Lorentzian structure for warped product spaces and causality theory

Author:

Georgios Tsimperis

Department of Mathematics
Cardiff University

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First supervisor: John Harvey
Second supervisor: Roger Behrend

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Abstract

Product spaces with Lorentzian structure are of great importance as they have been shown that they admit a very well-behaving causal picture, avoiding general pathologies of causal relations (such as spaces manifesting causal bubbling) and they also constitute a set of models upon which, many physical cosmological models are built. This study focuses on the emergence of causal structure stemming from a warped product space geometry, where the base is a non-regular Lorentzian length space of any dimension. We prove that such spaces can provide an analogue of length spaces for Lorentzian geometry and that they also manifest the nice-to-have properties of Lorentzian warped product spaces with one-dimensional base.

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1 [redacted]

Introduction

Although Lorentzian geometry has been the subject of constant development for more than a century now, we remain far from the point of a complete understanding of its scope. Much of the interest in the development of Lorentzian geometry is intertwined with open problems related to the content and structure of spacetimes that are frequently encountered in Physics. Such systems are those that describe massive stars, systems of stars or black holes (at least outside of the event horizon) and in some cases even the flat vacuum spacetime, i.e the Minkowski spacetime. This latter case is the one that we will invoke several times in this work to help us understand the more general picture. Despite the contribution General Relativity has made to the evolution of Lorentzian geometry (with the relation following the opposite direction in the beginning of General relativity, when a lot of inspiration was drawn by the tools of Lorentzian geometry in order for the basic tools of General Relativity to be conceptualized), there are still a lot of grey areas in its edifice.

In particular, the non-regularity of spacetimes are cases that are frequently encountered in the universe. Examples of physical systems wherein such problems manifest are in the study of the interiors of stars, where different types of matter are mixed, like neutron stars. Another area where low regularity poses an obstacle is cosmic strings. In some cases, even the very existence of any form of metric, that can be used to produce the causal structure is not possible. An approach to this kind of problems through the creation of a Lorentzian version of a length space, has been proven to be able to resolve some of them.

Another class of low regularity problems, where a length space perspective has led to generalisations of a few very important results of Physics are those that are related to singularities. In particular, as is shown in [KSSV14], for $C^{1,1}$ regularity class (i.e spacetime metrics with locally Lipschitz continuous first derivatives), there is a generalisation of Hawking's singularity theorem. Similar generalisations for singularity theorems are also shown in [AGKS19].

A very particular subclass of Lorentzian geometry is that of product spaces and in particular warped product spaces. Product spaces in Lorentzian geometry offer a very fertile ground for exploration, mainly due to the frequency that they appear in a large spectrum of General Relativity problems. From the Schwarzschild spacetime toy model to the $n+1$ (local) decomposition of spacetimes (where this is applicable, meaning the local decomposition of a spacetime M to a product space of the form $\mathbb{R} \times X$, with X an n-dimensional length space, which is spacelike wrt the inner product defined on the original manifold M), product spaces emerge in many tools we use to understand gravity and spacetime structure. In addition, as it was shown in [AGKS19], a particular subclass of product spaces, i.e the generalized cones, which are of great importance as they offer a generalization of frequently encountered cosmological models, have been shown to have a lot of the major causal structure properties of their smooth setting counterparts. Therefore, delving more into the different classes of product spaces one could expect to find the most general geometric characteristics of such spaces that preserve the key properties of the generalised cones and hence provide a general class of geometries with explicit physical importance.

To this end, a few important pieces of work are given in [AGKS19] (wherein a comprehensive study of warped products with one dimensional base is attempted), as well as [AB04], [AB15] and [AB98] (which instigated the establishment of the one-dimensional base Lorentzian warped product space). To the best of our knowledge, there haven't been any papers that provide a structured and methodical attempt to describe the causal structure of general warped product spaces. Hence, in this thesis, we were interested in the general properties the warped product spaces are required to possess with respect to their causal structure.

