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# Dynamic modelling for the family of 5-axis CNC milling machines with application to feed-rate optimization

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ABSTRACT

The 5-axis CNC milling machines are complex mechanical systems, and they can be designed with hundreds of 5 degrees of freedom mechanisms. The kinematic and dynamic modelling of machines is crucial for optimising the machine design and the milling process. However, formulating the dynamics model of 5-axis milling machines has never been proposed so far. Moreover, efforts have been made to develop kinematic models for the machines; however, formulating the generalized kinematic model for all possible 5-axis milling machines in closed form is challenging. In this paper, a novel method of kinematic and dynamic modelling for the entire family of 5-axis milling machines is developed. First, the special features of the machines family are fully studied to formulate effectively the generalized forward and inverse kinematic equations in closed-form. Second, by treating each 5-axis mechanism as a closed mechanism subject to a nonlinear kinematic constraint characterizing the machining process, the dynamics model for the whole family of 5-axis milling machines is successfully formulated, and the inverse dynamic equation is derived in closed-form. Finally, to demonstrate the importance and impact of the proposed method, useful applications were conducted; and these were supported by both numerical simulations and actual cutting experiments. Compared with previous works, our study includes new generalized equations of kinematics and dynamics for the family of 5-axis milling machines, and useful applications of the proposed equations for optimization of the 5-axis machining.

## 1. Introduction

The high-speed multi-axis CNC machining, especially 5-axis CNC milling or 5-axis milling, plays a leading role in the modern manufacturing industries [1–3]. The 5-axis CNC milling machines or 5-axis milling machines are complex mechanical systems, and they can be designed with hundreds of 5 degrees of freedom mechanisms [4,5]. Each 5-axis milling machine has a fixed base, 5 rigid links, 3 prismatic joints (3 linear axes) and 2 revolute joints (2 rotary axes). Fig. 1 presents a 3D model of the industrial 5-axis milling machine, Spinner U5.620 [1].

From the robotic views [37], a 5-axis milling machine can be considered as two cooperative robotic manipulators functioning together: (1) one manipulator holds the cutter (tool), and (2) the other clamps the workpiece (part). In other words, each 5-axis mechanism is a 5-joint 3D mechanism that comprises two cooperative kinematic chains: (1) the part kinematic chain, which includes the rigid links and joints (axes) from the fixed base to the part, and (2) the tool kinematic chain, encompassing the rigid links and joints (axes) from the fixed base to the cutter. Fig. 1 shows the 3D model and the kinematic chains of the 5-axis milling machine Spinner U5.620, in which the workpiece kinematic chain includes the two rotary axes, B and C, while the tool kinematic chain comprises the three linear axes, X, Y, and Z.

Understanding the kinematics and dynamics is important in design and operations of 5-axis milling machines, not only for the researchers and academia but also for the industrial practitioners who work in the areas of CAD/CAE and CAM/CAPP [38]. When working on CAM/CAPP activities and designing new 5-axis milling machines, it is required to

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**Nomenclature**

| | |
|---|---|
| $Oxyz$ | Reference frame |
| $O_T x_T y_T z_T$ | Tool coordinate system |
| $O_P x_P y_P z_P$ | Part kinematic system |
| $J_p$ | Primary rotary axis |
| $J_s$ | Secondary rotary axis |
| $\boldsymbol{A}_p$ | Rotation matrix characterizing the rotation of $J_p$ |
| $\boldsymbol{A}_s$ | Rotation matrix characterizing the rotation of $J_s$ |
| $\boldsymbol{q}_T = [x \quad y \quad z]^T$ | Vector of three linear axis variables |
| $\boldsymbol{q}_R = [s \quad p]^T$ | Vector of two rotary axis variables |
| $\boldsymbol{q} = [\boldsymbol{q}_T \quad \boldsymbol{q}_R]^T$ | Vector of all axis variables |
| $\boldsymbol{p}_T = [x_T \quad y_T \quad z_T]^T$ | Tooltip point |
| $\boldsymbol{p}_R = [\alpha \quad \beta]^T$ | Vector of two Euler angles of the tool axis |
| $\boldsymbol{p} = [\boldsymbol{p}_T \quad \boldsymbol{p}_R]^T$ | Vector of the tool pose |
| $\Omega, \Gamma, \Theta, \Lambda$ | Rotation matrices characterizing the rotation of the rotary axis couples |
| $\boldsymbol{w}$ | Vector from $O_P$ to a pivot point on the machine table |
| $\boldsymbol{t}$ | Vector from $O_T$ to a pivot point on the machine spindle head |
| $\boldsymbol{J}$ | Jacobian matrix (the relation between joint velocities and tool velocities) |
| $\boldsymbol{F}_R = [F_t \quad F_b \quad F_n]^T$ | Vector of resistance force |
| $\boldsymbol{\tau}$ | Vector of applied torques/forces |
| $L_i$ | Link $i$ |
| $m_i$ | Mass of a link $L_i$ |
| $\boldsymbol{I}_i$ | Inertia matrix of a link $L_i$ |
| $\boldsymbol{J}_{Ti}$ | Translational Jacobian matrix corresponding to a link $L_i$ |
| $\boldsymbol{J}_{Ri}$ | Rotational Jacobian matrix corresponding to a link $L_i$ |
| $\boldsymbol{r}_{Ci}$ | Center of mass of a link $L_i$ |
| $\boldsymbol{\omega}_i$ | Angular velocity of a link $L_i$ |
| $\boldsymbol{A}_i$ | Rotation matrix of a link $L_i$ |
| $m_w$ | Mass of the workpiece |
| $\boldsymbol{M}$ | Mass matrix |
| $\boldsymbol{C}$ | Matrix of the centrifugal and the Coriolis forces |
| $\boldsymbol{G}$ | Matrix of the gravity forces |

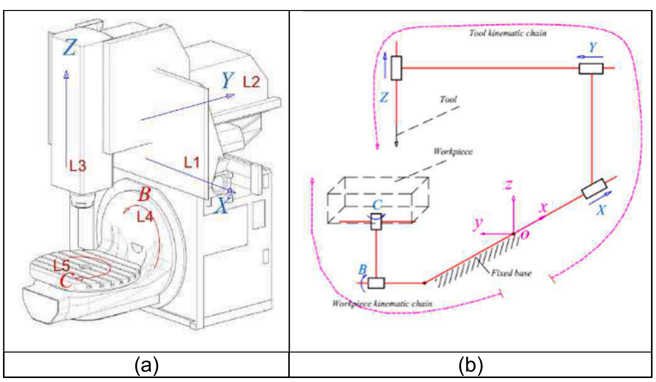


**Fig. 1.** A 3D model (a) and a kinematic chain (b) of a typical 5-axis CNC machine.

consider the factors that affect the machining process and the kinematic and dynamic performance of 5-axis milling machines, including the velocity, the acceleration, the mass of workpiece, the cutting forces, and the resistance forces imposing along the cutting trajectory, as well as the inertia, centrifugal, Coriolis, gravity and friction forces; and these factors can be determined and analyzed based on the dynamic equations of 5-axis milling machines. Furthermore, since the 5-axis milling machine models can be designed with hundreds of spatial 5 DoF mechanisms, there have been emerging interests in generalization of engineering computations and in development of algorithms in relevant CAD/CAE & CAM/CAPP packages. Therefore, the demand for generalization of the kinematic and dynamic modelling for the whole family of all 5-axis mechanisms is increasing.

In recent decades, studies have focused on the kinematic modeling of 5-axis milling machines and related systems for specific objectives and applications, including (1) machine configuration and kinematic performance analysis [4,5]; (2) toolpath planning optimization, such as toolpath interpolation [6–8], feed-rate planning [9–13], singular toolpath computation [14–16], kinematic error analysis and compensation [17–21], and (3) postprocessor development [22–29].

Additionally, there have been studies that focused on some specific dynamic effects of a machine-workpiece system on the milling performance and efficiency [30–33]. The influence of high-frequency excitation of a cutting tool during end milling of workpieces was studied in [30]. The dynamic characteristics of the cutter while milling a thin-walled workpiece were investigated in [31,32]. The vibration control of the milling machine was focused in [33]. The milling parameters optimization, the chatter stability, and other aspects of the machine dynamics have been also investigated in [34–38].

To analyze the kinematic performance and 5-axis configurations, the kinematic modelling of different 5-axis milling machines was developed in [4,5]. To optimize the feed-rate planning, the kinematic modeling of some individual 5-axis milling machines was investigated [6–13,39], in which the 5-axis kinematic formulas were used to derive the kinematic constraints for the optimization model. Besides, to evaluate the drive jerk limits of the 5-axis milling machines, the authors in [9–13] derived the third-order derivatives of the axis variables when formulating the kinematic models. In addition, the kinematic modelling of the 5-axis milling machines was investigated for the workpiece setup optimization [40] and the systematic errors compensation [42,43].

Particularly, the effect of the inertia, centrifugal, Coriolis, friction, and gravity forces of the machine-workpiece system on the feed-rate scheduling was investigate in [13]. However, the dynamic equation and the dynamic constraints of the feed-rate scheduling model have not been derived; instead, the dynamic equation of a Kuka robot arm was employed. It is noted that the dynamic modelling of a 5-axis milling machine is different from that of a single robot arm since each 5-axis mechanism is composed of two cooperative robotic manipulators working collaboratively in a single mechanical system.

To solve the kinematic singularity problem for the 5-axis machining optimization, the kinematic equations for individual machines with the nutating table were established in [14] and [16], where multiplications of homogeneous transformation matrices that characterize the kinematic relationships between consecutive joints of the kinematic chains were utilized. The kinematic modelling for the cases in which the machine has non-orthogonal rotary axes was also investigated [24–28]. In [24], the kinematic equations were established for a machine type, of which the axial line of the rotary axis B is inclined at 45°. In [25], the kinematic equations were derived for a machine, of which the rotary axes are arranged in the tool kinematic chain. The inverse kinematic solution for specific 5-axis milling machine types was investigated in [29] as well.

There have also been efforts to work on the generalization of kinematic modelling of 5-axis milling machines [4,15,18] and [22]. In [22], a generalized kinematic modelling of 5-axis milling machines was introduced for designing a generic 5-axis postprocessor; however, the proposed 7-axis mechanism includes two additional rotary axes. Based on the Screw theory, the kinematics model of a general 5-axis milling machine was formulated [15] and [18]. Similarly, by using the basic homogeneous transformation matrices and complex matrix transformations, the kinematic equations for different groups of machine configurations were derived in [4]; however, the derived kinematic equations [4,15] and [18] is still composed of the ellipsis marks which represent unlisted implicit components.

It is clearly seen that there has been an increasing demand for the kinematic and dynamic modeling of multi-axis machines to optimize machining processes and design optimal machines. Moreover, there has been an emerging need for the generalization of model formulations in kinematics and dynamics of multi-axis CNC machines [4,15,18] and [22]. However, most of the previous works related to the 5-axis kinematic modelling mainly focused on specific machines or some machine groups. Although there have been attempts at generalizing kinematic modeling, the formulation of the explicit forward kinematic equation and the closed form inverse kinematic solution for the entire family of possible 5-axis milling machines has received limited attention. The use of implicit kinematic models in previous works to explicitly derive the kinematic constraints, the Jacobian of the forward kinematic equation, and other components for the dynamic modeling of 5-axis milling machines in the generalized case is almost impossible.

It has also been revealed that there is a high demand for dynamic modeling of 5-axis milling machines. The mass of the workpiece, as well as centrifugal, Coriolis, inertia, resistance, and gravity forces, significantly affects the performance of the machine-workpiece system [13]. However, most related studies have specifically focused on machine spindle vibration, cutting tool chatter, single-axis dynamics, and cutting force [30–37]. To date, no study has proposed a dynamics model for the entire family of all possible 5-axis milling machines.

The critical issues outlined above motivate the development of a generalized mathematical method for formulating the kinematics and dynamics of all possible 5-axis mechanisms and applying this method to create valuable applications in the field of 5-axis machining.

This paper presents a novel method for kinematic and dynamic modeling of the entire family of 5-axis mechanisms, in which all the feasible configurations of the tool orientation as well as the special topology and kinematic characteristics of 5-axis milling machines are fully investigated. The kinematic modeling of all possible 5-axis mechanisms is generalized with explicit and closed form expressions of the kinematic equations. In particular, the dynamic equations governing the motion of 5-axis milling machines are successfully established. Analysis of relationships among the driving forces/torques, the mass of the workpiece and the feed-rate variations for 5-axis milling machines, and development of the feed-rate planning optimization model with dynamic constraints are also implemented to demonstrate the effectiveness of the proposed modelling method of kinematics and dynamics.

