# Pointwise decay of cumulants in chaotic states at low density

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**Abstract.** We study simple non-equilibrium distributions describing a classical gas of particles interacting via a pair potential $\phi(x/\varepsilon)$, in the Boltzmann-Grad scaling $\varepsilon \to 0$. We establish bounds for truncated correlations (cumulants) of arbitrary order as a function of the internal separation of particles in a cluster, showing exponential or polynomial decay, for finite or infinite range interactions respectively.

Keywords: Boltzmann-Grad limit; low density; kinetic theory; cluster expansion; correlation functions.

## 1. INTRODUCTION

This note reviews the cumulant method and discusses some of its implications from the point of view of kinetic theory. The cumulants allow a quantitative analysis of the small correlations in chaotic states of a rarefied gas. Typically, one assumes that particles have a little "core" of size $\varepsilon$, meaning that their interaction is repulsive at the origin, and negligible on scales larger than $\varepsilon$. The number of particles is increased with $\varepsilon$ in such a way that, letting the system evolve, we would see in average a finite number of interactions per unit of time, and a variation of the macroscopic density on distances of order one (order of the mean free path).

This is known as Boltzmann-Grad regime. Its interest arises in connection with validation issues in kinetic theory, as it is the regime in which the Boltzmann equation has been proved to hold rigorously, at least for short range potentials and small enough times; see [23] for a foundational paper and [8, 13, 21, 22, 33, 38] for surveys. In such works, the famous "Stosszahlansatz" of Boltzmann is ensured by the randomness of the initial conditions: one starts with a chaotic measure such that the joint $j$-particle distributions factorize as $\varepsilon \to 0$, and shows that the factorization is propagated at positive times.

It is difficult to establish the minimal set of probabilistic hypotheses on the initial conditions that allow the result to hold. Actually, one gives sufficient conditions, and verifies at the end that these conditions are valid for simple microscopic distributions. Therefore in practice, one starts with measures proportional to $f^{\otimes n} e^{-\beta U_n^\varepsilon}$, $\beta > 0$ where $f$ is an assigned (non-equilibrium) $1-$particle density, and $U_n^\varepsilon$ is the potential energy of the $n$-particle configuration. As $U_n^\varepsilon$ is concentrated on configurations of particles having mutual distance $\varepsilon$, this measure can be regarded as a quasi-Poisson, or "maximally factorized" state. The only source of correlation comes from the interaction at short scale.

Our aim here is to quantify such correlations in detail, and especially how they decay in space, passing from microscopic to macroscopic scales. To this end, a powerful tool at hand is provided by cumulants, or truncated correlation functions. Roughly, the $j$th cumulant measures the collective correlation in a cluster of $j$ particles. This is a standard notion in statistical mechanics; see for instance [15–17] where strong bounds on truncated functions are used to prove analyticity of thermodynamic functionals; or [14] for a recent investigation of the Ising model.

In the kinetic limit of non-equilibrium processes, truncated functions have been carefully studied in recent years. It has been argued that, as $\varepsilon \to 0$, the full collection of cumulants retains a complete information about the dynamical correlations [6] (see also [20, 26]); in particular, $L^1$ estimates on cumulants have been used to provide a rigorous theory of fluctuations and large deviations of a hard sphere gas on short time scales [7, 12, 37]. Moreover, $L^2$ estimates on similar truncated functions have been used to extend the equilibrium fluctuation theory up to hydrodynamic scales [5, 9–11, 24]. It seems however hard, in general, to obtain estimates in more accurate norms.

Most of the mathematical literature on the Boltzmann-Grad limit focuses on the simplest case of hard-sphere interactions (see [2, 18, 31] for exceptions). Nevertheless the main results are expected to hold as well for smooth short range interactions (for which they are proved in some cases), and even for power laws with fast enough decay (for which no result is available). Notice that, in the particular case of hard spheres, the maximally chaotic state (MCS from now on) is a pure product up to the excluded volume; while for long range interactions the product structure is perturbed by the tails.

In this paper we study grand canonical, maximally chaotic distributions for stable interactions with fast decay. We shall not be concerned with the dynamical problem, but only with the MCS itself. For such states, we prove decay of the $j$th truncated function, for arbitrary $j$, in terms of a suitable quantity measuring the internal spatial separation in a cluster of $j$ particles. This translates into an exponential decay in $\varepsilon$ at macroscopic distances in the case of compactly supported interactions (actually true for exponentially decaying interactions as well). Instead, the decay is in general polynomial in the case of interactions with an infinite range. We will deal in this paper only with integrable potential tails decaying as $|x|^{-s}$ with $s > d - 1$: for slower decay rates, screening-type effects need to be taken into account (see e.g. [1, 4, 40], and [27, 28] for predictions on the structure of the kinetic limit for long-range interactions).

Besides the role of "typical" initial state in the rigorous derivation of the Boltzmann equation, we conclude by mentioning yet another application of MCS to the dynamical problem. Of course the dynamics prevents the MCS structure to be propagated in time. Nevertheless, these states have been proved useful as an argument to obtain finite density corrections to the Boltzmann equation [35]. In the previous reference it is shown (restricting to hard spheres) that, if the ansatz of maximal chaoticity is made *at any time* (thus disregarding dynamical correlations), then the density evolves according to an Enskog equation, and an $H-$Theorem can be deduced for the associated entropy; see also [3, 25]. These remarks have not been object of a fully rigorous investigation so far; see however [19, 32, 34] for related discussions.

The rest of this work is organized in two parts. In Section 2 we set the assumptions on the interaction, introduce the maximally chaotic states, and state our main result (Theorem 4 below). Section 3 is devoted to the proof. This combines classical cluster expansion (c.f. [36]) with estimates on long chains of interacting particles.

## 2. ESTIMATES ON TRUNCATED CORRELATIONS

### A. Potentials and kinetic limit

We consider a system of identical point particles placed in an open connected set $\Lambda \subset \mathbb{R}^d$, $d \geq 1$ and we denote by

$$z_i = (x_i, v_i) \in \Lambda \times \mathbb{R}^d \ , \qquad i \in \mathbb{N} \ ,$$

the configuration of the $i$-th particle having position $x_i$ and velocity $v_i$. With

$$\boldsymbol{z}_j = (z_1, \dots, z_j) \ , \qquad j \in \mathbb{N}$$

we indicate the configuration of the first $j$ particles.

The grand canonical phase space is

$$\Omega = \bigcup_{n \geq 0} \Omega_n \ ,$$

where the canonical $n$-particle phase space is

$$\Omega_n = \left\{ \boldsymbol{z}_n \in \left( \Lambda \times \mathbb{R}^d \right)^n \mid x_i \neq x_k \ , \ i, k = 1, \dots, n, \ i \neq k \right\} \ .$$

The particles interact through a translation invariant pair potential given by a piecewise continuous function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$. As customary in statistical mechanics [36], we assume:

- *stability*: there exists a constant $B \geq 0$ such that

$$\sum_{1 \leq i < k \leq n} \phi(x_i - x_k) \geq -Bn \ , \tag{2.1}$$

  for all $n \geq 0$ and $x_1, \dots, x_n \in \mathbb{R}^d$;

- *decay*:

$$C_\beta := \int_{\mathbb{R}^d} \mathrm{d}x \, |\zeta(x, 0)| < +\infty \ , \qquad \beta > 0 \tag{2.2}$$

  where $\zeta(x_i, x_k) := \mathrm{e}^{-\beta \phi(x_i - x_k)} - 1$.

