Infinite Volume Ground States in the non-Abelian Quantum Double Model

On Anyon Excitations in the Plane

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Preface

This work represents my research at the Cardiff University, under the supervision of Dr. Pieter Naaijkens, over the past four years. I am thankful for Pieter's continued support, guidance, and patience, without which this thesis could not have been completed. Thanks to you, I have learned a lot about the interplay of operator algebras and quantum mechanics, and was able to experience the life of a researcher through you and the many conferences you urged me to attend, and the people you encouraged me to meet. This is an experience that will certainly stay with me for the rest of my life. I am also very grateful for your constructive criticism and thoughtful comments on my work, which have profoundly improved the quality of this thesis.

I started doing my Bachelor in physics because I was fascinated by the mysterious phenomena in modern physics, and in particular quantum field theory. I was intrigued by the idea that there are natural phenomena out there that can be repeatedly observed, yet not fully understood, and wanted to gain a deeper insight into these matters. After realizing that a deep mathematical understanding was necessary to really comprehend the natural laws governing physics, I decided to study mathematical physics at the Julius-Maximilians-University in Würzburg, where I first learned about operator algebras and their relation to quantum mechanics. I was happy to have been offered a scholarship at Cardiff University in a research area that allowed me to further my understanding of quantum field theory, through my own research as well as through the acquaintance with people in similar areas.

Doing a PhD in mathematical physics has been a deeply humbling experience. There were weeks, even months, I found myself lost in the intricate maze of solving a problem using algebraic and logical relations, not knowing how close or far off I was to a solution, only to arrive at the simplest and elegant answer, barely reflecting the countless hours spent in reaching those conclusions. Despite all that, there is something rewarding and exciting about working as a mathematician. There is a certain beauty in the abstract and oftentimes surprising connections between seemingly different mathematical ideas, and a mesmerizing feeling of satisfaction and enlightenment, once they finally reveal themselves.

Equally important to this journey has been the support of my friends and family. I thank in particular my parents, Mahmoud and Sakina. It would not be an exaggeration to say that none of my academic achievements would have been possible without you. Your sacrifice of all your belongings in your home country, enabled me to have a better future in Germany, away from war and political unrest, and I will forever appreciate that fact.

I would also like to thank my friends, Oscar, Sam and Valerio, with whom I shared an office in Cardiff, for their company during our shared struggle. I enjoyed the time we spent together greatly, and I regret it could not have been longer. I also thank my friends Ali, Ruqayah, Bayan, Yazan and Ahmed for the good times and many excursions we spent together. You all reminded me that life is not all about work, and I am happy that our paths have crossed.

Finally, I thank Rida, the love of my life, for supporting me throughout these past years. Life as a PhD student was certainly not always easy, but you were there for me and listened to the mathematical struggles I faced daily. You constantly encouraged me to follow my path, and I could not hope for a better partner in life.

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Summary

We study a particular class of ground states of the non-abelian Kitaev quantum double model on an infinite plane. These states are associated to the irreducible representations of the quantum double of a finite group G, and are called *anyonic excitations*. Anyonic excitations are a particular feature of topological phases of matter.

Recall that if G is a finite group, the irreducible representations of the quantum double $D(G)$ can be labelled by pairs $(\pi, \mathcal{C})$, where $\mathcal{C}$ is a conjugacy class of G and π is an irreducible representation of the centralizer subgroup of a fixed element $r \in \mathcal{C}$, the choice of which is irrelevant. Using the notion of ribbon operators as in [Kit03], we consider for each irreducible representation $\alpha := (\pi, \mathcal{C}) \in \widehat{D(G)}$, each label $I = 1, \dots, \dim_\alpha$ and semi-infinite ribbon ξ, the amplimorphisms $\chi_\xi^{II,\alpha}$ defined as in [Naa15, Eq 5.3] and show that the states $\omega_0 \circ \chi_\xi^{II,\alpha}$ define pure states, where ω_0 is the vacuum state of the model. Given two irreducible representations $\alpha, \beta \in \widehat{D(G)}$ and two semi-infinite ribbons ξ_1, ξ_2, we show that the GNS representations of $\omega_{\xi_1}^{II,\alpha}$ and $\omega_{\xi_2}^{JJ,\beta}$ are unitarily equivalent if and only if $\alpha \cong \beta$.

Furthermore, if either $\pi \neq \text{triv}$ or $|\mathcal{C}| = 1$ holds, then $\omega_\xi^{II,\alpha}$ is a ground state for a semi-infinite ribbon ξ in the infinite plane. We interpret $\omega_\xi^{II,\alpha}$ as a state creating a single localized excitation that cannot be removed by local observables. We also prove that the states $\omega_\xi^{II,\alpha}$ are indeed non-ground states in the other case and construct alternative non-pure ground states corresponding to these anyon sectors.

We conclude this work with an exposition on a work in progress. The amplimorphisms described in [Naa15] are transportable and localized, which is why they give rise to representations satisfying a superselection criterion for cones. These localized and transportable amplimorphisms form a category, and we conjecture that they are equivalent to the category $\text{rep}(D(G))$ of representations of $D(G)$ as a monoidal tensor category. In this thesis, we present some steps towards this conjecture.

Chapter 1

Introduction

In this work, we will present a family of pure ground states for the non-abelian quantum double model. The quantum double model is an example of a quantum many-body system, in which the ground state degeneracy is dependent on the topology of the surface on which the model is embedded. Such systems are called *topologically ordered* and the quantum phases of matter are named *topological phases of matter*, due to their topological dependence. The particular quantum double model is based on a group G, describing the symmetry and the interaction terms of the system, which in turn describe the dynamics of the model. If G is abelian, we call this model the *abelian* quantum double model.

The abelian quantum double model is already well studied [Kit03, Naa10, Naa15, FN15] and the infinite volume ground states of the non-abelian quantum double model on a plane are already well understood [CNN16]. The main goal of this work is to gain a better understanding of ground states of the non-abelian quantum double models.

In the infinite plane, there is a unique translational invariant ground state ω_0, called the *vacuum state* [Naa12]. It turns out that non-translational invariant ground states carry localized quasi particle excitations, called *anyons*, and the study of these anyons is closely related to the study of quantum phases of matter.

To understand what a quantum phase of matter is, let us first discuss phases of matter in more generality. While most are familiar with the classical 4 phases of matter - solid, liquid, gaseous and plasma - there are many more phases. Up until the late 1980s, physicists believed that all phases of matters could be described by the Ginzburg-Landau theory of symmetry breaking [ZCZW19]. Different phases were associated to different behaviour of the ground state space under the action of symmetries, and phase transitions were associated to symmetry breaking [LL80]. A symmetry breaking occurs, when ground states - or any equilibrium state at some temperature - are transformed to different ground states under the symmetry transformation, instead of being left invariant. A well known example of this effect is given by the Ising model, which has two distinct ground states that can be transformed into each other via a global symmetry transformations [ZCZW19].

However, in the late 1980s physicists realized that the symmetry breaking theory was not enough to fully describe all phases of matter. The chiral spin state, originally introduced to describe high-temperature superconductivity [KL87, WWZ89], could not be described by symmetry breaking alone [Wen89] as many different chiral spin states exhibited the same symmetry [Wen89]. It was suggested in [Wen90] to introduce a new order called *topological order*. Although it was later realized that chiral spin states do not describe high-temperature superconductors, similarities of chiral spin states with the fractional Hall effect [Lau83, TSG82] revitalized the notion of topological order as a means of describing different fractional quantum Hall states. We note also that fractional quantum Hall states were not the first phenomenon that realizes a topological order. Superconductors exhibit topological order as well [HOS04, Wen91a, Wen91b].

Our work focusses mostly on *quantum* phases of matter. Before the discovery of topological phases of matter, phase transitions were defined in terms of discontinuities in measurable physical quantities, called *order parameters*. When for example water enters the gaseous state from its liquid state, the local density takes a sudden drop, i.e. there is a discontinuity in the local density. The physical entity we are interested in here is the energy gap of our Hamiltonian, that is, the difference Δ between the lowest and second-lowest energy state of the Hamiltonian. We call a Hamiltonian *gapped*, if $\Delta > 0$, and two gapped ground states ω_1 and ω_2 are said to be in the same quantum phase if they are ground states of gapped Hamiltonians H_1 and H_2 that can be connected via a continuous path of gapped Hamiltonians, i.e., if there is a continuous map $t \mapsto H(t)$ of gapped Hamiltonians such that $H_0 = H(0)$ and $H_1 = H(1)$.

A new type of quantum phases are the topological quantum phases, which cannot be described by local order parameter [WN90]. It turns out that anyon states emerge in these topologically ordered systems only if the states are long-range entangled due to the localized nature of anyons [NO22].

Anyons were studied extensively by [Wil90, DPR91, BvDdWP92, dWPB99]. It is widely believed that the algebraic properties of anyons are described by a modular tensor category [Wan10, Kit06]. A good way of obtaining such a category is by applying methods from the superselection sector theory to anyons [Naa10, Naa15, Haa12]. We will explain this approach in more detail in Section 4.1, but the idea is to obtain ground states by pulling the vacuum state ω_0 back along some morphisms that satisfies suitable algebraic properties.

An important application, that makes the study of anyons attractive, lies in the possibility to construct quantum gates using braiding and fusion of anyons. Although this thesis does not study quantum gates, we want to briefly mention the idea. Until the discovery of the fractional quantum Hall effect in the early 1980s [TSG82, Lau83], physicists believed that bosonic and fermionic statistics were the only exchange statistics exhibited by particles, although purely theoretical observation of different statistics specific to the two-dimensional case have already been made [LM77]. Indeed, it is the exotic nature of the statistics of anyons that lend them their name [Wil82]. For abelian anyons, the exchange statistics, that is, the phase obtained by exchanging two anyons, can be an arbitrary root of the identity on the unit circle. For non-abelian anyons, the statistics are even more involved and the braiding of anyons has a direct impact on their fusion rules. The process of braiding anyons and performing a fusion could potentially be used to realize a quantum gate [Wan10].

Another important feature of anyons is that anyon ground states can only be transformed to other anyon ground states using global operators. The fact that it is not possible to move from one ground state to another via local operators only has potential use in implementing a quantum error correction code. The robustness against local perturbation is related to a stability assumption on the energy gap of the Hamiltonian against small perturbations [MZ13, BHM10]. We note however, that it was shown in [BT09] that error correction codes in 1 and 2 dimensions as in [CS96, Ste96] do not satisfy some necessary conditions to actually implement a self-correcting quantum memory. See also [AH06], which discusses the toric code as a particular example.

There are different techniques for studying topological order, and while this work will be focused on the operator algebraic approach, we want to briefly mention other viewpoints. One such viewpoint is the *string-net* picture, developed by Levin and Wen [LW05]. The Levin-Wen string-net model describes quantum spin systems, by describing the ground state as a superposition of so called *string-nets configurations*. These strings correspond to objects in a fusion category which encodes the fusion- and branching rules of the model.

Another closely related way of studying topological order is through *topological quantum field theory*(TQFT): Consider the category whose objects are $n-1$-dimensional smooth manifolds and whose morphisms are n-dimensional smooth manifolds, carrying the objects as surfaces, see Figure 1.1. There is a natural way of defining the composition of such morphisms by *gluing* the

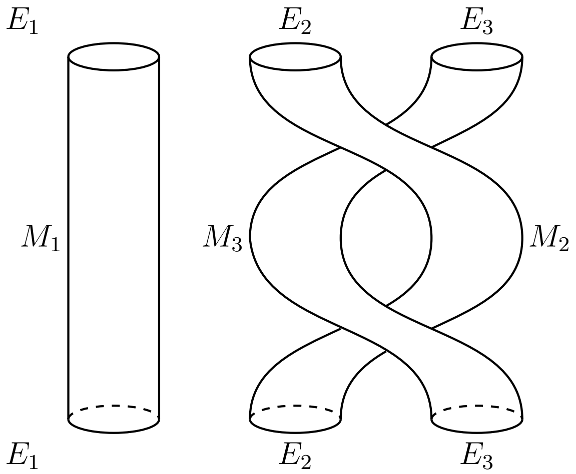


Figure 1.1: A depiction of 3-dimensional manifolds M_1, M_2, M_3 such that $E_i \coprod E_i$ is contained on the surface of M_i for $i = 1, 2, 3$. The cylinder M_1 describes the identity map on E_1 and M_2 and M_3 perform a double-braiding on the surfaces $E_2 \times E_3$. Under a functor F into $\mathrm{Vect}_{\mathbb{K}}$, this is identified by a map $\mathcal{H}_{E_1} \otimes \mathcal{H}_{E_2} \otimes \mathcal{H}_{E_3} \to \mathcal{H}_{E_1} \otimes \mathcal{H}_{E_2} \otimes \mathcal{H}_{E_3}$ performing a double braiding $\mathcal{H}_{E_2} \otimes \mathcal{H}_{E_3} \to \mathcal{H}_{E_3} \otimes \mathcal{H}_{E_2} \to \mathcal{H}_{E_2} \otimes \mathcal{H}_{E_3}$.

surfaces of two manifolds $M_1 : E_1 \to E_2$ and $M_2 : E_2 \to E_3$ and for every surface E, the cylinder $E \times [0, 1]$ can be chosen to be the identity morphism on E. Note that we always equate diffeomorphic structures in this setting. This category is called *bordism category*, and it can be equipped with a monoidal structure by defining $E_1 \otimes E_2 := E_1 \coprod E_2$ and $M_1 \otimes M_2 := M_1 \coprod M_2$ for $n-1$-dimensional manifolds E_1, E_2 and n- dimensional manifolds M_1, M_2. The idea is to identify an $n-1$-dimensional surface E with a Hilbert space $\mathcal{H}_E$, describing the states of the system. A manifold from E_1 to E_2 is then associated to a bounded linear map from $\mathcal{H}_{E_1}$ to $\mathcal{H}_{E_2}$. Broadly speaking, a topological field theory is then a functor from this bordism category to the category $\mathrm{Vect}_{\mathbb{K}}$ of $\mathbb{K}$-vector fields that respects the tensor products and their braidings in the respective symmetric monoidal category.

There is a huge area of research dedicated to the study of topological quantum field theory that cannot be done justice in this humble exposition. We refer to [CR18] for a nice introduction to the topic. Any spherical fusion category gives rise to a 3-dimensional TQFT [TV92, BW96] and the so constructed TQFT's afford a description equivalent to the string-net approach by Levin and Wen [KMR10, KKR10]. See also [Kir11] for a more mathematical approach.

Finally, we mention the tensor network approach, which employs a graphical calculus to describe states and transformations thereof using tensors. Broadly speaking, a tensor network is a family of tensors having virtual and physical indices, the latter describing the concrete physical subsystems. A state of the system can be described by a contraction of the tensor network along the virtual indices [AMN⁺23, SCPG10]. Projected entangled pair states (PEPS) are particularly interesting in the context of 2-dimensional quantum many-body system. Each tensor is associated to a site on the lattice, and the virtual indices describe the entanglement between adjacent sites. PEPS can be used to describe topologically ordered systems [FGSW⁺12, SPCPG13], as they naturally capture the notion of long-range entanglement [RDS15].

1.1 The Toric Code: An Example of a Topologically Ordered System

The most important example related to this work is the toric code [Kit03]. The toric code is a particular example of Shor's *stabilizer code*. These stabilizer codes are an example of a quantum error correcting code, as it was the initial hope that these codes perform a self-error correction on a quantum memory to some extend. This is a desirable feature in quantum information, as classical error correction cannot be established in the setting of quantum information. This is because classical methods for error corrections always involve copying data to some extent, but the *no cloning* theorem [Par70, WZ82, Die] forbids such a process entirely. We briefly discuss the main idea behind Shor's error correction code, but see also [CS96, Ste96]. Let $\mathcal{E} = \{E_k\}$ be a set of observables describing all possible noises the system can be exposed to. For example, for a spin system, this can mean a spin flip on one of the qubits. One important idea to realize error correction, is to store a single-qubit in a subspace of an N-qubit many-body system to mitigate the effect of the noises. These subspaces are what one usually calls the *stabilizer code* or in this case the *stabilizer code*. One then proceeds to find commuting projections $g_1, \ldots, g_n$ in the algebra generated by $\mathcal{E}$, chosen such that the stabilizer code lies in the image of these projections. Choosing the Hamiltonian

$$H = -\sum_i g_i,$$

the stabilizer code then re-emerges as the ground state space of H.

The specific toric code model is a quantum spin system on a $\mathbb{Z}^2$ lattice embedded on a torus, where each edge of $\mathbb{Z}^2$ is equipped with a qubit, i.e. the Hilbert space $\mathbb{C}^2$. This model exhibits a 4-dimensional ground state degeneracy. More generally, if g is the genus of the surface on which the lattice is embedded, the ground state degeneracy is 4^g [Kit03]. Let s and p describe a star- respectively plaquette shaped region of the lattice, see Figure 1.2. Then we define the star operators

$$A_s = \prod_{\mathfrak{e} \in s} \sigma_x^{\mathfrak{e}}$$

and plaquette operators

$$B_p = \prod_{\mathfrak{e} \in p} \sigma_z^{\mathfrak{e}},$$

where $\sigma_z^{\mathfrak{e}}$ and $\sigma_x^{\mathfrak{e}}$ are respectively defined to be the action by the Pauli matrix σ_z and σ_x at edge $\mathfrak{e}$, and the identity action everywhere else. The Hamiltonian of the system is described as a sum of the local stabilizer operators

$$H = -\sum_s A_s - \sum_p B_p,$$

where the sum is over all stars s and plaquettes p. The motivation behind these local stabilizer operators is that they detect errors [Kit03]. A ground states of the model can be characterized by the frustration freeness condition, that is, a state ω_0 is a ground state if and only if $\omega_0(A_s) = \omega_0(B_p) = 1$ for all stars s and plaquettes p. The term *frustration free* just means here that the state ω_0 minimizes each summand in the Hamiltonian separately, and ω_0 is called a frustration free ground state. The relevant strings in this model are given by paths and *dual paths*, that is, paths along the faces, of the lattice. Given a non-self-intersecting path $\xi = (\mathfrak{e}_1, \dots, \mathfrak{e}_n)$, along the edges $\mathfrak{e}_1, \dots, \mathfrak{e}_n$ we define the string operator F_ξ via

$$F_\xi = \prod_{i=1}^{n} \sigma_z^{\mathfrak{e}_i}.$$

Recall that a dual edge is a pair $(\mathfrak{f}_1, \mathfrak{f}_2)$, where $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are faces of the lattice. If $\xi^* = (\mathfrak{f}_1, \dots, \mathfrak{f}_m)$ is a non-self-intersecting dual path along dual edges $\mathfrak{f}_1, \dots, \mathfrak{f}_m$, i.e. edges connecting faces, and if $\mathfrak{e}_1, \dots, \mathfrak{e}_m$ denotes the edges of the lattice *crossing* dual edges $\mathfrak{f}_1, \dots, \mathfrak{f}_m$, we define the string operator F_{ξ^*} via

$$F_{\xi^*} = \prod_{i=1}^{m} \sigma_x^{\mathfrak{e}_i}.$$

Let Λ be the set of all edges of the lattice embedded on the torus. One can then verify that the state

$$\Omega_0 = \prod_s A_s \otimes_{\mathfrak{e}\in\Lambda} |0\rangle$$

defines a vector in $\otimes_{\mathfrak{e}\in\Lambda}\mathbb{C}^2$ that is invariant under the action of all star and plaquette operators, $|0\rangle, |1\rangle \in \mathbb{C}^2$ is the orthonormal basis in which the Pauli matrices are represented, corresponding to spin up and spin down states. Ω_0 can be identified with the vacuum vector. One can show, that for a path ξ and a dual path ξ^*, both non-self-intersecting, the vectors $F_\xi\Omega_0$, $F_{\xi^*}\Omega_0$ and $F_\xi F_{\xi^*}\Omega_0$ violate the frustration freeness condition precisely at the endpoints of ξ, respectively ξ^*, demonstrating how string operators create pairs of excitations. We also note that the algebra of all local observables is generated by the string operators, since the Pauli matrices σ_x and σ_z already generate the algebra $M_2(\mathbb{C})$. Hence, observables can only create pairs of excitations.

One can also directly observe how the topological ground state degeneracy is related to the genus of the surface in this model. One can show that the vectors $F_\xi\Omega_0, F_{\xi^*}\Omega_0$ and $F_\xi F_{\xi^*}\Omega_0$ are independent of the concrete shape of ξ and ξ^* and only depend on the endpoints of ξ and ξ^*, highlighting the topological features of these models further. Note also that by assuming the strings to be non-self-intersecting, we also excluded loops. If the strings are closed loops around the torus, however (and don't intersect anywhere else), the corresponding string operators map the ground state Ω_0 to different ground states. Since the non-trivial genus of the torus allows for two non-trivial homotopy classes and because we can choose between paths and dual paths, one may directly verify that additional 3 ground states can be obtained via the action of string operators on the ground state Ω_0.

1.2 The Quantum Double Model

The toric code is a special case of the more general quantum double model for groups. These models are surface code models on a lattice where each edge is decorated with the group Hilbert space $\mathbb{C}G$ of a given finite group G, instead of the Hilbert space $\mathbb{C}^2$ as in the case of spin systems. The group Hilbert space $\mathbb{C}G$ is defined by considering the group algebra of G as a vector space, and embedding it with a natural Hilbert space structure. The stabilizer operators A_s and B_s then turn out to be coming from a representation of the quantum double $D(G)$ of G at site s. The different anyons can then be associated to irreducible representation of the quantum double $D(G)$. As mentioned before, the infinite volume ground states of the abelian quantum

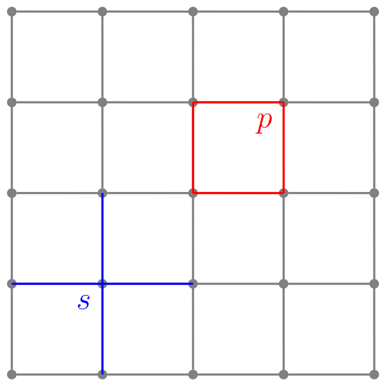


Figure 1.2: The figure depicts the star (blue) and plaquette (red) shaped regions on a $\mathbb{Z}^2$ lattice. On the edges of the star, the operator A_s acts with the pauli matrix σ_z at each edge, whereas the operator B_p acts with the Pauli matrix σ_x at each edge of the plaquette.

double model on a plane are already well understood. On the other hand, the non-abelian quantum double model is less well studied as the abelian model, but there still exists plenty of works [BMD08, CM22b, CM22a] and even generalizations to quantum double models stemming from semisimple Hopf algebras and their quantum doubles, as opposed to finite groups, were considered in the literature [CCY21]. A categorical generalization of Kitaev's quantum double model can be found in [HM23, Meu17].

In this work, we will only concern ourselves with the non-abelian quantum double model on a $\mathbb{Z}^2$-lattice embedded on an infinite plane. An important remark that we want to make at this point is that ground states in the infinite volume setting can also include single site excitations i.e. anyons, despite having higher energy than the vacuum state. In fact, if we consider a particle-antiparticle pair, created using operators similar to the string operators in the toric code, and sending one particle to infinity by considering a sequence of strings, then in many cases the resulting state will be a ground state, as we shall prove in this work. The reason these states are ground states, is that these single excitations turn out to be robust against local operators, i.e. they cannot be erased via local operators, although they can be moved around over finite distances. More formally, a ground state ω is defined as a state that satisfies the condition

$$-i\omega(A^*\delta(A)) \geq 0$$

for all observables A in the domain of the derivation δ, describing the dy-

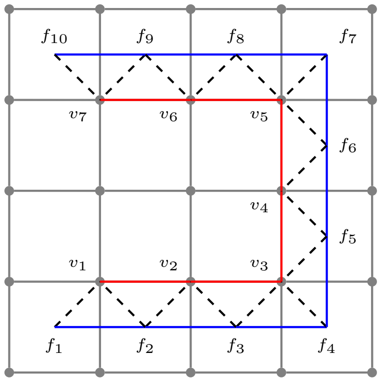


Figure 1.3: Depiction of a ribbon in the $\mathbb{Z}^2$-lattice. The vertices $(v_1, \ldots, v_7)$ form a path p (red) and the faces $(f_1, \ldots, f_{10})$ form a dual path p^* (blue). Every vertex of p forms a site (dashed) with a face of p^* and every face of p^* forms a site with a vertex of p. Note that every two consecutive sites form, together with either p or p^*, a triangle, and no two triangles overlap, which can be used to give an alternative definition of a ribbon.

namics of the system.

Following the construction presented in [Kit03], we will define the so-called *ribbon operators* $F_\xi^{IJ,\alpha}$, labelled by irreducible representations $\alpha \in \widehat{D(G)}$ of the quantum double $D(G)$ and $I, J = 1, \ldots, d_\alpha$, with $d_\alpha = \dim(\alpha)$ and *ribbons* ξ. Notice that one major difference to the abelian case becomes already apparent here: In the abelian case, all irreducible representations of $D(G)$ are one-dimensional, whereas the dimension of the irreducible representations of $D(G)$ are in general less trivial in the non-abelian case. To define a ribbon, recall that a *site* in a graph is a pair (v, s), where v is a vertex and f is a face having v in one of its corners. A ribbon is defined as a pair of a non-self-intersecting path $p = (v_1, \ldots, v_n)$, and a non-self-intersecting dual path $p^* = (f_1, \ldots, f_m)$ in the lattice model, such that each vertex in p forms a site with a face in p^*, and each face in p^* forms a site with a vertex in p such that p and p^* do not intersect, see Figure 1.3. This model admits a unique frustration free ground state ω_0 [Naa12], also called *vacuum state*, and a particle-antiparticle state can be created by considering states of the form

$$\omega_\xi^{II,\alpha} : A \mapsto \frac{1}{d_\alpha} \sum_{J=1}^{d_\alpha} \omega_0 \left(F_\xi^{IJ,\alpha} A (F_\xi^{IJ})^* \right). \tag{1.2.1}$$

This construction is motivated by [SV93], in the context of 1-dimensional quantum spin chains. The convergence of Equation (1.2.1) for infinite ribbons was already established in [Naa12], see also [FN15]. Note that if $(\mathbf{F}^\alpha_\xi)_{I,J=1,\dots,d_\alpha}$ is the matrix with entries $\frac{1}{\sqrt{d_\alpha}} F^{IJ,\alpha}_\xi$, then the above identity reads $\omega^{II,\alpha}_\xi = \omega_0 \circ \chi^{II,\alpha}_\xi$, where $\chi^{II,\alpha}_\xi$ is the entry in the I-th row and I-th column of the matrix $\chi^\alpha_\xi(A)$ defined via

$$\chi_\xi(A) = \mathbf{F}^\alpha_\xi (A \otimes \mathrm{id}_{\mathbb{C}^{d_\alpha}}) (\mathbf{F}^\alpha_\xi)^*, \tag{1.2.2}$$

where $\mathrm{id}_{\mathbb{C}^{d_\alpha}}$ is the identity matrix on $\mathbb{C}^{d_\alpha}$. χ^α_ξ is a unital *-homomorphism [Naa12] from $\mathfrak{A}$ to $\mathfrak{A} \otimes \mathrm{End}(\mathbb{C}^{d_\alpha})$, and such maps are called *amplimorphisms*. It was already shown [Naa12] that the amplimorphisms χ^α_ξ converge for infinite ribbons, even in the non-abelian case.

1.3 Main Results and Outline of the Thesis

Before we state our first main result, recall that the irreducible representations of the quantum double $D(G)$ are labelled by pairs $\alpha = (\pi_\alpha, \mathcal{C}_\alpha)$, where $\mathcal{C}_\alpha$ is a conjugacy class of G and π_α is an irreducible representation of the centralizer subgroup of an element $r_\alpha \in \mathcal{C}_\alpha$, the specific choice of which does not matter [Gou93]. We will give an overview of the quantum double and its representations in Section 2.4.1 and Section 2.4.2. Our first main result then reads:

Theorem A. *Let $\alpha = (\pi_\alpha, \mathcal{C}_\alpha) \in \widehat{D(G)}$ be an irreducible representation of the quantum double $D(G)$, ξ_n a sequence of ribbon extending to an infinite ribbon ξ with fixed starting site $\partial_0 \xi = \partial_0 \xi_n = s$ for all n, and $\omega^{II,\alpha}_\xi$ the states defined in Equation* (1.2.1)*. Furthermore, we define*

$$\omega^\alpha_\xi : X \mapsto \lim_{n\to\infty} \sum_{I,I',J} \omega_0 \left(F^{IJ,\alpha}_{\xi_n} X (F^{I'J,\alpha}_{\xi_n})^* \right),$$

which is well-defined, i.e. converges for each choice of $\alpha \in \widehat{D(G)}$ by [Naa12]. Then, if $\pi_\alpha \neq \mathrm{triv}$ is not the trivial representation, or $|\mathcal{C}_\alpha| = 1$, then the states $\omega^{II,\alpha}_\xi$ are infinite volume ground states of the non-abelian quantum double model on the plane. In all other cases, the states ω^α_ξ are ground states.

The reason for the anomaly for the case $\pi_\alpha = \mathrm{triv}$ is that in that case the states $\omega^{II,\alpha}_\xi$ are not orthogonal to the image of the stabilizing operators A_s any more, but are so in all other cases. The star operators A_s however, project into the equal-weighted superposition of all states within a sector, lowering the energy of the $\omega_{\xi}^{II,\alpha}$. The ground state must therefore be created using the equal-weighted superposition of all corresponding ribbon operators in that case.

Our second main result reads:

Theorem B. The states $\omega_{\xi}^{II,\alpha}$ are pure states for any irreducible representation α and their GNS representations are therefore irreducible.

We remark that this result was obtained independently by [BV23] using a different approach.

A natural question is whether different irreducible representations of the quantum double give rise to inequivalent pure states. This is answered in the following Theorem.

Theorem C. Let $\alpha, \beta \in \widehat{D(G)}$ be irreducible representation of the quantum double $D(G)$, $I \in \{1, \dots, \dim(\alpha)\}$, $J \in \{1, \dots, \dim(\beta)\}$ and ξ_1 and ξ_2 be two semi-infinite ribbons. Then the GNS representations of $\omega_{\xi_1}^{II,\alpha}$ and $\omega_{\xi_2}^{JJ,\beta}$ are equivalent if and only if $\alpha \cong \beta$.

Theorems A, B and C can be found as Theorems 3.5.4, 3.6.8 and 3.6.9 respectively in the main work.

It was shown in [Naa12, Naa15] that the amplimorphisms defined in Equation (1.2.2) are localized and transportable over finite regions in the setting of general finite groups. Here, localized means that there exists a region Λ such that for all observables supported outside of Λ, we have that $\chi_{\xi}(A) = A \otimes \mathrm{id}_{\mathbb{C}^{d_\alpha}}$, and being transportable translates to the existence of a unitary $V_{\Lambda'}$ for each region Λ' such that the amplimorphism $A \mapsto V_{\Lambda'}\chi(A)V_{\Lambda'}^{-1}$ is localized in Λ'. This terminology is motivated by the DHR analysis, of which we want to give a brief exposition here: Note first that for a general C*-algebra, there are many states that are considered physically irrelevant. One proposed criterion to sieve out those unphysical states in the framework of quantum field theory, is given by the superselection criterion by Doplicher, Haag and Roberts [DHR71]. If $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ is a local net of observables on a space time and π_0 the GNS representation of the vacuum state ω_0, then a state ω is said to satisfy the superselection criterion if

$$\pi_0 |_{\mathfrak{A}(\mathcal{O}')} \cong \pi |_{\mathfrak{A}(\mathcal{O}')}, \tag{1.3.1}$$

where $\mathcal{O}'$ denotes the causal complement of $\mathcal{O}$ and π is the GNS representation of ω. If $L_0(\mathcal{O})$ is the set of transportable endomorphisms localized in $\mathcal{O}$, the notions of transportability and localizability defined analogously to the amplimorphism setting, then by [DHR71, Proposition 1.2] a state π

satisfies the superselection criterion 1.3.1 for $\mathcal{O}$ if and only if there exists an endomorphism $\rho \in L_0(\mathcal{O})$ such that π is unitary equivalent to $\pi_0 \circ \rho$. In other words, the set

$$\{\pi_0 \circ \rho\}_{\rho \in L_0(\mathcal{O})}$$

consists of all representations satisfying the superselection criterion for the region $\mathcal{O}$, up to unitary equivalence.

In [SV93] it was shown that amplimorphisms on a quasilocal algebra form a braided monoidal category, and the authors showed that the subcategory of amplimorphisms constructed out of finite ribbons is equivalent to the category of representations of the quantum double $D(G)$ in the setting of 1-dimensional quantum spin chains. The analogue result was established in [Naa10] for the abelian quantum double model, emphasizing once again the role of the DHR-analysis in Kitaevs quantum double model. We conjecture that an analogue result can also be obtained in the more general non-abelian setting. We will present some steps towards this conjecture in Section 4.1, but note this is currently a work in progress.

Finally, an important topological feature possessed by the ribbon operators is that when acting on the vacuum, the action of the ribbon operators $F_\xi^{IJ,\alpha}$ only depends on the initial and final sites of the ribbon ξ, i.e., if Ω_0 is the cyclic vector of the GNS representation corresponding to the vacuum state ω_0, and ξ' any ribbon with the same endpoints as ξ, then $F_\xi^{IJ,\alpha}\Omega_0 = F_{\xi'}^{IJ,\alpha}\Omega_0$. To our knowledge, however, there is no complete or correct proof in the literature for the ribbon operators in the non-abelian quantum double model. In [BMD08], the definition of ribbon operators is not entirely correct [CCY21] and other authors [CM22b] often argue that because closed ribbon can be deformed within the vacuum to the empty ribbon, two ribbons ξ_1 and ξ_2 with the same endpoints give an identical action because following first ξ_1 and then the reverse of ξ_2 gives a closed ribbon again. While the former argument is certainly correct - a closed ribbon indeed deforms into the empty ribbon if no excitations are present, the latter fails because ξ_1 and the inverse of ξ_2 generally do not form a closed ribbon, even if they share the same endpoints, see Figure 1.4. We prove this important topological property in Corollary 3.3.11. This implies in particular that the states defined in Equation (1.2.1) for infinite ribbons, define charges localized at the beginning of ξ, independent of the chosen shape of the ribbon ξ.

This thesis is structured as follows: Chapter 2 introduces the necessary background for this work. We emphasize here that none of the statements derived in Chapter 2 are new results, even if it is not always explicitly stated. In Section 2.2, 2.3 and 2.4 we recall the basic notions from representation theory, the theory of Hopf algebras and define the quantum double construction

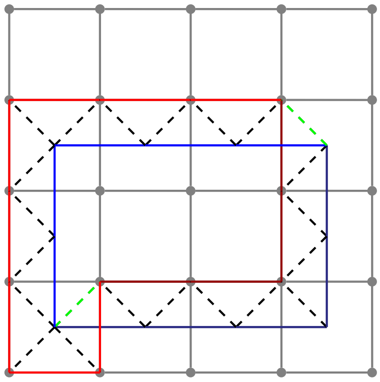


Figure 1.4: Depiction of two ribbons ξ_1 (red path and blue dual path) and ξ_2 (dark red path and dark blue dual path) starting and ending at the same site (indicated in green). However, they do not constitute a closed ribbon since the path of ξ_1 would intersect with the dual path of ξ_2 and vice versa.

respectively. In Section 2.5, we give a brief introduction to operator algebras and give a general definition of quantum spin systems and dynamics. We also introduce the notion of ground states and pure states in our particular setting. Chapter 3 contains the main work of this thesis. In Section 3.2, we define Kitaev's quantum double model and introduce necessary terminology. We also discuss the uniqueness of the vacuum state ω_0 and the notion of charges. In Section 3.3, we rigorously define ribbon operators and study their algebraic properties. We will demonstrate how these operators create pairs of excitations, and that the action of a ribbon operator on the vacuum state only depends on the endpoints of the ribbon. In Section 3.4, we draw the connection between excitations and the irreducible representations of the quantum double $D(G)$ by showing that the dynamics of the quantum double model is realized by a quantum double action, and that the excitation space can be decomposed into a direct sum of irreducible representations of $D(G)$. In Section 3.5, we state the first main theorem, Theorem 3.5.4 (Theorem A), providing a family of infinite volume ground states for the non-abelian quantum double model. In Section 3.6 we show Theorem B and also Theorem C. Finally, in Section 4.1 we discuss the possibility of generalizing the construction in [SV93] to our setting and the obstructions that one might encounter. The rest of Chapter 4 is devoted to discussing other possible generalizations and open questions.

Chapter 2

Preliminaries

2.1 Introduction

The dynamics of a quantum spin system can be described by the interaction terms between the individual spin systems. In the class of models that we are considering, these are given by two families of actions $A_s : h \mapsto A_s^h$ and $B_s : \delta_g \mapsto B_s^g$ of a group G and its dual $G^* = \{\varphi : G \to \mathbb{C}\}$, called electric and magnetic charge action respectively, with s describing the site on which the interaction is considered. As it turns out, in the quantum double model on an infinite plane, the ground states can be characterized by the values they take on these interaction terms, and so the actions A_s and B_s allow us to distinguish the different anyon sectors. Furthermore, the electric and magnetic charge action satisfy the commutation relation

$$B_s^g A_s^h = A_s^h B_s^{h^{-1}gh}. \tag{2.1.1}$$

In a more general framework, if H_1 and H_2 are Hopf algebras contained in a Hopf algebra H such that H_1 and H_2 satisfy a specific commutation relation within H, then H_1 and H_2 can be embedded in a universal algebra $H_1 \bowtie H_2$ whilst preserving said commutation relation. This construction is called the bicrossed product, and for the particular commutation relation given in Equation (2.1.1) it is called the quantum double of G.

We will discuss the quantum double and its representation theory in detail in Section 2.4 and the operator algebraic framework of quantum spin systems in Section 2.5. First, however, we start by giving a short overview on the necessary concepts of representation theory and Hopf algebras in Section 2.2 and Section 2.3 respectively.

We will assume familiarity with basic terminology from category theory, although we will not use any advanced results from that field. We refer to [Bor94, Ada09, Lan13] for a selection of literature on category theory and [EGNO16] for an exposition on fusion categories, which captures the behaviour of the representation category rep(G) on a categorical level.

2.2 Representations of Finite Groups

We cover some basic statements from the representation theory for finite groups in this section. We will not provide any proofs here as most of the following results are well known and refer to [Ser77, Hal13a, EGH$^+$11] for a detailed exposition.

Let G be a finite group. A **representation of G** is a pair (π, V_π) consisting of a group homomorphism $\pi : G \to \mathrm{Aut}\,(V_\pi)$ into the automorphism group of some vector space V_π. The space V_π is called a G-module. We will often just write either π or V_π for a representation (π, V_π), suppressing either the concrete G-module V_π or the concrete action π where no confusion arises. Furthermore, we will use adjectives for the action π interchangeably with the corresponding G-module V_π. For instance, we call π finite dimensional if V_π is finite dimensional.

A representation of G is called a **complex representation**, if the underlying vector space V_π is a complex vector space. If V_π is equipped with a sesqui-linear inner product $\langle\cdot,\cdot\rangle : V_\pi \times V_\pi \to \mathbb{C}$, then we call a representation π **unitary**, if $\pi(g)^{-1} = \pi(g)^*$ for all $g \in G$, where * denotes the adjoint. We will restrict ourselves to unitary representations only. We will argue later that the unitarity condition is not really a restriction.

For a fixed group G, we denote by rep(G) the set of finite dimensional unitary representations of G.

There are several ways to construct new representations out of given ones.

(a) **The dual representation:** If (π, V_π) is a representation of G, we may define a representation of G on $V_\pi^* = \mathrm{Hom}(V_\pi, \mathbb{C})$ via

$$(g \triangleright \varphi)(v) = \varphi(\pi(g^{-1})(v)) \tag{2.2.1}$$

for all $g \in G$, $\varphi \in \mathrm{Hom}(V_\pi, \mathbb{C})$ and $v \in V_\pi$. This is sometimes also called the **contragredient representation** of π.

(b) **The direct sum representation:** Given two representations (π_1, V_{π_1}) and (π_2, V_{π_2}), the action on the direct sum $V_{\pi_1} \oplus V_{\pi_2}$ is given by the liner extension of the mapping

$$(\pi_1 \oplus \pi_2)\,(g) : v_1 \oplus v_2 \mapsto \pi_1(g)(v_1) \oplus \pi_2(g)(v_2)$$

for all $g \in G$ and $v_1 \in V_{\pi_1}$, $v_2 \in V_{\pi_2}$.

(c) **The tensor product representation:** Given two representations (π_1, V_{π_1}) and (π_2, V_{π_2}), the action on the tensor product $V_{\pi_1} \otimes V_{\pi_2}$ is given by the linear extension of the mapping

$$(\pi_1 \otimes \pi_2)(g) : (v_1 \otimes v_2) \mapsto \pi_1(g)(v_1) \otimes \pi_2(g)(v_2) \tag{2.2.2}$$

for all $g \in G$ and $v_1 \in V_{\pi_1}, v_2 \in V_{\pi_2}$.

It is straightforward to verify that each of these examples yield indeed a representation.

A **morphism of representations** $f : (\pi_1, V_{\pi_1}) \to (\pi_2, V_{\pi_2})$ is a linear map $f : V_{\pi_1} \to V_{\pi_2}$ such that the diagram

$$\begin{array}{ccc} V_{\pi_1} & \xrightarrow{f} & V_{\pi_2} \\ {\scriptstyle \pi_1(g)}\downarrow & & \downarrow{\scriptstyle \pi_2(g)} \\ V_{\pi_1} & \xrightarrow[f]{} & V_{\pi_2} \end{array}$$

commutes for all $g \in G$. We call such a map f an **intertwiner** or sometimes a $G-$**equivariant** linear map, and often write $f : \pi_1 \to \pi_2$ instead of $f : (\pi_1, V_{\pi_1}) \to (\pi_2, V_{\pi_2})$. We call two representations π_1 and π_2 **equivalent** if there exists a unitary intertwiner $T : \pi_1 \to \pi_2$ and T is called an **isomorphism of representations**.

The reason we define equivalence in terms of intertwining unitaries, instead of (non-unitary) isomorphisms, is so that the intertwiner additionally respects the sesqui-linear products of the respective vector spaces. In particular, it maps orthogonal subspaces to orthogonal subspaces, which will become important once we study subrepresentations.

Let π_1 be a non-unitary representation on an inner product space $(V_\pi, \langle -, - \rangle)$. Then we may substitute V_π with the inner product space $(V_\pi, \langle -, - \rangle_0)$ with inner product defined via

$$\langle v_1, v_2 \rangle_0 = \sum_{g \in G} \langle \pi(g)v_1, \pi(g)v_2 \rangle.$$

We claim that G acts on $(V_\pi, \langle -, - \rangle_0)$ unitarily, and that $(V_\pi, \langle -, - \rangle_0)$ and $(V_\pi, \langle -, - \rangle)$ are equivalent. Indeed, we have

$$\begin{aligned} \langle \pi(h)v_1, \pi(h)v_2 \rangle_0 =& \sum_{g \in G} \langle \pi(g)\pi(h)v_1, \pi(g)\pi(h)v_2 \rangle \\ =& \sum_{g \in G} \langle \pi(gh)v_1, \pi(gh)v_2 \rangle = \sum_{g \in G} \langle \pi(g)v_1, \pi(g)v_2 \rangle \\ =& \langle v_1, v_2 \rangle_0 \end{aligned}$$

and the identity establishes an invertible intertwiner between these two spaces. This demonstrates that restricting to unitary representations does not concede any generality for all intents and purposes.

A **subrepresentation** of a representation (π, V_π) is a representation (ϑ, W_θ) with W_θ a subspace of V_π such that W_θ is invariant under the action of G under π. That is, for all $v \in W_\theta$ and $g \in G$ we always have $\pi(g)(v) \in W_\theta$. Clearly $\{0\}$ and V_π are subrepresentations of V_π. We call V_π **irreducible** if V_π and $\{0\}$ are the only subrepresentations of V_π. We denote by $\widehat{G}$ a choice of representatives of inequivalent irreducible representations of G.

Example 2.2.1. Take $G = S_3 = \{\mathrm{id}, (12), (13), (23), (123), (132)\}$ to be the symmetric group of permutations of the set $\{1, 2, 3\}$. Then G has the following inequivalent irreducible representations:

- The *trivial representation* $(\mathrm{triv}, \mathbb{C})$, sending each $\sigma \in S_3$ to the identity 1.
- The *sign representation* $(\mathrm{sign}, \mathbb{C})$, sending each $\sigma \in S_3$ to the signature of σ.
- The *standard representation* (stand, W), where W is the subspace of $\mathbb{R}^3$ defined via

$$W = \left\{ z_1 + z_2 + z_3 = 0 \mid z_1 e_1 + z_2 e_2 + z_3 e_3 \in \mathbb{C}^3 \right\}$$

and the action is defined via $\sigma \triangleright e_i = e_{\sigma(i)}$, where e_1, e_2, e_3 is a basis of $\mathbb{C}^3$.

Clearly, the trivial and sign representation are irreducible; They are one-dimensional and thus contain no non-trivial subspace. For the standard representation, note first that the identity $z_1 + z_2 + z_3 = 0$ is preserved under every permutation of the z_1, z_2, z_3, hence W is invariant under the standard representation. To see that it is also irreducible, note that for any vector $v \in W$ we have

$$(\mathrm{id} - (12))v \in \mathrm{span}_{\mathbb{C}} \left\{ \big(e_1 - e_2 \big) \right\}.$$

Similarly, the vector $\big(e_1 - e_2\big)$ can be mapped to any vector $v = z_1 e_1 + z_2 e_2 - (z_1 + z_2) e_3 \in W$ via $(z_1 - z_2)\,\mathrm{id} + z_2 (23) + z_3 (123)$. This implies that no one-dimensional subspace can be invariant and W must be irreducible.

Let $f : V_{\pi_1} \to V_{\pi_2}$ be a morphism between two representations V_{π_1} and V_{π_2} of a finite group G. Then $\ker(f)$ and $\mathrm{im}(f)$ are subrepresentations of V_{π_1} and V_{π_2} respectively. The following consequence is known as Schur's Lemma.

Theorem 2.2.2 (Schur's Lemma). *If V_{π_1} and V_{π_2} are irreducible representations of G and $f : V_{\pi_1} \to V_{\pi_2}$ a morphism of representations, then we have either $f = 0$ or f is an isomorphism. In the latter case, if we further have $V_{\pi_1} = V_{\pi_2}$, then f is a scalar multiple of the identity.*

Remark 2.2.3. Schur's Lemma holds more generally for algebraically closed fields. It fails however for general fields, see [EGH$^+$11].

A representation V is called **semisimple** if it can be written as the direct sum of irreducible representations. Every finite dimensional representation of a finite group is semisimple [EGH$^+$11, Ser77]. If V is semisimple and

$$V = \bigoplus_{\pi \in \hat{G}} n_\pi V_\pi \tag{2.2.3}$$

a decomposition of V into the direct summands $nV_\pi = \bigoplus_{k=1}^{n} V_\pi$, and if W is an irreducible subrepresentation of V, then W is isomorphic to one of the V_π appearing in Equation (2.2.3). More generally, if W is any subrepresentation of V, then W is semisimple and is isomorphic to a *subdecomposition* of V, i.e., there exists $0 \leq r_\pi \leq n_\pi$ such that

$$W \cong \bigoplus_{\pi \in \hat{G}} r_\pi V_\pi,$$

see [EGH$^+$11, Proposition 3.1.4]. It is well known [Ser77, Thm. 1 and Thm. 2] that any given finite-dimensional representation decomposes into a direct sum of irreducible representations. This is particularly true for tensor products: If (π_1, V_{π_1}) and (π_2, V_{π_2}) are irreducible representations, there exists coefficients $N_{\pi_1,\pi_2}^{\pi_3}$ such that

$$V_{\pi_1} \otimes V_{\pi_2} \simeq \bigoplus_{\pi \in \hat{G}} N_{\pi_1,\pi_2}^{\pi_3} V_{\pi_3}.$$

The coefficients $N_{\pi_1,\pi_2}^{\pi_3}$ are called **fusion coefficients**.

Remark 2.2.4. The representations of a finite group G form a category with $\mathrm{rep}(G)$ as objects and with intertwiners as morphisms, and we will denote this category by $\mathrm{rep}(G)$ again. This will create no confusion, as we will always equate the notations $\pi \in \mathrm{rep}(G)$ and $\pi \in \mathrm{obj}(\mathrm{rep}(G))$, where $\mathrm{obj}(\mathfrak{C})$ denotes the class of objects of a category $\mathfrak{C}$. In the language of category theory, a subrepresentation is simply a subobject and two representations are equivalent if they are equivalent as objects in $\mathrm{rep}(G)$. The latter can be seen as follows: If V is equipped with two Hermitian inner products $\langle \cdot, \cdot \rangle_1$ and

$\langle \cdot, \cdot \rangle_2$, then by the Riesz representation theorem there exists a unique positive definite and self-adjoint linear map $f : V \to V$ such that $\langle fv, w \rangle_1 = \langle v, w \rangle_2$. If $\pi : G \to \mathrm{Aut}(V)$ is a unitary representation, then for all $v, w \in V$ and $g \in G$ we have.

$$\langle gfv, w \rangle_1 = \langle fv, g^{-1}w \rangle_1 = \langle v, g^{-1}w \rangle_2 = \langle gv, w \rangle_2 = \langle fgv, w \rangle_1,$$

hence $gf = fg$ for all $g \in G$ and f must be an intertwiner. If V is irreducible, it follows that $\langle \cdot, \cdot \rangle_1$ is a positive and real scalar multiple of $\langle \cdot, \cdot \rangle_2$ by Schur's Lemma. It follows that if (π_1, V_{π_1}) and (π_2, V_{π_2}) are equivalent irreducible representations with intertwiner $f : V_{\pi_1} \to V_{\pi_2}$, that there exists a positive $\lambda \in \mathbb{R}$ with

$$\lambda \langle v, w \rangle_{\pi_1} = \langle f(v), f(w) \rangle_{\pi_2}.$$

and the map $\tilde{f} := \frac{1}{\sqrt{\lambda}} f$ affords a unitary intertwiner between V_{π_1} and V_{π_2}. Since every representation decomposes into a direct sum of irreducible ones, it follows that the categorical notion of equivalence coincides with the notion of unitary equivalence.

An important consequence of Schur's Lemma is the Peter-Weyl theorem. Let (π, V) be an irreducible representation of G and let $\dim_\pi = \dim(V_\pi)$ and $\Gamma_\pi : \dim_\pi \times \dim_\pi \times G \to \mathbb{C}$ denote an explicit unitary matrix representation for some appropriate basis $b_1, \ldots, b_n$. Then the Peter-Weyl theorem reads as follows.

Theorem 2.2.5 (Peter-Weyl)**.** *Let G be a finite group. Then the unitary matrix coefficients*

$$\left\{ \Gamma_\pi^{ij} \right\}_{\substack{\pi \in \hat{G} \\ i,j=1,\ldots,\dim(\pi)}}$$

defined via

$$\Gamma_\pi^{ij} : g \mapsto \langle b_i, \pi(g) b_j \rangle$$

for an orthonormal basis $\{b_i\}_{i=1,\ldots,\dim_\pi}$ for each irreducible module V_π, span $C(G)$, the set of complex valued functions on G. Furthermore, they are orthonormal with respect to the inner product

$$\begin{aligned} C(G) \times C(G) &\to \mathbb{C} \\ (f, h) \mapsto \langle f, h \rangle &= \frac{\dim_\pi}{|G|} \sum_{g \in G} \bar{f}(g) h(g). \end{aligned} \tag{2.2.4}$$

The orthogonality relation is shown in [Kna01, Corollary 1.10] and the density of the matrix coefficient is shown in [Kna01, Theorem 1.12]. There, Theorem 2.2.5 is derived for compact groups, with $C(G)$ being the space of continuous complex valued functions on G, and Equation (2.2.4) becomes an integral with respect to the so-called Haar measure. We will only be interested in the case where G is finite, and the special case described in Theorem 2.2.5 can then be derived by choosing the discrete topology on G.

For future reference, we want to list some identities that can easily be verified using Equation (2.2.4):

$$\sum_{g\in G}\bar{\Gamma}^{ij}_{\pi}(g)\Gamma^{kl}_{\pi'}(g) = \frac{\dim_\pi}{|G|}\delta_{i,k}\delta_{j,l}\delta_{\pi,\pi'} \tag{2.2.5}$$

$$\sum_{n,m\in G}\bar{\Gamma}^{ij}_{\pi'}(m)\bar{\Gamma}^{kl}_{\pi}(n)f(mn) = \sum_{m\in G}\frac{\dim_\pi}{|G|}\delta_{\pi,\pi'}\delta_{j,k}\bar{\Gamma}^{il}_{\pi'}(m)f(m), \tag{2.2.6}$$

where $f : G \to S$ is an arbitrary function from G to some set S and π, π' are irreducible representations of G.

Given an irreducible representation (π, V), we define the trace

$$g \mapsto \mathrm{tr}_\pi(g)$$

of an element $g \in G$ understood as an automorphism acting on V. The mapping $\mathrm{tr}_\pi : G \to \mathbb{C}$ is called the **character** of π. More generally, a **class function** is a map $\varphi : G \to \mathbb{C}$ such that $\varphi(h^{-1}gh) = \varphi(g)$ for all $g, h \in G$, hence, a character is a special case of a class function. It then follows from the Peter-Weyl theorem that the set of characters of the irreducible representations of G form an orthonormal set with respect to the inner product $\langle\varphi, \psi\rangle = \frac{1}{|G|}\sum_{g\in G}\overline{\varphi(g)}\psi(g)$, i.e. we have for all irreducible representations π, π'

$$\frac{1}{|G|}\sum_{g}\overline{\mathrm{tr}_{\pi_1}(g)}\,\mathrm{tr}_{\pi_2}(g) = \delta_{\pi_1,\pi_2}. \tag{2.2.7}$$

Furthermore, the space of class function is spanned by the characters [Ser77, Sec. 2.5, Thm. 6].

If (π_1, V_{π_1}) and (π_2, V_{π_2}) are two representations of G, the traces of $V_{\pi_1} \oplus V_{\pi_2}$ and $V_{\pi_1} \otimes V_{\pi_2}$ are respectively $\chi_{\pi_1\oplus\pi_2}(g) = \chi_{\pi_1}(g) + \chi_{\pi_2}(g)$ and $\chi_{\pi_1\otimes\pi_2}(g) = \chi_{\pi_1}(g)\chi_{\pi_2}(g)$ for all $g \in G$. As an important consequence of Equation (2.2.7) allows us to calculate the explicit multiplicities in Equation (2.2.3): If χ is the character of a representation V of G decomposed as in Equation (2.2.3), then $\chi(g) = \sum_{\pi\in\hat{G}} n_\pi\chi_\pi(g)$ and Equation (2.2.7) gives $n_\pi = \langle\chi, \chi_\pi\rangle$.

Given a finite group G, we may define the vector space

$$\mathbb{C}G = \text{span}\{h \mid h \in G\} \tag{2.2.8}$$

as the formal span of all group elements in G. This becomes a G-module via left-multiplication $g \triangleright h = gh \in \mathbb{C}G$ on the basis vectors $h \in \mathbb{C}G$. This representation is called the **regular representation**, and it contains all irreducible representations of G as subrepresentations:

Theorem 2.2.6. *Let G be a finite group and let $(\rho, \mathbb{C}G)$ be the regular representation of G, with ρ defining the regular left-action, $\rho(g)(h) = gh$. Then every finite-dimensional irreducible representation V of G is contained in $\mathbb{C}G$ as a subrepresentation with multiplicity being the dimension of V and $\mathbb{C}G$ can be decomposed into*

$$\mathbb{C}G \cong \bigoplus_{\pi \in \hat{G}} \dim(V_\pi) V_\pi.$$

In particular, G is semisimple, that is, the number of non-isomorphic finite-dimensional irreducible representations of G is finite.

A proof of this statement can be found in most standard books on representation theory, see e.g. [Ser77, Sec 6.2 Prop. 10]. We want to provide an explicit decomposition into the irreducible representations, since we will use similar arguments later on in Chapter 3. Let (π, V_π) be an irreducible representation of G and let Γ_π be a unitary matrix representation of π for some basis $\{b_s\}_{s=1\ldots n}$, where n is the dimension of V_π. Fix a number $s_2 \in \{1, \ldots, n\}$ and define for every $s_1 \in \{1, \ldots, n\}$ the vector $c_{s_1,s_2} \in \mathbb{C}G$ via

$$c_{s_1,s_2} = \sum_{g \in G} \overline{\Gamma}_\pi^{s_1 s_2}(g) g \tag{2.2.9}$$

Because $\Gamma_\pi(g)$ is unitary for each $g \in G$, the set $\{c_{s_1,s_2}\}_{s_1}$ is linearly independent, and we claim that the space

$$V_{G,\pi} := \text{span}_{s_1}\{c_{s_1,s_2}\}$$

is isomorphic to V_π as a G-module, and that the G-equivariant isomorphism is given via the linear extension of the mapping

$$\theta_{s_2} : c_{s_1,s_2} \mapsto b_{s_1},$$

for all $s_1 = 1, \dots, n$. Indeed, we have

$$\begin{aligned}
\theta_{s_2}(\rho(h)(c_{s_1,s_2})) &= \theta_{s_2}\left(\rho(h)\left(\sum_{g\in G}\overline{\Gamma}_\pi^{s_1s_2}(g)g\right)\right) \\
&= \sum_{g\in G}\overline{\Gamma}_\pi^{s_1s_2}(g)\theta_{s_2}(hg) \\
&\overset{g\mapsto h^{-1}g}{=} \sum_{g\in G}\overline{\Gamma}_\pi^{s_1s_2}(h^{-1}g)\theta_{s_2}(g) \\
&= \sum_{l=1}^{n}\sum_{g\in G}\overline{\Gamma}_\pi^{s_1l}(h^{-1})\overline{\Gamma}_\pi^{ls_2}(g)\theta_{s_2}(g).
\end{aligned}$$

Because $\Gamma_\pi(h)$ is unitary, we have $\overline{\Gamma}_\pi^{s_1,l}(h) = \Gamma_\pi^{l,s_1}(h^{-1})$. Furthermore, we have $\sum_{g\in G}\overline{\Gamma}_\pi^{ls_2}(g)\theta_{s_2}(g) = \sum_{g\in G}\theta_{s_2}(c_{l,s_2}) = b_l$, and the above expression becomes

$$\sum_{l=1}^{n}\Gamma_\pi^{ls_1}(h)b_l.$$

But his is just the action $\pi(h)b_{s_1}$ expressed in the basis $\{b_1, \dots, b_n\}$. Since this holds for every s_2, we find precisely $\dim(V_\pi)$ many copies of the submodule V_π inside $\mathbb{C}G$.

Remark 2.2.7. In the literature, as well as in this work, the expression $\mathbb{C}G$ usually refers to the group algebra, whose definition is identical to the one in Equation (2.2.8) when viewed as a vector space and whose algebra multiplication is given by the group multiplication.

2.3 Hopf Algebras

In this section, we will give an overview of some results for finite dimensional Hopf algebras. We define bialgebras in Section 2.3.1 and Hopf algebras in Section 2.3.2. We will mostly follow [Swe69, Kas12] and [Gou93].

2.3.1 An Introduction to Bialgebras

We can view an algebra A over a field $\mathbb{F}$ as a triple (A, μ, η), where A is a vector space over $\mathbb{F}$ together with linear maps $\mu : A \otimes A \to A$ and $\eta : \mathbb{F} \to A$, called multiplication and unit such that the associativity diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id}_A \otimes \mu} & A \otimes A \\ {\scriptstyle \mu \otimes \mathrm{id}_A} \downarrow & & \downarrow {\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\quad \mu \quad} & A \end{array}$$

and unitor diagram

$$\begin{array}{ccccc} \mathbb{F} \otimes A & \xrightarrow{\eta \otimes \mathrm{id}_A} & A \otimes A & \xleftarrow{\mathrm{id}_A \otimes \eta} & A \otimes \mathbb{F} \\ & {\scriptstyle \cong} \searrow & \downarrow {\scriptstyle \mu} & \swarrow {\scriptstyle \cong} & \\ & & A & & \end{array} \tag{2.3.1}$$

commute. We shall embrace this diagrammatic description, as it allows for a convenient way to define dual constructions by *reversing arrows*.

Definition 2.3.1 (Coalgebra)**.** A coalgebra over a field $\mathbb{F}$ is a triple (C, Δ, ε), where C is a vector space together with linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbb{K}$ called comultiplication and counit such that the coassociativity diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad \Delta \quad} & C \otimes C \\ {\scriptstyle \Delta} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_C \otimes \Delta} \\ C \otimes C & \xrightarrow[\Delta \otimes \mathrm{id}_C]{} & C \otimes C \otimes C \end{array} \tag{2.3.2}$$

and counitor diagram

$$\begin{array}{ccccc} \mathbb{F} \otimes C & \xleftarrow{\varepsilon \otimes \mathrm{id}_C} & C \otimes C & \xrightarrow{\mathrm{id}_C \otimes \varepsilon} & C \otimes \mathbb{F} \\ & {\scriptstyle \cong} \nwarrow & \uparrow {\scriptstyle \Delta} & \nearrow {\scriptstyle \cong} & \\ & & C & & \end{array} \tag{2.3.3}$$

commute.

Every field $\mathbb{F}$ is both an algebra and a coalgebra, with coproduct and counit given by $\Delta_{\mathbb{F}} = \varepsilon_{\mathbb{F}} = \mathrm{id}_{\mathbb{F}}$, with the identity $\mathbb{F} \otimes \mathbb{F} \cong \mathbb{F}$ in mind.

We introduce the Sweedler notation: Given an element $x \in C$, where C is a coalgebra, we write

$$\Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)} \tag{2.3.4}$$

for the coproduct of x in C. The associativity diagram, Diagram (2.3.2), then reads in Sweedler notation

$$\sum_{(x)} \left(\left(x^{(1)}\right)^{(1)} \otimes \left(x^{(1)}\right)^{(2)} \right) \otimes x^{(2)} = \sum_{(x)} x^{(1)} \otimes \left(\left(x^{(2)}\right)^{(1)} \otimes \left(x^{(2)}\right)^{(2)} \right).$$

If V is a vector space over some field $\mathbb{F}$, we denote by $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ the set of linear functionals from V to $\mathbb{F}$. Given a linear map $V \to W$ between vector spaces V and W, we define the dual map $f^* : W^* \to V^*$ as usual via

$$f^* : W^* \ni \varphi \mapsto \varphi \circ f \in V^*.$$

If (C, Δ, ε) is a coalgebra, then C^* becomes an algebra with multiplication $\Delta^* : C^* \times C^* \to C^*$ and unit $\varepsilon^* : \mathbb{F} \to C^*$ [Swe69, Proposition 1.1.1]. Furthermore, if (A, μ, η) is a finite dimensional algebra, (A^*, μ^*, η^*) becomes a coalgebra [Swe69, Proposition 1.1.2].