The remaining key sections of the paper are organized as follows. Section 2 presents the description and characteristics of the general 5-axis kinematics model. Section 3 presents the generalization of kinematic modelling for 5-axis milling machines. Section 4 presents the dynamic modelling for 5-axis milling machines. Section 5 presents case studies and the application of the proposed generalized model of kinematics and dynamics for 5-axis milling machines in process planning and optimization of CNC machining parameters, with a focus on feed-rate optimization. Finally, the summary and conclusions are presented in Section 6.

## 2. Description and special characteristics of all 5-axis mechanisms

### 2.1. Description of the general 5-axis kinematics model

As discussed earlier, each spatial 5-axis mechanism of the family is constructed with a fixed base, 5 rigid links, 3 linear axes and 2 rotary axes. The three linear axes are usually labeled as X, Y and Z, which move parallel with the three axes of the reference frame $\boldsymbol{Oxyz}$ fixed on the base of a 5-axis milling machine, respectively. The couple of rotary axes are usually denoted as (A-B), (A-C) and (B-C), where A, B and C represent the rotational angles about the axes $\boldsymbol{Ox}$, $\boldsymbol{Oy}$ and $\boldsymbol{Oz}$ of the reference frame $\boldsymbol{Oxyz}$, correspondingly. It is worth noting that all the links and joints of the 5-axis mechanisms are assumed to be manufactured ideally. Hence, all the geometric errors of the body structures including translation errors, rotational errors and squareness errors are assumed to be neglected in the kinematic and dynamic modelling of the 5-axis mechanisms. To improve the accuracy of machined parts, the calibration procedures and volumetric error compensation techniques presented in [41,42] could be used for individual machines. In addition, to simplify the dynamic modelling and computation, we assume that the friction forces at the joints are neglected in the dynamic equation of the 5-axis CNC machines.

It is worth noting that, different numbers of the axes that can be implemented on the two separated kinematic chains constitute different 5-axis mechanisms. Moreover, different sequences of the rotary axes adding to the linear axes constitute different groups of possible 5-axis mechanisms as well. In this sense, a general kinematic diagram for the family of all possible 5-axis mechanisms can be summarized and described in Fig. 2.

In Fig. 2, $\boldsymbol{Oxyz}$ represents the reference frame which is fixed to the machine base. $\boldsymbol{O_T x_T y_T z_T}$ and $\boldsymbol{O_P x_P y_P z_P}$ are the coordinate systems of the tool and the part, respectively. They are parallel to the reference frame, $\boldsymbol{O_P x_P y_P z_P} \uparrow\uparrow \boldsymbol{O_T x_T y_T z_T} \uparrow\uparrow \boldsymbol{Oxyz}$. The axis $\boldsymbol{O_T z_T}$ coincides with the tool axis.

Combining both the tool and the part kinematic chains through the fixed base yields a unique compound kinematic chain as shown in Fig. 2. Starting from the workpiece and ending at the tool, the combined kinematic chain can be considered as a single serial kinematic chain of a conventional industrial robot arm.

It is important to note that, due to the combination, the sign of the axis (joint) variables of all axes located on the part kinematic chain must be changed for all mathematical formulations of the kinematic and dynamic models presented in the next sections.

Note that for specific industrial 5-axis milling machines, the order of the 5 axes is usually as follows: X, Y, Z, the "primary" rotary axis and the "secondary" rotary axis. This axis sequence is mainly used for G-codes programming in NC control of a 5-axis milling machine. For examples,

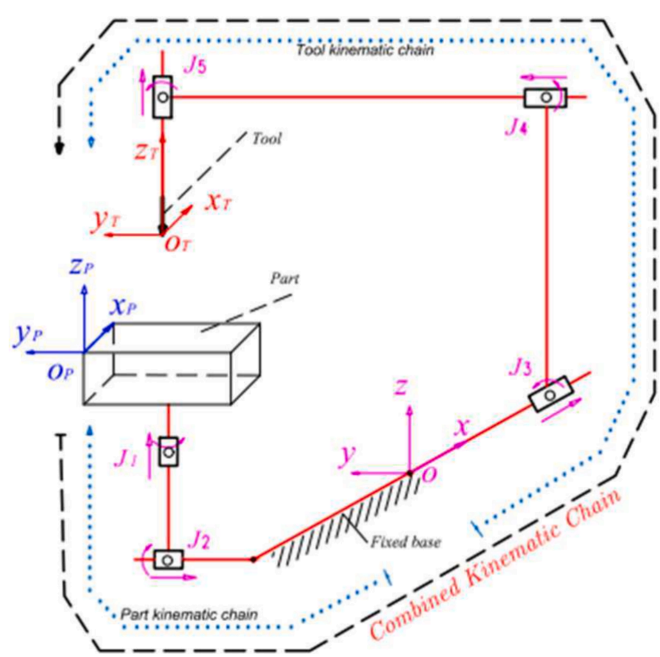


**Fig. 2.** Description of the general 5-axis kinematics model.

the axis sequence of the machine Maho 600e is {X,Y,Z,A,B}, and the "primary" and "secondary" rotary axes are A and B. The axis sequence of the machine DMU 70e is {X,Y,Z,B,C}, and the "primary" and "secondary" rotary axes are B and C.

However, in order to simplify and generalize the formulation of the general kinematics model, the axis sequence, the primary and secondary rotary, in this paper, are defined differently. Let J1, J2, J3, J4 and J5 be the axes in the combined kinematic chain, counting from the part to the tool of the 5-axis mechanisms. Each axis can be a linear axis or rotary axis.

For every 5-axis milling machine, with respect to the order of 5 axes (J1, J2, J3, J4 and J5) in the combined kinematic chain, which rotary axis is closest to the part is defined as the primary rotary axis $\boldsymbol{J_p}$, and the other rotary axis is the secondary rotary axis $\boldsymbol{J_s}$.

This definition is for the mathematical modelling and analysis of the machine kinematics and dynamics that does not change and affect the arrangement of rotary axes of the specific machine configurations in NC control.

Let L and R denote a linear axis and a rotary axis, respectively. Different combinations of L and R constitute 10 possible groups of 5-axis mechanisms as follows: (1) **RRLLL**, (2) **LLLRR**, (3) **RLLLR**, (4) **RLLRL**, (5) **RLRLL**, (6) **LRLLR**, (7) **LLRLR**, (8) **LLRRL**, (9) **LRRLL**, and (10) **LRLRL**.

The kinematic diagrams of the 10 mechanism groups are presented in Appendix A.

Most previous studies on the 5-axis kinematic modeling [5–14,16,17,19–21,23–29] have focused on three main groups of machines and mechanisms: (1) RRLLL (5-axis milling machines with the couple of rotary axes that rotates the machine table), (2) LLLRR (5-axis milling machines with the couple rotary axes that rotates the spindle head, and (3) RLLLR (5-axis milling machines with one rotary axis on the machine table and one rotary axis on the spindle head). Other possible machine types (4)-(10) have been discussed in some investigations, e.g. [4,15,18,22]. However, study on the formulation of the kinematics and dynamics models of these machine types is rare.

### *2.2. Special characteristics of all 5-axis mechanisms*

#### *2.2.1. Characteristics of the cutting tool orientation*

To generalize the modeling of kinematics and dynamics for all possible 5-axis mechanisms, it is necessary to mathematically describe the cutting tool orientation in a generalized manner which is presented in this section as follows.

**Lemma 1**. *Among 12 sets of Euler angles, there exist only 4 sets which can be used to describe the orientation of the cutting tool of all possible 5-axis milling machines in the part coordinate system* $O_Px_Py_Pz_P$.

**Proof.** Let's consider Euler angles that characterize the orientation of the cutting tool with respect to a fixed reference frame Oxyz. In Theory, these Euler angles are rotation angles around Ox, Oy, and Oz axes (x-y-z), and their values depend on the order of the rotation axes. In this manner, there are 12 sets of Euler angles as follows: (1) z-x-z, (2) x-y-x, (3) y-z-y, (4) z-y-z, (5) x-z-x, (6) y-x-y, (7) x-y-z, (8) y-z-x, (9) z-x-y, (10) x-z-y, (11) z-y-x, and (12) y-x-z.

However, during machining or cutting of the workpiece, the cutting tool on a 5-axis milling machine always rotates about its axis which is

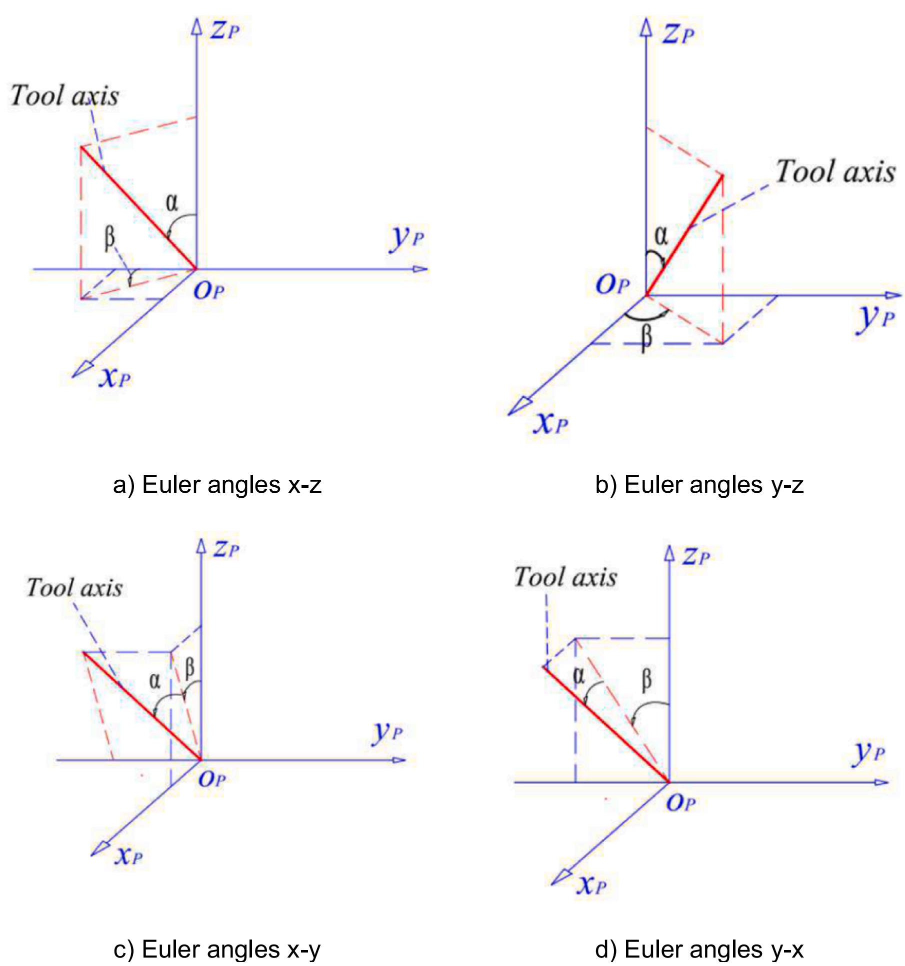


**Fig. 3.** The feasible sets of Euler angles of the tool in the part coordinate system.

normally the z-axis. In other words, the first Euler angle of the tool is always the rotation angle around the z-axis. Thus, among 12 sets of Euler angles, only 4 sets, namely (1) z-x-z, (4) z-y-z, (9) z-x-y, and (11) z-y-x, have the first rotation angle around the z-axis, and they can be used to characterize the tool orientation.

To simplify, we neglect the first component (the z-axis) of the 4 feasible sets of Euler angles, and they can be rewritten as (1) x-z, (4) y-z, (9) x-y, and (11) y-x. Fig. 3 shows the orientation angles α and β of the tool for the 4 corresponding sets of Euler angles.

As a consequence of Lemma 1, for the whole family of possible 5-axis mechanisms, there exist only 4 feasible combinations of the primary-secondary rotary axes ($J_p - J_s$), namely (C-A), (C-B), (B-A) and (A-B). It is impossible to design 5-axis milling machines that have the primary-secondary rotary axes (A-C) and (B-C).

It is clear that to obtain such 4 feasible sets of Euler angles, there are 4 corresponding combinations of the primary-secondary rotary axes $J_p$ and $J_s$ respectively. The rotary axis $J_p$ is synthesized to yield the rotation angle β, and the rotary axis $J_s$ is added to obtain the rotation angle α. In other words, there are only four feasible sets of the primary-secondary rotary axes for every 5-axis mechanism: (C-A), (C-B), (B-A) and (A-B).