Despite the great importance of understanding product Lorentzian length spaces (LLS) due to their relevance to Physics, there is another factor that dictates this approach from a mathematical point of view. In particular, for the spacetimes where the analytical properties of the metric are absent, or even the metric itself is ill-defined we require alternative tools that are able to replace the Einstein equations. In the language of metric space geometry (a detailed study of which is given in [BBI01]), a very interesting subclass of spaces is that of those admitting a curvature function which obeys certain convexity properties, an approach known as synthetic curvature. For this approach, the warped product spaces provide a less demanding setting on which to study synthetic curvature, e.g when we have information about the curvature bounds or the specific symmetries that the two spaces giving the warped product space possess. This was recognized by Alexander and Bishop, who provided a very detailed research of the synthetic curvature bounds in warped product metric spaces [AB04]. Therefore, motivated by these simplicities offered in studying synthetic curvature for warped product metric spaces, it is natural to ask for a framework capable to generalise into the Lorentzian setting the warped product metric spaces structure, with the anticipation that these spaces could also serve as a less demanding case study for a Lorentzian synthetic curvature (bounds) theory, that could potentially further our current knowledge of the topic ([AGKS19], [BS22], [CM20], [KS17], [MS18]).

Product spaces are of central importance from another mathematical point of view. In particular, they provide a very easy way of composing new spacetime geometries. From this point of view one might be interested in finding the general properties a Lorentzian and a metric space need to have, in order to synthesise a product space, which reduces down to a Lorentzian manifold in the smooth setting, but also provides a framework for causal structure in cases of lower regularity.

An obstacle of vital importance in providing a Lorentzian analogue of metric spaces has been that the Lorentz distance doesn't lead to a metric structure, as it is described in [V21]. An attempt to resolve this problem for a manifold was attempted in [SV15], where the notion of the null distance was established. However, this attempt later expanded into a lower regularity setting and within the context of a general Lorentzian length space in [KS21], wherein it was shown that warped product spaces with one dimensional base can admit an induced metric through their naturally occurring time function $\mathcal{T} : (t, x) \mapsto t$. Therefore, warped product spaces, which are of great importance to Physics and hence of practical interest, produce another case where the Lorentzian length space analogue of metric spaces can be meaningfully constructed.

1.1 Main open problems and developments in LLS

The motivation described above for the development of tools that could allow us to access spacetimes of low regularity has led to the direction of Lorentzian length spaces, as they were first introduced in [KS17]. However, to this date (despite the numerous efforts of researchers across the scientific community) there are still a lot of open problems in the field, most of which classify in one or more of the following categories:

- [AGKS19]: The development of warped product spaces of one-dimensional base, namely generalized cones, which is of particular importance to Physics, as there are several known cosmological models that follow this geometry.

- [CM20]: Introduces optimal transport methods in Lorentzian length spaces, defines timelike Ricci curvature bounds via suitable entropy conditions and gives applications to general relativity (synthetic singularity theorems).

- [SV15]: In here it was introduced for the first time the notion of the null distance, with the intention to create a notion of metric for a Lorentzian length space. This idea was fruther developed in [KS21], where authors studied Gromov-Hausdorff convergence, establishing first compatibility results with respect to curvature bounds.

- [BGH21]: This research attempted the study of existence of time functions in Lorentzian legth spaces.

- [KOV22]: This research studies the notion of distributional curvature bounds, generalising the theory of distributional curvature on manifolds as it was first developed in [LM07].

- [HPS20]: Attempts a generalisation of causal ladder for Lorentzian length spaces which was originally introduced in [KS17].

1.2 Results and structure

In section 1.3, we give an introduction to the theory of Lorentzian length spaces and lay down the motivation to pursue research on a more general structure of Lorentzian warped product spaces, by giving the example of the toy model of the Schwarzschild metric.

In section 2 we provide the geometrical tools for measuring lengths in our space and give some of its equivalent formulas that allow us access to more local features of the length-measuring formula in the absence of an analytic metric. Furthermore, in the same section, we give the description of the timelike and causal future/past for points in our space and we prove that the timelike future possesses the push-up property. Finally, we show that the "nice-to-have" properties for the one dimensional generalised cones of [AGKS19] generalise for a higher dimensional base.

Finally, in section 3 we expand our conversation to show that our structure of choice yields a Lorentzian length space.

1.3 Basics of Lorentzian length spaces

In order to be assisted in the understanding of the broader context under which the spaces studied in the rest of the thesis are important, we give here a very brief introduction to the theory of Lorentzian length spaces.