Example 2.3.2. Let G be a finite group and $\mathbb{C}G$ the regular representation introduced in Equation (2.2.8). $\mathbb{C}G$ becomes an algebra with the structure maps defined via the linear extension of the maps

$$\begin{aligned} \mu(g \otimes h) &= gh \\ \eta(1_{\mathbb{C}}) &= e. \end{aligned}$$

Let us denote the dual of $\mathbb{C}G$ by $\mathbb{C}(G)$ Then $\mathbb{C}(G)$ becomes a coalgebra by setting

$$\begin{aligned} \Delta(\varphi) &= \varphi \circ \mu : g \otimes h \mapsto \varphi(gh) \\ \varepsilon(\varphi) &= \varphi \circ \eta : \mathbb{C} \to \mathbb{C} \end{aligned}$$

for all $\varphi \in \mathbb{C}(G)$. Note that $\varepsilon(\varphi) \in \mathbb{C}$ since $\mathrm{End}(\mathbb{C}) \cong \mathbb{C}$. Furthermore, $\mathbb{C}G$ can be made a coalgebra by setting

$$\begin{aligned} \Delta(g) &= g \otimes g \\ \varepsilon(g) &= 1, \end{aligned}$$

which induces on $\mathbb{C}(G)$ the algebra structure

$$\mu(\varphi \otimes \psi) = (\varphi \otimes \psi) \circ \Delta : g \mapsto \varphi(g)\psi(g) \tag{2.3.5}$$

$$\eta(1_{\mathbb{C}}) = 1_{\mathbb{C}} \circ \varepsilon : g \mapsto 1 \in \mathbb{C}, \tag{2.3.6}$$

where we identified $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$ in Equation (2.3.5) and $1_{\mathbb{C}} \in \mathbb{C} \cong (\mathbb{C})^*$ is viewed as the constant one function in Equation (2.3.6).

Definition 2.3.3 (Algebra and Coalgebra morphisms)**.** A linear map $f : A \to B$ between algebras (A, μ_A, η_A) and (B, μ_B, η_B) is called an **algebra homomorphism** or **morphism of algebras** if the diagrams

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_B} \\ A & \xrightarrow[f]{} & B \end{array} \tag{2.3.7}$$

and

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle \eta_A}\nwarrow \quad \nearrow{\scriptstyle \eta_B} & \\ & \mathbb{F} & \end{array} \tag{2.3.8}$$

commute. In other words, if $f(ab) = f(a)f(b)$ and $f(1_A) = 1_B$ for all $a \in A, b \in B$. A linear map $g : C \to D$ between coalgebras $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ is called a **coalgebra morphism** or **morphism of coalgebras** if the diagrams

$$\begin{array}{ccc} C \otimes C & \xrightarrow{g \otimes g} & D \otimes D \\ {\scriptstyle \Delta_C}\uparrow & & \uparrow{\scriptstyle \Delta_D} \\ C & \xrightarrow[g]{} & D \end{array} \tag{2.3.9}$$

and

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ & {\scriptstyle \varepsilon_C}\searrow \quad \swarrow{\scriptstyle \varepsilon_D} & \\ & \mathbb{F} & \end{array} \tag{2.3.10}$$

commute. In other words, if $\sum_{(x)} g(x^{(1)}) \otimes g(x^{(2)}) = \sum_{(g(x))} (g(x))^{(1)} \otimes (g(x))^{(2)}$ and $\varepsilon_C(x) = \varepsilon_D(g(x))$ for all $x \in C$.

If $g : (C_1, \Delta_{C_1}, \varepsilon_{C_1}) \to (C_2, \Delta_{C_2}, \varepsilon_{C_2})$ is a morphism of coalgebras, then $g^* : C_2^* \to C_1^*$ becomes a morphism of algebras between the algebras $(C_2^*, \Delta_{C_2}^*, \varepsilon_{C_2}^*)$ and $(C_1^*, \Delta_{C_1}^*, \varepsilon_{C_1}^*)$ [Swe69, Proposition 1.4.1]. Similarly, if $(A_1, \mu_{A_1}, \eta_{A_1})$ and $(A_2, \mu_{A_2}, \eta_{A_2})$ are two finite dimensional algebras with algebra morphism $f : A_1 \to A_2$, then $f^* : A_2^* \to A_1^*$ becomes a morphism of the coalgebras from $(A_2^*, \mu_{A_2}^*, \eta_{A_2}^*)$ to $(A_1^*, \mu_{A_1}^*, \eta_{A_1}^*)$ [Swe69, Proposition 1.4.2].

2.3.2 An Introduction to Hopf Algebras

Given two algebras (A, μ_A, η_A) and (B, μ_B, η_B), we may equip $A \otimes B$ with an algebra structure by setting $\mu_{A \otimes B} = \mu_A \otimes \mu_B \circ (\mathrm{id}_A \otimes \tau_{A \otimes B} \otimes \mathrm{id}_B)$ and

$\eta_{A\otimes B} = \eta_A \otimes \eta_B$ as multiplication and unit on $A \otimes B$ respectively, where $\tau_{A\otimes B}$ is the twist sending $a \otimes b$ to $b \otimes a$ for all $(a, b) \in A \times B$. Similarly, given two coalgebras $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$, we may equip the tensor product $C \otimes D$ with a coalgebra structure by setting $\Delta_{C\otimes D} = (\mathrm{id}_C \otimes \tau_{C\otimes D} \otimes \mathrm{id}_D) \circ (\Delta_C \otimes \Delta_D)$ and $\varepsilon_C \otimes \varepsilon_D$.

Definition 2.3.4 (Bialgebra)**.** A **bialgebra** is a quintuple $(A, \mu, \eta, \Delta, \varepsilon)$ such that (A, μ, η) forms an algebra, (A, Δ, ε) forms a coalgebra and the structure maps $\mu : A \otimes A \to A$ and $\eta : \mathbb{F} \to A$ are morphisms of coalgebras, i.e. the diagrams

$$\begin{array}{ccc} (A\otimes A)\otimes(A\otimes A) & \xrightarrow{\mu\otimes\mu} & A\otimes A \\ {\scriptstyle \Delta_{A\otimes A}}\uparrow & & \uparrow{\scriptstyle \Delta} \\ A\otimes A & \xrightarrow[\mu]{} & A \end{array} \tag{2.3.11}$$

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\varepsilon\otimes\varepsilon} & \mathbb{F}\otimes\mathbb{F} \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{F}}} \\ A & \xrightarrow[\varepsilon]{} & \mathbb{F} \end{array} \tag{2.3.12}$$

and

$$\begin{array}{ccc} \mathbb{F} & \xrightarrow{\eta} & A \\ {\scriptstyle \mathrm{id}_{\mathbb{F}}}\downarrow & & \downarrow{\scriptstyle \Delta} \\ \mathbb{F}\otimes\mathbb{F} & \xrightarrow[\eta\otimes\eta]{} & A\otimes A \end{array} \tag{2.3.13}$$

$$\begin{array}{ccc} \mathbb{F} & \xrightarrow{\eta} & A \\ & {\scriptstyle \mathrm{id}_{\mathbb{F}}}\searrow \quad \swarrow{\scriptstyle \varepsilon} & \\ & \mathbb{F} & \end{array} \tag{2.3.14}$$

commute. If $\tau : A \otimes A \to A \otimes A, a \otimes b \mapsto b \otimes a$ denotes the natural flip, then we call A commutative, respectively cocommutative, if in addition the diagram

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\tau} & A\otimes A \\ & {\scriptstyle \mu}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & A & \end{array} \tag{2.3.15}$$

respectively the diagram

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\tau} & A\otimes A \\ & {\scriptstyle \Delta}\nwarrow \quad \nearrow{\scriptstyle \Delta} & \\ & A & \end{array} \tag{2.3.16}$$

commute.

It turns out that Diagram (2.3.11) to Diagram (2.3.14) are equivalent to saying that $\Delta : A \otimes A \to A$ and $\varepsilon : A \to \mathbb{F}$ are morphisms of algebras.

Now, if $(A, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra, we may define the **convolution**

$$\star : \mathrm{End}(A) \otimes \mathrm{End}(A) \to \mathrm{End}(A) \tag{2.3.17}$$

via

$$f \star g = \mu(f \otimes g)\Delta$$

for algebra morphisms $f, g \in \mathrm{End}(A)$. This pairing admits an identity given by $\varepsilon\eta$. To see this, we first note that Diagrams (2.3.3) and Diagram (2.3.1) can be written using Sweedler notation as

$$\sum_{(x)} \varepsilon(x^{(1)}) \otimes x^{(2)} = \sum_{(x)} x^{(1)} \otimes \varepsilon(x^{(2)}) = x$$

and

$$\mu(\eta(z) \otimes x) = \eta(z)x = \mu(x \otimes \eta(z)).$$

Then we obtain

$$\begin{aligned}(f \star (\eta\varepsilon))\,(x) &= \mu(f \otimes (\eta\varepsilon))\Delta(x)\\ &= \sum_{(x)} f\left(x^{(1)}\eta(\varepsilon(x^{(2)}))\right)\\ &= \sum_{(x)} f\left(x^{(1)}\varepsilon(x^{(2)})\right)\\ &= f(x).\end{aligned}$$

In fact, the triple $(\mathrm{End}(A), \star, \eta\varepsilon)$ even forms an algebra [Kas12, Proposition III.3.1].

Definition 2.3.5 (Hopf Algebra)**.** A **Hopf algebra** is a sixtuple $(H, \mu, \eta, \Delta, \varepsilon, S)$ such that $(H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra and $S : H \to H$ a linear map, called **antipode**, such that $\mathrm{id}_H \star S = S \star \mathrm{id}_H = \eta\varepsilon$, i.e. S is the inverse of id_H with respect to $\star$.

Note that the very definition of an antipode gives the equation

$$\sum_{(x)} x^{(1)} S(x^{(2)}) = \sum_{(x)} S(x^{(1)}) x^{(2)} = \varepsilon(x) 1_H,$$

where 1_H is the unit on H.

If $(H, \mu, \eta, \Delta, \varepsilon, S)$ is a finite dimensional Hopf algebra with antipode S, then $(H^*, \Delta^*, \varepsilon^*, \mu^*, \eta^*, S^*)$ becomes a Hopf algebra with antipode S^* [Kas12, Proposition III.3.3].

The example of interest for us is that of the group Hopf algebra $\mathbb{C}G$. We noted the algebraic and coalgebraic structures of $\mathbb{C}G$ in Example 2.3.2. It is straightforward to verify that $\mathbb{C}G$ indeed becomes a bialgebra with these structure maps. In fact, it is a Hopf algebra with antipode given by the linear extension of the mapping

$$S : g \mapsto g^{-1}.$$

We call this Hopf algebra the **group Hopf algebra** of G. The structure maps of the group Hopf algebra encode the structure of the representation category of G in the following way: If $(\pi_1, V_{\pi_1}), (\pi_2, V_{\pi_2})$ are representations of G, then the coproduct Δ describes the action on the tensor product $V_{\pi_1} \otimes V_{\pi_2}$ via

$$\begin{aligned} g \triangleright (v_1 \otimes v_2) :=& (\pi_1 \otimes \pi_2)(\Delta(g))(v_1 \otimes v_2) \\ =& \sum_{(g)} \pi_1(g^{(1)}) \otimes \pi_2(g^{(2)})(v_1 \otimes v_2) \\ =& \sum_{(g)} \pi_1(g^{(1)})(v_1) \otimes \pi_2(g^{(2)})(v_2) \\ =& (\pi_1(g)(v_1)) \otimes (\pi_2(g)(v_2)), \end{aligned}$$

which indeed coincides with our definition of the tensor product action given in Equation (2.2.2). Furthermore, the counit encodes the trivial representation on triv $\cong \mathbb{C}$. Finally, the antipode encodes the dual action: If (π, V_π) is a G-module, we may define the dual action on $V_\pi^* = \mathrm{Hom}(V, \mathbb{C})$ via

$$(g \triangleright \varphi)(v) = \varphi(\pi(S(g))(v)) = \varphi(\pi(g^{-1})(v)), \tag{2.3.18}$$

which again coincides with Equation (2.2.1). Note also that Equation (2.3.18) would not define a left action, were it not for

$$S(gh) = (gh)^{-1} = h^{-1}g^{-1} = S(h)S(g).$$

and

$$S(e) = e^{-1} = e.$$

Indeed, we have the following

Theorem 2.3.6. *Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra. Then*

$$S(ab) = S(b)S(a), \qquad S(1_H) = 1_H$$

and

$$(S \otimes S)\Delta = \tau_{H\otimes H}\Delta S, \qquad \varepsilon S = \varepsilon$$

for all $a, b \in H$ and if H is either commutative or cocommutative, then S is invertible with $S^2 = \mathrm{id}_H$.

See [Kas12, Theorem III.3.4] for a proof.

We want to conclude this section with an important property about Hopf algebras. An element x of a Hopf algebra H is called a **left integral** if

$$xa = \varepsilon(x)a$$

holds for all $a \in H$. We denote the set of left integrals by $\int_H$. The next theorem can be found in [Swe69, 5.1.8].

Theorem 2.3.7. *A finite dimensional Hopf algebra H is semisimple if and only if*

$$\varepsilon \mid_{\int_H} \neq 0$$

i.e., if there exist an element $x \in \int_H$ such that $\varepsilon(x) \neq 0$.

2.4 Crossed Products and the Quantum Double Construction

In this section, we will define the quantum double of a group and discuss their irreducible representations. In Section 2.4.1, we consider pairs of Hopf algebras A, B that satisfy certain compatibility conditions and define a universal object $A \bowtie B$ containing A and B as subalgebras. We then show that a particular case is given by a Hopf algebra H and its dual, and the constructed universal object $D(H) := H \bowtie H^*$ is called the *Quantum Double* of H. We then apply this construction to the group Hopf algebra $\mathbb{C}G$. All results can be found in more detail in [Kas12, Swe69]. In Section 2.4.2 we inspect the irreducible representations of the $D(G)$ following mostly [Gou93, DPR91].

2.4.1 The Quantum Double Construction

We now come to a very important construction, the bicrossed product of algebras and Hopf algebras. Let A and B be two algebras and assume $\mathcal{V}$ is both a left A-module and a right B module. Assume further that for all $a \in A$, $b \in B$ there exist elements denoted by $L_b(a) \in A$ and $R_a(b) \in B$, such that the commutation relation

$$ba = L_b(a)R_a(b) \tag{2.4.1}$$

holds, where the composition in Equation (2.4.1) is viewed as composition of elements in $\mathrm{End}(V)$. Finally, assume that $L : B \to \mathrm{End}(A), b \mapsto L_b$ is a left action of B on A and that $R : A \to \mathrm{End}(B), a \mapsto R_a$ a right action of A on B. Then a **bicrossed product** with respect to L and R is an algebra denoted by $A \bowtie B$ containing A and B as subalgebras such that Equation (2.4.1) holds for all $a \in A$ and $b \in B$, now viewed as elements of $A \bowtie B$, and such that $A \bowtie B \cong A \otimes B$ as vector spaces.

Example 2.4.1. Let $H, K \subset G$ be subgroups of a group G such that $H \cdot K = G$ and each element $g \in G$ factors uniquely into a product $g = hk$ with $h \in H$ and $k \in K$. This would for instance be given once $H \cap K = \{e\}$ since then for all $h, h' \in H$ and $k, k' \in K$ with $hk = h'k'$, we have

$$hk = h'k' \Leftrightarrow (h')^{-1}h = k^{-1}k' \in H \cap K,$$

implying $h = h'$ and $k = k'$. Then for each $h \in H$ and $k \in K$ there exist elements $L_k(h) \in H$ and $R_h(k) \in K$ such that

$$kh = L_k(h)R_h(k)$$

It is straightforward to verify that L and R satisfy the following identities for all $h, h' \in H$ and $k, k' \in K$:

$$\begin{aligned} L_{kk'} &= L_k L_{k'} \\ L_k(hh') &= L_k(h)L_{R_h(k)}(h') \\ R_{hh'} &= R_{h'}R_h \\ R_h(kk') &= R_{L_{k'}(h)}(k)R_h(k') \\ L_e(h) &= h \\ L_k(e) &= e \\ R_e(k) &= k \\ R_h(e) &= e. \end{aligned}$$

See the discussion at the beginning of Section IX.1 in [Kas12] for more details.

Definition 2.4.2. Let A and B be two algebras over some field $\mathbb{F}$ and $L : B \otimes A \to A$ and $R : B \otimes A \to B$ a left- respectively right action. Assume further that L and R are morphisms of comodules and that the diagrams

$$\begin{array}{ccc} B\otimes A\otimes A & \xrightarrow{(L\otimes R\otimes \mathrm{id}_A)(\Delta_{B\otimes A}\otimes \mathrm{id}_A)} & A\otimes B\otimes A \\ \Big\downarrow{\scriptstyle \mathrm{id}_B\otimes \mu_A} & & \Big\downarrow{\scriptstyle \mu_A(\mathrm{id}_A\otimes L)} \\ B\otimes A & \xrightarrow[L]{} & A \end{array} \tag{2.4.2}$$

$$\begin{array}{ccc} B\otimes B\otimes A & \xrightarrow{(\mathrm{id}_B\otimes L\otimes R)(\mathrm{id}_B\otimes \Delta_{B\otimes A})} & B\otimes A\otimes B \\ \Big\downarrow{\scriptstyle \mu_B\otimes \mathrm{id}_A} & & \Big\downarrow{\scriptstyle \mu_B(R\otimes \mathrm{id}_B)} \\ B\otimes A & \xrightarrow[R]{} & B \end{array} \tag{2.4.3}$$

$$\begin{array}{ccc} B\otimes \mathbb{F} & \xrightarrow{\mathrm{id}_B\otimes \eta_A} & B\otimes A \\ & \searrow{\scriptstyle \varepsilon_B\eta_A} & \Big\downarrow{\scriptstyle L} \\ & & A \end{array} \tag{2.4.4}$$

$$\begin{array}{ccc} \mathbb{F}\otimes A & \xrightarrow{\eta_B\otimes \mathrm{id}_A} & B\otimes A \\ & \searrow{\scriptstyle \eta_B\varepsilon_A} & \Big\downarrow{\scriptstyle R} \\ & & B \end{array} \tag{2.4.5}$$

and

$$\begin{array}{ccccc} B\otimes A & \xrightarrow{\Delta_{B\otimes A}} & B\otimes A\otimes B\otimes A & \xrightarrow{R\otimes L} & B\otimes A \\ & & \Big\downarrow{\scriptstyle L\otimes R} & \nearrow{\scriptstyle \tau_{B\otimes A}} & \\ & & A\otimes B & & \end{array} \tag{2.4.6}$$

commute. Then A and B are called a **matched pair of algebras** w.r.t L and R.

We have purposely chosen the same notation for the left- resp. right action in Definition 2.4.2 as in Example 2.4.1 to emphasize that Definition 2.4.2 offers a generalization of Example 2.4.1. Indeed, the group algebras of H and K from Example 2.4.1 form a matched pair of algebras.

Note that we use that $\mathbb{F}\otimes A\cong A$ and $B\otimes\mathbb{F}\cong B$ in Diagram (2.4.5) and Diagram (2.4.4). In components, Diagrams (2.4.2) to Diagram (2.4.6) read

$$L_b(a_1a_2) = \sum_{(b),(a_1)} L_{b^{(1)}}(a_1^{(1)})L_{R_{a_1^{(2)}}(b^{(2)}_1)}(a_2) \tag{2.4.7}$$

$$R_a(b_1b_2) = \sum_{(a),(b_2)} R_{L_{b_2^{(1)}}(a^{(1)})}(b_1)R_{a^{(2)}}(b_2^{(2)}) \tag{2.4.8}$$

$$R_a(1_B) = \varepsilon_A(a)1_B \tag{2.4.9}$$

$$\sum_{(b),(a)} R_{a^{(1)}}(b^{(1)})\otimes L_{b^{(2)}}(a^{(2)}) = \sum_{(b),(a)} R_{a^{(2)}}(b^{(2)})\otimes L_{b^{(1)}}(a^{(1)}). \tag{2.4.10}$$

Note in particular that Equation (2.4.10) is a weaker form of cocommutativity.

By [Kas12, Theorem IX.2.3], if A and B are matched, then there exists a unique bialgebra structure on $C =: A\otimes B$ with unit $1_A\otimes 1_B$ and product, coproduct and counit given via

$$(a_1\otimes b_1)(a_2\otimes b_2) = \sum_{(a),(b)} a_1L_{b_1^{(1)}}(a_2^{(1)})\otimes R_{a_2^{(2)}}(b_1^{(2)})b_2, \tag{2.4.11}$$

coproduct

$$\Delta(a\otimes b) = \sum_{(a),(b)} (a^{(1)}\otimes b^{(1)})\otimes(a^{(2)}\otimes b^{(2)})$$

and counit

$$\varepsilon(a\otimes b) = \varepsilon_A(a)\varepsilon_B(b),$$

and the embeddings $A\hookrightarrow C, a\mapsto a\otimes 1_B$ and $B\hookrightarrow C, b\mapsto 1_A\otimes b$ are morphisms of bialgebras. This algebra is called the **bicrossed product** of A and B with respect to L and R and is denoted by $A\bowtie B$. If A and B are in addition Hopf algebras with antipodes S_A and S_B, then C becomes a Hopf algebra as well with antipode given by

$$S(a\otimes b) = \sum_{(a),(b)} L_{S_B(b^{(2)})}(S_A(a^{(2)}))\otimes R_{S_A(a^{(1)})}\left(S_B(b^{(1)})\right). \tag{2.4.12}$$

Note also that Equation (2.4.11) describes the commutation relation in Equation (2.4.1) for the *group like* elements, that is, elements $a\in A$ and $b\in B$ with $\Delta_A(a) = a\otimes a$ and $\Delta_B(b) = b\otimes b$.

Remark 2.4.3. For group like elements, Diagrams (2.4.2), (2.4.3), (2.4.3) and (2.4.4) take the form of Equation (2.4.7), (2.4.8), (2.4.8) and (2.4.9) of Example 2.4.1 respectively, barring the tensor symbol.

We already saw that if $(H, \mu, \eta, \Delta, \varepsilon, S)$ is a finite dimensional Hopf algebra, the dual $(H^*, \Delta^*, \varepsilon^*, \mu^*, \eta^*)$ becomes a bialgebra again. Because

$$\eta_H \varepsilon_H = \mu_H (S \otimes \mathrm{id}_H) \Delta_H$$

immediately gives

$$\eta_{H^*} \varepsilon_{H^*} = \varepsilon_H^* \eta_H^* = \Delta_H^* (S \otimes \mathrm{id}_H)^* \mu_H^* = \mu_{H^*} (S^* \otimes \mathrm{id}_H^*) \Delta_H^*,$$

we see that S^* is an antipode of H^*, making $(H^*, \Delta^*, \varepsilon^*, \mu^*, \eta^*, S^*)$ a Hopf algebra again.

We may also consider the *opposite* bialgebra $(H^{op}, \mu_H^{op}, \eta_H, \Delta_H, \varepsilon_H)$ of H, i.e. the Hopf algebra H with multiplication $\mu_H^{op} = \mu_H \tau_{H \otimes H} : a \otimes b \mapsto \mu_H(b \otimes a)$. Straightforward calculations then show that if S is invertible, S^{-1} is an antipode of H^{op} and $(H^{op}, \mu_H^{op}, \eta_H, \Delta_H, \varepsilon_H, S^{-1})$ becomes a Hopf algebra again, and so does its dual $(H^{op})^*$ by our previous observations. It turns out that $(H^{op})^*$ and H can be made into a matched pair by defining the left- and right actions

$$L : H \otimes (H^{op})^* \to (H^{op})^*, \tag{2.4.13}$$

$$a \otimes \varphi \mapsto \left(L_a(\varphi) : x \mapsto \sum_{(a)} \varphi \left(S^{-1}(a^{(2)}) x a^{(1)} \right) \right) \tag{2.4.14}$$

$$R : H \otimes (H^{op})^* \to H, \tag{2.4.15}$$

$$a \otimes \varphi \mapsto R_\varphi(a) = \sum_{(a)} \varphi \left(S^{-1}(a^{(3)}) a^{(1)} \right) a^{(2)}. \tag{2.4.16}$$

The proof is straightforward and can be found after [Kas12, Theorem IX.3.5]. The bicrossed product $(H^{op})^* \bowtie H$ is called the **quantum double** of H and is denoted by $D(H)$. The structure maps are explicitly given as

$$1_{D(H)} = \mathrm{id}_H \otimes 1_H \tag{2.4.17}$$

$$\mu_{D(H)}((\varphi \otimes a) \otimes (\psi \otimes b)) = \sum_{(a)} \varphi\psi \left(S^{-1}(a^{(3)})(-)a^{(1)} \right) \otimes a^{(2)} b$$

$$\varepsilon_{D(H)}(\varphi \otimes a) = \varepsilon_H(a) \varphi(1_H) \tag{2.4.18}$$

$$\Delta(\varphi \otimes a) = \sum_{(a)(\varphi)} (\varphi^{(1)} \otimes a^{(1)}) \otimes (\varphi^{(2)} \otimes a^{(2)}). \tag{2.4.19}$$

As it turns out [Kas12, Proposition IX.4.3] if H is cocommutative, then the right action of $(H^{op})^*$ on (H) becomes trivial, i.e.

$$R_\varphi(a) = \varepsilon(\varphi)(a).$$

In this case, the bicrossed product is called a *semi-direct* product. Finally, if $H = \mathbb{C}G$ and $(H^{op})^* = (H)^* = (\mathbb{C}G)^*$ is the group Hopf algebra and its dual with structure maps given as in Example 2.3.2, then the quantum double of $D(H) = \mathbb{C}G \bowtie \mathbb{C}(G)$ is simply denoted by $D(G)$ and called the **quantum double of G**.

Proposition 2.4.4. *Let $D(G)$ be the quantum double of a finite group G. Then the left action $L : \mathbb{C}G \otimes \mathbb{C}(G) \to \mathbb{C}G$, right action $R : \mathbb{C}G \otimes \mathbb{C}(G) \to \mathbb{C}(G)$ and the structure maps of $D(G)$ take the explicit form*

$$L_h(\delta_g) = \delta_{hgh^{-1}} \tag{2.4.20}$$
$$R_{\delta_g}(h) = \delta_{g,e} h \tag{2.4.21}$$
$$1_{D(G)} = \sum_{g\in G} \delta_g \otimes e \tag{2.4.22}$$
$$(\delta_{g_1} \otimes h_1)(\delta_{g_2} \otimes h_2) = \delta_{g_1, h_1 g_2 h_1^{-1}} \delta_{g_1} \otimes h_1 h_2 \tag{2.4.23}$$
$$\varepsilon(\delta_g \otimes h) = \delta_{g,e} 1_{D(G)} \tag{2.4.24}$$
$$\Delta(\delta_g \otimes h) = \sum_{g_2 g_1 = g} (\delta_{g_1} \otimes h) \otimes (\delta_{g_2} \otimes h) \tag{2.4.25}$$
$$S(\delta_g \otimes h) = \delta_{h^{-1} g^{-1} h} \otimes h^{-1} \tag{2.4.26}$$

Where $\delta_g : G \to \mathbb{C}, h \mapsto \delta_{g,h}$ is the delta function and $\delta_{g,h}$ is the Kronecker delta.

Proof. Equation (2.4.20), (2.4.21), (2.4.22) and Equation (2.4.24) follow directly from Equation (2.4.15), (2.4.16), (2.4.17) and (2.4.18) respectively. To show Equation (2.4.25), note first that the coproduct Δ' on $\mathbb{C}(G)$ is given by the adjoint of the opposite multiplication μ^{op}. Then for $\delta_g \in \mathbb{C}(G)$ we obtain

$$\Delta'(\delta_g)(x \otimes y) = \delta_{g,yx} = \sum_{k_2 k_1 = g} \delta_{k_1}(x)\delta_{k_2}(y),$$

hence

$$\Delta'(\delta_g) = \sum_{k_2 k_1} \delta_{k_1} \otimes \delta_{k_2}$$

giving Equation (2.4.25) by using Equation (2.4.19). Furthermore, we have

$$\begin{aligned}(\delta_{g_1}\otimes h_1)(\delta_{g_2}\otimes h_2) &\overset{(2.4.11)}{=\!=} \sum_{k_2k_1=g_2}\delta_{g_1}L_{h_1}(\delta_{k_1})\otimes R_{\delta_{k_2}}(h_1)h_2\\ &=\sum_{k_2k_1=g_2}\delta_{g_1}\delta_{h_1k_1h_1^{-1}}\otimes\delta_{k_2,e}h_1h_2\\ &=\delta_{g_1,h_1g_2h_1^{-1}}\delta_{g_1}\otimes h_1h_2\end{aligned}$$

giving Equation (2.4.23). To see Equation (2.4.26) note that $S_{\mathbb{C}G}^{-1}=S_{\mathbb{C}G}: g\mapsto g^{-1}$ which gives $S_{\mathbb{C}(G)}(\delta_g)=\delta_g S^{-1}=\delta_{g^{-1}}$. Then Equation (2.4.12) gives

$$\begin{aligned}S(\delta_g\otimes h) &= \sum_{k_2k_1=g}L_{h^{-1}}(\delta_{k_2^{-1}})\otimes R_{\delta_{k_1^{-1}}}(h^{-1})\\ &=\sum_{k_2k_1=g}\delta_{h^{-1}k_2^{-1}h}\otimes\delta_{k_1^{-1},e}h^{-1}\\ &=\delta_{h^{-1}g^{-1}h}\otimes h^{-1}\end{aligned}$$

□

Remark 2.4.5. By viewing $\mathbb{C}G$ and $\mathbb{C}(G)$ as subalgebras of $D(G)$ via the inclusion $\delta_g\mapsto\delta_g\otimes e$ and $h\mapsto \mathrm{id}\otimes h$, (2.4.23) gives the commutation relation

$$h\delta_g=(\mathrm{id}_G\otimes h)(\delta_g\otimes e)=\delta_{hgh^{-1}}\otimes h=\delta_{hgh^{-1}}h.$$

Hence, within the quantum double $D(G)$, the elements in $\mathbb{C}G$ and $\mathbb{C}(G)$ are subject to the commutation relation

$$h\delta_g=\delta_{hgh^{-1}}h$$

and any representation of $D(G)$ will respect this identity. This observation will become important in Chapter 3.

The quantum double $D(G)$ admits the integral element

$$x=\sum_{h\in G}\delta_e\otimes h.$$

Indeed, applying Equation (2.4.23) we get for any element $\delta_{g_0}\otimes h_0\in D(G)$

$$(\delta_{g_0}\otimes h_0)x=\sum_{h\in G}\delta_{g_0,e}\delta_e\otimes h_0h=\delta_{g_0,e}\sum_{h\in G}\delta_e\otimes h=\delta_{g_0,e}x=\varepsilon(\delta_{g_0}\otimes h_0)x.$$

Therefore, $D(G)$ is semisimple by Theorem 2.3.7.

Remark 2.4.6. The bicrossed product for finite groups can alternatively be defined by a universal property on the embedding maps $i_A: A\to A\otimes B, a\mapsto a\otimes 1_B$ and $i_B: B\to A\otimes B, b\mapsto 1_A\otimes b$. See [ACIM07] for more details.

2.4.2 The irreducible representations of the quantum double

Let G be a finite group. In this section, we want to study the representations of its quantum double $D(G)$ following [Gou93] and [DPR91]. We start by explicitly constructing the irreducible representations of $D(G)$. To ease readability, we will denote the inverse g^{-1} of a group element $g \in G$ by $\bar{g}$ from now on. Let G_C denote the set of all conjugacy classes of G. For each $\mathcal{C} \in G_C$, we fix an element $r_\mathcal{C} \in \mathcal{C}$ and set $N_\mathcal{C} = \{n \in G \mid n r_\mathcal{C} \bar{n} = r_\mathcal{C}\}$ to be the centralizer subgroup of $r_\mathcal{C}$ in G. We also fix a set of representatives $Q_\mathcal{C} = \{q_c \mid c \in \mathcal{C}\}$ of $G/N_\mathcal{C}$ labelled such that

$$c = q_c r_\mathcal{C} \bar{q}_c.$$

Because $G = \dot{\bigcup\limits_{c\in\mathcal{C}}} q_c N_\mathcal{C}$, every element $g \in G$ can be factorized uniquely as

$$g = q_{g r_\mathcal{C} \bar{g}} n_g$$

with $q_{g r_\mathcal{C} \bar{g}} \in Q_\mathcal{C}$ and $n_g \in N_\mathcal{C}$, giving rise to maps $q : G \to Q_\mathcal{C}$ and $n : G \to N_\mathcal{C}$ for each fixed choice of $r_\mathcal{C} \in G$. We will use this observation quite frequently.

To ease readability, we will denote the inverse of a group element $g \in G$ by $\bar{g}$ instead of $\bar{g}$. Note that different choices of $r_\mathcal{C}$ merely lead to isomorphic centralizer subgroups.

Theorem 2.4.7. *Let G be a finite group, $\mathcal{C}$ a conjugacy class of G and $N_\mathcal{C}$ the centralizer subgroup of a fixed element $r_\mathcal{C} \in \mathcal{C}$. If (π, V_π) is an irreducible representation of $N_\mathcal{C}$, then the vector space*

$$\mathcal{V}^\alpha = \mathbb{C}\mathcal{C} \otimes V_\pi = \mathrm{span}_\mathbb{C} \{c \otimes v \mid c \in \mathcal{C}, v \in V_\pi\}$$

with label $\alpha = (\pi, \mathcal{C})$ becomes an irreducible representation of $D(G)$ with action given by

$$(\delta_g \otimes h) \triangleright (c \otimes v) = \delta_{g,hc\bar{h}} hc\bar{h} \otimes \pi(\bar{q}_{hc\bar{h}} h q_c)(v).$$

Proof. We will show that $D(G)u = \mathcal{V}^\alpha$ for any $u \in \mathcal{V}^\alpha$ non-zero, i.e., there is no non-trivial $D(G)$-invariant subspace of $\mathcal{V}^\alpha$. We will show this by demonstrating that u can be mapped to $c \otimes v$ for any choice of $(c, v) \in \mathcal{C} \times V_\pi$. It then follows that $c \otimes v$ can be mapped to any of the $c' \otimes v'$ for $(c', v') \in \mathcal{C} \times V_\pi$ as well, hence $D(G)u$ spans $\mathcal{V}^\alpha$.

First, observe that if $c_1, c_2, c_3 \in \mathcal{C}$ and $n \in N_\mathcal{C}$ are such that $q_{c_1} \bar{n} \bar{q}_{c_2} c_3 q_{c_2} n \bar{q}_{c_1} = c_1$, it follows that $c_2 = c_3$. This is because the mapping

$c \mapsto qc\bar{q}$ defines for each fixed $q \in G$ an automorphism on G, and because we already have $q_{c_1}\bar{n}\bar{q}_{c_2}c_2q_{c_2}n\bar{q}_{c_1} = q_{c_1}\bar{n}r_{\mathcal{C}}n\bar{q}_{c_1} = q_c r_{\mathcal{C}}\bar{q}_{c_1} = c_1$.

Now, let $0 \neq u = \sum_{k,l} \lambda_{kl} c_k \otimes v_l \in V_\alpha$ and $c \in \mathcal{C}$, $v \in V_\pi$ be fixed and let k_0 be such that not all $\lambda_{k_0,l}$ are zero. Because (π, V_π) is irreducible, there exists an element $n \in N_{\mathcal{C}}$ such that

$$\pi(n)\left(\sum_l \lambda_{k_0,l} v_l\right) = v$$

Then $a = \delta_c \otimes q_c n \bar{q}_{c_{k_0}}$ maps u to $c \otimes v$:

$$\begin{aligned}
(\delta_c \otimes q_c n \bar{q}_{c_{k_0}}) \triangleright u =& \sum_{k,l} \lambda_{k,l} \delta_{c, q_c n \bar{q}_{c_{k_0}} c_k q_{c_{k_0}} \bar{n} \bar{q}_c} c \otimes \pi(\bar{q}_{q_c n \bar{q}_{c_{k_0}} c_k q_{c_{k_0}} \bar{n} \bar{q}_c} q_c n \bar{q}_{c_{k_0}} q_{c_k})(v) \\
=& \sum_l \lambda_{k_0,l} c \otimes \pi(\bar{q}_c q_c n \bar{q}_{c_{k_0}} q_{c_{k_0}})(v) \\
=& \sum_l \lambda_{k_0,l} c \otimes \pi(n)(v) \\
=& c \otimes v.
\end{aligned}$$

□

We will often denote by $d_\alpha = \dim(\alpha)$ the dimension of an irreducible representation $\alpha \in \widehat{D(G)}$.

$D(G)$ can be made a *-algebra by defining the involution

$$(\delta_g \otimes h)^* = \delta_{\bar{h}gh} \otimes \bar{h}. \tag{2.4.27}$$

Definition 2.4.8 (Hopf *-algebra)**.** A **star involution on a Hopf algebra** H is an antilinear, involutive map $^* : H \to H$ such that

$$(^* \otimes ^*)(\Delta(a)) = \Delta(a^*), \tag{2.4.28}$$
$$(ab)^* = b^* a^*, \tag{2.4.29}$$
$$S(S(a)^*)^* = a, \tag{2.4.30}$$

for all $a, b \in H$. A Hopf algebra H together with a star involution is called a **Hopf *-algebra**.

Equation (2.4.28) reads in Sweedler notation:

$$\sum_{(a)} \left(a^{(1)}\right)^* \otimes \left(a^{(2)}\right)^* = \sum_{(a^*)} (a^*)^{(1)} \otimes (a^*)^{(2)}.$$

Proposition 2.4.9. *With the * involution defined in Equation* (2.4.27)*, $D(G)$ becomes a Hopf *-algebra.*

Proof. We have

$$(S(\delta_g \otimes h))^* = \left(\delta_{\bar{h}\bar{g}h} \otimes \bar{h}\right)^* = \delta_{h\bar{h}\bar{g}h\bar{h}} \otimes h = \delta_{\bar{g}} \otimes h,$$

and Equation (2.4.30) follows by applying $^*S$ a second time. For the other claims, we have

$$\begin{aligned}((\delta_{g_1} \otimes h_1)(\delta_{g_2} \otimes h_2))^* \overset{(2.4.23)}{=\!=\!=} &\delta_{g_1,h_1g_2\bar{h}_1}\left(\delta_{g_1} \otimes h_1h_2\right)^* \\ =&\delta_{g_1,h_1g_2\bar{h}_1}\delta_{\bar{h}_2\bar{h}_1g_1h_1h_2} \otimes \bar{h}_2\bar{h}_1 \\ =&\delta_{\bar{h}_2\bar{h}_1g_1h_1h_2,\bar{h}_2g_2h_2}\delta_{\bar{h}_2g_2h_2} \otimes \bar{h}_2\bar{h}_1 \\ =&(\delta_{\bar{h}_2g_2h_2} \otimes \bar{h}_2)(\delta_{\bar{h}_1g_1h_1} \otimes \bar{h}_1) \\ =&\left(\delta_{g_2} \otimes h_2\right)^*\left(\delta_{g_1} \otimes h_1\right)^*\end{aligned}$$

and

$$\begin{aligned}\Delta\left((\delta_g \otimes h)^*\right) =&\Delta(\delta_{\bar{h}gh} \otimes \bar{h}) \\ =&\sum_{g_1g_2=\bar{h}gh} (\delta_{g_2} \otimes \bar{h})(\delta_{g_1} \otimes \bar{h}) \\ =&\sum_{g_1g_2=g} (\delta_{\bar{h}g_2h} \otimes \bar{h})(\delta_{\bar{h}g_1h} \otimes \bar{h}) \\ =&\sum_{g_1g_2=g} (\delta_{g_2} \otimes h)^*(\delta_{g_1} \otimes h)^* \\ =&\sum_{g_1g_2=g} \left((\delta_{g_2} \otimes h)(\delta_{g_1} \otimes h)\right)^* \\ =&\left(\Delta(\delta_g \otimes h)\right)^*.\end{aligned}$$

By linear extension, (2.4.28) and Equation (2.4.29) hold for all $a, b \in D(G)$. □

Similar to the dual representation introduced in Section 2.2, we can define the dual representation of a Hopf-algebra representation.

Proposition 2.4.10 (Contragredient Representation)**.** *Let H be a Hopf algebra and V_π an H-module with representation given by $\pi : H \to \mathrm{End}(V_\pi)$. Then the maps $\pi^* : (H \otimes V_\pi)^* \to (V_\pi)^*$ and $\bar{\pi} : H \otimes V_\pi \to V_\pi$ defined via*

$$\begin{aligned}\pi^*(a) : (V_\pi)^* \ni \varphi \mapsto&\varphi\pi(S(a)), \\ \bar{\pi}(a) : V_\pi \ni v \mapsto&(\pi(a)^*)^t(v)\end{aligned}$$

*where $(-)^t$ denotes the transpose, define representations of H on $(V_\pi)^*$ and V_π, called the **contragredient representation** and **conjugated representation** respectively.*

It is straightforward to prove Proposition 2.4.10. Using similar arguments as in Section 2.2, we may assume that every representation of a Hopf algebra is unitary, that is, if $\Gamma_\pi^{IJ}(a)$ are the matrix coefficients of a representation $\pi \in \hat{H}$ with $I, J = 1, \ldots, d_\alpha$, then

$$\Gamma_\pi^{IJ}(a^*) = \bar{\Gamma}_\pi^{JI}(a),$$

with the *-involution defined as in Equation (2.4.27). The matrix coefficients of the conjugated and contragredient representation are respectively

$$\Gamma_{\bar{\pi}}^{IJ}(a) = \bar{\Gamma}_\pi^{IJ}(a) \tag{2.4.31}$$

and

$$\Gamma_{\pi^*}^{IJ}(a) = \Gamma_\pi^{JI}(S(a)). \tag{2.4.32}$$

Let $\alpha \in \widehat{D(G)}$ be an irreducible representation of $D(G)$ of the form $(\pi_\alpha, \mathcal{C}_\alpha)$, with π_α unitary and let $\{b_i\}_{i=1,\ldots,\dim_{\pi_\alpha}}$ be an orthonormal basis of the irreducible representation V_{π_α} associated to π_α. We write

$$I_\alpha = \{(i_1, i_2) \mid i_1 = 1, \ldots, |\mathcal{C}_\alpha|, i_2 = 1, \ldots, \dim_{\pi_\alpha}\}$$

to denote the labels of the basis $\{c_{i_1} \otimes b_{i_2} \mid (i_1, i_2) \in 1_\alpha\}$ of $\mathcal{V}^\alpha$, where $\{c_1, \ldots, c_{|\mathcal{C}_\alpha|}\} = \mathcal{C}_\alpha$. With the *-involution given in Equation (2.4.27), Γ_α becomes unitary with inner product given by

$$\langle c_{i_1} \otimes b_{i_2}, c_{j_1} \otimes b_{j_2} \rangle = \delta_{i_1,i_2}\delta_{i_2,j_2}$$

on $\mathbb{C}\mathcal{C}_\alpha \otimes V_{\pi_\alpha}$. Indeed, for $I, J \in 1_\alpha$ with $I = (i_1, i_2), J = (j_1, j_2)$, the matrix coefficients of $\Gamma_\alpha^{IJ}(\delta_g \otimes h)$ are given by

$$\begin{aligned}
\Gamma_\alpha^{IJ}(\delta_g \otimes h) &= \langle c_{i_1} \otimes b_{i_2}, (\delta_g \otimes h) \triangleright (c_{j_1} \otimes b_{j_2}) \rangle \\
&= \delta_{g,hc_{j_1}\bar{h}} \left\langle c_{i_1} \otimes b_{i_2}, (hc_{j_1}\bar{h}) \otimes \pi_\alpha(\bar{q}_{hc_{j_1}\bar{h}} h q_{c_{j_1}}) b_{j_2} \right\rangle \\
&= \delta_{g,c_{i_1}} \left\langle c_{i_1}, (hc_{j_1}\bar{h}) \right\rangle \left\langle b_{i_2}, \pi_\alpha(\bar{q}_{hc_{j_1}\bar{h}} h q_{c_{j_1}}) b_{j_2} \right\rangle \\
&= \delta_{g,hc_{j_1}\bar{h}} \delta_{c_{i_1},hc_{j_1}\bar{h}} \Gamma_{\pi_\alpha}^{i_2 j_2}(\bar{q}_{hc_{j_1}\bar{h}} h q_{c_{j_1}}).
\end{aligned} \tag{2.4.33}$$

Substituting $(\delta_g \otimes h)^* = \delta_{\bar{h}gh} \otimes \bar{h}$ gives

$$\begin{aligned}\Gamma_\alpha^{IJ}(\delta_{\bar{h}gh} \otimes \bar{h}) =& \delta_{\bar{h}gh,\bar{h}c_{j_1}h}\delta_{c_{i_1},\bar{h}c_{j_1}h}\Gamma_{\pi_\alpha}^{i_2j_2}(\bar{q}_{\bar{h}c_{j_1}h}\bar{h}q_{c_{j_1}})\\ =& \delta_{\bar{h}gh,\bar{h}c_{j_1}h}\delta_{c_{i_1},\bar{h}c_{j_1}h}\bar{\Gamma}_{\pi_\alpha}^{j_2i_2}(\bar{q}_{c_{j_1}}hq_{\bar{h}c_{j_1}h})\\ =& \delta_{g,hc_{i_1}\bar{h}}\delta_{c_{j_1},hc_{i_1}\bar{h}}\bar{\Gamma}_{\pi_\alpha}^{j_2i_2}(\bar{q}_{hc_{i_1}\bar{h}}hq_{c_{i_1}})\\ =& \bar{\Gamma}_\alpha^{JI}(\delta_g \otimes h).\end{aligned}$$

We want to calculate the trace explicitly in a fixed orthonormal basis $\{c_i \otimes b_j \mid c_i \in \mathcal{C}_\alpha, b_j \in \mathcal{V}^{\pi_\alpha}\}$ for an irreducible representation $\alpha \in \widehat{D(G)}$ with $\dim(\pi_\alpha) = n$. If $I = J$, then $c_{i_1} = hc_{j_1}\bar{h}$ gives $h \in N_G(c_{i_1})$ and therefore $h = q_{i_1}m\bar{q}_{i_1}$ for some $m \in N_G(r_\alpha)$. The character tr_α is therefore given by

$$\begin{aligned}\mathrm{tr}_\alpha(\delta_g \otimes h) &= \sum_{I\in 1_\alpha}\Gamma_\alpha^{II}(\delta_g \otimes h)\\ &= \sum_{i_1=1}^{|\mathcal{C}_\alpha|}\sum_{i_2=1}^{n}\delta_{g,c_{i_1}}\delta_{hg\bar{h},g}\Gamma_{\pi_\alpha}^{i_2i_2}(\bar{q}_g hq_g)\\ &= \delta_{g\in\mathcal{C}_\alpha}\delta_{h\in N_G(g)}\,\mathrm{tr}_{\pi_\alpha}(\bar{q}_g hq_g) \qquad (2.4.34)\end{aligned}$$

It follows from [Gou93] that the orthogonality relation for irreducible representations take the form

$$\sum_{g,h\in G}\mathrm{tr}_\alpha(\delta_g \otimes h)\,\mathrm{tr}_\beta(\delta_g \otimes h)^* = \delta_{\alpha,\beta}\,|G|\,.$$

Proposition 2.4.11. *The modules $\{V_\alpha\}_\alpha = \mathbb{C}\mathcal{C} \otimes V_\pi$ with $\alpha = (\pi, \mathcal{C})$ defined as in Theorem 2.4.7 form a complete set of inequivalent irreducible representations of the quantum double $D(G)$, and we have*

$$D(G) \cong \bigoplus_{\alpha\in\widehat{D(G)}}\dim_\alpha \mathcal{V}^\alpha.$$

Proof. We first show that the irreducible representations given in Theorem 2.4.7 are inequivalent for different choices of $\mathcal{C}$ and irreducible representations π of the centralizer $N_G(r_\mathcal{C})$. Using Equation (2.4.34), we obtain

$$\begin{aligned}\sum_{g,h\in G}\mathrm{tr}_\alpha(\delta_g \otimes h)\,\mathrm{tr}_\beta(\delta_{hg\bar{h}} \otimes \bar{h}) =& \sum_{g,h\in G}\delta_{g\in\mathcal{C}_\alpha}\delta_{h\in N_G(g)}\delta_{hg\bar{h}\in\mathcal{C}_\beta}\delta_{\bar{h}\in N_G(\bar{h}\bar{g}h)}\\ &\mathrm{tr}_{\pi_\alpha}(\bar{q}_g hq_g)\,\mathrm{tr}_{\pi_\beta}(\bar{q}_g\bar{h}q_g)\\ =& \sum_{g,h\in G}\delta_{g\in\mathcal{C}_\alpha}\delta_{g\in\mathcal{C}_\beta}\delta_{h\in N_G(g)}\,\mathrm{tr}_{\pi_\alpha}(\bar{q}_g hq_g)\,\mathrm{tr}_{\pi_\beta}(\bar{q}_g\bar{h}q_g).\end{aligned}$$

Note that $\mathcal{C}_\alpha \cap \mathcal{C}_\beta$ is either empty or $\mathcal{C}_\alpha = \mathcal{C}_\beta$. Using the orthogonality relation for irreducible characters applied to the irreducible representations π_α and π_β, this simplifies the above expression to

$$\begin{aligned} =&\delta_{\mathcal{C}_\alpha,\mathcal{C}_\beta}\delta_{\pi_\alpha,\pi_\beta} \left|\mathcal{C}_\alpha\right| \frac{|N_\alpha|}{\dim_{\pi_\alpha}} \\ =&\delta_{\alpha,\beta}\frac{|G|}{\dim_{\pi_\alpha}} \end{aligned}$$

Next, we will show that $D(G)$ and $\bigoplus_{\alpha\in\widehat{D(G)}} \dim_\alpha \mathcal{V}^\alpha$ are of the same dimension. By [EGH+11, Proposition 3.5.8], an algebra is semisimple if and only if $\dim(A) = \sum_{\pi\in\hat{A}} (\dim(V_\pi))^2$, where $\hat{A}$ is a set of representatives of irreducible representations (π, V_π) of A. Note that this is also true for every group algebra $\mathbb{C}N_\mathcal{C}$ of the centralizer subgroups, which is semisimple, as discussed in Section 2.2. Note also that $D(G)$ is semisimple by the discussion at the end of Section 2.4.1. Using that $|G| = |N_\mathcal{C}|\,|\mathcal{C}|$ and $\sum_{\mathcal{C}\in G_C} |\mathcal{C}| = |G|$ we see that

$$\begin{aligned} \dim(D(G)) =& |G|^2 = \sum_{\mathcal{C}\in G_C} |G|\,|\mathcal{C}| = \sum_{\mathcal{C}\in G_C} |N_\mathcal{C}|\,|\mathcal{C}|^2 \\ =& \sum_{\mathcal{C}\in G_C} \sum_{\pi\in\hat{N}_\mathcal{C}} |\mathcal{C}|^2 (\dim(V_\pi))^2 \\ =& \sum_{\mathcal{C}\in G_C} \sum_{\pi\in\hat{N}_\mathcal{C}} \left|\mathcal{V}^{(\pi,\mathcal{C})}\right|^2 . \end{aligned}$$

Finally, we will explicitly state an isomorphism $\phi : D(G) \to \bigoplus_{\alpha\in\widehat{D(G)}} \dim_\alpha \mathcal{V}^\alpha$ and show that ϕ is an isomorphism of modules. Let $\alpha = (\pi_\alpha, \mathcal{C}_\alpha)$ be an irreducible representation of the quantum double $D(G)$ and let $n_\alpha := \dim_{\pi_\alpha}$, $b_1, \ldots, b_{n_\alpha}$ be an orthonormal basis of π_α, such that the matrix representation $n \mapsto \Gamma_{\pi_\alpha}(n)$ is unitary for each $n \in N_\alpha$. For a fixed pair $J = (j_1, j_2) \in I_\alpha$, we define a map $\phi^{\alpha,J}$ by setting for each $(i_1, i_2) \in I_\alpha$

$$\phi^{\alpha,J}(c_{i_1} \otimes b_{i_2}) = \sum_{n\in N_\alpha} \bar{\Gamma}^{i_2 j_2}_{\pi_\alpha}(n) \delta_{c_{i_1}} \otimes q_{i_1} n \bar{q}_{j_1},$$

and extend linearly to $\mathcal{V}^\alpha$. We first show that this mapping realizes an intertwiner of representations. Given $(i_1, i_2) \in I_\alpha$ and $\delta_g \otimes h \in D(G)$, we have

$$(\delta_g \otimes h) \cdot \left(\sum_{n \in N_\alpha} \bar{\Gamma}^{i_2 j_2}_{\pi_\alpha}(n) \delta_{c_{i_1}} \otimes q_{i_1} n \bar{q}_{j_1} \right) = \delta_{g, h c_{i_1} \bar{h}} \sum_{n \in N_\alpha} \bar{\Gamma}^{i_2 j_2}_{\pi_\alpha}(n) \delta_g \otimes h q_{i_1} n \bar{q}_{j_1}. \tag{2.4.35}$$

From the coset decomposition $G = \dot{\bigcup}_{q \in Q_{\mathcal{C}_\alpha}} q_i N_\alpha$, it follows that there exists a unique pair $(q_k, m) \in Q_{\mathcal{C}_\alpha} \times N_\alpha$ such that $h q_{i_1} = q_k m$. We can then write

$$g = h c_{i_1} \bar{h} = q_k m \bar{q}_{i_1} c_{i_1} q_{i_1} \bar{m} \bar{q}_k = q_k m r_{\mathcal{C}_\alpha} \bar{m} \bar{q}_k = \bar{q}_k r_{\mathcal{C}_\alpha} q_k = c_k$$

and the right hand side of Equation (2.4.35) becomes

$$\begin{aligned}
&\delta_{g,c_k} \sum_{n \in N_\alpha} \bar{\Gamma}^{i_2 j_2}_{\pi_\alpha}(n) \delta_{c_k} \otimes q_k m n \bar{q}_{j_1} \\
\overset{n \mapsto \bar{m} n}{=} &\delta_{g,c_k} \sum_{n \in N_\alpha} \bar{\Gamma}^{i_2 j_2}_{\pi_\alpha}(\bar{m} n) \delta_{c_k} \otimes q_k n \bar{q}_{j_1} \\
= &\delta_{g,c_k} \sum_{n \in N_\alpha} \sum_{l=1}^{n_\alpha} \bar{\Gamma}^{i_2 t}_{\pi_\alpha}(\bar{m}) \bar{\Gamma}^{t j_2}_{\pi_\alpha}(n) \delta_{c_k} \otimes q_k n \bar{q}_{j_1} \\
= &\delta_{g,c_k} \sum_{l=1}^{n_\alpha} \bar{\Gamma}^{i_2 t}_{\pi_\alpha}(\bar{m}) \phi^{\alpha,J}(c_k \otimes b_t) \\
= &\delta_{g,c_k} \phi^{\alpha,J}(c_k \otimes \pi_\alpha(m) b_{i_2}) \\
= &\delta_{g,h c_{i_1} \bar{h}} \phi^{\alpha,J}(h c_{i_1} \bar{h} \otimes \pi_\alpha(\bar{q}_k h q_{i_1}) b_{i_2}) \\
= &\phi^{\alpha,J}((\delta_g \otimes h) \triangleright (c_{i_1} \otimes b_{i_2})).
\end{aligned}$$

Hence, $\phi^{\alpha,J}$ establishes an intertwiner. Next, we show that the images under the $\phi^{\alpha,J}$ yield orthogonal subspaces in $D(G)$, i.e. $\phi^{\alpha,J}(c \otimes b)$ is orthogonal to $\phi^{\beta,K}(c' \otimes b')$ for all $\alpha, \beta \in \widehat{D(G)}$, $c \in \mathcal{C}_\alpha, c' \in \mathcal{C}_\beta, b \in \mathcal{V}^\alpha, b' \in \mathcal{V}^\beta, J \in I_\alpha$ and $K \in I_\beta$ with $\alpha \neq \beta$ or $J \neq K$. Indeed, setting $I = (i_1, i_2), L = (l_1, l_2), J = (j_1, j_2), K = (k_1, k_2)$ with $I, J \in I_\alpha, K, L \in I_\beta$ and $c = c_{i_1}, c' = c_{l_1}, b = b_{i_2}, b' = b_{l_2}$ we have

$$\begin{aligned}
&\langle \phi^{\alpha,J}(c_{i_1} \otimes b_{i_2}), \phi^{\beta,K}(c_{l_1} \otimes b_{l_2}) \rangle \\
= &\sum_{\substack{n_1 \in N_\alpha \\ n_2 \in N_\beta}} \Gamma^{i_2 j_2}_{\pi_\alpha}(n_1) \bar{\Gamma}^{l_2 k_2}_{\pi_\beta}(n_2) \langle \delta_{c_{i_1}} \otimes q_{i_1} n_1 \bar{q}_{j_1}, \delta_{c_{l_1}} \otimes q_{l_1} n_2 \bar{q}_{k_1} \rangle \\
= &\sum_{\substack{n_1 \in N_\alpha \\ ne \in N_\beta}} \delta_{c_{i_1}, c_{l_1}} \delta_{q_{i_1} n_1 \bar{q}_{j_1}, q_{l_1} n_2 \bar{q}_{k_1}} \Gamma^{i_2 j_2}_{\pi_\alpha}(n_1) \bar{\Gamma}^{l_2 k_2}_{\pi_\beta}(n_2).
\end{aligned}$$

Since $c_{i_1} = c_{l_1}$ implies in particular $\mathcal{C}_\alpha = \mathcal{C}_\beta$ and hence $N_\alpha = N_\beta$, the above expression becomes

$$=\delta_{\mathcal{C}_\alpha,\mathcal{C}_\beta}\delta_{i_1,l_1}\delta_{q_{i_1}n_1\bar{q}_{j_1},q_{i_1}n_2\bar{q}_{k_1}}\sum_{n_1,n_2\in N_\alpha}\Gamma^{i_2j_2}_{\pi_\alpha}(n_1)\bar{\Gamma}^{l_2k_2}_{\pi_\beta}(n_2).$$

Using the unique coset factorization again, we see that $q_{i_1}n_1\bar{q}_{j_1} = q_{i_1}n_2\bar{q}_{k_1} \Leftrightarrow n_1 = n_2$ and $j_1 = k_1$, and due to the orthogonality relation for irreducible representations, the above expression becomes

$$= \delta_{\mathcal{C}_\alpha,\mathcal{C}_\beta}\delta_{\pi_\alpha,\pi_\beta}\delta_{i_1,l_1}\delta_{i_2,l_2}\delta_{j_1,k_1}\delta_{k_2,l_2} = \delta_{\alpha,\beta}\delta_{L,K}.$$

Note in particular that the linear independent vectors $\delta_{c_{i_1}} \otimes b_{i_2}$ are mapped to orthogonal, hence linear independent vectors again and $\phi^{\alpha,J}$ must be injective. It follows that the map

$$\phi := \bigoplus_{\alpha\in\widehat{D(G)}}\bigoplus_{J\in I_\alpha}\phi^{J,\alpha}$$

is an injective intertwiner into $D(G)$. By our calculations at the beginning of the proof, the codomain of ϕ has dimension equal to the dimension of $D(G)$. Hence, $\phi^{\alpha,J}$ must be bijective and therefore an isomorphism of representations. □

It is possible to construct projections into the irreducible submodules inside $D(G)$. By [Gou93, Equation (24)] these are given via

$$P_\alpha = \frac{d_\alpha}{|G|}\sum_{g,h\in G}\mathrm{tr}_\alpha(\delta_{\bar{h}gh}\otimes\bar{h})\delta_g\otimes h, \tag{2.4.36}$$

where $\alpha \in \widehat{D(G)}$ and P_α is viewed as operators via left-multiplication on $D(G)$. Using Equation (2.4.34), Equation (2.4.36) can be simplified to

$$P_\alpha = \frac{\dim_{\pi_\alpha}}{|N_\alpha|}\sum_{g\in\mathcal{C}_\alpha}\sum_{n\in N_\alpha}\mathrm{tr}_{\pi_\alpha}(\bar{n})\delta_g\otimes q_g n\bar{q}_g, \tag{2.4.37}$$

where we used that $d_\alpha = |\mathcal{C}_\alpha|\dim_{\pi_\alpha}$ and $|G| = |N_\alpha|\,|\mathcal{C}_\alpha|$. These operators are **central projections**, i.e. they lie in the centre of $D(G)$. Furthermore, the central projections P_α are mutually orthogonal, and we will use these projections in Chapter 3 to construct operators measuring charge excitations.

2.5 Operator Algebras

The theory of operator algebras was introduced by von Neumann and Murray in the early 1930s and serves to this date as a rigorous mathematical framework for describing quantum mechanics. The rough idea is to view observables as a set of bounded operators $\mathfrak{A}$ acting on a separable Hilbert space $\mathcal{H}$ [Seg47a]. A physical state is defined to be a positive linear and continuous functional $\omega : \mathfrak{A} \to \mathbb{C}$, understood to measure the expectation value $\omega(A)$ of an observable $A \in \mathfrak{A}$. Of particular interest are the pure states, which are defined to be states that cannot be expressed as a non-trivial convex combination of other states.

In an analogue to classical mechanics, the time evolution of a quantum mechanical system can be described via a continuous one-parameter group of automorphisms, describing either the time evolution of the observables, or equivalently, the time evolution of the states. The notion of time evolution allows us to define the notion of ground states as states whose total energy can at most grow under the action of local observables.

After introducing some basic terminology in Section 2.5.1 and Section 2.5.2, we shift our focus to infinite quantum spin systems in Section 2.5.3, which are obtained as a limit of finite tensor products of finite-dimensional matrix algebras, and are the main interest of this thesis. We will study the dynamics of these systems and the additional features of ground- and pure states in these settings. No result in Section 2.5 is new, and all details can be found in [Naa13, KR, KR86, BR12, BR03, Zhu93, Mur90, Hal13b, Rud91].

2.5.1 Basic Definitions

Let $\mathcal{H}$ be a Hilbert space, that is, a normed vector space equipped with inner product $\langle \cdot, \cdot \rangle$ such that $\|v\|^2 = \langle v, v \rangle$ and $\mathcal{H}$ is complete with respect to the topology induced by $\|\cdot\|$. Much like in the previous sections, we will only be concerned with complex vector spaces, and the inner product $\langle \cdot, \cdot \rangle$ is a sesquilinear form, where the complex conjugate is in the first argument, i.e. $\langle \lambda v, w \rangle = \bar{\lambda} \langle v, w \rangle$.