### 2.2.2. Orthogonal topology characteristic of 5-axis mechanisms

For the entire family of 5-axis CNC mechanism, the three linear axes of each mechanism are always parallel with the three orthogonal axes *Ox*, *Oy* and *Oz* of the reference frame *Oxyz*, respectively. In other words, the three translational links of each 5-axis CNC mechanism are moved orthogonally to each other in the 3D workspace. Also, by convention, the axial lines of two rotary axes (A, B or C) are parallel with three orthogonal axes *Ox*, *Oy* and *Oz* of *Oxyz*, correspondingly. This orthogonal topology characteristic is a special feature of all possible 5-axis mechanisms.

With respect to the orthogonal topology characteristic, the compound kinematic transformation can be used, instead of using the joint-by-joint kinematic transformations. For example, when modelling the kinematics of the last link of three consecutive translational links of a 5-axis milling machine, the compound transformation matrix is immediately calculated as $\boldsymbol{H}_{xyz} = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{q}_T \\ 0 & 1 \end{bmatrix}$ where $\boldsymbol{q}_T = [x \quad y \quad z]^T$, $\boldsymbol{E}$ is identity matrix, and x, y and z are the linear axis variables. Similarly, the compound transformation matrix for the couple of rotary axes is immediately calculated as $\boldsymbol{H}_{ps} = \begin{bmatrix} \boldsymbol{A}_p\boldsymbol{A}_s & 0 \\ 0 & 1 \end{bmatrix}$ where $\boldsymbol{A}_p$ and $\boldsymbol{A}_s$ are the two rotation matrices that characterize the rotation of the primary and secondary rotary axes $J_p$ and $J_s$ respectively.

For these reasons, all individual joint coordinate systems can be neglected when describing the kinematic diagram of the machine family, and only two coordinate systems $O_Tx_Ty_Tz_T$ and $O_Px_Py_Pz_P$ are sufficient for the description of the combined kinematic diagram as shown in Fig. 2.

In addition, with respect to the orthogonal topology characteristic of the machines family, the rotation of the rotary axes are elementary rotations about *Ox*, *Oy* and *Oz* of *Oxyz*. Hence, the rotation matrices $\boldsymbol{A}_p$ and $\boldsymbol{A}_s$ for the cases that $J_p$ and $J_s$ are the rotary axis A, B or C can be easily determined as follows:

$$\{\boldsymbol{A}_p, \boldsymbol{A}_s\}_{(J_p, J_s = A)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cosA & -\sin A \\ 0 & \sin A & \cos A \end{bmatrix} \tag{1}$$

$$\{\boldsymbol{A}_p, \boldsymbol{A}_s\}_{(J_p, J_s = B)} = \begin{bmatrix} \cos B & 0 & \sin B \\ 0 & 1 & 0 \\ -\sin B & 0 & \cos B \end{bmatrix} \tag{2}$$

$$\{\boldsymbol{A}_p, \boldsymbol{A}_s\}_{(J_p, J_s = C)} = \begin{bmatrix} \cos C & -\sin C & 0 \\ \sin C & \cos C & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

It is important to note that, for the 5-axis milling machines with the axial line of a rotary axis $J_p$ ($J_s$) that is inclined at an angle $\delta$, the matrix $\boldsymbol{A}_p$ ($\boldsymbol{A}_s$) must be additionally multiplied by $\boldsymbol{A}_\delta(\delta)$ and $\boldsymbol{A}_\delta(-\delta)$ on the left and right sides of $\boldsymbol{A}_p$ ($\boldsymbol{A}_s$), respectively, where $\boldsymbol{A}_\delta(\delta)$ represents the rotation of the axial line by the angle $\delta$.

## 3. Generalized closed-form formulation of kinematics model for all possible 5-axis mechanisms

Using the notations and definitions provided, we can derive explicitly the forward kinematic equation in a unique and compact form for the entire family of 5-axis milling machines in a generalized case as follows.

**Lemma 2.** *Let $\boldsymbol{q}_T = [x \quad y \quad z]^T$ be the vector of the three linear axis variables and $\boldsymbol{q}_R = [s \quad p]^T$ be the vector of the secondary and primary rotary axis variables, the forward kinematic equation for all possible 5-axis milling machines is given as follows*:

$$\begin{cases} \boldsymbol{p}_T = \Omega\Gamma\Theta\Lambda\boldsymbol{t} + \boldsymbol{w} + \left[ \Omega\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Omega\Gamma\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \Omega\Gamma\Theta\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]\boldsymbol{q}_T \\ \boldsymbol{p}_R = \boldsymbol{q}_R \end{cases} \tag{4}$$

where $\boldsymbol{p}_T = [x_T \quad y_T \quad z_T]^T$ is the tooltip point; $\boldsymbol{p}_R = [\alpha \quad \beta]^T$ is the Euler angles of the tool; Ω, Γ, Θ and Λ are the rotation matrices for the 10 groups of 5-axis milling machines as given in Table 1; $\boldsymbol{w}$ and $\boldsymbol{t}$ are the constant vectors that point from the point $O_P$ and $O_T$ to the pivot points on the machine table and the machine spindle head, respectively. For the machine types **RRLLL and LLLRR**, the pivot point is the intersection point of the axial lines of the two rotary axes. For the machine type **RLLLR**, the vectors $\boldsymbol{t}$ and $\boldsymbol{w}$ point to the direction which is perpendicular to the centerline of the rotary axes.

**Proof.** To compute the position and orientation of the cutting tool in the part coordinate system $O_Px_Py_Pz_P$, a matrix multiplication of the 5 transformation matrices which represent the kinematics of the 5 axes of a 5-axis mechanism is implemented. Obviously, when adding more rotary axes to the kinematic chain, such that the rotations of the added axes are characterized by identity matrices $\boldsymbol{E}_{3x3}$, the matrix multiplication result does not change. In this way, a general joint sequence (Fig. 4) can be considered, in which, in between every two linear axes L and L, one couple of rotary axes RR is added; two couple of rotary axes are also added in between the workpiece (W) and the linear axis, and the tool (T) and the linear axis, respectively.

It is seen that, for each possible 5-axis milling machine group, all rotation matrices representing the kinematics of the rotary axes equal to the identity matrix $\boldsymbol{E}_{3x3}$, except for the rotation matrices $\boldsymbol{A}_p$ and $\boldsymbol{A}_s$ of the primary rotary axis $J_p$ and the secondary rotary axis $J_s$.

Hence, for the entire family of 5-axis milling machines including 10 machine groups, the tool position $\boldsymbol{p}_T = [x_T \quad y_T \quad z_T]^T$ and rotation matrix $\boldsymbol{A}_T$ characterizing the tool orientation can be computed with the following kinematic transformation:

$$\begin{bmatrix} \boldsymbol{A}_T & \boldsymbol{p}_T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{w} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Omega & 0 \\ 0 & 1 \end{bmatrix}\boldsymbol{H}_x\begin{bmatrix} \Gamma & 0 \\ 0 & 1 \end{bmatrix}\boldsymbol{H}_y\begin{bmatrix} \Theta & 0 \\ 0 & 1 \end{bmatrix}\boldsymbol{H}_z\begin{bmatrix} \Lambda & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{E} & \boldsymbol{t} \\ 0 & 1 \end{bmatrix} \tag{5}$$

In Eq. (5), the transformation matrices $\begin{bmatrix} \Omega & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \Gamma & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \Theta & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} \Lambda & 0 \\ 0 & 1 \end{bmatrix}$ characterize the kinematics of the 4 couples RR in the joint sequence, correspondingly. The rotation matrices Ω, Γ, Θ and Λ for the 10 groups of 5-axis mechanisms are presented in Table 1.

The matrix $\begin{bmatrix} \boldsymbol{E} & \boldsymbol{w} \\ 0 & 1 \end{bmatrix}$ is the transformation matrix which represents

**Table 1**
The rotation matrices $\Omega$, $\Gamma$, $\Theta$ and $\Lambda$ for 10 groups of 5-axis milling machines.

| | | | | |
|---|---|---|---|---|
| **1. RRLLL** $\Omega = A_p A_s\{\Gamma,\Theta,\Lambda\} = \boldsymbol{E}$ | **2. LLLRR** $\Lambda = A_p A_s\{\Omega,\Gamma,\Theta\} = \boldsymbol{E}$ | **3. RLLLR** $\Omega = A_p\Lambda = A_s\{\Gamma,\Theta\} = \boldsymbol{E}$ | **4. RLLRL** $\Omega = A_p\Theta = A_s\{\Gamma,\Lambda\} = \boldsymbol{E}$ | **5. RLRLL** $\Omega = A_p\Gamma = A_s\{\Theta,\Lambda\} = \boldsymbol{E}$ |
| **6. LRLLR** $\Gamma = A_p\Lambda = A_s\{\Omega,\Theta\} = \boldsymbol{E}$ | **7. LLRLR** $\Theta = A_p\Lambda = A_s\{\Omega,\Gamma\} = \boldsymbol{E}$ | **8. LLRRL** $\Theta = A_p A_s\{\Omega,\Gamma,\Lambda\} = \boldsymbol{E}$ | **9. LRRLL** $\Gamma = A_p A_s\{\Omega,\Theta,\Lambda\} = \boldsymbol{E}$ | **10. LRLRL** $\Gamma = A_p\Theta = A_s\{\Omega,\Lambda\} = \boldsymbol{E}$ |

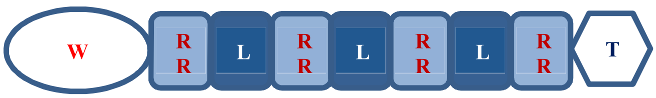


**Fig. 4.** A generic sequence of the rotary axes and linear axes.

the offset from $O_P x_P y_P z_P$ to the coordinate system of the first axis, and $\begin{bmatrix} \boldsymbol{E} & \boldsymbol{t} \\ 0 & 1 \end{bmatrix}$ is the transformation matrix that represents the offset from the coordinate system of the last axis to $O_T x_T y_T z_T$.

$\boldsymbol{H}_x = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{x} \\ 0 & 1 \end{bmatrix}$, $\boldsymbol{H}_x = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{y} \\ 0 & 1 \end{bmatrix}$, and $\boldsymbol{H}_x = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{z} \\ 0 & 1 \end{bmatrix}$, in which $\boldsymbol{x} = [x \quad 0 \quad 0]^T$, $\boldsymbol{y} = [0 \quad y \quad 0]^T$, and $\boldsymbol{z} = [0 \quad 0 \quad z]^T$, that characterize the kinematics of the 3 linear axes of every 5-axis CNC milling machine, correspondingly. Note that the constant offset vectors $\boldsymbol{w}$ and $\boldsymbol{t}$ point from the point $O_P$ or $O_T$ to a pivot point on the machine table or a pivot point on the machine spindle, respectively. For the groups of 5-axis CNC machines with two successive rotary axes implemented on the machine table or the machine spindle (RRLLL and LLLRR), the pivot point is the intersection point of the axial lines of the two rotary axes. For the groups of linkages with one rotary axis on the machine spindle or the machine table, the vectors $\boldsymbol{t}$ and $\boldsymbol{w}$ point to the direction which is perpendicular to the centerline of such rotary axis. For other cases, $\boldsymbol{t} = 0$ and $\boldsymbol{w} = 0$.

Transforming Eq. (5) yields

$$\begin{bmatrix} \boldsymbol{A}_T & \boldsymbol{p}_T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \Omega\Gamma\Theta\Lambda & (\Omega\Gamma\Theta\Lambda\boldsymbol{t} + \Omega\Gamma\Theta\boldsymbol{z} + \Omega\Gamma\boldsymbol{y} + \Omega\boldsymbol{x} + \boldsymbol{w}) \\ 0 & 1 \end{bmatrix} \tag{6}$$

$$\boldsymbol{p}_T = \Omega\Gamma\Theta\Lambda\boldsymbol{t} + \boldsymbol{w} + \Omega\Gamma\Theta\boldsymbol{z} + \Omega\Gamma\boldsymbol{y} + \Omega\boldsymbol{x} \tag{7}$$

$$\boldsymbol{p}_T = \Omega\Gamma\Theta\Lambda\boldsymbol{t} + \boldsymbol{w} + \begin{bmatrix} \Omega\begin{bmatrix}1\\0\\0\end{bmatrix} & \Omega\Gamma\begin{bmatrix}0\\1\\0\end{bmatrix} & \Omega\Gamma\Theta\begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix}\boldsymbol{q}_T \tag{8}$$

As shown in Lemma 1, for all 5-axis mechanisms, the primary rotary axis $J_p$ is designed to yield the rotation angle $\beta$, and the secondary rotary axis $J_s$ is added to obtain the rotation angle $\alpha$.

