Eigenvalue Bounds for Perturbed Periodic Dirac Operators

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Abstract

We characterise regions in the complex plane that contain all non-embedded eigenvalues of a perturbed periodic Dirac operator on the real line with real-valued periodic potential and a generally non-symmetric matrix-valued perturbation V. We show that the eigenvalues are located close to the end-points of the spectral bands for small $V \in L^1(\mathbb{R})^{2\times 2}$, but only close to the spectral bands as a whole for small $V \in L^p(\mathbb{R})^{2\times 2}$, $p > 1$. As auxiliary results, we prove the relative compactness of matrix multiplication operators in $L^{2p}(\mathbb{R})^{2\times 2}$ with respect to the periodic operator under minimal hypotheses, and find the asymptotic solution of the Dirac equation on a finite interval for spectral parameters with large imaginary part.

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1 Introduction

In the present paper, we consider the one-dimensional perturbed periodic Dirac operator

$$H = -i\,\sigma_2 \frac{d}{dx} + m\,\sigma_3 + q(x) + V(x) \qquad (x \in \mathbb{R}),$$

where σ_2 and σ_3 are Pauli matrices (see equation (18) below), $m \geq 0$ is the particle mass, $q : \mathbb{R} \to \mathbb{R}$ is a periodic potential and $V : \mathbb{R} \to \mathbb{C}^{2\times 2}$ is a matrix-valued perturbation. Although the unperturbed periodic operator

$$H_0 = -i\,\sigma_2 \frac{d}{dx} + m\,\sigma_3 + q(x) \qquad (x \in \mathbb{R}),$$

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is a self-adjoint operator in $L^2(\mathbb{R})^2$, the operator H is not self-adjoint in general as we do not assume that the matrix multiplication operator V is symmetric. We assume that V is bounded and that $V \in L^p(\mathbb{R})^{2\times 2}$ for some $p \geq 1$. Then H has the same essential spectrum as H_0, consisting of closed intervals on the real line (spectral bands), generally separated by spectral gaps, but may in addition have discrete eigenvalues in the complex plane (see Theorem 2 below).

Our aim is to find a priori enclosures for these eigenvalues, i.e. regions characterised in terms of the properties of the unperturbed periodic Dirac equation and the p-norm of V which contain all (non-embedded) eigenvalues of H. In the absence of a periodic background potential, $q = 0$, [5] proved that, for $V \in L^1(\mathbb{R})^{2\times 2}$ with $\|V\|_1 < 1$, the non-embedded eigenvalues of H lie within circles around (but not centred at) the points $\pm m$, the end-points of the two intervals of essential spectrum $\sigma_e(H) = (-\infty, -m] \cup [m, \infty)$. The radii of the circles tend to 0 as the 1-norm of V tends to 0, showing that when a coupling parameter ϵ is employed, the eigenvalues of $H_0 + \epsilon V$ emanate from the points $\pm m$ only as ϵ increases from 0.

In the present study, we extend this observation to the case where a periodic background potential q is present and allow V to be p-integrable with $p \geq 1$. Our main eigenvalue exclusion result (Theorem 3) states that a complex number λ outside the essential spectrum of H cannot be an eigenvalue of H if the p-norm of V (defined in equation (9) below) satisfies the inequality $\|V\|_p < F_p(\lambda)$, where F_p is some non-negative function determined completely in terms of solution properties of the unperturbed periodic equation. From our results, the following picture emerges. For $p = 1$, F_1 is bounded above by 1 and in fact tends to 1 as $|\operatorname{Im}\lambda| \to \infty$ (see Theorems 8, 9), so its level sets for levels < 1 lie in neighbourhoods of the real line. Moreover, F_1 tends to zero exactly at the end-points of spectral bands (Theorems 4, 5, 6). This means that for small $\|V\|_1 < 1$, the eigenvalues are confined to small neighbourhoods of the end-points of spectral bands and, when a coupling parameter is applied, will emerge from these end-points only. This behaviour appears to be a natural analogue to that observed in [5] and [6].

However, for $p > 1$, $F_p(\lambda)$ grows beyond all bounds as $|\operatorname{Im}\lambda| \to \infty$. Therefore the level sets of F_p will be in neighbourhoods of the real line for all positive levels, and we get eigenvalue enclosure regions for any size of $\|V\|_p$. However, F_p tends to 0 at all points of the essential spectrum of H, which means that for small $\|V\|_p$ the eigenvalues are confined to small neighbourhoods of the whole spectral bands. Although we do not show the actual appearance of eigenvalues in such position here, this opens up the possibility of eigenvalues approaching (or, with a coupling parameter, emerging from) any point of the essential spectrum of H, similar to the behaviour observed in [1] for Schrödinger operators.

We mention that in the recent study [2], a detailed spectral analysis of the different, but related Dirac operator where, instead of a real periodic potential, q is a purely imaginary jump potential was performed.

The present paper is structured as follows. In Section 2 we summarise the relevant results from Floquet theory of the periodic Dirac equation, describing in particular the definition of the complex quasimomentum used in this paper.

We also give a formula for the resolvent operator of H_0 and show that it is a bounded linear operator not only in $L^2(\mathbb{R})^2$, but also between a dual pair of non-Hilbert Lebesgue spaces (Theorem 1). In Section 3 we first prove that H has the same essential spectrum (for all five usual definitions for a non-selfadjoint operator) as H_0 and only discrete eigenvalues besides (Theorem 2). A key part of the proof is the observation that the operator of multiplication with a matrix-valued function in $L^{2p}(\mathbb{R})$ is H_0-relatively compact (Lemma 2), for which we provide a proof as it is not easily found in the literature in this generality, with locally integrable q, and hence may be of independent interest. We then proceed to the main eigenvalue exclusion theorem (Theorem 3) already described above. In Section 4, we show that the function determining the exclusion criterion for $p = 1$ tends to zero exactly at the end-points of the spectral bands. Finally, in Section 5, we show that this function tends to 1 as $\operatorname{Im}\lambda \to \infty$. This result is based on the general asymptotics of the fundamental system of the Dirac equation on a finite interval for this limit (Theorem 7), which is here obtained using a novel transformation of the Dirac equation into the pair of coupled differential equation systems (21) and may be of interest in its own right.

As a matter of notation, we write $|w|$ for the Euclidean norm $\sqrt{|w_1|^2 + |w_2|^2}$ of vectors $w \in \mathbb{C}^2$.

2 The periodic equation

Let $\Phi(\cdot, \lambda)$ be the canonical fundamental system of the periodic Dirac equation with spectral parameter $\lambda \in \mathbb{C}$, i.e. the solution of the (matrix) initial value problem

$$-i\sigma_2 \Phi'(x, \lambda) + (m\sigma_3 + q(x))\, \Phi(x, \lambda) = \lambda \Phi(x, \lambda) \quad (x \in \mathbb{R}), \quad \Phi(0, \lambda) = \mathbb{I}, \quad (1)$$

where $\mathbb{I}$ is the 2×2 unit matrix and q is a locally integrable, real-valued, periodic function. The qualitative behaviour of the solutions can be studied by means of Floquet theory considering the monodromy matrix $M(\lambda) := \Phi(a, \lambda)$ $(\lambda \in \mathbb{C})$, where $a > 0$ is the period of q, see [3]. As the (Wronskian) determinant of the monodromy matrix is equal to 1, its eigenvalues are inverses of each other. Their positions in the complex plane can be characterised in terms of the discriminant $\mathfrak{D}(\lambda) := \operatorname{Tr} M(\lambda)$. The characteristic equation for $M(\lambda)$,

$$\mu^2 - \mathfrak{D}(\lambda)\, \mu + 1 = 0,$$

shows that $M(\lambda)$ has two distinct eigenvalues if and only if $\mathfrak{D}(\lambda) \notin \{-2, 2\}$. In this case, either the eigenvalues lie on the unit circle and are complex conjugates of each other (this happens when $\mathfrak{D}(\lambda) \in (-2, 2)$), or one eigenvalue, $\rho(\lambda)$, lies outside, the other eigenvalue, $1/\rho(\lambda)$, lies inside the unit circle (this happens when $\mathfrak{D}(\lambda) \in \mathbb{C} \setminus [-2, 2]$). If $\mathfrak{D}(\lambda) \in \{-2, 2\}$, then either the geometric multiplicity of the eigenvalue ± 1 is 1 or $M(\lambda) = \pm\mathbb{I}$ (see [3, Section 1.4]).

If μ is an eigenvalue of $M(\lambda)$ and $v \in \mathbb{C}^2 \setminus \{0\}$ is a corresponding eigenvector, then $u(x) := \Phi(x, \lambda)\, v$ $(x \in \mathbb{R})$ is a *Floquet solution* of the Dirac equation

$$-i\sigma_2 u'(x) + (m\sigma_3 + q(x))\, u(x) = \lambda\, u(x) \qquad (x \in \mathbb{R}); \qquad (2)$$

clearly $u(0) = v$. Then the function $\varphi(x) := \mu^{-x/a}\, u(x)$ $(x \in \mathbb{R})$ is a-periodic. This shows that all solutions of the periodic Dirac equation are bounded if $\mathfrak{D}(\lambda) \in (-2, 2)$ and that there is one Floquet solution $u_+(\cdot, \lambda)$ exponentially small at $-\infty$ and one Floquet solution $u_-(\cdot, \lambda)$ exponentially small at ∞ if $\mathfrak{D}(\lambda) \in \mathbb{C} \setminus [-2, 2]$. If $\mathfrak{D}(\lambda) \in \{-2, 2\}$, then either one or all solutions are bounded. Hence we can deduce that $\sigma(H_0) = \{\lambda \in \mathbb{C} \mid \mathfrak{D}(\lambda) \in [-2, 2]\} \subset \mathbb{R}$ for the self-adjoint operator $H_0 = -i\sigma_2 \frac{d}{dx} + m\sigma_3 + q$ (see also [3, Theorem 4.7.1]).