We caution the reader that the $\bar{(\cdot)}$-notation in this context stands for the complex conjugate, i.e. $\bar{\lambda}$ is the complex conjugate of the complex number $\lambda \in \mathbb{C}$, and that $\bar{(\cdot)}$ is used to denote the inverse of a group element and a group elements only.

Recall that a linear operator $T : \mathcal{H} \to \mathcal{K}$ is continuous if and only if $\|T\| < \infty$, where

$$\|T\| = \sup_{\|v\|=1} \|T(v)\|$$

is the **norm** of T and T is called **bounded** in that case. The space $\mathfrak{B}(\mathcal{H})$ of bounded linear endomorphisms on $\mathcal{H}$ is a special case of a C^*-algebra.

Definition 2.5.1 (C^*-algebra)**.** Let $\mathfrak{A}$ be a complex algebra. We call $\mathfrak{A}$ a **normed algebra** if it is equipped with a norm $\|\cdot\|$ such that $(\mathfrak{A}, \|\cdot\|)$ becomes a normed vector space with $\|A \cdot B\| \leq \|A\| \cdot \|B\|$, and call it a **Banach algebra** if it is complete with respect to its norm.

A C^***-algebra** is a complex Banach algebra $\mathfrak{A}$ together with a map $^* : \mathfrak{A} \to \mathfrak{A}$ that is

- anti-linear: $(\lambda A)^* = \bar{\lambda} A^*$,
- involutive: $(A^*)^* = A$,
- and an anti ring homomorphism: $(AB)^* = B^* A^*$

such that the *-property

$$\|A^* A\| = \|A\|^2$$

is satisfied for all $A, B \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$. A C^*-algebra is called **unital** if it is unital as a ring.

The map * is called the **star involution** and, for an element $A \in \mathfrak{A}$, we call A^* the **adjoint** of A.

We will only concern ourselves with unital C^*-algebras. We note however that this is in many cases not a proper restriction, as one can always embed any C^*-algebra $\mathfrak{A}$ into a unital C^*-algebra $\tilde{\mathfrak{A}}$, see e.g. the discussion following [Mur90, Thm 1.2.9].

Example 2.5.2. (i) Let $\mathcal{H}$ be a complex Hilbert space. Then the set of bounded operators $\mathfrak{B}(\mathcal{H})$ equipped with the usual adjoint operation and norm given by the supremum norm

$$\|A\| := \sup_{x \in \mathcal{H}} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$$

forms a C^*-algebra as hinted before. Furthermore, every closed self-adjoint subalgebra $\mathfrak{A}$ of $\mathfrak{B}(\mathcal{H})$ is a C^*-algebra, where a subset of a C^*-algebra is called self-adjoint if $A \in S$ implies $A^* \in S$.

(ii) Let X be a compact Hausdorff space and $\mathfrak{A}$ the set of all complex-valued continuous functions on X with compact support. Then $\mathfrak{A}$ is a *-algebra via pointwise operations and the involution is given by the pointwise complex conjugation. It becomes a commutative C^*-algebra when equipped with the supremum norm

$$\|f\| = \sup_{x \in X} |f(x)| .$$

Many notions and properties of matrix algebras can be recovered in the abstract setting of C^*-algebras.

Definition 2.5.3. Let $\mathfrak{A}$ be a C^*-algebra and $A \in \mathfrak{A}$. We call A

(i) **self-adjoint** if $A = A^*$,

(ii) **normal** if $[A, A^*] = 0$, where $[A, B] = AB - BA$ denotes the commutator of A and B,

(iii) **unitary** if $A^{-1} = A^*$,

(iv) **orthogonal projection** if $A^2 = A = A^*$.

We will see later that every C^*-algebra is essentially of the form in Example 2.5.2(i). In fact, Segal defines a C^*-algebra as a closed, self-adjoint subalgebra of $\mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ [Seg47b]. Furthermore, every unital commutative C^*-algebra is of the form in Example 2.5.2 (ii)[Mur90, Thm 2.1.10].

Definition 2.5.4 (Spectrum)**.** Let $\mathfrak{A}$ be a C^*-algebra and $A \in \mathfrak{A}$. The **spectrum** $\sigma(A)$ of A is defined as the set

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not invertible}\} .$$

A is called **positive** if $\sigma(A) \subset \mathbb{R}^+$.

Definition 2.5.5 (State)**.** A **state** on a unital C^*-algebra $\mathfrak{A}$ is a bounded linear functional $\omega : \mathfrak{A} \to \mathbb{C}$ such that $\omega(1_{\mathfrak{A}}) = 1$ and $\omega(A) \geq 0$ for all positive elements $A \in \mathfrak{A}$. We will denote the set of all states of a C^*-algebras by $\mathcal{S}_{\mathfrak{A}}$.

We can define a norm on $\mathcal{S}_{\mathfrak{A}}$ by setting

$$\|\omega\| = \sup_{\|A\|=1} |\omega(A)|$$

for all $\omega \in \mathcal{S}_{\mathfrak{A}}$. By [Mur90, 3.3.4 Corollary] a bounded linear functional ω is positive if and only if $\omega(1_{\mathfrak{A}}) = \|\omega\|$.

The set of states forms a convex subset of the set of all linear functionals on $\mathfrak{A}$. A state is called **pure** if it is an extreme point of this convex set, i.e. if it cannot be expressed as a non-trivial convex combination of other states. A useful characterization of pure states is given by the following lemma.

Lemma 2.5.6. *Let $\mathfrak{A}$ be a C^*-algebra and ω a state on $\mathfrak{A}$. Then ω is pure if and only if for each positive linear functional $\psi : \mathfrak{A} \to \mathbb{C}$ with $\psi(A) \leq \omega(A)$ for all positive operators $A \in \mathfrak{A}$, it follows that ψ must be a scalar multiple of ψ.*

See [KR, Lem 3.4.6] for a proof.

There are several equivalent definitions for the positivity of an element.

Theorem 2.5.7. *For an element $A \in \mathfrak{A}$ of a C^*-algebra $\mathfrak{A}$, the following are equivalent*

(i) A is positive,

(ii) $A = C^2$ for some positive $C \in \mathfrak{A}$,

(iii) $A = B^*B$ for some $B \in \mathfrak{A}$,

(iv) $\omega(A) \geq 0$ for all states ω on $\mathfrak{A}$.

A proof can be found in [KR, 4.2.6 Thm and 4.3.4 Thm].

Given two C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, we call a linear map $\varphi : \mathfrak{A}_1 \to \mathfrak{A}_2$ a *-homomorphism if φ is an algebra-homomorphism that respects the star-involution, i.e.

$$\varphi(A^*) = \varphi(A)^*$$

for all $A \in \mathfrak{A}_1$. It follows from [Mur90, Thm 2.1.7] that every *-homomorphism is a contraction, hence continuous.

Definition 2.5.8 (Representation of a C^*-algebra)**.** Let $\mathfrak{A}$ be a C^*-algebra. By a **representation** of $\mathfrak{A}$ we mean a pair $(\pi, \mathcal{H}_\pi)$ where $\mathcal{H}$ is a Hilbert space and $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ is a *-homomorphism. We call the representation **irreducible** if it contains no non-trivial subspace invariant under the action of $\mathfrak{A}$, **faithful** if π is injective, and **cyclic** if there exists a vector $\Omega \in \mathcal{H}_\pi$ such that the space

$$\pi(\mathfrak{A})\Omega = \{\pi(A)\Omega \mid A \in \mathfrak{A}\}$$

lies dense in $\mathcal{H}_\pi$. Ω is called the **cyclic vector** of the cyclic representation and we write a cyclic representation as triple $(\pi, \mathcal{H}_\pi, \Omega_\pi)$. A bounded linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ between two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ is called an **intertwiner** if it commutes with the action of $\mathfrak{A}$, i.e.

$$T\pi_1(A) = \pi_2(A)T$$

for all $A \in \mathfrak{A}$. If T is in addition unitary, we call π_1 and π_2 **unitarily equivalent** and write $\pi_1 \simeq \pi_2$.

Representations of a C^*-algebra $\mathfrak{A}$ form a category rep($\mathfrak{A}$) with intertwiners as morphisms. Similarly to group representations, we often just write π instead of $(\pi, \mathcal{H})$ for the representation.

Remark 2.5.9. If G is a finite group, we can form the group algebra $\mathfrak{A} = \mathbb{C}G$ as before. Furthermore, by viewing $\mathbb{C}G$ as a vector space, we may equip it with the inner product

$$\langle g, h\rangle = \delta_{g,h}.$$

making $\mathbb{C}G$ a Hilbert space $\mathcal{H}$. If $L : \mathfrak{A} \otimes \mathcal{H} \to \mathcal{H}, g \otimes h \mapsto gh$ denotes the natural action on that Hilbert space via left multiplication, then we can equip $\mathfrak{A}$ with the supremum norm

$$\|A\| = \sup_{\|\varphi\|=1} \|L_A \varphi\|$$

induced by the action L. Equipping $\mathfrak{A}$ further with the *-involution $g^* = g^{-1}$ for all $g \in G$, extended antilinearly to all of $\mathfrak{A}$, it is straightforward to check that $\mathfrak{A}$ becomes a C^*-algebra with these conventions, called the **group C^*-algebra of G**, and every unitary G-module becomes a C^*-algebra representation. We remark that the construction of a group C^*-algebra is not limited to the setting of finite groups and can be extended to locally compact groups, see [Fol16, Sec. 7] for instance.

We will now make precise how every C^*-algebra can be seen as in Example 2.5.2(i).

Theorem 2.5.10 (GNS representation)**.** *Let ω be a state of a C^*-algebra $\mathfrak{A}$. Then there exists a cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$, with $\|\Omega_\omega\| = 1$ such that*

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$$

for all $A \in \mathfrak{A}$, called ***GNS representation****. Furthermore, if $(\tilde{\pi}, \tilde{\mathcal{H}}, \tilde{\Omega})$ is another cyclic representation of $\mathfrak{A}$ such that $\omega(A) = \left\langle \tilde{\Omega}, \tilde{\pi}(A)\tilde{\Omega} \right\rangle$, then there exists a unitary intertwiner $U : \mathcal{H}_\omega \to \tilde{\mathcal{H}}$ such that $U\Omega_\omega = \tilde{\Omega}$, i.e. $(\tilde{\mathcal{H}}, \tilde{\pi}, \tilde{\Omega})$ is unitarily equivalent to $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ and $\tilde{\Omega}$ is the image of Ω_ω under the unitary equivalence. The triple $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is called a* ***GNS triple****.*

See [BR03, Thm 2.3.16] for a proof. The GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of a state ω is viewed as the physical realisation of ω as a vector in the Hilbert space $\mathcal{H}_\omega$. The GNS representations of pure states are of particular interest.

Theorem 2.5.11. *Let ω be a state on a C^*-algebra $\mathfrak{A}$. Then the GNS representation of ω is irreducible if and only if ω is pure.*

See [KR86, Thm. 10.2.3] for a proof.

Before we conclude this section, we want to present some results about states that will come in handy later.

Proposition 2.5.12 (Cauchy-Schwartz inequality for states)**.** *Let $\mathfrak{A}$ be a C^*-algebra and ω a positive linear functional. Then we have for all $A, B \in \mathfrak{A}$*

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B). \tag{2.5.1}$$

Proof. This follows from the fact that $(A, B) \mapsto \omega(A^*B)$ defines a positive semi-definite inner product on the vector space $\mathfrak{A}$ [Zhu93, Prop 13.4]. □

Lemma 2.5.13. *Let $0 \leq X \leq I$ be a positive operator such that $\omega(X) = 1$ for some state ω on $\mathfrak{A}$. Then $\omega(XA) = \omega(AX) = \omega(A)$ for all $A \in \mathfrak{A}$.*

Proof. We can take the positive square root $\sqrt{1_{\mathfrak{A}} - X} = (\sqrt{1_{\mathfrak{A}} - X})^*$ and set $A' := \sqrt{1_{\mathfrak{A}} - X}A^*$. Then $\omega(A(I - X)) = \omega((A')^*\sqrt{1_{\mathfrak{A}} - X})$ and using the Cauchy-Schwartz inequality (2.5.1) we obtain

$$|\omega(A(1_{\mathfrak{A}} - X))|^2 \leq \omega(A'(A')^*)\omega(1_{\mathfrak{A}} - X) = 0,$$

implying $\omega(AX) = \omega(A)$. The other identity follows analogously. □

2.5.2 C^*-dynamical Systems

Let $\mathfrak{A}$ be a C^*-algebra. By a dynamics on $\mathfrak{A}$ we mean a strongly continuous one-parameter group of automorphisms τ, i.e. a map

$$\tau : \mathbb{R} \times \mathfrak{A} \to \mathfrak{A}, (t, A) \mapsto \tau_t(A)$$

that is continuous for each fixed $A \in \mathbb{R}$ such that $\tau_{t+s} = \tau_t \circ \tau_s$ and $\tau_0 = \mathrm{id}_{\mathfrak{A}}$. The pair $(\mathfrak{A}, \tau)$ is called a **C^*-dynamical system**.

Definition 2.5.14 (Infinitesimal generator)**.** Let $(\mathfrak{A}, \tau)$ be a C^*-dynamical system and define the set

$$D(\delta) = \left\{ A \in \mathfrak{A} \mid \lim_{\substack{t\to 0\\ t\in\mathbb{R}^+}} \frac{\tau_t(A) - A}{t} \text{ converges in norm} \right\}.$$

Then the operator

$$\delta : D(\delta) \to \mathfrak{A}, A \mapsto \lim_{t\to 0} \frac{\tau_t(A) - A}{t} \tag{2.5.2}$$

is called the **infinitesimal generator** of τ.

Remark 2.5.15. While it is possible to define the notion of an infinitesimal generator more generally for linear operators on Hilbert spaces (see [BR12, Definition 3.1.15]), it suffices for our purposes to define the notion of an infinitesimal generator for strongly continuous one-parameter groups of automorphisms τ only.

The map δ defined in Equation (2.5.2) is a **symmetric derivation**, i.e. it satisfies

$$\begin{aligned}\delta(AB) =& \delta(A)B + A\delta(B),\\ \delta(A^*) =& \delta(A)^*,\end{aligned}$$

for all elements $A, B \in D(\delta)$.

Suppose that $\mathfrak{A} \subset \mathfrak{B}(\mathcal{H})$ and that H is a self-adjoint operator in $\mathfrak{B}(\mathcal{H})$. A time evolution can then be defined via

$$\tau_t(A) = \exp(itH)\, A \exp(-itH)$$

and the operator H is called **Hamiltonian**. The infinitesimal generator can then be shown to be

$$\delta(A) = \lim_{t\to 0} \frac{\tau_t(A) - A}{t} = i\,[H, A]\,, \tag{2.5.3}$$

which can be verified by considering the first order terms of the exponential $\exp(itH)$. We note that H will often be unbounded. In many concrete quantum spin systems however, the ones considered in this work included, the right-hand side of Equation (2.5.3) will lie in $\mathfrak{A}$ regardless and the domain of δ is dense in $\mathfrak{A}$.

Next, we want to define the notion of a ground state. If $\mathcal{H}$ is a Hilbert space, a time evolution is usually understood as a strongly continuous one-parameter group of unitaries $\{U_t\}_{t\in\mathbb{R}}$. Time evolution of the physical system can then be viewed from two different perspectives. In the so-called *Schrödinger picture*, one views the observables as fixed, while the states, i.e., the vectors in $\mathcal{H}$, evolve over time, that is, if $\Omega_0 \in \mathcal{H}$ is a state at time $t = 0$, then $U_t\Omega_0$ is the state at time t. In this picture, a ground state is understood to be a vector Ω_0 such that $U_t\Omega_0 = \Omega_0$ for all times t and in physical applications, U_t is generated by a positive operator H, called Hamiltonian such that $U_t = e^{itH}$ and H describes the energy of the system. This Hamiltonian has eigenvalue 0, accounting for an energy minimum of the system.

The other perspective is called the *Heisenberg picture*. Here, time evolution is understood as the evolution of operators, as opposed to the evolution of vectors. That is, if $A \in \mathfrak{B}(\mathcal{H})$ is an observable at time $t = 0$, then U_tAU_{-t}

is that observable at time t. We will always consider time evolution in the Heisenberg picture. We want to relate our discussion so far with this physical viewpoint:

Let ω be a state in a C^*-dynamical system $(\mathfrak{A}, \tau)$ and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the GNS representation of ω. Then it is easy to see that $(\mathcal{H}, \pi_\omega \circ \tau_t, \Omega_\omega)$ is another cyclic representation for each t and also that $\omega_t = \omega \circ \tau_t$ is a state as well. If ω is invariant under the time evolution, that is, if $\omega_t = \omega$ for all times t, then because of

$$\omega(A) = \omega_t(A) = \langle \Omega_{\omega_t}, \pi_{\omega_t}(A)\Omega_{\omega_t} \rangle,$$

$(\mathcal{H}, \pi_\omega \circ \tau_t, \Omega_\omega)$ is a GNS representation of ω as well.

By Theorem 2.5.10, there exists a unitary $U_t : \mathcal{H}_\omega \to \mathcal{H}_\omega$ such that $U_t\Omega_\omega = \Omega_\omega$ and $\pi_\omega(\tau_t(A)) = U_t\pi_\omega(A)U_{-t}$ for all t. This demonstrates in particular that π_ω and $\pi_\omega \circ \tau_t$ are unitarily equivalent. It is straightforward to check that U is a strongly continuous one-parameter group of unitaries if τ is a strongly continuous one-parameter group of automorphisms. Finally, we have the following Theorem by Stone([Hal13b, Thm 10.15]):

Theorem 2.5.16 (Stone's Theorem)**.** *Let $U : \mathbb{R} \to \mathfrak{B}(\mathcal{H}), t \mapsto U_t$ be a strongly continuous one-parameter group of unitaries on a Hilbert space $\mathcal{H}$. Then there exists a densely defined, self-adjoint operator H such that*

$$U_t(A) = \exp(itH)\, A \exp(-itH) \tag{2.5.4}$$

holds for all $A \in \mathfrak{B}(\mathcal{H})$ and $t \in \mathbb{R}$.

Hence, the time evolution of a state ω can be understood via a mapping $\omega \mapsto \omega \circ \tau_t$ and it would make sense to define a ground state via the condition $\omega \circ \tau_t = \omega$ for all t.

Proposition 2.5.17 (Ground State)**.** *Let $(\mathfrak{A}, \tau)$ be a C^*-dynamical system and let ω be a state on $\mathfrak{A}$. Then the following are equivalent.*

(i) ω is invariant under the time evolution, i.e.

$$\omega \circ \tau_t = \omega$$

for all $t \in \mathbb{R}$ and if $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is the GNS representation of ω and if H_ω is the self-adjoint operator realizing the time evolution on $\mathcal{H}_\omega$ via

$$\pi_\omega(\tau_t(A))\Omega_\omega = \exp(itH_\omega)\, \pi_\omega(A)\Omega_\omega,$$

then H_ω is a positive.

(ii) If δ is the infinitesimal generator of τ, then

$$-i\omega(A^*\delta(A)) \geq 0.$$

*If either, and hence both, of the above conditions are fulfilled, ω is called a **ground state**.*

See [BR03, Prop 5.3.19] for a proof.

2.5.3 Quantum Spin Systems

We will now discuss an important construction in the framework of quantum spin systems: the *quasi-local algebra*.

Recall that a **pre-ordered set** is a set L together with a reflexive and transitive relation $\leq$. It is called **directed** if for all $\Lambda_1, \Lambda_2 \in L$ there exists an element $\Lambda_3 \in L$ with $\Lambda_1 \leq \Lambda_3$ and $\Lambda_2 \leq \Lambda_3$, i.e., if every finite subset of L has an upper bound. We call a symmetric relation $\perp$ an **orthogonality relation** if the following conditions hold

(i) for all $\Lambda \in L$ there exists a $\Lambda' \in L$ such that $\Lambda \perp \Lambda'$,

(ii) if $\Lambda_1, \Lambda_2, \Lambda_3 \in L$ with $\Lambda_1 \leq \Lambda_2$ and $\Lambda_2 \perp \Lambda_3$, then $\Lambda_1 \perp \Lambda_3$.

(iii) if $\Lambda_1, \Lambda_2, \Lambda_3 \in L$ with $\Lambda_1 \perp \Lambda_2$ and $\Lambda_1 \perp \Lambda_3$, then there exists a $\Lambda_4 \in L$ with $\Lambda_1 \perp \Lambda_4$ such that $\Lambda_2, \Lambda_3 \leq \Lambda_4$.

A prime example of a directed set with an orthogonality relation, and the only example we will be concerned with, is that of the set of subsets $L = P(Z)$ for a given set Z together with the subset relation as a pre-order and disjointness as orthogonality. Above properties are then quickly verified. In view of this example, we may also generalize the notion of a union. We assume that given two elements $\Lambda_1, \Lambda_2 \in L$, there exists a *least upper bound*, which we denote by $\Lambda_1 \cup \Lambda_2$. The following definition is taken from [BR12, Definition 2.6.3].

Definition 2.5.18 (Quasilocal Algebra)**.** Let $(L, \leq)$ be a directed set with an orthogonality relation $\perp$ and with least upper bounds $\Lambda_1 \cup \Lambda_2$ for all $\Lambda_1, \Lambda_2 \in L$ and let $\{\mathfrak{A}_\Lambda\}_{\Lambda \in L}$ be a family of C^*-algebras such that the following hold:

(i) For all $\Lambda_1 \leq \Lambda_2$ we have $\mathfrak{A}_{\Lambda_1} \subseteq \mathfrak{A}_{\Lambda_2}$,

(ii) The algebras $\mathfrak{A}_\Lambda$ share a common unit I,

(iii) If $\Lambda_1 \perp \Lambda_2$, then $[A, B] = 0$ for all $A \in \mathfrak{A}_{\Lambda_1}, B \in \mathfrak{A}_{\Lambda_2}$.

Then we can form the C^*-algebra

$$\mathfrak{A} = \overline{\bigcup_{\Lambda \in L} \mathfrak{A}_\Lambda},$$

where $\overline{(\cdot)}$ denotes the norm completion. We call the quadruple $(L, \leq, \{\mathfrak{A}_\Lambda\}_\Lambda, \mathfrak{A})$ a **quasilocal structure** and $\mathfrak{A}$ the **quasilocal algebra** generated by the $\{\mathfrak{A}_\Lambda\}_\Lambda$.

Let i_{Λ_1,Λ_2} and i_Λ denote the inclusion maps $i_{\Lambda_1,\Lambda_2} : \mathfrak{A}_{\Lambda_1} \to \mathfrak{A}_{\Lambda_2}$ and $i_\Lambda : \mathfrak{A}_\Lambda \to \mathfrak{A}$ for all $\Lambda_1, \Lambda_2, \Lambda \in L$. Then the quasilocal algebra $\mathfrak{A}$ is universal in the following sense: Let $\tilde{\mathfrak{A}}$ be another C^*-algebra together with a family of *-homomorphisms $j_\Lambda : \mathfrak{A}_\Lambda \to \tilde{\mathfrak{A}}$. Then there exists a unique map $J : \mathfrak{A} \to \tilde{\mathfrak{A}}$ such that the following diagram

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\exists! J} & \tilde{\mathfrak{A}} \\ {\scriptstyle i_\Lambda}\uparrow & \nearrow_{j_\Lambda} & \\ \mathfrak{A}_\Lambda & & \end{array} \tag{2.5.5}$$

commutes. Indeed, J is already uniquely determined by the condition $J \circ i_\Lambda = j_\Lambda$. More generally, a **directed family of C^*-algebras** is a family $\{\mathfrak{A}_\Lambda\}_{\Lambda \in Z}$ of C^*-algebras together with *-homomorphisms $i_{\Lambda_1\Lambda_2} : \mathfrak{A}_{\Lambda_1} \to \mathfrak{A}_{\Lambda_2}$ for all $\Lambda_1, \Lambda_2 \in Z$ such that

$$i_{\Lambda_2\Lambda_3} \circ i_{\Lambda_1\Lambda_2} = i_{\Lambda_1\Lambda_3}$$

holds for all $\Lambda_1, \Lambda_2, \Lambda_3 \in Z$. An algebra $\mathfrak{A}$ satisfying the universal property described in Diagram (2.5.5) is called a **directed limit** of the family $\{\mathfrak{A}_\Lambda\}_{\Lambda \in Z}$ and $\mathfrak{A}$ is defined by Diagram 2.5.5 uniquely up to isomorphism. We may define the algebra

$$\mathfrak{A}_{loc} = \bigcup_{\Lambda \in Z} \mathfrak{A}_\Lambda,$$

and call $\mathfrak{A}_{loc}$ the **algebra of local observables/operators** and its elements **local observables/operators**. For $a \in \mathfrak{A}_{loc}$, we say that a is **supported in** Λ if $a \in \mathfrak{A}_\Lambda$ and we denote the smallest subset with this property as the **support** of a. Finally, we will often write $\{\mathfrak{A}_\Lambda\}_\Lambda$ instead of the quadruple $(L, \leq, \{\mathfrak{A}_\Lambda\}_\Lambda, \mathfrak{A})$ to denote a quasilocal structure if no confusion arises.

Example 2.5.19. Let $L = P_0(\mathbb{Z}^\nu)$ be the set of all finite subsets of $\mathbb{Z}^\nu$ for some $\nu \in \mathbb{N}$ and let $\mathcal{H}_x$ be a fixed finite dimensional Hilbert space for each

$x \in \mathbb{Z}$. Then we define for $\{x\} \in L$ the finite dimensional matrix algebra $\mathfrak{A}_{\{x\}} = \mathfrak{B}(\mathcal{H}_x)$ and set

$$\mathfrak{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathfrak{A}_{\{x\}} \tag{2.5.6}$$

for each $\Lambda \in L$. If $\Lambda_1, \Lambda_2 \in L$ with $\Lambda_1 \subset \Lambda_2$, we can define an embedding $\iota_{\Lambda_1,\Lambda_2} : \mathfrak{A}_{\Lambda_1} \to \mathfrak{A}_{\Lambda_2}$ by setting

$$\iota_{\Lambda_1,\Lambda_2}(A) = A \otimes \mathrm{id}_{\Lambda_2\setminus\Lambda_1},$$

where $\mathrm{id}_{\Lambda_2\setminus\Lambda_1}$ is the identity map on the space

$$\bigotimes_{x\in\Lambda_2\setminus\Lambda_1} \mathcal{H}_x.$$

This allows us to view $\mathfrak{A}_{\Lambda_1}$ as a subset of $\mathfrak{A}_{\Lambda_2}$ for $\Lambda_1 \subset \Lambda_2$. By setting $\Lambda_1 \leq \Lambda_2 :\Leftrightarrow \Lambda_1 \subset \Lambda_2$, $\Lambda_1 \perp \Lambda_2 :\Leftrightarrow \Lambda_1 \cap \Lambda_2 = \emptyset$ and unions (least) as upper bounds, the quadruple $(L, \leq, \{\mathfrak{A}_\Lambda\}_\Lambda, \mathfrak{A})$ becomes a quasilocal structure.

We define a **quantum spin system** to be a quasilocal structure $\{\mathfrak{A}_\Lambda\}_\Lambda$, where all the $\mathfrak{A}_\Lambda$ are simple finite-dimensional algebras.

Proposition 2.5.20. *Let $\{\mathfrak{A}_\Lambda\}_\Lambda$ be a quantum spin system and let $\mathfrak{A}_\Lambda$ be simple for each $\Lambda \in L$. It follows that $\mathfrak{A}$ is simple as well.*

See [BR12, Corollary 2.6.19] for a proof.

Before we proceed, we want to discuss the typical dynamics of quantum spin systems. By an **interaction** we mean a self-adjoint map

$$\Phi : L \to \mathfrak{A}_{loc},$$
$$\Phi(\Lambda)^* = \Phi(\Lambda)$$

such that $\Phi(\Lambda) \in \mathfrak{A}_\Lambda$ for all regions Λ. Given a quasilocal algebra $\{\mathfrak{A}_\Lambda\}_\Lambda, \mathfrak{A}$ together with an interaction Φ, we define the **local Hamiltonian H_Λ^Φ in the region** Λ as the self-adjoint operator

$$H_\Lambda^\Phi = \sum_{\Lambda'\subset\Lambda} \Phi(\Lambda') \tag{2.5.7}$$

Let $L = P(\mathbb{Z}^2)$ and let $L_0 = P_0(\mathbb{Z}^2)$ denote the set of finite subsets of $\mathbb{Z}^2$. Then Equation (2.5.7) is a finite sum for $\Lambda \in L_0$ and H_Λ^Φ is a bounded operator. We then may define a **local time evolution** via the strongly continuous one-parameter group

$$\tau_t^{\Lambda,\Phi}(A) = \exp\left(itH_\Lambda^\Phi\right) A \exp\left(-itH_\Lambda^\Phi\right) \tag{2.5.8}$$

for $A \in \mathfrak{A}_\Lambda$ and $t \in \mathbb{R}$. We can explicitly calculate the infinitesimal generator in this case: Setting

$$B_A(t) = \sum_{n_1+n_2 \geq 2} \left(itH_\Lambda^\Phi\right)^{n_1} A \left(-itH_\Lambda^\Phi\right)^{n_2}$$

we obtain

$$\begin{aligned}\delta_\Lambda^\Phi(A) &= \lim_{t\to 0} \frac{\tau_t^{\Lambda,\Phi}(A) - A}{t} \\ &= \lim_{t\to 0} \frac{A + itH_\Lambda^\Phi A - itAH_\Lambda^\Phi - A}{t} + \underbrace{\frac{B_A(t)}{t}}_{\to 0} \\ &= iH_\Lambda^\Phi A - iAH_\Lambda^\Phi && (2.5.9) \\ &= i\left[H_\Lambda^\Phi, A\right]. && (2.5.10)\end{aligned}$$

In general, it is not clear whether $\tau^{\Lambda,\Phi}$ converges (in norm or otherwise) to a strongly continuous one-parameter group τ^Φ on $\mathfrak{A}$ as $\Lambda \to \infty$. In many cases, the local Hamiltonians H_Λ^Φ do not converge to a bounded operator. Rather, the sum $\sum_{\Lambda \in L} \Phi(\Lambda)$ is unbounded. We may still define a derivation δ as in Formula (2.5.9) on a self-adjoint subset $D(\delta) \subseteq \mathfrak{A}$ called domain of δ. We introduce a few notions and results on unbounded operators to shed some more light on these nuances.

In general, given an unbounded operator $T : D(T) \to \mathcal{H}_2$ with self-adjoint domain $D(T) \subset \mathcal{H}_1$ between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we say that $T_0 : D(T_0) \to \mathcal{H}_2$ is an **extension** of T if $D(T) \subseteq D(T_0)$ and $T = T_0 \mid_{D(T)}$. T is called **closed** if the graph of T is a closed subspace of $\mathcal{H}_1 \oplus \mathcal{H}_2$ and we say that T is **closable** if there exists a closed extension. If it exists, we denote by $\bar{T}$ the smallest closed extension of T, and call it the **closure of** T. We remark that if $D(T) = \mathcal{H}_1$ then T is bounded if [Rud91, Thm 2.15] and only if [Rud91, Prop 2.14] T is closed.

The following theorem follows from [BR03, Prop 6.2.3,Thm 6.2.4] and gives a sufficient condition for the convergence of a time-evolution and the existence of a closable infinitesimal generator.

Theorem 2.5.21. *Let $\{\mathfrak{A}_\Lambda\}_\Lambda, \mathfrak{A}$ be a quantum spin system with interaction Φ, $L = P(\mathbb{Z}^\nu)$, $\nu \in \mathbb{N}$ and assume there is a constant $\lambda > 0$ such that*

$$\|\Phi\|_\lambda = \sum_{n\geq 0} e^{\lambda n} \left(\sup_{x\in\mathbb{Z}^\nu} \sum_{\substack{x\in\Lambda \\ |\Lambda|=n+1}} \|\Phi(\Lambda)\| \right) < \infty. \qquad (2.5.11)$$

Let further $\delta^{\Lambda,\Phi}$ be the derivation defined via

$$\delta^{\Lambda,\Phi}(A) = i \sum_{\Lambda' \cap \Lambda \neq \emptyset} [\Phi(\Lambda'), A]$$

for all $A \in D(\delta^{\Lambda,\Phi})$ and $D(\delta) = \mathfrak{A}_{loc}$. Then $\delta^{\Lambda,\Phi}$ is norm-closable, and its closure $\bar{\delta}^{\Lambda,\Phi}$ is the infinitesimal generator of a strongly continuous one-parameter group τ on $\mathfrak{A}$. Moreover, if $\tau_\tau^{\Lambda,\Phi}$ is the family of local time evolutions defined via

$$\tau_t^{\Lambda,\Phi}(A) = \exp\left(itH_\Lambda^\Phi\right) A \exp\left(-itH_\Lambda^\Phi\right),$$

then $\tau_t^{\Lambda,\Phi}$ converges to τ_t^Φ for all $A \in \mathfrak{A}$ uniformly on compact sets in $\mathbb{R}$.

We call Φ **uniformly bounded** if

$$\sup_{\Lambda \in L} \|\Phi(\Lambda)\| \leq \infty.$$

If $L = P(\mathbb{Z}^2)$, we have a natural notion of a **translation symmetry** on L, that is an action of the abelian group $(\mathbb{R}, +)$ on L together with an action $T : \mathbb{R} \times \mathfrak{A} \to \mathfrak{A}, A \mapsto T_x(A)$ such $T_x(A) \in \mathfrak{A}_{x+\Lambda}$ for all $A \in \mathfrak{A}_\Lambda$, where $x + \Lambda = \{x + y \mid y \in \Lambda\}$. We call an interaction Φ **translationally invariant** if

$$\Phi(x + \Lambda) = T_x(\Phi(\Lambda)).$$

It is called a **finite range interaction** if there exists a $d \in \mathbb{N}$ such that $\Phi(\Lambda) = 0$ for all $\mathrm{diam}(\Lambda) \geq d$. For the lattice model, this implies in particular that there exists an $n_0 \in \mathbb{N}$ such that $|\Lambda| \geq n_0$ implies that $\Phi(\Lambda) = 0$ for all $n \geq n_0$. If Φ is in addition translationally invariant, (2.5.11) reads

$$\begin{aligned} \|\Phi\|_\lambda &= \sum_{n \geq 0} e^{\lambda n} \left(\sup_{x \in \mathbb{Z}^2} \sum_{\substack{x \in \Lambda \\ |\Lambda| = n+1}} \|\Phi(\Lambda)\| \right) \\ &= \sum_{n=0}^{n_0 - 1} e^{\lambda n} \sum_{\substack{0 \in \Lambda \\ |\Lambda| = n_0}} \|\Phi(\Lambda)\| \end{aligned}$$

which is in particular a finite sum, hence bounded and Theorem 2.5.21 guarantees the convergence of the local time evolutions $\tau^{\Lambda,\Phi}$ given in (2.5.7) by passing to some appropriate Λ_0 with

$$\Lambda_0 \supset \bigcup_{\Lambda' \cap \Lambda} \Lambda'.$$

This ascertains the existence of a global time evolution for quantum spin systems with translationally invariant finite range interactions.

We want to give an alternative definition of ground states in the setting of sufficiently well-behaved quantum spin systems. The statement can be found in more detail in [BR03, Thm 6.2.52]

Theorem 2.5.22. *Let* $\{\mathfrak{A}_\Lambda\}_\Lambda, \mathfrak{A}$ *be a quantum spin system with interaction* Φ *and local time evolutions* $\tau^{\Lambda,\Phi}$*. Assume that*

(i) $\tau^{\Lambda,\Phi}$ *converges strongly to an automorphism* τ^Φ*, i.e.,*

$$\lim_{\Lambda\to\infty} \left\|\tau_t^{\Lambda,\Phi}(A) \to \tau_t^\Phi(A)\right\| = 0$$

for all $A \in \mathfrak{A}$ *and* $t \in \mathbb{R}$*.*

(ii) The surface energies

$$W_\Lambda^\Phi = \sum_{\substack{X\cap\Lambda\neq\emptyset \\ X\cap\Lambda^c\neq\emptyset}} \Phi(X)$$

are well-defined elements of $\mathfrak{A}$ *for all regions* $\Lambda \in L$*.*

(iii) $\mathfrak{A}_{loc}$ *is a core for* δ*, i.e., the closure of the unbounded operator* $\delta \mid_{\mathfrak{A}_{loc}}$ *in the weak operator topology is equal to* δ*.*

Then the following are equivalent for a state ω *on* $\mathfrak{A}$*.*

(i) For all $\Lambda \subset L$ *we have*

$$\omega(\tilde{H}_\Lambda^\Phi) = \inf_{\omega'\in C_\Lambda^\omega} \omega'(\tilde{H}_\Lambda^\Phi)$$

where $\tilde{H}_\Lambda^\Phi = H_\Lambda^\Phi + W_\Lambda^\Phi$ *and* C_Λ^ω *is the set of all states* ω' *with* $\omega' \mid_{\mathfrak{A}_{\Lambda^c}} = \omega \mid_{\mathfrak{A}_\Lambda^c}$

(ii) ω *is a* τ*-ground state.*

Note that in the setting of Theorem 2.5.21, δ is norm-closable with domain $\mathfrak{A}_{loc}$, and with the norm-closure of $\mathfrak{A}_{loc}$ being $\mathfrak{A}$, Item (iii) of Theorem 2.5.21 is satisfied.

We want to study equivalence of states in the setting of quantum spin systems. We say that two states are **equivalent** if their corresponding GNS representations are unitarily equivalent. Two states ω_1 and ω_2 are called **quasi-equivalent** if there exists a cardinal n such that

$$n\pi_{\omega_1} \cong n\pi_{\omega_2},$$

i.e. π_{ω_1} and π_{ω_2} are unitarily equivalent up to a multiple. If π_{ω_1} and π_{ω_2} are irreducible, then quasi-equivalence implies unitary equivalence. One can show that two representations are quasi-equivalence, if and only if every irreducible subrepresentation appearing in π_1 also appears - up to unitary equivalence - in π_2 and vice versa [BR12, Thm 2.4.26].

Quasi-equivalence between π_{ω_1} and π_{ω_2} can be characterized using a particular class of states on $\mathcal{H}_{\omega_1}$ and $\mathcal{H}_{\omega_2}$. If $\mathcal{H}$ is a finite dimensional Hilbert space and $\mathfrak{A} \subset \mathfrak{B}(\mathcal{H})$ a finite dimensional C^*-algebra, then we can define a state via the mapping $\mathfrak{A} \ni X \mapsto \mathrm{tr}(\rho X) \in \mathbb{C}$, where $\rho \in \mathfrak{B}(\mathcal{H})$ is a density matrix, i.e. a positive matrix with $\mathrm{tr}(\rho) = 1$. We want to generalize this concept. If $\mathcal{H}$ is a Hilbert space, then a closed self-adjoint subalgebra $\mathfrak{N}$ of $\mathfrak{B}(\mathcal{H})$ is called **von Neumann algebra** if

$$\mathfrak{N}'' = \mathfrak{N},$$

where $\mathfrak{N}' = \{A \in \mathfrak{B}(\mathcal{H}) \mid [A, B] = 0 \text{ for all } B \in \mathfrak{N}\}$ is the commutant of $\mathfrak{N}$ in $\mathfrak{B}(\mathcal{H})$. The well known bicommutant theorem [BR12, Thm 2.4.11] states that $\mathfrak{N}$ is a von Neumann algebra if and only if $\mathfrak{N}$ is closed in the *weak topology*, that is, in the topology induced by the family of seminorms $\left\{ \|\cdot\|_{\psi,\varphi} \mid \psi, \varphi \in \mathcal{H} \right\}$ defined via

$$X \mapsto \|X\|_{\psi,\varphi} = \langle \varphi, X\psi \rangle. \tag{2.5.12}$$

Note that if $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$, then $\mathfrak{N}' = \{\lambda 1_{\mathfrak{A}} \mid \lambda \in \mathbb{C}\}$ and $\mathfrak{N}$ is already a von Neumann algebra. In general, we call a von Neumann algebra $\mathfrak{N}$ a **factor** if $\mathfrak{N}' = \mathbb{C}1_{\mathfrak{A}}$ consists of multiples of the identity only. Let $\mathfrak{A}$ be a C^*-algebra and $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ a representation of $\mathfrak{A}$ on $\mathcal{H}$. If $T \in \pi(\mathfrak{A})'$ is in the centre of $\mathfrak{A}$ in $\mathfrak{B}(\mathcal{H})$, then $T : \mathcal{H} \to \mathcal{H}$ is per definition an intertwiner from π to itself. If π is irreducible, it follows that T must be a multiple of the identity by Schur's Lemma. It follows that if π is an irreducible representation of $\mathfrak{A}$ on $\mathcal{H}$, then the von Neumann algebra $\pi(\mathfrak{A})''$ generated by $\pi(\mathfrak{A})$ must be a factor.

If $\mathcal{H}$ is a separable Hilbert space and $\{\xi_n\}_n \subset \mathcal{H}$ an orthonormal basis of $\mathcal{H}$, then we call an operator $X \in \mathfrak{B}(\mathcal{H})$ a **trace-class operator** if $\sum_n \langle \xi_n, X\xi_n \rangle$ converges. We call this sum the **trace of X** and denote it by $\mathrm{tr}(X)$. This definition is independent of the choice of the orthonormal basis $\{\xi_n\}_n \subset \mathcal{H}$ [KR, Prop 2.6.1]. Finally, we call a state ω on a von Neumann algebra $\mathfrak{N} \subset \mathfrak{B}(\mathcal{H})$ **normal** if there exists a positive trace-class operator $\rho \in \mathfrak{B}(\mathcal{H})$ with $\mathrm{tr}(\rho) = 1$ such that $\omega(X) = \mathrm{tr}(\rho X)$. If ω is a state on a C^*-algebra $\mathfrak{A}$ with GNS representation $(\mathcal{H}, \pi, \Omega)$, we can define a set

$$\mathcal{N}(\omega) = \left\{ \varphi \circ \omega \mid \varphi : \overline{\pi(\mathfrak{A})} \to \mathbb{C} \text{ is a normal state} \right\}.$$

By [KR86, Prop 10.3.13], two states ω_1 and ω_2 are quasi-equivalent if and only if $\mathcal{N}(\omega_1) = \mathcal{N}(\omega_2)$. Another useful characterization of quasi-equivalence in the setting of quantum spin systems is given by the following Lemma.

Lemma 2.5.23. *Let $\{\mathfrak{A}_\Lambda\}_\Lambda, \mathfrak{A}$ be a quantum spin system, and let ω_1, ω_2 be normal factor states on $\mathfrak{A}$. Assume further that $L = P(\mathbb{Z}^2)$. Then ω_1 and ω_2 are quasi-equivalent if and only if for all $\varepsilon > 0$ there exists a region Λ_0 such that*

$$|\omega_1(X) - \omega_2(X)| < \varepsilon \, \|X\|$$

for all $X \in \mathfrak{A}_\Lambda$ with Λ disjoint from Λ_0.

We refer to [BR12, Cor 2.6.11] in conjunction with [BR12, Thm 2.6.10] for a proof.

Chapter 3

Kitaev's Quantum Double Model

3.1 Introduction

Kitaev's Quantum Double Model was first introduced as a surface code model [Kit03]. The idea behind this model was to establish an error correction code using local *stabilizer operators* that create a situation resembling ferromagnetism, where perturbed spins are immediately corrected by their neighbours. The model consists of a quantum spin system on a lattice, where each edge is decorated with a finite dimensional Hilbert space, and the dynamics is described by a frustration free Hamiltonian, that is, the Hamiltonian is a sum of commuting projections, called *stabilizer operators*. In the infinite plane, this model admits a unique frustration free ground state, minimizing each stabilizer operator individually. We will define the model, the stabilizer operators and the frustration free ground state in detail in Section 3.2.

Another important concept introduced in [Kit03] is that of ribbon operators. A ribbon is a way of describing how excitations move within the lattice. Ribbon operators are constructed for each ribbon such that they create charges only on the endpoints of the ribbon. We will discuss this class of operators in detail in Section 3.3. Furthermore, ribbon operators have the following important topological property: Given two ribbons ξ and ξ' with the same endpoints, the action of the ribbon operators on the vacuum vector is the same. To our knowledge, there exists no correct proof of this statement for the non-abelian quantum double model, so we will provide a proof of this statement in Corollary 3.3.11. Ribbons afford a convenient way of describing charge creation in the quantum double model.

In Section 3.4, we draw the connection to the quantum double by demonstrating that the stabilizer operators are realized as an action of the quantum double $D(G)$ of an underlying group G. We identify excitations with irreducible representations of $D(G)$, and show how excitations can be created and measured with the use of ribbon operators.

Our study of anyon excitation is motivated by the DHR analysis in [DHR71], usually applied in the framework of quantum field theory. To apply these techniques here, we will show that the model admits a unique translational invariant ground state ω_0 which minimizes each term of the Hamiltonian in Equation (3.2.16) individually. Such a ground state is called **frustration free**.

While a C^*-algebra generally possesses many physically irrelevant states, the superselection criterion discussed in the introduction, gives a relatively good constraint on the set of states on $\mathfrak{A}$. However, the notion of a localized and transportable endomorphism as in [DHR71, Prop 1.2] has to be slightly tweaked for our purposes. Instead of considering endomorphisms $\chi : \mathfrak{A} \to \mathfrak{A}$, we will construct a suitable *-homomorphism of the form

$$\chi : \mathfrak{A} \to \mathfrak{A} \otimes \mathrm{End}(V) \tag{3.1.1}$$

for some finite dimensional vector space V. If ω_0 is the vacuum state of the model, we will then study states of the form

$$\omega^{\chi,II} : \mathfrak{A} \to \mathbb{C}, X \mapsto (\omega_0 \circ \chi^{II})(X) \tag{3.1.2}$$

where χ^{II} is the component in the I-th row and I-th column of χ for $I = 1, \ldots, \dim(V)$ in a chosen basis. The goal of the following sections is to construct these maps χ from ribbon operators, which can intuitively can be thought of as charge creating operators. We will then show in Section 3.5 that the states constructed as in Equation (3.1.2) are - up to some exceptions - ground states of the model (Theorem 3.5.4) and in Section 3.6 that their GNS representations are irreducible (Theorem 3.6.8). The latter statement was independently proven in [BV23] using different methods. We will also further demonstrate in Section 3.4 how these excitations are related to irreducible representations of the quantum double $D(G)$.

An important tool needed to show the ground state property are the *Wilson loop* operators defined in [BMD08, Eq. (B75)] for a ribbon σ enclosing a region. These operators can be used to measure the existence of charges in the region enclosed by the σ. Here too we are not aware of a rigorous proof of that statement, so we will provide one in Proposition 3.4.4.

We first recall some basic definitions from graph theory in Section 3.2 and general terminology and conventions to express the notion of regions and paths in our lattice model, and to discuss more formally the localized nature of anyonic excitations. This allows us already to show that the model admits a unique translationally invariant ground state. Although Section 3.2 provides no new insight that cannot be found in the existing literature [Kit03, Naa12, BMD08, CM22b], we provide careful proofs to some algebraic relations that are not always done in detail in the literature. In particular, Lemma 3.2.7 is a stronger version of [Naa12, Lem 12.1.2], which will be needed in Section 3.6.

3.2 Model and notation

We consider an oriented graph, whose vertices can be identified with $\mathbb{Z}^2$ and such that every two neighbouring vertices are connected by an edge. Let $V = \mathbb{Z}^2$ be the set of vertices and E the set of edges of this graph. While the concrete orientation of the graph does not matter, we will assume for simplicity that all horizontal edges are oriented such that they point *to the right* and that all vertical edges pointing *upwards* (see Figure 3.1). Now, let G be a finite group and consider the group algebra $\mathbb{C}G$. As a vector space, $\mathbb{C}G$ can be made into a Hilbert space $\mathcal{H}$ by introducing the scalar product

$$\langle g, h\rangle = \delta_{g,h}$$

for $g, h \in G$. At each edge $\mathfrak{e} \in E$, we define the Hilbert space $\mathcal{H}_\mathfrak{e} := \mathcal{H}$. Then by setting $L = P_0(E)$, we obtain a quasilocal structure exactly as we did in Example 2.5.19:

(i) Given $\mathfrak{e} \in \mathbb{Z}^2$, we have $\mathcal{H}_\mathfrak{e} := \mathbb{C}G$ viewed as a Hilbert space, and if $\Lambda \in L$, we set

$$\mathcal{H}_\Lambda = \bigotimes_{\mathfrak{e}\in\Lambda} \mathcal{H}_\mathfrak{e}$$

(ii) For $\Lambda_1, \Lambda_2 \in L$ with $\Lambda_1 \subset \Lambda_2$, we have $\mathfrak{A}_{\Lambda_1} \subset \mathfrak{A}_{\Lambda_2}$, with the inclusion maps given as in Equation (2.5.6).

(iii) For $\Lambda_1 \cap \Lambda_2 = \emptyset$, we have

$$[A, B] = 0$$

for all $A \in \mathfrak{A}_{\Lambda_1}$ and $B \in \mathfrak{A}_{\Lambda_2}$.

(iv) The quasilocal algebra generated by this quasilocal structure is given by

$$\mathfrak{A} = \overline{\bigcup_{\Lambda\in L} \mathfrak{A}_\Lambda}.$$

Since $\mathfrak{A}_\Lambda$ is a finite dimensional matrix algebra, and therefore simple, for each $\Lambda \in L$, the family $\{\mathfrak{A}_\Lambda\}_\Lambda$ forms a quantum spin system.

Remark 3.2.1. In Section 2.5, we were often times associating our Hilbert spaces with the vertices of the lattice, rather than the edges, but this does not create any significant changes.

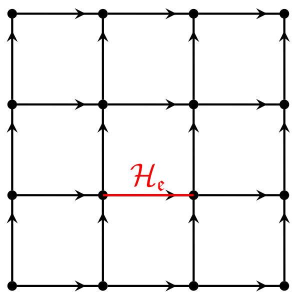


Figure 3.1: The lattice model $\mathbb{Z}^2$ depicted as a 2-dimensional graph. The orientations of the edges are indicated by the arrowheads and at each edge $\mathfrak{e}$ lives a Hilbert space $\mathcal{H}_\mathfrak{e}$.

Before we go further, we want to recall some basic terminology from graph theory which can be found in e.g. [BM07] and [GR01]. This is needed in order to properly define the notion of a ribbon and to understand how different charges can sit inside the lattice. Let $\mathscr{G} = (V, E)$ be an oriented graph with V the set of vertices and

$$E \subseteq \{(v_1, v_2) \mid v_1, v_2 \in V\}$$

the set of oriented edges. We will write $\bar{E}$ for the set of edges with opposite orientation, i.e.

$$\bar{E} = \{(v_1, v_2) \mid (v_2, v_1) \in E\}.$$

We will only consider graphs that have at most one edge between two vertices. We define a **path** p in such a graph as a sequence $p = (v_1, v_2, \ldots, v_n)$, where each pair (v_i, v_{i+1}), $i = 1, \ldots, n-1$ is connected by an edge. p is called **directed** or **oriented** if in addition $(v_i, v_{i+1}) \in E$ holds for all i. Unless explicitly stated otherwise, we will always assume that a path does not self-intersect. Given an edge $\mathfrak{e} = (v_1, v_2) \in E$, we denote by $\partial_0 \mathfrak{e} = v_1$ the starting vertex of $\mathfrak{e}$ and by $\partial_1 \mathfrak{e} = v_2$ the final vertex of $\mathfrak{e}$. We can extend the mappings ∂_0, ∂_1 for a path $p = (v_1, \ldots, v_n)$ by defining $\partial_0 p = v_1$ and $\partial_1 p = v_n$. We say a path p_1 can be **composed** with a path p_2 if $\partial_1 p_1 = \partial_0 p_2$ and p_1 and p_2 do not intersect anywhere but possibly at their endpoints. In that case, we denote by (p_1, p_2) the concatenation of these two paths, that is, if $p_1 = (v_1, \dots, v_n)$ and $p_2 = (w_1, \dots, w_m)$, then $(p_1, p_2) = (v_1, \dots, v_n, w_1, \dots, w_m)$. For a path $p = (v_1, \dots, v_n)$ we introduce the notations

$$(k : p) = (v_k, \dots, v_n)$$
$$(p : l) = (v_1, \dots, v_l)$$

and

$$(k : p : l) = (v_k, \dots, v_l). \tag{3.2.1}$$

See also Figure 3.2. If $p = (v_1, \dots, v_n)$, we may also write $(v_k : p)$, $(p : v_l)$ and $(v_k : p : v_l)$ instead of $(k : p)$, $(p : l)$ and $(k : p : l)$. A **subpath** of a path

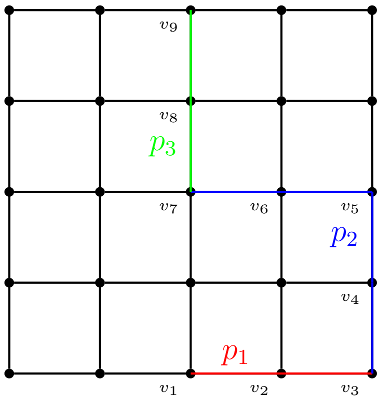


Figure 3.2: Depiction of a path $p = (v_1, \dots, v_9)$ with three subpaths $p_1 = (p : 3)$(red), $p_3 = (7 : p)$(green) and $p_2 = (3 : p : 7)$(blue)

p is a path of the form as in Equation (3.2.1). A **region** of $\mathscr{G}$ is a subset $\Lambda \subset E$ of edges in $\mathscr{G}$ and a **subgraph** of $\mathscr{G}$ is a graph $F = (V_F, E_F)$ such that $V_F \subset V$ and $E_F \subset E$. If Λ is a region of $\mathscr{G}$, we denote by $\mathscr{G}_\Lambda = (V_{\mathscr{G}_\Lambda}, E_{\mathscr{G}_\Lambda})$ the smallest subgraph of $\mathscr{G}$ such that $\Lambda = E_{\mathscr{G}_\Lambda}$, i.e. the smallest subgraph that has Λ as edges. A region is called **bounded** or **finite** if its intersection with the plane $\mathbb{R}^2$ is bounded.

A **planar graph** is a graph that can be embedded in $\mathbb{R}^2$ in such a manner, that no two edges intersect anywhere but at adjacent vertices and a **plane graph** is a planar graph with a chosen embedding. In the special case of the lattice graph, we call a region Λ a **square-shaped region of size** n if Λ is the intersection of a square of area n^2 with $\mathscr{G}$ in the plane $\mathbb{R}^2$. We denote the square-shaped regions of size n centred at the origin by Λ_n.

If $\mathscr{G}$ is a plane graph, we can define the dual graph $\mathscr{G}^*$ of $\mathscr{G}$, whose vertices consist of the faces of $\mathscr{G}$ and two faces are connected by an edge in $\mathscr{G}^*$ if they are separated by an edge in $\mathscr{G}$, see Figure 3.3. Recall that a **face** f of $\mathscr{G}$ is a connected component of the complement of $\mathscr{G}$ in $\mathbb{R}^2$.

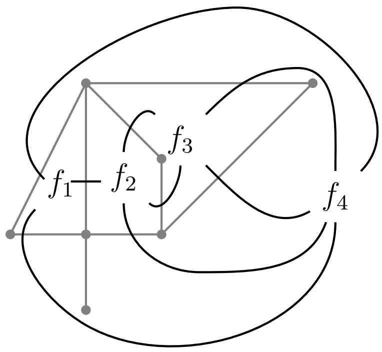


Figure 3.3: Depiction of a graph with 7 nodes. The complement of the graph as a subset of $\mathbb{R}^2$ has the four connected components f_1, f_2, f_3 and f_4 with f_1, f_2, f_3 bounded and f_4 unbounded. The graph $\mathscr{G}$ is drawn in grey to to highlight the edges of the dual graph $\mathscr{G}^*$ between the faces in full black.

If $\mathscr{G}$ is an oriented graph, then $\mathscr{G}^*$ becomes an oriented graph as follows: Let E^* denotes the set of dual edges and $\bar{E}^*$ the set of edges with opposite orientation. Then $\mathfrak{e} = (f_1, f_2) \in E^*$ if and only if rotating $\mathfrak{e}_1$ clockwise in the plane around its centre becomes an edge in E. Similarly, $\mathfrak{e} = (f_1, f_2) \in \bar{E}^*$ if rotating $\mathfrak{e}$ clockwise around its centre becomes an edge in $\bar{E}$, see also Figure 3.4. We note that this definition is opposite to the classical definition of the orientation of a dual graph, as found in standard literature like [BM07] or [GR01], but is in line with the convention chosen by many authors studying anyon excitations in lattice models, see [CCY21] or [Naa12] for instance.

A **site** s is a pair (v, f) where v is a vertex and f is a neighbouring face. For any given site $s = (v, f)$, we set $v(s) := v$ and $f(s) := f$ to be the face respectively vertex associated to s. If Λ is a region of $\mathscr{G}$, then we denote by $\mathcal{S}(\Lambda)$ the set of all sites of the subgraph $\mathscr{G}_\Lambda$, and we denote the set of all sites by $\mathcal{S}$. Similarly, we will say that a vertex v (a face f, a direct path p, a dual path p^*) lies in Λ if v, $(f,\ p,\ p^*)$ lies in $\mathscr{G}_\Lambda$. Note that if $s = (v_s, f_s) \in \mathcal{S}$ is a site in $\mathscr{G}$, then $s \in S(\Lambda)$ for a region Λ only if the four edges surrounding f_s are contained in Λ. In the particular case of the $\mathbb{Z}^2$ -lattice, we will call a face often times a **plaquette**.

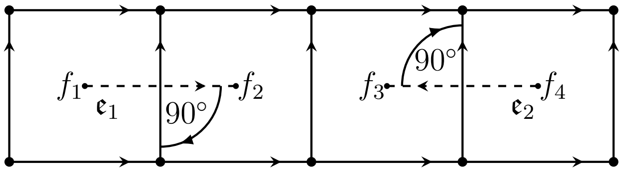


Figure 3.4: The figure depicts the two dual edges $\mathfrak{e}_1 = (f_1, f_2)$ and $\mathfrak{e}_2 = (f_4, f_3)$. $\mathfrak{e}_1$ is not aligned with the lattice, since rotating it by 90° clockwise about its centre makes it point downwards, whereas $\mathfrak{e}_2$ is aligned with the lattice since rotation by 90° clockwise about its centre makes it point upwards. With the specific orientation chosen in our setting for $\mathscr{G}$, i.e. horizontal edges pointing to the right and vertical edges pointing upwards, the edges of the dual graph $\mathscr{G}^*$ are oriented such that horizontal dual edges point to the left and vertical dual edges point upwards.

We will now introduce the notion of triangles in a graph. Triangles are used as atomic geometric shapes to define the dual and direct paths of a ribbon and we will define them using the notion of sites. The reader is advised to compare the definition of triangles with Figure 3.5.

Let $s_1, s_2 \in \mathcal{S}$ with $s_1 = (v_{s_1}, f_{s_1})$ and $s_2 = (v_{s_2}, f_{s_2})$ and write $\tau = (s_1, s_2)$. Then we call τ a **direct triangle** if $f_{s_1} = f_{s_2}$ and $(v_{s_1}, v_{s_2}) \in E \cup \bar{E}$ is an edge in $\mathscr{G}$ and we call τ a **dual triangle** if $v_{s_1} = v_{s_2}$ and $(f_{s_1}, f_{s_2}) \in E^* \cup \bar{E}^*$ forms an edge in the dual graph $\mathscr{G}^*$. If $\tau = (s_1, s_2)$ is direct (dual), we write $\mathfrak{e}_\tau = (v_{s_1}, v_{s_2})$ ($\mathfrak{e}_\tau = (f_{s_1}, f_{s_2})$) for the associated direct (dual) edge, and we call τ **aligned** (**not aligned**) if $\mathfrak{e}_\tau$ is oriented (not oriented). If $\tau = (s_1, s_2)$ is a direct triangle, we write $f(\tau) := f(s_1) = f(s_2)$. Similarly, if $\tau = (s_1, s_2)$ is a dual triangle, we write $v(\tau) = v(s_1) = v(s_2)$.

A direct (dual) triangle (s_1, s_2) is said to be oriented **locally counter-clockwise** if rotating the line segment of $s_1 = (v_{s_1}, f_{s_1})$ counter-clockwise around f_{s_1} in the plane swipes immediately through the interior of the triangle. Otherwise, we call its orientation **clockwise** and shall call this property **local orientation**. The local orientation of a triangle τ is independent of the alignment of the underlying edge $\mathfrak{e}_\tau$.

Remark 3.2.2. The notion of local orientation was introduced by [CCY21] and, to our knowledge, was not considered by previous authors, see e.g. [BMD08]. The importance of local orientation will become evident once we define ribbon operators, and we will discuss this issue in more detail then. For now, we only note that the definition in [BMD08] and previous authors lead in the setting of non-abelian Kitaev models occasionally to incorrect results.

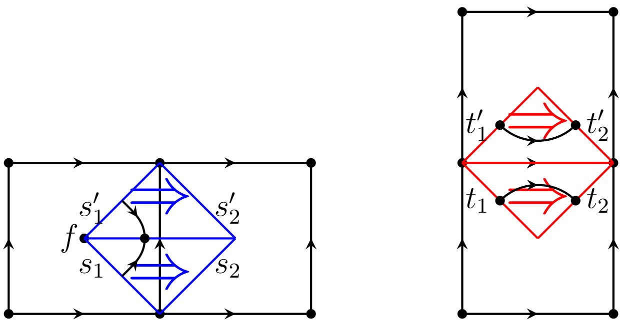


Figure 3.5: The figure depicts dual and direct triangles with different local orientations, even though all triangles have the same alignment with the lattice. The pairs (s_1, s_2) as well as $(s'_1 s'_2)$ both constitute non-aligned dual triangles (depicted in blue) which can be seen by rotating the blue $\Rightarrow$ -arrow clockwise by 90 degree. However, rotating the line segment given by s_1 around the face f must be performed counter-clockwise to swipe through the interior of the triangle (s_1, s_2) first. Similarly, (s'_1, s'_2) can be seen to be oriented locally clockwise, since s'_1 must be rotated clockwise around f to immediately swipe through the interior of (s'_1, s'_2) first. For the direct triangles (t_1, t_2) and (t'_1, t'_2) in red, we see similarly that (t_1, t_2) is oriented locally clockwise, while (t'_1, t'_2) is oriented locally counter-clockwise. Note that in all four cases the rotation is performed around the face associated to the initial site of the respective triangle. Finally, we remark that the line segments identified with the sites together with the direct/dual edge that connect the different vertices/faces of the sites indeed form the shape of a triangle for each of the cases drawn.