In other words,

$$p = \beta \tag{9}$$

$$s = \alpha \tag{10}$$

Finally,

$$\boldsymbol{p}_R = \boldsymbol{q}_R \tag{11}$$

Eqs. (8) and (11) complete the Proof.

As a consequence of Lemma 2, the generalized inverse kinematic solution for the whole family of 5-axis milling machines can be easily yielded in closed form as follows:

$$\begin{cases} \boldsymbol{q}_T = \begin{bmatrix} \Omega\begin{bmatrix}1\\0\\0\end{bmatrix} & \Omega\Gamma\begin{bmatrix}0\\1\\0\end{bmatrix} & \Omega\Gamma\Theta\begin{bmatrix}0\\0\\1\end{bmatrix} \end{bmatrix}^{-1}(\boldsymbol{p}_T - \Omega\Gamma\Theta\Lambda\boldsymbol{t} - \boldsymbol{w}) \\ \boldsymbol{q}_R = \boldsymbol{p}_R \end{cases} \tag{12}$$

Remarkably, the forward kinematic equation (4) and the inverse kinematic equation (12) can be rewritten in a compact form with Eqs. (13) & (14), respectively, as follow:

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) \tag{13}$$

$$\boldsymbol{q} = \boldsymbol{g}(\boldsymbol{p}) \tag{14}$$

where $\boldsymbol{p} = [\boldsymbol{p}_T \quad \boldsymbol{p}_R]^T$ and $\boldsymbol{q} = [\boldsymbol{q}_T \quad \boldsymbol{q}_R]^T$.

The soundness of the closed form formulation of the forward and inverse kinematics model was verified with several cases of specific 5-axis milling machines, including three machine types presented in [4] and the orientable-spindle machine and orientable-table 5-axis milling machine proposed in [18]. The verifications of these formulated kinematic equations were done as well for the individual cases of commercially available industrial 5-axis milling machines, including Mikron UCP 710 [5], DMU 50e [14], DMU 70e [16], DMU 340P [26], Spinner U5.620[29].

***Remarks:***

- To enhance the efficiency and effectiveness of 5-axis machining, the use of the kinematics model generalized for all 5-axis mechanisms makes it possible to generalize the formulation of several relevant problems in the field of 5-axis machining, including the generalization of 5-axis postprocessor, the feed-rate optimization, the workpiece setup optimization, regular/singular toolpath interpolation, the kinematic error compensation, etc. To design new 5-axis milling machines, the generalized kinematic equation proposed in this study is useful, especially when generating new machine concepts, and evaluating and comparing the kinematic performance of different machine architectures to select the optimal one.
- The explicit derivation of the forward kinematic equation (4) is essential for the dynamic modelling of the machine family, because without this closed form equation, deriving explicitly the components of the dynamic equation, i. e. the kinematic constraint equation $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$, and the Jacobian matrix of the constraint equation $\boldsymbol{J} = (\partial\boldsymbol{f}/\partial\boldsymbol{q})$, is impossible.
- The closed form inverse kinematic equation (12) is essential as well for the inverse dynamic computation and the dynamic modelling-based applications since the use of this closed form equation makes it feasible the derivation of the joint variable differentiation, i. e. $\dot{\boldsymbol{q}}$ and $\ddot{\boldsymbol{q}}$.
- When using the proposed kinematic formulas for the development of applications, practitioners must strictly follow the below notices: (1) the axes of $O_P x_P y_P z_P$, $O_T x_T y_T z_T$ and $Oxyz$ must point in the same positive direction of 3 linear axes of a 5-axis milling machine, respectively; (2) the sign of all axis variables on the part kinematic chain must be reversed.

In comparison with the previous methods, the proposed method in this study for the 5-axis kinematic modelling is advantageous and comprehensive.

As summarized in Table 2, the 5-axis kinematic formulas presented in [5–12,14,16], and [24–29] were derived just for individual 5-axis milling machines or some groups of the 5-axis mechanisms. These individual methods cannot be applied for the whole family of all possible 5-axis mechanisms, in the generalized case. Noticeable methods presented in [4,15,18], and [22] have attempted to generalization of the 5-axis kinematic modelling. However, the kinematic equations in closed

**Table 2**
Comparison of the kinematic modelling methods for 5-axis milling machines.

| Indicators | Previous methods | | Proposed method |
|---|---|---|---|
| | [5–12,14,16,24–29] | [4,15,18,22] | |
| Generalization for the whole family of 5-axis mechanisms | | * | ** |
| Explicit & closed form FK equation | * | | ** |
| Closed form IK solution | * | | ** |

form for the whole class of all possible 5-axis mechanisms have not been studied; the ellipsis marks representing unlisted implicit components were still existed in the kinematic transformations and equations. The challenge in generalizing these kinematic models is that the 5-axis CNC milling machines can be designed in a large variety of structures; each 5-axis mechanism is a 3D mechanism of 5 degrees of freedom; moreover, a 5-axis mechanism is composed of two cooperating kinematic chains, one kinematic chain carrying the part and one kinematic chain carrying the cutter. Different from the previous methods, the method proposed in this study proves the explicit forward kinematic equation and closed form inverse kinematic equations (4) and (12), which were generalized for the whole family of all possible 5-axis mechanisms.

## 4. Dynamic modelling of 5-axis milling machines

As discussed in Section 1, each 5-axis milling machine is considered as two collaborative robotic manipulators, one manipulator clamps the cutter, and the other manipulator clamps the workpiece. When a 5-axis milling machine is not engaged in machining or cutting the workpiece, the movements of the two manipulators are independent of each other. In this situation, the dynamic modelling of a 5-axis mechanism is equivalent to the dynamic modelling of two independent serial open-loop kinematic chains; and this case is not considered in this study.

In this study, we consider the case that the tool is engaged in machining or cutting the workpiece. Thus, the movement of the two kinematic chains of a 5-axis milling machine is dependent on each other during the process of machining the workpiece. In other words, at any time instant, the tool kinematic chain and the part kinematic chain are closed at a common cutting contact point on the cutting toolpath $\boldsymbol{p}$. For this reason, since $\boldsymbol{q} = \boldsymbol{g}(\boldsymbol{p})$, the values of the axis variables on both the kinematic chains depend on each other during the machining process. Therefore, the motion and the dynamic behavior of the two kinematic chains are subjected to the constraint $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$. In addition, due to the material removal process, there exists a resistance force $\boldsymbol{F_R} = [\boldsymbol{F_t} \quad \boldsymbol{F_b} \quad \boldsymbol{F_n}]^{\boldsymbol{T}}$ imposing on the tooltip. In this sense, the constraint equation of the machine system and the resistance force must be taken into account when formulating the dynamics model of the machines.

Let's consider the tool cutting the part at a cutter contact point as shown in Fig. 5. At this contact point, the resistance force $\boldsymbol{F_R} = [\boldsymbol{F_t} \quad \boldsymbol{F_b} \quad \boldsymbol{F_n}]^{\boldsymbol{T}}$ that acts on the cutting tool of a 5-axis milling machine,

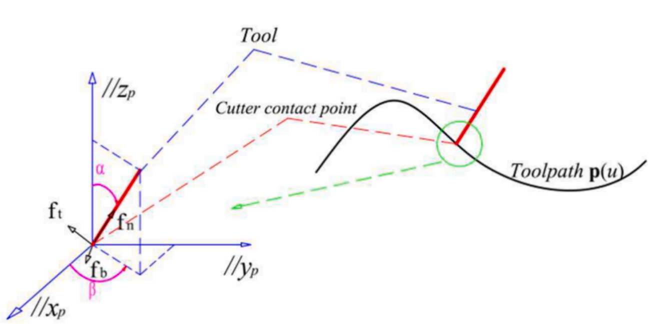


**Fig. 5.** The resistance force $\boldsymbol{F}_R = [F_t \quad F_b \quad F_n]^T$ acting on the cutting tool.

in which the cutting tool moves along a parametric toolpath $\boldsymbol{p}(\boldsymbol{u}) = [\boldsymbol{x_T}(\boldsymbol{u}) \quad \boldsymbol{y_T}(\boldsymbol{u}) \quad \boldsymbol{z_T}(\boldsymbol{u}) \quad \boldsymbol{\alpha}(\boldsymbol{u}) \quad \boldsymbol{\beta}(\boldsymbol{u})]^{\boldsymbol{T}}$, where $\boldsymbol{u} \in [0, 1]$ is a parametric domain. At the cuter contact point, the direction of the resistance force $\boldsymbol{F_R} = [\boldsymbol{F_t} \quad \boldsymbol{F_b} \quad \boldsymbol{F_n}]^{\boldsymbol{T}}$ are opposite to the directions of the driving force of the cutting tool. Where Ft is the force acting along the feed direction. Fn is the force acting along the tool axis direction. Fb is the force with the force vector that is perpendicular to the force vectors Ft and Fn. It is also remarkable that, the driving forces/torques of the 5 drives of a 5-axis milling machine are considered as the applied forces/torques at the 5 joints of a 5-axis mechanism. The action of the applied forces/torques at the 5 joints is to move the tool along the cutting trajectory. Hence, the dynamic modelling of the 5-axis mechanisms take into account only the movement of the 5 joints, not the rotation of the cutting tool about the tool axis. In this context, resistance forces are defined as those opposing the movement of the tool relative to the workpiece. In addition, the static and viscous friction forces are always existed at all 5 joints of a machine. However, to control all the 5 joints simultaneously with high sensitivity and accuracy to ensure high machining accuracy, the joints of industrial 5-axis milling machines are often manufactured with very high precision. In addition, lubricant solutions are usually used to minimize the effect of friction forces at the joints. Hence, to simplify the dynamic modelling and computation, we assume that the friction forces at the joints are neglected in the dynamic equation of the 5-axis milling machines.

Let $\boldsymbol{\tau} = [\tau_1 \quad \tau_2 \quad \tau_3 \quad \tau_4 \quad \tau_5]^T$ be the force vector of 5 applied torques/forces driving the 5 axes of the 5-axis milling machines. Let $\boldsymbol{q} = [q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5]^T = [x \quad y \quad z \quad s \quad p]^T$ be the vector of 5 generalized coordinates (the axis variables) of the general 5-axis CNC mechanism. Let $L_1$, $L_2$ and $L_3$ denote the three translational links that move with the axis $x$, $y$ and $z$, respectively. Let $L_4$ and $L_5$ denote the two rotational links that move with the axis $s$ and $p$, respectively.

For the kinematic constraint $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$, by using the Lagrangian formulation, the governing dynamic equation of the 5-axis milling machine-workpiece systems can be written as follows:

$$\begin{cases} M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + G(\boldsymbol{q}) = \tau + \boldsymbol{J}^T F \\ p = f(\boldsymbol{q}) \end{cases} \tag{15}$$

where $\boldsymbol{J} = (\partial \boldsymbol{f} / \partial \boldsymbol{q})$ is the Jacobian matrix, $\boldsymbol{F} = [F_t \quad F_b \quad F_n \quad 0 \quad 0]^T$ is the vector of the general resistance force. The mass matrix $\boldsymbol{M}$ in Equation (15) is given by the following Eq. (16):

$$\boldsymbol{M} = \sum\nolimits_{i=1}^{5} \left( m_i \boldsymbol{J}_{T_i}^T \boldsymbol{J}_{T_i} + \boldsymbol{J}_{R_i}^T \boldsymbol{I}_i \boldsymbol{J}_{R_i} \right) \tag{16}$$

where $m_i$ and $\boldsymbol{I}_i$ are the mass and the inertia matrix of a link $L_i$; $\boldsymbol{J}_{Ti} = \frac{\partial \boldsymbol{r}_{Ci}}{\partial \boldsymbol{q}}$ and $\boldsymbol{J}_{Ri} = \frac{\partial \omega_i}{\partial \boldsymbol{q}}$ are the translational and rotational Jacobian matrices corresponding to a link $L_i$, respectively; $\boldsymbol{r}_{Ci}$ is the center of mass of the link $L_i$ calculated in the reference frame; $\widetilde{\omega}_i = \dot{\boldsymbol{A}}_i \boldsymbol{A}_i^T$ is the skew matrix of the angular velocity, where $\boldsymbol{A}_i$ is the rotation matrix of the link $L_i$. Note that the mass of the link which clamps the workpiece includes the mass of the workpiece $m_w$. Using the Christoffel notations, the matrix of the centrifugal and the Coriolis forces $\boldsymbol{C}$ is given by

$$\boldsymbol{C}_{ij} = \frac{1}{2} \sum\nolimits_{j=1}^{5} \left( \sum\nolimits_{k=1}^{5} \left( \frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{kj}}{\partial q_i} \right) \dot{q}_j \dot{q}_k \right). \tag{17}$$

Gravity is represented by

$$\boldsymbol{G} = \left[ \frac{\partial \Pi}{\partial \boldsymbol{q}} \right]^T, \tag{18}$$

where the total potential energy of the machine-workpiece system is calculated as follows: $\Pi = -\sum_{i=1}^{5} m_i \boldsymbol{g}^T \boldsymbol{r}_{Ci}$, where $\boldsymbol{g}$ is the gravity vector. It is noted that, in Eq. (15), the nonlinear constraint $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ is the forward kinematic equation (13) which characterizes the mutual dependence of the 5 joint variables (the rotational angles of the rotary axes, and the axis extension of the linear axes) in both kinematic chains while the tool cutting the workpiece along the toolpath $\boldsymbol{p}(u)$.