The (entries of the) monodromy matrix M and hence also the discriminant $\mathfrak{D}$ are entire functions, cf. [7, Theorem 1.7.2]. Since $m > 0$ and q is real valued, it follows that $\Phi(x, \overline{\lambda}) = \overline{\Phi(x, \lambda)}$ $(x \in \mathbb{R})$ and so $M(\overline{\lambda}) = \overline{M(\lambda)}$ and $\mathfrak{D}(\overline{\lambda}) = \overline{\mathfrak{D}(\lambda)}$ for all $\lambda \in \mathbb{C}$. If $\mathfrak{D}(\lambda) \notin [-2, 2]$, let $v_+(\lambda)$ and $v_-(\lambda)$ be eigenvectors corresponding to the eigenvalues $\rho(\lambda)$ and $1/\rho(\lambda)$ of $M(\lambda)$, respectively. Then $\rho(\overline{\lambda}) = \overline{\rho(\lambda)}$ and we can choose the eigenvectors such that $v_\pm(\overline{\lambda}) = \overline{v_\pm(\lambda)}$. Therefore we focus on λ with $\operatorname{Im}\lambda \geq 0$ in the following.

The discriminant can be written in the form

$$
\begin{aligned}
\mathfrak{D}(\lambda) &= 2\cos k(\lambda)a \\
&= 2\cosh(a \operatorname{Im} k(\lambda)) \cos(a \operatorname{Re} k(\lambda)) - 2i \sinh(a \operatorname{Im} k(\lambda)) \sin(a \operatorname{Re} k(\lambda)) \qquad (3)
\end{aligned}
$$

$(\lambda \in \mathbb{C}, \operatorname{Im}\lambda \geq 0)$, where the (continuous) function k with $\operatorname{Im} k(\lambda) \geq 0$ is called the complex *quasimomentum* (see also e.g. [10]). As can be seen from equation (3), for $\lambda \in \mathbb{R}$, the quasimomentum $k(\lambda)$ is real; it is closely related to the rotation number (cf. [3, p.43]) in the intervals where $\mathfrak{D}(\lambda) \in [-2, 2]$ (stability intervals), whereas it has constant real part $\in \pi\mathbb{Z}$ and positive imaginary part in the intervals where $\mathfrak{D}(\lambda) \notin [-2, 2]$ (instability intervals). More generally, for $\lambda \in \mathbb{C}$ such that $\operatorname{Im}\lambda \geq 0$ and $\mathfrak{D}(\lambda) \notin [-2, 2]$, the eigenvalue of $M(\lambda)$ that lies outside the unit circle is $\rho(\lambda) = e^{-ik(\lambda)a}$, the other eigenvalue being $1/\rho(\lambda) = e^{ik(\lambda)a}$. Clearly $k(\lambda) \in \mathbb{R}$ implies that $\mathfrak{D}(\lambda) \in [-2, 2]$ and so $\lambda \in \mathbb{R}$. We also note the following.

Lemma 1. *Let* $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda \geq 0$. *Then*

$$
\operatorname{Im} k(\lambda) = \frac{1}{2a} \operatorname{Arcosh}\left(\frac{|\mathfrak{D}(\lambda)|^2}{4} + \sqrt{\left(1 - \frac{|\mathfrak{D}(\lambda)|^2}{4}\right)^2 + (\operatorname{Im}\mathfrak{D}(\lambda))^2} \right). \qquad (4)
$$

In particular,

$$
\lim_{\lambda \to \lambda_0} \operatorname{Im} k(\lambda) = 0 \qquad (5)
$$

if $\mathfrak{D}(\lambda_0) \in [-2, 2]$.

Proof. If $\operatorname{Im} k(\lambda) = 0$, then by equation (3) $\mathfrak{D}(\lambda) = 2\cos ak(\lambda) \in [-2, 2]$ and the right-hand side in equation (4) vanishes. If $\operatorname{Im} k(\lambda) > 0$, then by equation (3) we find

$$
\begin{aligned}
1 &= \frac{(\operatorname{Re}\mathfrak{D}(\lambda))^2}{4\cosh^2(a \operatorname{Im} k(\lambda))} + \frac{(\operatorname{Im}\mathfrak{D}(\lambda))^2}{4\sinh^2(a \operatorname{Im} k(\lambda))} \\
&= \frac{(\operatorname{Re}\mathfrak{D}(\lambda))^2(\cosh(2a \operatorname{Im} k(\lambda)) - 1) + (\operatorname{Im}\mathfrak{D}(\lambda))^2(\cosh(2a \operatorname{Im} k(\lambda)) + 1)}{2(\cosh^2(2a \operatorname{Im} k(\lambda)) - 1)}
\end{aligned}
$$

and hence by solving the quadratic equation

$$\begin{aligned}\cosh(2a\operatorname{Im}k(\lambda)) &= \frac{|\mathfrak{D}(\lambda)|^2}{4} \pm \sqrt{1+\frac{|\mathfrak{D}(\lambda)|^4}{16} - \frac{(\operatorname{Re}\mathfrak{D}(\lambda))^2-(\operatorname{Im}\mathfrak{D}(\lambda))^2}{2}} \\ &= \frac{|\mathfrak{D}(\lambda)|^2}{4} \pm \sqrt{\left(1-\frac{|\mathfrak{D}(\lambda)|^2}{4}\right)^2 + (\operatorname{Im}\mathfrak{D}(\lambda))^2}.\end{aligned}$$

Since

$$\frac{|\mathfrak{D}(\lambda)|^2}{4} - \sqrt{\left(1-\frac{|\mathfrak{D}(\lambda)|^2}{4}\right)^2 + (\operatorname{Im}\mathfrak{D}(\lambda))^2} \leq 1$$

and $\cosh(2a\operatorname{Im}k(\lambda)) > 1$ in the case under consideration, the square root must have the positive sign. □

For $\operatorname{Im}\lambda < 0$, the Floquet multiplier (eigenvalue) satisfies

$$\rho(\lambda) = \overline{\rho(\overline{\lambda})} = \overline{e^{-ik(\overline{\lambda})a}} = e^{-i(-\overline{k(\overline{\lambda})})a}.$$

This motivates the definition of the quasimomentum in the complex lower half-plane by setting $k(\lambda) := -\overline{k(\overline{\lambda})}$ $(\lambda \in \mathbb{C}, \operatorname{Im}\lambda < 0)$. Then we have $\rho(\lambda) = e^{-ik(\lambda)a}$ for all $\lambda \in \mathbb{C}$ such that $\mathfrak{D}(\lambda) \notin [-2, 2]$. Note that this extended quasimomentum function is not continuous at the real axis; nevertheless, its imaginary part is continuous as $\operatorname{Im}k(\lambda) = -(-\operatorname{Im}k(\overline{\lambda})) = \operatorname{Im}k(\overline{\lambda})$.

We now express the resolvent operator $(H_0-\lambda)^{-1}$ in terms of a fundamental system of Floquet solutions. Let $\lambda \in \mathbb{C}$ such that $\mathfrak{D}(\lambda) \notin [-2, 2]$. Then the Floquet solutions

$$\begin{aligned}u_+(x,\lambda) &= \Phi(x,\lambda)\,v_+(\lambda) = \rho(\lambda)^{x/a}\,\varphi_+(x,\lambda), \\ u_-(x,\lambda) &= \Phi(x,\lambda)\,v_-(\lambda) = \rho(\lambda)^{-x/a}\,\varphi_-(x,\lambda)\end{aligned} \tag{6}$$

with a-periodic functions $\varphi_\pm(\cdot,\lambda)$ are linearly independent and hence form a fundamental system of the Dirac equation. As $u_\pm(0,\lambda) = \varphi_\pm(0,\lambda) = v_\pm(\lambda)$, its Wronskian is $W(\lambda) = \det(v_+(\lambda), v_-(\lambda))$.

Theorem 1. *Let $\lambda \in \varrho(H_0)$. Then*

$$((H_0-\lambda)^{-1}f)(x) = \int_{\mathbb{R}} G(x,t,\lambda)\,f(t)\,dt \qquad (x \in \mathbb{R}; f \in L^2(\mathbb{R})^2)$$

with (matrix-valued) Green's function

$$G(x,t,\lambda) = -\frac{e^{ik(\lambda)\,|t-x|}}{\det(v_+(\lambda),v_-(\lambda))}\begin{cases}\varphi_+(x,\lambda)\,\varphi_-(t,\lambda)^T & \text{if } t > x \\ \varphi_-(x,\lambda)\,\varphi_+(t,\lambda)^T & \text{if } t < x\end{cases} \qquad (x,t \in \mathbb{R}).$$

For all $x,t \in \mathbb{R}$, $x \neq t$, the Frobenius norm of the matrix $G(x,t,\lambda)$ is

$$\|G(x,t,\lambda)\|_F = \frac{e^{-\operatorname{Im}k(\lambda)\,|t-x|}}{|\det(v_+(\lambda),v_-(\lambda))|}\begin{cases}|\varphi_+(x,\lambda)|\,|\varphi_-(t,\lambda)| & \text{if } t > x, \\ |\varphi_-(x,\lambda)|\,|\varphi_+(t,\lambda)| & \text{if } t < x.\end{cases}$$

Moreover, for any $r \in (1,2]$ *and conjugate exponent* $r' = 1/(1-\frac{1}{r}) \geq 2$*, the integral operator* $R_r(\lambda) : L^r(\mathbb{R})^2 \to L^{r'}(\mathbb{R})^2$,

$$(R_r(\lambda)f)(x) = \int_{\mathbb{R}} G(x,t,\lambda)\, f(t)\, dt \qquad (x \in \mathbb{R};\, f \in L^r(\mathbb{R})^2)$$

is a bounded linear operator with operator norm $\|R_r(\lambda)\| \leq C(\lambda) \left(\frac{4}{r' \operatorname{Im} k(\lambda)}\right)^{\frac{2}{r'}}$*, where*

$$C(\lambda) := \frac{\|\varphi_+(\cdot,\lambda)\|_\infty \, \|\varphi_-(\cdot,\lambda)\|_\infty}{|\det(v_+(\lambda), v_-(\lambda))|}. \tag{7}$$

Remarks. 1. The Green's function G is in fact independent of the choice of the eigenvectors $v_\pm(\lambda)$.

2. In the absence of a periodic background potential q, an operator norm bound for $R_r(\lambda)$ was obtained in [4, Theorem 3.1].

