)] = 0$, $n \neq l$, and

$$
\begin{aligned}
\sigma_n^2(T) &= [\eta_n^2(T)] \\
&= 2n! \int_0^T \int_0^T \int_{\mathbb{S}_{d-1}(1)\times\mathbb{S}_{d-1}(1)} \widetilde{C}^n \left(\|x - x'\|, |t - t'|\right) d\nu(x) d\nu(x') dt dt' \\
&= 2n! T |\mathbb{S}_{d-1}(1)|^2 \int_0^T \left(1 - \frac{\tau}{T}\right) E\left(\widetilde{C}^n(\|W_1 - W_2\|, \tau)\right) d\tau \\
&= 2n! T |\mathbb{S}_{d-1}(1)|^2 \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{d}{2}\right) \Gamma^{-1}\left(\frac{d-1}{2}\right) \\
&\quad \times \int_0^T \int_0^2 \left(1 - \frac{\tau}{T}\right) z^{d-2} \left(1 - \frac{z^2}{4}\right)^{\frac{d-3}{2}} C^n(z, \tau) dz d\tau \\
&= 2n! T 4\pi^{d-1/2} \Gamma\left(\frac{d}{2}\right) \left[\Gamma\left(\frac{d-1}{2}\right)\right]^{-1} \\
&\quad \times \int_0^T \int_0^2 \left(1 - \frac{\tau}{T}\right) z^{d-2} \left(1 - \frac{z^2}{4}\right)^{\frac{d-3}{2}} C^n(z, \tau) dz d\tau,
\end{aligned}
\tag{69}
$$

where C^n denotes, as before, the nth power of the covariance function C. Here, W_1 and W_2 are two independent uniformly distributed random vectors on $\mathbb{S}_{d-1}(1)$ with probability density of their Euclidean distance given by, for $0 \leq z \leq 2$ (see Lemma 1.4.4 in Ivanov and Leonenko [19])

$$
\frac{d}{dz} P\left[\|W_1 - W_2\| \leq z\right] = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{d}{2}\right) \Gamma^{-1}\left(\frac{d-1}{2}\right) z^{d-2} \left(1 - \frac{z^2}{4}\right)^{\frac{d-3}{2}}. \tag{70}
$$

Condition L.

(i) Assume that $\sup_{z\in[0,2]} C(z, \tau) \to 0$ as $\tau \to \infty$.
(ii) There exists $\delta \in (0, 1)$, such that

$$
\lim_{T\to\infty} \frac{1}{T^{\delta}} \int_0^T \left(1 - \frac{\tau}{T}\right) \int_0^2 z^{d-2} C(z, \tau) \left(1 - \frac{z^2}{4}\right)^{\frac{d-3}{2}} dz d\tau = \infty.
$$

Remark 7 Note that, for the restriction to the sphere of a STGRF with covariance function as in **Example 1**, Condition L holds if $d \geq 3$.

Under *Conditions K and L*, applying the corresponding reduction theorem, the following central limit result follows.

Theorem 5 *Assume that* Conditions K *and* L *hold. Then, the random variable*

$$
\frac{N_T - T(1 - \Phi(u))|\mathbb{S}_{d-1}(1)|}{\phi(u)\left[\frac{8\pi^{d-\frac{1}{2}}}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d-1}{2}\right)} T \int_0^T \left(1 - \frac{\tau}{T}\right) \int_0^2 C(z, \tau) z^{d-2} \left(1 - \frac{z^2}{4}\right)^{\frac{d-3}{2}} dz d\tau\right]^{1/2}}
$$

has asymptotically standard normal distribution as $T \to \infty$.

The following central limit result is obtained for T-varying thresholds.