The component $\boldsymbol{J}^T\boldsymbol{F}$ characterizes the influence of the resistant forces imposing on the mechanical system during the cutter travels along the cutting path $\boldsymbol{p}(u)$. Since the forward kinematic equation (13) was derived in a closed form expression (4), the Jacobian matrix $\boldsymbol{J} = \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}\right)_{5\times5}$, and then the component $\boldsymbol{J}^T\boldsymbol{F}$ can be easily calculated in a generalized case as follows:

$$\boldsymbol{J}^T\boldsymbol{F} = \begin{bmatrix} \Delta^T[F_t \quad F_b \quad F_n]^T \\ 0 \\ 0 \end{bmatrix} \Delta = \left[ \Omega \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Omega\Gamma \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \Omega\Gamma\Theta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

Note that Eq. (15) is the dynamic equation of motion of 5-axis CNC mechanisms that includes both the second order differential equation and the kinematical constraint equation. In comparison with the previous studies, the proposed dynamic equation in this study is advantageous and comprehensive. In [30–33], the dynamic modelling of the machine-workpiece system have been studied. However, these works mainly focused on formulation of the vibration model, in order to analyze some dynamic characteristics of the cutter while milling a thin-walled workpiece, and the influence of the dynamic vibration during end milling of workpieces. In [13], a noticeable method was proposed for the dynamic modelling of a 5-axis CNC milling machine. However, since the dynamic modelling method of the single open serial kinematic chain (industrial robot arm) was employed for the dynamic modelling of 5-axis CNC milling machine, the dynamics model proposed in [13] neglects the kinematic constraint equation $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ and the resistance force $\boldsymbol{J}^T\boldsymbol{F}$ that characterize the milling process in time. It is noted that the dynamic modelling of a 5-axis milling machine is different from that of a single robot arm because each 5-axis mechanism is composed of two cooperative robotic manipulators working collaboratively in a single closed system.

## 5. Applications of the generalized model of kinematics and dynamics of 5-axis milling machines

### 5.1. Introduction of applications (case studies)

As discussed earlier, the modelling of kinematics and dynamics of 5-axis milling machines could be used for development of several useful applications in practice. By using the generalized kinematic and dynamic equations in closed form, the kinematic performance and dynamic performance of all possible 5-axis mechanisms can be evaluated and analyzed effectively, that is very useful for the machine design optimization. In particular, the generalization of the 5-axis kinematic modelling makes it possible the design of a generic real time post-processor for all kinds of 5-axis milling machines. A generic post-processor is able to transfer any common CL data produced by CAM systems to NC programs (G-code files) for any 5-axis milling machines. Also, the generalized kinematic equations in closed form are helpful for the kinematic errors modelling and the kinematic singularity analysis for different 5-axis milling machines. Thus, the kinematic errors and the tool path error near singular points of any 5-axis milling machines can be analyzed and compensated effectively. In addition, the dynamic modelling of the 5-axis milling machines plays a very important role in investigating the dynamic factors that affect the machining process and the processing parameters of 5-axis milling machines, such as the mass of workpiece, the driving forces/torques, the cutting forces, and the inertia, centrifugal, Coriolis, gravity and friction forces of the machine-workpiece system.

To demonstrate the merits of the generalized model of kinematics and dynamics newly proposed and presented in this study, two applications (case studies) are investigated for the industrial 5-axis milling machine, which are presented in the following sections.

### 5.2. Analysis of relationships among the driving forces/torques, the mass of the workpiece and the feed-rate for 5-axis milling machines

This case study aims to demonstrate the effective applications of the proposed dynamic modelling solution, the generalized model of kinematics, in the design and development of 5-axis milling machines and mechanisms, and process planning and optimization of CNC machining parameters. Specifically, the case study focuses on the analysis of relationships among the driving forces/torques, the mass of the workpiece and the feed-rate for 5-axis milling machines, using the 5-axis milling machine "Spinner U5-620" [1] for demonstrations. The machine "Spinner U5-620" is an industrial 5-axis CNC milling machine of the type RRLLL, which has three linear axes x, y and z on the spindle kinematic chain, and two rotary axes B and C on the workpiece kinematic chain.

As shown in Fig. 1, the generalized coordinates of "Spinner U5-620" are given as $\boldsymbol{q} = [\boldsymbol{x} \quad \boldsymbol{y} \quad \boldsymbol{z} \quad \boldsymbol{B} \quad \boldsymbol{C}]^T$. The driving forces/torques are $\boldsymbol{\tau} = [\boldsymbol{F_x} \quad \boldsymbol{F_y} \quad \boldsymbol{F_z} \quad \boldsymbol{\tau_B} \quad \boldsymbol{\tau_C}]^T$. The resistance forces are chosen by experience ($\boldsymbol{F_t} = 150\boldsymbol{N}$, $\boldsymbol{F_n} = 70\boldsymbol{N}$ and $\boldsymbol{F_b} = 50\boldsymbol{N}$). The dynamic parameters of the 5-axis milling machine "Spinner U5-620" are listed in Table 3. The forward kinematic equations of the 5-axis milling machine "Spinner U5-620" are calculated as follows:

$$x_T = x\cos B\cos C - y\sin C + z\sin B\cos C - d\sin B\cos C \tag{19}$$

$$y_T = x\cos B\sin C + y\cos C + z\sin B\cos C - d\sin B\sin C \tag{20}$$

$$z_T = -x\sin B + z\cos B - d\cos B + d \tag{21}$$

$$\alpha = B \tag{22}$$

$$\beta = C \tag{23}$$

The mass matrix $\boldsymbol{M}$ is yielded

$$\boldsymbol{M} = \begin{bmatrix} m_{11} & 0 & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 & 0 \\ 0 & 0 & m_{33} & 0 & 0 \\ 0 & 0 & 0 & m_{44} & m_{45} \\ 0 & 0 & 0 & m_{54} & m_{55} \end{bmatrix}, \tag{24}$$

where $\{m_{ij}\}_{5x5}$ are given in Appendix B, and the translational and rotational Jacobian matrices $\{\boldsymbol{J}_{Ti}\}_{i=1\div5}$ and $\{\boldsymbol{J}_{Ri}\}_{i=1\div5}$ are presented in Appendix C. Invoking Equation (17) yields matrix $\boldsymbol{C}$ in terms of the elements of matrix $\boldsymbol{M}$. $\boldsymbol{G}$ is easily calculated with Eq. (18).

To analyze the relationships among the driving forces/torques, the mass of the workpiece and the feed-rate, the toolpath $\boldsymbol{p}(u)$ is prescribed as a Bezier curve, $\boldsymbol{p}(u) = (1-u)^3\boldsymbol{P}_0 + 3(1-u)^2u\boldsymbol{P}_1 + 3(1-u)u^2\boldsymbol{P}_2 + u^3\boldsymbol{P}_3$, which is constructed through the following control points: $\boldsymbol{P}_0 = [0 \quad 0.05 \quad 0.05]^T$, $\boldsymbol{P}_1 = [0.1 \quad 0.05 \quad 0.15]^T$, $\boldsymbol{P}_2 = [0.2 \quad 0.05 \quad 0.05]^T$, and $\boldsymbol{P}_3 = [0.3 \quad 0.05 \quad 0.15]^T$ as shown in Fig. 6. Based on the given toolpath $\boldsymbol{p}(u)$ and the feed-rate, where $f = \dot{s}(u)$ and $s(u)$ is the arc length of $\boldsymbol{p}(u)$, the value of the joint variables $\boldsymbol{q}$ are

**Table 3**
Dynamic parameters of the 5-axis CNC machine "Spinner U5-620".

| Link $L_i$ | Center of mass $r_{Ci}$ (m) | | | Mass $m_i$ (kg) | Moments of inertia $I_i$ (kg.m$^2$) | | |
|---|---|---|---|---|---|---|---|
| | $x_C$ | $y_C$ | $z_C$ | | $I_{xx}$ | $I_{yy}$ | $I_{zz}$ |
| X | 0.0 | −0.05 | 0.0 | 330 | 7.5 | 9.2 | 9.85 |
| Y | 0.0 | 0.0 | 0.0 | 225 | 1.9 | 1.9 | 3.2 |
| Z | 0.0 | 0.0 | 0.15 | 105 | 1.9 | 1.9 | 3.2 |
| B | 0.0 | −0.35 | −0.2 | 290 | 7.5 | 9.9 | 9.85 |
| C | 0.0 | 0.0 | 0.25 | 120 | 1.9 | 1.9 | 3.2 |

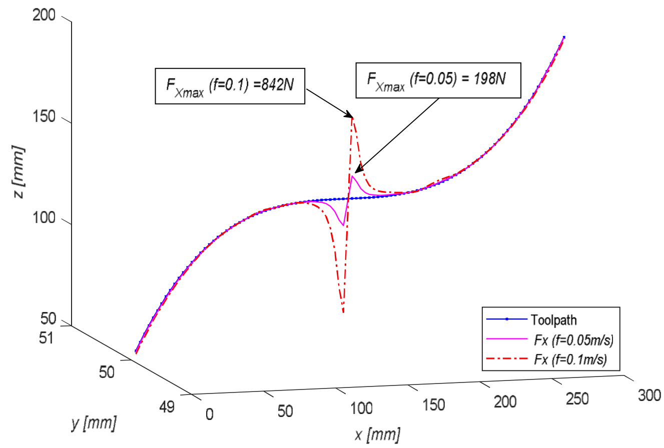


**Fig. 6.** The toolpath $\boldsymbol{p}(u)$ and the driving force $F_x$ for the cases $f = 0.1$ m/s and $f = 0.05$ m/s.

calculated with the inverse kinematic equation (12); then the velocity $\dot{\boldsymbol{q}}$ and the acceleration $\ddot{\boldsymbol{q}}$ are determined accordingly. The driving forces/torques $\boldsymbol{\tau} = \begin{bmatrix} F_x & F_y & F_z & \tau_B & \tau_C \end{bmatrix}^T$ are computed with the following inverse dynamic equation:

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) - \boldsymbol{J}^T\boldsymbol{F} \tag{25}$$

Fig. 6 presents the detailed results of the driving force $F_x$, when the machine cuts a workpiece of 150 kg with the two cases of proposed feed-rates $f = 0.1$ m/s and 0.05 m/s. It is seen that, the driving force $F_x$ increases four times when the feed-rate increases by a factor of two.

Fig. 7 presents the driving torques $\tau_B$ and $\tau_C$ which were calculated for an identical cutting path $\boldsymbol{p}(u)$ on two workpieces with different weights $m_w = 300$ kg and $m_w = 150$ kg. The feed-rate $f = 0.05$ m/s is applied for both cases. It is shown that the maximum torques that are required to drive axes B and C are almost tripled when the mass of the workpiece is doubled.

The above-presented investigation and analysis of the dynamic and machining parameter are very important in the design and optimizations of 5-axis milling machines, as well as machining process improvement and enhancement of the machining quality and productivity of 5-axis milling machines that are currently used in the manufacturing factories. It has shown that the driving forces/torques are significantly affected by the feed-rate and the mass of the workpiece. Hence, the relationships among the following parameters need to be carefully considered: the driving forces/torques, the mass of the workpiece and the feed-rate, especially for the case of high-speed CNC machining applications and for the case of CNC machining of workpieces with heavy weights.

### 5.3. Novel feed-rate optimization for 5-axis milling machines with dynamic constraints

In CNC machining, the feed-rate is a very important process parameter that has a direct influence on the productivity and efficiency of the machining process. Determination of the optimal process parameters for a machining process, especially the optimal feed-rate, is an important task of Computer Aided Process Planning (CAPP). In the current practice of high-speed 5-axis machining, most practitioners choose a constant feed-rate for selected parts or even for the entire machining toolpath, based on their experiences. However, this often results in an unnecessarily long machining time, since the maximum allowable capacity of the 5 drives of a 5-axis milling machine has never been reached. Hundreds of hours of CNC machining may need to be required to complete the required machining operations for a part; therefore, optimization of the feed-rate is very important.