We write $\partial_0 \tau = s_1$ and $\partial_1 \tau_1 = s_2$ for a triangle $\tau = (s_1, s_2)$. Given two triangles τ_1 and τ_2, we say that τ_1 **is composable with** τ_2 if $\tau_1 \neq \tau_2$, $\partial_1 \tau_1 = \partial_0 \tau_2$ and if τ_1 and τ_2 have the same local orientation. It follows that τ_1 is composable with τ_2 only if τ_1 and τ_2 intersect at their sites, and non-composable triangles τ_1 and τ_2 with $\partial_1 \tau_1 = \partial_0 \tau_2$ always intersect non trivially. This can be seen as follows: If $\tau_1 = (s_1, s_2)$ is locally clockwise oriented, then rotating s_1 clockwise around its face swipes first through the interior of the triangle τ_1. One may check in that case that rotating s_2 counter-clockwise around its face then swipes first through the interior of τ_1. If s_2 was the initial site of a triangle τ_2 that was oriented locally counter-clockwise, then by definition the counter-clockwise rotation of s_2 around its face swipes through the interior of τ_2 as well, implying that τ_1 and τ_2 overlap. See also Figure 3.6.

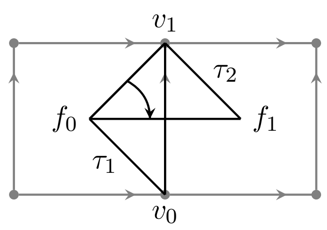


Figure 3.6: Depiction of two overlapping triangles $\tau_1 = ((v_0, f_0), (v_1, f_0))$ and $\tau_2 = ((v_1, f_0), (v_1, f_1))$.

A similar argument demonstrates that if τ_1 and τ_2 had the same local orientation, that then τ_1 and τ_2 shared no common area.

A **ribbon** is a concatenation of composable triangles, $(\tau_1, \ldots, \tau_n)$ such that $\mathfrak{e}_{\tau_i} \neq \mathfrak{e}_{\tau_k}$ for all $i, k = 1, \ldots, n$ with $i \neq k$. We say that a ribbon $\xi = (\tau_1, \ldots, \tau_n)$ is contained in a region Λ if $\mathscr{G}_\Lambda$ contains all edges $\mathfrak{e}_{\tau_i}$, $i = 1, \ldots, n$. A ribbon $\xi_1 = (\tau_1^{(1)}, \ldots, \tau_n^{(1)})$ is said to be **composable** with a ribbon $\xi_2 = (\tau_1^{(2)}, \ldots, \tau_n^{(2)})$ if $(\tau_1^{(1)}, \ldots, \tau_n^{(1)}, \tau_2^{(1)}, \ldots, \tau_2^{(n)})$ forms a ribbon again.

We may occasionally write a ribbon as $\xi = (s_1, \ldots, s_n)$, by which we mean $\xi = (\tau_1, \ldots, \tau_{n-1})$, where s_i are sites and $\tau_i = (s_i, s_{i+1})$. The sites s_1 and s_n of a ribbon $\xi = (s_1, \ldots, s_n)$ are called **endpoints** of ξ, and we write $\partial_0 \xi = s_1$ and $\partial_1 \xi = s_n$. We call a ribbon $(\tau_1, \ldots, \tau_n)$ **locally clockwise** (**locally counter-clockwise**) oriented if one (and hence all) of its triangles are locally clockwise (locally counter-clockwise) oriented. If a ribbon ξ consists only of direct (dual) triangles, we call ξ **direct** (**dual**).

A ribbon can be seen as a pair of a direct path and a dual path: Given a ribbon $\xi = (\tau_1, \dots, \tau_n)$, we call the path obtained from all edges of the direct triangles as **direct path of** ξ and denote it by ξ^{di}. Similarly, we call the concatenation of the faces of the dual triangles of ξ the **dual path of** ξ and denote it by ξ^{du}, see Figure 3.7. As always, we say that a ribbon (direct

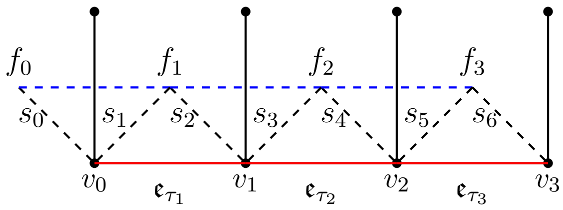


Figure 3.7: Depiction of a ribbon $\xi = (s_0, s_1, s_2, s_3, s_4, s_5, s_6)$ with sites $s_0 = (v_0, f_0), s_1 = (v_0, f_1), s_2 = (v_1, f_1)$, $s_3 = (v_1, f_2)$, $s_4 = (v_2, f_2)$, $s_5 = (v_2, f_3)$, $s_6 = (v_3, f_3)$, dual path $\xi^{du} = (f_0, f_1, f_2, f_3)$ (blue and dashed) and direct path $\xi^{di} = (v_0, v_1, v_2, v_3)$ (red).

triangle, dual triangle) lies in a region Λ if the ribbon (direct triangle, dual triangle) lies in $\mathscr{G}_\Lambda$.

Finally, we will denote the inverse of a ribbon, triangle, path, edge or group element of some finite group G by a bar $\bar{\cdot}$, i.e.

(i) if $\tau = (s_1, s_2)$ is a triangle, then $\bar{\tau} = (s_2, s_1)$,

(ii) if $\xi = (s_1, \dots, s_n)$ is a ribbon, then $\bar{\xi} = (s_n, \dots, s_1) = (\bar{\tau}_{n-1}, \dots, \bar{\tau}_1)$ with $\tau_i = (s_i, s_{i+1})$,

(iii) if $p = (v_1, \dots, v_n)$ is a path, then $\bar{p} = (v_n, \dots, v_1)$,

(iv) if $\mathfrak{e} = (v_1, v_2)$, then $\bar{\mathfrak{e}} = (v_2, v_1)$,

(v) if G is a finite group and $g \in G$, then $\bar{g} = g^{-1}$.

Having established this terminology, we return to the quantum spin system introduced at the beginning of this section. We can define actions of the group algebra $\mathbb{C}G$ and its dual $\mathbb{C}(G) = (\mathbb{C}G)^*$ on the lattice model for each triangle. For $h \in G$, $\delta_k \in \mathbb{C}(G)$ and τ a direct triangle, we define the projections

$$T_\tau^k(g) = \begin{cases} \delta_{\bar{k},g}\, g & \text{if } \tau \text{ is aligned} \\ \delta_{k,g}\, g & \text{otherwise} \end{cases} \tag{3.2.2}$$

acting on the edge $\mathfrak{e}_\tau$ and if $\tilde{\tau}$ is a dual triangle, we define left actions on the edge intersecting the dual edge $\mathfrak{e}_{\tilde{\tau}}$ via:

$$L^h_{\tilde{\tau}}(g) = \begin{cases} L_h : g \mapsto hg & \text{if } \tilde{\tau} \text{ is aligned} \\ R_{\bar{h}} : g \mapsto g\bar{h} & \text{otherwise} \end{cases} \tag{3.2.3}$$

if $\tilde{\tau}$ is locally clockwise oriented, and right actions

$$L^h_{\tilde{\tau}}(g) = \begin{cases} R^h : g \mapsto gh & \text{if } \tilde{\tau} \text{ is aligned} \\ L^{\bar{h}} : g \mapsto \bar{h}g & \text{otherwise} \end{cases} \tag{3.2.4}$$

If $\tilde{\tau}$ is locally counter-clockwise oriented. See also Figure 3.8

Remark 3.2.3. We will see later that dual- and direct triangles create pairs of excitations at the two sites defining the respective triangle. The reason for defining the action of the triangles differently for different local orientations, is to ensure that the type of excitation created is always the same, regardless of the orientation of the triangle as long as the sites on which the charges are considered are the same.

One might wonder if such a distinction is necessary for direct triangles as well. This matter is discussed in [CCY21] in more detail, but it is essentially related to the fact that the dual $\mathbb{C}(G)$ of the group algebra $\mathbb{C}G$ is commutative (since $\mathbb{C}G$ is cocommutative) making a distinction between direct triangles of different local orientation redundant for the quantum double model based on groups.

As mentioned, the triangle operators are the atomic components for constructing charges in our quantum spin system. It is therefore worthwhile to inspect their commutation relations more closely. Let τ_1 and τ_2 be triangles, such that the corresponding triangle operators act on the same edge. Clearly, $\left[T^{k_1}_{\tau_1}, T^{k}_{\tau_2}\right] = 0$ for all $k_1, k_2 \in G$, regardless of the alignments of τ_1 and τ_2. Assume τ_1 and τ_2 are dual triangles that have the same local orientation and opposite alignment. Then either L_{τ_1} acts via left-inverse multiplication and L_{τ_2} acts via right-inverse multiplication or vice versa. In either case, we have $\left[L^{h_1}_{\tau_1}, L^{h_2}_{\tau_2}\right] = 0$ for all $h_1, h_2 \in G$ due to the associativity of G. Similarly, if τ_1 and τ_2 are either both aligned with the lattice or both not aligned with the lattice, but have opposite local orientation, then $\left[L^{h_1}_{\tau_1}, L^{h_2}_{\tau_2}\right] = 0$, again due to the associativity of G.

The only constellations not yet considered are the ones where τ_1 and τ_2 have the same alignment and the same local orientation, and where τ_1 and τ_2 have opposite alignment and opposite local orientation. In the first case, $\tau := \tau_1 = \tau_2$ must already be identical, and we have $L^{h_1}_{\tau_1} L^{h_2}_{\tau_2} = L^{h_1 h_2}_{\tau}$. If the local orientation and the alignment are opposite, then L_{τ_1} and L_{τ_2} act from

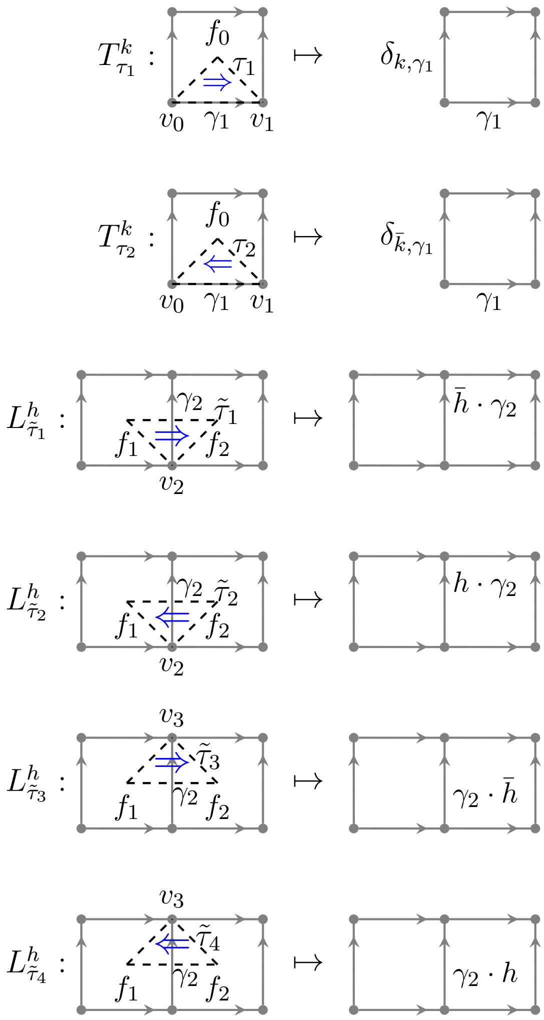


Figure 3.8: Graphical depiction of the action of the triangle operators defined in Equation (3.2.2), Equation (3.2.3) and Equation (3.2.4). The direct triangle $\tau_1 = ((v_0, f_0), (v_1, f_0))$ is aligned, whereas the direct triangle $\tau_2 = ((v_1, f_0), (v_0, f_0))$ is not. The dual triangles $\tilde{\tau}_1 = ((v_2, f_1), (v_2, f_2))$, $\tilde{\tau}_2 = ((v_2, f_2), (v_2, f_1))$, $\tilde{\tau}_3 = ((v_3, f_1), (v_3, f_2))$ and $\tilde{\tau}_4 = ((v_3, f_2), (v_3, f_1))$ are respectively locally counter-clockwise oriented and not aligned, locally clockwise oriented and aligned, locally clockwise oriented and not aligned and locally counter-clockwise oriented and aligned.

the same side, but one acts via inverse multiplication and the other one does not. In that case, one may verify that $L^{h_1}_{\tau_1} L^{h_2}_{\tau_2} = L^{h_1\bar{h}_2}_{\tau_1}$ for $h_1, h_2 \in G$ if τ_1 is locally clockwise oriented, and $L^{h_1}_{\tau_1} L^{h_2}_{\tau_2} = L^{\bar{h}_1 h_2}_{\tau_2}$ if τ_2 is locally clockwise oriented. We summarize these identities in a Lemma.

Lemma 3.2.4. *Let τ_1, τ_2 be dual triangles and $h_1, h_2 \in G$. If τ_1, τ_2 have either the same local orientation and opposite alignment or opposite local orientation and the same alignment, then*

$$\left[L^{h_1}_{\tau_1}, L^{h_2}_{\tau_2}\right] = 0.$$

If τ_1 and τ_2 have the same local orientation and the same alignment, then $\tau_1 = \tau_2$ and we have

$$L^{h_1}_{\tau_1} L^{h_2}_{\tau_2} = L^{h_1 h_2}_{\tau_1}.$$

Finally, if τ_1 and τ_2 have opposite local orientation and opposite alignment, we have $\tau_1 = \bar{\tau}_2$ and

$$L^{h}_{\tau_1} L^{h}_{\tau_2} = \begin{cases} L^{h_1 \bar{h}_2}_{\tau_1} & \text{if } \tau_1 \text{ is locally clockwise oriented} \\ L^{\bar{h}_1 h_2}_{\tau_1} & \text{otherwise,} \end{cases} \tag{3.2.5}$$

for all $h_1, h_2 \in G$.

This leaves the commutation relations between dual- and direct triangle operators to inspect, of which there are still 8 possible combinations to consider: Two scenarios for the direct triangles and four scenarios for the dual triangles. However, some of these cases can be dealt with simultaneously by considering the relative alignment of the triangles:

Let τ_1 and τ_2 be a direct and a dual triangle respectively, with either $\mathfrak{e}_{\tau_1} = \mathfrak{e}_{\tau_1}$ or $\mathfrak{e}_{\tau_1} = \bar{\mathfrak{e}}_{\tau_1}$. Then we have the following commutation relations:

$$L^{h}_{\tau_1} T^{g}_{\tau_2} = \begin{cases} T^{hg}_{\tau_2} L^{h}_{\tau_1} & \text{if } \mathfrak{e}_{\tau_1} = \mathfrak{e}_{\tau_2} \text{ and } \tau_1 \text{ is oriented clockwise,} \\ T^{g\bar{h}}_{\tau_2} L^{h}_{\tau_1} & \text{if } \mathfrak{e}_{\tau_1} = \mathfrak{e}_{\tau_2} \text{ and } \tau_1 \text{ is oriented counter-clockwise,} \\ T^{gh}_{\tau_2} L^{h}_{\tau_1} & \text{if } \mathfrak{e}_{\tau_1} = \bar{\mathfrak{e}}_{\tau_2} \text{ and } \tau_1 \text{ is oriented clockwise,} \\ T^{\bar{h}g}_{\tau_2} L^{h}_{\tau_1} & \text{if } \mathfrak{e}_{\tau_1} = \bar{\mathfrak{e}}_{\tau_2} \text{ and } \tau_1 \text{ is oriented counter-clockwise.} \end{cases}$$

Proof. Let $p : G \to G$ be a map and denote by $\tilde{p} : \mathbb{C}G \to \mathbb{C}G$ its linear extension to $\mathbb{C}G$. If τ_2 is aligned, then

$$(\tilde{p} \circ T^{g}_{\tau_2})(k) = \delta_{g,k} p(k) = \delta_{p(g),p(k)} p(k) = (T^{p(g)}_{\tau_2} \circ \tilde{p})(k),$$

and if τ_2 is not aligned and $q : G \to G$, a map with extension $\tilde{q} : \mathbb{C}G \to \mathbb{C}G$ such that it satisfies $p(\bar{k}) = \overline{q(k)}$, we have

$$(\tilde{p} \circ T^{g}_{\tau_2})(k) = \delta_{\bar{g},k} p(k) = \delta_{p(\bar{g}),p(k)} p(k) = \delta_{\overline{q(g)},p(k)} p(k) = (T^{q(g)}_{\tau_2} \circ \tilde{p})(k).$$

If τ_2 is aligned, the result then follows by substituting $p = L^h_{\tau_1}$. If τ_2 is not aligned note that

$$\overline{L^h_{\tau_1}(\bar{k})} = \begin{cases} \overline{\bar{k}\bar{h}} = hk = L^h(k) & \text{if } \tau_1 \text{ is not aligned and locally clockwise oriented,} \\ \overline{\bar{h}\bar{k}} = kh = R^h(k) & \text{if } \tau_1 \text{ is not aligned and locally counter-clockwise oriented,} \\ \overline{h\bar{k}} = k\bar{h} = R^{\bar{h}}(k) & \text{if } \tau_1 \text{ is aligned and locally clockwise oriented,} \\ \overline{\bar{k}h} = \bar{h}k = L^{\bar{h}}(k) & \text{if } \tau_1 \text{ is aligned and locally counter-clockwise oriented,} \end{cases} \tag{3.2.6}$$

and we can choose $q(k)$ to be the right-hand side of Equation (3.2.6) in each case to obtain the result.

□

So far, we have been focusing on describing the geometric structure of the lattice. To inspect how the operators just defined can be used to create charges in the model, we need to shift our attention to the Hilbert spaces $\mathcal{H}_\Lambda$ of the regions $\Lambda \subset E$. A useful basis is given by the elementary tensors

$$C_G(\Lambda) := \left\{ \bigotimes_{\mathfrak{e}\in\Lambda} \gamma_\mathfrak{e} \mid \gamma_\mathfrak{e} \in G \right\} \subset \mathcal{H}_\Lambda.$$

Elements in $C_G(\Lambda)$ are in one-to-one correspondence with maps $\Lambda \to G$; Clearly, an elementary tensor $\gamma \in C_G(\Lambda)$ defines a map $\gamma : \Lambda \to G$ by setting $\gamma(\mathfrak{e}) = \gamma_\mathfrak{e}$ for an edge $\mathfrak{e} \in E$. On the other hand, given a map $\gamma : \Lambda \to G$, we can identify γ as an element of $C_G(\Lambda)$ via the identification

$$\gamma \mapsto \bigotimes_{\mathfrak{e}\in\Lambda} \gamma(\mathfrak{e}).$$

We will interchangeably view $\gamma \in C_G(\Lambda)$ either as an elementary tensor or a map from Λ to G. We call the elements in $C_G(\Lambda)$ G-**connections**. If $\mathfrak{e}^*$ is a dual edge and $\mathfrak{e}$ the unique edge crossing $\mathfrak{e}$, we define

$$\gamma(\mathfrak{e}^*) := \gamma(\mathfrak{e}). \tag{3.2.7}$$

This definition is useful when considering the value of an expression of the form $L^h_\tau\gamma$ at the edge crossing the dual edge $\mathfrak{e}_\tau$, since we can then simply write $(L^h_\tau\gamma)(\mathfrak{e}_\tau)$ without passing to the corresponding edge crossing $\mathfrak{e}_\tau$ first.

We now want to define a map that *measures* the charge of a G-connection.

Definition 3.2.5. Let $p = (v_1, \ldots, v_n)$ be a path. Then we define the **charge measure** $\beta^{(p)}$ as follows: If $p = \{v_1, v_2\}$ consists of a single edge $\mathfrak{e} := (v_1, v_2)$, we set

$$\beta^{(p)}(\gamma) = \begin{cases} \gamma(\mathfrak{e}) & \text{if } \mathfrak{e} \in E, \\ \overline{(\gamma(\bar{\mathfrak{e}}))} & \text{if } \mathfrak{e} \in \bar{E}. \end{cases} \tag{3.2.8}$$

If $p = (p_1, p_2)$ is a concatenation of two composable paths p_1 and p_2, we set

$$\beta^{(p)}(\gamma) = \beta^{(p_1)}(\gamma)\beta^{(p_2)}(\gamma), \tag{3.2.9}$$

and extend $\beta^{(p)}$ linearly to $\mathcal{H}_\Lambda$. If p is a closed path and $\psi \in \mathcal{H}_\Lambda$, then we call $\beta^{(p)}(\psi)$ the **monodromy** or **magnetic flux/charge** of ψ in the area enclosed by p. Otherwise, we call $\beta^{(p)}(\psi)$ the **β-value of ψ along** p. If p is closed, we call the monodromy of a G-connection γ **trivial** if $\beta^{(p)}(\psi)$ is the identity element $e \in G$.

Note that β is independent of the choice of Λ, that is, if $\Lambda \subset \Lambda'$ for some larger bounded region Λ', we can extend γ arbitrarily to a G-connection on Λ' without changing the value of $\beta^{(p)}(\gamma)$ for any path p contained in Λ.

We are now in a position to define the interaction terms of the quantum spin system. The electric and magnetic charge operators of the system are defined as follows: Let Λ be a finite region of the lattice and $\mathfrak{A}_\Lambda = \mathfrak{B}(\mathcal{H}_\Lambda)$ as before and let v be a vertex such that all edges connecting to v in $\mathscr{G}$ are contained in Λ, i.e. for all edges $\mathfrak{e} \in E$ with $v \in \partial\mathfrak{e}$ we have $\mathfrak{e} \in \Lambda$. If $s_1 = (v, f) \in \mathcal{S}$ is a site with vertex v, we let $\xi_{s_1} = (\tau_1, \ldots, \tau_4)$ be the smallest clockwise oriented ribbon around v starting and ending at the site s_1 (See Figure 3.10) and we define the operator

$$A^k_{s_1} = \bigotimes_{i=1}^{4} L^k_{\tau_i}. \tag{3.2.10}$$

We call the operator $A^k_{s_1}$ **vertex operator**, **star operator** or **electric charge operator** at site s_1. The term *star* is related to the star-shaped domain of $A^k_{s_1}$, see also Figure 3.9. We will also define the operator

$$A_s = \frac{1}{|G|}\sum_{k\in G} A^k_s. \tag{3.2.11}$$

for any site $s \in \mathcal{S}$ and call it the **projection into the trivial electric charge** at the site s.

Next, let f be a face in Λ, i.e. $f \in \mathscr{G}_\Lambda$, and s_2 a site with f as a face. Set $\xi_{s_2} = (\tau_1, \ldots, \tau_4)$ to be the smallest locally counter-clockwise oriented direct ribbon around f starting and ending at s_2, and let $\varphi \in \mathbb{C}(G)$ be an element in the dual of $\mathbb{C}G$. The **plaquette operator** or **magnetic charge operator** B^φ_s at site $s = (v, f)$ is defined as

$$B^\varphi_s = \sum_{(\varphi)} T^{\varphi^{(1)}}_{\tau_1} \otimes T^{\varphi^{(2)}}_{\tau_2} \otimes T^{\varphi^{(3)}}_{\tau_3} \otimes T^{\varphi^{(4)}}_{\tau_4}, \tag{3.2.12}$$

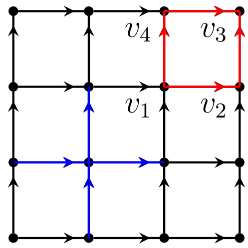


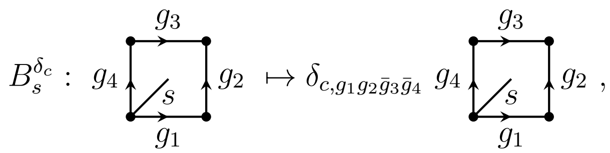
Figure 3.9: Depiction of the star(blue) and plaquette(red) shaped domain of the star and plaquette operator. If γ is a G-connection on this region and $\gamma_1, \ldots, \gamma_4 \in G$ such that $\gamma_i := \gamma(\mathfrak{e}_i)$ and edges $\mathfrak{e}_1 = (v_1, v_2), \mathfrak{e}_2 = (v_2, v_3), \mathfrak{e}_3 = (v_3, v_3)$ and $\mathfrak{e}_4 = (v_4, v_1)$, then the action of the plaquette operator can alternatively be described via $\gamma \mapsto \delta_e(\beta^p(\gamma)) = \delta_e(\gamma_1\gamma_2\bar{\gamma}_3\bar{\gamma}_4) = \delta_{e,\gamma_1,\gamma_2,\bar{\gamma}_3\bar{\gamma}_4}$.

where we used the Sweedler notation introduced in Equation (2.3.4).

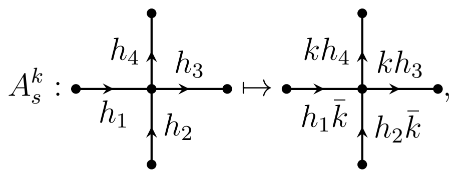
We want to add an alternative way of defining the plaquette operator B_s^φ: Let $\gamma \in G_C(\Lambda)$ be a G-connection on some bounded region Λ containing the site $s = (v, f)$ and p_s the smallest path starting and ending at v and moving counter-clockwise around f (compare with Figure 3.10 again). Then B_s^φ coincides with the linear extension of the mapping

$$B_s^\varphi : \gamma \mapsto \varphi(\beta^{(p_s)}(\gamma))\gamma. \tag{3.2.13}$$

If $\varphi = \delta_c$ for some $c \in G$ and $k \in G$, we can depict the plaquette operator as



and the star operator as



where $h_i, g_i \in G$ are the group elements sitting at the depicted edges. Let $\mathcal{C} \subset G$ be a conjugacy class of G and $c \in \mathcal{C}$. We say that an element $\psi \in \mathcal{H}_\Lambda$ has a **magnetic flux of type** $\mathcal{C}$ at site s if $B_s^{\delta_c}(\psi) \neq 0$. It is easy to verify that the star operators A_s^k leave the type of magnetic charge at the site s invariant. Indeed, we have the following commutation relation

$$B_s^{\delta_g} A_s^k = A_s^k B_s^{\delta_{\bar{k}gk}}. \tag{3.2.14}$$

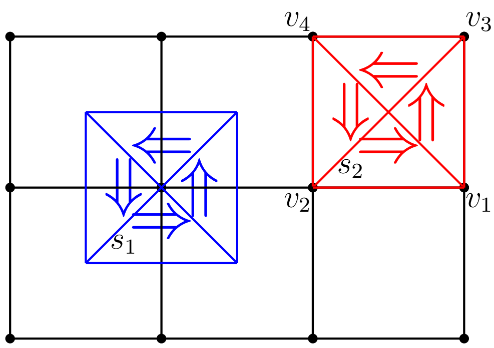


Figure 3.10: Picture of a locally clockwise oriented dual ribbon around a vertex starting and ending at site s_1 (blue) and a locally counter-clockwise oriented direct ribbon around a face starting and ending at site s_2(red). The $\Rightarrow$ symbol is used to indicate the direction of the edges associated to the triangles, as explained at the beginning of this Section, see also Figure 3.8. The ribbon operator associated to the blue ribbon depicts the star operator at site s_1, whereas the ribbon operator associated to the red ribbon depicts the plaquette operator at site s_2. The plaquette operator can alternatively be described by $\gamma \mapsto \delta_e(\beta^p(\gamma))\gamma$ for the path $p = (v_1, v_2, v_3, v_4, v_1)$.

Hence, if $\mathcal{C}$ is a conjugacy class in G and $\chi_{\mathcal{C}} : G \to \mathbb{C}$ the characteristic function on $\mathcal{C}$, then

$$\left[A_s^k, B_s^{\chi_{\mathcal{C}}}\right] = 0$$

for all $k \in G$.

Since we will rarely use any functions on G other than delta functions in the argument of the plaquette operators, we will ease our notation by writing B_s^g instead of $B_s^{\delta_g}$ from now on.

Given a site $s = (v, f)$, we will write star(s) to denote the unique star shaped region with the vertex v at its centre. Similarly, plaq(s) denotes the unique plaquette shaped region with f at its centre.

Note that the star operator, unlike the plaquette operator, really only depends on the vertex v and not on the full data of the site $s = (v, f)$, and we may identify $A_s^k = A_v^k$ at times. For the plaquette operator, however, we indeed need to know the site and not just the face to determine its action. If v and v' are different vertices such that $s = (v, f)$ and $s' = (v', f)$ share the same face, one can verify that $B_s^c = B_{s'}^{c'}$ for some c' in the conjugacy class of c. The reason for labelling both the star- and plaquette operator with the site s will become clear in Section 3.4. See in particular Proposition 3.4.1

We want to pay special attention to the trivial conjugacy class $\mathcal{C} = \{e\}$ and call the operator

$$B_s := B_s^e \tag{3.2.15}$$

the **projection into the trivial magnetic charge** at site s. If $\gamma \in C_G(\Lambda)$ is a G-connection, then we call it **flat** if the monodromy around each plaquette in Λ is trivial, and we denote the set of flat G-connections on Λ by $C_G^f(\Lambda)$. Note also that the definition of the star operator still makes sense when not all four edges of the star are inside the region Λ, whereas the plaquette operator only acts on sites where the full plaquette is contained in Λ. Since we will only really be concerned with square-shaped regions, this will pose no restriction.

The dynamics of the quantum double model are given by the interaction

$$\phi(\Lambda) = \begin{cases} 1 - A_s & \text{if } \Lambda = \text{star}(s) \text{ for some site } s \in \mathcal{S}, \\ 1 - B_s & \text{if } \Lambda = \text{plaq}(s) \text{ for some site } s \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

For a finite region $\Lambda \subset E$, we define the local Hamiltonian H_Λ as in Equation (2.5.7) to be:

$$H_\Lambda = \sum_{\text{star}(s) \subset \Lambda} (1 - A_s) + \sum_{\text{plaq}(s) \subset \Lambda} (1 - B_s). \tag{3.2.16}$$

The local time evolution is given by Equation (2.5.8) as

$$\tau_\Lambda^t : \mathfrak{A} \ni A \mapsto \exp(itH_\Lambda) A \exp(-itH_\Lambda).$$

Clearly, the interaction Φ is translationally invariant and uniformly bounded. By Theorem 2.5.21 and subsequent discussion, it follows that $\tau_t^{\Lambda,\phi}$ converges for all $A \in \mathfrak{A}$ uniformly on compact sets in $\mathbb{R}$ to a time evolution τ_t^ϕ, which we will just denote by τ, with infinitesimal generator δ. This model is called **Kitaev's Quantum double model of** G.

The rest of this section is dedicated to showing the existence of a unique frustration free ground state ω_0. We remind the reader that frustration freeness means here that $\omega_0(A_s) = \omega_0(B_s) = 1$.

Proposition 3.2.6 ([Naa12], see also [FN15, CNN16])**.** *Kitaev's Quantum double model admits a translational invariant ground state $\omega_0 : \mathfrak{A} \to \mathbb{C}$ uniquely defined by the equations*

$$\omega_0(A_s) = 1 \tag{3.2.17}$$

$$\omega_0(B_s) = 1 \tag{3.2.18}$$

on all sites s and the corresponding GNS representation $(\pi_{\omega_0}, \mathcal{H}_{\omega_0}, \Omega_{\omega_0})$ is faithful. Furthermore, if Λ_n is a square-shaped region of size $n \in \mathbb{N}$, then

$$\omega_0(T^\gamma) = \begin{cases} \frac{1}{|C_G^f(\Lambda_n)|} & \text{if } \gamma \in C_G^f(\Lambda_n), \\ 0 & \text{otherwise,} \end{cases}$$

where T^γ is the projection defined in Equation (3.2.22).

Note that being translationally invariant means in particular that $\omega_0(A_{s_1}) = \omega_0(A_{s_2})$ and $\omega_0(B_{s_1}) = \omega_0(B_{s_2})$ for all sites $s_1, s_2 \in \mathcal{S}$ since A_{s_1} and B_{s_1} can be transformed into A_{s_2} and B_{s_2} by translation. Hence, Equation (3.2.17) and (3.2.18) are equivalent to saying that ω_0 is translationally invariant and that $\omega_0(H_\Lambda) = 0$ for every bounded region Λ.

Equations (3.2.17) and (3.2.17) imply together with Lemma 2.5.13

$$\omega_0(X) = \omega_0(A_s X) = \omega_0(X A_s) \tag{3.2.19}$$

$$\omega_0(X) = \omega_0(B_s X) = \omega_0(X B_s) \tag{3.2.20}$$

for all sites s and $X \in \mathfrak{A}$. Furthermore, because of $A_s^k A_s = A_s$, we also have

$$\omega_0(X) = \omega_0(A_s^k X) = \omega_0(X A_s^k). \tag{3.2.21}$$

We will use these identities frequently throughout this work.

Although a proof of Proposition 3.2.6 can already be found in [Naa12], we will include a sketch of the proof here, as we will use similar arguments later in the proof of Theorem 3.6.7.

Before we attempt to prove Proposition 3.2.6, we want to develop some more physical intuition. In view of Equation (3.2.20), we should think of a ground state as a state that has trivial magnetic charge around each plaquette. In view of Equation (3.2.19), which is equivalent to Equation (3.2.21), and knowing that star operators A_s^k for $k \in G$ permute G-connections without changing the type of the magnetic charge at each plaquette, we would like to think of the vacuum state as the equal weighted superposition of all states with trivial magnetic charges. To really understand what this means, we need to lift our notions of magnetic charges to that of operators on $\mathfrak{A}$ first.

Given a G-connection $\gamma : \Lambda \to G$, we define an operator $T^\gamma \in \mathfrak{A}_\Lambda$ via

$$T_\Lambda^\gamma = \prod_{\mathfrak{e} \in \Lambda} T_{\tau_\mathfrak{e}}^{\gamma(\mathfrak{e})}, \tag{3.2.22}$$

where $\tau_{\mathfrak{e}}$ is the direct triangle aligned with $\mathfrak{e}$. If $p = (v_1, \ldots, v_n)$ is a path in Λ and $\mathfrak{e}_i = (v_i, v_{i+1})$, $i = 1, \ldots, n-1$, we set similarly

$$T_p^\gamma = \prod_{\mathfrak{e}_i} T_{\tau_{\mathfrak{e}_i}}^{\gamma(\mathfrak{e}_i)}. \tag{3.2.23}$$

Note that in the latter case, $\mathfrak{e}_i$ might be an edge with reversed direction if p is not aligned. Finally, if $g \in G$, and $p, \mathfrak{e}_i$ as before, we define

$$T_p^g = \sum_{\substack{\gamma \in C_G(\Lambda) \\ \beta^{(p)}(\gamma) = g}} T_p^\gamma. \tag{3.2.24}$$

Note that $T_p^g \gamma = \gamma$ only if γ has β-value g along p and $T_p^g \gamma = 0$ otherwise.

We want to distinguish G-connections by the magnetic charges they represent at each site. To that end, let Λ be a bounded region and $\kappa : S(\Lambda) \to G$ be a map that associates to each site some element in G. Given a G-connection $\gamma \in C_G(\Lambda)$, we can construct a map $\kappa_\gamma : S(\Lambda) \to G$ via

$$\kappa_\gamma(s) = \beta^{(p_s)}(\gamma) \in G,$$

where p_s is the path defined as in Equation (3.2.13). We say that two G-connections $\gamma_1, \gamma_2 \in C_G(\Lambda)$ exhibit the **same magnetic charges at site** s, written $\gamma_1 \sim_s \gamma_2$, if $\kappa_{\gamma_1}(s) = \kappa_{\gamma_2}(s)$. This is an equivalence relation, and the next lemma states that we can transform equivalent G-connections to one another using star operators only.

Lemma 3.2.7. *Let Λ_n be a square-shaped region of size $n \geq 1$, $s_0 = (v_0, f_0) \in \Lambda$ a site and $\gamma_1, \gamma_2 \in C_G(\Lambda_{n+1})$ G-connections with $\gamma_1 \sim_{s_0} \gamma_2$. If the magnetic flux of γ_1 and γ_2 is trivial on all sites $s = (v, f)$ with $f \neq f_0$, then there exists a finite sequence of star operators $\left\{A_{s_i}^{k_i}\right\}_i$ with $k_i \in G$ and $s_i \in \Lambda_{n+1}$ such that*

$$\prod_i A_{s_i}^{k_i} \gamma_1(\mathfrak{e}) = \gamma_2(\mathfrak{e}).$$

for all $\mathfrak{e} \in \Lambda_n$. Furthermore, the sequence $\left\{A_{s_i}^{k_i}\right\}_i$ can be chosen such that $s_i \neq s_0$.

This lemma and its proof is motivated by [Naa12, Lem 12.1.2], and the intuition behind the lemma in [Naa12] is that G-connections that have trivial magnetic charge at each plaquette can be permuted into one another using star operators only. What is new here, and perhaps somewhat surprising at first glance, is that we claim such a permutation can be achieved even when refraining from using vertex operators at a fixed vertex v_0. The implication is that if a G-connection has a single non-trivial magnetic charge at some site s_0, then this G-connection can be permuted to any other G-connection with the same charges at each plaquette, since such a permutation can be achieved while using star operators A_s^k with $s \neq s_0$. This will be used in the proof of Theorem 3.6.7 later as well.

Proof of Lemma 3.2.7. We will provide an algorithmic construction to transform all edges in Λ_n using star operators at sites $s \neq s_0$. Let $s_0 = (v_0, f_0)$ and v w.l.o.g. in the *lower-right* corner of f_0 see Figure 3.11.

Let $\{v_i\}_i$ be a finite sequence of vertices constructed as follows: Set p_1 to be a path starting at v_0 and moving horizontally to the right until we reach the right boundary of Λ_n and write $p_1 = (v_0, v_1, \ldots, v_{i_1})$ with some $i_1 \in \mathbb{N}$. Set v_{i_1+1} to be the vertex right above v_{i_1} and let $p_2 = (v_{i_1+1}v_{i_1+2}, \ldots, v_{i_2})$ be the path going horizontally to left until the left boundary of Λ_n is reached again. If v_{i_2} is right above v_0, we stop. Otherwise, let v_{i_2+1} be the vertex below v_{i_2} and let $p_3 = (v_{i_2+1}, \ldots, v_0)$ be the path moving horizontally to the right until we reach v_0 again. This forms a closed direct path $p = (p_1, p_2, p_3) = (v_0, v_1, \ldots, v_j, v_0)$ starting and ending at v_0. We will first show that using vertex operators at $v_1, \ldots, v_j$ we can transform the values of the G-connection γ_1 along the path p to the values of γ_2. Let $\mathfrak{e}_i = (v_{i-1}, v_i)$ for $i = 1, \ldots, j$ and define $k_i \in G$ recursively as follows: If we set $k_1 = \overline{\gamma_2(\mathfrak{e}_1)}\gamma_1(\mathfrak{e}_1)$, then

$$(A_{v_1}^{k_1}\gamma_1)(\mathfrak{e}_1) = \gamma_1(\mathfrak{e}_1)\bar{k}_1 = \gamma_2(\mathfrak{e}_1)\overline{\gamma_1(\mathfrak{e}_1)}\gamma_2(\mathfrak{e}_1) = \gamma_2(\mathfrak{e}_1).$$

Giving $\gamma_1^{(1)}(\mathfrak{e}_1) = \gamma_2(\mathfrak{e}_1)$ for $\gamma_1^{(1)} := A_{v_1}^{k_1}\gamma_1$. We then proceed inductively by setting $k_i = \overline{\gamma_2(\mathfrak{e}_i)}\gamma_1^{(i-1)}(\mathfrak{e}_i)$, $\mathfrak{e}_i \in E$ and $k_i = \gamma_2(\mathfrak{e}_i)\overline{\gamma_1^{(i-1)}}$ if $\mathfrak{e} \in \bar{E}$. Then $\gamma_1^{(j)}$ coincides with γ_2 on all edges $\mathfrak{e}_i$ for $i = 1, \ldots, j$. Note also, that star operators leave the flux of γ_1 at each site trivial, and if $\gamma_1^{(j)}$ coincides with γ_2 on three out of four edges $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3, \mathfrak{f}_4$ forming a plaquette, then conditions like

$$\gamma_1^{(j)}(\mathfrak{f}_1) \cdot \gamma_1^{(j)}(\mathfrak{f}_2) \cdot \overline{\gamma_1^{(j)}(\mathfrak{f}_3)} \cdot \overline{\gamma_1^{(j)}(\mathfrak{f}_4)} = \gamma_2(\mathfrak{f}_1) \cdot \gamma_2(\mathfrak{f}_2) \cdot \overline{\gamma_2(\mathfrak{f}_3)} \cdot \overline{\gamma_2(\mathfrak{f}_4)} \quad (3.2.25)$$

implies that they must coincide on the last edge as well. Figure 3.11 demonstrates that then $\gamma_1^{(j)}$ and γ_2 coincide on all edges enclosed by p as well. For the other edges, note that we can use similar techniques *walking down* from the vertices $v_1, \ldots, v_{i_1}, v_{i_2+1}, \ldots, v_j, v_0$ and *walking up* from the vertices $v_{i_1+1}, \ldots, v_{i_2}$ and deducing the value from the remaining edges from conditions as in Equation (3.2.25). This process is independent of the concrete position of s_0 in the lattice, and the proof is concluded. □

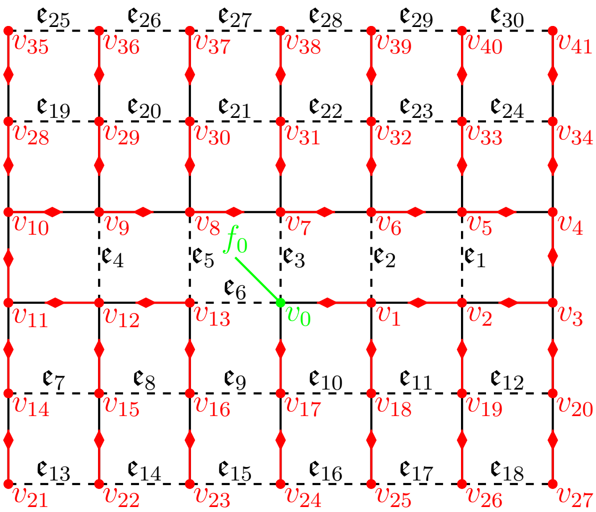


Figure 3.11: Depiction of the algorithm on a square lattice of size 6 with $s_0 = (v_0, f_0)$, f_0 in the centre. The red arrow ⟝◆ symbolizes that we act with a star operator at the tail of the arrow with the intention of transforming the edge under the diamond-shaped arrowhead. In the notation of the proof of Lemma 3.2.7, we have $p_1 = (v_0, v_1, v_2, v_3)$, $p_2 = (v_4, v_5, \ldots, v_{10})$ and $p_3 = (v_{11}, v_{12}, v_{13}, v_0)$. Acting with star operators in the order indicated by the vertices, i.e. acting first with $A^{k_1}_{v_1}$, then with $A^{k_2}_{v_2}$ etc., we can transform the G-values at the edges under the diamond-shaped arrowhead to arbitrary values. The dashed edges indicate that the value on the respective edge is already uniquely determined by the values of the surrounding edges. The reader may verify, by considering the edges in the order they are labelled, that each G-value is uniquely determined and that no consecutive star operator influences the intended transformation of a preceding star operator.

Proof of Proposition 3.2.6. Note first that if ω_0 is a state satisfying Equation (3.2.17) and (3.2.18), Equation (3.2.19) and (3.2.20) imply that ω_0 is left invariant by the time evolution, and it follows that ω_0 is indeed a ground state by Proposition 2.5.17. It then follows from [BR03, Prop. 6.2.17] that the GNS representation of ω_0 is faithful.

Next, let Λ_n be a square-shaped region of size $n \geq 1$ and $\gamma \in C_G(\Lambda_{n+1})$ be a G-connection on a square-shaped region of size $n+1$. Then because of

$$\omega_0(T^{\gamma}_{\Lambda_n}) \overset{(3.2.20)}{=\!=\!=} \omega_0(B_s T^{\gamma}_{\Lambda_n})$$

for each site $s \in \Lambda_n$, it follows that $\omega_0(T^{\gamma}_{\Lambda_n}) = 0$ unless γ is flat. By Lemma 3.2.7, we have $\omega_0(T^{\gamma_1}_{\Lambda_n}) = \omega_0(T^{\gamma_2}_{\Lambda_n})$ for all $\gamma_1, \gamma_2 \in C^f_G(\Lambda_n)$, since we can commute with arbitrary star operators according to Equation (3.2.21) and Lemma 3.2.7. Since ω_0 is a state, we have

$$1 = \omega_0(1_{\mathfrak{A}}) = \sum_{\gamma \in C_G(\Lambda_n)} \omega_0(T^{\gamma}_{\Lambda_n}) = \sum_{\gamma \in C^f_G(\Lambda_n)} \omega_0(T^{\gamma}_{\Lambda_n}) = |C_G(\Lambda_n)|\, \omega_0(T^{\gamma_0}_{\Lambda_n}),$$

where γ_0 is an arbitrary but fixed flat G-connection $\gamma_0 \in C^0_G(\Lambda_n)$. This shows that ω_0 takes the fixed value $\omega_0(T^{\gamma}_{\Lambda_n}) = \frac{1}{\left|C^f_G(\Lambda_n)\right|}$ for all $\gamma \in C_G(\Lambda_n)$.

To show that ω_0 is uniquely determined, we show that for observables $X \in \mathfrak{A}_{loc}$ supported in Λ_n, the value of $\omega_0(X)$ is already determined. Similarly to how we defined $T^{\gamma}_{\Lambda_n}$ in Equation (3.2.22), we may define the operators

$$L^{\gamma} = \prod_{\mathfrak{e} \in \Lambda_n} L^{\gamma(\mathfrak{e})}_{\mathfrak{e}}. \tag{3.2.26}$$

Then it is enough to consider operators of the form

$$L^{\gamma_1}_{\Lambda_n} T^{\gamma_2}_{\Lambda_n}$$

since these operators span all local operators supported in Λ_n. Using plaquette operators, one can show that either $\omega_0(L^{\gamma}_{\Lambda_n} T^{\gamma}_{\Lambda_n}) = 0$ or $B_s L^{\gamma_1}_{\Lambda_n} T^{\gamma_2}_{\Lambda_n} B_s = L^{\gamma_1}_{\Lambda_n} T^{\gamma_2}_{\Lambda_n}$, implying that $T^{\gamma_2}_{\Lambda_n}$ must have a flat G-connection everywhere and if γ_2' is such that $L^{\gamma_1}_{\Lambda_n} T^{\gamma_2}_{\Lambda_n} = T^{\gamma_2'}_{\Lambda_n} L^{\gamma_1}_{\Lambda_n}$, then $\gamma_2' \in C_G(\Lambda_n)$ as well. Since such a transformation can already be performed by star operators and because the action of $L^{\gamma_1}_{\Lambda_n}$ on operators of the form $T^{\gamma_2}_{\Lambda_n}$ is faithful, it follows that $L^{\gamma_1}_{\Lambda_n}$ is given by a sequence of star operators, giving

$$\omega_0(L^{\gamma_1}_{\Lambda_n} T^{\gamma_2}_{\Lambda_n}) \overset{(3.2.21)}{=} \omega_0(T^{\gamma_2}_{\Lambda_n}) = \frac{1}{\left|C^f_G(\Lambda_n)\right|}.$$

□

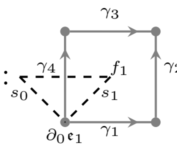
We will pay special attention to the representation $(\mathcal{H}_{\omega_0}, \pi_{\omega_0}, \Omega_{\omega_0})$ and simply write $\mathcal{H}$ instead of $\mathcal{H}_{\omega_0}$, Ω_0 instead of Ω_{ω_0} and identify $\pi_{\omega_0}(\mathfrak{A})$ with $\mathfrak{A}$, suppressing the representation map under exploitation of the faithfulness. It follows from (3.2.17) and (3.2.18) that Ω_0 lies in the image of A_s and B_s for all sites s, i.e.

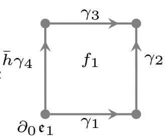
$$A_s\Omega_0 = B_s\Omega_0 = \Omega_0. \tag{3.2.27}$$

3.3 Ribbon Operators and Excitations

In this section, we will explicitly construct operators in $\mathfrak{A}$ that can be used to describe charges of the quantum double model, and study their properties. These operators are called ribbon operators, and are defined for each ribbon ξ. We will construct them using the triangle operators introduced in the previous section in such a way that they create charges at the endpoints of the ribbon ξ. In view of Equation (3.2.27), such an operator F_ξ should be such that the condition in Equation (3.2.27) is violated at $\partial_0\xi$ and $\partial_1\xi$. We say an operator X creates an **electric excitation** at s if $A_sX\Omega_0 \neq X\Omega_0$ and a **magnetic excitation** at s if $B_sX\Omega_0 \neq X\Omega_0$.

We will demonstrate how $L^h_{\tau_0}$ changes the magnetic flux of a flat G-connection for a dual triangle τ_0 and $h \in G$, thus creating a magnetic charge. Let $\tau_0 = (s_0, s_1)$ be a dual triangle. Let γ be a flat G-connection on some region Λ and $\mathfrak{e}_1, \ldots, \mathfrak{e}_4 \in \Lambda$ a plaquette enclosing the face $f_1 = f(s_1)$ such that $\partial_0\mathfrak{e}_1 = v(s_1)$, i.e. $s_1 = (\partial_0\mathfrak{e}_1, f_1)$. If $h \in G$, $\gamma_i := \gamma(\mathfrak{e}_i)$ for $i = 1, \ldots, 4$ and τ_0 a locally counter-clockwise oriented, then we have

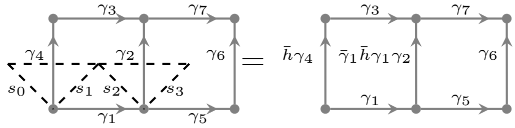
$B^c_{s_1} L^h_{\tau_0}$: 

$= \delta_{\gamma_1\gamma_2\bar{\gamma}_3\bar{\gamma}_4 h, c}$ 

.

If γ is flat, then $\gamma_1\gamma_2\bar{\gamma}_3\bar{\gamma}_4 = e$, and the above expression is non-zero if and only if $h = c$. Hence, the operator $L^h_{\tau_0}$ creates a magnetic charge of type $\mathcal{C}_h = \{gh\bar{g} \mid g \in G\}$ at site s_1 when acting on a flat G-connection. Similar calculations demonstrate that $L^h_{\tau_0}$ creates a charge of type $\mathcal{C}_{\bar{h}} = \left\{g\bar{h}\bar{g} \mid g \in G\right\}$ at site s_0 and one may repeat the calculation for locally clockwise oriented triangles to arrive at the same result, i.e. all dual triangle operators create the same charge-type regardless of their choice of local orientation and this is precisely the reason for having different definitions for $L^h_{\tau_0}$ for different local orientations of τ_0.

A systematic way to create such pairs of excitations can now be found as follows: Let $\tau_1 = (s_1, s_2)$ be a locally counter-clockwise direct triangle and let $\tau_2 = (s_2, s_3)$ be a locally counter-clockwise oriented dual triangle such that (τ_0, τ_1, τ_2) forms a ribbon. Then

$L^h_{\tau_0} L^{\bar{\gamma}_1 h \gamma_1}_{\tau_2}$  ,

and if γ is flat, the magnetic flux of $L^h_{\tau_0} L^{\bar{\gamma}_1 h \gamma_1}_{\tau_2}\gamma$ around the site s_2 is given by

$$\bar{\gamma}_1 \bar{h} \gamma_1 \gamma_2 \bar{\gamma}_3 \bar{\gamma}_4 h \gamma_1 = \bar{\gamma}_1 \bar{h} h \gamma_1 = e.$$

As we have discussed before, switching sites within the same plaquette only conjugates the flux of a G-connection, and it follows that the flux at s_1 is trivial as well. Furthermore, the calculation from before shows that the flux at s_3 is $\bar{\gamma}_1 h \gamma_1 \in \mathcal{C}_h$ and the flux at s_0 is still $\bar{h} \in \mathcal{C}_{\bar{h}}$. Thus, $L^h_{\tau_1} L^{\bar{\gamma}_1 h \gamma_1}_{\tau_2}$ creates a magnetic flux of type $\mathcal{C}_{\bar{h}}$ at site s_0, of type $\mathcal{C}_h$ at site s_3 and leaves the magnetic charge trivial at all other sites when acting on γ. For an arbitrary flat G-connection γ, the same can be achieved with the operator

$$\sum_{g \in G} L^h_{\tau_0} T^g_{\tau_1} L^{\bar{g} h g}_{\tau_2},$$

since $T^g_{\tau_1}\gamma \neq 0$ only if $\gamma_1 = g$. This construction can be repeated recursively to obtain operators that create charges at the endpoints of any ribbon $\xi = (s_1, \ldots, s_n)$ without creating charges at the sites in between. The idea behind this construction is due to [Kit03].

Definition 3.3.1. For each ribbon ξ, we define a family of operators $\left\{F^{h,g}_\xi\right\}_{h,g \in G}$ as follows: If $\xi = \emptyset$ is the empty ribbon, we set

$$F^{h,g}_\emptyset = \delta_{e,g} 1_{\mathfrak{A}}, \tag{3.3.1}$$

where $1_{\mathfrak{A}}$ is the unit in $\mathfrak{A}$. If $\xi = \tau$ consists of a single triangle, we set

$$F^{h,g}_\tau = \begin{cases} \delta_{e,g} L^h_\tau & \text{if } \tau \text{ is dual,} \\ T^g_\tau & \text{if } \tau \text{ is direct.} \end{cases}$$

Finally, let $\xi = (\xi_1, \xi_2)$ be the composition of two ribbons ξ_1 and ξ_2. Then

$$F^{h,g}_\xi = \sum_{k \in G} F^{h,k}_{\xi_1} F^{\bar{k} h k, \bar{k} g}_{\xi_2}. \tag{3.3.2}$$

$F^{h,g}_\xi$ is called a **ribbon operator with ribbon** ξ.

The well-definedness of Equation (3.3.2) can be shown by decomposing $F_\xi^{h,g}$ inductively into a composition of all triangles which make out $\xi = (\tau_1, \ldots, \tau_n)$. Let $\{\tau_1, \ldots, \tau_n\}$ be all direct- and $\{\tilde{\tau}_1, \ldots, \tilde{\tau}_l\}$ be all dual triangles of which ξ consists, ordered such that the edges $(\mathfrak{e}_{\tau_1}, \ldots, \mathfrak{e}_{\tau_n})$ and $(\mathfrak{e}_{\tilde{\tau}_1}, \ldots, \mathfrak{e}_{\tilde{\tau}_l})$ form the direct path $\xi^{di} = (\partial_0\mathfrak{e}_{\tau_1}, \partial_0\mathfrak{e}_{\tau_2}, \ldots, \partial_0\mathfrak{e}_{\tau_l}, \partial_1\mathfrak{e}_{\tau_{l+1}})$, respectively dual path $\xi^{du} = (\partial_0\mathfrak{e}_{\tilde{\tau}_1}, \partial_0\mathfrak{e}_{\tilde{\tau}_2}, \ldots, \partial_0\mathfrak{e}_{\tilde{\tau}_l}, \partial_1\mathfrak{e}_{\tilde{\tau}_l})$ of ξ. Let furthermore $\xi_{d,j}$ be the direct path of ξ up until we reach the vertex of the dual triangle $\tilde{\tau}_j$ (see Figure 3.12). Then for a G-connection $\gamma \in C_G(\Lambda)$ defined for a finite region

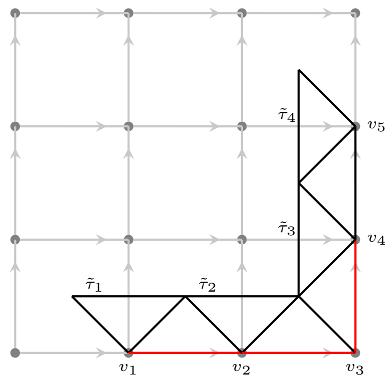


Figure 3.12: The figure demonstrates the action of a dual triangle operator of a ribbon operator at the end of a subpath of the direct path of that ribbon. The depicted ribbon ξ contains the dual triangles $\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4$. The direct path up to the third dual triangle $\tilde{\tau}_3$ is given as $\xi_{d,3} = (v_1, v_2, v_3, v_4)$ (red). If γ is a G-connection with $\beta^{(\xi_{d,3})}(\gamma) = k$, then the ribbon operator $F_\xi^{h,g}$ acts on the edge $\mathfrak{e}_{\tilde{\tau}_3}$ with $L_{\tilde{\tau}_3}^{\bar{k}hk}$.

Λ containing ξ, we obtain

$$F_\xi^{h,g}\gamma = \prod_{j=1}^{l} \sum_{k\in G} T^k_{\xi_{d,j}} L^{\bar{k}hk}_{\tilde{\tau}_j}\gamma, \tag{3.3.3}$$

where $T^k_{\xi_{d,j}}$ is defined as in Equation (3.2.24).

We will omit the details of verifying the above formula, as it is a straightforward application of Equation (3.3.2). We refer to Figure 3.13 for a depiction of the action of the ribbon operator $F_\xi^{h,g}$ on a G-connection γ. We note that ribbon operators span the space of all local operators $\mathfrak{A}_{loc}$. This trivially follows by considering single triangle ribbons $\xi = \{\tau\}$, since these already provide all left multiplications and delta projections at each edge.

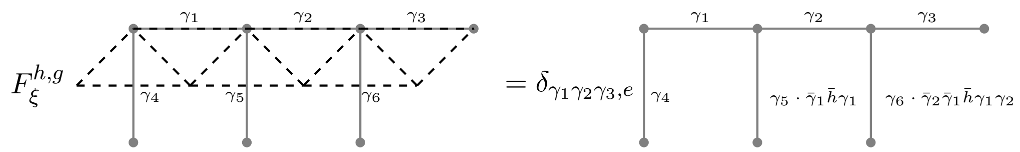


Figure 3.13: Depiction of the action of the ribbon operator $F_\xi^{h,g}$ on a G-connection γ with values $\gamma(\mathfrak{e}_i) := \gamma_i \in G$.

Remark 3.3.2. In a more general setting, formula (3.3.2) can be expressed in terms of the quantum double $D(H)$ of a general Hopf algebra H and the well-definedness can then be reduced to the coassociativity of H, see [CCY21] for more details.

Given a ribbon ξ, we define the operator L_ξ^h to be

$$L_\xi^h = \sum_{k\in G} F_\xi^{h,k}. \tag{3.3.4}$$

L_ξ^h is essentially the sum of the dual operators appearing in Equation (3.3.3). A more compact way of writing down a ribbon operator that reflects the formula (3.3.3) can then be given by

$$F_\xi^{h,g} = T_{\xi^{di}}^g L_\xi^h. \tag{3.3.5}$$

As mentioned before, the most important property of ribbon operators is that they create charges only at the endpoints of the ribbon. Let $s_i = \partial_i\xi$, $i = 0, 1$. Then $F_\xi^{h,g}\Omega_0$ is a vector with magnetic flux $\bar{h}$ at site s_0 and magnetic flux $\bar{g}hg$ at s_1, whilst projecting into G-connections with $\beta^{\xi^{(di)}}(\gamma) = g$, ξ^{di} being the direct path of ξ and β defined in Definition 3.2. This is obvious in the case where $\xi = \emptyset$. To see that this is also true for a general ribbon ξ, we may decompose $\xi = (\xi_1, \xi_2)$ into an arbitrary concatenation of ribbons ξ_1 and ξ_2. Using formula (3.3.2) then allows for an inductive argument using similar calculations as the ones preceding Definition 3.3.1. See also [Kit06].

The property that ribbon operators create charges only at their endpoints can be characterized by the commutation relations between ribbon operators and star- and plaquette operators.

Lemma 3.3.3. *Let ξ be a ribbon with $s_1 = \partial_0\xi$ and $s_2 = \partial_1\xi$. Then the* *following commutation relations hold for all $h, g, k, c \in G$ and $s \notin \{s_1, s_2\}$:*

$$A^k_{s_0} F^{h,g}_\xi = F^{kh\bar{k},kg}_\xi A^k_{s_0}, \tag{3.3.6}$$
$$A^k_{s_1} F^{h,g}_\xi = F^{h,g\bar{k}}_\xi A^k_{s_1},$$
$$\left[F^{h,g}_\xi A^k_s\right] = \left[F^{h,g}_\xi, B^c_s\right] = 0. \tag{3.3.7}$$

If ξ is locally clockwise oriented, then we have

$$B^c_{s_0} F^{h,g}_\xi = F^{h,g}_\xi B^{ch}_{s_0}, \tag{3.3.8}$$
$$B^c_{s_1} F^{h,g}_\xi = F^{h,g}_\xi B^{\bar{g}\bar{h}gc}_{s_1}. \tag{3.3.9}$$

If ξ is locally counter-clockwise oriented, we have

$$B^c_{s_0} F^{h,g}_\xi = F^{h,g}_\xi B^{hc}_{s_0}, \tag{3.3.10}$$
$$B^c_{s_1} F^{h,g}_\xi = F^{h,g}_\xi B^{c\bar{g}\bar{h}g}_{s_1}. \tag{3.3.11}$$

See [CCY21] for a proof.

Remark 3.3.4. We emphasize that the commutation relations in Equation (3.3.10) and (3.3.11) differ from the old literature, see e.g. [BMD08, (B42)]. This is because different local orientations were not taken into account in these sources, choosing the same action for ribbon operators regardless of their local orientation. In that case, however, it is neither guaranteed that a ribbon operator of the form $F^{h,g}_\xi$ always creates the magnetic flux $\bar{h}$ at $\partial_0\xi$, nor are the commutation relations given in [BMD08] entirely correct. See [CCY21, Sec 3.3] for a more detailed exposition on this subject.

It is important to note, that we always have

$$B_{s_0} F^{h,g}_\xi = F^{h,g}_\xi B^h_{s_0} \tag{3.3.12}$$
$$B_{s_1} F^{h,g}_\xi = F^{h,g}_\xi B^{\bar{g}\bar{h}g}_{s_1} \tag{3.3.13}$$

for $s_0 = \partial_0\xi$ and $s_1 = \partial_1\xi$ regardless of the local orientation of ξ. This will be useful when dealing with the interaction terms $(1_{\mathfrak{A}} - B_s)$ of the local Hamiltonian.