Proof. Let $f \in L^2(\mathbb{R})^2$; then solving the inhomogeneous Dirac equation

$$-i\sigma_2 u'(x) + (m\sigma_3 + q(x) - \lambda)\, u(x) = f(x) \qquad (x \in \mathbb{R})$$

by the variation of constants method on the basis of the fundamental system $(u_+(\cdot,\lambda), u_-(\cdot,\lambda))$ gives

$$u(x) = \int_{\mathbb{R}} G(x,t,\lambda)\, f(t)\, dt \qquad (x \in \mathbb{R}).$$

For $x \neq t$, the Frobenius norm of the matrix $G(x,t,\lambda)$ is

$$\begin{aligned}
\|G(x,t,\lambda)\|_F &= \sqrt{\operatorname{Tr}(G(x,t,\lambda)^*\ G(x,t,\lambda))} \\
&= \frac{|e^{ik(\lambda)\,|t-x|}|}{|\det(v_+(\lambda), v_-(\lambda))|} \sqrt{\operatorname{Tr}(\overline{\varphi_\mp(t,\lambda)}\, \varphi_\pm(x,\lambda)^*\, \varphi_\pm(x,\lambda)\, \varphi_\mp(t,\lambda)^T)} \\
&= \frac{e^{-\operatorname{Im} k(\lambda)\,|t-x|}}{|\det(v_+(\lambda), v_-(\lambda))|} \sqrt{\operatorname{Tr}(\varphi_\pm(x,\lambda)^*\, \varphi_\pm(x,\lambda)\, \varphi_\mp(t,\lambda)^T\, \overline{\varphi_\mp(t,\lambda)})} \\
&= \frac{e^{-\operatorname{Im} k(\lambda)\,|t-x|}}{|\det(v_+(\lambda), v_-(\lambda))|} \sqrt{|\varphi_\pm(x,\lambda)|^2\, |\varphi_\mp(t,\lambda)|^2}
\end{aligned}$$

with the sign in the index depending on whether $t > x$ or $t < x$. Setting $\|\varphi_\pm(\cdot,\lambda)\|_\infty := \sup_{x\in\mathbb{R}} |\varphi_\pm(\cdot,\lambda)|$, we can estimate the operator norm

$$\|G(x,t,\lambda)\| \leq \|G(x,t,\lambda)\|_F \leq C(\lambda)\, e^{-\operatorname{Im} k(\lambda)|t-x|} \tag{8}$$

$(x, t \in \mathbb{R}, t \neq x)$ with $C(\lambda)$ defined in equation (7). Now let $f \in L^r(\mathbb{R})$; then

$$\begin{aligned}
\|R_r(\lambda) f\|_{r'} &= \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} G(x,t,\lambda)\, f(t)\, dt \right|^{r'} dx \right)^{\frac{1}{r'}} \\
&\leq \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \|G(x,t,\lambda)\|\, |f(t)|\, dt \right)^{r'} dx \right)^{\frac{1}{r'}} \\
&\leq C(\lambda) \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-\operatorname{Im} k(\lambda)\, |t-x|}\, |f(t)|\, dt \right)^{r'} dx \right)^{\frac{1}{r'}} \\
&\leq C(\lambda) \left(\int_{\mathbb{R}} e^{-\operatorname{Im} k(\lambda)\, |s|\, \frac{r'}{2}}\, ds \right)^{\frac{2}{r'}} \left(\int_{\mathbb{R}} |f(x)|^r\, dx \right)^{\frac{1}{r}} \\
&= C(\lambda) \left(\frac{4}{r' \operatorname{Im} k(\lambda)} \right)^{\frac{2}{r'}} \|f\|_r
\end{aligned}$$

by Young's inequality, noting that $\frac{1}{r} + \frac{2}{r'} = \frac{1}{r'} + 1$. This shows that the integral operator $R_r(\lambda)$ (and in particular the resolvent operator $(H_0 - \lambda)^{-1} = R_2(\lambda)$) is well-defined and bounded, with the stated operator norm estimate. □

3 Eigenvalue exclusion

We now consider the Dirac operator with an additional non-periodic perturbation, $H := H_0 + V$, where V is the operator of multiplication with the matrix-valued function $V : \mathbb{R} \to \mathbb{C}^{2\times 2}$. We assume that V is bounded and, for some $p \geq 1$, $V \in L^p(\mathbb{R})^{2\times 2}$, which means that the norm (cf. [6])

$$\|V\|_p := \left(\int_{\mathbb{R}} \|V(x)\|^p\, dx \right)^{\frac{1}{p}} \tag{9}$$

is finite. Here $\|V(x)\|$ is the operator norm of the matrix $V(x)$, $x \in \mathbb{R}$. This is different from the operator norm $\|V\|$ of the multiplication operator V in $L^2(\mathbb{R})^2$.

For each $x \in \mathbb{R}$, we use the polar decomposition of $V(x)$,

$$V(x) = B(x)A(x), \qquad B(x) = U(x)\, |V(x)|^{\frac{1}{2}}, \qquad A(x) = |V(x)|^{\frac{1}{2}}, \tag{10}$$

where $|V(x)| = (V(x)^*\ V(x))^{\frac{1}{2}}$ and $U(x)$ is a partial isometry of $\mathbb{C}^2$, cf. [12, Theorem VI.10]; then

$$\|A(x)\| = \sqrt{\|V(x)\|}, \quad \|B(x)\| \leq \sqrt{\|V(x)\|} \qquad (x \in \mathbb{R}). \tag{11}$$

Thus we have matrix-valued functions $A, B \in L^{2p}(\mathbb{R})^{2\times 2}$ that give rise to bounded multiplication operators A, B on $L^2(\mathbb{R})^2$.

As we don't assume that V is symmetric, the operator H is not self-adjoint in general; however, as a sum of a closed (self-adjoint) operator and a bounded operator, it is closed (cf. [15, Theorem 5.5]). Moreover, we have the following statement about its essential spectrum, using any of the 5 usual definitions (cf. [8, Section I.4]), e.g. the third,

$$\sigma_e(H) := \{\lambda \in \mathbb{C} \mid H - \lambda \text{ is not a Fredholm operator}\}.$$

Theorem 2. $\sigma_e(H) = \sigma_e(H_0) = \{\lambda \in \mathbb{R} \mid \mathfrak{D}(\lambda) \in [-2, 2]\}$. *The spectrum of H outside $\sigma_e(H)$ only consists of isolated eigenvalues of finite multiplicity.*

In the proof of this theorem, we use the relative compactness of the multiplication operator A with respect to H_0. We give a full proof of this statement (which holds for any $A \in L^{2p}(\mathbb{R})^{2\times 2}$), as it does not seem to be easily available in the literature; note that we only assume that the periodic potential q is locally integrable, so the results of e.g. [14, Theorem 4.1] or [4, Theorem 4.1] are not directly applicable. We remark that in the case $p = 1$ the relative compactness can be shown more easily by proving that $A\,(H_0 - \lambda)^{-1}$, an integral operator with kernel $A(x)\,G(x, t, \lambda)$, is a Hilbert-Schmidt operator, using the Frobenius norm estimate (8).

Lemma 2. *Let $\lambda \in \varrho(H_0)$. Then the operator $A\,(H_0 - \lambda)^{-1}$ is compact.*

Proof. (a) We first show that the statement is true for $A \in C_0^\infty(\mathbb{R})^{2\times 2}$. Let $a < b$ be such that $\operatorname{supp} A \subset [a, b]$. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(\mathbb{R})^2$, $\|u_n\|_2 \le K$ $(n \in \mathbb{N})$, and set $y_n := A\,(H_0 - \lambda)\,u_n$ $(n \in \mathbb{N})$. Then, for all $x \in [a, b]$ and $n \in \mathbb{N}$, we have by Theorem 1

$$\begin{aligned}|y_n(x)| &\le \|A(x)\| \left| \int_{\mathbb{R}} G(x, t, \lambda)\, u_n(t)\, dt \right| \le \|A(x)\| \left(\int_{\mathbb{R}} \|G(x, t, \lambda)\|^2\, dt \right)^{\frac{1}{2}} \|u_n\|_2 \\ &\le \sup_{z\in\mathbb{R}} \|A(z)\| \left(\int_{\mathbb{R}} e^{-2\operatorname{Im} k(\lambda)\,|t|}\, dt \right)^{\frac{1}{2}} C(\lambda)\, K < \infty\end{aligned}$$

and $y_n(x) = 0$ for all $x \in \mathbb{R} \setminus [a, b]$, so the sequence of functions $(y_n)_{n\in\mathbb{N}}$ is uniformly bounded. Also, for $a \le x < z \le b$ we find, using the estimate (8),

$$\begin{aligned}|y_n(x) - y_n(z)| &= \left| A(x) \int_{\mathbb{R}} G(x, t, \lambda)\, u_n(t)\, dt - A(z) \int_{\mathbb{R}} G(z, t, \lambda)\, u_n(t)\, dt \right| \\ &\le \frac{1}{W} \Bigg(\int_{-\infty}^{x} \left| \left(A(x)\, e^{ik(\lambda)\,(x-t)} \varphi_-(x) - A(z)\, e^{ik(\lambda)\,(z-t)} \varphi_-(z) \right) \varphi_+(t)^T\, u_n(t) \right| dt \\ &+ \int_{x}^{z} \left| \left(A(x) e^{ik(\lambda)(t-x)} \varphi_+(x) \varphi_-(t)^T - A(z) e^{ik(\lambda)(z-t)} \varphi_-(z) \varphi_+(t)^T \right) u_n(t) \right| dt \\ &+ \int_{z}^{\infty} \left| \left(A(x)\, e^{ik(\lambda)\,(t-x)}\, \varphi_+(x) - A(z)\, e^{ik(\lambda)\,(t-z)}\, \varphi_+(z) \right) \varphi_-(t)^T\, u_n(t) \right| dt \Bigg),\end{aligned}$$

where we abbreviated $W := \det(v_+(\lambda), v_-(\lambda))$. Here the first integral is less than or equal to