The feed-rate optimization for high-speed 5-axis milling machines has been extensively studied in recent years [9–13]. However, little attention has been paid to the dynamics of the machine-workpiece components i.e. the workpiece mass and inertia; the resistance force; gravity; centrifugal and Coriolis forces. Therefore, in this case study, a new feed-rate optimization model which includes not only conventional kinematic constraints but also the constraints relevant to driving forces/torques, is presented. It has been shown that the dynamic effects of the machine-workpiece system are significant. The efficiency of the proposed optimization model has been proven by numerical examples and by actual machining.

Let's denote the limits of the velocity, acceleration, jerk and driving forces/torques as $\boldsymbol{V}_{max}$, $\boldsymbol{A}_{\max}$,$\boldsymbol{J}_{max}$ and $\boldsymbol{\tau}_{max}$ respectively. Note that the above are the 5-component vectors, for instance, $\boldsymbol{V}_{max} = \begin{bmatrix} V_{x_{max}} & V_{y_{max}} & V_{z_{max}} & V_{A_{max}} & V_{C_{max}} \end{bmatrix}^T$. Hence, there are 20 scalar constraints. The following are the objective function and the constraints of the feed-rate optimization model:

$$\underset{u\in[0,1]}{\text{maximize}} f(u) \tag{26}$$

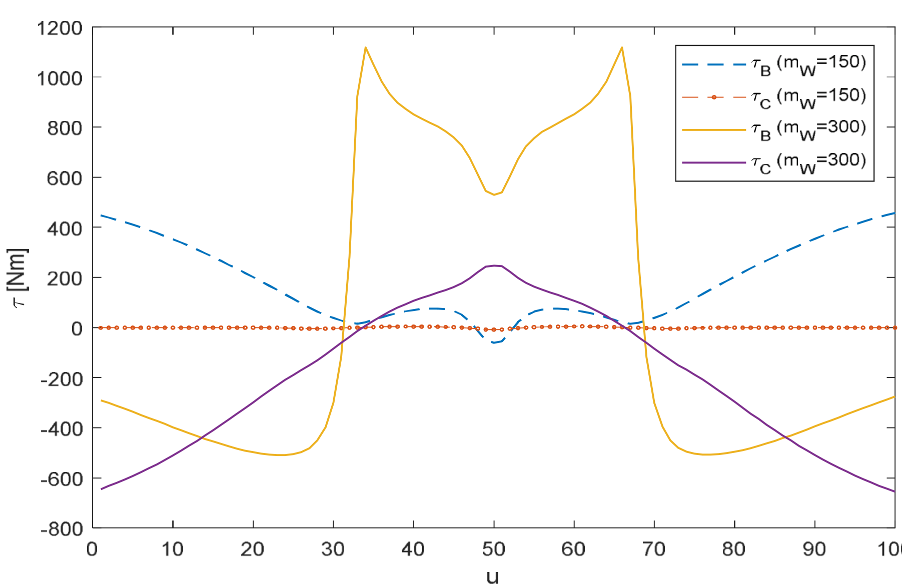


**Fig. 7.** The driving torques $\tau_B$ and $\tau_C$ computed for $m_w = 300$ kg and $m_w = 150$ kg.

**Table 4**
The drive limits of the 5-axis CNC machine "Spinner U5-620".

| Axes | Velocity $\boldsymbol{V}_{max}$m/s (rad/s) | Acceleration $\boldsymbol{A}_{max}$ m/s$^2$ (rad/s$^2$) | Jerk $\boldsymbol{J}_{max}$ m/s$^3$ (rad/s$^3$) | Force/Torque $\boldsymbol{\tau}_{max}$N (N.m) |
|---|---|---|---|---|
| X | 0.25 | 2.5 | 6 | 250 |
| Y | 0.25 | 2.5 | 6 | 250 |
| Z | 0.25 | 2.1 | 12 | 1250 |
| B | 2.5 | 15.0 | 4.5 | 1050 |
| C | 4.1 | 11.0 | 3.3 | 450 |

subject to

$$|\dot{\boldsymbol{q}}(u)| \le \boldsymbol{V}_{max}, \tag{27}$$

$$|\ddot{\boldsymbol{q}}(u)| \le \boldsymbol{A}_{max}, \tag{28}$$

$$\left|\overset{[C3Dabv]}{\boldsymbol{q}}(u)\right| \le \boldsymbol{J}_{max}, \tag{29}$$

$$|\boldsymbol{\tau}(u)| \le \boldsymbol{\tau}_{max}. \tag{30}$$

It is noted that Eqs. (27)–(29) are kinematic constraints, and Eq. (30) is the dynamic constraint which is newly developed for the application of feed-rate optimization. The kinematic constraints Eqs. (27)–(29) imply that, the joint velocity $\dot{\boldsymbol{q}}(u)$, joint acceleration $\ddot{\boldsymbol{q}}(u)$ and joint jerk $\overset{[C3Dabv]}{\boldsymbol{q}}(u)$ must be less than the allowable limits [9–13]. The dynamic constraint Eq. (30) implies that the driving torques/forces $\boldsymbol{\tau}(u)$ at all joints of a machine must not be greater than the allowable values. The left-hand sides of the constraints (27)-(30) are represented by the inverse kinematic and dynamic equations. As shown in Section 5.1, the driving forces/torques are significantly affected by the feed-rate and the mass and inertia of the workpiece. Hence, the dynamic constraint plays an important role in calculating the optimal the feed-rate, especially for the case of 5-axis CNC machining of workpieces with heavy weights. The feed-rate optimization model with the dynamic constraint implies that the same toolpath for workpieces with a different mass and geometry requires a different feed-rate.

Note that the maximum feed-rate $f_{max}$ is indicated in the specifications of the 5-axis milling machines. For example, for the 5-axis milling machine "Spinner U5-620", the maximum feed-rate $f_{max} = 0.1$ m/s, and for the 5-axis milling machine "Mikron UCP 710", the maximum feed-rate $f_{max} = 0.083$ m/s. However, in industrial practices, a suitable feed-rate $f_0$ for particular parts and materials is usually selected based on experiences or suggestions from the machining handbook. For example, a feed-rate $f_0 = 0.02$ m/s is often selected for the 5-axis milling machine "Spinner U5-620". Hence, $[f_0, f_{max}]$ is considered as a feasible range of $f(u)$ when solving the feed-rate optimization model.

To demonstrate the efficiency of the proposed optimization model, the numerical examples and the actual machining tests were carried out on the 5-axis milling machine "Spinner U5-620". Table 4 presents the limits of the drives of the 5-axis milling machine "Spinner U5-620". The maximum allowable tangential acceleration $a_{\max} = 1000\,\mathrm{mm/s^2}$.

In this case study, the identical tool path $\boldsymbol{p}(u)$ presented in Fig. 6 is considered for two cases $m_w = 100$ kg and $m_w = 300$ kg. The optimal feed-rate is calculated by using iterative computational procedure with a time complexity of $O(\boldsymbol{n})$ for the two cases that is shown in Fig. 8. The two solutions are significantly different. When the mass and the inertia of the workpiece increase, the driving forces/torques change. To satisfy the dynamic constraints, the optimal feed-rate has been reduced for $u \in [0.4, 0.65]$..

To demonstrate the advantages of the proposed dynamic method, let us compare it with the kinematic methods of feed-rate optimization. The tool path $\boldsymbol{p}(u)$ presented in Fig. 6 is considered. The feed-rate optimized with respect to only the kinematic constraints and for both the kinematic and dynamic constraints is shown in Fig. 9. The difference between the two curves clearly shows the impact of the dynamic constraints on the feed-rate.

Now let us consider a numerical demonstration that the feed-rate optimization model is implemented for machining a complex parametric surface which is given by

$$\boldsymbol{S}(u, v) = \left(30u, 30v, 8\frac{uv(u^2 - v^2)}{u^2 + v^2}\right), \ -1 \le u, v \le 1 \tag{31}$$

The surface is shown in Fig. 10. The zigzag tool path is generated in the parametric domain. The feed-rate along the tool path is optimized by the proposed optimization model. Fig. 11 shows the tool path and the optimal feed-rate. The estimated machining time using the practical feed-rate $f_0 = 0.02\,\mathrm{m/s}$ is approximately 31.2 min, whereas machining with the optimal feed-rate requires 19.8 min. Therefore, the optimization reduces the machining time by 36.1 %.

To demonstrate the efficiency and the effectiveness of the proposed optimization model, real cutting experiments were carried out with the 5-axis milling machine "Spinner U5-620". The Bezier tool path is given

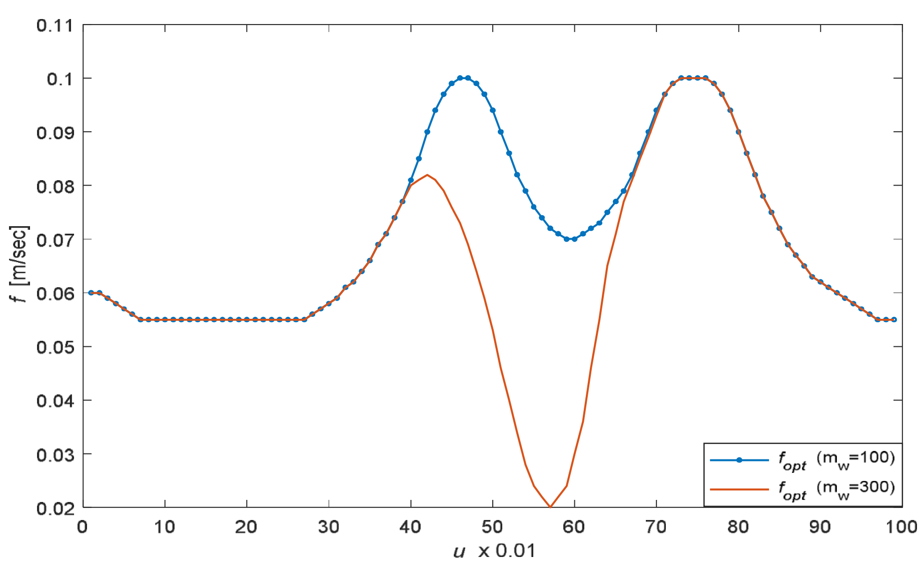


**Fig. 8.** The optimal feed-rates for the two study cases with the mass of the workpiece $m_w = 100\,\mathrm{kg}$ and $m_w = 300\,\mathrm{kg}$.