Star and plaquette operators are ribbon operators as well. Note that the sites of a direct (dual) ribbon share a common face (vertex), see also Figure 3.10 again. Let s be a site, and ξ^{du}_s the unique locally clockwise oriented dual ribbon starting and ending at s. Then $F^{h,g}_{\xi^{du}_s}$ is the product of four dual triangle operators acting precisely as the star operator A^h_s for $g = e$, i.e.

$$F^{h,g}_{\xi^{du}_s} = \delta_{g,e} A^h_s. \tag{3.3.14}$$

Similarly, if ξ_s^{di} is the smallest closed locally counter-clockwise oriented direct ribbon starting and ending at s, then $F_{\xi_s^{di}}^{h,g}$ is a sum of the product over all direct triangles operators measuring a magnetic flux g around s, i.e.

$$F_{\xi_s^{di}}^{h,g} = B_s^g. \tag{3.3.15}$$

It is straightforward to verify that if $\bar{\xi}_s^{di}$ and $\bar{\xi}_s^{du}$ are the unique direct, respectively dual locally counter-clockwise oriented ribbon surrounding s, then

$$F_{\bar{\xi}_s^{du}}^{h,g} = \delta_{g,e} A_s^{\bar{h}}, \tag{3.3.16}$$

$$F_{\bar{\xi}_s^{di}}^{h,g} = B_s^{\bar{g}}. \tag{3.3.17}$$

We caution the reader not to confuse the local orientation of a ribbon with the way it *encircles* an area. Indeed, a locally clockwise oriented dual ribbon moves *counter-clockwise* around its vertex, whereas a locally counter-clockwise oriented dual ribbon moves *clockwise* around its vertex, which can also be seen in Figure 3.10.

We derive some useful algebraic relations for ribbon operators.

Lemma 3.3.5. *We have the following identities for any ribbon ξ:*

(1)

$$F_\xi^{h_1,g_1} F_\xi^{h_2,g_2} = \delta_{g_1,g_2} F_\xi^{h_1 h_2,g_1} \tag{3.3.18}$$

for all $h_1, h_2, g_1, g_2 \in G$.

(2)

$$\left(F_\xi^{h,g}\right)^* = F_\xi^{\bar{h},g}$$

for all $h, g \in G$.

(3) If $\bar{\xi}$ is the obtained by inverting the direction of ξ, we have

$$F_\xi^{h,g} = F_{\bar{\xi}}^{\bar{g}\bar{h}g,\bar{g}}. \tag{3.3.19}$$

(4) Finally, we have

$$\sum_{k \in G} F_\xi^{h,k} F_{\bar{\xi}}^{\bar{k}\bar{h}k,\bar{k}g} = F_\emptyset^{h,g} = \delta_{e,g} 1_{\mathfrak{A}} \tag{3.3.20}$$

for all $h, g \in G$.

Proof. Part (1) is a simple application of Equation (3.3.3), using that $T_\tau^{k_1} T_\tau^{k_2} = \delta_{k_1,k_2} T_\tau^{k_1}$ and $L_{\tau'}^{k_1} L_{\tau'}^{k_2} = L_{\tau'}^{k_1 k_2}$ for all $k_1, k_2 \in G$. Note that no two triangles overlap in ξ, hence all triangle operators appearing in Equation (3.3.3) commute. Similarly, Part (2) follows from $(L_\tau^h)^* = L_\tau^{\bar{h}}$. To see part (3), we first make a few observations: If $\xi^{di} = (\partial_0 \mathfrak{e}_1, \ldots, \partial_0 \mathfrak{e}_n, \partial_1 \mathfrak{e}_n)$ is the direct path of $\xi = (s_1, \ldots, s_n)$ and $\xi_{d,j}$ the direct path of ξ up to the dual triangle $\tilde{\tau}_j$ (see discussion after Definition 3.3.1), and $\bar{\xi} = (s_n, \ldots, s_1)$ the ribbon with direction inverted, then if γ is a G-connection with $\beta^{(\xi^{di})}(\gamma) = g$, we have

$$\beta^{(\bar{\xi}_{d,j})}(\gamma)\overline{\beta^{(\xi_{d,l-j+1})}(\gamma)} = \bar{g}$$

where β is the map defined in Definition 3.2. Using (3.3.5) and (3.3.3) we obtain

$$\begin{aligned} F_{\bar{\xi}}^{\bar{g}\bar{h}g,\bar{g}} &= \prod_{j=1}^{l} \sum_{k \in G} T_{\bar{\xi}_{d,j}}^{k} L_{\bar{\tilde{\tau}}_j}^{\bar{k}\bar{g}\bar{h}gk} \gamma \\ &= \prod_{j=1}^{l} \sum_{k \in G} T_{\bar{\xi}_{d,j}}^{k} L_{\bar{\tilde{\tau}}_j}^{\overline{\beta^{\bar{\xi}_{d,j}}(\gamma)}\bar{g}\bar{h}g\beta^{\bar{\xi}_{d,j}}(\gamma)} \gamma \\ &= \prod_{j=1}^{l} \sum_{k \in G} T_{\bar{\xi}_{d,j}}^{k} L_{\bar{\tilde{\tau}}_j}^{\overline{\beta^{(\xi_{d,l-j+1})}(\gamma)}\bar{h}\beta^{(\xi_{d,l-j+1})}(\gamma)} \gamma \\ &\overset{j \mapsto l-j+1}{=} \prod_{j=1}^{l} \sum_{k \in G} T_{\xi_{d,j}}^{k} L_{\tilde{\tau}_j}^{\overline{\beta_{(\xi_{d,j})}(\gamma)}h\beta_{(\xi_{d,j})}(\gamma)} \gamma = F_\xi^{h,g}, \end{aligned}$$

where we used that $L_{\tilde{\tau}}^{k} = L_{\tau}^{\bar{k}}$ for all $k \in G$ at the end. Part (4) now follows from part (3):

$$\sum_k F_\xi^{h,k} F_{\bar{\xi}}^{\bar{k}hk,\bar{k}g} = \sum_k F_\xi^{h,k} F_{\bar{\xi}}^{\bar{g}\bar{h}g,\bar{g}k} = \delta_{g,e} \sum_k F_\xi^{e,k} = \delta_{g,e} 1_{\mathfrak{A}}.$$

□

We say that two ribbons ξ_1 and ξ_2 **have the same endpoints** if $\partial_0 \xi_1 = \partial_0 \xi_2$ and $\partial_1 \xi_1 = \partial_1 \xi_2$. If in addition $\xi_1 \neq \xi_2$, we call ξ_2 a **deformation** of ξ_1. We want to show that if ξ_2 is a deformation of ξ_1 and $\psi \in \mathcal{H}$ a state that has no excitations in the region between ξ_1 and ξ_2, i.e. if $A_s \psi = B_s \psi = \psi$ for all sites enclosed by the ribbons ξ, then $F_{\xi_1}^{h,g} \psi = F_{\xi_2}^{h,g} \psi$. In other words, we can deform ribbons in the absence of excitations. This statement is Corollary 3.3.11 and the following discussion leading up to that corollary serves to give the necessary insight to prove this result.

Lemma 3.3.6. *Let p_1, p_2 be two paths, with $\partial_0 p_1 = \partial_0 p_2$ and $\partial_1 p_1 = \partial_1 p_2$ contained in a region Λ and define*

$$B_\Lambda = \prod_{s \in \mathcal{S}(\Lambda)} B_s. \tag{3.3.21}$$

Then we have for all $g \in G$

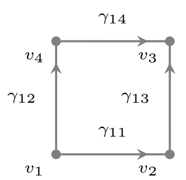
$$T^g_{p_1} B_\Lambda = T^g_{p_2} B_\Lambda. \tag{3.3.22}$$

It follows that

$$T^g_p B_\Lambda = \delta_{g,e} B_\Lambda$$

for any closed path p and $g \in G$.

Proof. Equation (3.3.22) can be verified by acting on an arbitrary flat G-connection. If γ is a flat G-connection depicted as



on some plaquette in Λ, then $\gamma_1\gamma_2 = \gamma_3\gamma_4$. In other words, we have

$$T^{\mathfrak{e}_1}_{p_1}\gamma = T^{\mathfrak{e}_2}_{p_2}\gamma,$$

for all $\gamma \in \mathrm{im}(B_\Lambda)$, where $p_1 = (v_1, v_2, v_3)$ and $p_2 = (v_1, v_4, v_3)$. Using these atomic deformations, we can transform any path p_1 to any path p_2 by changing the way p_1 traverses each plaquette step by step, while keeping the starting and endpoint of p_1 and p_2 fixed. This shows Equation (3.3.22). If p is a closed path, then p can be decomposed into two subpaths p_1 and p_2 with $p = (p_1, p_2)$. Since $\partial_1 p_1 = \partial_0 p_2$ and $\partial_0 p_1 = \partial_1 p_2$, we can deform p_2 into $\bar{p}_1$ and we have

$$T^g_p\gamma = \sum_{k\in G} T^g_{p_1} T^{\bar{k}g}_{p_2}\gamma = \sum_{k\in G} T^g_{p_1} T^{\bar{k}g}_{\bar{p}_1}\gamma = T^g_{p_1} T^{\bar{g}k}_{p_1}\gamma = \sum_{k\in G} \delta_{g,\bar{g}k}\gamma.$$

But $\delta_{g,\bar{g}k} = 0$ unless $g = e$. □

Before we continue, we want to introduce some terminology that allows us to separate the action of a ribbon operator $F^{h,g}_\xi$ into an action on the direct path ξ^{di} and an action on the dual ξ^{du}.

Definition 3.3.7 (Dual Action)**.** Let ξ be a ribbon with dual triangles $(\tau_1, \ldots, \tilde{\tau}_l)$ and $h, g \in G$. If γ is a G-connection with $F_\xi^{h,g}\gamma \neq 0$, we say an operator $X \in \mathfrak{A}$ has the **same dual action as $\mathbf{F_\xi^{h,g}}$ on γ at edge $\mathfrak{e}$** if $X\gamma$ is a G-connection as well, $\mathfrak{e}$ intersects $\mathfrak{e}_{\tau_t}$ for some $t = 1, \ldots, n$ and

$$(X\gamma)(\mathfrak{e}_{\tau_t}) = (F_\xi^{h,g}\gamma)(\mathfrak{e}_{\tau_t}),$$

where we identified $\gamma(\mathfrak{e}_{\tau_t}) = \gamma(\mathfrak{e})$, see Equation (3.2.7). We say **X has the same dual action as $\mathbf{F_\xi^{h,g}}$ on** γ if X has the same dual action as $F_\xi^{h,g}$ on γ for each edge intersecting $\mathfrak{e}_{\tau_t}$ for $t = 1, \ldots, n$. If X has the same dual action as $F_\xi^{h,g}$ for each G-connection, we simply say **X has the same dual action as $F_\xi^{h,g}$**. Finally, we call the transformations performed on the values $\gamma(\mathfrak{e}_{\tau_t})$ the **dual action of $\mathbf{F_\xi^{h,g}}$**.

Write

$$F_\xi^{h,g}\gamma = \prod_{j=1}^{l} \sum_{k \in G} T_{\xi_{d,j}}^k L_{\tilde{\tau}_j}^{\bar{k}hk}\gamma,$$

using the same notation as in Equation (3.3.3). Then an operator X has the same dual action as $F_\xi^{h,g}$ on $\gamma \notin \ker(F_\xi^{h,g})$ at $\mathfrak{e}_{\tau_j}$ if and only if

$$(X\gamma)(\mathfrak{e}_{\tau_j}) = (L_{\tilde{\tau}_j}^{\overline{\beta^{\xi_{d,j}}(\gamma)}h\beta^{\xi_{d,j}}(\gamma)}\gamma)(\mathfrak{e}_{\tau_j}), \tag{3.3.23}$$

and it has the same dual action as $F_\xi^{h,g}$ on γ if Equation (3.3.23) holds for all $j = 1, \ldots, l$. Trivially, each of the operators $L_{\tilde{\tau}_j}^{\overline{\beta^{\xi_{d,j}}(\gamma)}h\beta^{\xi_{d,j}}(\gamma)}$ appearing in Equation (3.3.3) has the same dual action as $F_\xi^{h,g}$ at $\mathfrak{e}_{\tau_j}$ on each G-connection $\gamma \notin \ker(F_\xi^{h,g})$, and the operator L_ξ^h defined in Equation (3.3.4) has the same dual action as $F_\xi^{h,g}$.

We have shown in Lemma 3.3.6 that for ribbons ξ_1 and ξ_2 with the same endpoints, the operators $T_{\xi_1^{di}}$ and $T_{\xi_2^{di}}$ coincide when acting on a flat G-connection, giving us some freedom of choice for the direct path of a ribbon when acting on the vacuum. What may be less obvious is that the dual action of a ribbon operator is independent of the choice of the direct path when acting on a flat G-connection as well. First, note that if p^* is a dual path in Λ, then we can find a ribbon ξ in Λ such that $\xi_{du} = p^*$. This is always possible, because for a face f to be contained in Λ, we must always have that the surrounding edges are contained in Λ as well. We would like to have a definition of the form $L_{p^*}^h = L_\xi^h$, where the right-hand side is defined in Equation (3.3.4). Although this definition is dependent on the choice of ξ, it becomes well-defined when restricted to the image of B_Λ defined in Equation (3.3.21).

Proposition 3.3.8. *Let $p^* = (f_1, \dots, f_n)$ be a dual path in Λ and let $(\tau_1, \dots, \tau_{n-1})$ such that $\tau_i = (s_1^{(i)}, s_2^{(i)})$ with $s_1^{(i)} = (v_i, f_i)$ and $s_2^{(i)} = (v_i, f_{i+1})$ for $i = 1, \dots, n-1$. Furthermore, let $\mathfrak{e}_i$ be the edge that intersects with the dual edge (f_i, f_{i+1}) for each $i = 1, \dots, n-1$ and $(u_1, \dots, u_{n-1})$ a collection of vertices such that $u_i \in \partial\mathfrak{e}_i$ for $i = 1, \dots, n-1$. If $p = (w_1, \dots, w_k)$ is any path that contains $\{u_1, \dots, u_{n-1}\}$ in any order, i.e. there exists a map $t : \{1, \dots, n-1\} \to \{1, \dots, k\}$ such that $u_i = w_{t_i}$ (see Figure 3.14), then for any ribbon ξ with dual triangles $(\tau_1, \dots, \tau_{n-1})$ the operator*

$$\sum_{k\in G} \prod_i T^k_{p_i} L^{\bar{k}hk}_{\tau_i} \tag{3.3.24}$$

with $p_i = (w_1, \dots, w_{t_i})$ has the same dual action as $F^{h,g}_\xi$ on each flat G-connection $\gamma \notin \ker(F^{h,g}_\xi)$. It follows that

$$\sum_{k\in G} \prod_i T^k_{p_i} L^{\bar{k}hk}_{\tau_i} T^g_{p_0} \gamma = F^{h,g}_\xi \gamma \tag{3.3.25}$$

for all flat G-connections $\gamma \in C_G(\Lambda)$ and any path p_0 in Λ with $\partial_0 p_0 = \partial_0 \xi^{di}$ and $\partial_1 p_1 = \partial_1 \xi^{di}$, where ξ^{di} is the direct path of ξ.

Proof. Note that for each i, both v_i and u_i are vertices at the boundary of the same edge $\mathfrak{e}_i$ and we either have $u_i = v_i$ or u_i and v_i are at different ends of the same edge $\mathfrak{e}_i$. The action of a dual triangle operator appearing in Equation (3.3.3) for some ribbon operator $F^{h,g}_\xi$ on a G-connection γ is determined by the group element h and the β-value of γ along the subpath of ξ^{di} that starts at $\partial_0\xi^{di}$ and ends at the vertex $v(\tau_i) = v_i$ of the dual triangle τ_i. By Lemma 3.3.6, this is independent of the path chosen between $\partial_0\xi^{di}$ and $v(\tau_i)$, covering the case $v_i = u_i$. So we assume $v_i \neq u_i$ and let $\tau_i = (s_i, s_{i+1})$ be the unique dual triangle in ξ with $v(\tau_i) = v_i$. We define $p_1 := (\xi^{di} : v_i)$ and $p_2 := (p : u_i)$ to be the path ξ^{di}, respectively p *cut off* at v_i respectively u_i, and let $\tilde{\tau}_i = (\tilde{s}_i, \tilde{s}_{i+1})$ where $\tilde{s}_i = (u_i, f(s_i))$ and $\tilde{s}_{i+1} = (u_i, f(s_{i+1}))$. $\tilde{\tau}_i$ is just a *mirrored* version of τ_i, see also figure 3.15. In view of Equation (3.3.3), we must show that

$$(L_{\tau_i}^{\overline{\beta^{(p_1)}(\gamma)}h\beta^{(p_1)}(\gamma)}\gamma)(\mathfrak{e}_i) = (L_{\tilde{\tau}_i}^{\overline{\beta^{(p_2)}(\gamma)}h\beta^{(p_2)}(\gamma)}\gamma)(\mathfrak{e}_i) \tag{3.3.26}$$

holds. Setting $p_{\mathfrak{e}_i} := (v_i, u_i)$, $k_{\mathfrak{e}_i} := \beta^{(p_{\mathfrak{e}_i})}(\gamma)$, $k_1 := \beta^{(p_1)}(\gamma)$ and $k_2 := \beta^{(p_2)}(\gamma)$, we have by Lemma 3.3.6

$$k_1 k_{\mathfrak{e}_i} = k_2. \tag{3.3.27}$$

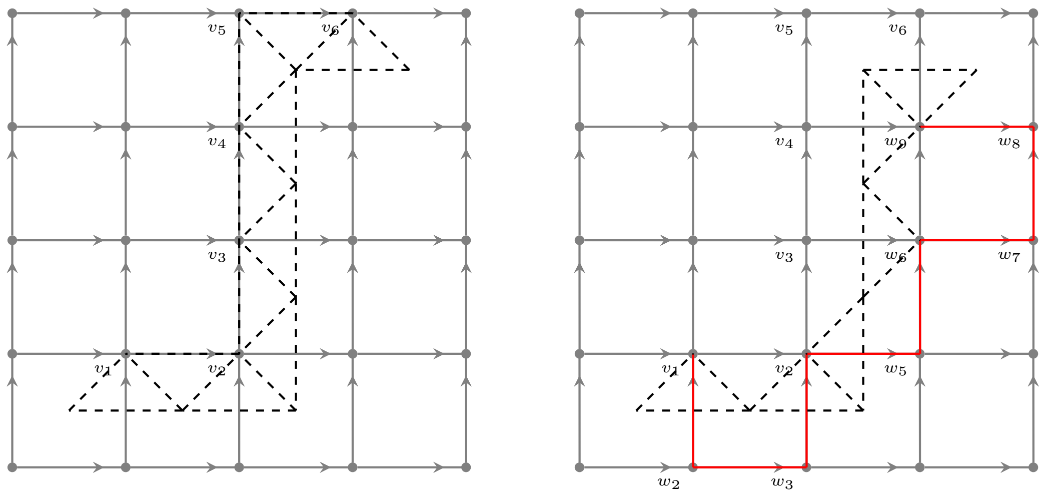


Figure 3.14: Depiction of the scenario described in Proposition 3.3.8. On the left, a ribbon ξ is drawn with direct path $\xi^{di} = (v_1, \ldots, v_6)$ and dual path $\xi^{du} = (f_1, \ldots, f_7)$. In the right figure, no direct triangles are drawn, but the dual path is still covered by the depicted dual triangles. The red path $p = (w_1, \ldots, w_9)$ covers the vertices $w_1 = v_1$ and $w_4 = v_2$. The vertices w_5, w_6 and w_9 lie at the opposite endpoint of the edge shared with v_2, v_3 and v_6 respectively. In either case, the path p covers at least one vertex at the boundary of every edge that intersects with the dual path. Note also that p starts at $v_1 = v(\partial_0 \xi)$, but does not necessarily have to end at $v(\partial_1 \xi)$ for the dual action to be the same.

We distinguish four cases.
Case 1: τ_i is aligned and locally clockwise oriented
In this case, we have $k_{\mathfrak{e}i} = \gamma(\mathfrak{e}_i)$ giving $k_1\gamma(\mathfrak{e}_i) = k_2$ and $\gamma(\mathfrak{e}_i)\bar{k}_2 = \bar{k}_1$. Then we have

$$L_{\tau_i}^{\bar{k}_1 h k_1}(\gamma)(\mathfrak{e}_i) = \bar{k}_1 h k_1 \cdot \gamma(\mathfrak{e}_i) = \gamma(\mathfrak{e}_i) \cdot \bar{k}_2 h k_2 = L_{\tau_i}^{\bar{k}_2 h k_2}(\gamma)(\mathfrak{e}_i),$$

since $\tilde{\tau}_i$ is aligned and locally counter-clockwise oriented.
Case 2: τ_i is not aligned and locally clockwise oriented
In this case, we have $k_{\mathfrak{e}i} = \overline{\gamma(\mathfrak{e}_i)}$ giving $\gamma(\mathfrak{e}_i)\bar{k}_1 = \bar{k}_2$ and $k_1 = k_2\gamma(\mathfrak{e}_i)$. Then we have

$$L_{\tau_i}^{\bar{k}_1 h k_1}(\gamma)(\mathfrak{e}_i) = \gamma(\mathfrak{e}_i) \cdot \bar{k}_1 \bar{h} k_1 = \bar{k}_2 \bar{h} k_2 \cdot \gamma(\mathfrak{e}_i) = L_{\tau_i}^{\bar{k}_2 h k_2}(\gamma)(\mathfrak{e}_i),$$

since $\tilde{\tau}_i$ is not aligned and locally counter-clockwise oriented.
Case 3: τ_i is aligned and locally counter-clockwise oriented
In this case we have $k_{\mathfrak{e}i} = \overline{\gamma(\mathfrak{e}_i)}$ giving $\gamma(\mathfrak{e}_i)\bar{k}_1 = \bar{k}_2$ and $k_1 = k_2\gamma(\mathfrak{e}_i)$ as before. Then we have

$$L_{\tau_i}^{\bar{k}_1 h k_1}(\gamma)(\mathfrak{e}_i) = \gamma(\mathfrak{e}_i) \cdot \bar{k}_1 h k_1 = \bar{k}_2 h k_2 \cdot \gamma(\mathfrak{e}_i) = L_{\tau_i}^{\bar{k}_2 h k_2}(\gamma)(\mathfrak{e}_i),$$

since $\tilde{\tau}_i$ is aligned and locally clockwise oriented.
Case 4: τ_i is not aligned and locally counter-clockwise oriented
In this case we have $k_{\mathfrak{e}i} = \gamma(\mathfrak{e}_i)$ giving $k_1\gamma(\mathfrak{e}_i) = k_2$ and $\gamma(\mathfrak{e}_i)\bar{k}_2 = \bar{k}_1$ as in the first case. Then we have

$$L_{\tau_i}^{\bar{k}_1 h k_1}(\gamma)(\mathfrak{e}_i) = \bar{k}_1 \bar{h} k_1 \cdot \gamma(\mathfrak{e}_i) = \gamma(\mathfrak{e}_i) \cdot \bar{k}_2 \bar{h} k_2 = L_{\tau_i}^{\bar{k}_2 h k_2}(\gamma)(\mathfrak{e}_i),$$

since $\tilde{\tau}_i$ is not aligned and locally counter-clockwise oriented.

□

This shows that the dual action of a ribbon operator $F_\xi^{h,g}$ is determined by the dual path ξ^{du} in the following sense: Let ξ be a ribbon in some bounded region Λ with dual path $\xi^{du} = (f_1, \ldots, f_n)$ and let $\{s_1, \ldots, s_n\}$ be any choice of sites, not necessarily of ξ such that $f(s_i) = f_i$. Let further $\{\mathfrak{e}_1, \ldots, \mathfrak{e}_n\}$ be such that $\mathfrak{e}_i$ intersects the dual edge (f_i, f_{i+1}) for $i = 1, \ldots, n-1$ and p_i any path contained in Λ starting at $v_0 = v(\partial_0\xi)$ and ending at $v_i := v(f_i) = v(f_{i+1})$. By Lemma 3.3.6, the value $k_{i,\gamma} := \beta^{(p_i)}(\gamma)$ is independent of the choice of p_i for any flat G-connection γ, as long as the endpoints of the path stay fixed. Then for any flat G-connection $\gamma \notin \ker(F_\xi^{h,g})$ and $i = 1, \ldots, n-1$ we have

$$(F_\xi^{h,g}\gamma)(\mathfrak{e}_i) = (L_{\tau_i}^{\bar{k}_{i,\gamma} h k_{i,\gamma}}\gamma)(\mathfrak{e}_i).$$

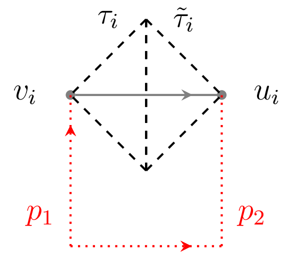


Figure 3.15: Depiction of two vertices v_i and u_i at opposite sides of the same edge $\mathfrak{e}_i = (v_i, u_i)$. The dual triangles τ_i and $\tilde{\tau}_i$ cover the same faces, but $v(\tau_i) = v_i$ and $v(\tilde{\tau}_i) = u_i$. The dotted paths depict different arbitrary paths p_1 and p_2 having the same starting point $v(\partial_0(\xi))$ but with $\partial_1 p_1 = v_i$ and $\partial_1 p_2 = u_i$

It follows that the operator

$$\prod_{i=1}^{n-1} L_{\tau_i}^{\bar{k}_{i,\gamma} h k_{i,\gamma}} \tag{3.3.28}$$

has the same dual action as $F_\xi^{h,g}$ on γ. In the following, we want to demonstrate that by commuting with star operators, dual paths can be transformed in the sense that the expression in Equation (3.3.28) transforms to expressions corresponding to different dual paths.

Recall that a dual ribbon is a ribbon that consists only of dual triangles.

Lemma 3.3.9. *Let ξ_1, ξ_2 be dual ribbons sharing the same endpoints $s_0 := \partial_0\xi_1 = \partial_0\xi_2$ and $s_1 := \partial_1\xi_1 = \partial_1\xi_2$ and let ξ_1 be locally clockwise oriented and ξ_2 locally counter-clockwise oriented. Then*

$$A_{s_0}^h F_{\xi_1}^{h,g} B_\Lambda = F_{\xi_2}^{h,g} B_\Lambda. \tag{3.3.29}$$

Proof. Let $\xi_1^{du} = (f_1, \dots, f_n)$ and $\xi_2^{du} = (e_1, \dots, e_n)$ be the dual paths of ξ_1 and ξ_2. By assumption, we have $f_1 = e_1$ and $f_n = e_n$, and because consecutive dual ribbons share the same vertex, all dual triangles appearing in the dual ribbons ξ_1 and ξ_2 must share the same vertex v_1. Also note that there generally exists only two dual ribbons from s_0 to s_1, which differ in local orientation. If $\bar{\xi}_1$ and $\bar{\xi}_2$ are the ribbons ξ_1 and ξ_2 with directions reversed, then the ribbon $(\xi_1\bar{\xi}_2)$ is the unique locally clockwise oriented dual ribbon starting and ending at s_0 and $(\xi_2, \bar{\xi}_1)$ is the unique locally counter-clockwise oriented dual ribbon starting and ending at s_1, i.e.

$$\begin{aligned}\xi_{s_0}^{du} =&(\xi_1, \bar{\xi}_2)\\ \bar{\xi}_{s_0}^{du} =&(\xi_2, \bar{\xi}_1),\end{aligned}$$

where $\xi_{s_0}^{du}$ is the ribbon associated with the star operator A_{s_0} cf. Equation (3.3.14) and accompanied discussion. Write $\xi_{s_0}^{du} = (\tau_1, \dots, \tau_4)$ and let $0 \leq t \leq 4$ such that

$$\begin{aligned}F_{\xi_1}^{h,g} =&\delta_{g,e} \prod_{i=1}^{t} L_{\tau_i}^{h},\\ F_{\xi_2}^{h,g} =&\delta_{g,e} \prod_{i=t+1}^{4} L_{\bar{\tau}_i}^{h}.\end{aligned}$$

Note that the cases $t = 0$ and $t = 4$ are indeed covered, because of Equation (3.3.1). Then we have

$$A_{s_0}^{\bar{h}} F_{\xi_1}^{h,g} = \left(\prod_{i=1}^{4} L_{\bar{\tau}_i}^{h}\right) \delta_{g,e} \left(\prod_{i=1}^{t} L_{\tau_i}^{h}\right) \overset{(3.2.5)}{=\!=\!=} \delta_{g,e} \prod_{i=t+1}^{4} L_{\bar{\tau}_i}^{h} = F_{\xi_2}^{h,g}$$

□

Proposition 3.3.10. *Let ξ_1 and ξ_2 be two ribbons contained in some bounded region Λ such that $s_0 := \partial_0\xi_1 = \partial_0\xi_2$ and $s_1 := \partial_1\xi_1 = \partial_1\xi_2$. Then there exists a finite set of star operators*

$$\left\{A_s^{k_{s,\gamma}} \mid s \in \mathcal{S}(\Lambda) \setminus \{s_0, s_1\}, \gamma \in C_G(\Lambda), k_{s,\gamma} \in G\right\} \tag{3.3.30}$$

such that

$$F_{\xi_1}^{h,g} B_\Lambda = \sum_{\gamma \in C_G(\Lambda)} \left(\prod_{s \in \mathcal{S}(\Lambda) \setminus \{s_0, s_1\}} A_s^{k_{s,\gamma}}\right) F_{\xi_2}^{h,g} B_\Lambda T^\gamma. \tag{3.3.31}$$

Proof. Let p^* be a dual path that differs from ξ^{du} only at one dual plaquette $\mathfrak{p}$ and let s be a site with $v(s)$ centred at that dual plaquette. Let furthermore $(\tau_1, \dots, \tau_n)$ be the dual ribbon that is aligned with the intersection of ξ^{du} with $\mathfrak{p}$ and $(\tilde{\tau}_1, \dots, \tilde{\tau}_{n-4})$ the *complement* of $(\tau_1, \dots, \tau_n)$, that is, $(\tau_1, \dots, \tau_n)$ is the unique dual ribbon such that $\xi_s := (\bar{\tau}_n, \dots, \bar{\tau}_1\tilde{\tau}_1, \dots, \tilde{\tau}_n)$ is a closed dual ribbon, see also Figure 3.16.

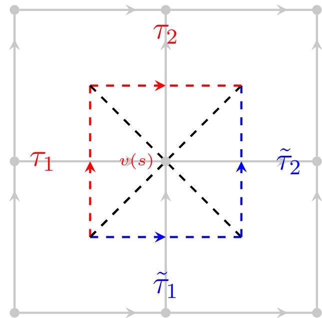


Figure 3.16: Depiction of a dual ribbon (τ_1, τ_2) together with a complementary dual ribbon $(\tilde{\tau}_1, \tilde{\tau}_2)$ such that $(\bar{\tau}_2, \bar{\tau}_1, \tilde{\tau}_1, \tilde{\tau}_2)$ form a closed dual ribbon around the common vertex $v(s)$.

By Proposition 3.3.8, the dual action of $F_\xi^{h,g}$ along $\mathfrak{p}$ is for each $\gamma \notin \ker(F_\xi^{h,g})$ is given by

$$\prod_{i=1}^{n} L_{\tau_i}^{\bar{k}hk},$$

where $k \in G$ is the β-value of γ along any path starting at $\partial_0 \xi^{di}$ and ending at $v(s)$. By Lemma 3.3.9, there is a closed dual ribbon ξ_s such that

$$\left(F_{\xi_s}^{\bar{k}hk,e} \prod_{i=1}^{n} L_{\tau_i}^{\bar{k}hk} \gamma \right)(\mathfrak{e}) = \left(\prod_{i=1}^{n-4} L_{\tilde{\tau}_i} \gamma \right)(\mathfrak{e}).$$

for all edges $\mathfrak{e}$ intersecting $\mathfrak{p}$. It follows for any ribbon ξ' with dual path p^* the identity

$$(F_{\xi_s}^{\bar{k}hk,e} F_\xi^{h,g} \gamma)(\mathfrak{e}) = (F_{\xi'}^{h,g} \gamma)(\mathfrak{e})$$

for all $\mathfrak{e}$ intersecting $\mathfrak{p}$ and G-connection γ with $\gamma \notin \ker(F_\xi^{h,g}) \cup \ker(F_{\xi'}^{h,g})$. Recall also that the operators $F_{\xi_s}^{\bar{k}hk}$ are just star operators. Repeating this process, we see that for any dual path p^* and flat G-connection $\gamma \notin \ker(F_\xi^{h,g})$ we can find appropriate star operators $\left\{ A_s^{k_{s,\gamma}} \right\}_{s \in \mathcal{S}(\Lambda) \setminus \{s_0, s_1\}, k \in G}$ such that the operator

$$\prod_{s \in \mathcal{S}(\Lambda) \setminus \{s_0, s_1\}} A_s^{k_{s,\gamma}} F_\xi^{h,g}$$

has the same dual action as any ribbon operator $F_{\xi_2}^{h,g}$ with $\xi_2^{du} = p^*$ by Proposition 3.3.8.

□

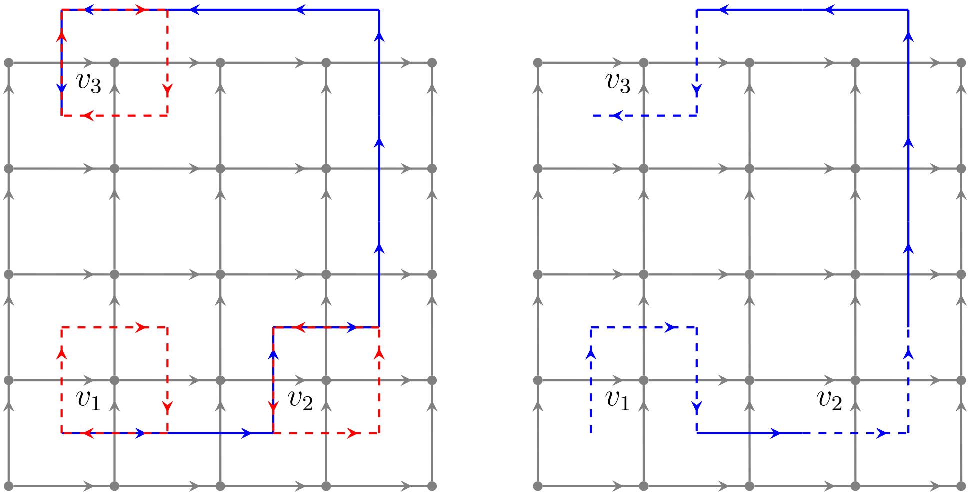


Figure 3.17: Depiction of the deformation process performed on the dual path alone. On the left picture, we see a dual path (blue) and three vertex operators at v_1, v_2 and v_3, depicted by drawing only the dual path (red and dashed). The dual path of the vertex operators is such that it is aligned opposite to the original dual path. On the right, the deformed dual path is shown in blue, where the dashed segments highlight the performed deformation. Note that the endpoints of the dual paths stay fixed, as we do not act with star operators at the endpoints of the ribbon.

This completes the following picture: Just like we were able to *choose* between different direct paths when measuring the β-value of a flat G-connection by changing the way we traverse each plaquette individually, we can *choose* up to multiplication with star operators between different dual paths by changing the way we traverse each dual plaquette individually. We refer to Figure 3.17 for a visualization of the deformation process for dual paths using star operators.

As a final result, we have the following corollary.

Corollary 3.3.11. *Let ξ_1 and ξ_2 be two ribbons sharing the same endpoints and contained in a bounded region Λ, and let $\mathcal{H}$ be the GNS representation of the vacuum state ω_0. If $\psi \in \mathcal{H}$ is such that $A_s\psi = B_s\psi = \psi$ for all $s \in \mathcal{S}(\Lambda)$,* *then*

$$F_{\xi_1}^{h,g}\psi = F_{\xi_2}^{h,g}\psi \tag{3.3.32}$$

for all $h, g \in G$. Furthermore, if σ is a closed ribbon, then we have

$$F_{\sigma}^{h,g}\psi = \delta_{g,e}\psi \tag{3.3.33}$$

Proof. Note first that $A_s^k A_s = A_s$, implies $A_s^k\psi = \psi$ for all $k \in G$. We set $s_0 = \partial_0\xi_1$ and $s_1 = \partial_1\xi_1$. By Proposition 3.3.10, we have

$$F_{\xi_1}^{h,g}\psi = F_{\xi_1}^{h,g}B_\Lambda\psi = \sum_{\gamma\in C_G(\Lambda)}\left(\prod_{s\in\mathcal{S}\setminus\{s_0,s_1\}} A_s^{k_{s,\gamma}}\right)F_{\xi_2}^{h,g}B_\Lambda T^\gamma\psi$$

For a family of star operators $A_s^{k_{s,\gamma}}$ as in Equation (3.3.30). By Equation 3.3.7, we can commute $F_{\xi_2}^{h,g}$ with each $A_s^{k_{s,\gamma}}$, since $s \notin \{\partial_0\xi_2, \partial_1\xi_2\}$. Also note that commuting star operators with products of triangle operators of the form T^γ simply affords a bijection on the set of G-connections. We denote by $\gamma_{k_{s,\gamma}}$ the G-connection that satisfies $(\prod_s A_s^{k_{s,\gamma}})T^\gamma = T^{\gamma_{k_{s,\gamma}}}(\prod_s A_s^{k_{s,\gamma}})$. Then the above expression becomes

$$\begin{aligned}
&\sum_{\gamma\in C_G(\Lambda)} F_{\xi_2}^{h,g}T^{\gamma_{k_{s,\gamma}}}\left(\prod_{s\in\mathcal{S}\setminus\{\partial_0\xi_1,\partial_1\xi_1\}} A_s^{k_{s,\gamma}}\right)\psi\\
=&\sum_{\gamma\in C_G(\Lambda)} F_{\xi_2}^{h,g}T^{\gamma_{k_{s,\gamma}}}\psi\\
=&\sum_{\gamma\in C_G(\Lambda)} F_{\xi_2}^{h,g}T^{\gamma}\psi\\
=&F_{\xi_2}^{h,g}\psi.
\end{aligned}$$

To see (3.3.33), we deploy an analogue argument as in the proof of Lemma 3.3.6. Let $\sigma = (\xi_1, \xi_2)$. Since σ is closed, ξ_2 is a deformation of $\bar{\xi}_1$ and we have

$$F_\sigma\psi \overset{(3.3.2)}{=} \sum_{k\in G} F_{\xi_1}^{h,k}F_{\xi_2}^{\bar{k}hk,\bar{k}g}\psi = \sum_{k\in G} F_{\xi_1}^{h,k}F_{\bar{\xi}_1}^{\bar{k}hk,\bar{k}g}\psi \overset{(3.3.20)}{=} \psi$$

□

The next proposition describes the energy of the local observables $F_\xi^{h,g}$ in states that have no excitations at the endpoints of the ribbon ξ.

Proposition 3.3.12. *Let ξ be a ribbon and let ω be a linear functional with $\omega(F_\xi^{h,g}) = \omega(A_s^k F_\xi^{h,g} A_s^k) = \omega(B_s F_\xi^{h,g} B_s)$ for all $k, h, g \in G$ and $s \in \partial\xi$. Then*

$$\omega\left(F_\xi^{h,g}\right) = \delta_{h,e}\frac{1}{|G|} \tag{3.3.34}$$

For all $h, g \in G$. In particular, the above holds for the translational invariant ground state ω_0 of the quantum double model.

Proof. Let $s = \partial_0\xi$ be the initial site of the ribbon ξ. Then

$$\omega(F_\xi^{h,g}) = \omega(B_s F_\xi^{h,g} B_s) \overset{(3.3.12)}{=} \omega\left(F_\xi^{h,g} B_s^h B_s\right) = \delta_{h,e}\omega\left(F_\xi^{h,g}\right)$$

If $h \neq e$, (3.3.34) is trivially true. Otherwise, we have

$$\omega\left(F_\xi^{e,g}\right) = \omega\left(A_{s_1}^k F_\xi^{e,g} A_{s_1}^k\right) \overset{(3.3.6)}{=} \omega\left(A_{s_1}^k A_{s_1}^{\bar{k}} F_\xi^{e,gk}\right) = \omega\left(F_\xi^{e,gk}\right)$$

for all $g, k \in G$, giving $\omega\left(F_\xi^{e,g_1}\right) = \omega\left(F_\xi^{e,g_2}\right)$ for all $g_1, g_2 \in G$. The result then follows from $\sum_{g\in G} F_\xi^{e,g} = 1_{\mathfrak{A}}$. □

We conclude this section by showing how ribbon operators can be used to generate orthogonal subspaces in the GNS representation of the vacuum state ω_0.

Proposition 3.3.13. *Let ξ be a ribbon, $h, g \in G$ and ω_0 the vacuum state and $(\mathcal{H}, \pi, \Omega)$ the corresponding GNS representation. Then the set $\left\{F_\xi^{h,g}\right\}_{h,g\in G}$ is linearly independent and the vectors $\left\{F_\xi^{h,g}\Omega_0\right\}_{h,g\in G}$ are mutually orthogonal. Furthermore, the vacuum state Ω_0 is separating for the algebra generated by the ribbon operators $\left\{F_\xi^{h,g}\right\}_{h,g\in G}$.*

Proof. From

$$\left\|F_\xi^{h,g}\Omega_0\right\|^2 = \omega\left(F_\xi^{\bar{h},g} F_\xi^{h,g}\right) = \omega(F_\xi^{e,g}) = \frac{1}{|G|}$$

it follows that $F_\xi\Omega_0 \neq 0$, where we used Proposition 3.3.12 for ω_0. The orthogonality of the vectors $F_\xi^{h,g}\Omega_0$ follows by direct calculation:

$$\begin{aligned}\left\langle F_\xi^{h_1,g_1}\Omega_0, F_\xi^{h_2,g_2}\Omega_0\right\rangle &= \omega_0\left(\left(F_\xi^{h_1,g_1}\right)^* F_\xi^{h_2,g_2}\right)\\ &\overset{(3.3.18)}{=} \delta_{g_1,g_2}\omega_0\left(F^{\bar{h}_1 h_2, g_1}\right)\\ &\overset{(3.3.34)}{=} \delta_{g_1,g_2}\delta_{h_1,h_2}\omega_0(F^{e,g_1}).\end{aligned}$$

This also shows that the operators $F_\xi^{h,g}$ are orthogonal with respect to the sesquilinear map $(A, B) \mapsto \omega_0(A^*B)$ and hence linearly independent. Finally, let $g_1, g_2, h_1, h_2 \in G$ such that $F_\xi^{h_1,g_1}\Omega_0 = F_\xi^{h_2,g_2}\Omega_0$. Then

$$\begin{aligned}
0 = & \left\| \left(F_\xi^{h_1,g_1} - F_\xi^{h_2,g_2}\right)\Omega_0 \right\|^2 = \langle (F_\xi^{h_1,g_1} - F_\xi^{h_2,g_2})\Omega_0, \left(F_\xi^{h_1,g_1} - F_\xi^{h_2,g_2}\right)\Omega_0 \rangle \\
= & \omega\left(\left(F_\xi^{h_1,g_1}\right)^* F_\xi^{h_1,g_1}\right) + \omega\left(\left(F_\xi^{h_2,g_2}\right)^* F_\xi^{h_2,g_2}\right) \\
& - \left[\omega\left(\left(F_\xi^{h_1,g_1}\right)^* F_\xi^{h_2,g_2} + \left(F_\xi^{h_2,g_2}\right)^* F_\xi^{h_1,g_1}\right)\right] \\
= & \left\| F_\xi^{h_1,g_1}\Omega \right\|^2 + \left\| F_\xi^{h_2,g_2}\Omega_0 \right\|^2 - \delta_{g_1,g_2}\left[\omega\left(F_\xi^{\bar{h}_1 h_2, g_1}\right) + \omega\left(F_\xi^{\bar{h}_2 h_1, g_2}\right)\right] \\
\Leftrightarrow & 2\left\| F_\xi^{h_1,g_1}\Omega_0 \right\|^2 = \delta_{g_1,g_2}\left[\omega\left(F_\xi^{\bar{h}_1 h_2, g_1}\right) + \omega\left(F_\xi^{\bar{h}_2 h_1, g_2}\right)\right]
\end{aligned}$$

If $g_1 \neq g_2$ then $\left\| F^{h_1,g_1}\Omega_0 \right\| = 0$ follows, contradicting $F_\xi^{h,g}\Omega_0 \neq 0$, hence $g_1 = g_2$. Similarly, the right-hand side becomes zero, unless $h_1 = h_2$ by Proposition 3.3.12. That Ω_0 is separable for linear combinations of ribbon operators as well follows from the linear independence of the $F_\xi^{h,g}\Omega_0$. Finally, note that by Equation (3.3.18), the algebra generated by the ribbon operators along a fixed ribbon ξ is already spanned by the set $\left\{F_\xi^{h,g}\right\}_{h,g\in G}$ □

A ribbon operator $F_\xi^{h,g}$ is understood to create anyonic excitations at the endpoints of the ribbon ξ, and extending the ribbon ξ corresponds to moving the anyons at the endpoints around. As explained in the introduction of this work, an anyonic excitation is characterized by a non-trivial exchange statistics, that is, an exchange statistic that does not simply correspond to a sign but rather to a complex phase for abelian anyons and unitary transformations in the more general case of non-abelian anyons. To see that ribbon operators indeed create anyonic excitations at the endpoints of a ribbon, one can calculate the commutation relation of two ribbon operators, $F_{\xi_1}^{h_1,g_1}$ and $F_{\xi_2}^{h_2,g_2}$, whose ribbons ξ_1 and ξ_2 *split*, i.e. there exists ribbons ξ', ξ_1', ξ_2' such that $\xi_1 = (\xi', \xi_1')$ and $\xi_2 = (\xi', \xi_2')$, see Figure 3.18. The study of these braid relations lies outside the scope of this work, but the interested reader may consult [Kit03], particularly [Kit03, Section 5.3]. We will, however, examine these braid relations in greater detail in [BHNVBA].

In the next section, we will concretely study these anyonic states created by ribbon operators and unfold their relation to the representation theory of the quantum double $D(G)$ of the group G.

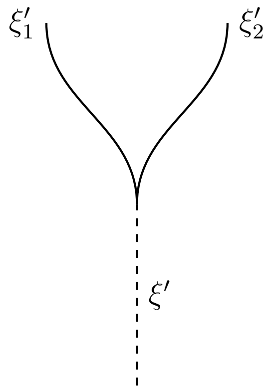


Figure 3.18: Depiction of two initially overlapping ribbons $\xi_1 = (\xi', \xi'_1)$ and $\xi_2 = (\xi', \xi'_2)$ that split. For simplicity, the ribbons are depicted as curves, rather than concatenation of sites. The dashed line ξ' indicates that the ribbons ξ_1 and ξ_2 start at some arbitrary site in the distance.

3.4 Anyon excitations

We have seen in the previous section that we can construct vectors in the GNS representation $(\mathcal{H}, \pi, \Omega)$ of the vacuum state ω_0 that violate the ground state conditions in Equation (3.2.27) by means of ribbon operators. In this section, we will see that the different charges are closely related to the irreducible representations of the quantum double $D(G)$.

We mentioned that the star operators A_s^k for $k \in G$ at site $s = (v, f)$ was independent of the concrete choice of the face f as long as the vertex v stays the same. The next proposition says that star- and plaquette operators realize an action of the quantum double for each site s, justifying the symmetric notation for both plaquette- and star operators.

Proposition 3.4.1. *Let Λ be a finite region containing the site s, and let $D(G)$ be the quantum double of G. Then the map*

$$U_s : D(G) \to \mathfrak{A}_\Lambda, \delta_g \otimes k \mapsto B_s^g A_s^k \tag{3.4.1}$$

*establishes a faithful representation of $D(G)$ on $\mathcal{H}_\Lambda$ for each site $s \in \mathcal{S}$. Furthermore, if $D(G)$ is equipped with the star involution $(\delta_g \otimes h)^* = \delta_{hg\bar{h}} \otimes \bar{h}$ as defined in Equation (2.4.27), U_s becomes a *-homomorphism from $D(G)$ to $\mathfrak{A}_\Lambda$.*

Proof. Clearly, the set $\left\{B_s^g A_s^h \mid h, g \in G\right\}$ is linearly independent. The rest follows from

$$
\begin{aligned}
&U_s\left[(\delta_{g_1}\otimes h_1)\cdot(\delta_{g_2}\otimes h_2)\right]=\delta_{g_1,h_1g_2\bar{h_1}}U_s\left[\delta_{g_1}\otimes h_1h_2\right]=B_s^{g_1}B_s^{h_1g_2\bar{h_1}}A_s^{h_1h_2}\\
=&B_s^{g_1}B_s^{h_1g_2\bar{h_1}}A_s^{h_1}A_s^{h_2}=B_s^{g_1}A_s^{h_1}B_s^{g_2}A_s^{h_2},
\end{aligned}
$$

and

$$
\begin{aligned}
(U_s(\delta_g\otimes h))^*=&\left(B_s^gA_s^h\right)^*=A_s^{\bar{h}}B_s^g=B_s^{\bar{h}gh}A_s^{\bar{h}}=U_s(\delta_{\bar{h}gh}\otimes\bar{h})\\
=&U_s\left((\delta_g\otimes h)^*\right).
\end{aligned}
$$

□

Thus, $\mathcal{H}$ becomes a $D(G)$-module under the action. We want to identify the irreducible submodules at each site with the aid of ribbon operators.

Before we continue, we review the notation given in Section 2.4.2. Denote by $\widehat{D(G)}$ a fixed set of inequivalent representatives of the set of irreducible representations of $D(G)$. Recall that elements $\alpha\in\widehat{D(G)}$ can be labelled by pairs $\alpha=(\pi_\alpha,\mathcal{C}_\alpha)$, where $\mathcal{C}_\alpha\in G_C$ is a conjugacy class of G and π_α an irreducible representation of the centralizer subgroup N_α of a fixed element $r_\alpha\in\mathcal{C}_\alpha$. Writing $\mathcal{C}_\alpha=\left\{c_1,\dots,c_{|\mathcal{C}_\alpha|}\right\}$, we fix for each $\alpha\in\widehat{D(G)}$ elements $q_i\in G/N_\alpha$ such that $c_i=q_ir_\alpha\bar{q}_i$ for all $c_i\in\mathcal{C}_\alpha$. The irreducible modules of the quantum double are concretely given by

$$
\mathcal{V}^\alpha:=\mathbb{C}\mathcal{C}_\alpha\otimes V_{\pi_\alpha}, \tag{3.4.2}
$$

where V_{π_α} is the irreducible module associated to π_α, and the concrete action on $\mathcal{V}^\alpha$ by elements in $D(G)$ is given via

$$
(\delta_g\otimes h)\rhd(c\otimes v)=\delta_{g,hc\bar{h}}hc\bar{h}\otimes\pi\left(\bar{q}_{hc\bar{h}}hq_c\right)(v). \tag{3.4.3}
$$

Finally, we set

- $I_{\mathcal{C}_\alpha}=\{1,\dots,|\mathcal{C}_\alpha|\}$
- $I_{\pi_\alpha}=\{1,\dots,\dim_{\pi_\alpha}\}$
- $I_\alpha=I_{\mathcal{C}_\alpha}\times I_{\pi_\alpha}$

so that $\mathcal{C}_\alpha=\{c_k\}_{k\in I_{\mathcal{C}_\alpha}}$. Motivated by the arguments in the proof of Theorem 2.2.6, see in particular Equation (2.2.9), we choose the following new basis for the ribbon operators constructed in Section 3.3: Given index pairs $I,J\in I_\alpha$ with $I=(i_1,i_2)$ and $J=(j_1,j_2)$ and a ribbon ξ, we define the operator

$$
F_\xi^{IJ,\alpha}=\sqrt{d_\alpha}\sum_{g,h\in G}\bar{\Gamma}_\alpha^{IJ}(\delta_g\otimes h)F_\xi^{\bar{g},h}, \tag{3.4.4}
$$

where Γ_α^{IJ} are the unitary matrix coefficients of α as in Equation (2.4.33) and $d_\alpha = |\mathcal{C}_\alpha| \dim_{\pi_\alpha}$ is the dimension of α.

It follows from Equation (2.4.33) that the summation over $g \in G$ is non-zero only if $g = c_{i_1} = hc_{j_1}\bar{h}$. Since for each $h \in N_\alpha$ there exists a unique $n \in N_\alpha$ such that $h = q_{i_1} n \bar{q}_{j_1}$, the above formula reads

$$F_\xi^{IJ,\alpha} = \sqrt{d_\alpha} \sum_{n \in N_\mathcal{C}} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(n) F_\xi^{\bar{c}_{i_1}, q_{i_1} n \bar{q}_{j_1}}, \tag{3.4.5}$$

where Γ_{π_α} is a unitary matrix representation of π_α. The reason for choosing $\bar{g}$ instead of g in Equation (3.4.4) is so that the type of the magnetic charge at site $\partial_0\xi$ inserted by $F_\xi^{IJ,\alpha}$ would be $\mathcal{C}_\alpha$, since $F_\xi^{g,h}$ inserts the flux $\bar{g}$ at site $\partial_0\xi$. The constant $\sqrt{d_\alpha}$ serves as a normalization factor, as we will see later.

These operators indeed give a decomposition of the GNS representation $\mathcal{H}$ into irreducible components.

Proposition 3.4.2. *Let ξ be an open ribbon, i.e. $\partial_0\xi \neq \partial_1\xi$. Then the vectors*

$$\left\{ F_\xi^{IJ,\alpha}\Omega_0 \mid \alpha \in \widehat{D(G)}, I, J \in I_\alpha \right\}$$

form an orthonormal set of vectors in $\mathcal{H}$. Furthermore, for each fixed $\alpha \in \widehat{D(G)}$ and fixed $J \in I_\alpha$, the subspaces

$$\mathcal{W}^{\alpha,J} = \operatorname{span}_{I \in I_\alpha} \left\{ F_\xi^{IJ,\alpha}\Omega_0 \right\} \tag{3.4.6}$$

are mutually orthogonal and irreducible $D(G)$-submodules of $\mathcal{H}$ with action given as in Proposition 3.4.1 at site $s = \partial_0\xi$ and are isomorphic to the irreducible modules $\mathcal{V}^\alpha$ given in Equation (3.4.2) as $D(G)$ representations for each fixed choice of $J \in I_\alpha$. It follows that the space

$$\left\{ F_\xi^{h,g} \mid g, h \in G \right\}$$

is isomorphic to the regular representation of $D(G)$.

Proof. We show the orthonormality by direct calculation: Let $\alpha, \beta \in \widehat{D(G)}$ and write $I, J \in I_\alpha$ with $I = (i_1, i_2)$, $J = (j_1, j_2)$ and $K, L \in I_\beta$ with

$K = (k_1, k_2)$, $L = (l_1, l_2)$. Then

$$\begin{aligned}\left\langle F_\xi^{IJ,\alpha}\Omega_0, F_\xi^{KJ,\beta}\Omega_0\right\rangle =& \sqrt{d_\alpha d_\beta}\sum_{\substack{n\in N_\alpha\\ m\in N_\beta}}\Gamma_{\pi_\alpha}^{i_2,j_2}(n)\bar{\Gamma}_{\pi_\beta}^{k_2,l_2}(m)\omega_0\left(F_\xi^{c_{i_1},q_{i_1}n\bar{q}_{j_1}}F_\xi^{\bar{c}_{k_1},q_{k_1}m\bar{q}_{l_1}}\right)\\ \overset{(3.3.18)}{=}& \sqrt{d_\alpha d_\beta}\sum_{\substack{n\in N_\alpha\\ m\in N_\beta}}\Gamma_{\pi_\alpha}^{i_2,j_2}(n)\bar{\Gamma}_{\pi_\beta}^{k_2,l_2}(m)\delta_{q_{i_1}n\bar{q}_{j_1},q_{k_1}m\bar{q}_{l_1}}\omega_0\left(F_\xi^{c_{i_1}\bar{c}_{k_1},q_{i_1}n\bar{q}_{j_1}}\right)\\ \overset{(3.3.34)}{=}& \delta_{i_1,k_1}\frac{\sqrt{d_\alpha d_\beta}}{|G|}\sum_{\substack{n\in N_\alpha\\ m\in N_\beta}}\delta_{q_{i_1}n\bar{q}_{j_1},q_{k_1}m\bar{q}_{l_1}}\Gamma_{\pi_\alpha}^{i_2,j_2}(n)\bar{\Gamma}_{\pi_\beta}^{k_2,l_2}(m)\end{aligned}$$

The identity $i_1 = k_1$ gives $n\bar{q}_{j_1} = m\bar{q}_{l_1}$ for the above expression to not be zero, and because each element $g \in G$ can be uniquely factorized as $g = n_0\bar{q}_s$ for some $n_0 \in N_\alpha$ and $q_s \in \mathcal{C}_\alpha$, it follows that $m = n$ and $l_1 = j_1$. Above expression becomes

$$\begin{aligned}&\delta_{i_1,k_1}\delta_{j_1,l_1}\frac{\sqrt{d_\alpha d_\beta}}{|G|}\sum_{\substack{n\in d_\alpha\\ m\in n_\beta}}\delta_{n,m}\Gamma_{\pi_\alpha}^{i_2,j_2}(n)\bar{\Gamma}_{\pi_\beta}^{k_2,l_2}(n)\\ \overset{(2.2.5)}{=\!=}&\delta_{i_1,k_1}\delta_{j_1,l_1}\delta_{i_2,k_2}\delta_{j_2,l_2}\delta_{\alpha,\beta}\frac{\sqrt{d_\alpha d_\beta}}{|G|}\frac{|N_\alpha|}{\dim_{\pi_\alpha}}\\ =&\delta_{I,K}\delta_{J,L}\delta_{\alpha,\beta}\frac{|\mathcal{C}_\alpha|\,|N_\alpha|\dim_{\pi_\alpha}}{|G|\dim_{\pi_\alpha}}\\ =&\delta_{I,K}\delta_{J,L}\delta_{\alpha,\beta},\end{aligned}$$

where we used that $|G| = |\mathcal{C}_\alpha|\,|N_\alpha|$. Let $\{b_j\}_{j=1,\ldots,\dim_{\pi_\alpha}}$ be an orthonormal basis of the irreducible representation V_{π_α} of N_α such that $\Gamma_{\pi_\alpha}^{i_2j_2}(n) = \langle b_{i_2}, \pi_\alpha(n)b_{j_2}\rangle$ is a unitary matrix representation of π_α. Then it follows that a linear map $\phi : \mathcal{V}^\alpha \to \mathcal{W}^{\alpha,J}$ can be defined as the linear extension of

$$c_{i_1}\otimes b_{i_2} \mapsto \sum_{n\in N_\alpha}\bar{\Gamma}_{\pi_\alpha}^{i_2j_2}(n)F_\xi^{\bar{c}_{i_1},q_{i_1}n\bar{q}_{j_1}}\Omega_0 \tag{3.4.7}$$

and that $\dim(\mathcal{W}^{\alpha,J}) = |N_\alpha|\cdot|\mathcal{C}_\alpha| = \dim(\mathcal{V}^\alpha)$. Hence, ϕ is an isomorphism of vector spaces. Moreover, if ξ is locally clockwise oriented, we have

$$\begin{aligned}U_s(\delta_g\otimes h)F_\xi^{IJ,\alpha}\Omega_0 =& \sqrt{d_\alpha}\sum_{n\in N_\alpha}B_s^gA_s^h\bar{\Gamma}_{\pi_\alpha}^{i_2,j_2}(n)F_\xi^{\bar{c}_{i_1},q_{i_1}n\bar{q}_{j_1}}\Omega_0\\ \overset{(3.3.6)}{=\!=}& \sqrt{d_\alpha}\sum_{n\in N_\alpha}\bar{\Gamma}_{\pi_\alpha}^{i_2,j_2}(n)B_s^gF_\xi^{h\bar{c}_{i_1}\bar{h},hq_{i_1}n\bar{q}_{j_1}}\Omega_0\\ \overset{(3.3.8)}{=\!=}& \sqrt{d_\alpha}\sum_{n\in N_\alpha}\bar{\Gamma}_{\pi_\alpha}^{i_2,j_2}(n)F_\xi^{h\bar{c}_{i_1}\bar{h},hq_{i_1}n\bar{q}_{j_1}}B_s^{gh\bar{c}_{i_1}\bar{h}}\Omega_0.\end{aligned}$$

and if ξ was locally counter-clockwise oriented, (3.3.10) gives

$$\sqrt{d_\alpha}\sum_{n\in N_\alpha}\bar{\Gamma}^{i_2,j_2}_{\pi_\alpha}(n)F_\xi^{h\bar{c}_{i_1}\bar{h},hq_{i_1}n\bar{q}_{j_1}}B_s^{h\bar{c}_{i_1}\bar{h}g}\Omega_0.$$

In either case, we have $B_s^{gh\bar{c}_{i_1}\bar{h}}\Omega_0 = B_s^{h\bar{c}_{i_1}\bar{h}g}\Omega_0 = \delta_{g,h\bar{c}_{i_1}\bar{h}}\Omega_0$, and the above expression becomes

$$\delta_{g,h\bar{c}_{i_1}\bar{h}}\sqrt{d_\alpha}\sum_{n\in N_\alpha}\bar{\Gamma}^{i_2,j_2}_{\pi_\alpha}(n)F_\xi^{h\bar{c}_{i_1}\bar{h},hq_{i_1}n\bar{q}_{j_1}}\Omega_0.$$

From the coset decomposition $G=\dot{\bigcup}_{i\in I_{\mathcal{C}_\alpha}} q_iN_\alpha$, it follows that there exists a unique pair $(k,m)\in I_{\mathcal{C}_\alpha}\times N_\alpha$ such that $hq_{i_1}=q_km$. This implies that

$$h\bar{c}_{i_1}\bar{h} = q_km\bar{q}_{i_1}(q_{i_1}\bar{r}\bar{q}_{i_1})q_{i_1}\bar{m}\bar{q}_k = q_km\bar{r}\bar{m}\bar{q}_k = q_k\bar{r}\bar{q}_k = \bar{c}_k. \tag{3.4.8}$$

Hence, we obtain

$$\begin{aligned}
&\delta_{g,hc_{i_1}\bar{h}}\sqrt{d_\alpha}\sum_{n\in N_\alpha}\bar{\Gamma}^{i_2,j_2}_{\pi_\alpha}(n)F_\xi^{\bar{c}_k,q_kmn\bar{q}_{j_1}}\Omega_0\\
\overset{n\mapsto\bar{m}n}{}&\delta_{g,hc_{i_1}\bar{h}}\sqrt{d_\alpha}\sum_{n\in N_\alpha}\bar{\Gamma}^{i_2j_2}_{\pi_\alpha}(\bar{m}n)F_\xi^{\bar{c}_k,q_kn\bar{q}_{j_1}}\Omega_0\\
=&\delta_{g,hc_{i_1}\bar{h}}d_\alpha\sum_{n\in N_\alpha}\sum_{t=1}^{\dim_{\pi_\alpha}}\bar{\Gamma}^{i_2t}_{\pi_\alpha}(\bar{m})\bar{\Gamma}^{tj_2}_{\pi_\alpha}(n)F_\xi^{\bar{c}_k,q_kn\bar{q}_{j_1}}\Omega_0
\end{aligned}$$

Writing $I_t=(k,t)$, the above expression simplifies to

$$\begin{aligned}
\delta_{g,hc_{i_1}\bar{h}}\sum_{t=1}^{\dim_{\pi_\alpha}}\bar{\Gamma}^{i_2t}_{\pi_\alpha}(\bar{m})F_\xi^{I_tJ,\alpha}\Omega_0 =&\delta_{g,hc_{i_1}\bar{h}}\sum_{t=1}^{\dim_{\pi_\alpha}}\Gamma^{ti_2}_{\pi_\alpha}(m)\phi(c_k\otimes b_t).\\
=&\phi\left(\delta_{g,hc_{i_1}\bar{h}}hc_{i_1}\bar{h}\otimes\pi_\alpha(m)(b_{i_2})\right).\\
\overset{(3.4.8)}{=\!=}&\phi\left(\delta_{g,hc_{i_1}\bar{h}}hc_{i_1}\bar{h}\otimes\pi_\alpha(\bar{q}_khq_{i_1})(b_{i_2})\right).\\
\overset{(3.4.3)}{=\!=}&\phi((\delta_g\otimes h)\triangleright(c_{i_1}\otimes b_{i_2})).
\end{aligned}$$

where we used that $hc_{i_1}\bar{h}=c_k \Leftrightarrow q_k=q_{hc_{i_1}\bar{h}}$. Hence, ϕ is an isomorphism of $D(G)$-modules.