$$\left| e^{ik(\lambda)x} A(x)\, \varphi_-(x) - e^{ik(\lambda)z} A(z)\, \varphi_-(z) \right| \left(\int_{-\infty}^{b} e^{2\,\mathrm{Im}\, k(\lambda) t} |\varphi_+(t)|^2 \, dt \right)^{\frac{1}{2}} K,$$

and as A and φ_- are continuous and hence uniformly continuous on $[a,b]$, this integral tends to 0 as $|x-z| \to 0$ uniformly on $[a,b]$ and in $n \in \mathbb{N}$. Analogous reasoning applies to the third integral. The second integral can be estimated by

$$\left(\int_x^z \|A(x) e^{ik(\lambda)\,|t-x|} \varphi_+(x) \varphi_-(t)^T - A(z) e^{ik(\lambda)\,|t-z|} \varphi_-(z) \varphi_+(t)^T \|^2 dt \right)^{\frac{1}{2}}$$

$$\times \left(\int_x^z |u_n(t)|^2 dt \right)^{\frac{1}{2}} \leq 2 \sup_{t\in[a,b]} \|A(t)\| \sup_{t\in[a,b]} |\varphi_+(t)| \sup_{t\in[a,b]} |\varphi_-(t)|\, K \sqrt{z-x},$$

which tends to 0 as $|x-z| \to 0$ uniformly on $[a,b]$ and in $n \in \mathbb{N}$. Consequently, the sequence of functions $(y_n)_{n\in\mathbb{N}}$ is also equicontinuous. By the Arzelà-Ascoli Theorem, it has a subsequence that is uniformly convergent and hence, in view of the compact support, also converges in $L^2(\mathbb{R})^2$.

(b) Now let $A \in L^{2p}(\mathbb{R})^{2\times 2}$. Let $u, v \in L^2(\mathbb{R})^2$, $\|u\|_2 = \|v\|_2 = 1$. Then

$$\begin{aligned}
|(A\,(H_0-\lambda)^{-1}u, v)| &= \left| \int_{\mathbb{R}} A(x) \int_{\mathbb{R}} G(x,t,\lambda)\, u(t)\, dt\, \overline{v(x)}\, dx \right| \\
&\leq C(\lambda) \int_{\mathbb{R}} |u(t)| \left(\int_{\mathbb{R}} e^{-\,\mathrm{Im}\, k(\lambda)\,|t-x|} \, \|A(x)\|\, |v(x)|\, dx \right) dt \\
&= C(\lambda) \int_{\mathbb{R}} \overline{F(|u|)}\, F\left(e^{-\,\mathrm{Im}\, k(\lambda)\,|\cdot|} * (\|A(\cdot)\|\, |v|) \right) \\
&= C(\lambda) \int_{\mathbb{R}} \overline{F(|u|)} \sqrt{2\pi}\, F(e^{-\,\mathrm{Im}\, k(\lambda)\,|\cdot|})\, F(\|A(\cdot)\|\, |v|) \\
&= 2\,\mathrm{Im}\, k(\lambda)\, C(\lambda) \int_{\mathbb{R}} F(\|A(\cdot)\|\, |v|)(\xi)\, \frac{\overline{F(|u|)(\xi)}}{\xi^2 + (\mathrm{Im}\, k(\lambda))^2}\, d\xi \\
&\leq 2\,\mathrm{Im}\, k(\lambda)\, C(\lambda) \left(\int_{\mathbb{R}} |F(\|A(\cdot)\|\, |v|)|^{r'} \right)^{\frac{1}{r'}} \left(\int_{\mathbb{R}} \left| \frac{F(|u|)(\xi)}{\xi^2 + (\mathrm{Im}\, k(\lambda))^2} \right|^r d\xi \right)^{\frac{1}{r}},
\end{aligned}$$

where we used the Plancherel identity for the Fourier transform F and then Hölder's inequality with exponent $r := \frac{2p}{p+1} \in [1,2)$ and conjugate exponent r'. (In the case $p = 1$, where $r = 1$, the above and the following estimates hold with $\left(\int_{\mathbb{R}} |F(\|A(\cdot)\|\, |v|)|^{r'} \right)^{1/r'}$ replaced with $\sup_{x\in\mathbb{R}} |F(\|A(\cdot)\|\, |v|)(x)|$.) By the

Hausdorff-Young inequality,

$$\begin{aligned}\left(\int_{\mathbb{R}} |F(\|A(\cdot)\|\,|v|)|^{r'}\right)^{\frac{1}{r'}} &\le \sqrt{2\pi}^{1-\frac{2}{r}} \left(\int_{\mathbb{R}} \|A(\cdot)\|^r\,|v|^r\right)^{\frac{1}{r}}\\ &= \frac{1}{\sqrt{2\pi}^{\frac{1}{p}}} \left(\int_{\mathbb{R}} \left(\|A(\cdot)\|^{2p}\right)^{\frac{1}{q}} \left(|v|^2\right)^{\frac{1}{q'}}\right)^{\frac{1}{r}}\\ &\le \frac{1}{\sqrt{2\pi}^{\frac{1}{p}}} \left(\int_{\mathbb{R}} \|A(\cdot)\|^{2p}\right)^{\frac{1}{rq}} \left(\int_{\mathbb{R}} |v|^2\right)^{\frac{1}{rq'}} = \frac{1}{\sqrt{2\pi}^{\frac{1}{p}}}\|A\|_{2p},\end{aligned}$$

using Hölder's inequality with exponents $q := p+1 = \frac{2p}{r}$ and $q' = \frac{p+1}{p} = \frac{2}{r}$. The same Hölder inequality gives

$$\left(\int_{\mathbb{R}} \left|\frac{F(|u|)(\xi)}{\xi^2 + (\operatorname{Im} k(\lambda))^2}\right|^r d\xi\right)^{\frac{1}{r}} \le \left(\int_{\mathbb{R}} (\xi^2 + (\operatorname{Im} k(\lambda))^2)^{-2p} d\xi\right)^{\frac{1}{2p}} \|F(|u|)\|_2.$$

As $\|F(|u|)\|_2 = \|u\|_2 = 1$, taking the supremum over u, v gives the bound for the operator norm

$$\|A\,(H_0 - \lambda)^{-1}\| \le 2 \operatorname{Im} k(\lambda)\, C(\lambda) \left(\frac{1}{2\pi}\int_{\mathbb{R}} (\xi^2 + (\operatorname{Im} k(\lambda))^2)^{-2p}\, d\xi\right)^{\frac{1}{2p}} \|A\|_{2p}.$$

As $C_0^\infty(\mathbb{R})$ is dense in $L^{2p}(\mathbb{R})$, there is a sequence $(A_n)_{n\in\mathbb{N}}$ in $C_0^\infty(\mathbb{R})^{2\times 2}$ that converges to A in $\|\cdot\|_{2p}$; by the above estimate, $A_n\,(H_0 - \lambda)^{-1}$ converges to $A\,(H_0-\lambda)^{-1}$ in operator norm and the statement of the lemma follows from (a) and the fact that the space of compact operators is closed in the operator norm. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. The resolvent set of H, $\varrho(H)$, contains points in the upper and the lower complex half-planes, as $\lambda \in \varrho(H)$ if $|\operatorname{Im}\lambda| > \|V\|$. By the resolvent identity, we find for $\lambda \in \varrho(H) \cap \varrho(H_0)$

$$(H_0 - \lambda)^{-1} - (H - \lambda)^{-1} = (H - \lambda)^{-1}\, B\, A\,(H_0 - \lambda)^{-1}.$$

As $(H - \lambda)^{-1}$ and B are bounded operators, this resolvent difference is compact by Lemma 2. We can now apply Theorem IX.2.4 of [8] to conclude the equality of the essential spectra (all five types) of H and of H_0.

The complement of the essential spectrum of H, $\mathbb{C} \setminus \sigma_e(H)$, is open and either connected (if H_0 has at least one spectral gap) or has the upper and lower complex half-planes as connected components (if $\sigma(H_0) = \mathbb{R}$ — this happens if $m = 0$, see [13, Proposition 1]). In either case, each component of the complement of $\sigma_e(H)$ contains points of the resolvent set $\varrho(H)$, and we can therefore apply Theorem XVII.2.1 of [9] to conclude that the spectrum of H outside $\sigma_e(H)$ only consists of isolated eigenvalues of finite multiplicity. □

In the statement of the eigenvalue exclusion theorem, we use the function $\Gamma : D(\Gamma) \to (0,1]$, $D(\Gamma) = \{A \in \mathbb{C}^{2\times 2} \mid A \text{ has two distinct eigenvalues}\}$,

$$\Gamma(A) = \frac{|\det(v_+, v_-)|}{|v_+|\,|v_-|}, \tag{12}$$

where $v_\pm \in \mathbb{C}^2 \setminus \{0\}$ are eigenvectors of A for the two different eigenvalues. As the eigenvectors are uniquely determined up to a complex factor, $\Gamma(A)$ does not depend on the choice of eigenvectors and is therefore well-defined. The domain $D(\Gamma)$ is an open subset of $\mathbb{C}^{2\times 2}$ and Γ is continuous. However, Γ cannot be continuously extended to all of $\mathbb{C}^{2\times 2}$; for example,

$$\lim_{\varepsilon\to 0} \Gamma \begin{pmatrix} 1 & 0 \\ 0 & 1+\varepsilon \end{pmatrix} = 1 \neq 0 = \lim_{\varepsilon\to 0} \Gamma \begin{pmatrix} 1 & \varepsilon \\ \varepsilon^2 & 1 \end{pmatrix},$$

so Γ has no continuous extension at the unit matrix. For the monodromy matrix M of equation (1), we have the following statement.

Lemma 3. *For all $\lambda \in \mathbb{C}$, $M(\lambda) \in D(\Gamma)$ if and only if $\mathfrak{D}(\lambda) \notin \{-2, 2\}$.*

We can now state the main eigenvalue exclusion theorem.