Stability is usually required to ensure the existence of a partition function. In our setting, the stability property is motivated by [31], where the validity theorem for the Boltzmann equation has been extended to forces including an attractive part provided that condition (2.1) holds.

Since the potential is bounded below, (2.2) corresponds to a fast enough decrease at infinity. Although we will be actually be interested in finite range interactions (for which (2.2) is trivially true), our result could be extended to any potential satisfying the above decay assumption.

We are interested here in a low density regime where particles interact strongly at a short distance $\varepsilon > 0$. We will use then the potential in the rescaled form $\phi\,(x/\varepsilon)$. In the Boltzmann-Grad scaling of a constant mean free path, the average number of particles should diverge as

$$\mu_\varepsilon := \varepsilon^{-(d-1)}\ . \tag{2.3}$$

We introduce next a class of measures describing this asymptotic.

### B. Grand canonical chaotic states

By definition, a *state* of the system is an absolutely continuous measure on $\Omega$ admitting a set of densities $\{W_n^\varepsilon\}_{n\geq 0}$, where $W_n^\varepsilon : \Omega_n \to \mathbb{R}^+$ are positive Borel functions. For each $n$, $W_n^\varepsilon$ is assumed to be invariant under permutations of the particle labels. The quantity $(1/n!)W_n^\varepsilon(\boldsymbol{z}_n)$ represents the probability density of finding $n$ particles in the configuration $\boldsymbol{z}_n = (z_1, \ldots, z_n)$. In particular the distribution of the number of particles is $(1/n!)\int W_n^\varepsilon$ and the normalization condition reads

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Omega_n} W_n^\varepsilon(\boldsymbol{z}_n)\, \mathrm{d}\boldsymbol{z}_n = 1\ .$$

The collection of *correlation functions* $\{\rho_j^\varepsilon\}_{j\geq 0}$ with $\rho_j^\varepsilon : \Omega_j \to \mathbb{R}^+$ is given by

$$\rho_j^\varepsilon(\boldsymbol{z}_j) := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Omega_n} W_{j+n}^\varepsilon(\boldsymbol{z}_{j+n})\, \mathrm{d}z_{j+1} \ldots \mathrm{d}z_{j+n}\ . \tag{2.4}$$

Note that $\rho_0^\varepsilon = 1$. We say that the state admits correlation functions when the series in the right hand side is convergent, together with the series in the inverse formula

$$W_j^\varepsilon(\boldsymbol{z}_j) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\Omega_n} \rho_{j+n}^\varepsilon(\boldsymbol{z}_{j+n})\, \mathrm{d}z_{j+1} \ldots \mathrm{d}z_{j+n}\ .$$

In this case the correlation functions are an alternative description of all the statistical properties of the system. After suitable normalization, the function $\rho_j^\varepsilon$ provides the joint distribution of $j$ distinct particles.

Since the average number of particles is equal to the integral of $\rho_1^\varepsilon$, the *Boltzmann-Grad* condition can be formulated by requiring that

$$\lim_{\varepsilon\to 0} \mu_\varepsilon^{-1} \int_{\Omega_1} \rho_1^\varepsilon(z)\,\mathrm{d}z = 1\ . \tag{2.5}$$

Therefore the correlation functions $\rho_j^\varepsilon$ will diverge as $\mu_\varepsilon^j$.

In order to state the aforementioned validity theorem for the Boltzmann equation ([18, 23, 31]), it is essential to assume that the rescaled correlation functions $\mu_\varepsilon^{-j}\rho_j^\varepsilon$ are bounded, uniformly in $\varepsilon$, by the correlation functions of an equilibrium state. In fact, the invariance of the energy is used to rule out the emergence of too strong correlations. Moreover, a strong chaos property is assumed to the extent that

$$\lim_{\varepsilon\to 0} \mu_\varepsilon^{-j}\rho_j^\varepsilon = f^{\otimes j}\ , \tag{2.6}$$

uniformly on compact sets of $\Omega_j$. Here $f^{\otimes j}(\boldsymbol{z}_j) = f(z_1)f(z_2)\dots f(z_j)$ and $f$ is a continuous probability density on $\Omega_1$, playing the role of the Boltzmann density at a given time.

To fulfill such hypotheses, one resorts to the following grand canonical non-equilibrium prescription.

**Definition 1.** *Let $\varepsilon > 0$ and $f$ be a continuous probability density on $\Omega_1$ with spatial density $\rho(x) := \int_{\mathbb{R}^d} f(x,v)dv$ such that $\bar\rho := \|\rho\|_{L^\infty} < \infty$ . A "maximally chaotic" state (MCS) is a grand canonical measure with densities*

$$\frac{1}{n!}W_n^\varepsilon(\boldsymbol{z}_n) := \frac{1}{\mathcal{Z}_\varepsilon}\frac{\mu_\varepsilon^n}{n!} f^{\otimes n}(\boldsymbol{z}_n)\,\psi_n^\varepsilon(\boldsymbol{x}_n)\ , \qquad n\geq 0, \tag{2.7}$$

*where $\mathcal{Z}_\varepsilon$ is the partition function*

$$\mathcal{Z}_\varepsilon := \sum_{n\geq 0}\frac{\mu_\varepsilon^n}{n!}\int_{\Omega_n} f^{\otimes n}(\boldsymbol{z}_n)\,\psi_n^\varepsilon(\boldsymbol{x}_n)\,d\boldsymbol{z}_n, \tag{2.8}$$

*and $\{\psi_n^\varepsilon\}_{n\geq 0}$ are the Boltzmann factors*

$$\psi_n^\varepsilon(\boldsymbol{x}_n) := \prod_{1\leq\alpha<\alpha'\leq n} e^{-\beta\phi(\frac{x_\alpha - x_{\alpha'}}{\varepsilon})}\ , \qquad \beta > 0, \tag{2.9}$$

*$(\psi_0^\varepsilon = \psi_1^\varepsilon = 1)$.*

The equilibrium is recovered by the choice $f(x,v) = (z/|\Lambda|)M_\beta(v)$, where $z > 0$ and $M_\beta$ is the normalized Maxwellian with inverse temperature $\beta$. Hence Definition 1 resembles an equilibrium distribution for particles interacting by means of $\phi$ in an external field $-\beta^{-1}\log\rho(x)$. This allows to apply known cluster expansion methods to analyze its properties in the limit $\varepsilon \to 0$.

### C. Main result

We first introduce some notation.