Finally, since the isomorphism $\mathcal{W}^{\alpha,J}\cong\mathcal{V}^\alpha$ holds for each fixed $J\in I_\alpha$, we have at each site $\dim_\alpha=|\mathcal{C}_\alpha|\cdot|N_\alpha|$ many copies of the irreducible representation $\mathcal{V}^\alpha$. Using Proposition 2.4.11, we obtain

$$\left\{F_\xi^{h,g}\mid g,h\in G\right\}\cong\bigoplus_{\alpha\in\widehat{D(G)}}d_\alpha\mathcal{W}^{\alpha,J}\cong\bigoplus_{\alpha\in\widehat{D(G)}}d_\alpha\mathcal{V}^\alpha\simeq D(G). \tag{3.4.9}$$

□

The fact that the $\mathcal{W}^{\alpha,J}$ can be identified with subrepresentations of the regular representation of $D(G)$ was already shown in [CM22b, Proposition 3.10] using the same techniques, i.e. the commutation relations for ribbon operators and star- and plaquette operators. However, there the local orientation of the ribbons were not taken into account.

Since ribbon operators generate dual and direct triangle operators, they generate the algebra $\mathfrak{A}_{loc}$ of local observables. Thus, $\mathcal{H}$ contains a copy of $D(G)$ as a submodule for each site s of the lattice. The projections into the corresponding irreducible submodules given in Equation (3.4.6) are given by the central projections in Equation (2.4.37). Under the action U_s given in Equation (3.4.1), these projections take the form

$$P_s^\alpha = \frac{\dim_{\pi_\alpha}}{|N_\alpha|} \sum_{n\in N_\alpha} \sum_{i\in I_{\mathcal{C}_\alpha}} \overline{\mathrm{tr}}_{\pi_\alpha}(n) A_s^{q_{c_i} n \bar{q}_{c_i}} B_s^{c_i}. \tag{3.4.10}$$

We can interpret these operators as charge detectors at site s, but there is a different way of detecting charges. Another class of projections is given by

$$P_\sigma^\alpha = \frac{\dim_{\pi_\alpha}}{|N_\alpha|} \sum_{n\in N_\alpha} \sum_{i\in I_{\mathcal{C}_\alpha}} \overline{\mathrm{tr}}_{\pi_\alpha}(n) F_\sigma^{q_{c_i} n \bar{q}_{c_i}, c_i}. \tag{3.4.11}$$

In either case, the projections given in Equation (3.4.11) respectively Equation (3.4.10) are mutually orthogonal, i.e. $P_\sigma^{\alpha_1} P_\sigma^{\alpha_2} = P_\sigma^{\alpha_2} P_\sigma^{\alpha_1} = \delta_{\alpha_1,\alpha_2} P_\sigma^{\alpha_1}$ and $P_s^{\alpha_1} P_s^{\alpha_2} = P_s^{\alpha_2} P_s^{\alpha_1} = \delta_{\alpha_1,\alpha_2} P_s^{\alpha_1}$. We claim that these two types of projections operators coincide on states that have a single excitation at some site and we will show this using the deformation property for ribbon operators. This is particularly surprising, since the domain of P_σ^α would be disjoint of the site s. In fact, the closed ribbon can be arbitrarily large and far away from the site s. This property will make these operators particularly useful later on.

Before we attempt to prove our claim, we want to show that the operators defined in Equation (3.4.11) are rotationally invariant.

Proposition 3.4.3. *Let $\sigma = (s_1, \ldots, s_n)$ be a closed ribbon, that is, $s_n = s_0$, and let $\sigma' = (s_k, \ldots, s_{n-1}, s_0, \ldots, s_{k-1})$ be a rotation of σ. Then*

$$P_\sigma^\alpha = P_{\sigma'}^\alpha$$

for all irreducible representations $\alpha \in \widehat{D(G)}$.

Proof. Let $\sigma_1 = (s_0, \ldots, s_{k-1})$ and $\sigma_2 = (s_k, \ldots, s_{n-1})$ such that $\sigma = (\sigma_1\sigma_2)$

and $\sigma' = (\sigma_2\sigma_1)$. We calculate

$$\frac{|N_\alpha|}{\dim_{\pi_\alpha}} P_\sigma^\alpha = \sum_{n\in N_\alpha}\sum_{i\in I_{\mathcal{C}_\alpha}} \overline{\mathrm{tr}}(n) F_\sigma^{q_i n\bar{q}_i, c_i}$$
$$\overset{(3.3.2)}{=} \sum_{n\in N_\alpha}\sum_{i\in I_{\mathcal{C}_\alpha}}\sum_{k\in G} \overline{\mathrm{tr}}(n) F_{\sigma_1}^{q_i n\bar{q}_i, k} F_{\sigma_2}^{\bar{k}q_i n\bar{q}_i k, \bar{k}c_i}.$$

For every fixed $i \in I_{\mathcal{C}_\alpha}$ we can find some $j \in I_{\mathcal{C}_\alpha}$ and $m \in N_\alpha$ such that $k = q_i m\bar{q}_j$. Writing further $c_i = q_i r_\alpha \bar{q}_i$, the sum can alternatively be written as

$$\sum_{n\in N_\alpha}\sum_{i\in I_{\mathcal{C}_\alpha}}\sum_{j\in I_{\mathcal{C}_\alpha}, m\in N_\alpha} \overline{\mathrm{tr}}(n) F_{\sigma_1}^{q_i n\bar{q}_i, q_i m\bar{q}_j} F_{\sigma_2}^{q_j\bar{m}nm\bar{q}_j, q_j\bar{m}\bar{q}_i q_i r_\alpha \bar{q}_i}$$
$$\overset{n\mapsto mn\bar{m}}{=\!=\!=} \sum_{n\in N_\alpha}\sum_{i\in I_{\mathcal{C}_\alpha}}\sum_{j\in I_{\mathcal{C}_\alpha}, m\in N_\alpha} \overline{\mathrm{tr}}(n) F_{\sigma_2}^{q_j n\bar{q}_j, q_j\bar{m}r_\alpha\bar{q}_i} F_{\sigma_1}^{q_i mn\bar{m}\bar{q}_i, q_i m\bar{q}_j}$$
$$\overset{m\mapsto r_\alpha m}{=\!=\!=} \sum_{n\in N_\alpha}\sum_{i\in I_{\mathcal{C}_\alpha}}\sum_{j\in I_{\mathcal{C}_\alpha}, m\in N_\alpha} \overline{\mathrm{tr}}(n) F_{\sigma_2}^{q_j n\bar{q}_j, q_j\bar{m}\bar{q}_i} F_{\sigma_1}^{q_i r_\alpha mn\bar{m}\bar{r}_\alpha\bar{q}_i, q_i r_\alpha m\bar{q}_j}.$$

Note that $m, n \in Z_G(r_\alpha)$, giving $r_\alpha mn\bar{m}\bar{r}_\alpha = mn\bar{m}$. Note also that we can substitute $q_i m$ with $q_i m q_j$ since by summing over all $i \in I_\alpha$ and $m \in N_\alpha$ the expression $q_i m$ runs through all group elements in G. The expression then simplifies to

$$\sum_{n\in N_\alpha}\sum_{i\in I_{\mathcal{C}_\alpha}}\sum_{j\in I_{\mathcal{C}_\alpha}, m\in N_\alpha} \overline{\mathrm{tr}}(n) F_{\sigma_2}^{q_j n\bar{q}_j, \bar{m}\bar{q}_i} F_{\sigma_1}^{q_i m q_j n\bar{q}_j\bar{m}\bar{q}_i, q_i m q_j r_\alpha\bar{q}_j}$$
$$\overset{q_j r_\alpha\bar{q}_j = c_j}{=\!=\!=} \sum_{n\in N_\alpha}\sum_{i\in I_{\mathcal{C}_\alpha}}\sum_{j\in I_{\mathcal{C}_\alpha}, m\in N_\alpha} \overline{\mathrm{tr}}(n) F_{\sigma_2}^{q_j n\bar{q}_j, \overline{(q_i m)}} F_{\sigma_1}^{(q_i m) q_j n\bar{q}_j (\overline{q_i m}), q_i m c_j}$$

Finally, by writing $k = \overline{(q_i m)}$ the sum becomes

$$\sum_{j\in I_{\mathcal{C}_\alpha}, m\in N_\alpha}\sum_{n\in N_\alpha}\sum_{k\in G} \overline{\mathrm{tr}}(n) F_{\sigma_2}^{q_j n\bar{q}_j, k} F_{\sigma_1}^{\bar{k}q_j n\bar{q}_j k, \bar{k}c_j}$$
$$\overset{(3.3.2)}{=\!=\!=} \sum_{j\in I_{\mathcal{C}_\alpha}, m\in N_\alpha}\sum_{n\in N_\alpha} \overline{\mathrm{tr}}(n) F_{\sigma'}^{q_j n\bar{q}_j, c_j}$$
$$= \frac{|N_\alpha|}{\dim_{\pi_\alpha}} P_{\sigma'}^\alpha.$$

□

Note that rotationally invariant ribbon operators automatically commute with all star and plaquette operators. This is because star and plaquette operators A_s^h and B_s^g commute with P_σ^α at all sites $s \notin \{\partial_0\sigma, \partial_1\sigma\}$ by Equation (3.3.7) but by Proposition 3.4.3, the endpoints of σ can be changed without changing the operator P_σ^α. If σ is a closed ribbon, we define the **interior of** σ to be the connected component bounded by the direct and dual path of σ in $\mathbb{R}^2$, see also Figure 3.19. It turns out that in the absence of excitations in the interior of σ, the projection operators given in Equation (3.4.11) and Equation (3.4.10) coincide.

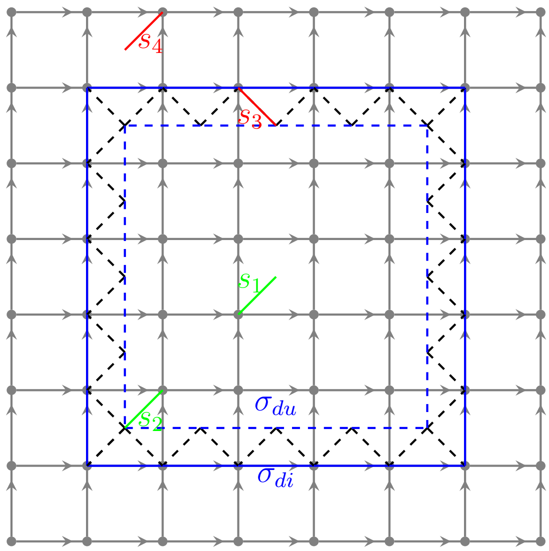


Figure 3.19: Depiction of a closed ribbon σ inside a square-shaped region of size 7. In the scenario depicted here, the interior is bounded by the dual ribbon σ_{du} (blue, dashed line). The sites s_1 and s_2 (green) are in the interior of σ, but the sites s_3 and s_4 (red) are not.

Proposition 3.4.4. *Let σ be a closed ribbon and Λ a square-shaped region containing σ, that is, the graph $\mathscr{G}_\Lambda$ contains all sites of σ. Let furthermore s_0 be a site in the interior of σ and $\psi \in \mathcal{H}_\Lambda$ be such that $A_s\psi = B_s\psi = \psi$ for all $s \neq s_0$. If the direct and dual path are oriented such that they point counter-clockwise around the interior of σ, we have*

$$P_\sigma^\alpha \psi = P_{s_0}^\alpha \psi,$$

otherwise we have

$$P_\sigma^\alpha \psi = P_{s_0}^{\alpha^*} \psi$$

for all $\alpha \in \widehat{D(G)}$, where α^* denotes the dual representation of α defined in Proposition 2.4.10.

Proof. We will only sketch the proof, but the arguments are exactly the same as the ones in Lemma 3.3.6 and Corollary 3.3.11. Because P^α_σ is rotationally invariant, it commutes with all star operators everywhere, the endpoints of σ included. We can then use star operators, to change the dual path around each dual plaquette in the lattice, including those at the endpoints of σ, until the dual path encloses the unique dual plaquette enclosing $s_0 = (v_0, f_0)$, described by some dual ribbon $\xi_{v_0} = (\tau_1, \dots, \tau_4)$. We refer to Figure 3.17 for a visualization of the general deformation procedure for the dual path, with emphasize on the fact that we are here allowed to commute with star operators at the endpoints of σ as well. Similarly, P^α_σ commutes with all plaquette operators everywhere, which allows deformations that include the endpoints as well, until the direct path encloses the unique smallest plaquette containing f_0, described by some direct path $p = (v_1, v_2, v_3, v_0, v_1)$. See also Figure 3.20. Note, that if σ moves clockwise, respectively counter-clockwise around the site s_0, then so do p and ξ_{v_0}. We set $p_\Delta = (v_1, v_0)$ and set $k = \beta^{(p_\Delta)}(\gamma)$ for some fixed G-connection $\gamma \notin \ker(P^\alpha_\sigma)$ with flat monodromy around each face $f \neq f_0$. Using Equation (3.3.25), we have

$$
\begin{aligned}
P^\alpha_\sigma \gamma &= \sum_{n \in N_\alpha} \overline{\mathrm{tr}_{\pi_\alpha}(n)} \prod_{i=1}^{4} L^{\bar{k} q_i n \bar{q}_i k}_{\tau_i} T^{c_i}_{p} \gamma \\
&= \sum_{n \in N_\alpha} \overline{\mathrm{tr}_{\pi_\alpha}(n)} \prod_{i=1}^{4} L^{\bar{k} q_i n \bar{q}_i k}_{\tau_i} T^{\bar{k} c_i k}_{p'} \gamma,
\end{aligned}
$$

where $p' = (\bar{p}_\Delta, p, p_\Delta)$. We can apply the exact same substitutions as in the proof of Proposition 3.4.3 to see that the above expression becomes

$$
P^\alpha_\sigma \gamma = \sum_{n \in N_\alpha} \overline{\mathrm{tr}_{\pi_\alpha}(n)} \prod_{i=1}^{4} L^{q_i n \bar{q}_i}_{\tau_i} T^{c_i}_{p'} \gamma,
$$

regardless of the choice of γ. If σ moves clockwise around s_0, then so do ξ_{v_0} and p' and ξ_{v_0} is locally counter-clockwise oriented while $T^{c_i}_{p'} = B^{\bar{c}_i}_{s_0}$ for all $i \in I_{\mathcal{C}_\alpha}$. See also the discussion after Equation (3.3.17). It follows that

$$
\begin{aligned}
P^\alpha_\sigma &= \sum_{n \in N_\alpha} \overline{\mathrm{tr}_{\pi_\alpha}(n)} A^{q_i \bar{n} \bar{q}_i}_{s_0} B^{\bar{c}_i}_{s_0} \\
&= \sum_{n \in N_\alpha} \overline{\mathrm{tr}_{\pi_\alpha}(\bar{n})} A^{q_i n \bar{q}_i}_{s_0} B^{\bar{c}_i}_{s_0} \\
&= P^{\alpha^*}_{s_0}.
\end{aligned}
$$

Similarly, if σ moves counter-clockwise around s_0, then so do ξ_{v_0} and p', and we have

$$\begin{aligned} P_\sigma^\alpha =& \sum_{n\in N_\alpha} \overline{\mathrm{tr}_{\pi_\alpha}(n)} A_{s_0}^{q_i n \bar{q}_i} B_{s_0}^{c_i} \\ =& P_{s_0}^\alpha . \end{aligned}$$

□

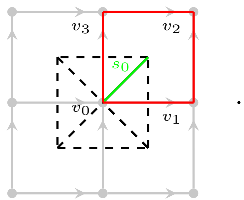


Figure 3.20: Depiction of the site $s_0 = (v_0, f_0)$ (green). The face v_0 is encircled by a dual plaquette aligned with a dual ribbon ξ_{v_0} (dashed) and the face f_0 is encircled by a direct path $p = (v_0, v_1, v_2, v_3, v_0)$ (red).

This shows that the projection operators P_σ^α measure the existence of an anyonic excitation in the interior of σ.

Remark 3.4.5. If $s_1, \dots, s_n$ is a collection of sites enclosed by a closed ribbon σ moving counter-clockwise around its interior, and $\psi \in \mathcal{H}$ such that $A_s\psi = \psi$ and $B_s\psi = \psi$ for all $s \notin \{s_1, \dots, s_n\}$, then we have more generally

$$P_\sigma^\alpha \psi = \sum_{i\in I_{\mathcal{C}_\alpha}} \sum_{n\in N_\alpha} \sum_{\substack{k_1,\dots,k_{n-1}\in G \\ c_1\cdot c_2\cdot\dots\cdot c_{n-1}=c_i}} A_{s_1}^{q_i n \bar{q}_i} B_{s_1}^{c_1} \left(\prod_{l=1}^{n-1} A_{s_l}^{\bar{k}_l q_i n \bar{q}_i k_l} B_{s_l}^{\bar{k}_l c_l k_l} T_{p_l}^{k_l} \right) \psi,$$

where p_l are paths starting at $v(s_1)$ and ending at $v(s_l)$ such that p_l is contained in p_{l+1} for all $l = 1, \dots, n-1$ and p_{n-1} is a closed path that moves clockwise around all sites. This can be shown using the same arguments as in the proof of Proposition 3.4.4 by choosing convenient deformations of the direct- and dual path of σ. A similar result is true for the case where σ encircles its interior clockwise.

We will verify directly that the projection operators P_σ^α can indeed measure the irreducible components $\mathcal{W}^{j_1 j_2,\alpha}$ in Proposition 3.4.2.

Proposition 3.4.6. *Let σ be a closed ribbon that is moving counter-clockwise around its interior, and ξ a ribbon with $s_0 := \partial_0\xi$ in the interior of σ and $\partial_1\xi$ outside σ, see Figure 3.21. Then we have*

$$P_\sigma^\alpha F_\xi^{IJ,\beta}\Omega_0 = \delta_{\alpha,\beta} F_\xi^{IJ,\beta}\Omega_0 \tag{3.4.12}$$

for all index pairs I and J and irreducible representations $\alpha = (\pi_\alpha, \mathcal{C}_\alpha)$ and $\beta = (\pi_\beta, \mathcal{C}_\beta)$ of the quantum double $D(G)$.

Proof. Let U_{s_0} be the action of the quantum double $D(G)$ on $\mathcal{H}$ as in Equation (3.4.1). By Proposition 3.4.2, the space

$$\left\{F_\xi^{h,g}\Omega_0 \mid h, g \in G\right\}$$

viewed as a $D(G)$-module under the action of U_{s_0} is isomorphic to the left-regular representation of $D(G)$. By Proposition 3.4.4, we have

$$P_\sigma^\alpha F_\xi^{h,g}\Omega_0 = P_{s_0}^\alpha F_\xi^{h,g}\Omega_0$$

for all $h, g \in G$. But the operators $P_{s_0}^\alpha$ are the central projections onto the irreducible submodules in $D(G)$ isomorphic to $\mathcal{V}^\alpha$ under the action of U_{s_0}. By Proposition 3.4.2, we have $\mathcal{V}^\alpha \cong \mathcal{W}^{\alpha,J}$ for each $J \in I_\alpha$, where $\mathcal{W}^{\alpha,J}$ is defined as in Equation (3.4.6) as the linear span of the operators $F_\xi^{IJ,\alpha}$. This implies the claim. □

Note that Proposition 3.4.6 implies in particular

$$P_\sigma^\alpha\Omega_0 = \delta_{\alpha,\mathrm{triv}}\Omega_0, \tag{3.4.13}$$

since $\Omega_0 = \sum_{g\in G} \Gamma_{\mathrm{triv}}^{1,1}(g) F_\xi^{e,g}\Omega_0 = F_\xi^{(1,1)(1,1),\mathrm{triv}}\Omega_0$. This result will become useful later.

We will now use the operators $\left\{F_\xi^{IJ,\alpha}\right\}$ to define *-homomorphisms, as in Equation (3.1.1). For $\alpha \in \widehat{D(G)}$ and ξ a ribbon, let $\mathbf{F}_\xi^\alpha \in \mathfrak{A} \otimes M_{I_\alpha}(\mathbb{C})$ be an $I_\alpha \times I_\alpha$ matrix with coefficients in the quasilocal algebra $\mathfrak{A}$, and entries given via

$$(\mathbf{F}_\xi^\alpha)^{IJ} = \frac{1}{\sqrt{d_\alpha}} F_\xi^{IJ,\alpha} = \sum_{n\in N_\alpha} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(n) F_\xi^{\bar{c}_{i_1}, q_{i_1} n \bar{q}_{j_1}}.$$

Then we define a map $\chi_\xi^\alpha : \mathfrak{A} \to \mathfrak{A} \otimes M_{I_\alpha}(\mathbb{C})$ via

$$\chi_\xi^\alpha : X \mapsto \mathbf{F}_\xi^\alpha (X \otimes \mathrm{id})(\mathbf{F}_\xi^\alpha)^*, \tag{3.4.14}$$

where id denotes the identity on the vector space $\mathbb{C}^{I_\alpha}$. These maps satisfy the properties in the following definition.

Definition 3.4.7 (Amplimorphism)**.** An **amplifying morphism** or **amplimorphism** of a C^*-algebra $\mathfrak{A}$ is a *-homomorphism $\chi : \mathfrak{A} \to \mathfrak{A} \otimes \mathrm{End}(V)$ where V is a finite dimensional vector space. If $\mathfrak{A}$ is a quasilocal algebra generated by a net $\{\mathfrak{A}_\Lambda\}_{\Lambda \in I}$, then χ is called **localized in** Λ if

$$\chi(A) = \chi(1_{\mathfrak{A}})(A \otimes \mathrm{id})$$

for all observables $A \in \mathfrak{A}_{\Lambda^c}$ supported outside Λ. An amplimorphism is called **localized** if it is localized in some $\Lambda \in I$. χ is called **unital** if $\chi(1_{\mathfrak{A}}) = 1_{\mathfrak{A}} \otimes \mathrm{id}_V$.

The idea to consider amplimorphisms instead of endomorphisms is due to [SV93] and the construction of these amplimorphisms in the setting of Kitaev's quantum double model has already been performed in [Naa12, Naa15].

Proposition 3.4.8. *The matrices* $\mathbf{F}^\alpha_\xi$ *are unitary for each ribbon* ξ *and irreducible representation* $\alpha \in \widehat{D(G)}$ *and satisfy*

$$\left(\mathbf{F}^\alpha_\xi\right)^* = \mathbf{F}^\alpha_{\bar{\xi}}, \tag{3.4.15}$$

where $\bar{\xi}$ *is the ribbon* ξ *with inverted orientation. If* ξ_1, ξ_2 *are two ribbons such that* ξ_1 *is composable with* ξ_2*, then*

$$\mathbf{F}^\alpha_{\xi_1}\mathbf{F}^\alpha_{\xi_2} = \mathbf{F}^\alpha_{(\xi_1,\xi_2)}. \tag{3.4.16}$$

It follows that the maps $\chi^\alpha_\xi : \mathfrak{A} \to \mathfrak{A} \otimes M_{I_\alpha}(\mathbb{C})$ *given in Equation* (3.4.14) *are unital and localized amplimorphisms.*

Proof. We will start by verifying Equation (3.4.15) and Equation (3.4.16) component wise:

$$\begin{aligned}
\left((\mathbf{F}^\alpha_\xi)^*\right)^{IJ} = \left((\mathbf{F}^\alpha_\xi)^{JI}\right)^* &= \sum_{n \in N_\alpha} \Gamma^{j_2 i_2}_{\pi_\alpha}(n) F_\xi^{c_{j_1}, q_{j_1} n \bar{q}_{i_1}} \\
&\overset{(3.3.19)}{=\!=} \sum_{n \in N_\alpha} \bar{\Gamma}^{i_2 j_2}_{\pi_\alpha}(\bar{n}) F_{\bar{\xi}}^{q_{i_1}\bar{n}\bar{q}_{j_1}\bar{c}_{j_1} q_{j_1} n \bar{q}_{i_1}, \bar{q}_{i_1}\bar{n} q_{j_1}} \\
&= \sum_{n \in N_\alpha} \bar{\Gamma}^{i_2 j_2}_{\pi_\alpha}(\bar{n}) F_{\bar{\xi}}^{\bar{c}_{i_1}, \bar{q}_{i_1}\bar{n} q_{j_1}} \\
&\overset{n \mapsto \bar{n}}{=} (\mathbf{F}^\alpha_{\bar{\xi}})^{IJ},
\end{aligned}$$

giving Equation (3.4.15). To see Equation (3.4.16), let ξ_1, ξ_2 be ribbons such that ξ_1 is composable with ξ_2. Then

$$\begin{aligned}\sum_{K\in I_\alpha} F^{IK,\alpha}_{\xi_1} F^{KJ,\alpha}_{\xi_2} &= \sum_{K\in I_\alpha}\sum_{n_1,n_2\in N_\alpha} \bar{\Gamma}^{i_2k_2}_{\pi_\alpha}(n_1)\bar{\Gamma}^{k_2j_2}_{\pi_\alpha}(n_2) F^{\bar{c}_{i_1},q_{i_1}n_1\bar{q}_{k_1}}_{\xi_1} F^{\bar{c}_{k_1},q_{k_1}n_2\bar{q}_{j_1}}_{\xi_2} \\ &= \sum_{k_1\in I_\alpha}\sum_{n_1,n_2\in N_\alpha} \bar{\Gamma}^{i_2j_2}_{\pi_\alpha}(n_1n_2) F^{\bar{c}_{i_1},q_{i_1}n_1\bar{q}_{k_1}}_{\xi_1} F^{\bar{c}_{k_1},q_{k_1}n_2\bar{q}_{j_1}}_{\xi_2} \\ &\xlongequal{n_2\mapsto\bar{n}_1n_2} \sum_{k_1\in I_\alpha}\sum_{n_1,n_2\in N_\alpha} \bar{\Gamma}^{i_2j_2}_{\pi_\alpha}(n_2) F^{\bar{c}_{i_1},q_{i_1}n_1\bar{q}_{k_1}}_{\xi_1} F^{\bar{c}_{k_1},q_{k_1}\bar{n}_1n_2\bar{q}_{j_1}}_{\xi_2}.\end{aligned}$$

Noting that every element $h \in G$ can uniquely be expressed in the form $h = n_1q_{k_1}$, we realize that the summation over k_1 and n_1 can alternatively be performed over all $h = q_{i_1}n_1\bar{q}_{k_1} \in G$. In that case, we have $c_{k_1} = \bar{h}c_{i_1}h$ and $q_{k_1}\bar{n}_1n_2\bar{q}_{j_1} = \bar{h}q_{i_1}n_2\bar{q}_{j_1}$. The above expression becomes

$$\sum_{h\in G}\sum_{n_2\in N_\alpha} \bar{\Gamma}^{i_2j_2}_{\pi_\alpha}(n_2) F^{\bar{c}_{i_1},h}_{\xi_1} F^{\bar{h}\bar{c}_{i_1}h,\bar{h}q_{i_1}n_2\bar{q}_{j_1}}_{\xi_2} = F^{IJ,\alpha}_{(\xi_1,\xi_2)}.$$

We verify that the matrix $\mathbf{F}^\alpha_\xi$ is unital:

$$(\mathbf{F}^\alpha_\xi(\mathbf{F}^\alpha_\xi)^*)^{IJ} \xlongequal{(3.4.15)} \sum_{K\in I_\alpha} (\mathbf{F}^\alpha_\xi)^{IK}(\mathbf{F}^\alpha_{\bar{\xi}})^{KJ} \xlongequal{(3.4.16)} (\mathbf{F}^\alpha_{\xi\bar{\xi}})^{IJ} \xlongequal{(3.3.20)} (\mathbf{F}^\alpha_\emptyset)^{IJ}.$$

In view of Equation (3.3.1), $(\mathbf{F}^\alpha_\emptyset)^{IJ}$ becomes

$$(\mathbf{F}^\alpha_\emptyset)^{IJ} = \sum_{n\in N_\alpha} \bar{\Gamma}^{i_2j_2}_{\pi_\alpha}(n) F^{\bar{c}_{i_1},q_{i_1}n\bar{q}_{j_1}}_\emptyset = \sum_{n\in N_\alpha} \delta_{q_{i_1}n\bar{q}_{j_1},e}\bar{\Gamma}^{i_2j_2}_{\pi_\alpha}(n)1_{\mathfrak{A}}.$$

Now, $q_{i_1}n\bar{q}_{j_1} = e$ implies $q_{j_1} = q_{i_1}n$. Since the right-hand side of this decomposition is unique, it follows that $i_1 = j_1$ and $n = e$. Because $\bar{\Gamma}^{i_2j_2}_{\pi_\alpha}(e) = \delta_{i_2,j_2}$, it follows that $(\mathbf{F}^\alpha_\emptyset)^{IJ} = \delta_{I,J}1_{\mathfrak{A}}$, and we have

$$\mathbf{F}^\alpha_\xi(\mathbf{F}^\alpha_\xi)^* = 1_{\mathfrak{A}} \otimes \mathrm{id}\,. \tag{3.4.17}$$

Using similar arguments, we can show that

$$(\mathbf{F}^\alpha_\xi)^*\mathbf{F}^\alpha_\xi = 1_{\mathfrak{A}} \otimes \mathrm{id} \tag{3.4.18}$$

holds as well. For the other claims, notice that the map χ^α_ξ is unital if and only if

$$\mathbf{F}^\alpha_\xi(\mathbf{F}^\alpha_\xi)^* = 1_{\mathfrak{A}} \otimes \mathrm{id}\,.$$

Furthermore, if $X, Y \in \mathfrak{A}$, then

$$\begin{aligned}\chi_\xi^\alpha(XY) =&\mathbf{F}_\xi^\alpha(XY \otimes \mathrm{id})(\mathbf{F}_\xi^\alpha)^* = \mathbf{F}_\xi^\alpha(X \otimes \mathrm{id})(\mathbf{F}_\xi^\alpha)^*\mathbf{F}_\xi^\alpha(Y \otimes \mathrm{id})(\mathbf{F}_\xi^\alpha)^* \\ =&\chi_\xi^\alpha(X)\chi_\xi^\alpha(Y)\end{aligned}$$

and

$$(\chi_\xi^\alpha(X))^* = \mathbf{F}_\xi^\alpha(X \otimes \mathrm{id})^*(\mathbf{F}_\xi^\alpha)^* = \chi_\xi^\alpha(X^*).$$

Clearly, χ_ξ^α is a linear map, and it follows that χ_ξ^α is a *-homomorphism.

Finally, χ_ξ^α is localized in any region Λ containing the ribbon ξ: For any operator X supported outside Λ we have $\left[X, F_\xi^{IJ,\alpha}\right] = 0$ for all $I, J \in I_\alpha$ and therefore

$$\chi_\xi^\alpha(X) = \mathbf{F}_\xi^\alpha(X \otimes \mathrm{id})(\mathbf{F}_\xi^\alpha)^* = (X \otimes \mathrm{id})\mathbf{F}_\xi^\alpha(\mathbf{F}_\xi^\alpha)^* = (X \otimes \mathrm{id}).$$

□

The matrices $\mathbf{F}_\xi^{IJ,\alpha}$ are not yet suitable enough to describe ground states of the quantum double model. The physical interpretation is that ribbon operators create pairs of excitations, while a ground state should intuitively describe a single charge only. Otherwise, we would be able to move one charge back to the other using local observables, and the fusion would lower the energy. The idea is to take the limits of ribbons ξ_n and sending the endpoint $\partial_1\xi$ to infinity. Let $\{\xi_n\}$ be a sequence of ribbons with fixed initial site $s_0 = \partial_0\xi_n$ such that $\xi_n \subset \xi_{n+1}$ for all $n \in \mathbb{N}$ and such that the number of sites $S(n)$ of $\xi_n = (s_0, \ldots, s_{S(n)})$ strictly grows with n. We call such a sequence a **ribbon extending to infinity from** s_0 and call the infinite ribbon $\xi := (s_0, \ldots)$ the **limit** of $\{\xi_n\}$, denoted by $\lim\limits_{n\to\infty} \xi_n := \xi$.

Proposition 3.4.9. *Let $\{\xi_n\}$ be a sequence of ribbons extending to infinity from some site s_0, and let ξ be the limit of ξ_n. Denote by $\chi_{\xi_n}^{IJ,\alpha}$ the component in the I-th row and J-th column of the map $\chi_{\xi_n}^\alpha$ defined in Equation* (3.4.14) *for $I, J \in I_\alpha$, that is*

$$\begin{aligned}\chi_{\xi_n}^{IJ,\alpha}(X) =& \sum_{K\in I_\alpha} (\mathbf{F}_{\xi_n}^\alpha)^{IK}(X \otimes \mathrm{id})((\mathbf{F}_{\xi_n}^\alpha)^*)^{KJ} \\ =&\frac{1}{d_\alpha}\sum_K F_{\xi_n}^{IK,\alpha}(X \otimes \mathrm{id})\left(F_{\xi_n}^{JK,\alpha}\right)^*.\end{aligned}$$

Then for each $\alpha \in \widehat{D(G)}$, $I, J \in I_\alpha$ and $X \in \mathfrak{A}_{loc}$ the limit

$$\chi_\xi^{IJ,\alpha}(X) := \lim_{n\to\infty} \chi_{\xi_n}^{IJ,\alpha}(X)$$

exists and extends to a bounded linear map $\chi_\xi^{IJ}: \mathfrak{A} \to \mathfrak{A}$*. Furthermore, the amplimorphism* χ_ξ^α *exhibits the following properties:*

(i) $\chi_\xi^\alpha(1_\mathfrak{A}) = 1_\mathfrak{A} \otimes \mathrm{id}$

(ii) $\chi_\xi^\alpha(A^*) = \chi_\xi^\alpha(A)^*$

(iii) $\chi_\xi^\alpha(ABC) = (A \otimes \mathrm{id})\chi_\xi^\alpha(B \otimes \mathrm{id})(C \otimes \mathrm{id})$ *for all* $A, B, C \in \mathfrak{A}$ *with* A *and* C *supported outside* ξ*.*

(iv) For all $X \in \mathfrak{A}_{loc}$ *there exists an* $n_0 \in \mathbb{N}$ *such that* $\chi_n^\alpha(X) = \chi_{\xi_{n_0}}^\alpha(X)$ *for all* $n \geq n_0$*.*

Proof. We start by showing (iv). Let $X \in \mathfrak{A}_{loc}$ and n_0 such that $\mathrm{supp}(A) \cap (\xi_n \setminus \xi_{n_0}) = \emptyset$ for all $n \geq n_0$. Writing $\tilde{\xi_n} = \xi_n \setminus \xi_{n_0}$ to denote the ribbon starting at the endpoint of ξ_{n_0} such that $\xi_n = (\xi_{n_0}, \tilde{\xi_n})$, we obtain for $n \geq n_0$

$$\begin{aligned}\chi_{\xi_n}^\alpha(X) =& \mathbf{F}_{\xi_n}^\alpha (X \otimes \mathrm{id})(\mathbf{F}_{\xi_n}^\alpha)^* \\ \overset{(3.4.16)}{=\!=\!=}& \mathbf{F}_{\xi_{n_0}}^\alpha \mathbf{F}_{\tilde{\xi}_n}^\alpha (X \otimes \mathrm{id})(\mathbf{F}_{\tilde{\xi}_n}^\alpha)^*(\mathbf{F}_{\xi_{n_0}}^\alpha)^* \\ =& \mathbf{F}_{\xi_{n_0}}^\alpha (X \otimes \mathrm{id})(\mathbf{F}_{\xi_{n_0}}^\alpha)^* \mathbf{F}_{\tilde{\xi}_n}^\alpha (\mathbf{F}_{\tilde{\xi}_n}^\alpha)^* \\ =& \chi_{\xi_{n_0}}^{II,\alpha}(X).\end{aligned}$$

Note that this already implies convergence, since $\chi_{\xi_n}(X)$ becomes eventually constant for each $X \in \mathfrak{A}_{loc}$. Item (i) and (ii) follow from Proposition 3.4.8 and (iii) is clear. □

Remark 3.4.10. The map $\chi_\xi^{\mathcal{C}_\alpha,\pi_\alpha}$ defined in [Naa12, Lem 12.2.3] is related to the map $\chi_\xi^{II,\alpha}$ via $\chi_\xi^{\mathcal{C}_\alpha,\pi_\alpha} = \sum_I \chi_\xi^{II,\alpha}$.

Define

$$\omega_\xi^{IJ,\alpha} := \omega_0 \circ \chi_\xi^{IJ,\alpha}.$$

It is easy to see that for $I = J$ the composition $\omega_0 \circ \chi_\xi^{II,\alpha}$ becomes a state for all $\alpha \in \widehat{D(G)}$ and $I \in I_\alpha$.

Remark 3.4.11. By Corollary 3.3.11, the map $\omega_\xi^{IJ,\alpha}$ for some semi-infinite ribbon ξ only depends on the initial site $s = \partial_0\xi$. Indeed, we have for all $X \in \mathfrak{A}_{loc}$ and $n \in \mathbb{N}$ large enough

$$\omega_\xi^{II,\alpha}(X) = \frac{1}{d_\alpha}\sum_J \langle F_{\xi_n}^{IJ,\alpha}\Omega_0, XF_{\xi_n}^{IJ,\alpha}\Omega_0\rangle = \frac{1}{d_\alpha}\sum_J \langle F_\zeta^{IJ,\alpha}\Omega_0, XF_\zeta^{IJ,\alpha}\Omega_0\rangle,$$

for all ribbons ζ with the same endpoints as ξ_n. Furthermore, if ξ_1, ξ_2 are two semi-infinite ribbons with starting sites $s_1 := \partial_0\xi_1$ and $s_2 := \partial_0\xi_2$, then $\omega_{\xi_1}^{II,\alpha}(X) = \omega_{\xi_2}^{II,\alpha}(X)$ for all X supported outside a region containing a ribbon ζ with $\partial_0\zeta = s_1$ and $\partial_1\zeta = s_2$. This follows because $(\mathbf{F}_\zeta^\alpha)^*$ is a unitary matrix commuting with $X \otimes \mathrm{id}$, and $(\mathbf{F}_\zeta^\alpha \mathbf{F}_{\xi_2'}^\alpha)^{IJ}\Omega_0 = (\mathbf{F}_{(\zeta\xi_2')}^\alpha)^{IJ}\Omega_0 = (\mathbf{F}_{\xi_1'}^\alpha)^{IJ}\Omega_0$, where $\xi_1' \subset \xi_1$ and $\xi_2' \subset \xi_2$ are arbitrary finite ribbons. Hence, $\omega_{\xi_1}^{II,\alpha}$ and $\omega_{\xi_2}^{II,\alpha}$ *look the same from afar.* This observation will become important in Theorem 3.6.9.

Lemma 3.4.12. *The states $\omega_\xi^{II,\alpha}$ coincide with the states*

$$\rho_\xi^\alpha : X \mapsto \frac{1}{d_\alpha} \sum_{J,I \in I_\alpha} \omega_0(F_\xi^{IJ,\alpha} X (F_\xi^{IJ,\alpha})^*) \tag{3.4.19}$$

for all $X \in \mathfrak{A}_{loc}$ supported on a region disjoint from $\partial_0\xi$ and each $\alpha \in \widehat{D(G)}$, $I \in I_\alpha$ and semi-infinite ribbon ξ.

Proof. It was already shown in [Naa12] that the map defined in Equation (3.4.19) defines a state for each semi-infinite ribbon ξ and irreducible representation $\alpha \in \widehat{D(G)}$. Let X be such that $s_0 := \partial_0\xi \notin \mathrm{supp}(X)$ and write $A_{s_0} = \frac{1}{|G|} \sum_{m \in N_\alpha} \sum_{l \in I_{\mathcal{C}_\alpha}} A_{s_0}^{q_l m \bar{q}_{i_1}}$ for the star operator at site s_0. Then

$$\begin{aligned}
\omega_\xi^{II,\alpha}(X) =& \sum_J \sum_{n_1,n_2 \in N_\alpha} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(n_1)\Gamma_{\pi_\alpha}^{j_2 i_2}(n_2)\omega_0(F_\xi^{\bar{c}_{i_1}, q_{i_1} n_1 \bar{q}_{j_1}} X F_\xi^{c_{i_1}, q_{i_1} n_2 \bar{q}_{j_1}}) \\
=& \frac{1}{|G|} \sum_J \sum_{l \in I_{\mathcal{C}_\alpha}} \sum_{n_1,n_2,m \in N_\alpha} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(n_1)\Gamma_{\pi_\alpha}^{i_2 j_2}(n_2)\omega_0(A_{s_0}^{q_l m \bar{q}_{i_1}} F_\xi^{\bar{c}_{i_1}, q_{i_1} n_1 \bar{q}_{j_1}} X F_\xi^{c_{i_1}, q_{i_1} n_2 \bar{q}_{j_1}}) \\
=& \frac{1}{|G|} \sum_J \sum_{l \in I_{\mathcal{C}_\alpha}} \sum_{n_1,n_2,m \in N_\alpha} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(n_1)\Gamma_{\pi_\alpha}^{i_2 j_2}(n_2)\omega_0(F_\xi^{\bar{c}_{l_1}, q_{l_1} m n_1 \bar{q}_{j_1}} X F_\xi^{c_{l_1}, q_{l_1} m n_2 \bar{q}_{j_1}} A_{s_0}^{q_l m \bar{q}_{i_1}})
\end{aligned}$$

Substituting $n_1 \mapsto \bar{m} n_1$ and $n_2 \mapsto \bar{m} n_2$:

$$= \frac{1}{|G|} \sum_J \sum_{l \in I_{\mathcal{C}_\alpha}} \sum_{n_1,n_2,m \in N_\alpha} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(\bar{m} n_1)\Gamma_{\pi_\alpha}^{i_2 j_2}(\bar{m} n_1)\omega_0(F_\xi^{\bar{c}_l, q_l n_1 \bar{q}_{j_1}} X F_\xi^{c_l, q_l n_2 \bar{q}_{j_1}})$$

For the matrix coefficients, we obtain:

$$
\begin{aligned}
&= \sum_{j_2 \in I_{\pi_\alpha}} \sum_{m \in N_\alpha} \bar{\Gamma}^{i_2 j_2}_{\pi_\alpha}(\bar{m} n_1) \Gamma^{i_2 j_2}_{\pi_\alpha}(\bar{m} n_2) \\
&= \sum_{j_2 \in I_{\pi_\alpha}} \sum_{m \in N_\alpha} \Gamma^{j_2 i_2}_{\pi_\alpha}(\bar{n}_1 m) \Gamma^{i_2 j_2}_{\pi_\alpha}(\bar{m} n_2) \\
&= \sum_{j_2, t_1, t_2 \in I_{\pi_\alpha}} \sum_{m \in N_\alpha} \Gamma^{j_2 t_1}_{\pi_\alpha}(\bar{n}_1) \Gamma^{t_1 i_2}_{\pi_\alpha}(m) \Gamma^{i_2 t_2}_{\pi_\alpha}(\bar{m}) \Gamma^{t_2 j_2}_{\pi_\alpha}(n_2) \\
&= \sum_{j_2, t_1, t_2 \in I_{\pi_\alpha}} \sum_{m \in N_\alpha} \Gamma^{t_2 j_2}_{\pi_\alpha}(n_2) \Gamma^{j_2 t_1}_{\pi_\alpha}(\bar{n}_1) \Gamma^{t_1 i_2}_{\pi_\alpha}(m) \bar{\Gamma}^{t_2 i_2}_{\pi_\alpha}(m) \\
&= \frac{|N_\alpha|}{\dim_{\pi_\alpha}} \sum_{j_2, t \in I_{\pi_\alpha}} \Gamma^{t j_2}_{\pi_\alpha}(n_2) \Gamma^{j_2 t}_{\pi_\alpha}(\bar{n}_1) \\
&= \frac{|N_\alpha|}{\dim_{\pi_\alpha}} \operatorname{tr}(\bar{n}_1 n_2)
\end{aligned}
$$

and the above expression becomes

$$
\begin{aligned}
&= \frac{|N_\alpha|}{|G| \dim_{\pi_\alpha}} \sum_{j_1} \sum_{l \in I_{\mathcal{C}_\alpha}} \sum_{n_1, n_2 \in N_\alpha} \operatorname{tr}(\bar{n}_1 n_2) \omega_0(F_\xi^{\bar{c}_l, q_l n_1 \bar{q}_{j_1}} X F_\xi^{c_l, q_l n_2 \bar{q}_{j_1}}) \\
&\overset{|G|=|N_\alpha||\mathcal{C}_\alpha|}{=} \frac{1}{d_\alpha} \sum_{j_1} \sum_{l \in I_{\mathcal{C}_\alpha}} \sum_{n_1, n_2 \in N_\alpha} \operatorname{tr}(\bar{n}_1 n_2) \omega_0(F_\xi^{\bar{c}_l, q_l n_1 \bar{q}_{j_1}} X F_\xi^{c_l, q_l n_2 \bar{q}_{j_1}}) \\
&= \frac{1}{d_\alpha} \sum_{j_1} \sum_{l \in I_{\mathcal{C}_\alpha}} \sum_{i_2, j_2 \in I_{\pi_\alpha}} \sum_{n_1, n_2 \in N_\alpha} \bar{\Gamma}^{i_2 j_2}_{\pi_\alpha}(n_1) \Gamma^{i_2 j_2}_{\pi_\alpha}(n_2) \omega_0(F_\xi^{\bar{c}_l, q_l n_1 \bar{q}_{j_1}} X F_\xi^{c_l, q_l n_2 \bar{q}_{j_1}}) \\
&= \rho_\xi^\alpha(X).
\end{aligned}
$$

This shows that $\omega_\xi^{II,\alpha} |_{\Lambda^c} = \rho_\xi^\alpha |_{\Lambda^c}$ for each region Λ containing s_0. □

Remark 3.4.13. We emphasize that the ribbon operators $F_{\xi_n}^{IJ,\alpha}$ themselves don't converge in the operator norm as n tends to ∞. This can be seen as follows: If $F_{\xi_n}^{h,g}$ was converging to some operator in the norm topology for a sequence of ribbons ξ_n extending to a semi-infinite ribbon ξ, then $F_{\xi_n}^{h,g}$ would form a Cauchy-sequence. Then the sequence $F_{\xi_n}^{h,g}\Omega_0$ would form a Cauchy-sequence as well, but we claim that for some $n, m \in \mathbb{N}$ with $m > n$ the expression

$$
\left\| F_{\xi_n}^{h,g}\Omega_0 - F_{\xi_m}^{h,g}\Omega_0 \right\|
$$

is constant and non-zero. Indeed, we have

$$\begin{aligned}\left\|F^{h,g}_{\xi_n}\Omega_0 - F^{h,g}_{\xi_m}\Omega_0\right\|^2 =&\omega_0\left(\left[F^{\bar{h},g}_{\xi_n} - F^{\bar{h},g}_{\xi_m}\right]\left[F^{h,g}_{\xi_n} - F^{h,g}_{\xi_m}\right]\right)\\ =&\omega_0(F^{e,g}_{\xi_n} + F^{e,g}_{\xi_m}) - \omega_0(F^{\bar{h},g}_{\xi_m}F^{h,g}_{\xi_n}) - \omega_0(F^{\bar{h},g}_{\xi_n}F^{h,g}_{\xi_m}).\end{aligned}$$

Using Equation (3.3.2), we write $F^{h,g}_{\xi_m} = \sum_{k\in G} F^{h,k}_{\xi_n}F^{\bar{k}hk,\bar{k}g}_{\xi'}$, where ξ' is such that $(\xi_n, \xi') = \xi_m$. Furthermore, Lemma 3.3.12 gives $\omega_0(F^{e,g}_{\xi_n}) = \omega_0(F^{e,g}_{\xi_m}) = \frac{1}{|G|}$. We obtain

$$\begin{aligned}&\frac{2}{|G|} - \sum_{k\in G}\omega_0(F^{\bar{h},k}_{\xi_n}F^{\bar{k}\bar{h}k,\bar{k}g}_{\xi'}F^{h,g}_{\xi_n}) - \sum_{k\in G}\omega_0(F^{\bar{h},g}_{\xi_n}F^{h,k}_{\xi_n}F^{\bar{k}hk,\bar{k}g}_{\xi'})\\ \overset{(3.3.18)}{=}&\frac{2}{|G|} - \omega_0(F^{e,g}_{\xi_n}F^{\bar{g}\bar{h}g,e}_{\xi'}) - \omega_0(F^{e,g}_{\xi_n}F^{\bar{g}hg,e}_{\xi'}).\end{aligned}$$

Applying Proposition 3.3.12 to the linear functionals $X \mapsto \omega_0(X(-)F^{\bar{g}h\bar{g}g,e}_{\xi'})$ and $X \mapsto \omega_0(X(-)F^{\bar{g}hg,e}_{\xi'})$ we see that

$$\omega_0(F^{e,g}_{\xi_n}F^{\bar{g}hg,e}_{\xi'}) = \frac{1}{|G|}\omega_0(F^{\bar{g}hg,e}_{\xi'}) = \frac{\delta_{\bar{g}hg,e}}{|G|^2} = \frac{\delta_{h,e}}{|G|^2}$$

and similarly, $\omega_0(F^{e,g}_{\xi_n}F^{\bar{g}\bar{h}g,e}_{\xi'}) = \frac{\delta_{h,e}}{|G|^2}$ holds as well. It follows that

$$\left\|F^{h,g}_{\xi_n}\Omega_0 - F^{h,g}_{\xi_m}\Omega_0\right\| = \frac{2}{|G|} - \delta_{h,e}\frac{2}{|G|^2},$$

which is in particular constant for all $h, g \in G$ and $n, m \in \mathbb{N}$ with $m > n$.

Given a ribbon ξ extending to infinity, we define the state $\omega^{II,\alpha}_{\xi_n}$ via

$$\omega^{II,\alpha}_{\xi_n} = \omega_0 \circ \chi^{II,\alpha}_{\xi_n} \tag{3.4.20}$$

and set

$$\omega^{II,\alpha}_{\xi} = \lim_{n\to\infty}\omega^{II,\alpha}_{\xi_n}. \tag{3.4.21}$$

In the next lemma, we calculate the energy of these diagonal states $\omega^{II,\alpha}_{\xi}$.

Lemma 3.4.14. *Let $s_0 = \partial_0\xi$. Then we have*

$$\omega^{II,\alpha}_{\xi}(1_{\mathfrak{A}} - A_{s_0}) = 1 - \delta_{\pi_\alpha,\mathrm{triv}}\frac{1}{|\mathcal{C}_\alpha|}, \tag{3.4.22}$$

$$\omega^{II,\alpha}_{\xi}(1_{\mathfrak{A}} - B_{s_0}) = 1 - \delta_{\mathcal{C}_\alpha,\{e\}}, \tag{3.4.23}$$

$$\omega^{II,\alpha}_{\xi}(P^{\rho}_{\sigma}) = \delta_{\rho,\alpha}, \tag{3.4.24}$$

for all closed ribbons σ enclosing s_0, and

$$\omega_\xi^{II,\alpha}(1_{\mathfrak{A}} - A_{s'}) = \omega_\xi^{II,\alpha}(1_{\mathfrak{A}} - B_{s'}) = 0 \tag{3.4.25}$$

for all $s' \neq s$.

Proof. Let $I \in I_\alpha$ with $I = (i_1, i_2)$. To verify Equation (3.4.22), we may write the trivial electric charge operator as

$$A_{s_0} = \frac{1}{|G|} \sum_{n \in N_\alpha} \sum_{i \in I_{\mathcal{C}_\alpha}} A_{s_0}^{q_i m \bar{q}_{i_1}},$$

using again that for every element $k \in G$ there exists a unique pair $(i, m) \in I_{\mathcal{C}_\alpha} \times N_\alpha$ such that $g q_{i_1} = q_i m$. We then have for all $K = (k_1, k_2) \in I_\alpha$ and $n_1, n_2 \in N_\alpha$:

$$\begin{aligned}
&\omega_0 \left(F_\xi^{\bar{c}_{i_1}, q_{i_1} n_1 \bar{q}_{k_1}} A_{s_0} \left(F_\xi^{c_{i_1}, q_{i_1} n_2 \bar{q}_{k_1}} \right)^* \right) \\
=&\frac{1}{|G|} \sum_{m \in N_\alpha} \sum_{i=1}^{|\mathcal{C}_\alpha|} \omega_0 (F_\xi^{\bar{c}_{i_1}, q_{i_1} n_1 \bar{q}_{k_1}} A_{s_0}^{q_i m \bar{q}_{i_1}} F_\xi^{c_{i_1}, q_{i_1} n_2 \bar{q}_{k_1}}) \\
=&\frac{1}{|G|} \sum_{m \in N_\alpha} \sum_{i=1}^{|\mathcal{C}_\alpha|} \omega_0 (F_\xi^{\bar{c}_{i_1}, q_{i_1} n_1 \bar{q}_{k_1}} F_\xi^{c_i, q_i m n_2 \bar{q}_{k_1}}) \\
=&\frac{1}{|G|} \sum_{m \in N_\alpha} \sum_{i=1}^{|\mathcal{C}_\alpha|} \delta_{q_{i_1} n_1 \bar{q}_{k_1}, q_i m n_2 \bar{q}_{k_1}} \omega_0 (F_\xi^{\bar{c}_{i_1} c_i, q_{i_1} n_1 \bar{q}_{k_1}}) \\
=&\sum_{m \in N_\alpha} \sum_{i=1}^{|\mathcal{C}_\alpha|} \delta_{n_1, m n_2} \delta_{i_1, i} \frac{1}{|G|^2} = \frac{1}{|G|^2}
\end{aligned}$$

Then we have

$$\begin{aligned}
&\omega_0 \left((\mathbf{F}_\xi^\alpha)^{IK} A_{s_0} ((\mathbf{F}_\xi^\alpha)^*)^{KI} \right) \\
=&\frac{1}{d_\alpha} \omega_0 \left(F_\xi^{IK,\alpha} A_{s_0} (F_\xi^{IK,\alpha})^* \right) \\
=&\sum_{n_1, n_2 \in N_\alpha} \bar{\Gamma}_{\pi_\alpha}^{i_2 k_2}(n_1) \Gamma_{\pi_\alpha}^{i_2 k_2}(n_2) \omega_0 (F_\xi^{\bar{c}_{i_1}, q_{i_1} n_1 \bar{q}_{k_1}} A_{s_0} F_\xi^{c_{i_1}, q_{i_1} n_2 \bar{q}}) \\
=&\frac{1}{|G|^2} \sum_{n_1, n_2 \in N_\alpha} \bar{\Gamma}_{\pi_\alpha}^{i_2 k_2}(n_1) \Gamma_{\pi_\alpha}^{i_2 k_2}(n_2)
\end{aligned}$$

By the Peter-Weyl Theorem, we have

$$\sum_{n\in N_\alpha}\Gamma_{\pi_\alpha}^{i_2j_2}(n)=\sum_{n\in N_\alpha}\Gamma_{\pi_\alpha}^{i_2j_2}(n)\bar{\Gamma}_{\mathrm{triv}}^{1,1}(n)=|N_\alpha|\,\delta_{\pi_\alpha,\mathrm{triv}},$$

and using again that $|N_\alpha|\,|\mathcal{C}_\alpha|=|G|$, the above expression becomes

$$\frac{1}{|G|^2}\,|N_\alpha|^2\,\delta_{\pi_\alpha,\mathrm{triv}}=\frac{\delta_{\pi_\alpha,\mathrm{triv}}}{|C_\alpha|^2}.$$

Thus, we have

$$\begin{aligned}\omega_\xi^{II,\alpha}(1_{\mathfrak{A}}-A_{s_0})=&1_{\mathfrak{A}}-\sum_{K\in I_\alpha}\omega_0\left((F_\xi^\alpha)^{IK}A_{s_0}((F_\xi^\alpha)^*)^{KI}\right)\\=&1_{\mathfrak{A}}-\sum_{k_1=1}^{|\mathcal{C}_\alpha|}\delta_{\pi_\alpha,\mathrm{triv}}\frac{1}{|\mathcal{C}_\alpha|^2}\\=&1_{\mathfrak{A}}-\frac{1}{|\mathcal{C}_\alpha|}\end{aligned}$$

showing Equation (3.4.22) For Equation (3.4.23), note that we have

$$\begin{aligned}\omega_0(F_\xi^{\bar{c}_{i_1},q_{i_1}n_1\bar{q}_{k_1}}B_{s_0}F_\xi^{c_{i_1},q_{i_1}n_2\bar{q}_{k_1}})\overset{(3.3.8)}{=\!=}&\omega_0(F_\xi^{\bar{c}_{i_1},q_{i_1}n_1\bar{q}_{k_1}}F_\xi^{c_{i_1},q_{i_1}n_2\bar{q}_{k_1}}B_{s_0}^{c_{i_1}})\\\overset{(2.5.13)}{=\!=}&\omega_0(F_\xi^{\bar{c}_{i_1},q_{i_1}n_1\bar{q}_{k_1}}F_\xi^{c_{i_1},q_{i_1}n_2\bar{q}_{k_1}}B_{s_0}^{c_{i_1}}B_{s_0})\\=&\delta_{c_{i_1},e}\omega_0(F_\xi^{\bar{c}_{i_1},q_{i_1}n_1\bar{q}_{k_1}}F_\xi^{c_{i_1},q_{i_1}n_2\bar{q}_{k_1}})\\=&\delta_{\mathcal{C}_\alpha,\{e\}}\omega_0(F_\xi^{\bar{c}_{i_1},q_{i_1}n_1\bar{q}_{k_1}}F_\xi^{c_{i_1},q_{i_1}n_2\bar{q}_{k_1}}).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{K\in I_\alpha}\omega_0((\mathbf{F}_\xi^\alpha)^{IK}B_{s_0}((\mathbf{F}_\xi^\alpha)^*)^{KI})=&\sum_{K\in I_\alpha}\delta_{\mathcal{C}_\alpha,\{e\}}\omega_0((\mathbf{F}_\xi^\alpha)^{IK}((\mathbf{F}_\xi^\alpha)^*)^{KI})\\=&\delta_{\mathcal{C}_\alpha,\{e\}}\omega_\xi^{II,\alpha}(1_{\mathfrak{A}})\\=&\delta_{\mathcal{C}_\alpha,\{e\}}.\end{aligned}$$

Finally, (3.4.24) follows from Proposition 3.4.6 and Equation (3.4.25) follows from Equation (3.3.7). □

Remark 3.4.15. Lemma 3.4.14 implies that for $\pi_\alpha=\mathrm{triv}$, $F_\xi^{IJ,\alpha}\Omega_0$ is in general *not* an eigenvector of A_s for $s=\partial_0\xi$. Indeed, as a projection, A_s has eigenvalues 0 or 1. But for $|\mathcal{C}_\alpha|\neq 1$, the expectation value of A_s in that state would not be an integer. This problem can be resolved by choosing a different linear combination in this case, which we will present in the next section.

3.5 Ground States

Recall that ω_0 is the unique translation invariant ground state of the non-abelian quantum double model. This ground state is a pure state, and is completely determined by the condition that $\omega_0(A_s) = \omega_0(B_s) = 1$ for all s by Proposition 3.2.6. It is furthermore distinguished from the other ground states in that it is the only frustration free ground state. The aim of this section is to find other non-frustration free ground states. We will show, that for semi-infinite ribbons σ, the states $\omega_\xi^{II,\alpha}$ are ground states for the case $\pi_\alpha \neq$ triv and non-ground states in the case $\pi_\alpha =$ triv. We can still find ground states corresponding to the case $\pi_\alpha =$ triv by taking appropriate linear combinations of the ribbon operators $F_\xi^{IJ,\alpha}$. These states are constructed in Lemma 3.5.3, and the main theorem of this section is stated as Theorem 3.5.4. Recall that if ω is a ground state on $\mathfrak{A}$, then for any local observable $X \in \mathfrak{A}_{loc}$ we have

$$-i\delta(X^*\delta(X)) = \omega(X^*\left[H_{\Lambda_n}, X\right]),$$

where Λ_n is a square-shaped region containing $\mathrm{supp}(X)$ as well as the support of all interaction terms $(1 - A_s), (1 - B_s)$ whose support intersect with $\mathrm{supp}(X)$, H_{Λ_n} is the local Hamiltonian defined in Equation (3.2.16), and δ is the infinitesimal generator of the time evolution τ. See also Section 2.5.2 for a reminder on C^*-dynamical systems and the discussion following Equation (3.2.16).

One physical intuition that we want to utilize is that a state that minimizes the Hamiltonian locally for each fixed region, must be a ground state, i.e., minimizes the Hamiltonian globally. This intuition is captured by the following lemma.

Lemma 3.5.1. *Let $(\mathfrak{A}, \tau)$ be a C^*-dynamical system and H_n a sequence of positive elements in $\mathfrak{A}$ such that the sequence of maps*

$$\delta_n : \mathfrak{A}_{loc} \to \mathfrak{A}, X \mapsto i\left[H_n, X\right] \tag{3.5.1}$$

converges pointwise in the strong topology to the infinitesimal generator δ of the time evolution τ. If $\omega : \mathfrak{A} \to \mathbb{C}$ is a state such that $\omega(H_n) = 0$ for all n, then ω is a ground state.