Theorem 3. *Let $p \geq 1$ and let $V \in L^p(\mathbb{R})^{2\times 2} \cap L^\infty(\mathbb{R})^{2\times 2}$. Then $\lambda \in \mathbb{C} \setminus \sigma_e(H)$ is not an eigenvalue of H if*

$$\|V\|_1 < \Gamma(M(\lambda))\,\gamma_+(\lambda)\,\gamma_-(\lambda) \qquad (\textit{if } p = 1),$$

$$\|V\|_p < \Gamma(M(\lambda))\,\gamma_+(\lambda)\,\gamma_-(\lambda)\,(\operatorname{Im} k(\lambda))^{\frac{p-1}{p}} \left(\frac{p}{2(p-1)}\right)^{\frac{p-1}{p}} \qquad (\textit{if } p > 1),$$

where

$$\gamma_\pm(\lambda) = \frac{|\varphi_\pm(0,\lambda)|}{\sup_{x\in[0,a]} |\varphi_\pm(x,\lambda)|} \tag{13}$$

and $\varphi_\pm$ are the periodic functions in equation (6).

Remark. The additional factor that appears on the right-hand side of the inequality in Theorem 3 for $p > 1$ tends to 1 as $p \to 1$, so the exclusion criterion is formally continuous in p.

Proof. By the Birman-Schwinger Principle (see e.g. [2, Theorem B.2]), λ is an eigenvalue of $H_0 + V$ if and only if -1 is an eigenvalue of $A\,(H_0 - \lambda)^{-1}\,B$, where A, B are as in equation (10).

Case 1: $p = 1$. For $u, v \in L^2(\mathbb{R})^2$, we obtain from Theorem 1 and the estimate (8), noting that $e^{-\operatorname{Im} k(\lambda)\,|t-x|} \leq 1$,

$$\begin{aligned}
\left|(A(H_0-\lambda)^{-1}Bu, v)\right| &= \left|\int_{-\infty}^{\infty}\left(A(x)\int_{-\infty}^{\infty} G(x,y,\lambda)B(y)u(y)\,dy\right)^T \overline{v(x)}\,dx\right| \\
&\leq \int_{-\infty}^{\infty} \|A(x)\|\,\|G(x,y,\lambda)\|\,\|B(y)\|\,|u(y)|\,|v(x)|\,dy\,dx \\
&\leq C(\lambda)\left(\int_{-\infty}^{\infty}\|A(x)\|\,|v(x)|\,dx\right)\left(\int_{-\infty}^{\infty}\|B(y)\|\,|u(y)|\,dy\right) \\
&\leq C(\lambda)\,\|V\|_1\,\|v\|_2\,\|u\|_2,
\end{aligned}$$

where we used Hölder's inequality and the estimate (11) in the last step. Setting $v := A(H_0-\lambda)^{-1}Bu$ and taking the supremum over u, we hence find the estimate for the operator norm of the Birman-Schwinger kernel

$$\|A(H_0-\lambda)^{-1}B\| \leq C(\lambda)\,\|V\|_1.$$

Case 2: $p > 1$. Let $r := \frac{2p}{p+1} \in (1,2)$ with conjugate exponent $r' = \frac{2p}{p-1} > 2$. We now associate the matrix-valued functions A and B with multiplication operators $A_{r',2} : L^{r'}(\mathbb{R})^2 \to L^2(\mathbb{R})^2$, $B_{2,r} : L^2(\mathbb{R})^2 \to L^r(\mathbb{R})^2$ and write the Birman-Schwinger kernel as $A(H_0-\lambda)^{-1}B = A_{r',2}R_r(\lambda)B_{2,r}$, where the operator $R_r(\lambda) : L^r(\mathbb{R})^2 \to L^{r'}(\mathbb{R})^2$ is defined as in Theorem 1. For $u \in L^{r'}(\mathbb{R})^2$, we find

$$\begin{aligned}
\|A_{r',2}u\|_2 &= \left(\int_{\mathbb{R}} |A(x)\,u(x)|^2\,dx\right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}} \|A(x)\|^2 \left(|u(x)|^{r'}\right)^{\frac{2}{r'}}\,dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{R}} \|V(x)\|^{\frac{r'}{r'-2}}\,dx\right)^{\frac{r'-2}{2r'}} \|u\|_{r'},
\end{aligned}$$

using Hölder's inequality with exponents $\frac{r'}{2}$ and $\frac{r'}{r'-2}$. As $\frac{r'}{r'-2} = p$, we obtain the operator norm estimate $\|A_{r',2}\| \leq \|V\|_p^{\frac{1}{2}}$. Similarly, we find for $u \in L^2(\mathbb{R})^2$

$$\begin{aligned}
\|B_{2,r}u\|_r &= \left(\int_{\mathbb{R}} |B(x)\,u(x)|^r\,dx\right)^{\frac{1}{r}} \leq \left(\int_{\mathbb{R}} \|B(x)\|^r \left(|u(x)|^2\right)^{\frac{r}{2}}\,dx\right)^{\frac{1}{r}} \\
&\leq \left(\int_{\mathbb{R}} \|V(x)\|^{\frac{r}{2-r}}\,dx\right)^{\frac{2-r}{2r}} \|u\|_2,
\end{aligned}$$

using Hölder's inequality with exponents $\frac{2}{r}$ and $\frac{2}{2-r}$, and hence, as $\frac{r}{2-r} = p$, the operator norm estimate $\|B_{2,r}\| \leq \|V\|_p^{\frac{1}{2}}$. In conjunction with Theorem 1, we obtain

$$\begin{aligned}
\|A(H_0-\lambda)^{-1}B\| = \|A_{r',2}R_r(\lambda)B_{2,r}\| &\leq \|A_{r',2}\|\,\|R_r(\lambda)\|\,\|B_{2,r}\| \\
&\leq C(\lambda)\left(\frac{2}{\operatorname{Im} k(\lambda)}\,\frac{p-1}{p}\right)^{\frac{p-1}{p}} \|V\|_p,
\end{aligned}$$

since $\frac{2}{r'} = \frac{p-1}{p}$.

Now if λ is an eigenvalue of H, then -1 is an eigenvalue of $A(H_0 - \lambda)B$ and therefore $\|A(H_0 - \lambda)^{-1}B\| \geq 1$; noting that $\frac{1}{C(\lambda)} = \Gamma(M(\lambda))\,\gamma_+(\lambda)\,\gamma_-(\lambda)$ since $\varphi_\pm(0,\lambda) = v_\pm(\lambda)$, we obtain the eigenvalue exclusion criteria in the theorem by contraposition. □

4 Behaviour near the essential spectrum

In this section we study the behaviour of the right-hand side of the inequalities in Theorem 3, in particular as λ approaches the essential spectrum $\sigma_e(H)$. We begin by finding a positive lower bound for the factors $\gamma_\pm(\lambda)$ defined in equation (13).

Theorem 4. *(a) Let $\lambda \in \mathbb{C}$ such that $\mathfrak{D}(\lambda) \notin [-2,2]$. Then*

$$e^{-a\left(\operatorname{Im} k(\lambda) + \sqrt{m^2 + (\operatorname{Im}\lambda)^2}\right)} \leq \gamma_\pm(\lambda) \leq 1.$$

(b) Let $\lambda_0 \in \mathbb{R}$ be such that $\mathfrak{D}(\lambda_0) \in [-2,2]$. Then

$$e^{-am} \leq \liminf_{\lambda \to \lambda_0} \gamma_\pm(\lambda) \leq 1.$$

Proof. (a) The upper bound is immediate from the definition of $\gamma_\pm$. For the lower bound, we note that $|\varphi_\pm|^2$ satisfies the differential equation

$$\frac{d}{dx}|\varphi_\pm(x,\lambda)|^2 = \varphi_\pm(x,\lambda)^T \mathcal{B}_\pm(\lambda)\overline{\varphi_\pm(x,\lambda)} \qquad (x \in \mathbb{R}), \tag{14}$$

with

$$\mathcal{B}_\pm(\lambda) = 2\begin{pmatrix} \mp \operatorname{Im} k(\lambda) & m - i\operatorname{Im}\lambda \\ m + i\operatorname{Im}\lambda & \mp\operatorname{Im} k(\lambda)\end{pmatrix}.$$

Indeed, the Floquet solutions $u_\pm(\cdot,\lambda)$ are solutions of the differential equation (2), which can be rewritten in the form

$$u'(x,\lambda) = \begin{pmatrix} 0 & m - q(x) + \lambda \\ m + q(x) - \lambda & 0 \end{pmatrix} u(x,\lambda) \qquad (x \in \mathbb{R});$$

so by differentiation of $\varphi_\pm(x,\lambda) = u_\pm(x,\lambda)\,e^{\pm ik(\lambda)x}$ $(x \in \mathbb{R})$ we find that $\varphi'_\pm(x,\lambda) = B_\pm(x,\lambda)\,\varphi_\pm(x,\lambda)$, where

$$B_\pm(x,\lambda) = \begin{pmatrix} \pm i\,k(\lambda) & m - q(x) + \lambda \\ m + q(x) - \lambda & \pm i\,k(\lambda)\end{pmatrix} \qquad (x \in \mathbb{R}).$$

Hence

$$\begin{aligned}\frac{d}{dx}\,|\varphi_\pm(x,\lambda)|^2 &= \varphi_\pm(x,\lambda)^T\,\overline{\varphi'_\pm(x,\lambda)} + \varphi'_\pm(x,\lambda)^T\,\overline{\varphi_\pm(x,\lambda)} \\ &= \varphi_\pm(x,\lambda)^T\left(\overline{B_\pm(x,\lambda)} + B_\pm(x,\lambda)^T\right)\overline{\varphi(x,\lambda)}\end{aligned}$$

and equation (14) follows noting that $\mathcal{B}_\pm(\lambda) = \overline{B_\pm(x,\lambda)} + B_\pm(x,\lambda)^T$ does not depend on x. From (14),

$$\begin{aligned}\frac{d}{dx}|\varphi_\pm(x,\lambda)|^2 \le \left|\frac{d}{dx}|\varphi_\pm(x,\lambda)|^2\right| &= |\varphi_\pm(x,\lambda)^T \mathcal{B}_\pm(\lambda)\overline{\varphi_\pm(x,\lambda)}| \\ &\le \|\mathcal{B}_\pm(\lambda)\|\,|\varphi_\pm(x,\lambda)|^2.\end{aligned}$$