Before we introduce any kind of Lorentzian structure, we start by determining a topology on which to build our space. For the purposes of the spaces studied in this thesis, there are two natural choices of topologies. Before we introduce them though, along the lines of [KS17], Definition 2.4, we need to introduce another primary feature of the Lorentzian structure, i.e the ordering of points in the space. This is done through the relations between points (Definition 2.1, [KS17]):

Definition 1.3.1. *Let $(Y, \leq, \ll)$ be a tuple, where Y is a set endowed with a reflexive and transitive relation $\leq$ (pre-order) and a transitive relation $\ll$ contained in $\leq$. Then $(Y, \leq, \ll)$ is called a causal space. If $x, y \in Y$ and $x \ll y$ or $x \leq y$, we call x and y timelike or causally related, respectively.*

From Definition 1.3.1, we can now give the two natural topologies in a causal space that were mentioned above, as they are defined in [KS17], Definition 2.4. First we set that:

Definition 1.3.2. *For Y a causal space and $p \in Y$:*

- $J^+(p) = \{q \in Y : p \leq q\}$ *and* $J^-(p) = \{q \in Y : q \leq p\}$
- $I^+(p) = \{q \in Y : p \ll q\}$ *and* $I^-(p) = \{q \in Y : q \ll p\}$

and from the sets $J^{\pm}, I^{\pm}$ the two topologies are given to be:

Definition 1.3.3. *Let $(Y, \leq, \ll)$ be a causal space. Then:*

1. *Define a topology $\mathcal{A}$ on Y by using $S := \{I^{+}(p) \cap I^{-}(q); p, q \in Y\}$ as a subbase. We name this the Alexandrov topology on Y wrt $\ll$.*
2. *Define a topology $\mathcal{I}$ on Y by using $P := \{I^{\pm}(p); p \in Y\}$ as a subbase. We name this the chronological topology on Y.*

In what follows though, we want to use the metric topology ($\mathcal{D}$). This approach requires the use of a metric that serves as the means to distinguish points in the space. The use of the metric is not important and it can be freely chosen, as long as it describes a space that is topologically equivalent to the causal space Y we want to study. Further below, we will see the extent of agreement between one or more of the topologies introduced above and what kind of structure emerges when certain topologies agree.

So far, we have set after describing the global structure of a Lorentzian space. The features of causal relations and the topology are two of the most fundamental characteristics of such a space. But they are not the only ones (Definition 2.8, [KS17]).

Definition 1.3.4. *If Y is a causal space that is equipped with a metric d and a lower semicontinuous map $\tau : Y \times Y \to [0, \infty) \cup \{\infty\}$ that satisfies the reverse triangle inequality:*

$$\tau(x, z) \geq \tau(x, y) + \tau(y, z)$$

(for $x \leq y \leq z$), as well as $\tau(x, y) = 0$ if $x \not\leq y$ and $\tau(x, y) > 0 \Leftrightarrow x \ll y$, then $(Y, d, \ll, \leq, \tau)$ is called a Lorentzian pre-length space and τ is called the time separation function (or Lorentzian distance) of Y.

Due to [KS17], Example 2.11, any smooth spacetime is a Lorentzian pre-length space (where the distance metric is induced by some Riemannian background metric, used only to describe the topological properties of the space).

With the Lorentzian pre-length spaces we basically give the global structure of a space, part of which global structure are the causal relations that were defined above. Therefore, with these relations in hand, it is only natural to ask for a definition of causal paths between points. As we would expect, all (topological) curves in a Lorentzian pre-length space are classified into two different categories wrt the relations of Definition 1.3.2: causal and non-causal. The former are defined as follows (Definition 2.18 in [KS17]):

Definition 1.3.5. *Let $(Y, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space: a locally Lipschitz curve $\gamma : [a, b] \to Y$ is called future-directed causal, if $\gamma(s) \leq \gamma(t)$ for all $s, t \in [a, b]$, $s < t$. Past-directed curves are defined analogously.*

Causal curves are further classified in null and timelike (in analogy to manifolds). Specifically, if all pairs of points $\gamma(s), \gamma(t)$, with $s < t$, comply with the relation $\gamma(s) \ll \gamma(t)$, then the curve is called *future-directed timelike* (analogously for *past-directed timelike*). If a causal curve has no pair of points related with $\ll$-relation, then the curve is called *null*.