Proof. We want to show that $-i\delta(X^*\delta(X)) \geq 0$ for all $X \in \mathfrak{A}_{loc}$. Because H_n is positive, there exists some element $B \in \mathfrak{A}$ such that $H_n = B_n^* B_n$ by Theorem 2.5.7. Using Proposition 2.5.12, we see that

$$\begin{aligned} \left|\omega\left(XH_n\right)\right|^2 = &\left|\omega(XB_n^*B_n)\right|^2 \leq \omega\left(XB_n^*B_nX^*\right)\omega\left(B_n^*B_n\right) \\ =&\omega\left(XH_nX^*\right)\omega\left(H_n\right) = 0 \end{aligned}$$

for all operators X. In particular, if $X = A^*A$ for some local operator A, we see that $\omega(A^*AH_n) = 0$. Therefore

$$\begin{aligned}-i\omega\left(A^*\delta_n(A)\right) =&\omega\left(A^*\left[H_n, A\right]\right)\\ =&\omega\left(A^*H_nA\right) - \omega\left(A^*AH_n\right) = \omega\left(A^*H_nA\right) \geq 0,\end{aligned}$$

since A^*H_nA is positive. Since this holds for all n, it holds in particular in the limit $n \to \infty$, and because δ_n converges to δ pointwise, the result follows. □

We have seen in Lemma 3.4.14 that for $\alpha \in \widehat{D(G)}$ and index pair $I \in I_\alpha$, the states $\omega_\xi^{II,\alpha}$ seem to describe a single excitation at $\partial_0\xi$, measured by the projection operators P_σ^α given in Equation (3.4.11) for any closed ribbon σ enclosing the site $\partial_0\xi$. The idea is to consider Hamiltonians with boundary terms of the form

$$H_n^\alpha = H_{\Lambda_n} - \varepsilon^\alpha P_{\sigma_n}^\alpha,$$

where σ_n is a closed ribbon encircling the square-shaped region Λ_n counter-clockwise and ε^α is a constant chosen such that it suitably accounts for the energy of the states $\omega_\xi^{II,\alpha}$ at site $\partial_0\xi$, see Figure 3.21. More precisely, we will choose ε^α in the hopes that $\omega_\xi^{II,\alpha}(H_n^\alpha)$ becomes zero for each $n \in \mathbb{N}$. The expression $\left[(H_{\Lambda_n} - \varepsilon^\alpha P_{\sigma_n}^\alpha), X\right]$ will converge to $[H, X]$ for each fixed $X \in \mathfrak{A}_{loc}$, and Lemma 3.5.1 then implies that $\omega_\xi^{II,\alpha}$ is a ground state if H_n^α is positive.

Lemma 3.5.2. *Let Λ_n be a square-shaped region of size $n \in \mathbb{N}$ and σ_n a closed ribbon whose direct path forms the boundary of Λ_n such that σ_n bounds its interior by its dual path and such that the direct and dual path encircle Λ counter-clockwise, see also Figure 3.21. For every irreducible $\alpha = (\pi_\alpha, \mathcal{C}_\alpha) \in \widehat{D(G)}$ write*

$$\varepsilon^\alpha := 2 - \delta_{\pi_\alpha,\mathrm{triv}} - \delta_{\mathcal{C}_\alpha,\{e\}}.$$

Then the operator

$$H_n^\alpha = H_{\Lambda_n} - \varepsilon^\alpha P_{\sigma_n}^\alpha \tag{3.5.2}$$

is positive, and we have

$$\omega_\xi^{II,\alpha}\left(H_n^\alpha\right) = \begin{cases} 0 & \text{if } \pi_\alpha \neq \mathrm{triv}, \\ 1 - \frac{1}{|\mathcal{C}_\alpha|} & \text{otherwise} \end{cases} \tag{3.5.3}$$

for all $n > 1$.

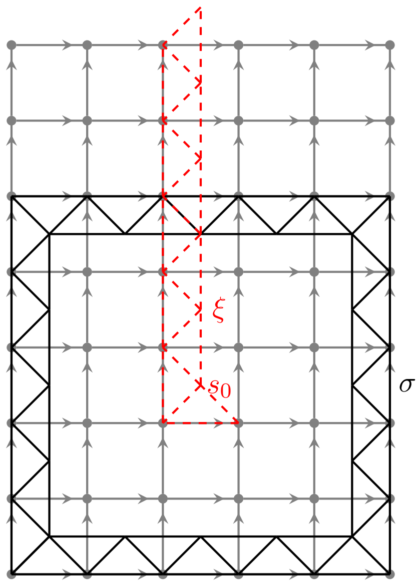


Figure 3.21: Depiction of the scenario described in Lemma 3.5.2 for $n = 5$. The initial site s_0 of ξ lies in the interior of σ and the direct path of σ marks the boundary of Λ_5 in this example.

Proof. By Proposition 3.4.3 and subsequent discussion, $P^{\alpha}_{\sigma_n}$ commutes with all star and plaquette operators. Therefore, the operator $H_{\Lambda_n} - \varepsilon^{\alpha} P^{\mathcal{C}}_{\sigma_n}$ is a sum of the commuting projections A_s, B_s and $P^{\alpha}_{\sigma_n}$, $s \in S(\Lambda_n)$, and there exists a common family of orthonormal eigenvectors $\left\{\psi^k\right\}_k$ in the finite dimensional Hilbert space $\mathcal{H}_{\Lambda_n}$ for these operators. We will show that all eigenvalues of H^{α}_n are non-negative. Write

$$\left\langle \psi^k, \left(H_{\Lambda_n} - \varepsilon^{\alpha} P^{\alpha}_{\sigma_n}\right) \psi^k \right\rangle = \lambda_k - \varepsilon^{\alpha} \rho_k$$

where $\lambda_k \geq 0$ is the eigenvalue of ψ^k as the eigenvector of the non-negative operator H_{Λ_n}, and $\rho_k \in \{0, 1\}$ is the eigenvalue of ψ^k as an eigenvector of $P^{k}_{\sigma_n}$. If $\psi^k \in \ker(P^{\alpha}_{\sigma_n})$, then $\langle \psi^k, \left(H_{\Lambda_n} - \varepsilon^{\alpha} P^{\alpha}_{\sigma_n}\right) \psi^k \rangle = \lambda_k$ is already non-negative. We will therefore show that either $\lambda_k \geq \varepsilon^{\alpha}$ or $\psi^k \in \ker(P^{\alpha}_{\sigma_n})$.

Notice first that we have either $A^h_s \psi^k = A_s \psi^k = \psi^k$ or $A_s \psi^k = 0$ and similarly, we have either $B_s \psi^k = \psi^k$ or $B_s \psi^k = 0$ at any site s, since ψ^k is an eigenvector of each of the operators $(1 - A_s)$ and $(1 - B_s)$ and because $A^h_s A_s = A_s$ for all $h \in G$. If $\lambda_k \geq 2$ there is nothing to show. If $\lambda_k = 0$, then $A_s \psi^k = B_s \psi^k = \psi^k$ for all sites s and ψ^k must be the vacuum state Ω_0. But then Equation (3.4.13) implies either $P^{\alpha}_{\sigma} \Omega_0 = 0$ or $\alpha = \text{triv}$, i.e. $\varepsilon^{\alpha} = 0$. As all other cases are now exhausted, we finally assume that $\lambda_k = 1$. This implies that at most one of the conditions

$$\begin{aligned} A_s\psi^k =& \psi^k \\ B_s\psi^k =& \psi^k \end{aligned}$$

is violated at some sites. Assume that there exists a site s_0 such that $B_s\psi^k = \psi^k$ for all sites s but $A_s\psi^k = \psi^k$ for all $s \neq s_0$. Then we have by Proposition 3.4.4

$$\begin{aligned} P_\sigma^\alpha\psi^k =& P_{s_0}^\alpha\psi^k \\ =& \frac{\dim_{\pi_\alpha}}{|N_\alpha|}\sum_{n\in N_\alpha}\overline{\operatorname{tr}(n)}A_{s_0}^{q_i n\bar{q}_i}B_{s_0}^{c_i}\psi^k. \end{aligned}$$

Because $B_{s_0}\psi^k = \psi^k$, we have $B_{s_0}^{c_i}\psi^k = \delta_{c_i,e}B_{s_0}^{c_i}B_{s_0}\psi^k$ and $\psi^k \in \ker(P_\sigma^\alpha)$ or $\mathcal{C}_\alpha = \{e\}$. In the latter case $\varepsilon^\alpha = 1-\delta_{\pi_\alpha,\text{triv}} \leq 1 = \lambda_k$. Similarly, if $A_s\psi^k = \psi^k$ for all sites s but $B_s\psi^k = \psi^k$ for all $s \neq s_0$, then the orthogonality relations for irreducible characters give

$$\begin{aligned} P_\sigma^\alpha\psi^k =& \frac{\dim_{\pi_\alpha}}{|N_\alpha|}\sum_{n\in N_\alpha}\overline{\operatorname{tr}_{\pi_\alpha}(n)}B_{s_0}^{c_i}\psi^k \\ =& \frac{\dim_{\pi_\alpha}}{|N_\alpha|}\sum_{n\in N_\alpha}\overline{\operatorname{tr}_{\pi_\alpha}(n)}\operatorname{tr}_{\text{triv}}(n)B_{s_0}^{c_i}\psi^k = \delta_{\pi_\alpha,\text{triv}}B_{s_0}^{c_i}\psi^k, \end{aligned}$$

where we used that $A_{s_0}^{q_i n\bar{q}_i}B_{s_0}^{c_i} = B_{s_0}^{c_i}A_{s_0}^{q_i n\bar{q}_i}$ for all i, n since $q_i n\bar{q}_i$ commutes with c_i. This implies again that either $\psi^k \in \ker(P_\sigma^\alpha)$ or $\varepsilon^\alpha = 1 - \delta_{\mathcal{C}_\varepsilon,\{e\}} \leq 1 = \lambda_k$.

We proceed to show Equation (3.5.3). Let $s_0 = \partial_0\xi$. Using Lemma 3.4.14, we get

$$\begin{aligned} \omega_\xi^{II,\alpha}(H_n^\alpha) =& \sum_{s\in S(\Lambda)}\omega_\xi^{II,\alpha}(1-A_s) + \sum_{s\in S(\Lambda)}\omega_\xi^{II,\alpha}(1-B_s) - \varepsilon^\alpha\omega_\xi^{II,\alpha}(P_{\sigma_n}^\alpha) \\ =& \omega_\xi^{II,\alpha}(1-A_{s_0}) + \omega_\xi^{II,\alpha}(1-B_{s_0}) - \varepsilon^\alpha \\ =& 2 - \frac{\delta_{\pi_\alpha,\text{triv}}}{|\mathcal{C}_\alpha|} - \delta_{\mathcal{C}_\alpha,\{e\}} - (2 - \delta_{\pi_\alpha,\text{triv}} - \delta_{\mathcal{C}_\alpha,\{e\}}) \\ =& \delta_{\pi_\alpha,\text{triv}}\left(1 - \frac{1}{|\mathcal{C}_\alpha|}\right) = \begin{cases} 0 & \text{if } \pi_\alpha \neq \text{triv}, \\ 1 - \frac{1}{|\mathcal{C}_\alpha|} & \text{otherwise.} \end{cases} \end{aligned}$$

□

We note that the concrete choice of the ribbons σ_n is irrelevant as long as they move counter-clockwise, enclose the site $\partial_0\xi$ and grow to infinity with $n \to \infty$.

The anomaly in Equation (3.5.3) for the case $\pi_\alpha = \text{triv}$ is due to Equation (3.4.22). This can be solved by redefining the state ω_ξ^α for the case $\pi_\alpha = \text{triv}$ for finite ribbons ξ to be

$$\begin{aligned}\omega_\xi^\alpha(X) =& \frac{1}{|\mathcal{C}_\alpha|}\sum_{i_1,i_2,j}\sum_{n_1,n_2}\omega_0(F_\xi^{c_{i_1},q_{i_1}n_1\bar{q}_j} X F_\xi^{\bar{c}_{i_2},q_{i_2}n_2\bar{q}_j}) \\ =& \frac{1}{|\mathcal{C}_\alpha|}\sum_j \omega_0((F_\xi^j)^* X F_\xi^j)\end{aligned} \tag{3.5.4}$$

for all $X \in \mathfrak{A}_{loc}$, with F_ξ^j defined as

$$F_\xi^j = \sum_{n,i} F_\xi^{\bar{c}_i,q_i n\bar{q}_j}.$$

Notice that F_ξ^j commutes with the star operator A_s at $s = \partial_0\xi$. To see this, let us write the star operator again in the form

$$A_s = \frac{1}{|G|}\sum_{m\in N_\alpha}\sum_{l\in I_{\mathcal{C}_\alpha}} A_s^{q_l m\bar{q}_i}.$$

Then

$$\begin{aligned}A_s F_\xi^j =& \frac{1}{|G|}\sum_{n,m\in N_{\mathcal{C}_\alpha}}\sum_{i,l} A_s^{q_l m\bar{q}_i} F_\xi^{\bar{c}_i,q_i n\bar{q}_j} \\ =& \frac{1}{|G|}\sum_{n,m\in N_{\mathcal{C}_\alpha}}\sum_{i,l} F_\xi^{\bar{c}_l,q_l mn\bar{q}_j} A_s^{q_l m\bar{q}_i} \\ \overset{n\mapsto\bar{m}n}{=}& \frac{1}{|G|}\sum_{n,m\in N_{\mathcal{C}_\alpha}}\sum_{i,l} F_\xi^{\bar{c}_l,q_l n\bar{q}_j} A_s^{q_l m\bar{q}_i} \\ =& \sum_{n\in N_\alpha}\sum_{i\in I_{\mathcal{C}_\alpha}} F_\xi^{\bar{c}_l,q_l n\bar{q}_j} A_s \\ =& F_\xi^j A_s.\end{aligned}$$

It follows for $s_0 = \partial_0\xi$ that $\omega_\xi^\alpha(H_n^\alpha) = 0$ with H_n^α defined as in Lemma 3.5.2. However, we need to verify that ω_ξ^α is a well-defined state even for semi-infinite ribbons.

Lemma 3.5.3. *Let $\{\xi_l\}$ be a sequence of ribbons extending to infinity from $s_0 = \partial_0 \xi_l$ and ξ be the semi-infinite ribbon arising as limit from $\{\xi_l\}$. Set*

$$\omega_{\xi_l}^{\alpha}(X) := \frac{1}{d_\alpha} \sum_{I,I',J \in I_\alpha} \omega_0(F_{\xi_l}^{IJ\alpha} X (F_{\xi_l}^{I'J,\alpha})^*)$$

for each $X \in \mathfrak{A}_{loc}$ and $l \in \mathbb{N}$. Then the limit

$$\omega_\xi^\alpha := \lim_{l\to\infty} \omega_{\xi_l}^{\alpha}(X) = \lim_{l\to\infty} \frac{1}{d_\alpha} \sum_{I,I',J \in I_\alpha} \omega_0(F_{\xi_l}^{IJ\alpha} X (F_{\xi_l}^{I'J,\alpha})^*) \tag{3.5.5}$$

exists for each $X \in \mathfrak{A}_{loc}$ and extends to a state on $\mathfrak{A}$.

Proof. We will show convergence by showing that the expression in Equation (3.5.5) eventually becomes constant. If $X \in \mathfrak{A}_{loc}$, then there exists an $n_0 \in \mathbb{N}$ such that $\text{supp}(X) \cap \xi_l \setminus \xi_{l_0} = \emptyset$ for all $l \geq l_0$. Using Equation (3.3.2), we obtain for $l \geq l_0$ and $\xi' := \xi_l \setminus \xi_{l_0}$

$$\begin{aligned}
&\frac{1}{d_\alpha} \sum_{I,I',J \in I_\alpha} \omega_0(F_{\xi_l}^{IJ,\alpha} X (F_{\xi_l}^{I'J,\alpha})^*) \\
=&\frac{1}{d_\alpha} \sum_{I,I',J \in I_\alpha} \sum_{K,K'} \omega_0(F_{\xi_{l_0}}^{IK,\alpha} F_{\xi'}^{KJ,\alpha} X (F_{\xi_{l_0}}^{I'K',\alpha} F_{\xi'}^{K'J,\alpha})^*) \\
=&\frac{1}{d_\alpha} \sum_{I,I',J \in I_\alpha} \sum_{K,K'} \omega_0(F_{\xi'}^{KJ,\alpha} (F_{\xi'}^{K'J,\alpha})^* F_{\xi_{l_0}}^{IK,\alpha} X (F_{\xi_{l_0}}^{I'K',\alpha})^*) \\
=&\frac{1}{d_\alpha} \sum_{I,I',J \in I_\alpha} \sum_{K,K'} \sum_{n,n' \in N_\alpha} \bar{\Gamma}_{\pi_\alpha}^{k_2 j_2}(n) \Gamma_{\pi_\alpha}^{k'_2 j_2}(n') \\
&\omega_0(F_{\xi'}^{\bar{c}_{k_1}, q_{k_1} n \bar{q}_{j_1}} F_{\xi'}^{c_{k'_1}, q_{k'_1} n' \bar{q}_{j_1}} F_{\xi_{l_0}}^{IK,\alpha} X (F_{\xi_{l_0}}^{I'K',\alpha})^*) \\
=&\frac{1}{d_\alpha} \sum_{I,I',J \in I_\alpha} \sum_{K,K'} \sum_{n,n' \in N_\alpha} \bar{\Gamma}_{\pi_\alpha}^{k_2 j_2}(n) \Gamma_{\pi_\alpha}^{k'_2 j_2}(n') \\
&\delta_{q_{k_1} n \bar{q}_{j_1}, q_{k'_1} n' \bar{q}_{j_1}} \omega_0(F_{\xi'}^{\bar{c}_{k_1} c_{k'_1}, q_{k_1} n \bar{q}_{j_1}} F_{\xi_{l_0}}^{IK,\alpha} X (F_{\xi_{l_0}}^{I'K',\alpha})^*).
\end{aligned}$$

Like before, $q_{k_1} n \bar{q}_{j_1} = q_{k'_1} n' \bar{q}_{j_1}$ implies $k_1 = k'_1$ and $n = n'$. Furthermore, by commuting with star operators at the endpoint of ξ', we can see that the expression

$$\omega_0(F_{\xi'}^{\bar{c}_{k_1} c_{k'_1}, q_{k_1} n \bar{q}_{j_1}} F_{\xi_{l_0}}^{IK,\alpha} X (F_{\xi_{l_0}}^{I'K',\alpha})^*) = \frac{1}{|G|} \omega_0(F_{\xi_{l_0}}^{IK,\alpha} X (F_{\xi_{l_0}}^{I'K',\alpha})^*)$$

is independent of $q_{k_1}n\bar{q}_{j_1}$ and in particular independent of n. This allows us to use the orthogonality relation for irreducible representations to obtain

$$\begin{aligned}
&\frac{1}{d_\alpha\,|G|}\sum_{I,I',J\in I_\alpha}\sum_{\substack{k_1,k_1'\in I_{\mathcal{C}_\alpha}\\ k2,k_2'\in I_{\pi_\alpha}}}\sum_{n\in N_\alpha}\bar{\Gamma}^{k_2j_2}_{\pi_\alpha}(n)\Gamma^{k_2'j_2}_{\pi_\alpha}(n)\delta_{k_1,k_1'}\omega_0(F^{IK,\alpha}_{\xi_{l_0}}X(F^{I'K',\alpha}_{\xi_{l_0}})^*)\\
=&\frac{1}{d_\alpha\,|G|}\frac{|N_\alpha|}{\dim_{\pi_\alpha}}\sum_{I,I',J\in I_\alpha}\sum_{\substack{k_1\in I_{\mathcal{C}_\alpha}\\ k_2\in I_{\pi_\alpha}}}\omega_0(F^{IK,\alpha}_{\xi_{l_0}}X(F^{I'K,\alpha}_{\xi_{l_0}})^*)\\
=&\frac{1}{|G|}\frac{|N_\alpha|}{\dim_{\pi_\alpha}}\sum_{I,I',K\in I_\alpha}\omega_0(F^{IK,\alpha}_{\xi_{l_0}}X(F^{I'K,\alpha}_{\xi_{l_0}})^*)\\
=&\omega^\alpha_{\xi_{l_0}}(X),
\end{aligned}$$

where we used $|G| = |\mathcal{C}_\alpha|\,|N_\alpha|$ and $d_\alpha = |\mathcal{C}_\alpha|\dim_{\pi_\alpha}$ in the last step. This shows that $\omega^\alpha_\xi(X)$ exists for each $X \in \mathfrak{A}_{loc}$ and extends to a linear functional on $\mathfrak{A}$. Clearly, ω^α_ξ is positive. It is left to show that $\omega^\alpha_\xi(1_{\mathfrak{A}}) = 1$:

$$\begin{aligned}
\omega^\alpha_\xi(1_{\mathfrak{A}}) =&\frac{1}{d_\alpha}\sum_{I,I',J\in I_\alpha}\sum_{n,n'\in N_\alpha}\bar{\Gamma}^{i_2j_2}_{\pi_\alpha}(n)\Gamma^{i_2'j_2}_{\pi_\alpha}(n')\omega_0(F^{\bar{c}_{i_1},q_{i_1}n\bar{q}_{j_1}}_\xi F^{c_{i_1'},q_{i_1'}n'\bar{q}_{j_1}}_\xi)\\
=&\frac{1}{d_\alpha}\sum_{I,I',J\in I_\alpha}\sum_{n,n'\in N_\alpha}\bar{\Gamma}^{i_2j_2}_{\pi_\alpha}(n)\Gamma^{i_2'j_2}_{\pi_\alpha}(n')\delta_{q_{i_1}n\bar{q}_{j_1},q_{i_1'}n'\bar{q}_{j_1}}\omega_0(F^{\bar{c}_{i_1}c_{i_1'},q_{i_1}n\bar{q}_{j_1}}_\xi)\\
\stackrel{(*)}{=}&\frac{1}{d_\alpha}\sum_{I,I',J\in I_\alpha}\sum_{n,n'\in N_\alpha}\bar{\Gamma}^{i_2j_2}_{\pi_\alpha}(n)\Gamma^{i_2'j_2}_{\pi_\alpha}(n')\delta_{i_1,i_1'}\delta_{n,n'}\frac{1}{|G|}\\
=&\frac{1}{d_\alpha}\sum_{I,J\in I_\alpha}\frac{|N_\alpha|}{\dim_{\pi_\alpha}}\frac{1}{|G|}\\
=&1,
\end{aligned}$$

where we used Proposition 3.3.12 again in $(*)$. $\square$

Note that the states ω^α_ξ defined in Equation (3.5.5) indeed coincide with the states defined in Equation (3.5.4) for the special case $\pi_\alpha = \text{triv}$.

Theorem 3.5.4. *Let $\xi = \lim_{n\to\infty}\xi_n$ be a semi-infinite ribbon with $\{\xi_n\}$ a sequence of ribbons extending to infinity from some site $s_0 = \partial_0\xi$. Then the states $\omega^{II,\alpha}_\xi$ as defined in Equation (3.4.21) for $\alpha \in \widehat{D(G)}$ with either $\pi_\alpha \neq \text{triv}$ or $|\mathcal{C}_\alpha| > 1$ and any index pair $I \in I_\alpha$ as well as the states ω^α_ξ defined in Equation (3.5.4) for $\pi_\alpha = \text{triv}$ and $|\mathcal{C}_\alpha| > 1$ form a family of ground states of the non-abelian quantum double model.*

Proof. Let Λ_n be a sequence of growing square-shaped regions, $H_n^\alpha = H_{\Lambda_n}^\alpha - \varepsilon^\alpha P_{\sigma_n}^\alpha$ as in Lemma 3.5.2 and ξ a ribbon with $s = \partial_0 \xi$ extending to infinity. Given a local operator $A \in \mathfrak{A}_{loc}$, there always exists an n_0 such that $\mathrm{supp}(A) \subset \Lambda_n$ and $\mathrm{supp}(A) \cap \sigma_n = \emptyset$ for all $n \geq n_0$. But then, setting $\delta_n(A) := i\,[H_n^\alpha, A]$, we have $\delta_n(A) = \delta(A)$, implying that δ_n converges pointwise to δ on $\mathfrak{A}_{loc}$ in the strong operator topology, and by extension on $\mathfrak{A}$. If either $\pi_\alpha \neq \mathrm{triv}$ or $|\mathcal{C}_\alpha| = 1$ or both, $\omega_\xi^{II,\alpha}(H_n^\alpha) = 0$ for all $n > 1$ by Lemma 3.5.2 If $\pi_\alpha = \mathrm{triv}$ and $|\mathcal{C}_\alpha| > 1$, then $\omega_\xi^\alpha(H_n^\alpha) = 0$ by the discussion preceding Lemma 3.5.3. In either case, Lemma 3.5.1 then implies the claim. □

Remark 3.5.5. The states $\omega_\xi^{II,\alpha}$ are indeed non-ground states in case $\pi_\alpha = \mathrm{triv}$ and $|\mathcal{C}_\alpha| > 1$. Fix an irreducible representation $\alpha \in \widehat{D(G)}$ with $\pi_\alpha = \mathrm{triv}$. We claim that

$$\omega_\xi^{II,\alpha} \mid_{\mathfrak{A}_{\Lambda^c}} = \omega_\xi^\alpha \mid_{\mathfrak{A}_{\Lambda^c}}$$

for any region Λ containing $s = \partial_0 \xi$ and any $\alpha \in \widehat{D(G)}$. Indeed, let X be a local operator with $\partial_0 \xi \notin \mathrm{supp}(X)$. By Lemma 3.4.12, $\omega_\xi^{II,\alpha}(X)$ coincides with

$$\rho_\xi^\alpha(X) = \frac{1}{d_\alpha} \sum_{J,I \in I_\alpha} \omega_0(F_\xi^{IJ,\alpha} X (F_\xi^{IJ,\alpha})^*).$$

Furthermore, because of $\left[X, B_{s_0}^c\right] = \left[X, A_{s_0}^h\right] = 0$ for all $c, h \in G$, we can decompose $\xi = \xi_1 \xi_2$ such that $\xi_1 \cap \mathrm{supp}(X) = \emptyset$ and because $F_\xi^{IJ,\alpha} = \sum_{K \in I_\alpha} F_{\xi_1}^{IK,\alpha} F_{\xi_2}^{KJ,\alpha}$ and the $\left\{F_{\xi_1}^{IK,\alpha} \Omega_0\right\}_{I \in I_\alpha}$ form an orthogonal set, we have

$$\omega_0(F_\xi^{IJ,\alpha}, X F_\xi^{I'J,\alpha}) = \delta_{I,I'} \omega_0(F_\xi^{IJ,\alpha}, X F_\xi^{IJ,\alpha}).$$

It follows that

$$\omega_\xi^{II,\alpha}(X) = \frac{1}{d_\alpha} \sum_{I,I',J} \omega_0(F_\xi^{IJ,\alpha} X F_\xi^{I'J,\alpha}) = \rho_\xi^\alpha(X)$$

for all X with $s_0 \notin \mathrm{supp}(X)$. Now, if $\omega_\xi^{II,\alpha}$ was a ground state, it would follow from Theorem 2.5.22 that $\omega(H_n^\alpha) \geq \omega_\xi^{II,\alpha}(H_n^\alpha)$ for all states ω with $\omega \mid_{(\Lambda_n)^c} = \omega_\xi^\alpha \mid_{(\Lambda_n)^c}$. But if $|\mathcal{C}_\alpha| > 1$, we have $\omega_\xi^{II,\alpha}(H_{\Lambda_n}^\alpha) = 1 - \frac{1}{|\mathcal{C}_\alpha|} > 0 = \omega_\xi^\alpha(H_{\Lambda_n}^\alpha)$ for all n.

3.6 Irreducibility of the GNS representations of the states $\omega_\xi^{II,\alpha}$

In this section, we will show that the states $\omega_\xi^{II,\alpha}$ are pure states, implying that their GNS representations are irreducible by Theorem 2.5.11. The idea is to find analogues to the condition $\omega_0(A_s) = \omega_0(B_s) = 1$ for all sites s, as in Equation (3.2.17) and Equation (3.2.18) for the ground states $\omega_\xi^{II,\alpha}$. By Lemma 2.5.6, $\omega_\xi^{II,\alpha}$ is pure if and only if any positive linear functional majorized by $\omega_\xi^{II,\alpha}$ is a scalar multiple of $\omega_\xi^{II,\alpha}$. If we find a family of projections $\{P_k\}_{k\in I}$, with I being some index set such that $\omega_\xi^{II,\alpha}$ is uniquely defined by the condition $\omega_\xi^{II,\alpha}(P_k) = 1$ for all $k \in I$, then for any non-zero positive linear functional ψ with $\psi \leq \omega_\xi^{II,\alpha}$, it follows that

$$0 \leq \psi(1_{\mathfrak{A}} - P_k) \leq \omega_\xi^{II,\alpha}(1_{\mathfrak{A}} - P_k) = 0.$$

Hence, $\psi(P_k) = \psi(1_{\mathfrak{A}})$ and $\tilde{\psi} := \frac{1}{\psi(1_{\mathfrak{A}})}\psi$ is a state with $\tilde{\psi}(P_k) = 1$ for all k. If these conditions fix $\tilde{\psi}$ uniquely, it follows that $\tilde{\psi}$ is equal to $\omega_\xi^{II,\alpha}$, implying that $\psi = \psi(1_{\mathfrak{A}})\omega_\xi^{II,\alpha}$. It then follows that $\omega_\xi^{II,\alpha}$ is pure by Lemma 2.5.6.

We first have to construct such projection operators. Given an irreducible representation $\alpha \in \widehat{D(G)}$, we may define at each site s the operators $A_s^{IJ,\alpha}$ via

$$A_s^{IJ,\alpha} = \frac{\dim_{\pi_\alpha}}{|N_\alpha|} \sum_{n\in N_\alpha} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(n) A_s^{q_{i_1} n \bar{q}_{j_1}} \tag{3.6.1}$$

for each $I, J \in I_\alpha$.

These operators exhibit the following properties.

Lemma 3.6.1. *Let $\alpha, \beta \in \widehat{D(G)}$ and $I, J \in I_\alpha$. Then*

$$\left(A_s^{IJ,\alpha}\right)^* = A_s^{JI,\alpha}.$$

If furthermore $J = (j_1, j_2)$ and $K = (k_1, k_2) \in I_\beta$ with $j_1 = k_1$ and $\mathcal{C} := \mathcal{C}_\alpha = \mathcal{C}_\beta$, then

$$A_s^{IJ,\alpha} A_s^{KL,\beta} = \delta_{\alpha,\beta}\delta_{J,K} A_s^{IL,\alpha}. \tag{3.6.2}$$

In particular, the operators $A_s^{II,\alpha}$ are projections.

Proof. We have $(A_s^k)^* = A_s^{\bar{k}}$ for all $k \in G$, giving

$$\begin{aligned}\frac{|N_\alpha|}{\dim_{\pi_\alpha}} \left(A_s^{IJ,\alpha}\right)^* &= \sum_{n\in N_\alpha} \Gamma_{\pi_\alpha}^{i_2 j_2}(n) A_s^{q_{j_1}\bar{n}\bar{q}_{i_1}} \\ &\overset{n\mapsto\bar{n}}{=\!=} \sum_{n\in N_\alpha} \Gamma_{\pi_\alpha}^{i_2 j_2}(\bar{n}) A_s^{q_{j_1} n\bar{q}_{i_1}}. = \qquad \sum_{n\in N_\alpha} \bar{\Gamma}_{\pi_\alpha}^{j_2 i_2}(n) A_s^{q_{j_1} n\bar{q}_{i_1}} \\ &= \frac{|N_\alpha|}{\dim_{\pi_\alpha}} A_s^{JI,\alpha},\end{aligned}$$

showing the first claim. For the second claim, note first that $\mathcal{C}_\alpha = \mathcal{C}_\beta$ implies $N_\alpha = N_\beta$. It follows that

$$\begin{aligned}\frac{|N_\alpha|}{\dim_{\pi_\alpha}} \frac{|N_\beta|}{\dim_{\pi_\beta}} A_s^{IJ,\alpha} A_s^{KL,\beta} &\overset{j_1=k_1}{=} \sum_{m,n\in N} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(m) \bar{\Gamma}_{\pi_\beta}^{k_2 l_2}(n) A_s^{q_{i_1} m\bar{q}_{j_1}} A_s^{q_{j_1} n\bar{q}_{l_1}} \\ &= \sum_{m,n\in N} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(m) \bar{\Gamma}_{\pi_\beta}^{k_2 l_2}(n) A_s^{q_{i_1} mn\bar{q}_{l_1}} \\ &\overset{(2.2.6)}{=\!=} \delta_{\alpha,\beta}\delta_{j_2,k_2} \frac{\dim_{\pi_\alpha}}{|N_\alpha|} \sum_{m\in N} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(m) A_s^{q_{i_1} m\bar{q}_{l_1}} \\ &= \frac{|N_\alpha|}{\dim_{\pi_\alpha}} \delta_{\alpha,\beta}\delta_{j_2,k_2} A_s^{IL,\alpha} \\ &= \frac{|N_\alpha|}{\dim_{\pi_\alpha}} \delta_{\alpha,\beta}\delta_{J,K} A_s^{IL,\alpha},\end{aligned}$$

since $j_1 = k_1$ already holds by assumption. □

Remark 3.6.2. If α is the trivial representation of $D(G)$, i.e. $\mathcal{C}_\alpha = \{e\}$ and $\pi_\alpha = \text{triv}$, then

$$A_s^{(1,1)(1,1),\alpha} = \frac{\dim_{\text{triv}}}{|Z_G(e)|} \sum_{n\in Z_G(e)} \bar{\Gamma}_{\text{triv}}^{11}(n) A_s^n = \frac{1}{|G|} \sum_{k\in G} A_s^k = A_s$$

becomes the projection into the trivial electric charge.

We have seen in Proposition 3.4.2 that the operators $F_\xi^{IJ,\alpha}$ can be used to decompose the GNS representation $\mathcal{H}$ into irreducible representations at site $s = \partial_0\xi$. The operators in Equation (3.6.1) can be used to permute the vectors $F_\xi^{IJ,\alpha}\Omega_0$ to one another. In [BV23], these operators were therefore called *label changers*.

Lemma 3.6.3. *Let $\alpha, \beta \in \widehat{D(G)}$ with $\mathcal{C} = \mathcal{C}_\alpha = \mathcal{C}_\beta$ and $N = N_\alpha = N_\beta$ and let ξ be an open ribbon with $s = \partial_0\xi$. Then we have*

$$A_s^{IJ,\alpha} B_s^{c_{j_1}} F_\xi^{KL,\beta}\Omega_0 = \delta_{\alpha,\beta}\delta_{J,K} F_\xi^{IL,\beta}\Omega_0 \tag{3.6.3}$$

for all labels $I, J \in I_\alpha$ and $K, L \in I_\beta$.

Proof. First note that because of Equation (3.3.12), we always have

$$B_s^c F_\xi^{h,g}\Omega = B_s^c F_\xi^{h,g} B_s \Omega = B_s^c B_s^{\bar{h}} F_\xi^{h,g}\Omega_0 = \delta_{c,\bar{h}} B_s^c F_\xi^{h,g}\Omega_0$$

regardless of the local orientation of ξ. This gives

$$\begin{aligned}
A_s^{IJ,\alpha} B_s^{c_{j_1}} F_\xi^{KL,\beta}\Omega_0 &= \sum_{n,m\in N} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(m)\bar{\Gamma}_{\pi_\beta}^{k_2 l_2}(n) A_s^{q_{i_1} m \bar{q}_{j_1}} B_s^{c_{j_1}} F_\xi^{\bar{c}_{k_1}, q_{k_1} n \bar{q}_{l_1}}\Omega_0 \\
&= \delta_{k_1,j_1} \sum_{n,m\in N} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(m)\bar{\Gamma}_{\pi_\beta}^{k_2 l_2}(n) A_s^{q_{i_1} m \bar{q}_{j_1}} F_\xi^{\bar{c}_{j_1}, q_{j_1} n \bar{q}_{l_1}}\Omega_0 \\
&= \delta_{k_1,j_1} \sum_{n,m\in N} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(m)\bar{\Gamma}_{\pi_\beta}^{k_2 l_2}(n) F_\xi^{\bar{c}_{i_1}, q_{i_1} m n \bar{q}_{l_1}} A_s^{q_{i_1} m \bar{q}_{j_1}}\Omega_0 \\
&= \delta_{k_1,j_1} \sum_{n,m\in N} \bar{\Gamma}_{\pi_\alpha}^{i_2 j_2}(m)\bar{\Gamma}_{\pi_\beta}^{k_2 l_2}(n) F_\xi^{\bar{c}_{i_1}, q_{i_1} m n \bar{q}_{l_1}}\Omega_0 \\
&\overset{(2.2.6)}{=\!=} \delta_{\alpha,\beta}\delta_{k_1,j_1}\delta_{k_2,j_2} \sum_{n\in N_{\mathcal{C}_\beta}} \bar{\Gamma}_{\pi_\beta}^{i_2 l_2}(n) F_\xi^{\bar{c}_{i_1}, q_{i_1} n \bar{q}_{l_1}}\Omega_0 \\
&= \delta_{\alpha,\beta}\delta_{K,J} F_\xi^{IL,\beta}\Omega_0.
\end{aligned}$$

□

Corollary 3.6.4. *The operators $A_s^{II,\alpha}$ form a family of mutually orthogonal projections. Furthermore, if ξ is a ribbon extending to infinity, $s = \partial_0\xi$ and $\omega_\xi^{II,\alpha}$ the states defined in Equation* (3.4.21)*, then we have*

$$\omega_\xi^{II,\alpha}(A_{s'}) = 1, \tag{3.6.4}$$

$$\omega_\xi^{II,\alpha}(B_{s'}) = 1, \tag{3.6.5}$$

for all $s \neq s'$ and

$$\omega_\xi^{II,\alpha}(B_s^c) = \delta_{c,c_{i_1}}, \tag{3.6.6}$$

$$\omega_\xi^{II,\alpha}(A_s^{LL,\alpha}) = \delta_{I,L}. \tag{3.6.7}$$

Proof. Let $\alpha \in \widehat{D(G)}$. That the $A_s^{II,\alpha}$ are mutually orthogonal projections follows from Equation (3.6.2). Let ξ be a ribbon with $\partial_0\xi = s$. Equation (3.6.7) follows from Equation (3.6.3):

$$\begin{aligned}
\omega_\xi^{II,\alpha}(A_s^{LL,\alpha}) &= \sum_{J\in I_\alpha} \omega_0\left(\left(F_\xi^{IJ,\alpha}\right)^* A_s^{LL,\alpha} F_\xi^{IJ,\alpha}\right) \\
&= \delta_{I,L} \sum_{J\in I_\alpha} \omega_0\left(\left(F_\xi^{IJ,\alpha}\right)^* F_\xi^{IJ,\alpha}\right) \\
&= \delta_{I,L}\,\omega_\xi^{II,\alpha}(1_{\mathfrak{A}}) \\
&= 1.
\end{aligned}$$

To see Equation (3.6.6) note first that

$$\begin{aligned}\omega_\xi^{II,\alpha}(B_s^{c_{i_1}}) &= \sum_{J\in I_\alpha}\sum_{n_1,n_2}\Gamma_{\pi_\alpha}^{i_2j_2}(n_1)\bar{\Gamma}_{\pi_\alpha}^{i_2j_2}(n_2)\langle F_\xi^{c_{i_1},q_{i_1}n_2\bar{q}_{j_1}}\Omega_0, B_s^{c_{i_1}}F_\xi^{\bar{c}_{i_1},q_{i_1}n_2\bar{q}_{j_1}}\Omega_0\rangle\\ &= \sum_{J\in I_\alpha}\sum_{n_1,n_2}\Gamma_{\pi_\alpha}^{i_2j_2}(n_1)\bar{\Gamma}_{\pi_\alpha}^{i_2j_2}(n_2)\langle F_\xi^{c_{i_1},q_{i_1}n_2\bar{q}_{j_1}}\Omega_0, F_\xi^{\bar{c}_{i_1},q_{i_1}n_2\bar{q}_{j_1}}\Omega_0\rangle\\ &=1.\end{aligned}$$

Using Lemma 2.5.13, we obtain

$$\omega_\xi^{II,\alpha}(B_s^c) = \omega_\xi^{II,\alpha}(B_s^c B_s^{c_{i_1}}) = \delta_{c,c_{i_1}}.$$

For the other equations, note that $\left[A_{s'}, F_\xi^{h,g}\right] = \left[B_{s'}, F_\xi^{h,g}\right] = 0$ for all $s' \neq s$ and with $A_{s'}\Omega_0 = B_{s'}\Omega_0 = \Omega_0$, Equation (3.6.4) and Equation (3.6.5) follow. □

Thus, the operators $A_s^{II,\alpha}$ allow us to indeed define a generalized version of the stabilizer conditions Equation (3.2.17) and Equation (3.2.18). We claim that Equation (3.6.4) to Equation (3.6.7) already determine $\omega_\xi^{II,\alpha}$.

Before we show this, it is worthwhile to decompose the star operators into a different basis.

Proposition 3.6.5. *Let $\alpha \in \widehat{D(G)}$ be an irreducible representation and fix an index $i_1 \in I_{\mathcal{C}_\alpha}$. Then the family*

$$\left\{A_s^{K,(i_1,l_2),\beta} \mid \beta \in \widehat{D(G)}, \mathcal{C}_\alpha = \mathcal{C}_\beta, K \in I_\beta, l_2 \in I_{\pi_\beta}\right\}, \tag{3.6.8}$$

where

$$A_s^{K,(i_1,l_2),\beta} := \frac{\dim_{\pi_\beta}}{|N_\beta|}\sum_{n\in N_\beta}\bar{\Gamma}_{\pi_\beta}^{k_2l_2}(n)A_s^{q_{k_1}n\bar{q}_{i_1}}$$

constitutes a new basis for the star operators $\left\{A_s^k\right\}_{k\in G}$ at site s. Furthermore, we have the identities

$$B_s^{c_{i_1}}A_s^{K(i_1l_2),\beta}B_s^{c_{i_1}} = \delta_{i_1,k_1}A_s^{(k_1,k_2)(i_1l_2),\alpha}B_s^{c_{i_1}} = \delta_{i_1,k_1}B_s^{c_{i_1}}A_s^{(k_1,k_2)(i_1l_2),\alpha} \tag{3.6.9}$$

$$A_s^{II,\alpha}A_s^{(i_1,k_2)(i_1l_2),\beta} = \delta_{\alpha,\beta}\delta_{i_2,k_2}A_s^{(i_1,k_2)(i_1l_2),\beta} \tag{3.6.10}$$

and

$$A_s^{(i_1,k_2)(i_1l_2),\beta}A_s^{II,\alpha} = \delta_{i_2,l_2}\delta_{\alpha,\beta}A_s^{(i_1,k_2)(i_1l_2),\beta}. \tag{3.6.11}$$

for any $\beta \in \widehat{D(G)}$ with $\mathcal{C}_\beta = \mathcal{C}_\alpha$.

Proof. Note that Equation (3.6.10) and Equation (3.6.11) are special cases of Equation (3.6.2) and are just listed here for later reference. For any element $n \in N_\alpha$, we have by Equation (3.2.14)

$$A_s^{q_{k_1} n \bar{q}_{i_1}} B_s^{c_{i_1}} = B_s^{q_{k_1} n \bar{q}_{i_1} c_{i_1} q_{i_1} \bar{n} \bar{q}_{k_1}} A_s^{q_{k_1} n \bar{q}_{i_1}} = B_s^{c_{k_1}} A_s^{q_{k_1} n \bar{q}_{i_1}}$$

for all $k_1 \in I_{\mathcal{C}_\alpha}$. This implies

$$\begin{aligned} B_s^{c_{i_1}} A_s^{K(i_1 l_2),\beta} B_s^{c_{i_1}} =& B_s^{c_{i_1}} B_s^{c_{k_1}} A_s^{K(i_1 l_2),\alpha} \\ =& \delta_{i_1,k_1} B_s^{c_{i_1}} A_s^{K(i_1 l_2),\alpha} \\ =& \delta_{i_1,k_1} A_s^{K(i_1 l_2),\alpha} B_s^{c_{i_1}}. \end{aligned}$$

To see that the operators given in Equation (3.6.8) form a new basis for the star operators A_s^k acting at site s, consider the matrix Γ with entries $\Gamma_{\pi_\alpha}^{k_2 l_2}(n)$ whose rows are labelled by $n \in N_\alpha$ and columns are labelled by triples $(i_2, j_2, \pi_\alpha) \in I_\alpha \times I_\alpha \times \widehat{N_\alpha}$. Note that due to the identity

$$\sum_{\pi_\alpha \in \widehat{N_\alpha}} \dim_{\pi_\alpha}^2 = |N_\alpha|,$$

Γ is a square matrix. For $\beta_1, \beta_2 \in \widehat{D(G)}$ with $C_{\beta_1} = C_{\beta_2} = \mathcal{C}_\alpha$ and triples $(i_2, j_2, \pi_{\beta_1}), (k_2, l_2, \pi_{\beta_2})$, the Peter-Weyl Theorem gives

$$(\Gamma\Gamma^*)^{(i_2,j_2,\pi_{\beta_1})(k_2,l_2,\pi_{\beta_2})} = \sum_{n \in N_\alpha} \bar{\Gamma}_{\beta_1}^{i_2 j_2}(n) \Gamma_{\beta_2}^{k_2 l_2}(n) = \frac{|N_{\beta_1}|}{\dim_{\pi_{\beta_1}}} \delta_{\beta_1,\beta_2} \delta_{i_2,k_2} \delta_{l_2,j_2}.$$

This implies in particular that the matrix Γ is regular and represents therefore a base change on $\mathcal{A}_s^{k_1} := \operatorname{span}\left\{ A_s^{q_{k_1} n \bar{q}_{i_1}} \mid n \in N_{\mathcal{C}} \right\}$ and since G factorizes into its cosets $G = \coprod_{c_{k_1} \in \mathcal{C}} q_{k_1} N_{\mathcal{C}}$, every A_s^k lies in $\mathcal{A}_s^{k_1}$ for a unique k_1.

□

This proposition allows us to show that Equation (3.6.4) to Equation (3.6.7) uniquely determine a state on all star and plaquette operators.

Lemma 3.6.6. *Let ψ be a state on $\mathfrak{A}$ such that*

$$\psi\left(A_{s_0}^{II,\alpha}\right) = \psi\left(B_{s_0}^{c_{i_1}}\right) = 1 \tag{3.6.12}$$

for some $\alpha \in \widehat{D(G)}$, $I = (i_1, i_2) \in I_\alpha$ and site s_0. Then

$$\psi\left(B_{s_0}^{c}\right) = \delta_{c,c_{i_1}} \tag{3.6.13}$$

and

$$\psi\left(A_{s_0}^{K(i_1 l_2),\rho}\right) = \delta_{\alpha,\rho} \delta_{K,I} \delta_{l_2,i_2} \tag{3.6.14}$$

holds. In particular, the above holds for the state $\omega_\xi^{II,\alpha}$ at $s_0 = \partial_0 \xi$.

Proof. Using Lemma 2.5.13 we, obtain

$$\begin{aligned}\psi(A_{s_0}^{K(i_1l_2),\rho}) =&\psi\left(A^{II,\alpha}B_{s_0}^{c_{i_1}}A_{s_0}^{K(i_1l_2),\rho}B_{s_0}^{c_{i_1}}A^{II,\alpha}\right)\\ \overset{\text{Lem 3.6.3}}{=}&\delta_{\pi_\alpha,\pi_\rho}\delta_{i_1,k_1}\delta_{i_2,k_2}\delta_{i_2,l_2}\psi\left(A^{II,\alpha}\right)\\ =&\delta_{\pi_\alpha,\pi_\rho}\delta_{i_1,k_1}\delta_{i_2,k_2}\delta_{i_2,l_2}.\end{aligned}$$

For the plaquette operator, we obtain similarly

$$\psi\left(B_{s_0}^{c}\right) = \psi\left(B_{s_0}^{c_{i_1}}B_{s_0}^{c}\right) = \delta_{c_{i_1},c} \tag{3.6.15}$$

□

Theorem 3.6.7. *Let ξ be a ribbon extending to infinity and α an irreducible representation of $D(G)$. Then the state $\omega_\xi^{II,\alpha}$ is uniquely determined by Equation* (3.6.7) *and Equation* (3.6.6) *at site $s_0 = \partial_0\xi$ and*

$$\omega_\xi^{II,\alpha}(A_s) = \omega_\xi^{II,\alpha}(B_s) = 1$$

for all sites $s \neq s_0$.

Proof. We will use similar arguments as in the proof of [Naa12, Thm 12.1.3]. Let ψ be a state on $\mathfrak{A}$ such that

$$\psi(A_s^k) = \omega_\xi^{II,\alpha}(A_s^k), \tag{3.6.16}$$

$$\psi(B_s^k) = \omega_\xi^{II,\alpha}(B_s^k) \tag{3.6.17}$$

holds for all sites s and $k \in G$. Let Λ_n be a square-shaped region of size n centred at s_0. We know that the space of local observables supported on Λ_n is spanned by operators of the form

$$L_{\Lambda_n}^{\lambda}T_{\Lambda_n}^{\gamma}$$

for G-connections $\lambda, \gamma \in C_G(\Lambda)$, where $T_{\Lambda_n}^{\gamma}$ and $L_{\Lambda_n}^{\lambda}$ are defined as in Equation (3.2.22) and Equation (3.2.26) respectively and Λ is any region containing Λ_n. Note that we have the commutation relation $L_{\Lambda_n}^{\lambda}T_{\Lambda_n}^{\gamma} = T_{\Lambda_n}^{\lambda\gamma}L_{\Lambda_n}^{\lambda}$, where $\lambda\gamma : \mathfrak{e} \mapsto \lambda(\mathfrak{e})\gamma(\mathfrak{e})$ is the pointwise product of λ and γ, which we will denote by $\gamma' := \lambda\gamma$ from now on. Because of

$$\psi(L_{\Lambda_n}^{\lambda}T_{\Lambda_n}^{\gamma}) = \psi(B_sL_{\Lambda_n}^{\lambda}T_{\Lambda_n}^{\gamma}B_s) = \psi(B_sT_{\Lambda_n}^{\gamma'}L_{\Lambda_n}^{\lambda}B_s)$$

for all $s \neq s_0$ and

$$\psi(L_{\Lambda_n}^{\lambda}T_{\Lambda_n}^{\gamma}) = \psi(B_{s_0}^{c_{i_1}}L_{\Lambda_n}^{\lambda}T_{\Lambda_n}^{\gamma}B_{s_0}^{c_{i_1}}) = \psi(B_{s_0}^{c_{i_1}}T_{\Lambda_n}^{\gamma'}L_{\Lambda_n}^{\lambda}B_{s_0}^{c_{i_1}}),$$

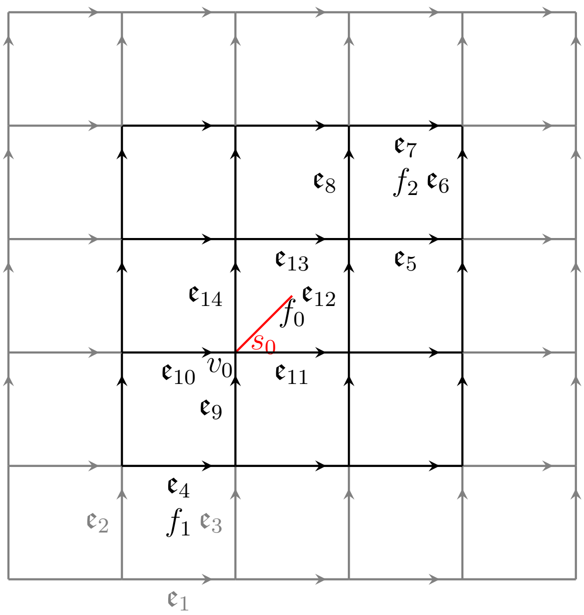


Figure 3.22: Depiction of the square-shaped regions Λ_3 and Λ_4, with Λ_3 in black and $\Lambda_4 \setminus \Lambda_3$ in gray. The magnetic charge is sitting at site $s_0 = (v_0, f_0)$ (red) surrounded by the edges $\mathfrak{e}_{11}, \mathfrak{e}_{12}, \mathfrak{e}_{13}$ and $\mathfrak{e}_{14}$. The magnetic flux of the G-connection γ at s_0 is $\beta_{s_0}(\gamma) = \gamma_{11}\gamma_{12}\bar{\gamma}_{13}\bar{\gamma}_{14} = c_{i_1}$ and the flux at all other sites is trivial. The local operators $L^\lambda T^\gamma$ act only on Λ_3. The only dual triangle operators that can not be eliminated by star operators A_v with $v \neq v_0$ are the ones acting on $\mathfrak{e}_9, \mathfrak{e}_{10}, \mathfrak{e}_{11}$ and $\mathfrak{e}_{14}$.

it follows that γ and γ' must both have a trivial magnetic flux at each site $s = (v, f)$ with $f \neq f_0$, where f_0 is the face associated to s_0 and that γ and γ' have the same flux at s_0 for the above expressions to be non-zero. We want to show that we can transform $L^\lambda_{\Lambda_n}$ to some $L^{\lambda'}_{\Lambda_n}$, where $\lambda'(\mathfrak{e}_i) = e \in G$ for all edges $\mathfrak{e}_i$ with $v_0 \notin \partial\mathfrak{e}_i$ by commuting with star operators at sites other than s_0. In other words, $L^{\lambda'}_{\Lambda_n}$ acts non trivially only on the star at site s_0. We will apply our arguments to the square-shaped region Λ_3 of size $n = 3$ with s_0 being at the plaquette in the centre and $\Lambda = \Lambda_{n+1} = \Lambda_4$, i.e. γ is defined on a slightly bigger square shaped region, with labellings chosen as in Figure 3.22. Note however that this is only done to ease readability of this proof, and all our arguments can straightforwardly be applied to the case where n is arbitrary and s_0 is positioned anywhere.

Before we continue, note that we always have

$$L^k_{\mathfrak{e}}\gamma = R^{\overline{\gamma(\mathfrak{e})}k\gamma(\mathfrak{e})}_{\mathfrak{e}}\gamma$$

for any G-connection $\gamma \in C_G(\Lambda)$, $k \in G$ and edge $\mathfrak{e} \in E$, where $R^k_{\mathfrak{e}}$ denotes the right-multiplication with k at edge $\mathfrak{e}$. In other words, left-multiplication of some $g \in G$ with k is the same as right multiplication of g with of $\bar{g}kg$.

Let $\mathfrak{e}_i$ be labelled as in Figure 3.22, $\gamma \in C_G(\Lambda_{n+1})$ and write $\gamma_i := \gamma(\mathfrak{e}_i)$, $\gamma'_i = \gamma'(\mathfrak{e}_i)$ and $\lambda_i = \lambda(\mathfrak{e}_i)$ and let further $\beta_s(\gamma)$ denote the magnetic flux of γ at site s. We argue first that $\lambda_i = e$ if $\mathfrak{e}_i$ is on the boundary of Λ_3. Indeed, this follows because γ as well as γ' have trivial magnetic charge for all faces in Λ_{n+1} that are not in Λ_n and $L^\lambda_{\Lambda_n}$ can only act on exactly one of the edges surrounding those faces. For instance, for the site $s_1 = (\partial_0\mathfrak{e}_4, f_1)$ we have

$$e = \beta_{s_1}(\gamma') = \bar{\gamma}_2\gamma_1\gamma_3\bar{\gamma}_4\bar{\lambda}_4 = \beta_{s_1}(\gamma)\bar{\lambda}_4 = \bar{\lambda}_4,$$

since $\beta_{s_1}(\gamma) = e$ as well.

Next, consider f_2 and its surrounding edges. The magnetic charge at site $s_2 = (\partial_0\mathfrak{e}_5, f_2)$ is

$$\lambda_5\gamma_5\gamma_6\bar{\gamma}_7\bar{\gamma}_8\bar{\lambda}_8 = \lambda_5\bar{\lambda}_8 \tag{3.6.18}$$

which is equal to e if and only if $\lambda_5 = \lambda_8$. Write $v_2 = \partial_0\mathfrak{e}_5$. Then the star operator $A^{\bar{\lambda}_5}_{v_2} = L^{\bar{\lambda}_5}_{\mathfrak{e}_5}L^{\bar{\lambda}_5}_{\mathfrak{e}_8}R^{\lambda_5}_{\mathfrak{e}_{13}}R^{\lambda_5}_{\mathfrak{e}_{12}}$ will cancel the effect of $L^{\lambda_5}_{\mathfrak{e}_5}L^{\lambda_5}_{\mathfrak{e}_8}$. Furthermore, we have

$$\begin{aligned} R^{\lambda_5}_{\mathfrak{e}_{13}}L^{\lambda_{13}}_{\mathfrak{e}_{13}}T^{\gamma_{13}}_{\mathfrak{e}_{13}} =& R^{\lambda_5}_{\mathfrak{e}_{13}}T^{\lambda_{13}\gamma_{13}}_{\mathfrak{e}_{13}}L^{\lambda_{13}}_{\mathfrak{e}_{13}} \\ =& L^{\lambda_{13}\gamma_{13}\lambda_5\bar{\gamma}_{13}\bar{\lambda}_{13}}_{\mathfrak{e}_{13}}T^{\lambda_{13}\gamma_{13}}_{\mathfrak{e}_{13}}L^{\lambda_{13}}_{\mathfrak{e}_{13}} \\ =& L^{\lambda_{13}\gamma_{13}\lambda_5\bar{\gamma}_{13}\bar{\lambda}_{13}}_{\mathfrak{e}_{13}}L^{\lambda_{13}}_{\mathfrak{e}_{13}}T^{\gamma_{13}}_{\mathfrak{e}_{13}} \\ =& L^{\lambda_{13}\gamma_{13}\lambda_5\bar{\gamma}_{13}}_{\mathfrak{e}_{13}}T^{\gamma_{13}}_{\mathfrak{e}_{13}}. \end{aligned}$$

Similar arguments hold for the edge $\mathfrak{e}_{12}$ and it follows that $A^{\bar{\lambda}_8}_{v_1}L^\lambda_{\Lambda_n}T^\gamma_{\Lambda_n}$ can be written as $L^{\lambda_1}_{\Lambda_n}T^\gamma_{\Lambda_n}$ for some suitable λ_1, and $L^{\lambda_1}_{\Lambda_n}$ acts trivially on $\mathfrak{e}_5$ and $\mathfrak{e}_8$, i.e. $\lambda_1(\mathfrak{e}_5) = \lambda_1(\mathfrak{e}_8) = e$. Because $v_1 \neq v_0$, we further have

$$\psi(L^\lambda_{\Lambda_n}T^\gamma_{\Lambda_n}) = \psi(A^{\bar{\lambda}_5}_{v_1}L^\lambda_{\Lambda_n}T^\gamma_{\Lambda_n}) = \psi(L^{\lambda_1}_{\Lambda_n}T^\gamma_{\Lambda_n}).$$

We can repeat this argument until we transformed $L^\lambda_{\Lambda_n}T^\gamma_{\Lambda_n}$ to some $L^{\lambda'}_{\Lambda_n}T^\gamma_{\Lambda_n}$ with all $\lambda'(\mathfrak{e}_i) = e$ unless $\mathfrak{e}_i$ is an edge of the star-shaped region centred at v_0. We assume that $\lambda = \lambda'$ already holds for the rest of the proof.

We want to show that $L^\lambda_{\Lambda_n}\gamma = A^k_{v_0}\gamma$ and that k commutes with c_{i_1}. To see this, let f_3, f_4 and f_5 be as in Figure 3.23, and let $s_i = (v_0, f_i)$ for $i = 3, 4, 5$. Then because of $\beta_{s_i}(\gamma) = \beta_{s_i}(\gamma') = e$ for all $i = 3, 4, 5$, we have

$$\beta_{s_3}(\gamma') = e = \bar{\gamma}_9\bar{\lambda}_9\gamma_{16}\gamma_{19}\bar{\gamma}_{11}\bar{\lambda}_{11} = \bar{\gamma}_9\bar{\lambda}_9\gamma_9\bar{\gamma}_9\gamma_{16}\gamma_{19}\bar{\gamma}_{11}\bar{\lambda}_{11} = \bar{\gamma}_9\bar{\lambda}_9\gamma_9\bar{\lambda}_{11},$$

which implies $\lambda_{11} = \bar{\gamma}_9\bar{\lambda}_9\gamma_9$. Similarly calculations show that the identities $\beta_{s_4}(\gamma') = \beta_{s_5}(\gamma') = e$ imply $\bar{\gamma}_{10}\lambda_{10}\gamma_{10} = \bar{\gamma}_9\lambda_9\gamma_9$ and $\lambda_{14} = \bar{\gamma}_{10}\bar{\lambda}_{10}\gamma_{10}$. This implies in particular that $\lambda_{11} = \lambda_{14}$ and because $\lambda_9\gamma_9 = \gamma_9\bar{\gamma}_9\lambda_9\gamma_9 = \gamma_9\bar{\lambda}_{11}$ and $\lambda_{10}\gamma_{10} = \gamma_{10}\bar{\gamma}_{10}\lambda_{10}\gamma_{10} = \gamma_{10}\bar{\lambda}_{14} = \gamma_{10}\bar{\lambda}_{11}$, we see that the action of $L^\lambda_{\Lambda_n}$ is given by the star operator $A^{\lambda_{11}}_{v_0}$ on γ, i.e.

$$L^\lambda_{\Lambda_n}\gamma = L^{\lambda_{11}}_{\mathfrak{e}_{11}}L^{\lambda_9}_{\mathfrak{e}_9}L^{\lambda_{10}}_{\mathfrak{e}_{10}}L^{\lambda_{14}}_{\mathfrak{e}_{14}}\gamma = L^{\lambda_{11}}_{\mathfrak{e}_{11}}R^{\bar{\lambda}_{11}}_{\mathfrak{e}_9}R^{\bar{\lambda}_{11}}_{\mathfrak{e}_{10}}L^{\lambda_{11}}_{\mathfrak{e}_{14}}\gamma = A^{\gamma_{11}}_{v_0}\gamma.$$

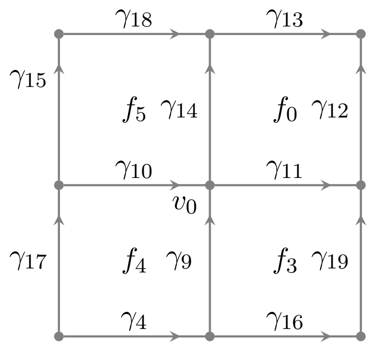


Figure 3.23: Depiction of the γ-values for the four plaquettes bordering the star at v_0. The edges $\mathfrak{e}_i$ are labelled with the values of the G-connection γ. Although the labelling may seem arbitrary, it is still consistent with Figure 3.22

It follows that by commuting with star operators at vertices other than v_0, the operator $L^\lambda_{\Lambda_n}T^\gamma_{\Lambda_n}$ can be transformed into an operator of the form $A^k_{s_0}T^\gamma_{\Lambda_n}$ for some $k \in G$. Furthermore, because of

$$c_{i_1} = \beta_{s_0}(\gamma') = k\gamma_{11}\gamma_{12}\bar{\gamma}_{13}\bar{\gamma}_{14}\bar{k} = k\beta_{s_0}(\gamma)\bar{k} = kc_{i_1}\bar{k}$$

we must have that k lies in the centralizer subgroup of c_{i_1}, hence $k = q_{i_1}nq_{i_1}$ for some $n \in N_\alpha$ and $A^k_{v_0}$ must lie in $\mathcal{A}^{i_1}_s = \operatorname{span}\left\{A^{q_{i_1}n\bar{q}_{i_1}}_{v_0} \mid n \in N_\alpha\right\}$. By Proposition 3.6.5 it is therefore enough to evaluate ψ on operators of the form

$$A^{K,(i_1,l_2),\beta}_{s_0}T^\gamma_{\Lambda_n}$$

for $\beta \in \widehat{D(G)}$, $K = (k_1, k_2) \in I_\beta$, $k_1 = i_1$ and $l_2 \in I_{\pi_\beta}$. Because of

$$\begin{aligned}\psi(A^{K,(i_1,l_2),\beta}_{s_0}T^\gamma_{\Lambda_n}) &= \psi(A^{II,\alpha}_s A^{K,(i_1,l_2),\beta}_{s_0}T^\gamma_{\Lambda_n}) \\ &\overset{(3.6.2)}{=\!=} \delta_{\alpha,\beta}\delta_{K,I}\psi(A^{I,(i_1,l_2),\alpha}_{s_0}T^\gamma_{\Lambda_n})\end{aligned}$$

We therefore may assume that $K = I$ and $\alpha = \beta$.