To find the operator norm of the symmetric matrix $\mathcal{B}_\pm(\lambda)$, we calculate its eigenvalues $\mp 2\operatorname{Im} k(\lambda) + 2\sqrt{m^2 + (\operatorname{Im}\lambda)^2}$ and $\mp 2\operatorname{Im} k(\lambda) - 2\sqrt{m^2 + (\operatorname{Im}\lambda)^2}$, and hence the spectral radius

$$\|\mathcal{B}_\pm(\lambda)\| = 2\operatorname{Im} k(\lambda) + 2\sqrt{m^2 + (\operatorname{Im}\lambda)^2}.$$

Hence the above differential inequality gives

$$|\varphi_\pm(x,\lambda)|^2 \le |\varphi_\pm(0,\lambda)|^2\, e^{(2\operatorname{Im} k(\lambda) + 2\sqrt{m^2+(\operatorname{Im}\lambda)^2})\,x} \qquad (x \in [0,a])$$

and so the lower bound in the Theorem.
(b) By part (a), we have

$$\gamma_\pm(\lambda) - e^{-a\left(\operatorname{Im} k(\lambda) + \sqrt{m^2+(\operatorname{Im}\lambda)^2}\right)} \ge 0$$

for all $\lambda \in \mathbb{C}$ such that $\mathfrak{D}(\lambda) \notin \{-2, 2\}$, so using equation (5), we find

$$\begin{aligned}0 &\le \liminf_{\lambda\to\lambda_0}\left(\gamma_\pm(\lambda) - e^{-a\left(\operatorname{Im} k(\lambda) + \sqrt{m^2+(\operatorname{Im}\lambda)^2}\right)}\right) \\ &\le \liminf_{\lambda\to\lambda_0}\gamma_\pm(\lambda) - \lim_{\lambda\to\lambda_0} e^{-a\left(\operatorname{Im} k(\lambda) + \sqrt{m^2+(\operatorname{Im}\lambda)^2}\right)} = \liminf_{\lambda\to\lambda_0}\gamma_\pm(\lambda) - e^{-am}.\end{aligned}$$

□

We now consider the function $\Gamma(M(\lambda))$, which, as a composition of a continous and an entire function, is continuous. By Lemma 3 and the definition of Γ, we see that $\Gamma(M(\lambda)) > 0$ for all $\lambda \in \mathbb{C}$ for which $\mathfrak{D}(\lambda) \notin \{-2, 2\}$. However, at the points where $\mathfrak{D}(\lambda) \in \{-2, 2\}$, $\Gamma(M(\lambda))$ is not defined. These points are the real values of λ where either $M(\lambda) = \pm\mathbb{I}$ — then $\mathfrak{D}'(\lambda) = 0$ and λ is an interior point of a spectral band where two instability intervals meet — or $M(\lambda) \ne \pm\mathbb{I}$ has eigenvalue ± 1 with algebraic multiplicity 2, but geometric multiplicity 1 — then $\mathfrak{D}'(\lambda) \ne 0$ and λ is an end-point of a spectral band (cf. [3, Theorem 1.6.1]). In the following we investigate the limiting behaviour of $\Gamma(M(\lambda))$ at these exceptional points. We can characterise the derivative of M at such points as follows. Let $\lambda \in \mathbb{R}$. Denoting the columns of the canonical fundamental system Φ of equation (1) by u and v, we have

$$M'(\lambda) = M(\lambda)\begin{pmatrix} I_1(\lambda) & I_2(\lambda) \\ -I_3(\lambda) & -I_1(\lambda)\end{pmatrix} \tag{15}$$

where

$$I_1(\lambda) := \int_0^a (u_1(x,\lambda)v_1(x,\lambda) + u_2(x,\lambda)v_2(x,\lambda))\,dx,$$
$$I_2(\lambda) := \int_0^a (v_1(x,\lambda)^2 + v_2(x,\lambda)^2)\,dx,$$
$$I_3(\lambda) := \int_0^a (u_1(x,\lambda)^2 + u_2(x,\lambda)^2)\,dx$$

(see [3, eq. (1.6.4), (1.6.6)]). Note that $I_1(\lambda)$, $I_2(\lambda)$ and $I_3(\lambda)$ are complex in general; however, for real spectral parameter they are real and have the following property.

Lemma 4. *For any $\lambda \in \mathbb{R}$, $I_1(\lambda)^2 < I_2(\lambda)\, I_3(\lambda)$. In particular, $I_2(\lambda), I_3(\lambda) \neq 0$.*

Proof. Since $\lambda \in \mathbb{R}$, u and v are $\mathbb{R}^2$-valued continuous functions. The Cauchy-Schwarz inequality in $L^2(0,a)^2$ then gives

$$\begin{aligned} I_1(\lambda)^2 &= \left(\int_0^a u(x,\lambda)^T v(x,\lambda)\,dx\right)^2 \\ &\leq \int_0^a |u(x,\lambda)|^2\,dx \int_0^a |v(x,\lambda)|^2\,dx = I_2(\lambda)\, I_3(\lambda) \end{aligned}$$

with equality if and only if u and v are linearly dependent functions; however, the latter is impossible as $u(0,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v(0,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. □

We can now find the limit of $\Gamma(M(\lambda))$ at a point $\lambda_0 \in \mathbb{R}$ where $\mathfrak{D}(\lambda_0) = \pm 2$, distinguishing the cases of $M(\lambda_0) = \pm\mathbb{I}$ and of $M(\lambda_0) \neq \pm\mathbb{I}$. Note that in the following theorems 5 and 6, the limits $\lambda \to \lambda_0$ allow complex λ.

Theorem 5. *Let $\lambda_0 \in \mathbb{R}$ be such that $M(\lambda_0) = s\mathbb{I}$, where $s \in \{-1, 1\}$. Then*

$$\lim_{\lambda \to \lambda_0} \Gamma(M(\lambda)) = 2\,\frac{\sqrt{I_2(\lambda_0)\, I_3(\lambda_0) - I_1(\lambda_0)^2}}{I_2(\lambda_0) + I_3(\lambda_0)} > 0.$$

Proof. Since M is entire, we have by equation (15) for $\lambda \in \mathbb{C}$, abbreviating $I_j := I_j(\lambda_0)$, $j \in \{1,2,3\}$,

$$\begin{aligned} M(\lambda) &= M(\lambda_0) + M'(\lambda_0)\,(\lambda - \lambda_0) + R(\lambda - \lambda_0) \\ &= s\mathbb{I} + s\mathbb{I}\begin{pmatrix} I_1 & I_2 \\ -I_3 & -I_1 \end{pmatrix}(\lambda - \lambda_0) + R(\lambda - \lambda_0) \\ &= s\mathbb{I} + s\,(\lambda - \lambda_0)\, N(\lambda - \lambda_0), \end{aligned}$$

where

$$N(\Lambda) := \begin{pmatrix} I_1 & I_2 \\ -I_3 & -I_1 \end{pmatrix} + \frac{R(\Lambda)}{\Lambda} \qquad (\Lambda \in \mathbb{C}).$$

Here $R(\Lambda)/\Lambda$ is analytic with $\lim_{\Lambda\to 0} R(\Lambda)/\Lambda = 0$. Clearly, $w \in \mathbb{C}^2$ is an eigenvector of $N(\lambda-\lambda_0)$ for eigenvalue $\mu \in \mathbb{C}$ if and only if it is an eigenvector of $M(\lambda)$ for eigenvalue $s\,(1+(\lambda-\lambda_0)\,\mu)$. Therefore $\Gamma(M(\lambda)) = \Gamma(N(\lambda-\lambda_0))$ and we only need to find $\lim_{\Lambda\to 0}\Gamma(N(\Lambda))$.

Using Lemma 4, we see that the matrix $N(0) = \begin{pmatrix} I_1 & I_2 \\ -I_3 & -I_1 \end{pmatrix}$ has distinct purely imaginary eigenvalues $\pm i\sqrt{I_2I_3 - I_1^2}$ with corresponding eigenvectors

$$w_+ = \begin{pmatrix} I_2 \\ -I_1 + i\sqrt{I_2I_3 - I_1^2} \end{pmatrix}, \qquad w_- = \begin{pmatrix} -I_1 + i\sqrt{I_2I_3 - I_1^2} \\ I_3 \end{pmatrix}.$$

Hence by equation (12)

$$\Gamma(N(0))^2 = \frac{|2(I_2I_3 - I_1^2 + iI_1\sqrt{I_2I_3 - i_1^2})|^2}{I_2\,(I_2+I_3)\,I_3\,(I_2+I_3)} = 4\,\frac{I_2I_3 - I_1^2}{(I_2+I_3)^2}.$$

By analytic perturbation theory (see [11, Theorem II.1.8]), the eigenspaces of $N(\Lambda)$ converge to those spanned by w_+ and w_- as $\Lambda \to 0$, and we conclude that

$$\lim_{\lambda\to\lambda_0}\Gamma(M(\lambda)) = \lim_{\Lambda\to 0}\Gamma(N(\Lambda)) = \Gamma(N(0)) = 2\,\frac{\sqrt{I_2I_3 - I_1^2}}{I_2 + I_3},$$

which is positive by Lemma 4. □

Theorem 6. *Let $\lambda_0 \in \mathbb{R}$ be such that $\mathfrak{D}(\lambda_0) = \pm 2$, but $M(\lambda_0) \neq \pm\mathbb{I}$. Then*

$$\lim_{\lambda\to\lambda_0}\Gamma(M(\lambda)) = 0.$$

Proof. We can write the monodromy matrix as

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & \mathfrak{D}(\lambda) - a(\lambda) \end{pmatrix} \qquad (\lambda \in \mathbb{C})$$

with entire functions a, b, c (and $\mathfrak{D}$). As the (Wronskian) $\det M(\lambda) = 1$ for all λ, we find

$$cb = a\mathfrak{D} - a^2 - 1. \tag{16}$$

Therefore, if $b(\lambda_0) = c(\lambda_0) = 0$, then $a(\lambda_0) = \mathfrak{D}(\lambda_0)/2$ and hence $M(\lambda_0) = \pm\mathbb{I}$, contradicting the hypotheses. So $b(\lambda_0) \neq 0$ or $c(\lambda_0) \neq 0$.