In comparison to the theory of length spaces, an equivalent notion of an intrinsic Lorentzian structure is not very easily established. With the causal curves defined the next step in the creation of an analogue of an intrinsic Lorentzian space is the use of a length-measuring formula for Lorentzian pre-length spaces. Using the time separation function, we can define such a formula, which we call the τ-length of a causal curve (Definition 2.24, [KS17]):

Definition 1.3.6. *For $\gamma : [a, b] \to Y$ future-directed causal we define the τ-length of a curve:*

$$L_\tau(\gamma) = \inf\{\sum_{i=0}^{N-1} \tau(\gamma(t_i), \gamma(t_{i+1})) \text{:}\ \ a = t_0 < t_1 < ... < t_N = b\}$$

From the τ-length, in analogy with intrinsic metric spaces, we can provide an intrinsic time separation function:

$$\mathcal{T}(p, q) = \sup\{L_\tau(\gamma) : \gamma \text{ future directed causal from } p \text{ to } q \text{ in } Y\} \quad (1.3.1)$$

Before, we proceed further, in order to understand the meaning and importance of the τ-length formula and connect it with the case of Lorentzian manifolds, we need to consider [KS17], Proposition 2.32, wherein it is stated that for any smooth, strongly causal spacetime (M), the time separation function between two points $p, q \in M$ stems from the length-measuring formula occurring from the metric, for the causally related points, i.e

$$\tau(p=\gamma(a), q=\gamma(b)) = \sup_{\gamma}\{L_g(\gamma);\, L_g(\gamma) = \int_a^b g_{\mu\nu}u^{\mu}u^{\nu}ds,\ \forall \text{ causal curve } \gamma \text{ from } p \text{ to } q\} \tag{1.3.2}$$

where $g_{\mu\nu}$ is the metric, $\gamma : [a,b] \to M$ a causal curve and u^{μ} the tangent vector of γ. If σ is a maximal curve (i.e a geodesic) in M, then:

$$\tau(\sigma(a), \sigma(b)) = L_g(\sigma)$$

which is an equality that follows from Definition 2.33 in [KS17]:

Definition 1.3.7. *Let $(Y, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space. A future-directed causal curve $\gamma : [a,b] \to Y$ is maximal if $L_{\tau}(\gamma) = \tau(\gamma(a), \gamma(b))$, and analogously for past-directed causal curves.*

The maximality of causal curves is a particularly significant constituent of a length space. Two more important constituents that are related to particular properties of causal relations and causal curves in Y are the causal path connectedness and causal closure of a space. On one hand, the importance of causal path connectedness lies in the general property of Lorentzian manifolds to always admit continuous paths between pairs of causally related points. Along these lines, for the causal path connectedness, we have from Definition 3.1 in [KS17]:

Definition 1.3.8. *A Lorentzian pre-length space $(Y, d, \ll, \leq, \tau)$ is called causally path-connected if for all $x, y \in Y$ with $x \ll y$, there is a future-directed timelike curve from x to y and for $x \leq y$ there is a future-directed causal curve from x to y.*

On the other hand, the causal closure is an important property that gives us the continuity of the causal relations, which will be shown to be rather useful on many occasions, when trying to deal with particular sequences of curves, while it is also a very useful property of a space, when demanding a local structure of a space close to that of a Lorentzian manifold. Therefore, from Definition 3.4 in [KS17]:

Definition 1.3.9. *Let $(Y, d, \ll, \leq, \tau)$ be a Lorentzian pre-length space and let $x \in Y$. A neighbourhood U of x is called causally closed if $\leq$ is closed in $\bar{U} \times \bar{U}$, i.e., if $p_n, q_n \in U$ with $p_n \leq q_n$ for all $n \in N$ and $p_n \to p \in \bar{U}$, $q_n \to q \in \bar{U}$, then $p \leq q$. A Lorentzian pre-length space $(Y, d, \ll, \leq, \tau)$ is called locally causally closed if every point has a causally closed neighbourhood.*

At this point and before we conclude this section negotiating the global structure of a causal space, we give a much-coveted property for a Lorentzian space/manifold that is related to the exclusion of the unwanted phenomenon of causal bubbling, whereby there are causally related points in the interior of the region $J(p) \setminus I(p)$. This is called the push-up property and it is described in the following way ([KS17], Lemma 2.10):