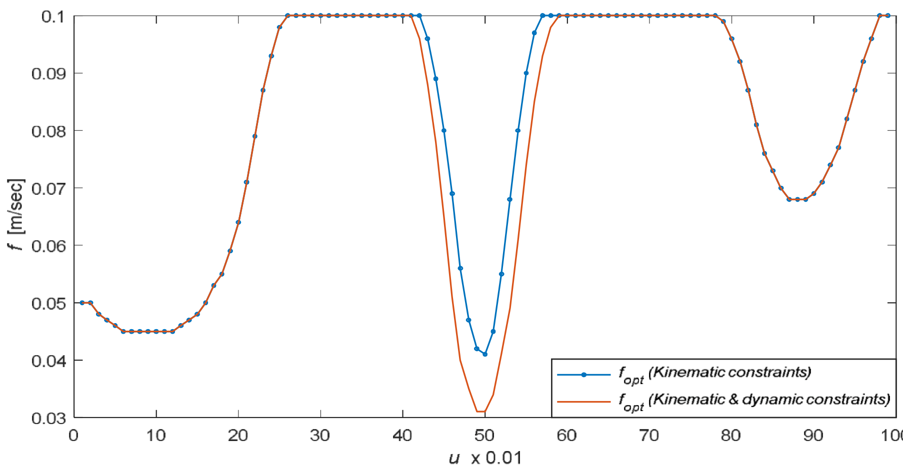


**Fig. 9.** The optimal feed-rate with and without the dynamic constraints.

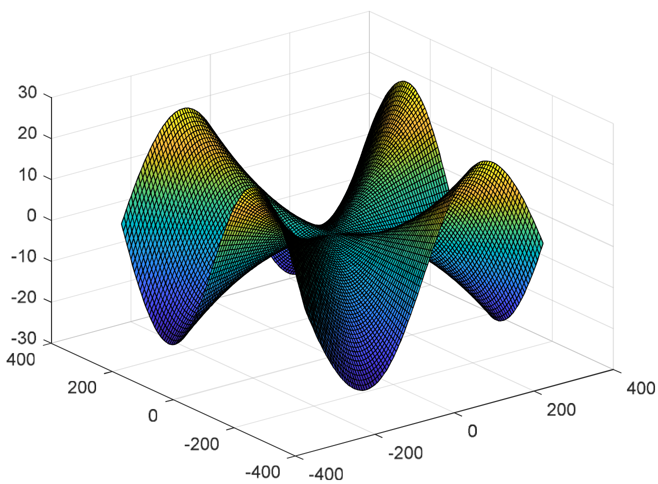

**Fig. 10.** A complex machining surface.

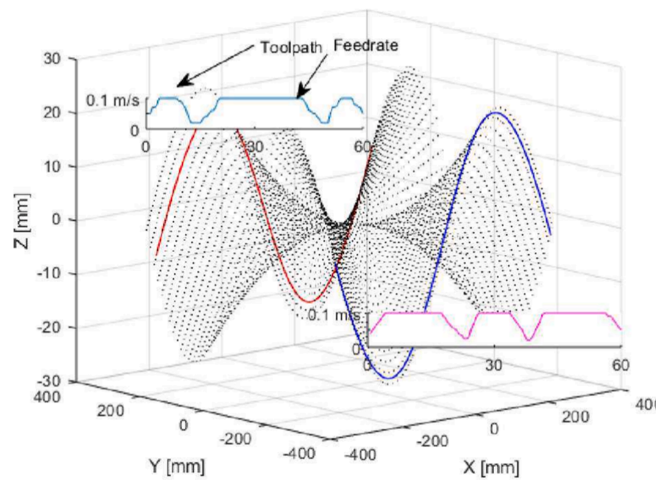


**Fig. 11.** The tool path and optimal feed-rate for a sculptured surface.

as shown in Fig. 6. The workpiece (Steel C45) is well prepared for a finishing operation. The ball nose end cutter with diameter of 12 mm is employed for the cuts. For the machined part shown in Fig. 12a, the actual machining time is approximately 76sec. The optimal feed-rate reduces the machining time by 39.5 % compared to an experimentally established feed-rate $f_0 = 0.02$ m/s. It is noted that by multiplying the feed-rate value with the tool path length step-by-step, the machining time is 62.3sec that is less the actual machining time 76sec. This discrepancy between the calculated and the actual machining of the machining process may be due to the setting time and the delay time of the NC controller of the machine Spinner U5.620 while performing several consecutive G01 commands for cutting along the given tool path.

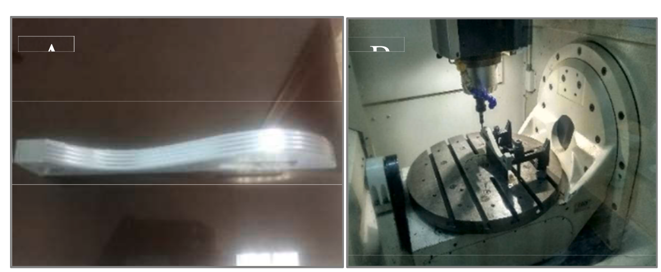

**Fig. 12.** The experiments performed on the 5-axis CNC machine “Spinner U5-620”. (A): The machined part. (B) The machined part which is fixed on the machine table.

## 6. Discussions

By using the proposed kinematic and dynamic equations, a case study of a new feed-rate optimization model for the 5-axis machining has been developed, which considers the dynamics effects of the machine-workpiece system, i.e., the mass and inertia of the machine-workpiece components; the resistance forces acting on the tool and reacting on the workpiece; as well as the gravity; centrifugal and Coriolis forces.

The impacts of the dynamics model of the system on the feed-rate optimization are analyzed effectively. It has shown that the dynamic performance of the machine-workpiece system affects significantly the feed-rate optimization.

In comparison with the conventional feed-rate optimization model subject to only kinematic constraints [9–13], the feed-rate optimization model with both kinematic and dynamic constraints proposed in this paper is more advantageous and comprehensive. The numerical examples and experiments performed on “Spinner U5-620” demonstrate the effectiveness of the proposed optimization model. Machining time is reduced by up to 39.5 % compared to the experimental feed rate.

It has also shown that the generalized kinematic equations for the 5-

axis machines family newly proposed in this study is useful, especially for development of new machine architectures, generalization of 5-axis postprocessor, optimization of the workpiece setup, compensation of the kinematic errors, etc. Since the forward and inverse kinematic equations are explicitly expressed in compact forms, the kinematical analyses, and especially the construction of kinematics-based computational algorithms for the whole family of machines are convenient, accurate and highly efficient.

By using the generalized kinematic and dynamic equations in closed-form, the kinematic and dynamic performance of all possible 5-axis mechanisms can be evaluated and analyzed effectively, that is very useful for the machine design optimization.

In particular, the generalization of the 5-axis kinematic modelling makes it possible the design of a generic real time postprocessor for all kinds of 5-axis CNC machines. A generic postprocessor is able to transfer any common CL data produced by CAM systems to NC programs (G-code files) for any 5-axis CNC machines.

Also, the generalized kinematic equations in closed-form are helpful for the kinematic errors modelling and the kinematic singularity analysis for different 5-axis CNC machines. Thus, the kinematic errors and the tool path error near singular points of any 5-axis CNC machines can be analyzed and compensated effectively.

## 7. Conclusions

A novel method for formulating the model of kinematics and dynamics for the whole family of possible 5-axis milling machines was presented, including the demonstrations of effectiveness and significance of the proposed method via useful applications with numerical simulation and cutting experiments. The generalized kinematic and dynamic modeling proposed in this study has been shown to be useful for the design optimization of multi-axis machines and machining processes.

In the study, the generalized form of kinematic formulas for all possible 5-axis mechanisms were explicitly derived and expressed in closed form. Particularly, by taking into account the special characteristics of the generalized kinematic model, it has proven that there exist only 4 feasible combinations of the primary-secondary rotary axes, i.e. A-B, B-A, C-A and C-B. Hence, designing 5-axis milling machines with the primary-secondary rotary axes A-C and B-C is infeasible. These findings are useful when synthesizing the 5-axis CNC kinematic chains to design 5-axis milling machines.

It has been shown that the explicit derivation of the forward kinematic equation and the closed-form expression of the inverse kinematic equation are essential for the dynamic modelling and the inverse dynamic computation since the use of the explicit kinematic formulas makes it possible to derive explicitly the kinematic constraint equation $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$, and the Jacobian matrix of the constraint function $\boldsymbol{J} = (\partial \boldsymbol{f}/\partial \boldsymbol{q})$ Additionally, the use of the closed form inverse kinematic equation makes it more applicable to derive the joint variable differentiations $\dot{\boldsymbol{q}}$ and $\ddot{\boldsymbol{q}}$ for the inverse dynamic computation.

To our knowledge, this is the first time a dynamic equation of the complex 5-axis milling machine–workpiece system has been established. In particular, the formulation of the dynamics model takes into account the collaboration of the workpiece and the cutting tool during the machining process.

In the future, we will continue the work to study the generalization of 5-axis postprocessor, the generalization of regular/singular toolpath interpolation, the machining energy optimization, and the effect of the translation errors, rotational errors and squareness errors on the kinematic and dynamic performance of the machines.

## CRediT authorship contribution statement

**Anh-My Chu:** Conceptualization, Methodology, Software, Writing – original draft, Funding acquisition. **Van-Cong Nguyen:** Validation, Software, Formal analysis, Data curation. **Chi-Hieu Le:** Writing – review & editing, Visualization, Investigation. **James Gao:** Resources, Formal analysis. **Michael Packianather:** Validation, Resources. **Shwe Soe:** Writing – review & editing, Validation.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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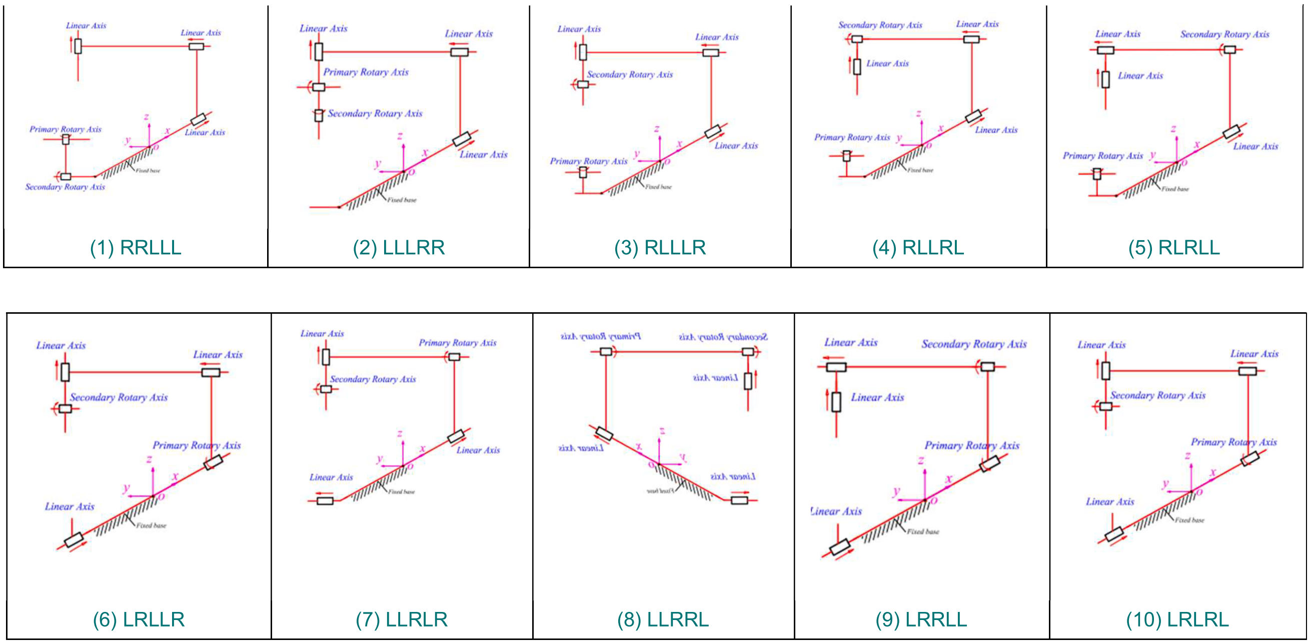
### *Declaration of Generative AI and AI-assisted technologies in the writing process*

During the preparation of this work the authors do not use generative AI and AI – assisted technologies in the writing process.

## Appendix A. The kinematic diagrams of all possible machine groups of the machine family



(1) RRLLL (2) LLLRR (3) RLLLR (4) RLLRL (5) RLRLL

(6) LRLLR (7) LLRLR (8) LLRRL (9) LRRLL (10) LRLRL

## Appendix B. The entries of the matrix$M$

$$m_{11} = m_1 + m_2 + m_3; m_{22} = m_2 + m_3; m_{33} = m_3 \tag{B.1}$$

$$\begin{aligned} m_{44} = {} & I_{4y} + m_4 (y_{C4}cosB + x_{C4}sinB)^2 + m_4 (x_{C4}cosB + y_{C4}sinB)^2 + m_5 (z_{C5}cosB + dcosB + x_{C5}cosCsinB + y_{C5}sinCsinB)^2 \\ & + m_5 (z_{C5}cosBcosC + y_{C5}sinCcosB + z_{C5}sinB + dsinB)^2 + I_{5y} \end{aligned} \tag{B.2}$$

$$\begin{aligned} m_{45} & = m_{54} \\ & = I_{5y} + m_5 (x_{C5}cosC + y_{C5}sinC)^2 - m_5[(x_{C5}sinBsinC - y_{C5}cosCsinB)(x_{C5}cosBcosC + y_{C5}cosBsinC - z_{C5}sinB + dsinB)] - m_5[(y_{C5}cosBcosC \\ & \quad - x_{C5}sinCcosB)(z_{C5}cosB + y_{C5}sinBsinC + x_{C5}sinBcosC - dcosB)] \end{aligned} \tag{B.3}$$

$$m_{55} = m_5 (x_{C5}sinBsinC - y_{C5}cosCsinB)^2 + m_5 (z_{C5}cosBcosC - x_{C5}cosBsinC)^2 + I_{5y} + m_5 (x_{C5}cosC + y_{C5}sinC)^2 \tag{B.4}$$