**Definition 2.** *(i)* $\mathcal{G}_k$ *is the set of all graphs* $G$ *(unoriented, with no loops and no multiple edges) with vertices* $V(G) = \{1, \dots, k\}$ *and edges* $E(G) \subset \{\{\alpha, \alpha'\} \mid \alpha, \alpha' \in V(G), \alpha \neq \alpha'\}$.
*(ii)* $\mathcal{C}_k \subset \mathcal{G}_k$ *is the set of all connected graphs on* $\{1, \dots, k\}$*, i.e. graphs* $G$ *such that for any* $\alpha, \alpha' \in V(G)$ *with* $\alpha \neq \alpha'$ *there exist* $\alpha_1, \cdots, \alpha_\ell$ *with* $\{\alpha, \alpha_1\}, \{\alpha_1, \alpha_2\}, \cdots \{\alpha_\ell, \alpha'\}$ *in* $E(G)$.
*(iii)* $\mathcal{T}_k \subset \mathcal{C}_k$ *is the set of trees on* $\{1, \dots, k\}$*, i.e. connected graphs with no cycles.*

We shall adopt the following abbreviations. For $S = \{i_1, i_2, \dots, i_s\}$ a set of indices of cardinality $s = |S|$, and $g_s$ a generic function of $s$ variables, we write

$$\begin{aligned} \boldsymbol{y}_S &= (y_{i_1}, \cdots, y_{i_s}) , \\ g_S &= g_s(\boldsymbol{y}_S) = g_s(y_{i_1}, \cdots, y_{i_s}) . \\ g^{\otimes S} &= g(y_{i_1}) g(y_{i_2}) \cdots g(y_{i_s}) . \end{aligned} \tag{2.10}$$

Moreover, we indicate by $(S_1, \dots, S_k)_S$ a generic non trivial partition of $S$ into subsets $S_1, \dots, S_k$. We shall use later on the superscript zero, $(S_1, \dots, S_k)^0_S$, for partitions of $S$ into possibly trivial subsets.

**Definition 3.** *The truncated correlation (or cluster, cumulant) functions* $\{\rho_j^{\varepsilon,T}\}_{j \geq 0}$ *are defined recursively on* $\Omega_j$ *by* $\rho_0^{\varepsilon,T} = 0$,

$$\begin{cases} \rho_1^{\varepsilon,T} := \rho_1^\varepsilon , \\ \rho_J^{\varepsilon,T} := \rho_J^\varepsilon - \displaystyle\sum_{k=2}^{j} \sum_{(J_1,\dots,J_k)_J} \prod_{i=1}^{k} \rho_{J_i}^{\varepsilon,T} , \qquad j \geq 2 . \end{cases} \tag{2.11}$$

Our main result is that truncated correlations behave like "stretched trees". More precisely, their magnitude is controlled by the length of a tree graph constructed on the space configuration $(x_1, \cdots, x_j)$ and measuring the collective separation between particles; which we define next.

Given a tree $T \in \mathcal{T}_j$ and a configuration of particles $\boldsymbol{x}_j$, the *length* of $T$ on $\boldsymbol{x}_j$ is

$$\mathcal{L}_T(\boldsymbol{x}_j) := \sum_{\{\alpha,\alpha'\} \in E(T)} |x_\alpha - x_{\alpha'}| . \tag{2.12}$$

The length of the cluster $\mathcal{L}(\boldsymbol{x}_j)$ is the minimal length of all tree graphs connecting the vertices $x_1, \dots x_j$, and possibly other vertices:

$$\mathcal{L}(\boldsymbol{x}_j) := \min_n \min_{T \in \mathcal{T}_{j+n}} \min_{x_{j+1}, \cdots, x_{j+n}} \mathcal{L}_T(\boldsymbol{x}_{j+n}) . \tag{2.13}$$

Finally, we introduce a rescaled decay function as in Eq. (2.2) by

$$\zeta^\varepsilon(x_i, x_k) := \mathrm{e}^{-\beta\phi(\frac{x_i - x_k}{\varepsilon})} - 1 . \tag{2.14}$$

Due to the scaling (2.6), we are interested in estimating the truncated functions multiplied by the factor $\mu_\varepsilon^{-j}$. We then obtain the following result. We use

$$\Omega_j^\varepsilon := \{\boldsymbol{x}_j \mid |x_i - x_k| > \varepsilon\}. \tag{2.15}$$

Furthermore we assume that $\phi$ satisfies one of the following hypotheses.

**(H1):** $\phi$ is compactly supported, without loss of generality

$$\operatorname{supp}\phi \subset B_1(0) \tag{2.16}$$

**(H2):** For some $s > 0$ we have

$$|\zeta(x,y)| \leq C_0 \tilde{\zeta}_s(x,y), \quad \text{where} \quad \tilde{\zeta}_s(x,y) := \frac{1}{(1+|x-y|)^s}. \tag{2.17}$$

**Theorem 4.** *(A) Suppose that the pair potential satisfies* (2.1) *and Hypothesis H1 (cf.* (2.16)*). Then there exist positive constants* $\varepsilon_0 = \varepsilon_0(\phi, \bar{\rho}, \beta, B)$ *and* $A, c$ *such that the truncated two-particle correlation function of the maximally chaotic state* (2.7) *satisfies the bound:*

$$\frac{|\rho_2^{\varepsilon,T}(\boldsymbol{z}_2)|}{\mu_\varepsilon^2} \leq \frac{(Af)^{\otimes 2}}{1-(\varepsilon/\varepsilon_0)}\,(\varepsilon/\varepsilon_0)^{\lfloor \frac{|x_1-x_2|}{\varepsilon} \rfloor}, \tag{2.18}$$

*for all* $\boldsymbol{z}_2 \in \Omega_2$ *and* $\varepsilon < \varepsilon_0$*. More generally, the* $j$*-th truncated correlation function satisfies:*

$$\frac{|\rho_j^{\varepsilon,T}(\boldsymbol{z}_j)|}{\mu_\varepsilon^j} \leq \frac{(Af)^{\otimes j}}{1-(\varepsilon/\varepsilon_0)}\, j^{j-2}\,(\varepsilon/\varepsilon_0)^{(\mathcal{L}(\boldsymbol{x}_j)/\varepsilon-(j-1))_+} \tag{2.19}$$

*for all* $j$, $\boldsymbol{z}_j \in \Omega_j$ *and* $\varepsilon < \varepsilon_0$.

*(B) Let the potential* $\phi$ *satisfy Hypothesis H2 (cf.* (2.17)*) with* $s > (d-1)$*, , and assume* $\Lambda \subset B(0; L/2)$, $L > 0$*. Then there exist positive constants* $\varepsilon_0 = \varepsilon_0(\phi, \bar{\rho}, \beta, B)$ *and* $A, c$ *such that for* $\varepsilon$ *small enough:*

$$\frac{|\rho_2^{\varepsilon,T}(\boldsymbol{z}_2)|}{\mu_\varepsilon^2} \leq \frac{(Af)^{\otimes 2}}{(1-(\mu_\varepsilon \varepsilon^s/\varepsilon_0'))}\left(\sum_{T\in\mathcal{T}_2}\prod_{\{\alpha,\alpha'\}\in E(T)}\left|\tilde{\zeta}_s^\varepsilon(x_\alpha, x_{\alpha'})\right|\right). \tag{2.20}$$