Definition 1.3.10. *For a causal space $(Y, \ll, \leq)$ if for $p \ll q \leq z$ or $p \leq q \ll z$ we get $p \ll z$, then we say that Y has the push-up property.*

With all the definitions that preceded we have introduced the basic global causal structure of a space, that we called a Lorentzian pre-length space. Essentially, these are the necessary global features that can be used to generalise a Lorentzian manifold. In order to sum up the global structure of such a generalised space, we use the background metric d that describes the topology and we define a quantity named the d-arclength of a causal curve γ, which is denoted as $L_d(\gamma) = \sup\{\sum_{i=0}^{N-1} d(\alpha(t_i), \alpha(t_{i+1})) \colon \ a = t_0 < t_1 < ... < t_n = b\}$ and we use it, in order to restrict the topology of a Lorentzian pre-length space in such a way that we retrieve some very frequently encountered properties of Lorentzian manifolds ([KS17], Definition 2.35):

Definition 1.3.11. *Let Y be a causal space. Then Y is:*

- *non-totally imprisoning if for every compact set $K \Subset Y$ there is a $C > 0$ such that the d-arclength of all causal curves contained in K is bounded by C,*
- *strongly causal if the Alexandrov topology $\mathcal{A}$ agrees with the metric topology $\mathcal{D}$ (and hence also with the chronological topology $\mathcal{I}$), and*
- *globally hyperbolic if $(Y, d, \ll, \leq, \tau)$ is non-totally imprisoning and for every $p, q \in Y$ the set $J^+(p) \cap J^-(q)$ is compact in Y.*

With Definition 1.3.11, we conclude the main aspects of the global causal structure of a space that generalises the main and fundamental properties of Lorentzian length spaces, which we want to provide a generalisation of the corresponding manifolds. In order though to properly perform this generalisation, we need to establish some very particular local structure. In order to do so, we need to establish the notion of localisability of a Lorentzian pre-length space ([KS17], Definition 3.16):

Definition 1.3.12. *A Lorentzian pre-length space $(Y, d, \ll, \leq, \tau)$ is called localisable if $\forall x \in Y$ there is an open neighbourhood Ω_x of x in Y with the following properties:*

1. *There is a $C > 0$ such that $L_d(\gamma) \leq C$ for all causal curves γ contained in Ω_x (Y is d-compatible).*
2. *There is a continuous map $\omega_x : \Omega_x \times \Omega_x \to [0, \infty)$, such that $(\Omega_x, d_{\Omega_x \times \Omega_x}, \ll_{\Omega_x \times \Omega_x}, \leq_{\Omega_x \times \Omega_x}, \omega_x)$ is a Lorentzian pre-length space with the following non-triviality condition: For every $y \in \Omega_x$ we have $I^{\pm}(y) \cap \Omega_x \neq \emptyset$.*
3. *For all $p, q \in \Omega_x$ with $p \leq q$, there is a future-directed causal curve $\gamma_{[p,q]}$ from p to q that is maximal in Ω_x and satisfies:*

$$L_\tau(\gamma_{[p,q]}) = \omega_x(p, q) \leq \tau(p, q) \tag{1.3.3}$$

(The curve $\gamma_{[p,q]}$ being maximal in Ω_x means that for every other future-directed causal curve λ, connecting p and q with image contained in Ω_x, we have that $L_\tau(\gamma_{[p,q]}) \geq L_\tau(\lambda)$.)

If we combine now, the definitions 1.3.11 and 1.3.12, as long as the definition for the τ-length, we obtain the following proposition ([KS17], Proposition 3.17):

Proposition 1.3.13. *Let $(Y, d, \ll, \leq, \tau)$ be a strongly causal and localizable Lorentzian pre-length space. Then, L_τ is upper semicontinuous, i.e., if $(\gamma_n)_n : [a, b] \to Y$ is a sequence of future-directed causal curves, converging uniformly to a future-directed causal curve $\gamma : [a, b] \to Y$, then:*

$$L_\tau(\gamma) \geq \limsup_n L_\tau(\gamma_n) \tag{1.3.4}$$

Relying on the definitions above concerning the causal path connectedness and closure, along with the localisability, we can give the definition of a Lorentzian length space as in [KS17]:

Definition 1.3.14. *If $(Y, d, \ll, \leq, \tau)$ is a Lorentzian pre-length space and additionally:*

- *locally causally closed*
- *causally path connected*
- *localisable*
- *and $\mathcal{T}(p, q) = \tau(p, q)$*

then Y is a Lorentzian length space.