## Appendix C. Derivation of the Jacobian matrices $\{J_{Ti}\}_{i=1\div5}$ and $\{J_{Ri}\}_{i=1\div5}$

$$\boldsymbol{J}_{T1} = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}_{5\times5} \tag{C.1}$$

$$\boldsymbol{J}_{T2} = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}_{5\times5} \tag{C.2}$$

$$\boldsymbol{J}_{T3} = \boldsymbol{E}_{5x5} \tag{C.3}$$

$$\boldsymbol{J}_{T4} = \frac{\partial(\boldsymbol{A}_s^T \boldsymbol{r}_{C4})}{\partial \boldsymbol{q}} \tag{C.4}$$

$$\boldsymbol{J}_{T5} = \frac{\partial\left(\boldsymbol{A}_s^T \boldsymbol{A}_p^T \boldsymbol{r}_{C5}\right)}{\partial \boldsymbol{q}} \tag{C.5}$$

$\boldsymbol{J}_{R1} = \boldsymbol{J}_{R2} = \boldsymbol{J}_{R3} = 0$; (C.6)

$$J_{R4} = \frac{\partial\left(\dot{A}_s^T A_s\right)}{\partial \dot{q}} \tag{C.7}$$

$$J_{R5} = \frac{\partial\left(\left(\dot{A}_p A_s + A_p \dot{A}_s\right)^T (A_p A_s)\right)}{\partial \dot{q}} \tag{C.8}$$

where $A_s = \begin{bmatrix} \cos B & 0 & \sin B \\ 0 & 1 & 0 \\ -\sin B & 0 & \cos B \end{bmatrix}$ and $A_p = \begin{bmatrix} \cos C & -\sin C & 0 \\ \sin C & \cos C & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

## References


[1] Spinner U5-620, Machining-centers with the 5-axis machining capability. www.spinner.eu.com. Access April 2023.
[2] L. Zhang, et al., Intelligent integrated framework towards high-accuracy machining, Eng. Sci. Technol. Int. J. 40 (2023) 101359, https://doi.org/10.1016/j.jestch.2023.101359.
[3] B.I. Oladapo, et al., Experimental analytical design of CNC machine tool SCFC based on electro-pneumatic system simulation, Eng. Sci. Technol. Int. J. 19 (4) (2016) 1958–1965, https://doi.org/10.1016/j.jestch.2016.08.010.
[4] C. My, E. Bohez, A novel differential kinematics model to compare the kinematic performances of 5-axis milling machines, Int. J. Mech. Sci. 163 (2019) 105117, https://doi.org/10.1016/j.ijmecsci.2019.105117.
[5] S. Lavernhe, C. Tournier, C. Lartigue, Kinematic performances in 5-axis machining, in: Advances in Integrated Design and Manufacturing in Mechanical Engineering II, Springer, 2007, pp. 489–503, https://doi.org/10.1007/978-1-4020-6761-7_33.
[6] H. Liu, Q. Liu, P. Sun, Q. Liu, S. Yuan, The optimal feed-rate planning on five-axis parametric tool path with geometric and kinematic constraints for CNC machine tools, Int. J. Prod. Res. 55 (2017) 3715–3731, https://doi.org/10.1080/00207543.2016.1254357.
[7] Q.B. Xiao, M. Wan, X.B. Qin, Y. Liu, W.-H. Zhang, Real-time smoothing of G01 commands for five-axis machining by constructing an entire spline with the bounded smoothing error, Mech. Mach. Theory 161 (2021) 104307, https://doi.org/10.1016/j.mechmachtheory.2021.104307.
[8] P.Y. Tang, M.T. Lin, M.S. Tsai, C.-C. Cheng, Toolpath interpolation with novel corner smoothing technique, Rob. Comput. Integr. Manuf. 78 (2022) 102388, https://doi.org/10.1016/j.rcim.2022.102388.
[9] S.D. Timar, R.T. Farouki, T.S. Smith, C.L. Boyadjieff, Algorithms for time–optimal control of CNC machines along curved tool paths, Rob. Comput. Integr. Manuf. 21 (2005) 37–53, https://doi.org/10.1016/j.rcim.2004.05.004.
[10] X. Beudaert, S. Lavernhe, C. Tournier, Feed-rate interpolation with axis jerk constraints on 5-axis NURBS and G1 tool path, Int. J. Mach. Tool Manuf. 57 (2012) 73–82, https://doi.org/10.1016/j.ijmachtools.2012.02.005.
[11] Y. Zhang, T. Wang, P. Peng, J. Dong, L. Cao, C. Tian, Feed-rate blending method for five-axis linear tool path under geometric and kinematic constraints, Int. J. Mech. Sci. 195 (2021) 106262, https://doi.org/10.1016/j.ijmecsci.2020.106262.
[12] K. Erkorkmaz, S.E. Layegh, I. Lazoglu, H. Erdim, Feed-rate optimization for freeform milling considering constraints from the feed drive system and process mechanics, CIRP Ann. 62 (2013) 395–398, https://doi.org/10.1016/j.cirp.2013.03.084.
[13] M. Vulliez, S. Lavernhe, O. Bruneau, Dynamic approach of the feed-rate interpolation for trajectory planning process in multi-axis machining, Int. J. Adv. Manuf. Technol. 88 (2017) 2085–2096, https://doi.org/10.1007/s00170-016-8903-y.
[14] K. Sørby, Inverse kinematics of five-axis machines near singular configurations, Int. J. Mach. Tool Manuf. 47 (2007) 299–306, https://doi.org/10.1016/j.ijmachtools.2006.03.011.
[15] J. Yang, Y. Altintas, Generalized kinematics of five-axis serial machines with non-singular tool path generation, Int. J. Mach. Tool Manuf. 75 (2013) 119–132, https://doi.org/10.1016/j.ijmachtools.2013.09.002.
[16] C.A. My, E.L.J. Bohez, New algorithm to minimise kinematic tool path errors around 5-axis machining singular points, Int. J. Prod. Res. 54 (2016) 5965–5975, https://doi.org/10.1080/00207543.2015.1134838.
[17] Q. Hu, Y. Chen, J. Yang, On-line contour error estimation and control for corner smoothed five-axis tool paths, Int. J. Mech. Sci. 171 (2020) 105377, https://doi.org/10.1016/j.ijmecsci.2019.105377.
[18] Y. Liu, M. Wan, W.-J. Xing, Q.-B. Xiao, W.-H. Zhang, Generalized actual inverse kinematic model for compensating geometric errors in five-axis machine tools, Int. J. Mech. Sci. 145 (2018) 299–317, https://doi.org/10.1016/j.ijmecsci.2018.07.022.
[19] R.T. Farouki, C.Y. Han, S. Li, Inverse kinematics for optimal tool orientation control in 5-axis machining, Comput. Aided Geom. Des. 31 (2014) 13–26, https://doi.org/10.1016/j.cagd.2013.11.002.
[20] L. Zhang, Y. Bian, H. Chen, K. Wang, Implementation of a CNC NURBS curve interpolator based on control of speed and precision, Int. J. Prod. Res. 47 (2009) 1505–1519, https://doi.org/10.1080/00207540701491108.
[21] B. Yang, G. Zhang, Y. Ran, H. Yu, Kinematic modeling and machining precision analysis of multi-axis CNC machine tools based on screw theory, Mech. Mach. Theory 140 (2019) 538–552, https://doi.org/10.1016/j.mechmachtheory.2019.06.021.
[22] C.-H. She, C.-C. Chang, Design of a generic five-axis postprocessor based on generalized kinematics model of machine tool, Int. J. Mach. Tool Manuf. 47 (2007) 537–545, https://doi.org/10.1016/j.ijmachtools.2006.06.002.
[23] J.D. Hwang, J.H. Kim, H.J. Son, Y.T. Cho, Y.G. Jung, A study on the development of post processor for five-axis machining using angle head spindle, Int. J. Precis. Eng. Manuf. 16 (2015) 2683–2689, https://doi.org/10.1007/s12541-015-0343-3.
[24] C.A. My, Integration of CAM systems into multi-axes computerized numerical control machines, in: 2010 Second Int. Conf. Knowl. Syst. Eng., 2010, pp. 119–124, https://doi.org/10.1109/KSE.2010.30.
[25] H.-J. Son, J.-D. Hwang, Y.-T. Cho, Y.-G. Jung, Development of post processor for five-axis machine of non-orthogonal head tilting type, Int. J. Precis. Eng. Manuf. 17 (2016) 189–194, https://doi.org/10.1007/s12541-016-0024-x.
[26] C.-H. She, Z.-T. Huang, Postprocessor development of a five-axis machine tool with nutating head and table configuration, Int. J. Adv. Manuf. Technol. 38 (2008) 728–740, https://doi.org/10.1007/s00170-007-1126-5.
[27] Y.H. Jung, D.W. Lee, J.S. Kim, H.S. Mok, NC post-processor for 5-axis milling machine of table-rotating/tilting type, J. Mater. Process. Technol. 130 (2022) 641–646, https://doi.org/10.1016/S0924-0136(02)00725-2.
[28] Y. Boz, I. Lazoglu, A postprocessor for table-tilting type five-axis machine tool based on generalized kinematics with variable feed-rate implementation, Int. J. Adv. Manuf. Technol. 66 (2013) 1285–1293, https://doi.org/10.1007/s00170-012-4406-7.
[29] C.A. My, et al., A framework for practically effective creation of postprocessors for 5-axis milling machines with all possible configurations and working mechanisms, in: Proceedings of the Institution of Mechanical Engineers, Part B: Journal of Engineering Manufacture, 2023, https://doi.org/10.1177/09544054231179314.
[30] V. Ostasevicius, R. Gaidys, R. Dauksevicius, S. Mikuckyte, Study of vibration milling for improving surface finish of difficult-to-cut materials, Strojniški Vestnik-J. Mech. Eng. 59 (2013) 351–357, https://doi.org/10.5545/sv-jme.2012.856.
[31] Y. Dun, Z. Lida, S. Wang, Multi-modal method for chatter stability prediction and control in milling of thin-walled workpiece, Appl. Math. Model. 80 (2020) 602–624, https://doi.org/10.1016/j.apm.2019.12.003.
[32] J. Ma, D. Zhang, Y. Liu, B. Wu, M. Luo, Tool posture dependent chatter suppression in five-axis milling of thin-walled workpiece with ball-end cutter, Int. J. Adv. Manuf. Technol. 91 (2017) 287–299, https://doi.org/10.1007/s00170-016-9748-0.
[33] D.G. Ford, A. Myers, F. Haase, S. Lockwood, A. Longstaff, Active vibration control for a CNC milling machine, Proc. Inst. Mech. Eng. Part C J. Mech. Eng. Sci. 228 (2014) 230–245, https://doi.org/10.1177/0954406213484224.
[34] B. Das, et al., Application of grey fuzzy logic for the optimization of CNC milling parameters for Al–4.5% Cu–TiC MMCs with multi-performance characteristics, Eng. Sci. Technol. Int. J. 19 (2) (2016) 857–865, https://doi.org/10.1016/j.jestch.2015.12.002.
[35] C.C. Hong, Cheng-Long Chang, Chien-Yu Lin, Static structural analysis of great five-axis turning–milling complex CNC machine, Eng. Sci. Technol. Int. J. 19 (4) (2016) 1971–1984, https://doi.org/10.1016/j.jestch.2016.07.013.
[36] Y. Altintas, M. Weck, Chatter stability of metal cutting and grinding, CIRP Ann. Manuf. Technol. 53 (2) (2004) 619–642, https://doi.org/10.1016/S0007-8506(07)60032-8.
[37] Q. Huang, L. Jinlong, Practical limitations of lane detection algorithm based on Hough transform in challenging scenarios, Int. J. Adv. Rob. Syst. 18 (2) (2021), 17298814211008752.
[38] Q. Huang, J. Liu, C. Ulishney, C. Dumitrescu, On the use of artificial neural networks to model the performance and emissions of a heavy-duty natural gas spark ignition engine, Int. J. Engine Res. 23 (11) (2022) 1879–1898.

[39] Y. Zhang, T. Wang, P. Peng, J. Dong, L. Cao, C. Tian, Feedrate blending method for five-axis linear tool path under geometric and kinematic constraints, Int. J. Mech. Sci. 195 (2021) 106262.
[40] C.C. Wei, W. Lee, Optimization of the setup position of a workpiece for five-axis machining to reduce machining time, Adv. Mech. Eng. 12 (12) (2020) 1687814020975544.
[41] J. Chen, S. Lin, B. He, Geometric error compensation for multi-axis CNC machines based on differential transformation, Int. J. Adv. Manuf. Technol. 71 (2014) 635–642.
[42] M. Givi, J. Mayer, Volumetric error formulation and mismatch test for five-axis CNC machine compensation using differential kinematics and ephemeral G-code, Int. J. Adv. Manuf. Technol. 77 (2015) 1645–1653.