*More generally, if the potential* $\phi$ *satisfies Hypothesis H2 with* $s > 2(d-1)$*, then we can bound the* $j$*-th truncated correlation functon by*

$$\frac{|\rho_j^{\varepsilon,T}(\boldsymbol{z}_j)|}{\mu_\varepsilon^j} \leq \frac{(Af)^{\otimes j}}{j!(1-(\mu_\varepsilon \varepsilon^{s/2}/\varepsilon_0'))}\left(\sum_{T\in\mathcal{T}_j^c}\prod_{\{\alpha,\alpha'\}\in E(T)}\left|\tilde{\zeta}_s^\varepsilon(x_\alpha, x_{\alpha'})\right|^{1/2}\right), \tag{2.21}$$

*for all* $j$*, and* $\varepsilon$ *small enough. Here we denote by* $\mathcal{T}_j^c \subset \mathcal{T}_j$ *the set of linear tree graphs.*

In particular, if the particles are at macroscopic distance, the minimal distance between them is $a > 0$ and the estimate provides the exponential decay:

$$\frac{|\rho_j^{\varepsilon,T}(\boldsymbol{z}_j)|}{\mu_\varepsilon^j} \le (Af)^{\otimes j}\, j^{j-2} e^{-(j-1)a'\varepsilon^{-1}\log\varepsilon^{-1}}$$

for some $a' > 0$.

**Remark.** Estimates similar to (2.19) and (2.21) have been shown in [15–17], under the name of 'strong cluster property', for the thermodynamic limit of equilibrium states. For $j > 2$, the precise decay of the truncated correlation function depends on the geometry of the positions $\boldsymbol{x}_j$. As in [15–17], this complex problem is circumvented by means of the reduction to linear tree graphs, which accounts for the weaker decay in (2.21) compared to (2.20).

**Theorem 5.** *Let $\Lambda \subset B(0; L/2)$, $L \in (0,\infty)$. Suppose that the pair potential $\phi$ is even, satisfies* (2.1) *and* (2.17) *with $s > d-1$. In particular:*

$$C'_{\beta,L} := \lim_{\varepsilon\to 0} \varepsilon^{d-s} \int_{\{|x|<L/\varepsilon\}} |\zeta(x,0)|\, dx < +\infty. \tag{2.22}$$

*We set*

$$\varepsilon'_0 = \frac{1}{2\bar\rho\, C'_{\beta,L} e^{2\beta B+1}}\ .$$

*Then the truncated correlation functions of the maximally chaotic state* (2.7) *with density $\mu_\varepsilon$ satisfy the bounds*

$$\left\| \mu_\varepsilon^{-j} \rho_j^{\varepsilon,T} \right\|_{L^1(\Omega_j)} \le j^{j-2} \frac{(A'\mathrm{e})^{2j\beta B}}{1-(\mu_\varepsilon\varepsilon^s/\varepsilon'_0)} \left(\varepsilon^{s\wedge d} \bar\rho\, C'_\beta\right)^{j-1}$$

*for all $j$ and $\varepsilon$ such that $\mu_\varepsilon \varepsilon^s < \varepsilon'_0$.*

**Remark.** In the Boltzmann-Grad scaling (2.3), the above result seems to allow power tails $s > d-1$. It also allows $s = d-1$ if the mean free path is small enough. Moreover the latter case should provide fluctuations with long range correlations (instead of the usual white noise), of the same order of those produced by collisions. On the other hand in order to go beyond $d-1$, one should take densities even smaller than (2.3) to get a similar picture. This leads to perplexities on the validity of the Boltzmann equation for such potentials.

The results above can be used to characterize the fluctuation field associated to the maximally chaotic states. Let $\mathbb{P}_\varepsilon$ and $\mathbb{E}_\varepsilon$ denote the probability of an event and the expectation with respect to the MCS. We denote by $\pi_\varepsilon$ the rescaled empirical measure associated to $\boldsymbol{z}_n \in \Omega$, as well as its action on test functions $h \in C(\Omega_1)$

$$\pi_\varepsilon(dxdv) = \frac{1}{\mu_\varepsilon} \sum_{i=1}^n \delta(x-x_i), \delta(v-v_i) \tag{2.23}$$

$$\pi_\varepsilon(h) = \int_{\Omega_1} h(x,v)\pi_\varepsilon(dxdv). \tag{2.24}$$

Further, we define the fluctuation field $\zeta_\varepsilon$ applied to a test function $h \in C(\Omega_1)$ as:

$$\zeta_\varepsilon(h) = \sqrt{\mu_\varepsilon}\left(\pi_\varepsilon(h) - \mathbb{E}_\varepsilon[\pi_\varepsilon(h)]\right). \tag{2.25}$$

Using the results above, we can characterize the covariance of the MCS.

**Corollary 6.** *Let $\phi$ satisfy Hypothesis H2* (2.17) *with $s > d-1$. Then the fluctuation field $\zeta_\varepsilon$ converges in law to a Gaussian fluctuation field $\zeta(h)$ with*

$$\mathbb{E}[\zeta(h)\zeta(g)] = \int_{\Omega_2} f(z_1)h(z_1)g(z_2)\delta(z_1 - z_2)dz_1dz_2. \tag{2.26}$$

*Let $\phi$ satisfy Hypothesis* (2.17) *with $s = d-1$, and more precisely*

$$|\phi(x) - \frac{A}{|x|^{d-1}}| + |\nabla\phi(x) + \frac{(d-1)Ax}{|x|^{d+1}}| \leq \frac{C}{|x|^d}, \tag{2.27}$$

*for some constants $A, C > 0$. Then the same result holds provided that $C_b'$ in* (2.22) *is small enough. In this case, $\zeta$ is not given by white noise, but instead the covariance has a long-range part:*

$$\mathbb{E}[\zeta(h)\zeta(g)] = \int_{\Omega_2} f(z_1)h(z_1)g(z_2)\delta(z_1 - z_2)dz_1dz_2 + \int_{\Omega_2} \overline{\rho}_2^T(z_1, z_2)h(z_1)g(z_2)dz_1dz_2, \tag{2.28}$$

*where $\overline{\rho}_2^T$ is given by*

$$\overline{\rho}_2^T = \lim_{\varepsilon\to 0} \mu_\varepsilon^{-2}\rho_2^{\varepsilon,T}. \tag{2.29}$$

**Remark.** Below the threshold $s = d-1$, the maximally factorized states no longer satisfy the Boltzmann-Grad condition (2.5). To demonstrate this, let $\phi$ for simplicity be given by

$$\phi(x) = |x|^{-s}. \tag{2.30}$$

We compute the expected number of particles

$$\mathcal{N} = \int_{\Omega_1} \rho_1 dz_1 \tag{2.31}$$

under the rescaling

$$\mu_\varepsilon = \varepsilon^{-2}.$$

To simplify the argument, let $f(z) = f_0(v)$ be a spatially homogeneous density, and $\Lambda = T_1$ be the unit torus. Let us define

$$A(n, \varepsilon) = \int_{\Omega_n} f^{\otimes n}\psi_n^\varepsilon(x_n)dz_n. \tag{2.32}$$