We want to show that also $l_2 = i_2$ holds by inserting the operator $A_{s_0}^{II,\alpha}$ on the right of the argument of ψ. At first, this does not seem to lead anywhere due to the fact that the different summands $A_{s_0}^{q_{i_1}m\bar{q}_{i_1}}$ for $m \in N_\alpha$ commute differently with $T_{\Lambda_n}^{\gamma}$, i.e.

$$A_{s_0}^{II,\alpha} T_{\Lambda_n}^{\gamma} = \sum_{m \in N_\alpha} \frac{\dim_{\pi_\alpha}}{|N_\alpha|} \bar{\Gamma}_{\pi_\alpha}^{i_2 i_2}(m) T_{\Lambda_n}^{\gamma_m} A_{s_0}^{q_{i_1}m\bar{q}_{i_1}}$$

with suitable G-connections γ_m. However, by Lemma 3.2.7, we can transform each $T_{\Lambda_n}^{\gamma_m}$ individually to some constant $T_{\Lambda_n}^{\gamma}$ using star operators at vertices other than v_0, commuting with each star operator at s_0. This gives indeed

$$\begin{aligned}
\delta_{K,I}\delta_{\alpha,\beta}\psi(A_{s_0}^{I,(i_1,l_2),\alpha} T_{\Lambda_n}^{\gamma}) =& \psi(A_{s_0}^{I,(i_1,l_2),\alpha} T_{\Lambda_n}^{\gamma} A_{s}^{II,\alpha}) \\
=& \delta_{K,I}\delta_{\alpha,\beta}\psi(A_{s_0}^{I,(i_1,l_2),\alpha} A_{s_0}^{II,\alpha} T_{\Lambda_n}^{\gamma}) \\
=& \delta_{K,I}\delta_{\alpha,\beta}\delta_{l_2,i_2}\psi(A_{s_0}^{II,\alpha} T_{\Lambda_n}^{\gamma}) \\
=& \delta_{K,I}\delta_{\alpha,\beta}\delta_{l_2,i_2}\psi(T_{\Lambda_n}^{\gamma})
\end{aligned}$$

Let $C_G^{s_0,c_{i_1}}(\Lambda_n)$ denote the space of G-connections that are flat for all $s = (v, f)$ with $f \neq f_0$ and have magnetic flux c_{i_1} at site s_0. As mentioned before, Lemma 3.2.7 allows us to commute with star operators at sites other than s_0 to commute $T_{\Lambda_n}^{\gamma}$ to any $T_{\Lambda_n}^{\gamma'}$ with $\gamma' \in C_G^{s_0,c_{i_1}}(\Lambda_n)$, hence $\psi(T_{\Lambda_n}^{\gamma}) = \psi(T_{\Lambda_n}^{\gamma'})$ for all $\gamma' \in C_G^{s_0,c_{i_1}}(\Lambda_n)$, and because of $\psi(1_{\mathfrak{A}}) = 1$ we have

$$\psi\left(A_{s_0}^{K,(i_1,l_2)}, \beta T_{\Lambda_n}^{\gamma}\right) = \frac{\delta_{K,I}\delta_{\alpha,\beta}\delta_{l_2,i_2}}{\left|C_G^{s_0,c_{i_1}}(\Lambda_n)\right|}.$$

We conclude that ψ is uniquely determined on all operators of the form $L_{\Lambda_n}^{\lambda} T_{\Lambda_n}^{\gamma}$, hence on all of $\mathfrak{A}_{loc}$ and by continuity on $\mathfrak{A}$. □

Theorem 3.6.8. *$\omega_\xi^{II,\alpha}$ is pure for each irreducible representation $\alpha \in \widehat{D(G)}$ and index pair $I = (i_1, i_2) \in I_\alpha$, and the corresponding GNS representations is therefore irreducible.*

Proof. We will repeat the arguments from the beginning of this section in more detail. Write $s_0 = (v_0, f_0) = \partial_0\xi$. By Lemma 2.5.6, $\omega_\xi^{II,\alpha}$ is pure if and only if for every positive linear functional $\psi : \mathfrak{A} \to \mathbb{C}$ with $\psi \leq \omega_\xi^{II,\alpha}$, it follows that ψ is a multiple of $\omega_\xi^{II,\alpha}$. If ψ is non-zero and $\psi(X) \leq \omega_\xi^{II,\alpha}(X)$ for all positive $X \in \mathfrak{A}$, then

$$0 \leq \psi(1_{\mathfrak{A}} - A_{s_0}^{II,\alpha}) \leq \omega_\xi^{II,\alpha}(1_{\mathfrak{A}} - A_{s_0}^{II,\alpha}) = 0$$

and

$$0 \leq \psi(1_{\mathfrak{A}} - A_s) \leq \omega_\xi^{II,\alpha}(1_{\mathfrak{A}} - A_s) = 0$$

for all $s = (v, f)$ with $f \neq f_0$, implying $\psi(A_s) = \psi(A_{s_0}^{II,\alpha}) = 1_{\mathfrak{A}}$. Similarly, $\psi(B_s) = \psi(B_{s_0}^{c_{i_1}}) = \psi(1_{\mathfrak{A}})$. Then the map $\tilde{\psi} = \frac{1}{\psi(1_{\mathfrak{A}})}\psi$ defines a state with $\tilde{\psi}(A_{s_0}^{II,\alpha}) = \tilde{\psi}(A_s) = 1$ and $\tilde{\psi}(B_s) = \tilde{\psi}(B^{c_{i_1}}) = 1$ It follows from Theorem 3.6.7 that $\tilde{\psi} = \omega_\xi^{II,\alpha}$, implying $\psi = \psi(1_{\mathfrak{A}})\omega_\xi^{II,\alpha}$, hence $\omega_\xi^{II,\alpha}$ is pure.

Finally, by [KR86, Thm 10.2.3] a state is pure if and only if its corresponding GNS representation is irreducible. □

Theorem 3.6.9. *Let $\alpha, \beta \in \widehat{D(G)}$, $I \in I_\alpha$, $J \in I_\beta$ and ξ_1, ξ_2 be semi-infinite ribbons. Then the GNS representations of $\omega_{\xi_1}^{II,\alpha}$ and $\omega_{\xi_2}^{JJ,\beta}$ are equivalent if and only if $\alpha \cong \beta$.*

Proof. By construction, $\omega_{\xi_1}^{II,\alpha}$ and $\omega_{\xi_2}^{JJ,\beta}$ are normal states. By Theorem 3.6.8, the GNS representations $\pi_{\xi_1}^{II,\alpha}$ respectively $\pi_{\xi_2}^{JJ,\beta}$ of $\omega_{\xi_1}^{II,\alpha}$ respectively $\omega_{\xi_2}^{JJ,\beta}$ are irreducible. It follows that the commutants $\pi_{\xi_1}^{II,\alpha}(\mathfrak{A})'$ and $\pi_{\xi_2}^{JJ,\beta}(\mathfrak{A})'$ must be multiples of the identity, implying that $\omega_{\xi_1}^{II,\alpha}$ and $\omega_{\xi_2}^{JJ,\beta}$ are factor states, and by Lemma 2.5.23, $\omega_{\xi_1}^{II,\alpha}$ and $\omega_{\xi_2}^{JJ,\beta}$ are quasi-equivalent if and only if for each $\varepsilon > 0$ there exists a region Λ such that

$$\left|\omega_{\xi_1}^{II,\alpha}(X) - \omega_{\xi_2}^{JJ,\beta}(X)\right| < \varepsilon \|X\| \tag{3.6.19}$$

for all $X \in \mathfrak{A}_{loc}$ supported on Λ^c. In Lemma 3.4.12, we saw that if $\alpha = \beta$, we even have $\omega_{\xi_1}^{II,\alpha}(X) = \omega_{\xi_1}^{\alpha}(X)$ for any region disjoint from $\partial_0\xi$, where $\omega_{\xi_1}^{\alpha}$ are the states defined in Lemma 3.5.3, giving in particular $\omega_{\xi_1}^{II,\alpha}(X) = \omega_{\xi_1}^{JJ,\alpha}(X)$ in that case and Equation (3.6.19) is trivially satisfied for all ε and $\omega_{\xi_1}^{II,\alpha}$ and $\omega_{\xi_1}^{JJ,\alpha}$ must be quasi-equivalent. Since pure states are quasi-equivalent if and only if they are equivalent, $\omega_{\xi_1}^{II,\alpha}$ and $\omega_{\xi_1}^{JJ,\alpha}$ must be equivalent. By our discussion in Remark 3.4.11, we also have $\omega_{\xi_1}^{II,\alpha}(X) = \omega_{\xi_2}^{II,\alpha}(X)$ for all $X \in \mathfrak{A}_{loc}$ supported outside of a region containing any ribbon ζ connecting $\partial_0\xi_1$ and $\partial_0\xi_2$. This implies that $\omega_{\xi_1}^{II,\alpha}$ and $\omega_{\xi_2}^{JJ,\alpha}$ must be quasi-equivalent, and hence equivalent, as well.

For the other direction, let $\alpha \ncong \beta$. Then we can choose for any finite region Λ a closed ribbon σ supported outside Λ with s_1 and s_2 in the interior of σ to obtain

$$\omega_{\xi_1}^{II,\alpha}(P_\sigma^\alpha) = 1,$$
$$\omega_{\xi_2}^{JJ,\beta}(P_\sigma^\alpha) = 0,$$

by Equation (3.4.24). But then Equation (3.6.19) is violated for any $\varepsilon < 1$ and $X = P^{\alpha}_{\sigma}$ and $\omega^{II,\alpha}_{\xi_1}$ and $\omega^{JJ,\beta}_{\xi_2}$ must be inequivalent. □

Remark 3.6.10. The proof of Theorem 3.6.7 is a generalization of the proof of uniqueness of the frustration free ground state for the quantum double model [FN15]. Indeed, as indicated before, choosing $\alpha = (\mathrm{triv}, \{e\})$ in Equation (3.4.20), we get the frustration free ground state.

Chapter 4

Outlook and Discussion

In this chapter, we want to discuss some open question that this work can branch out to. In Section 3.4 we introduced the notion amplimorphisms as a means to defining the states $\omega_\xi^{II,\alpha}$. In Section 4.1, we inspect these amplimorphisms in further detail. It turns out that the matrices $\mathbf{F}_\xi^\alpha$ carry natural transformation rules under the irreducible representation $\alpha \in \widehat{D(G)}$. Furthermore, the amplimorphisms obtained from these matrices form objects of a category that can be related to the representation category $\text{rep}(D(G))$. What makes these amplimorphisms interesting to study is that they give rise to states satisfying the superselection criterion for cones: If C is a cone in the plane, then the GNS representation π_0 of the vacuum state ω_0 is quasi-equivalent to the representation $\chi_\xi^\alpha \circ \pi_0$ when restricted to the complement of C in $\mathbb{Z}^2$. In other words, we have

$$\pi_0 \mid_{C^c} \cong_{q.e.} (\chi_\xi^\alpha \circ \pi_0) \mid_{C^c},$$

where the composition on the right-hand side is understood component wise. We remind the reader that the classical superselection criterion is defined in the setting of quantum field theory for light-cones, and the complement considered is the space-like complement. Furthermore, we eased the criterion to demand only quasi-equivalence, whereas in [DHR71] one demands unitary equivalence.

A similar class of amplimorphism was analysed in [SV93] and a categorical equivalence was established between this category of amplimorphisms and the category $\text{rep}(D(G))$. We investigate this relation in the context of the non-abelian quantum double[1]. The biggest difference in our work is that we are dealing with infinite ribbons, while in [SV93], all operators were localized

[1]This is a work in progress in collaboration with Alex Bols, Pieter Naaijkens and Siddharth Vadnerkar

in a finite region of the spin chain model. Recall also that the operators $\mathbf{F}_\xi^\alpha$ defined in Chapter 3 are not well-defined for infinite ribbons, even though the amplimorphisms $\chi_\xi^{II,\alpha}$ are.

There is another important difference between our setting and the scenario in [SV93]: Morphisms in the category of amplimorphism are defined as unitaries in $\mathfrak{A} \otimes \mathrm{Hom}(V, W)$ for amplimorphisms $\chi_1 \in \mathfrak{A} \otimes \mathrm{End}(V)$ and $\chi_2 \in \mathfrak{A} \otimes \mathrm{End}(W)$. In our case, however, the entries of such a unitary will not live in $\mathfrak{A}$, but rather in the von Neumann algebra generated by the GNS representation of ω_0.

In Section 4.2, we discuss open questions and possible generalizations from our work. In Section 4.2.1, we mention briefly mention the possibility of other ground state in the non-abelian quantum double model. In Section 4.2.2 we consider quantum double models stemming from general Hopf algebras. Finally, we discuss ideas to extend the quantum double construction to compact groups in Section 4.2.3.

4.1 The category of Amplimorphisms

First, we return to the analysis of superselection sectors. We have mentioned at the beginning of this thesis that it is widely believed that anyons are described by a modular tensor category, and that we would like to describe the algebraic properties of anyons using the theory of superselection sectors. We explained in the beginning of Chapter 3 that the construction of the amplimorphisms $\chi_\xi^\alpha : \mathfrak{A} \to \mathfrak{A} \otimes \mathrm{End}(\mathcal{V}^\alpha)$ is motivated by the DHR theory, which associates the superselection sectors of physical states with localized and transportable endomorphisms. A representation π is said to satisfy the **superselection criterion for cones**, if for every cone C we have

$$\pi_0 \mid_{\mathfrak{A}(C^c)} \cong_{q.e.} \pi \mid_{\mathfrak{A}(C^c)}, \tag{4.1.1}$$

where $\cong_{q.e.}$ denotes quasi-equivalence. If χ is an amplimorphism localized in some cone C, then for all A supported outside of C we have $\pi_0 \circ \chi(A) = \pi_0(A) \otimes \mathrm{id} = \bigoplus_{I \in I_\alpha} \pi_0(A)$. As we have seen, the amplimorphisms χ_ξ^α are localized as well (Proposition 3.4.8). Furthermore, it was shown in [BV23] that the representations $\chi_\xi^{II,\alpha} \circ \pi_0$ are unitarily equivalent to the GNS representations $\pi_\xi^{II,\alpha}$ of the pure states $\omega_\xi^{II,\alpha}$ for each $I \in I_\alpha$ and regardless of the choice of the initial site of ξ, and with Theorem 3.6.9 it follows that $\chi_\xi^{II,\alpha} \circ \pi_0$ is unitarily equivalent to $\chi_{\xi'}^{JJ,\alpha} \circ \pi_0$ for any ribbon ξ' and index $J \in I_\alpha$, which is why we may write π^α instead of $\pi_\xi^{II,\alpha}$. It follows that the states $\chi_\xi^\alpha \circ \pi_0$ satisfy the superselection criterion for cones.

We summarize our observation in the following theorem.

Theorem 4.1.1. *Let ξ be a semi-infinite ribbon extending to infinity from some initial site s_0. Then the mapping*

$$\widehat{D(G)} \to \operatorname{rep}(\mathfrak{A}), \alpha \mapsto \chi_\xi^\alpha \circ \pi_0, \tag{4.1.2}$$

maps each irreducible representation of the quantum double $D(G)$ to a state satisfying the superselection criterion Equation (4.1.1)*.*

Whether transportability for cones holds was left as an open question in [Naa15], but is now answered in Theorem 4.1.1

Transportable and localized amplimorphisms form a category, and we anticipate that this category has the same structure as the representation category of $D(G)$ as a modular tensor category. In [SV93], a 1-dimensional quantum spin chain is explored within the setting just described. There, it was shown that all anyon sectors can be obtained via localized and transportable amplimorphism $\chi : \mathfrak{A} \to \mathfrak{A} \otimes M_n(\mathbb{C})$ on the quasilocal algebra $\mathfrak{A}$, and a fixed vacuum representation π_0. As mentioned before, there are certain additional subtleties in the setting of the infinite plane. One such subtlety is that the unitary transporters $\mathbf{V}$ are not defined in $\pi^\alpha(\mathfrak{A})$, but rather in the von Neumann algebra generated by $\pi(\mathfrak{A})$ [Naa15]. Another difference is that the transformation rule satisfied by the matrices $\mathbf{F}_\xi^\alpha$ are slightly different from the ribbon operators defined in [SV93]. In the following, we will explore the difficulties and possibilities to generalize their result to the two-dimensional non-abelian quantum double model on an infinite lattice.

Let H be a Hopf *-algebra with unitary action on a module V via some map $U : H \otimes V \to V$. Then we can define the **coadjoint action** of H on $\operatorname{End}(V)$ via

$$\gamma_a^U(A) = \sum_{(a)} U(a^{(1)}) A U(S(a^{(2)})). \tag{4.1.3}$$

Lemma 4.1.2. *The map γ^U defined in Equation* (4.1.3) *defines indeed an action of H on* $\operatorname{End}(V)$ *and satisfies the identities*

$$\gamma_a^U(AB) = \sum_{(a)} \gamma_{a^{(1)}}^U(A) \gamma_{a^{(2)}}^U(B) \tag{4.1.4}$$

and

$$\gamma_a^U(A^*) = \gamma_{S(a)^*}^U(A)^*. \tag{4.1.5}$$

Proof. That γ defines an action follows from

$$\begin{aligned}\gamma^U_{ab}(A) =& \sum_{(ab)} U((ab)^{(1)})AU(S(ab)^{(2)})\\ =& \sum_{(a)}\sum_{(b)} U(a^{(1)})U(b^{(1)})AU(S(b^{(2)}))U(S(a^{(2)}))\\ =& \gamma^U_a \circ \gamma^U_b(A).\end{aligned}$$

We proceed to show the other identities:

$$\begin{aligned}\gamma^U_a(AB) =& \sum_{(a)} U(a^{(1)})ABU(S(a^{(2)}))\\ =& \sum_{(a)} U(a^{(1)})AU(S(a^{(2)}))U(a^{(3)})BU(S(a^{(4)}))\\ =& \sum_{(a)} \gamma^U_{a^{(1)}}(A)\gamma^U_{a^{(2)}}(B)\end{aligned}$$

$$\begin{aligned}\gamma^U_{S(a)^*}(A) =& \sum_{(a)} \left[U((S(a)^*)^{(1)})AU(S((S(a)^*)^{(2)}))\right]^*\\ =& \sum_{(a)} \left[U((S(a^{(2)})^*))AU(S(S(a^{(1)})^*))\right]^*\\ =& \sum_{(a)} \left[U((S(a^{(2)})^*))AU((a^{(1)})^*)\right]^*\\ =& \sum_{(a)} \left[U(a^{(1)})A^*U(S(a^{(2)}))\right]\\ =& \gamma^U_a(A^*).\end{aligned}$$

□

Let U_s be the action of the quantum double $D(G)$ at site s defined in Proposition 3.4.1 via $U_s(\delta_g \otimes h) = B^g_s A^h_s$ and let $\gamma^s := \gamma^{U_s}$ be the corresponding coadjoint action on $\mathfrak{A}$. Given a ribbon ξ and an irreducible representation $\alpha \in \widehat{D(G)}$, the elements $(\mathbf{F}^\alpha_\xi)^{IJ}$ in Equation (3.4.14) form the entries of an element $\mathbf{F} \in \mathfrak{A} \otimes \mathrm{End}(V)$, with $V \cong \mathbb{C}^{|I_\alpha|}$. For an arbitrary element $\mathbf{F} \in \mathfrak{A} \otimes \mathrm{Hom}(V, W)$ we say that $\mathbf{F}$ satisfies the **F-algebra relation** [SV93] if

$$\mathbf{F}^*\mathbf{F} = 1_{\mathfrak{A}} \otimes \mathrm{id}_V, \tag{4.1.6}$$

i.e. if

$$\sum_K (\mathbf{F}^{KI})^* \mathbf{F}^{KJ} = \delta_{IJ} 1_{\mathfrak{A}}$$

for all $I = 1, \ldots, \dim_W$ and $J = 1, \ldots, \dim_V$. We call $\mathbf{F}$ **non-degenerate** if $\mathbf{F}$ satisfies the F-algebra relation, Equation (4.1.6), and

$$\mathbf{F}\mathbf{F}^* = 1_{\mathfrak{A}} \otimes \mathrm{id}_W, \tag{4.1.7}$$

i.e. if

$$\sum_K \mathbf{F}^{IK} (\mathbf{F}^{JK})^* = \delta_{I,J} 1_{\mathfrak{A}}.$$

For any element $\mathbf{F} \in \mathfrak{A} \otimes \mathrm{Hom}(V, W)$ we define a map $\chi_{\mathbf{F}}$ via

$$\chi_{\mathbf{F}} : \mathfrak{A} \to \mathfrak{A} \otimes \mathrm{End}(V), \quad X \mapsto \mathbf{F} X \mathbf{F}^*. \tag{4.1.8}$$

If $\mathbf{F}$ satisfies the F-algebra relation, then

$$\begin{aligned} \chi_{\mathbf{F}}(AB) =& (\mathbf{F})^* (AB \otimes \mathrm{id}_V)\mathbf{F} = (\mathbf{F})^* (A \otimes \mathrm{id}_V)\mathbf{F}\mathbf{F}^*(B \otimes \mathrm{id}_V)\mathbf{F} \\ =& \chi_{\mathbf{F}}(A)\chi_{\mathbf{F}}(B) \end{aligned}$$

and $\chi_{\mathbf{F}}$ becomes an amplimorphism that is unital if and only if $\mathbf{F}$ is in addition non-degenerate.

We can extend the coadjoint action of $D(G)$ to $\mathfrak{A} \otimes \mathrm{Hom}(V, W)$ via

$$\gamma_a^s(\mathbf{F}) = \sum_{(a)} (U_s[a^{(1)}] \otimes \mathrm{id})\mathbf{F}(U_s[S(a^{(2)})] \otimes \mathrm{id}). \tag{4.1.9}$$

Before we continue, we introduce a few notations to ease readability. If $X \in \mathfrak{A}$, we set

$$X\mathbf{F} := (X \otimes \mathrm{id}_W)\mathbf{F},$$

i.e., we view X as acting component wise. Similarly, if $T \in \mathrm{End}(W)$, we may identify $T = 1_{\mathfrak{A}} \otimes T$ and write $T\mathbf{F}$ instead of $(1_{\mathfrak{A}} \otimes T)\mathbf{F}$. Note that we have $[T, X] = 0$ for all $X \in \mathfrak{A}$ and $T \in \mathrm{End}(V)$ with this notation. Equation (4.1.9) reads in this notation

$$\gamma_a^s(\mathbf{F}) = \sum_{(a)} U_s[a^{(1)}]\mathbf{F} U_s[S(a^{(2)})]. \tag{4.1.10}$$

If $\alpha \in \widehat{D(G)}$ is an irreducible representation of the quantum double G, then we call $\mathbf{F}$ an **α-multiplet at site** s if $\mathbf{F}$ satisfies the F-algebra relation and if

$$\gamma_a^s(\mathbf{F}) = \Gamma_{\alpha^*}(S(a))\mathbf{F}, \tag{4.1.11}$$

where α^* is the contragredient representation of α, (cf. Proposition 2.4.10) and Γ_{α^*} is the unitary matrix representation of α^* as given in Equation (2.4.33). Note that we identified $\Gamma_\alpha(a)$ with $1_{\mathfrak{A}} \otimes \Gamma_\alpha(a)$. The motivation behind these definitions can be found in the following Proposition.

Proposition 4.1.3. *Let $\alpha \in \widehat{D(G)}$ be an irreducible representation of the quantum double $D(G)$ and ξ a ribbon with initial site $s = \partial_0\xi$. Then the matrix $\mathbf{F}_\xi^\alpha \in \mathfrak{A} \otimes M_{d_\alpha}(\mathbb{C})$ defined in Equation* (3.4.14) *is non-degenerate and satisfies the F-algebra relations. If ξ is locally clockwise oriented, then $\mathbf{F}_\xi^\alpha$ is a non-degenerate α-multiplet at site s.*

Proof. The F-algebra relation and non-degeneracy has been shown in the proof of Proposition 3.4.8, see in particular Equation (3.4.18) and Equation (3.4.17). We verify the multiplet property for elements of the form $\delta_g \otimes h \in$

$D(G)$ with $g, h \in G$ and consider the I-th row and J-th column of $\gamma^s(a)(\mathbf{F}^\alpha_\xi)$:

$$
\begin{aligned}
\left(\gamma^s_a(\mathbf{F}^\alpha_\xi)\right)^{IJ} &= \sum_{n\in N_\alpha}\sum_{(a)} \bar\Gamma^{i_2j_2}_{\pi_\alpha}(n)U_s(a^{(1)})F_\xi^{\bar c_{i_1},q_{i_1}n\bar q_{j_1}}U_s(S(a^{(2)}))\\
&\overset{(2.4.25),(2.4.26)}{=\!=} \sum_{n\in N_\alpha}\sum_{(a)} \bar\Gamma^{i_2j_2}_{\pi_\alpha}(n)U_s(\delta_{g_2}\otimes h)F_\xi^{\bar c_{i_1},q_{i_1}n\bar q_{j_1}}U_s(\delta_{\bar h\bar g_1 h}\otimes \bar h)\\
&\overset{(3.4.1)}{=\!=} \sum_{n\in N_\alpha}\sum_{g_1g_2=g} \bar\Gamma^{i_2j_2}_{\pi_\alpha}(n)B_s^{g_2}A_s^hF_\xi^{\bar c_{i_1},q_{i_1}n\bar q_{j_1}}B_s^{\bar h\bar g_1 h}A_s^{\bar h}\\
&\overset{(3.3.6),(3.3.8)}{=\!=} \sum_{n\in N_\alpha}\sum_{g_1g_2=g} \bar\Gamma^{i_2j_2}_{\pi_\alpha}(n)F_\xi^{h\bar c_{i_1}\bar h,hq_{i_1}n\bar q_{j_1}}B_s^{g_2h\bar c_{i_1}\bar h}A_s^hB_s^{\bar h\bar g_1 h}A_s^{\bar h}\\
&\overset{(3.2.14)}{=\!=} \sum_{n\in N_\alpha}\sum_{g_1g_2=g} \bar\Gamma^{i_2j_2}_{\pi_\alpha}(n)F_\xi^{h\bar c_{i_1}\bar h,hq_{i_1}n\bar q_{j_1}}B_s^{g_2h\bar c_{i_1}\bar h}B_s^{\bar g_1}A_s^hA_s^{\bar h}\\
&= \sum_{n\in N_\alpha}\sum_{g_1g_2=g} \delta_{\bar g_1,g_2h\bar c_{i_1}\bar h}\bar\Gamma^{i_2j_2}_{\pi_\alpha}(n)F_\xi^{h\bar c_{i_1}\bar h,hq_{i_1}n\bar q_{j_1}}B_s^{\bar g_1}\\
&\overset{g_1g_2=g}{=\!=} \sum_{n\in N_\alpha}\sum_{g_1g_2=g} \delta_{\bar g,h\bar c_{i_1}\bar h}\bar\Gamma^{i_2j_2}_{\pi_\alpha}(n)F_\xi^{h\bar c_{i_1}\bar h,q_{hc_{i_1}\bar h}\bar q_{hc_{i_1}\bar h}hq_{i_1}n\bar q_{j_1}}B_s^{\bar g_1}\\
&\overset{n\mapsto \bar q_{i_1}\bar h q_{hc_{i_1}\bar h}n}{=\!=} \sum_{n\in N_\alpha} \delta_{\bar g,h\bar c_{i_1}\bar h}\bar\Gamma^{i_2j_2}_{\pi_\alpha}(\bar q_{i_1}\bar h q_{hc_{i_1}\bar h}n)F_\xi^{h\bar c_{i_1}\bar h,q_{hc_{i_1}\bar h}n\bar q_{j_1}}\\
&= \sum_{n\in N_\alpha}\sum_{k_2=1}^{\dim_{\pi_\alpha}}\sum_{k_1=1}^{|\mathcal{C}_\alpha|} \delta_{g,hc_{i_1}\bar h}\delta_{c_{k_1},hc_{i_1}\bar h}\Gamma^{k_2i_2}_{\pi_\alpha}(\bar q_{hc_{i_1}\bar h}hq_{i_1})\\
&\quad\bar\Gamma^{k_2j_2}_{\pi_\alpha}(n)F_\xi^{\bar c_{k_1},q_{h\bar c_{i_1}\bar h}n\bar q_{j_1}}\\
&\overset{(2.4.33)}{=\!=} \sum_{K\in I_\alpha}\Gamma^{KI}_\alpha(\delta_g\otimes h)(\mathbf{F}^\alpha_\xi)^{KJ}\\
&\overset{(2.4.32)}{=\!=} \sum_{K}\Gamma^{IK}_{\alpha^*}(S(\delta\otimes h))(\mathbf{F}^\alpha_\xi)^{KJ}\\
&= \Gamma_{\alpha^*}(S(\delta_g\otimes h))(\mathbb{F}^\alpha_\xi)
\end{aligned}
$$

□

In components, Equation (4.1.11) takes the form

$$\sum_{K\in I_\alpha}\Gamma^{KI}_\alpha(\delta_g\otimes h)(\mathbf{F}^\alpha_\xi)^{KJ}, \tag{4.1.12}$$

as seen in the third from last step of the proof of Proposition 4.1.3. We will denote the space of α-multiplets by Mult_α and write RMult_α for the set of multiplets of the form $\mathbf{F}^\alpha_\xi$. We also write

$$\mathrm{Mult} = \bigcup_{\alpha\in\mathrm{Reps}(D(G))} \mathrm{Mult}_\alpha \tag{4.1.13}$$

$$\mathrm{RMult} = \bigcup_{\alpha\in\mathrm{Reps}(D(G))} \mathrm{RMult}_\alpha \tag{4.1.14}$$

to denote the set of all multiplets, respectively ribbon multiplets.

Lemma 4.1.4. *If $\mathbf{F} \in \mathfrak{A} \otimes \mathrm{Hom}(V, W)$ is an α-multiplet for $\alpha \in \widehat{D(G)}$ at site s, then we have*

$$\gamma^s_a(\mathbf{F}^*) = \mathbf{F}^*\Gamma_{\alpha^*}(a) \tag{4.1.15}$$

and

$$\gamma^s_a(\mathbf{F}^t) = \mathbf{F}^t\Gamma_\alpha(a) \tag{4.1.16}$$

for the adjoint $\mathbf{F}^*$ and the transpose $\mathbf{F}^t$ of $\mathbf{F}$.

Proof. Note that $(\mathbf{F}^*)^{IJ} = (\mathbf{F}^{JI})^*$. Using that $\mathbf{F}$ is an α-multiplet, we obtain

$$\begin{aligned}(\gamma_a(\mathbf{F}^*))^{IJ} \overset{(4.1.5)}{=\!=} \gamma_{S(a)^*}(\mathbf{F}^{JI})^* &\overset{(4.1.12)}{=\!=} \sum_{K\in I_\alpha} \left(\Gamma_{\alpha^*}(S^2(a)^*)^{JK}\mathbf{F}^{KI}\right)^* \\ =(\Gamma_{\alpha^*}(a^*)\mathbf{F})^* &\overset{(2.4.31)}{=\!=} \mathbf{F}^*\Gamma_{\alpha^*}(a)\end{aligned}$$

and

$$\begin{aligned}\gamma^s_a(\mathbf{F}^t)^{IJ} =& \gamma^s_a(\mathbf{F}^{JI}) = \Gamma^{JK}_{\alpha^*}(S(a))\mathbf{F}^{KI} \\ =& \Gamma^{KJ}_\alpha(a)(\mathbf{F}^t)^{IK} = \left(\mathbf{F}^t\Gamma_\alpha(a)\right)^{IJ}\end{aligned}$$

□

Remark 4.1.5. Equation (4.1.16) is the original version of the α-multiplet property given in [SV93, Eq. (4.1)] in the setting of one-dimensional quantum spin chains.

Given an irreducible representation $\alpha \in \widehat{D(G)}$, we can always construct a ribbon multiplet $\mathbf{F}^\alpha_\xi$ at site s for some finite ribbon ξ with $s = \partial_0\xi$. As we have seen in Proposition 3.4.9, if ξ is an infinite ribbon, the amplimorphism

$$\chi_\xi : \mathfrak{A} \to \mathfrak{A} \otimes M_{n_\alpha}(\mathbb{C}), X \mapsto \mathbf{F}^\alpha_\xi X(\mathbf{F}^\alpha_\xi)^*$$

understood as limits of finite ribbons $\xi_n \in \xi$ converges for each $X \in \mathfrak{A}$ even though $\mathbf{F}_\xi^\alpha$ does not. This allows us to choose for each irreducible representation $\alpha \in \widehat{D(G)}$ an amplimorphism χ_{ξ^α} by choosing a sequence of ribbons ξ_n^α converging to some infinite ξ_α. Note that each of the individual $\chi_{\xi_n^\alpha}$ are already amplimorphisms and each of the $\mathbf{F}_{\xi_n^\alpha}^\alpha$ is already a non-degenerate α-multiplet at site $s = \partial_0 \xi_n^\alpha$. There is a connection between the category $\mathrm{rep}(D(G))$ and amplimorphisms, and we will see that the ribbon multiplets play a special role. But first, we need a few more definitions.

Definition 4.1.6 (Morphisms of amplimorphism)**.** Let $\chi_1 : \mathfrak{A} \to \mathrm{Hom}(V_1, W_1)$ and $\chi_2 : \mathfrak{A} \to \mathrm{Hom}(V_2, W_2)$ be two amplimorphisms. A **morphism of amplimorphism** from χ_1 to χ_2 is an element $U \in \mathfrak{A} \otimes \mathrm{Hom}(W_1, W_2)$ such that

$$U\chi_1(A) = \chi_2(A)U \tag{4.1.17}$$

$$U\chi_1(1_\mathfrak{A}) = U = \chi_2(1_\mathfrak{A})U \tag{4.1.18}$$

for all $A \in \mathfrak{A}$ and we denote the space of morphisms by $\mathrm{Hom}(\chi_1, \chi_2)$. We call $U \in \mathrm{Hom}(\chi_1, \chi_2)$ a **unitary equivalence** if U is a partial isometry with $UU^* = \chi_2(1_\mathfrak{A})$ and $U^*U = \chi_1(1_\mathfrak{A})$ and write $\chi_1 \sim \chi_2$ if a unitary equivalence exists.

Amplimorphisms form together with their morphisms a category, which we shall denote by Amp. Given two morphisms $U \in \mathrm{Hom}(\chi_1, \chi_2)$ and $V \in \mathrm{Hom}(\chi_2, \chi_3)$, their composition is given via $VU \in \mathrm{Hom}(\chi_1, \chi_2)$ and the identity morphism on $\mathrm{Hom}(\chi, \chi)$ is given by $\chi(1_\mathfrak{A})$ for any amplimorphism χ.

The following proposition draws an important connection between morphisms of amplimorphisms and the underlying linear spaces.

Proposition 4.1.7. *Let $\mathbf{F}_i \in \mathfrak{A} \otimes \mathrm{Hom}(V_i, W_i)$, $i = 1, 2$ satisfy the F-algebra relation, i.e.*

$$\mathbf{F}_i^* \mathbf{F}_i = 1_\mathfrak{A} \otimes \mathrm{id}$$

and let $\chi_{\mathbf{F}_1}, \chi_{\mathbf{F}_2}$ be the corresponding amplimorphisms. If $T \in \mathrm{Hom}(V_1, V_2)$ is viewed as an element in $1_\mathfrak{A} \otimes \mathrm{Hom}(V_1, V_2)$, then the matrix

$$U_T = \mathbf{F}_2 T \mathbf{F}_1^* \in \mathfrak{A} \otimes \mathrm{Hom}(W_1, W_2) \tag{4.1.19}$$

is a morphism from $\chi_{\mathbf{F}_1}$ to $\chi_{\mathbf{F}_2}$. On the other hand, given any matrix $U \in \mathfrak{A} \otimes \mathrm{Hom}(W_1, W_2)$, we set

$$T_U = \mathbf{F}_2^* U \mathbf{F}_1 \in \mathfrak{A} \otimes \mathrm{Hom}(V_1, V_2). \tag{4.1.20}$$

Then the following hold:

(1) $U \in \mathrm{Hom}(\chi_{\mathbf{F}_1}, \chi_{\mathbf{F}_2})$ *if and only if* $T_U \in 1_{\mathfrak{A}} \otimes \mathrm{Hom}(V_1, V_2)$ *and* $U\mathbf{F}_1\mathbf{F}_1^* = U = \mathbf{F}_2\mathbf{F}_2^*U$.

(2) $U \in \mathrm{Hom}(\chi_{\mathbf{F}_1}, \chi_{\mathbf{F}_2})$ *is an equivalence if and only if* $T_U \in 1_{\mathfrak{A}} \otimes \mathrm{Hom}(V_1, V_2)$ *is unitary and* $U\mathbf{F}_1\mathbf{F}_1^* = U = \mathbf{F}_2\mathbf{F}_2^*U$.

Proof. If $T \in \mathrm{Hom}(V_1, V_2) \cong 1_{\mathfrak{A}} \otimes \mathrm{Hom}(V_1, V_2)$, then T commutes with $A \otimes \mathrm{id}_{V_1}$ for each $A \in \mathfrak{A}$. This gives

$$\begin{aligned} U_T\chi_{\mathbf{F}_1}(A) =& \mathbf{F}_2 T \mathbf{F}_1^* \mathbf{F}_1 (A \otimes \mathrm{id}_{V_1})\mathbf{F}_1^* = \mathbf{F}_2 T (A \otimes \mathrm{id}_{V_1})\mathbf{F}_1^* = \mathbf{F}_2 (A \otimes \mathrm{id}_{V_1}) T \mathbf{F}_1^* \\ =& \mathbf{F}_2 (A \otimes \mathrm{id}_{V_1})\mathbf{F}_2^* \mathbf{F}_2 T \mathbf{F}_1^* = \chi_{\mathbf{F}_2}(A) U_T, \end{aligned}$$

and we also have

$$\begin{aligned} U_T \mathbf{F}_1\mathbf{F}_1^* &= \mathbf{F}_2 T \mathbf{F}_1^* \mathbf{F}_1 \mathbf{F}_1^* = \mathbf{F}_2 T \mathbf{F}_1^* = U_T \\ \mathbf{F}_2\mathbf{F}_2^* U_T &= \mathbf{F}_2\mathbf{F}_2^*\mathbf{F}_2 T \mathbf{F}_1^* = \mathbf{F}_2 T \mathbf{F}_1^* = U_T. \end{aligned}$$

Next, let $U \in \mathrm{Hom}(\chi_{\mathbf{F}_1}, \chi_{\mathbf{F}_2})$ be given and $T_U = \mathbf{F}_2^* U \mathbf{F}_1$. Then

$$\begin{aligned} T_U(A \otimes \mathrm{id}) =& \mathbf{F}_2^* U \mathbf{F}_1 (A \otimes \mathrm{id}) = \mathbf{F}_2^* U \mathbf{F}_1 (A \otimes \mathrm{id}) \mathbf{F}_1^* \mathbf{F}_1 = \mathbf{F}_2^* \mathbf{F}_2 (A \otimes \mathrm{id}) \mathbf{F}_2^* U \mathbf{F}_1 \\ =& (A \otimes \mathrm{id}) T_U \end{aligned}$$

giving $T_U^{IJ} \in \mathfrak{A}'$ for each matrix entry of T_U for some chosen basis. By Proposition 2.5.20, we must have $T_U^{IJ} \in 1_{\mathfrak{A}} \cdot \mathbb{C}$, hence $T_U \in 1_{\mathfrak{A}} \otimes \mathrm{Hom}(V_1, V_2)$. If U is in addition an equivalence, then $U^*U = \mathbf{F}_1\mathbf{F}_1^*$ and $UU^* = \mathbf{F}_2\mathbf{F}_2^*$ are projections, hence

$$\begin{aligned} T_U T_U^* =& (\mathbf{F}_2^* U \mathbf{F}_1)(\mathbf{F}_2^* U \mathbf{F}_1)^* = \mathbf{F}_2^* U \mathbf{F}_1 \mathbf{F}_1^* U^* \mathbf{F}_2 = \mathbf{F}_2^* U U^* \mathbf{F}_2 = \mathbf{F}_2^* \mathbf{F}_2 \mathbf{F}_2^* \mathbf{F}_2 \\ =& 1_{\mathfrak{A}} \otimes \mathrm{id}, \end{aligned}$$

and $T_U^* T_U = 1_{\mathfrak{A}} \otimes \mathrm{id}$ follows similarly. On the other hand, if $\mathbf{F}_2^* U \mathbf{F}_1 \in 1_{\mathfrak{A}} \otimes \mathrm{Hom}(V_1, V_2)$ and $\mathbf{F}_2\mathbf{F}_2^* U = U = U\mathbf{F}_1\mathbf{F}_1^*$, then

$$U\mathbf{F}_1(A \otimes \mathrm{id})\mathbf{F}_1^* = \mathbf{F}_2\mathbf{F}_2^* U \mathbf{F}_1 (A \otimes \mathrm{id}) \mathbf{F}_1^* = \mathbf{F}_2 (A \otimes \mathrm{id}) \mathbf{F}_2^* U \mathbf{F}_1 \mathbf{F}_1^* = \mathbf{F}_2 (A \otimes \mathrm{id}) \mathbf{F}_2^* U$$

hence, $U \in (\chi_{\mathbf{F}_2}, \chi_{\mathbf{F}_1})$. If $\mathbf{F}_2^* U \mathbf{F}_1$ is in addition unitary, then

$$1_{\mathfrak{A}} \otimes \mathrm{id} = \mathbf{F}_2^* U \mathbf{F}_1 \mathbf{F}_1^* U^* \mathbf{F}_2 = \mathbf{F}_2^* U U^* \mathbf{F}_2$$

multiplying both sides from the left with $\mathbf{F}_2$ and from the right with $\mathbf{F}_2^*$ and using that $\mathbf{F}_2\mathbf{F}_2^* U = U$, respectively $U^*\mathbf{F}_2\mathbf{F}_2^* = U^*$ gives

$$\mathbf{F}_2\mathbf{F}_2^* = UU^*.$$

Similarly, the identity

$$1_{\mathfrak{A}} \otimes \mathrm{id} = \mathbf{F}_1^* U^* \mathbf{F}_2 \mathbf{F}_2^* U \mathbf{F}_1$$

gives $U^*U = \mathbf{F}_1\mathbf{F}_1^*$. This concludes the proof. □

In [SV93], 1-dimensional spin models are considered, and the field algebra operators $\mathbf{F}_\alpha$ considered there satisfy the relation

$$\gamma_a^s(\mathbf{F}_\alpha) = \mathbf{F}_\alpha \Gamma_\alpha(a), \tag{4.1.21}$$

which is different from the transformation rule that applies in our case. While conceptually, this should not pose a problem, the convenient placement of the Γ matrix on the right of $\mathbf{F}_\alpha$ allows the following nice additional characterization of intertwiners:

Proposition 4.1.8. *Let $\mathbf{F}_i \in \mathfrak{A} \otimes \mathrm{Hom}(V_i, W_i)$ satisfy Equation (4.1.21) for irreducible representations $\alpha_i \in \widehat{D(G)}$. If $T \in (\alpha_2, \alpha_1)$ is an intertwiner from α_2 to α_1, then the operator*

$$U_T := \mathbf{F}_2 T \mathbf{F}_1^* \in (\chi_1, \chi_2) \tag{4.1.22}$$

is an ε-multiplet. Conversely, if $U \in (\chi_1, \chi_2)$ is an ε multiplet, then the operator

$$T_U = \mathbf{F}_2^* U \mathbf{F}_1 \tag{4.1.23}$$

is an intertwiner from α_1 to α_2.

Proof. Much like in Lemma 4.1.4, one can show that

$$\gamma_a^s(\mathbf{F}_i^*) = \mathbf{F}_i \Gamma_\alpha(S(a)).$$

Let $U \in \mathrm{Hom}(\chi_{\mathbf{F}_1}, \chi_{\mathbf{F}_2})$ be such that $\gamma_a^s(U) = \varepsilon(a)U$. Since U is in particular a morphism of amplimorphisms, we still have $T_U \in 1_{\mathfrak{A}} \otimes \mathrm{Hom}(V_1, V_2)$, implying $\gamma_a(T_U) = \varepsilon(a) T_U$. This gives

$$\begin{aligned}
\varepsilon(a) T_U =& \gamma_a(\mathbf{F}_2^* U \mathbf{F}_1) \\
=& \sum_{(a)} \gamma^s_{a^{(1)}}(\mathbf{F}_2^*) \gamma^s_{(a^{(2)})}(U) \gamma^s_{a^{(3)}}(\mathbf{F}_1) \\
=& \sum_{(a)} \varepsilon(a^{(2)}) \gamma_{a^{(1)}}(\mathbf{F}_2^*) U \gamma_{a^{(3)}}(\mathbf{F}_1) \\
=& \sum_{(a)} \varepsilon(a^{(2)}) \Gamma_{\alpha_1}(S(a^{(1)})) \mathbf{F}_2^* U \mathbf{F}_1 \Gamma_{\alpha_2}(a^{(3)}) \\
=& \sum_{(a)} \Gamma_{\alpha_1}(S(a^{(1)})) T_U \Gamma_{\alpha_2}(a^{(2)})
\end{aligned} \tag{4.1.24}$$

for all $a \in D(G)$. But then

$$\begin{aligned} T_U \Gamma_{\alpha_2}(a) &= \sum_{(a)} \Gamma_{\alpha_1}(a^{(1)}) \Gamma_{\alpha_1}(S(a^{(2)})) T_U \gamma_{\alpha_2}(a^{(3)}) \\ &\overset{(4.1.24)}{=} \sum_{(a)} \Gamma_{\alpha_1}(a^{(1)}) \varepsilon(a^{(2)}) T_U = \Gamma_{\alpha_1}(a) T_U, \end{aligned}$$

hence, T_U is an intertwiner from α_2 to α_1. For the other direction, we evaluate

$$\begin{aligned} \gamma_a(U_T) =& \gamma_a(\mathbf{F}_2 \mathbf{F}_2^* U_T \mathbf{F}_1 \mathbf{F}_1^*) \\ =& \sum_{(a)} \gamma_{a^{(1)}}(\mathbf{F}_2) \varepsilon(a^{(2)}) \mathbf{F}_2^* U_T \mathbf{F}_1 \gamma_{a^{(3)}}(\mathbf{F}_1^*) \\ =& \sum_{(a)} \mathbf{F}_2 \Gamma_{\alpha_1}(a^{(1)}) T \Gamma_{\alpha_2}(S(a^{(2)})) \mathbf{F}_1^* \\ =& \sum_{(a)} \mathbf{F}_2 \Gamma_{\alpha_1}(a^{(1)}) \Gamma_{\alpha_1}(S(a^{(2)})) T \mathbf{F}_1^* \\ =& \varepsilon(a) \mathbf{F}_2 T \mathbf{F}_1^* = \varepsilon(a) U_T. \end{aligned}$$

□

In view of Lemma 4.1.4 one approach could be to substitute the matrices $\mathbf{F}_i$ in Proposition 4.1.8 with the transposes of α-multiplets. Intertwiners of representation T would then give rise to ε-multiplets $\mathbf{F}_2^t T (\mathbf{F}_1^t)^*$ for α-multiplets $\mathbf{F}_1, \mathbf{F}_2$ understood as in Equation (4.1.11). Note also that $U_T \mathbf{F}_1 = \mathbf{F}_2 T \mathbf{F}_1^* \mathbf{F}_1 = \mathbf{F}_2 T$ already gives

$$\gamma_a^s(U_T \mathbf{F}_1) = \gamma_a^s(\mathbf{F}_2 T) = \Gamma_{\alpha^*}(S(a)) \mathbf{F}_2 T, \tag{4.1.25}$$

showing that $U_T \mathbf{F}_1$ is an α-multiplet regardless of whether $\mathbf{F}_2$ had any multiplet structure and regardless of the choice of the linear map T. This suggests that an intertwiner of amplimorphisms given by multiplets should do more than just satisfying Equation (4.1.25). We propose the following definition:

Definition 4.1.9 (Intertwiners)**.** Let $\mathbf{F}_1, \mathbf{F}_2$ be an α_1-, respectively α_2-multiplet and $\chi_{\mathbf{F}_1}, \chi_{\mathbf{F}_2}$ the corresponding amplimorphism. Then we call a morphism $U \in \mathrm{Hom}(\chi_{\mathbf{F}_1}, \chi_{\mathbf{F}_2})$ an **intertwiner from $\chi_{\mathbf{F}_1}$ to $\chi_{\mathbf{F}_2}$** if

$$(\mathbf{F}_2)^t \mathbf{F}_2^* U \mathbf{F}_1 (\mathbf{F}_1^t)^* \tag{4.1.26}$$

is an ε-multiplet.

If furthermore T is a linear map between the modules $\mathcal{V}^{\alpha_1}, \mathcal{V}^{\alpha_2}$ associated to representations $\alpha_1, \alpha_2 \in \text{rep}(D(G))$, then U_T defined in Equation (4.1.22) becomes a morphism of amplimorphisms in the sense of Definition 4.1.6 and is unitary if and only if T is unitary. If T is in addition an intertwiner, then U_T becomes an intertwiner in the sense of Definition 4.1.9:

$$\begin{aligned}\gamma_a^s(\mathbf{F}_2^t \mathbf{F}_2^* U_T \mathbf{F}_1 (\mathbf{F}_1^t)^*)(\mathbf{F}_1^t)^* =& \sum_{(a)} \mathbf{F}_2^t \Gamma_{\alpha_2}(a^{(1)}) \gamma^s_{a^{(2)}}(T) \Gamma_{\alpha_1}(S(a^{(3)})) \\ =& \sum_{(a)} \mathbf{F}_2^t \Gamma_{\alpha_2}(a^{(1)}) \varepsilon(a^{(2)}) T \Gamma_{\alpha_1}(S(a^{(3)}))(\mathbf{F}_1^t)^* \\ =& \varepsilon(a) \mathbf{F}_2^t T (\mathbf{F}_1^t)^*.\end{aligned}$$

On the other hand, $T_U = \mathbf{F}_2^* U \mathbf{F}_1$ is always a c-number matrix if U is a morphism of amplimorphisms, and hence an ε-multiplet. If $\mathbf{F}_1$ and $\mathbf{F}_2$ are in addition non-degenerate α_1- respectively α_2-multiplets, then

$$\begin{aligned}\varepsilon(a)\mathbf{F}_2^t \mathbf{F}_2^* U \mathbf{F}_1 (\mathbf{F}_1^t)^* =& \sum_{(a)} \gamma^s_{a^{(1)}}(\mathbf{F}_2^t) \gamma^s_{a^{(2)}}(T_U) \gamma^s_{a^{(3)}}((\mathbf{F}_1^t)^*) \\ =& \sum_{(a)} \mathbf{F}_2^t \Gamma_\alpha(a^{(1)}) \varepsilon(a^{(2)}) T_U \Gamma_\alpha(S(a^{(3)}))(\mathbf{F}_1^t)^*.\end{aligned}$$

Multiplying both sides from the left with $(F_2^t)^*$ and from the right with F_1^t gives

$$\varepsilon(a) T_U = \sum_{(a)} \Gamma_\alpha(a^{(1)}) T_U \Gamma_\alpha(S(a^{(2)}))$$

and similar calculations as in the proof of Proposition 4.1.7 show that T_U is an intertwiner from α_1 to α_2. This establishes a functor between the category R-Amp and rep$(D(G))$ as follows: For each site s, we choose a sequence of ribbons ξ_n with fixed initial site $s = \partial_0 \xi_n$ and extending to a semi-infinite ribbon ξ as explained in Proposition 3.4.9. Then for every representation $\alpha \in \widehat{D(G)}$, we obtain a non-degenerate α-multiplet $\mathbf{F}_\xi^\alpha$, giving rise to an amplimorphism $\chi_{\mathbf{F}_\xi^\alpha}$. By taking direct sums, this extends to arbitrary representations $\alpha \in \text{rep}(D(G))$.

It is straightforward to see that this mapping is indeed functorial. In [SV93], an equivalence of fusion categories between the category rep$(D(G))$ and RMult was established, and we are fairly certain that the functor just introduced for the non-abelian quantum double will exhibit a similar behaviour.

4.2 Possible Generalizations and Open Questions

4.2.1 The Complete Set of Infinite Volume Ground States

In Theorem 3.5.4 we have provided a set of infinite volume ground states for the non-abelian quantum double, similar to the ones in [CNN16]. It is however still an open question whether there are other ground stats of the model. In [CNN16] it was shown that any arbitrary ground state minimizes some Hamiltonian with a boundary condition similar to the operator H_n^α we inspected. One could then analyse infinite volume ground states with the use of finite volume ground states. Translating the arguments to the non-abelian setting may be possible, but the calculations become much more involved in our setting. It would be interesting to see if the family of ground states obtained in Theorem 3.5.4 give indeed a complete set, or whether there are other non-equivalent ground states admitted in our setting.

4.2.2 Generalization to Hopf algebras

Many concepts in this work are independent of the particular structure of the quantum double $D(G)$. The question naturally arises if the discussion of anyons can be based on the quantum double $D(H)$ of a general semisimple Hopf* algebra H. A detailed description can be found in [BMCA13] but we will give a brief overview here.

We discussed in Section 2.4 how the quantum double $D(H)$ can be constructed from a finite-dimensional Hopf algebra H and that $D(G)$ is a special case for the Hopf algebra $\mathbb{C}G$. To generalize the quantum double model to finite dimensional Hopf algebras, we would like to decorate the edges of our lattice with the Hopf algebra H as a vector space at each edge and give the Hamiltonian in terms of quantum double action. First, we would have to establish how the Hilbert space H could be turned into a vector space. If ϕ is the Haar integral of H^* and H a Hopf *-algebra, an inner product can be defined via

$$\langle a, b \rangle = \phi(a^* b) \tag{4.2.1}$$

for $a, b \in H$.

The triangle operators can mostly be straightforwardly generalized: The operators L_τ^h act either via left- respectively right multiplication with the element $h \in H$ if $\mathfrak{e}_\tau$ is aligned, or with the antipode of h if $\mathfrak{e}_\tau$ is not aligned

[CCY21]. For the direct triangle operator T^{φ}_{τ} act on an element $k \in H$ at the edge crossing $\mathfrak{e}_{\tau}$ via

$$k \mapsto \begin{cases} \sum_{(k)} \varphi(k^{(1)})k^{(2)} & \text{If } \tau \text{ is aligned and locally clockwise oriented} \\ \sum_{(k)} \varphi(S(k^{(2)}))k^{(1)} & \text{If } \tau \text{ is not aligned and locally clockwise oriented} \\ \sum_{(k)} \varphi(k^{(2)})k^{(1)} & \text{If } \tau \text{ is aligned and locally counter-clockwise oriented} \\ \sum_{(k)} \varphi(S(k^{(1)}))k^{(2)} & \text{If } \tau \text{ is not aligned and locally counter-clockwise oriented} \end{cases} \tag{4.2.2}$$

Most notably, the direct triangle operator now depend on the local orientation, unless H is cocommutative. From the construction of triangle operators, we can construct star and plaquette operators as before. For the Hamiltonian, the projection into the trivial electric- and magnetic charge operators are given via the action of the Haar integrals h_0 and ϕ of H and H^* respectively. The trivial electric charge operator then becomes $A_s^{h_0}$, and the trivial magnetic charge operator becomes B_s^{ϕ}. A local Hamiltonian can then be defined analogously to the local Hamiltonian for the non-abelian quantum double model:

$$H_{\Lambda} = \sum_{s \in \mathcal{S}} \left(1 - A_s^{h_0}\right) + \sum_{s \in \mathcal{S}} \left(1 - B_s^{h_0}\right). \tag{4.2.3}$$

Finally, the recursive formula for ribbon operators take the form [CCY21]

$$F^{h \otimes f} := \sum_{(h \otimes f)} F_{\xi_1}^{(h \otimes f)^{(1)}} F_{\xi_2}^{(h \otimes f)^{(2)}}. \tag{4.2.4}$$

Equation (3.4.4) would allow a straightforward generalization to ribbon-multiplets and it would be interesting to study anyon excitations in this setting. We believe that our results and methods in Chapter 3 should generalize to the setting of Hopf algebras.

4.2.3 Generalizations to continuous Groups

In [KM96] a construction of the quantum double for locally compact groups is presented. If G is a Hausdorff locally compact group, we may identify the quantum double $D(G)$ with $C(G \times G)$, the space of continuous, complex valued maps on $G \times G$. Given elements $\varphi, \varphi_1, \varphi_2 \in D(G)$, the structure maps of $D(G)$ are presented in [KM96] as

$$\varphi_1 \cdot \varphi_2(g,h) = \int_G \varphi_1(g,k)\varphi_2(\bar{k}gk, \bar{k}h)dk \tag{4.2.5}$$

$$1_{D(G)}(g,h) = \delta_{e,h} \tag{4.2.6}$$

$$\Delta(\varphi)(g_1,h_1,g_2,h_2) = \varphi(g_1g_2,h_1)\delta_{h_1,h_2} \tag{4.2.7}$$

$$\varepsilon(\varphi)(g,h) = \int_G \varphi(e,h)dh \tag{4.2.8}$$

$$S(\varphi)(g,h) = \varphi(\bar{h}\bar{g}h,\bar{h}) \tag{4.2.9}$$

$$\varphi^*(g,h) = \overline{\varphi(\bar{h}gh,\bar{h})} \tag{4.2.10}$$

Note, that the unit is not an element of $D(G)$ nor is $\Delta(\varphi)$ an element of $D(G)\otimes D(G)$. However, expressions of the form $1_{D(G)}\cdot\varphi$ and $\varepsilon(\varphi_1)\varphi_2$ are still well-defined. The irreducible representations of the quantum double are given as follows: Let $r\in G$ be a fixed element and $\mathcal{C}_r$ the conjugacy class of r and let further N_r be the centralizer subgroup of r and $\pi\in\hat{N}_r$ an irreducible representation of N_r. We define the set $L^2(G,\mathcal{V}^\pi)$ to be the space of all square integrable maps $\phi: G\to\mathcal{V}^\pi$, such that

$$\phi(gn) = \pi(\bar{n})g \tag{4.2.11}$$

for all $n\in N_r$ and almost all $g\in G$. If $\sim$ denotes the equivalence relation

$$\phi_1\sim\phi_2 :\Leftrightarrow \phi_1=\phi_2 \text{ almost everywhere,} \tag{4.2.12}$$

then the space $L^2_\pi(G,\mathcal{V}^\pi) := L^2(G,\mathcal{V}^\pi)\big/\sim$ becomes an irreducible representation of the quantum double under the action

$$(\varphi\triangleright\phi)(g) = \int_G \varphi(gr\bar{g},k)\phi(\bar{k}g)dk \tag{4.2.13}$$

For the lattice model, we may choose the following approach:[2]

- At each edge $\mathfrak{e}$, we attach the Hilbert space $L^2(G)$ of continuous and square integrable complex valued functions on G with inner product defined via

$$\langle f_1, f_2\rangle = \int_G f_1^*(g)f_2(g)dg \tag{4.2.14}$$

[2] I thank Dr. Christiaan van de Ven for his work and the many discussions.

- Given a path p in the lattice, we redefine the map $\beta^{(p)}$ in Equation (3.2.9) by setting

$$\beta^p(f) = \begin{cases} f & \text{if } \mathfrak{e} \in E \\ f \circ S & \text{if } \mathfrak{e} \in \bar{E} \end{cases} \tag{4.2.15}$$

If $p = \{\mathfrak{e}\}$ is a singleton path and

$$\beta^{(p)}(f) = \beta^{(p_1)}(f) \star \beta^{(p_2)}(f) \tag{4.2.16}$$

if $p = (p_1, p_2)$, where

$$(f_1 \star f_2)(g) = \int_G f_1(h\bar{k}) f_2(k) dk \tag{4.2.17}$$

is the convolution of f_1 with f_2.

One can check that Equation (4.2.15) and Equation (4.2.16) coincides with (3.2.8) and Equation (3.2.9) respectively for finite G by identifying $g = \delta_g$ as a continuous map on G under the discrete topology.

Attempts to obtain a quantum double model that generalizes the familiar model has not been successful so far, and remains a work in progress. The biggest challenge in this endeavour has proven to be the construction of the plaquette operator. Morally, one could try to find an analogue by identifying $\delta_g = G = C(G)$ like previously noted, and realizing that the plaquette operator *measures* the domain of δ_g. However, the desired projective nature of a magnetic flux operator seems to be incompatible with a continuous action. It is not even clear, how an action of the quantum double $C(G \times G)$ could be defined on this lattice model, which would already simplify the task significantly. Alternatively, one could explore other notions of the quantum double in the continuous group case. For instance, in [LZ13] a quantum double construction is established for compact groups. In this approach, the resulting quantum double $D(G)$ becomes a unital Hopf*-algebra.

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