We first consider the case $b(\lambda_0) \neq 0$. Then $b \neq 0$ in a neighbourhood of λ_0. For λ in this neighbourhood, we can write, using equation (16),

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ \frac{a(\lambda)\mathfrak{D}(\lambda) - a(\lambda)^2 - 1}{b(\lambda)} & \mathfrak{D}(\lambda) - a(\lambda) \end{pmatrix},$$

with eigenvalues $\frac{\mathfrak{D}(\lambda)\pm\sqrt{\mathfrak{D}(\lambda)^2-4}}{2}$ and eigenvectors

$$w_+(\lambda) = \begin{pmatrix} b(\lambda) \\ \frac{\mathfrak{D}(\lambda)+\sqrt{\mathfrak{D}(\lambda)^2-4}}{2} - a(\lambda) \end{pmatrix}, \qquad w_-(\lambda) = \begin{pmatrix} b(\lambda) \\ \frac{\mathfrak{D}(\lambda)-\sqrt{\mathfrak{D}(\lambda)^2-4}}{2} - a(\lambda) \end{pmatrix}.$$

Then $|w_\pm(\lambda)| \geq |b(\lambda)|$ and, by equation (12),

$$\Gamma(M(\lambda)) \leq \frac{|-b(\lambda)\sqrt{\mathfrak{D}(\lambda)^2-4}|}{|b(\lambda)|^2} = \frac{|\sqrt{\mathfrak{D}(\lambda)^2-4}|}{|b(\lambda)|} \to 0 \qquad (\lambda \to \lambda_0).$$

If $b(\lambda_0) = 0$, then $c \neq 0$ in a neighbourhood of λ_0, and for λ in this neighbourhood we can write

$$M(\lambda) = \begin{pmatrix} a(\lambda) & \frac{a(\lambda)\mathfrak{D}(\lambda)-a(\lambda)^2-1}{c(\lambda)} \\ c(\lambda) & \mathfrak{D}(\lambda)-a(\lambda) \end{pmatrix};$$

this matrix has the same eigenvalues as above and eigenvectors

$$w_\pm(\lambda) = \begin{pmatrix} \frac{\mathfrak{D}(\lambda)\pm\sqrt{\mathfrak{D}(\lambda)^2-4}}{2} - \mathfrak{D}(\lambda) + a(\lambda) \\ c(\lambda) \end{pmatrix},$$

so $|w_\pm(\lambda)| \geq |c(\lambda)|$ and

$$\Gamma(M(\lambda)) \leq \frac{|c(\lambda)\sqrt{\mathfrak{D}(\lambda)^2-4}|}{|c(\lambda)|^2} = \frac{|\sqrt{\mathfrak{D}(\lambda)^2-4}|}{|c(\lambda)|} \to 0 \qquad (\lambda \to \lambda_0).$$

□

5 Asymptotics for large $\operatorname{Im}\lambda$

The results of the preceding section show that the functions $\gamma_\pm$ do not tend to zero at any point in the complex plane and that $\Gamma \circ M$ tends to zero only (and exactly) at the end-points of the spectral bands. However, they do not yet preclude the possibility that these functions become small for λ far away from the real axis; in fact, the lower bound in Theorem 4 (a) tends to zero as $|\operatorname{Im}\lambda| \to \infty$ and hence is not very good in this respect. In the present section, we show that in fact $\Gamma(M(\lambda))$ and $\gamma_\pm(\lambda)$ tend to 1 as $|\operatorname{Im}\lambda| \to \infty$, which implies that the level sets of $\Gamma(M(\lambda))\gamma_+(\lambda)\gamma_-(\lambda)$ are located in strip neighbourhoods of the real axis. The basis for this is provided by the following asymptotic of the canonical fundamental system of a Dirac system with real-valued potential on a bounded interval; this result may be of interest in its own right.

We focus on the case $\operatorname{Im}\lambda > 0$, as the asymptotics for $\operatorname{Im}\lambda \to -\infty$ are the same due to the symmetry of the Dirac equation (1) with real-valued q.

Theorem 7. *Let $q \in L^1[0,a]$ be real-valued and $m \geq 0$. Let $Q(x) = \int_0^x q$ ($x \in [0,a]$) and let $\mu \in \mathbb{R}$, $\alpha > 0$. Then, for each $x \in [0,a]$, the solution of the initial value problem (1) with $\lambda = \mu + i\alpha$ has the asymptotic for $\alpha \to \infty$*

$$e^{-\alpha x}\,\Phi(x, \mu+i\alpha) = \frac{1}{2}\left(e^{i(Q(x)-\mu x)} + e^{-2\alpha x}e^{-i(Q(x)-\mu x)}\right)\mathbb{I} + \frac{1}{2}\left(-e^{i(Q(x)-\mu x)} + e^{-2\alpha x}e^{-i(Q(x)-\mu x)}\right)\sigma_2 + O_{\text{unif}}(\tfrac{1}{\alpha}). \qquad (17)$$

Here O_{unif} means that the bound is uniform in $x \in [0,a]$.

Corollary 1. *Under the hypotheses of Theorem 7,*

$$e^{-\alpha x}\,\Phi(x,\mu+i\alpha) = \frac{1}{2}\,e^{i(Q(x)-\mu x)}\,(\mathbb{I}-\sigma_2) + O(\tfrac{1}{\alpha}) \qquad (\alpha\to\infty)$$

for each $x \in (0,a]$.

Proof of Theorem 7. Write $\Phi = \sum_{j=0}^{3} \sigma_j\,\phi_j$ with complex-valued functions ϕ_0, ϕ_1, ϕ_2, ϕ_3 and the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}. \tag{18}$$

Then the initial value problem (1) is equivalent to the system

$$\begin{aligned} \begin{pmatrix} \phi_0 \\ \phi_2 \end{pmatrix}' &= (-\alpha - iq + i\mu)\sigma_1 \begin{pmatrix} \phi_0 \\ \phi_2 \end{pmatrix} + m \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_3 \end{pmatrix}, \\ \begin{pmatrix} \phi_1 \\ \phi_3 \end{pmatrix}' &= (-\alpha - iq + i\mu)(-\sigma_2) \begin{pmatrix} \phi_1 \\ \phi_3 \end{pmatrix} + m \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_2 \end{pmatrix}, \end{aligned} \tag{19}$$

with initial values $\phi_0(0) = 1$, $\phi_1(0) = \phi_2(0) = \phi_3(0) = 0$. We now make the ansatz

$$\begin{pmatrix} \phi_0 \\ \phi_2 \end{pmatrix} = \frac{e^r}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} u_1 + \frac{e^{-r}}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} u_2, \quad \begin{pmatrix} \phi_1 \\ \phi_3 \end{pmatrix} = \frac{e^r}{2} \begin{pmatrix} 1 \\ -i \end{pmatrix} u_3 + \frac{e^{-r}}{2} \begin{pmatrix} 1 \\ i \end{pmatrix} u_4, \tag{20}$$

with functions u_1, u_2, u_3, u_4 and given $r(x) = -\alpha x - i(Q(x) - \mu x)$ ($x \in [0,a]$). This is motivated by the fact that, with *constants* u_1, u_2, u_3, u_4, the above are the general solution of the decoupled equation system when $m = 0$. The initial conditions translate to $u_1(0) = u_2(0) = 1$, $u_3(0) = u_4(0) = 0$. Using this ansatz in the coupled differential equation system (19) and then multiplying the first equation (from the left) with the row vectors $(1,1)$ and $(1,-1)$, the second equation with the vectors $(1,i)$ and $(1,-i)$, respectively yields the two separate differential equation systems

$$\begin{cases} u_1' = m\,e^{-2r}\,u_4 \\ u_4' = m\,e^{2r}\,u_1 \end{cases} \qquad \begin{cases} u_3' = m\,e^{-2r}\,u_2 \\ u_2' = m\,e^{2r}\,u_3 \end{cases} \tag{21}$$

which are in fact the same system, but with different initial values. Focusing on the system on the left-hand side first, we observe that

$$|u_1|'(x) \le |u_1'(x)| = m\,e^{2\alpha x}\,|u_4|(x), \qquad |u_4|'(x) \le |u_4'(x)| = m\,e^{-2\alpha x}\,|u_1|(x).$$

The solutions exist on $[0,a]$ and, as continuous functions, are bounded. By an integration by parts and using the initial values,

$$\begin{aligned}|u_4|(x) &= |u_4(0)| + \int_0^x |u_4|' \le \int_0^x m\, e^{-2\alpha t}\, |u_1|(t)\, dt \\ &= -\frac{m}{2\alpha}\, (e^{-2\alpha x}\, |u_1|(x) - 1) + \frac{m}{2\alpha} \int_0^x e^{-2\alpha t}\, |u_1|'(t)\, dt \\ &\le \frac{m}{2\alpha} - \frac{m}{2\alpha}\, e^{-2\alpha x}\, |u_1|(x) + \frac{m^2}{2\alpha} \int_0^x e^{-2\alpha t} e^{2\alpha t}\, |u_4|(t)\, dt \qquad (x \in [0,a]),\end{aligned}$$

so

$$\sup_{x\in[0,a]} |u_4(x)| \le \frac{m}{2\alpha} + \frac{m^2 a}{2\alpha} \sup_{x\in[0,a]} |u_4(x)|$$

and hence

$$\sup_{x\in[0,a]} |u_4(x)| \le \frac{m}{2\alpha}\, \frac{1}{1 - \frac{m^2 a}{2\alpha}} \le \frac{m}{\alpha}$$

for $\alpha > m^2 a$. Consequently,

$$|u_1(x) - 1| \le \int_0^x |u_1'| = \int_0^x m\, e^{2\alpha t}\, |u_4(t)|\, dt \le \frac{m^2}{2\alpha^2}\, (e^{2\alpha x} - 1) \qquad (x \in [0,a]).$$