Definition 1.3.14 offers an analogue of length spaces in metric geometry. From (1.3.1) we see that (considering that the time separation function plays the role of the metric function in metric geometry) the internal geometry of the space Y is connected to the equivalent of the metric in the Lorentzian setting. In addition, from (1.3.1) we see that the existence of Lorentzian length spaces is interconnected to the notion of maximal curves. This implies that even in the case that they don't exist directly in the space, they serve as a limit for the length of causal curves in it. In the case they do exist, we get the equivalent notion of geodesic for a Lorentzian length space:

Definition 1.3.15. *A Lorentzian pre-length space $(Y, d, \ll, \leq, \tau)$ is called geodesic if for all $x, y \in Y$ with $x \leq y$ there is a future-directed causal curve γ from x to y with $\tau(x, y) = L_\tau(\gamma)$ (hence maximizing).*

The existence of geodesics in a length space can be associated with global hyperbolicity ([KS17], Theorem 3.30), which is an important constituent of any space the geometry of which is of physical meaning, since global hyperbolicity in its turn is associated with the existence of temporal functions.

The main topic of study in this thesis is the warped product spaces. As it was mentioned earlier, a seminal piece of work for these spaces is [AGKS19]. One of the most important results in it is that there are no pathologies in the causal structure of warped product spaces with one dimensional base and in particular they exhibit the push-up property, which leads in its turn to a strong causally connected space. Moreover, these spaces are shown in [AGKS19] to have global hyperbolicity. Spacetimes of low regularity (below Lipschitz) can exhibit the phenomenon of causal bubbling, as shown in [CG11] for spacetimes with continuous metrics. However, the additional structure of a warped product space excludes such pathology.

1.4 The Schwarzschild metric

Before any definitions for the geometrical description of our warped product space, it is useful to understand the motivation that led to such a consideration of space construction. Such a motivation is provided by the product spaces with one-dimensional base (cones) on metric spaces. These spaces have a lot of interest, as several of their features and properties are expected to be similar or require similar treatment to Lorentzian product spaces with one-dimensional base. In chapter 2 of [AGKS19], we see such an example, where the Minkowski cone is introduced through a quotient space structure of $[0, \infty) \times X$, resulting from identifying all points of the form $(0, p)$. This cone, denoted as $Y = Cone(X)$, is also equipped with the cone metric d_c as in [BBI01], Def. 3.6.16. This structure, following the steps of chapter 2 in [AGKS19], is shown to be capable to describe a $1 + n$ dimensional spacetime, with the one-dimensional base corresponding to the time component and thus the overall space being capable to support causal structure, alongside an appropriate time separation function (equation (1), [AGKS19]). However, the Minkowski cone is a very simple and special case of our physical world. There are other metrics in the neighbourhoods of the universe that have a more rich content, first and foremost in the presence of gravitational field. A very well-known and established toy model for a spacetime with gravity is that of Schwarzschild spacetime, i.e the spacetime the metric of which describes the presence of a gravitational field with a rotational symmetry in the vacuum spacetime:

$$-d\tau^2 = -(1 - \frac{r_S}{r})dt^2 + \frac{1}{1 - \frac{r_S}{r}}dr^2 + r^2 d\Omega^2 \tag{1.4.1}$$

where the metric is taken in the usual Schwarzschild coordinates t, r, θ, ϕ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. In what follows, we will focus only on the Schwarzschild spacetime for $r > r_S$. This subspace of the Schwarzschild spacetime can be considered as a subspace of:

$$Z = \mathbb{R} \times (0, \infty) \times S^2 \tag{1.4.2}$$

with S^2 being the 2-sphere, $\mathbb{R}$ the set of real numbers corresponding to the coordinate of time and $(0, \infty)$ stands for the set where the radial coordinate takes its values. Now, having identified the split in (1.4.2), we want to consider the other equivalent decomposition of the Schwarzschild spacetime, meaning as a spacetime with one dimensional base in the form of $\mathbb{R} \times X'$, so that we take advantage of the (variational) length-measuring formula suggested in Definition 3.9, [AGKS19] and motivate the discussion of the creation of a corresponding (variational) length-measuring formula for the case of a base with more structure.