Then we can write

$$\frac{\mathcal{N}}{\mu_\varepsilon} = \frac{\sum_{n=0}^\infty \frac{1}{n!\mu_\varepsilon}\mu_\varepsilon^{n+1}A(n+1,\varepsilon)}{\sum_{n=0}^\infty \frac{1}{n!}\mu_\varepsilon^n A(n,\varepsilon)}. \tag{2.33}$$

Since the denominator is bounded below by 1, we can bound this by

$$\frac{\mathcal{N}}{\mu_\varepsilon} \leq \frac{\sum_{n=0}^{M_\varepsilon} \frac{1}{n!\mu_\varepsilon} \mu_\varepsilon^{n+1} A(n+1,\varepsilon)}{\sum_{n=0}^{\infty} \frac{1}{n!} \mu_\varepsilon^n A(n,\varepsilon)} + \sum_{n=M_\varepsilon}^{\infty} \frac{1}{n!\mu_\varepsilon} \mu_\varepsilon^{n+1} A(n+1,\varepsilon), \tag{2.34}$$

for any sequence $M_\varepsilon$. We now observe that

$$A(n,\varepsilon) = \int_{\Omega_n} f^{\otimes n} \psi_n^\varepsilon(x_n) dz_n \leq e^{-\beta \varepsilon^s n^2}. \tag{2.35}$$

By Jensen's inequality, we have on the other hand, for some $\kappa_0 > 0$

$$A(n,\varepsilon) = \int_{\Omega_n} f^{\otimes n} \psi_n^\varepsilon(x_n) dz_n \geq e^{-n^2 \kappa_0 \varepsilon^s}. \tag{2.36}$$

Now we choose $M_\varepsilon = \lceil -d/\beta \log(\varepsilon) \varepsilon^{-s} \rceil$ to find:

$$\begin{aligned} \frac{\mathcal{N}}{\mu_\varepsilon} &\leq \frac{\sum_{n=0}^{M_\varepsilon} \frac{1}{n!\mu_\varepsilon} \mu_\varepsilon^{n+1} A(n+1,\varepsilon)}{\sum_{n=0}^{\infty} \frac{1}{n!} \mu_\varepsilon^n A(n,\varepsilon)} + C\varepsilon^d & (2.37) \\ &= \frac{\sum_{n=1}^{M_\varepsilon} \frac{n}{\mu_\varepsilon} \frac{1}{n!} \mu_\varepsilon^n A(n,\varepsilon)}{\sum_{n=0}^{M_\varepsilon} \frac{1}{n!} \mu_\varepsilon^n A(n,\varepsilon)} + C\varepsilon^d. & (2.38) \end{aligned}$$

Now since $s < d-1$, we conclude

$$\mathcal{N}/\mu_\varepsilon \leq C\varepsilon^{(d-1)-s} \log \varepsilon \to 0. \tag{2.39}$$

For a lower bound, we observe for $m_\varepsilon = \lceil \frac{1}{2} \varepsilon^{-s} \rceil$ and $\varepsilon > 0$ small enough:

$$\sum_{n=m_\varepsilon}^{2m_\varepsilon} \frac{1}{n!} \mu_\varepsilon^n A(n,\varepsilon) \geq \sum_{n=m_\varepsilon}^{2m_\varepsilon} \frac{1}{n!} (e^{-\beta} \mu_\varepsilon)^n \geq \sum_{n=0}^{m_\varepsilon} \frac{1}{n!} \mu_\varepsilon^n A(n,\varepsilon). \tag{2.40}$$

Then, for such values of $\varepsilon$:

$$\frac{\mathcal{N}}{\mu_\varepsilon} \geq \frac{1}{3} \frac{\sum_{n=m_\varepsilon}^{M_\varepsilon} \frac{n}{\mu_\varepsilon} \frac{1}{n!} \mu_\varepsilon^n A(n,\varepsilon)}{\sum_{n=m_\varepsilon}^{M_\varepsilon} \frac{1}{n!} \mu_\varepsilon^n A(n,\varepsilon)} \geq \frac{1}{6} \varepsilon^{(d-1)-s}, \tag{2.41}$$

choosing $\varepsilon > 0$ small enough.

## 3. PROOF OF THEOREM 4

*Proof of Theorem 4.* The correlation functions (2.4) of an MCS (see Def. 1) are given by the following explicit expression:

$$\frac{\rho_j^\varepsilon(\boldsymbol{z}_j)}{\mu_\varepsilon^j} = \frac{f^{\otimes j}(\boldsymbol{z}_j)}{\mathcal{Z}_\varepsilon} \sum_{n=0}^{\infty} \frac{\mu_\varepsilon^n}{n!} \int_{\Lambda^n} \psi_{j+n}^\varepsilon(\boldsymbol{x}_{j+n}) \, \mathrm{d}\rho(x_{j+1}) \cdots \mathrm{d}\rho(x_{j+n}) \, . \tag{3.1}$$

Expanding the Boltzmann factor

$$\psi_k^\varepsilon(\boldsymbol{x}_k) = \prod_{1\le\alpha<\alpha'\le k} (1+\zeta^\varepsilon(x_\alpha,x_{\alpha'}))\,,$$

one finds a sum over all possible graphs on $k$ vertices:

$$\psi_k^\varepsilon(\boldsymbol{x}_k) = \sum_{G\in\mathcal{G}_k}\ \prod_{\{\alpha,\alpha'\}\in E(G)} \zeta^\varepsilon(x_\alpha,x_{\alpha'})\ .$$

Truncated functions have a similar expression, the main difference being that generic graphs are replaced by connected graphs. We introduce Ursell functions $\{u_k^\varepsilon\}_{k\ge 0}$ by $u_0^\varepsilon=0$, $u_1^\varepsilon = 1$ and

$$u_k^\varepsilon(\boldsymbol{x}_k) := \sum_{G\in\mathcal{C}_k}\ \prod_{\{\alpha,\alpha'\}\in E(G)} \zeta^\varepsilon(x_\alpha,x_{\alpha'})\ , \qquad k>1\ .$$

Then we have the following classical result.

**Lemma 7.** *Suppose that conditions* (2.1) *and* (2.2) *are satisfied. Then the MCS has partition function*

$$\mathcal{Z}_\varepsilon = \exp\left\{\mu_\varepsilon \sum_{m=1}^\infty \frac{\mu_\varepsilon^{m-1}}{m!}\int u_m^\varepsilon(\boldsymbol{x}_m)\, d\rho^{\otimes m}(\boldsymbol{x}_m)\right\}\,, \tag{3.2}$$

*and the truncated correlation functions are given by*

$$\frac{\rho_j^{\varepsilon,T}(\boldsymbol{z}_j)}{\mu_\varepsilon^j} = f^{\otimes j}(\boldsymbol{z}_j)\sum_{n=0}^\infty \frac{\mu_\varepsilon^n}{n!}\int_{\Lambda^n} u_{j+n}^\varepsilon(\boldsymbol{x}_{j+n})\,\mathrm{d}\rho(x_{j+1})\dots\mathrm{d}\rho(x_{j+n})\quad, \tag{3.3}$$

*where the series and integrals are absolutely convergent, uniformly for*

$$\varepsilon < \frac{1}{\bar\rho\, C_\beta e^{2\beta B+1}}\ . \tag{3.4}$$

The result follows directly from the following subtle estimate, known as *tree-graph inequality.*

**Lemma 8.** *Suppose that condition* (2.1) *is satisfied. Then one has*

$$\left|\sum_{G\in\mathcal{C}_k}\ \prod_{\{\alpha,\alpha'\}\in E(G)} \zeta^\varepsilon(x_\alpha,x_{\alpha'})\right| \le \mathrm{e}^{2k\beta B}\sum_{T\in\mathcal{T}_k}\ \prod_{\{\alpha,\alpha'\}\in E(T)} |\zeta^\varepsilon(x_\alpha,x_{\alpha'})|\ . \tag{3.5}$$

This bound goes back to work of Penrose for positive interactions [29], later extended to stable interactions [30] (see also [39] for an improvement).