Theorem 6 *Assume that* Conditions *K and L hold, and there exists* $\beta \in (0, \delta)$ *such that as* $T \to \infty$,

$$u^2(T) \sup_{u\in(0,2)} C(u, T^{\beta}) \to 0, \quad u^2(T) = o(\log(T)).$$

Then, the random variable

$$\frac{N_T^{\star} - T(1 - \Phi(u(T))|\mathbb{S}_{d-1}(1)|}{\phi(u(T))\left[\frac{8\pi^{d-\frac{1}{2}}}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d-1}{2}\right)} T\int_0^T \left(1-\frac{\tau}{T}\right)\int_0^2 C(z,\tau)z^{d-2}\left(1-\frac{z^2}{4}\right)^{\frac{d-3}{2}} dzd\tau\right]^{1/2}},$$

has asymptotically a standard normal distribution, as $T \to \infty$, *where*

$$N_T^{\star} = |\{0 \le t \le T;\ T_R(x,t) \ge u(T),\ x \in \mathbb{S}_{d-1}(1)\}|\,.$$

The proofs of Theorems 5 and 6 can be obtained from Eqs. (69) and (70), in a similar way to the proofs of Theorems 2 and 4, respectively.

7.1 Spherical Spatiotemporal Covariance Functions

Special cases of stationary covariance functions on spheres cross time have recently been analyzed in White and Porcu [46]. In our case, we pay special attention to the family of nonseparable covariance functions introduced in Eq. (11) in White and Porcu [46], since its restriction to $\mathbb{S}_{d-1}(1)$ can be considered as proposed here. In addition, in Theorem 2 in White and Porcu [46], competitive models of spatiotemporal spherical covariance functions are proposed for real data analysis. In particular, the covariance function family

$$C(\theta, u) = \frac{\sigma^2}{\psi(u^2)}\varphi\left(\frac{\theta}{\psi(u^2)}\right), \quad \theta \in [0,\pi],\ u \in \mathbb{R},$$

is considered for surface air temperature reanalysis data. These covariance modes capture the strong spatial structure displayed by data given by daily temperature averages over a global grid. Since the overall temperature distribution is similar across days displaying a clear spatial structure, the implemented spatiotemporal spherical covariance models that rescale space with time, and are expressed in terms of the geodesic spherical distance instead of the Euclidean distance, allow to effectively capture the strong spatial structure in this type of data.

Note also that, one can consider the restriction to the sphere of the covariance function family considered in Eqs. (6) and (11) in White and Porcu [46], and beyond. Equations (64) and (67) then hold for such a restriction. Specifically, in Eq. (30), we consider, for every $u \ge 0$,

$$\begin{aligned}\varphi(u) &= \left(2^{\nu-1}\Gamma(\nu)\right)^{-1}\left(cu^{1/2}\right)^{\nu} K_{\nu}\left(cu^{1/2}\right), \quad c>0,\ \nu>0\\ \psi(u) &= (1+au^{\alpha})^{\beta}, \quad a>0,\ 0<\alpha\le 1, \quad 0<\beta\le 1,\end{aligned} \tag{71}$$

where $K_{\nu}(z)$ is the modified Bessel function of the second kind of order ν, or MacDonald function (see, e.g., Gradshteyn and Ryzhik [16]). Thus,

$$\varphi(\|z\|^2) = \sigma^2\left(2^{\nu-1}\Gamma(\nu)\right)^{-1}(c\,\|z\|)^{\nu} K_{\nu}(c\,\|z\|)\,, \quad \sigma^2>0, \quad c>0, \quad \nu>0,\ z \in \mathbb{R}^d \tag{72}$$

with associated Fourier transform

$$\widehat{\varphi}(\lambda) = \mathcal{M}\left(c^2 + \|\lambda\|^2\right)^{-\left(\nu+\frac{d}{2}\right)}, \quad \lambda \in \mathbb{R}^d, \quad \mathcal{M} > 0, \tag{73}$$

that is involved in the definition of the spectral measure F in Eq. (57). Its restriction to $\mathbb{S}_{d-1}(1)$ then leads to an alternative family of spherical covariance functions to the ones considered in White and Porcu [46].

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Data Availability No data were created during preparation of the present paper.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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