From the split in (1.4.2), by identifying the base B with $\mathbb{R} \times (0, \infty)$, we can obtain the time separation function for it, denoted as τ_b, which will essentially correspond to:

$$\tau_b = L_g(\sigma) \tag{1.4.3}$$

where $\sigma : [a, b] \to \mathbb{R} \times (0, \infty)$ is a geodesic in $B = \mathbb{R} \times (0, \infty)$, defined on the interval $[a, b]$ and L_g is given by:

$$L_g(\sigma) = \int_a^b \sqrt{g_{\mu\nu} u^\mu u^\nu} ds \tag{1.4.4}$$

with $g_{\mu\nu}$ the metric on B, $\mu, \nu = 0, 1$ and u^μ the tangent vector on the curve σ. Now, for a general causal curve $\gamma = (\alpha, \beta) : [a, b] \to Z$, where α is the projection of γ to B and β the projection of γ to S^2, by setting $d\Omega$ as the metric on S^2, $d_{S^2}(\beta(s), \beta(s')) = d\Omega^2$ and using also (1.4.3), we get the expression:

$$\Pi_\gamma(\gamma(s), \gamma(s')) := \tau_b^2(\alpha(s), \alpha(s')) - r_m^2 d_{S^2}^2(\beta(s), \beta(s')) \tag{1.4.5}$$

where $r_m = \min(r(s) : s \in [a, b])$ and $s, s' \in [a, b] \subset \mathbb{R}$ and the components (α, β) are also described by the corresponding coordinates in the Schwarzschild space, given by $\alpha(s) = (t, r)$, $\beta(s) = (\theta, \phi)$. However, from (1.4.2) we see that Schwarzschild spacetime for $r > r_S$ is a subspace of Z, which can also be interpreted as a Lorentzian space with one dimensional base $\mathbb{R}$ and a fibre $X' = (0, \infty) \times S^2$, which has a metric $d_{X'}$ that can be given (in infinitesimal form) by:

$$dl^2 = \frac{1}{1 - \frac{r_S}{r}} dr^2 + r^2 d\Omega^2 \tag{1.4.6}$$

Therefore, by setting $\beta' = (r, \beta) : [a', b'] \to X'$, we reparametrise γ as $\gamma' = (h, \beta') : [a', b'] \to Z$ and we choose:

$$h(\rho) := \int_{a'}^{\rho} \sqrt{(1 - \frac{r_S}{r(\tilde{\rho})})\frac{dt}{d\tilde{\rho}}} d\tilde{\rho} \tag{1.4.7}$$

Consequently, this way, we have managed to view the original space Z as a Lorentzian space with one-dimensional base and thus we can use the (variational) length-measuring tool in [AGKS19] (subroot expression in Definition 3.9):

$$\Pi'_{\gamma}(\gamma'(\rho), \gamma'(\rho')) := (h(\rho') - h(\rho))^2 - d^2_{X'}(\beta'(\rho), \beta'(\rho'))$$

Hence, by taking two appropriate values for each parametrisation γ, γ', i.e s, s' and ρ, ρ', we can write:

$$\sqrt{\Pi_{\gamma}(\gamma(s), \gamma(s'))} = \sqrt{\Pi'_{\gamma}(\gamma'(\rho), \gamma'(\rho'))} \tag{1.4.8}$$

where $\rho, \rho', \tilde{\rho} \in [a', b'] \subset \mathbb{R}$ (here we note that negative values of h correspond to past points in time and a negative derivative of h implies past-directed curve, while a positive derivative of h implies a future-directed curve). From Lemma 3.11 of [AGKS19], we know that the right hand side of (1.4.8) obeys the main properties of the time separation function (in that it complies with the reverse triangle inequality and it is non-negative), which leads the left hand side to comply with the same properties (later we will prove this in detail and show these properties for the left hand side by employing a similar concept in creating comparison triangles between a general spacetime and a model space). Moreover, the right hand side of (1.4.8) is used in [AGKS19] to obtain a variational length formula for an 1-dimensional base Lorentzian space, which is also shown in Proposition 3.14 to agree with the actual length of the curve. Hence, we are motivated here to consider an expression of the form given in (1.4.5) in order to derive a length-measuring formula for the case that our space has a higher dimensional base.