Now applying an analogous procedure to the right-hand side system in equation (21), we find

$$\begin{aligned}|u_2|(x) &= |u_2|(0) + \int_0^x |u_2|' \le 1 + m \int_0^x e^{-2\alpha t}\, |u_3|(t)\, dt \\ &= 1 - \frac{m}{2\alpha}\, (e^{-\alpha x}\, |u_3|(x) - 0) + \frac{m}{2\alpha} \int_0^x e^{-2\alpha t}\, |u_3|'(t)\, dt \\ &\le 1 - \frac{m}{2\alpha}\, e^{-2\alpha x}\, |u_3(x)| + \frac{m^2}{2\alpha} \int_0^x e^{-2\alpha t} e^{2\alpha t}\, |u_2|(t)\, dt,\end{aligned}$$

so

$$\sup_{x\in[0,a]} |u_2(x)| \le 1 + \frac{m^2 a}{2\alpha} \sup_{x\in[0,a]} |u_2(x)|$$

and hence

$$\sup_{x\in[0,a]} |u_2(x)| \le \frac{1}{1 - \frac{m^2 a}{2\alpha}} < 2$$

for $\alpha > m^2 a$. Also,

$$\begin{aligned}|u_3|(x) &= |u_3|(0) + \int_0^x |u_3|' \le m \int_0^x e^{2\alpha t}\, |u_2|(t)\, dt \\ &= \frac{m}{2\alpha}\, (e^{2\alpha x}\, |u_2|(x) - 1) - \frac{m}{2\alpha} \int_0^x e^{2\alpha t}\, |u_2|'(t)\, dt \\ &\le \frac{m}{2\alpha}\, (e^{2\alpha x}\, |u_2|(x) - 1) + \frac{m^2}{2\alpha} \int_0^x e^{2\alpha t} e^{-2\alpha t}\, |u_3|(t)\, dt\end{aligned}$$

and therefore

$$\sup_{t\in[0,x]} |u_3(t)| \le \frac{m}{2\alpha}\,\frac{e^{2\alpha x}}{1-\frac{m^2 a}{2\alpha}} - \frac{m}{2\alpha} + \frac{m^2 a}{2\alpha} \sup_{t\in[0,x]} |u_3(t)|,$$

which gives

$$\sup_{t\in[0,x]} |u_3(t)| \le \frac{1}{1-\frac{m^2 a}{2\alpha}} \left(\frac{m\, e^{2\alpha x}}{2\alpha - m^2 a} - \frac{m}{2\alpha} \right) \le \frac{2m}{\alpha}\,(e^{2\alpha x} - \tfrac{1}{2}) \le \frac{2m}{\alpha}\, e^{2\alpha x}$$

for all $x \in [0, a]$ and $\alpha > m^2 a$. Consequently,

$$|u_2(x) - 1| \le m \int_0^x e^{-2\alpha t}\, |u_3(t)|\, dt \le \frac{2m^2 a}{\alpha} \qquad (x \in [0, a]).$$

By equation (20), we have thus obtained the asymptotics

$$\begin{aligned}\begin{pmatrix}\phi_0\\ \phi_2\end{pmatrix}(x) &= \frac{e^{-\alpha x}}{2}\, e^{-i(Q(x)-\mu x)} \begin{pmatrix}1\\ 1\end{pmatrix} (1 + O(\frac{e^{2\alpha x}}{\alpha^2}))\\ &\quad + \frac{e^{\alpha x}}{2}\, e^{i(Q(x)-\mu x)} \begin{pmatrix}1\\ -1\end{pmatrix} (1 + O_{\text{unif}}(\tfrac{1}{\alpha})),\\ \begin{pmatrix}\phi_1\\ \phi_3\end{pmatrix}(x) &= \frac{e^{-\alpha x}}{2}\, e^{-i(Q(x)-\mu x)} \begin{pmatrix}1\\ -i\end{pmatrix} O(\frac{e^{2\alpha x}}{\alpha}) + \frac{e^{\alpha x}}{2}\, e^{i(Q(x)-\mu x)} \begin{pmatrix}1\\ i\end{pmatrix} O_{\text{unif}}(\tfrac{1}{\alpha}),\end{aligned}$$

and equation (17) follows. □

On the basis of the preceding theorem, we now find the asymptotics of the quasimomentum and of $\Gamma \circ M$ (in Theorem 8) and of $\varphi_\pm$ and $\gamma_\pm$ (in Theorem 9).

Theorem 8. *Let $\mu \in \mathbb{R}$, $\alpha > 0$. Then we have the following asymptotics as $\alpha \to \infty$ for the monodromy matrix of the periodic Dirac equation* (1)

$$e^{-\alpha a}\, M(\mu + i\alpha) = \frac{1}{2}\, e^{i(Q(a)-\mu a)}(\mathbb{I} - \sigma_2) + O(\tfrac{1}{\alpha}),$$

the discriminant

$$e^{-\alpha a}\, \mathfrak{D}(\mu + i\alpha) = e^{i(Q(x)-\mu a)} + O(\tfrac{1}{\alpha}),$$

the quasimomentum

$$k(\mu + i\alpha) = \mu + i\alpha - \frac{Q(a)}{a} + O(\tfrac{1}{\alpha})$$

and the eigenvectors of $M(\mu + i\alpha)$

$$v_+(\mu + i\alpha) = \begin{pmatrix}1\\ -i\end{pmatrix} + O(\tfrac{1}{\alpha}), \qquad v_-(\mu + i\alpha) = \begin{pmatrix}1\\ i\end{pmatrix} + O(\tfrac{1}{\alpha}).$$

Consequently,

$$\Gamma(M(\mu + i\alpha)) = 1 + O(\tfrac{1}{\alpha}).$$

Proof. The asymptotics of the monodromy matrix and hence the discriminant follow directly from Corollary 1. As $e^{-\alpha a}M(\mu+i\alpha)$ has determinant $e^{-2\alpha a}$, solving its characteristic equation shows that the two eigenvalues of this matrix have asymptotics $e^{i(Q(a)-\mu a)}+O(\frac{1}{\alpha})$ and $e^{-2\alpha a}\,(e^{-i(Q(a)-\mu a)}+O(\frac{1}{\alpha}))$, respectively. Hence the larger eigenvalue of $M(\mu+i\alpha)$ is

$$e^{\alpha a}\,(e^{i(Q(a)-\mu a)}+O(\frac{1}{\alpha}))=e^{-i(\mu+i\alpha-\frac{Q(a)}{a}+O(\frac{1}{\alpha}))a},$$

and we can read off the asymptotics for the quasimomentum. The asymptotic form of the eigenvectors follows from that of the matrix $e^{-\alpha a}M(\mu+i\alpha)$ and its eigenvalues. □

Theorem 9. *Let $\mu\in\mathbb{R}$, $\alpha>0$. For the periodic Dirac equation* (1)*, the periodic functions $\varphi_\pm$ of equation* (6) *have asymptotics*

$$|\varphi_\pm(x,\mu+i\alpha)|=|\varphi_\pm(0,\mu+i\alpha)|\,(1+O_{\mathrm{unif}}(\tfrac{1}{\alpha}))\qquad(\alpha\to\infty)$$

uniformly in $x\in[0,a]$. Consequently, the functions $\gamma_\pm$ of equation (13) *satisfy*

$$\gamma_\pm(\mu+i\alpha)=1+O(\tfrac{1}{\alpha})\qquad(\alpha\to\infty).$$

Proof. By Theorem 8 and equation (17), observing that

$$\tfrac{1}{2}\,(\mathbb{I}-\sigma_2)\,v_+(\mu+i\alpha)=v_+(\mu+i\alpha)+O(\tfrac{1}{\alpha}),\quad \tfrac{1}{2}\,(\mathbb{I}+\sigma_2)\,v_+(\mu+i\alpha)=O(\tfrac{1}{\alpha}),$$

we obtain

$$\begin{aligned}\varphi_+(x,\mu+i\alpha)&=e^{ik(\mu+i\alpha)x}\Phi(x,\mu+i\alpha)\,v_+(\mu+i\alpha)\\&=e^{i(\mu-\frac{Q(a)}{a}+O(\frac{1}{\alpha}))x}\,e^{-\alpha x}\,\Phi(x,\mu+i\alpha)\,v_+(\mu+i\alpha)\\&=e^{i(Q(x)-\frac{Q(a)}{a}\,x+O_{\mathrm{unif}}(\frac{1}{\alpha}))}\,(v_+(\mu+i\alpha)+O(\tfrac{1}{\alpha}))\\&\qquad+e^{-2\alpha x}e^{-i(Q(x)+\frac{Q(a)}{a}\,x-2\mu x+O_{\mathrm{unif}}(\frac{1}{\alpha}))}\,O(\tfrac{1}{\alpha})+O_{\mathrm{unif}}(\tfrac{1}{\alpha})\end{aligned}$$

and hence

$$\begin{aligned}|\varphi_+(x,\mu+i\alpha)|&=(1+O_{\mathrm{unif}}(\tfrac{1}{\alpha}))\,(v_+(\mu+i\alpha)+O(\tfrac{1}{\alpha}))+O_{\mathrm{unif}}(\tfrac{1}{\alpha})\\&=|v_+(\mu+i\alpha)|+O_{\mathrm{unif}}(\tfrac{1}{\alpha})=|\varphi_+(0,\mu+i\alpha)|(1+O_{\mathrm{unif}}(\tfrac{1}{\alpha})).\end{aligned}$$

Analogous reasoning for $\varphi_-(x,\mu+i\alpha)$ does not work since the exponentially large factor $e^{-ik(\mu+i\alpha)x}$ ($=\rho(\lambda)^{x/a}$ in equation (6)) leads to uncontrolled amplification of the $O(\frac{1}{\alpha})$ error term. However, the Floquet solution $u_-(x,\lambda)$ is equal, up to a constant factor, to $\sigma_3\tilde{u}_+(a-x,\lambda)$ $(x\in\mathbb{R})$, where $\tilde{u}_+$ is the Floquet solution corresponding to the eigenvalue of modulus greater than 1 of the periodic Dirac equation with potential $\tilde{q}(x):=q(a-x)$ $(x\in\mathbb{R})$. Therefore the corresponding periodic functions satisfy (up to a constant factor) $|\varphi_-(x,\lambda)|=|\tilde{\varphi}_+(a-x,\lambda)|$, so we obtain the asymptotics of $|\varphi_-|$ by applying the above reasoning to $|\tilde{\varphi}_+|$. □

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