*Proof of Lemma 7.* Given Lemma 8, the proof reduces to simple combinatorics. Since

$$\psi_n^\varepsilon(\boldsymbol{x}_n) = \sum_{k=1}^n\ \sum_{(J_1,\dots,J_k)_J}\ \prod_{i=1}^k u_{J_i}^\varepsilon$$

where $J = \{1, \cdots, j\}$ (and using the notations introduced in (2.10)) one gets from (2.8) that

$$\mathcal{Z}_\varepsilon = 1 + \sum_{n\geq 1} \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{m_1,\dots,m_k>0\\ m_1+\ldots+m_k=n}} \prod_{i=1}^{k} \left( \frac{\mu_\varepsilon^{m_i}}{m_i!} \int_{\Lambda^{m_i}} u_{m_i}^\varepsilon(\boldsymbol{x}_{m_i})\, \mathrm{d}\rho^{\otimes m_i}(\boldsymbol{x}_{m_i}) \right) ,$$

from which interchanging the sums we find

$$\mathcal{Z}_\varepsilon = 1 + \sum_{k=1}^{n} \frac{1}{k!} \left( \sum_{m=1}^{\infty} \frac{\mu_\varepsilon^m}{m!} \int_{\Lambda^m} u_m^\varepsilon(\boldsymbol{x}_m)\, \mathrm{d}\rho^{\otimes m}(\boldsymbol{x}_m) \right)^k .$$

Thus (3.2) holds.

Similarly, given a partition $(J_1, \dots, J_k)_J$ of $J = \{1, \dots, j\}$, a subset $I_0 \subset J^c = \{j + 1, \dots, j+n\}$ and a partition of $J^c \setminus I_0$ into (possibly trivial) subsets $I_1, \dots, I_k$, we expand $\psi_{j+n}^\varepsilon$ in (3.1) into corresponding connected components

$$\psi_{j+n}^\varepsilon(\boldsymbol{x}_{j+n}) = \sum_{I_0\subset J^c} \psi_{I_0}^\varepsilon \sum_{k=1}^{j} \sum_{(J_1,\dots,J_k)_J} \sum_{(I_1,\dots,I_k)^0_{J^c\setminus I_0}} \prod_{i=1}^{k} u_{J_i\cup I_i}^\varepsilon$$

and obtain

$$\begin{aligned}
\frac{\rho_j^\varepsilon(\boldsymbol{z}_j)}{\mu_\varepsilon^j} &= \frac{f^{\otimes j}(\boldsymbol{z}_j)}{\mathcal{Z}_\varepsilon} \sum_{k=1}^{j} \sum_{(J_1,\dots,J_k)_J} \sum_{n=0}^{\infty} \frac{\mu_\varepsilon^n}{n!} \sum_{(I_0,I_1,\dots,I_k)^0_{J^c}} \left( \int_{\Lambda^{i_0}} \psi_{I_0}^\varepsilon \, \mathrm{d}\rho^{\otimes I_0} \right) \prod_{\ell=1}^{k} \left( \int_{\Lambda^{i_\ell}} u_{J_\ell\cup I_\ell}^\varepsilon \, \mathrm{d}\rho^{\otimes I_\ell} \right) \\
&= \frac{f^{\otimes j}(\boldsymbol{z}_j)}{\mathcal{Z}_\varepsilon} \sum_{k=1}^{j} \sum_{(J_1,\dots,J_k)_J} \sum_{n=0}^{\infty} \sum_{\substack{m_0+\ldots+m_k=n\\ m_0,\dots,m_k\geq 0}} \left( \frac{\mu_\varepsilon^{m_0}}{m_0!} \int_{\Lambda^{m_0}} \psi_{m_0}^\varepsilon(\boldsymbol{y}_{m_0})\, \mathrm{d}\rho^{\otimes m_0}(\boldsymbol{y}_{m_0}) \right) \\
&\qquad \prod_{\ell=1}^{k} \left( \frac{\mu_\varepsilon^{m_\ell}}{m_\ell!} \int_{\Lambda^{m_\ell}} u_{j_\ell+m_\ell}^\varepsilon(\boldsymbol{x}_{J_\ell}, \boldsymbol{y}_{m_\ell})\, \mathrm{d}\rho^{\otimes m_\ell}(\boldsymbol{y}_{m_\ell}) \right) .
\end{aligned}$$

By comparing with (2.11), we get (3.3).

Absolute convergence in the region (3.4) follows from (3.5), recalling that $|\mathcal{T}_k| = k^{k-2}$ (Cayley's formula), that $\mu_\varepsilon \varepsilon^d = \varepsilon$, and using (2.2) to control the decay functions. A more explicit bound for the truncated functions is provided in the rest of this section. □

We turn to the proof of (2.19). From (3.3) and (3.5) we get

$$\left| \frac{\rho_j^{\varepsilon,T}(\boldsymbol{z}_j)}{\mu_\varepsilon^j} \right| \leq f^{\otimes j}(\boldsymbol{z}_j) e^{2j\beta B} \sum_{n=0}^{\infty} \frac{\mu_\varepsilon^n}{n!} e^{2n\beta B} \int_{\Lambda^n} \bar{u}_{j+n}^\varepsilon(\boldsymbol{x}_{j+n})\, \mathrm{d}\rho(x_{j+1}) \dots \mathrm{d}\rho(x_{j+n}) \tag{3.6}$$

where

$$\bar{u}_{j+n}^\varepsilon(\boldsymbol{x}_{j+n}) := \sum_{T\in\mathcal{T}_{j+n}} \prod_{\{\alpha,\alpha'\}\in E(T)} |\zeta^\varepsilon(x_\alpha, x_{\alpha'})| .$$

For given $n$ and $T \in \mathcal{T}_{j+n}$, we study the integral

$$I^\varepsilon_T(\boldsymbol{x}_j) := \int_{\Lambda^n} \prod_{\{\alpha,\alpha'\}\in E(T)} |\zeta^\varepsilon(x_\alpha, x_{\alpha'})| \, \mathrm{d}\rho(x_{j+1}) \dots \mathrm{d}\rho(x_{j+n}) \, .$$

Since the potential is compactly supported in the ball of radius 1, we can restrict the integration region to the set

$$S^\varepsilon_T(\boldsymbol{x}_j) := \Big\{ |x_\alpha - x_{\alpha'}| < \varepsilon \;\; \forall \{\alpha, \alpha'\} \in E(T) \Big\} \, .$$

In particular $I^\varepsilon_T(\boldsymbol{x}_j) = 0$ certainly holds unless $n \geq n_0 = n_0(\boldsymbol{x}_j)$, where $n_0$ is defined as the minimum number of open balls of radius $\varepsilon$, $B_\varepsilon(x_{j+1}), \cdots, B_\varepsilon(x_{j+n})$, necessary to realize a connection between the points $(x_1, \cdots, x_j)$; meaning that $\cup_{1\leq i\leq j+n} B_\varepsilon(x_i)$ is a connected set. By (2.12)-(2.13), we observe that $n_0$ cannot be smaller than $\boldsymbol{\mathcal{L}}(\boldsymbol{x}_j)/\varepsilon - j$.

Given $\boldsymbol{x}_j$, $n \geq n_0$ and $T \in \mathcal{T}_{j+n}$, we apply Fubini and perform the integrations in the following ordered way:

(i) first, we integrate with respect to all the available leaves (vertices of degree 1), thus obtaining a smaller tree;

(ii) we iterate the procedure and prune the tree graph leaf by leaf, until we are left with a union of $j-1$ linear chains joining $x_1, \cdots, x_j$;

(iii) we integrate with respect to the remaining variables forming the chains.

Each one of the elementary integrals in steps (i) and (ii) is bounded by (cf. (2.2))

$$\int |\zeta^\varepsilon(0,x)| \, \mathrm{d}\rho(x) \leq \varepsilon^d \bar\rho \, C_\beta \, .$$

Similarly using (2.1) (with $n = 2$), a chain made of $m+1$ edges is bounded by $(1+e^{2\beta B}) \left(\varepsilon^d \bar\rho \, C_\beta\right)^m$. We conclude that

$$I^\varepsilon_T(\boldsymbol{x}_j) \leq (1+e^{2\beta B})^{j-1} \left(\varepsilon^d \bar\rho \, C_\beta\right)^n \; .$$

Thus

$$\left| \frac{\rho^{\varepsilon,T}_j(\boldsymbol{z}_j)}{\mu^j_\varepsilon} \right| \leq f^{\otimes j}(\boldsymbol{z}_j) e^{2j\beta B} (1+e^{2\beta B})^{j-1} \sum_{n=n_0}^\infty \frac{\varepsilon^n}{n!} e^{2n\beta B} (j+n)^{j+n-2} \left(\bar\rho \, C_\beta\right)^n$$

where we used again $\mu_\varepsilon \varepsilon^d = \varepsilon$. As

$$\frac{(j+n)^{j+n-2}}{n!} \leq j^{j-2} (A')^j (2e)^n \tag{3.7}$$

for some pure constant $A' > 0$, we obtain the final result by choosing $A = A' e^{2\beta B}(1+e^{2\beta B})$ and

$$\varepsilon_0 = \frac{1}{2\bar\rho \, C_\beta e^{2\beta B+1}} \, .$$

For the proof of (2.21), we start from the representation

$$\left|\frac{\rho_j^{\varepsilon,T}(\boldsymbol{z}_j)}{\mu_\varepsilon^j}\right| \le f^{\otimes j}(\boldsymbol{z}_j) e^{2j\beta B} \sum_{n=0}^{\infty} \frac{\mu_\varepsilon^n}{n!} e^{2n\beta B} \int_{\Lambda^n} \tilde{u}_{j+n}^\varepsilon(\boldsymbol{x}_{j+n})\, \mathrm{d}\rho(x_{j+1}) \ldots \mathrm{d}\rho(x_{j+n}), \tag{3.8}$$

where $\tilde{u}_{j+n}^\varepsilon(\boldsymbol{x}_{j+n})$ is given by

$$\tilde{u}_{j+n}^\varepsilon(\boldsymbol{x}_{j+n}) := \sum_{T\in\mathcal{T}_{j+n}} \prod_{\{\alpha,\alpha'\}\in E(T)} \left|\tilde{\zeta}_s^\varepsilon(x_\alpha, x_{\alpha'})\right| . \tag{3.9}$$

Now we use the following result which can be found as Lemma 5 in [17]:

$$\sum_{T\in\mathcal{T}_{j+n}} \prod_{\{\alpha,\alpha'\}\in E(T)} \left|\tilde{\zeta}_s^\varepsilon(x_\alpha, x_{\alpha'})\right| \le 2\frac{(j+n)^{j+n-2}}{(j+n)!} \sum_{T\in\mathcal{T}_{j+n}^c} \prod_{\{\alpha,\alpha'\}\in E(T)} \left|\tilde{\zeta}_s^\varepsilon(x_\alpha, x_{\alpha'})\right|^{\frac{1}{2}} . \tag{3.10}$$

Now the variables $x_j+1, \ldots, x_j+n$ can be integrated using the estimates:

$$\int_\Lambda \left|\tilde{\zeta}_s^\varepsilon(x,y)\right|^{\frac{1}{2}} \mathrm{d}\rho(y) \le C_s \varepsilon^{s/2} \tag{3.11}$$

$$\int_\Lambda \left|\tilde{\zeta}_s^\varepsilon(x,y)\right|^{\frac{1}{2}} \left|\tilde{\zeta}_s^\varepsilon(y,z)\right|^{\frac{1}{2}} \mathrm{d}\rho(y) \le C_s \varepsilon^{s/2} \left|\tilde{\zeta}_s^\varepsilon(x,z)\right|^{\frac{1}{2}} . \tag{3.12}$$

Inserting these estimates back into (3.8) yields:

$$\begin{aligned}\left|\frac{\rho_j^{\varepsilon,T}(\boldsymbol{z}_j)}{\mu_\varepsilon^j}\right| &\le f^{\otimes j}(\boldsymbol{z}_j) \left(\sum_{T\in\mathcal{T}_j^c} \prod_{\{\alpha,\alpha'\}\in E(T)} \left|\tilde{\zeta}_s^\varepsilon(x_\alpha, x_{\alpha'})\right|^{1/2}\right) \sum_{n=0}^{\infty} \frac{(C_s\varepsilon^{s/2}\mu_\varepsilon)^n (j+n)^{j+n-2} e^{2(j+n)\beta B}}{n!j!} \\ &\le f^{\otimes j}(\boldsymbol{z}_j) \left(\sum_{T\in\mathcal{T}_j^c} \prod_{\{\alpha,\alpha'\}\in E(T)} \left|\tilde{\zeta}_s^\varepsilon(x_\alpha, x_{\alpha'})\right|^{1/2}\right) \sum_{n=0}^{\infty} \frac{j^{j-2}(A')^j (2eC_s\varepsilon^{s/2}\mu_\varepsilon)^n e^{2(j+n)\beta B}}{j!} .\end{aligned}$$

By assumption, $s > 2(d-1)$, and $\mu_\varepsilon = \varepsilon^{-(d-1)}$, the series is absolutely convergent for $\varepsilon$ small enough and the claim follows. □

*Proof of Theorem 5.* We proceed as in the proof of Theorem 4 up to formula (3.6), of which we want to control the right hand side for $j \ge 0$ in norm $L^1$, for $\Lambda$ bounded and $\beta, B$ fixed by the assumptions on $\phi$. By (2.22) we have that

$$\sup_{x_0\in\Lambda} \int_\Lambda |\zeta^\varepsilon(x_0,x)|\, \mathrm{d}\rho(x) = \sup_{x_0\in\Lambda} \varepsilon^d \int_{\frac{\Lambda-x_0}{\varepsilon}} |\zeta(x,0)|\, \rho(x_0+\varepsilon x)\, dx \le \varepsilon^s \bar{\rho}\, C'_{\beta,L} .$$

For given $T \in \mathcal{T}_{j+n}$, we apply Fubini and perform the integrations by pruning the tree graph leaf by leaf (in an order which is otherwise arbitrary):

$$\int_{\Lambda^{j+n}} \prod_{\{\alpha,\alpha'\}\in E(T)} |\zeta^\varepsilon(x_\alpha, x_{\alpha'})|\, \mathrm{d}\rho^{\otimes(j+n)}(\boldsymbol{x}_{j+n}) \le \left(\varepsilon^s \bar{\rho}\, C'_{\beta,L}\right)^{j+n-1} .$$

Therefore

$$
\begin{aligned}
& e^{2j\beta B} \sum_{n=0}^{\infty} \delta_{j+n>0} \frac{\mu_\varepsilon^n}{n!} e^{2n\beta B} \sum_{T\in\mathcal{T}_{j+n}} \int_{\Lambda^{j+n}} \prod_{\{\alpha,\alpha'\}\in E(T)} |\zeta^\varepsilon(x_\alpha, x_{\alpha'})| \, \mathrm{d}\rho^{\otimes(j+n)}(\boldsymbol{x}_{j+n}) \\
& \leq \left(\varepsilon^s \bar{\rho}\, C'_{\beta,L}\right)^{j-1} e^{2j\beta B} \sum_{n=0}^{\infty} \frac{(j+n)^{j+n-2}}{n!} \left(\mu_\varepsilon \varepsilon^s e^{2\beta B} \bar{\rho}\, C'_{\beta,L}\right)^n \delta_{j+n>0} \\
& \leq j^{j-2} \left(\varepsilon^s \bar{\rho}\, C'_{\beta,L}\right)^{j-1} (A' e)^{2j\beta B} \sum_{n\geq 0} \left(\mu_\varepsilon \varepsilon^s 2 e^{1+2\beta B} \bar{\rho}\, C'_{\beta,L}\right)^n \delta_{j+n>0}
\end{aligned}
$$

where in the last step we used (3.7), which gives the result for $j > 0$. For $j = 0$ the partition function (3.2) is obtained with the bound

$$
\mathcal{Z}_\varepsilon \leq \exp\left\{ \mu_\varepsilon \frac{2e^{1+2\beta B}}{1 - (\mu_\varepsilon \varepsilon^s / \varepsilon'_0)} \right\} .
$$

□

*Proof of Corollary 6.* The result follows if we can show that the limit in (2.29) exists. We have shown that the series expansion

$$
\frac{\rho_2^{\varepsilon,T}(\boldsymbol{z}_2)}{\mu_\varepsilon^2} = f^{\otimes 2}(\boldsymbol{z}_2) \sum_{n=0}^{\infty} \frac{\mu_\varepsilon^n}{n!} \sum_{G\in\mathcal{C}_{2+n}} \int_{\Lambda^n} \prod_{\{\alpha,\alpha'\}\in E(G)} \zeta^\varepsilon(x_\alpha, x_{\alpha'}) \, d\rho(x_{2+1}) \ldots d\rho(x_{2+n}) \quad , \tag{3.13}
$$

is absolutely convergent. Therefore, it remains to prove term-by-term convergence of the expansion above. It is easy to see that

$$
\lim_{\varepsilon\to 0} \int f^{\otimes 2}(\boldsymbol{z}_2) \frac{\mu_\varepsilon^n}{n!} \int_{\Lambda^n} \prod_{\{\alpha,\alpha'\}\in E(G)} \zeta^\varepsilon(x_\alpha, x_{\alpha'}) \, d\rho(x_{2+1}) \ldots d\rho(x_{2+n}) h(z_1, z_2) dz_1 dz_2 = 0,
$$

for each $G \in \mathcal{C}_{2+n} \setminus \mathcal{T}_{2+n}$ and continuous function $h$. On the other hand, for $G \in \mathcal{T}_{2+n}$ a tree graph, the limit

$$
C_G(h) := \lim_{\varepsilon\to 0} \int f^{\otimes 2}(\boldsymbol{z}_2) \frac{\mu_\varepsilon^n}{n!} \int_{\Lambda^n} \prod_{\{\alpha,\alpha'\}\in E(G)} \zeta^\varepsilon(x_\alpha, x_{\alpha'}) \, d\rho(x_{2+1}) \ldots d\rho(x_{2+n}) h(z_1, z_2) dz_1 dz_2
$$

exists. This can be seen by observing that the claim is true for $n = 0$, and remains true when a leaf is added to the tree, courtesy of (2.27). □

We define a tree partition scheme, i.e. a map $\pi : \mathcal{C}_k \to \mathcal{T}_k$ such that for any $T \in \mathcal{T}_k$, there is a graph $R(T) \in \mathcal{C}_k$ satisfying

$$
\pi^{-1}(\{T\}) = \left\{ G \in \mathcal{C}_k \,/\, E(T) \subset E(G) \subset E(R(T)) \right\} .
$$

Such a partition scheme can be obtained in the following way. We first order the set of all possible edges. Then, given a graph $G$, we define its image $T$ iteratively. We start by drawing the two smallest edges of $G$. Then we proceed iteratively following the order of edges, and adding the edge at each step provided that it does not create a cycle, until all edges of $G$

have been checked. The procedure ends obviously with a unique tree $T \in T_n$. In order to characterize $R(T)$, we then have to investigate which edges of $G$ have been discarded. These are the edges $\{\alpha, \alpha'\}$ such that walking on $T$ from $\alpha$ to $\alpha'$ we always see edges of smaller order.

We therefore get for positive potentials that

$$\sum_{G\in\mathcal{C}_k} \prod_{\{\alpha,\alpha'\}\in E(G)} \zeta^\varepsilon(x_\alpha, x_{\alpha'}) = \sum_{T\in\mathcal{T}_k} \sum_{G\in\pi^{-1}(T)} \prod_{\{\alpha,\alpha'\}\in E(G)} (-\zeta^\varepsilon(x_\alpha, x_{\alpha'}))$$
$$= \sum_{T\in\mathcal{T}_n} \left( \prod_{\{\alpha,\alpha'\}\in E(T)} (-\zeta^\varepsilon(x_\alpha, x_{\alpha'}) \right) \left( \prod_{\{\alpha,\alpha'\}\in E'(T)} (1-\zeta^\varepsilon(x_\alpha, x_{\alpha'})) \right).$$

and the conclusion follows from the fact that $(1 - \zeta^\varepsilon(x_\alpha, x_{\alpha'})) \in [0, 1]$.

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