2

Warped product pre-length space

2.1 Outline of the new space: Topology and lengths

The formula given in (1.4.5), which is introduced as a measuring tool for the Schwarzschild spacetime, motivates us to generalize its use as a measuring tool for a more general structure of a warped product space. The first step in understanding this space is to specify its topology. In this research we are interested in obtaining a length space analogue for the warped product geometry that comes from (1.4.5). Basically this implies that we want to abandon the condition of smoothness of our spacetime. In such cases of low-regularity, the product topology remains the same. Consequently, from the product split of the Schwarzschild spacetime it is natural to consider a generalized base for the new space that we want to construct in the form of a Lorentzian length space and similarly for the fibre we initially allow it to be merely a metric space, with the intention to investigate what further properties are required for these two constituent spaces. In what follows these notions, along with the ones explicitly associated with the existence of such spaces, will be used repeatedly.

Since there are a few notions that are used in several parts throughout the following sections, before we introduce the topology of the new space, we begin by introducing some notation that should be considered to have global effect in anything that follows. In particular, we denote the base of the new space with $(B, d_B, \ll, \leq, \tau_b)$, where B is for a Lorentzian length space that has a metric (d_B) describing its topology, a pre-order $\leq$ and a transitive relation $\ll \subset \leq$, which describe the causal relations between any two pairs of points in B. Moreover, the space B has a time separation function denoted as τ_b. Similarly for the fibre, we have a space X, that will always be assumed to be at least a metric space, with the metric being denoted as d_X. Moreover, when we define a function f, we will write I for the interval where f is valued, meaning $I = (0, +\infty)$, unless stated otherwise. Finally, when we want to refer to the causal future/past of a point $\bar{p} \in B$ we write $J_B^{\pm}(\bar{p})$, whereas for the timelike future/past we write $I_B^{\pm}(\bar{p})$.

Having established a given notation and by having motivated the use of our selected topology for the new space, we define the following:

Definition 2.1.1. *For (X, d_X) a metric space and $(B, d_B, \ll, \leq, \tau_b)$ a Lorentzian length space, $Y := B \times X$ is defined to be the product space with the metric:*

$$d(x, x') = \sqrt{d_B(\bar{x}, \bar{x'})^2 + d_X(\tilde{x}, \tilde{x}')^2} \tag{2.1.1}$$

for $x = (\bar{x}, \tilde{x}), x' = (\bar{x}', \tilde{x}') \in Y$

Adding a few extra notations to the globally used ones, as introduced above, from now on any time we write Y, we imply a product space given by the Definition 2.1.1. Moreover, any point $p \in Y$ will have a projection $\bar{p}$ on B and $\tilde{p}$ on X and will be written as $p = (\bar{p}, \tilde{p})$. Additionally, in what follows we will very often require the projection of the curve γ onto the base B and the fibre X. For this purpose we introduce here two more functions that project the curve to the corresponding spaces and in what follows, their operation shall be implied whenever a projection of a curve in Y to either B or X is mentioned. Specifically, we have that a curve γ is projected to B via the function π_B, or onto X via the function π_X and we write that:

$$\pi_B \circ \gamma = \alpha,$$

$$\pi_X \circ \gamma = \beta$$

The definition of the above topology serves as the instigating factor to establish a measuring tool for this kind of topology and which encapsulates the equation for the Schwarzschild metric as a special case. In order to attain this, we reckon with the equivalent reasoning behind the simple and well-known smooth spacetime models for warped product spaces, like the

FRWL spacetime or like models with lower regularity as in [AGKS19]. In both cases the main idea remains the same: construct a space admitting the topology $B \times X$, with B a Lorentzian length space of dimension $n \geq 1$ and X a metric space of dimension $m \geq 1$. This kind of structure, for $m, n \geq 2$ resembles the Schwarzschild metric split described above. Therefore, an intutitive first step towards generalising the (smooth) Schwarzschild case is to allow the base to be a Lorentzian length space and the fibre a general metric space. In this case, the tools that become immediately available are the time separation function for the base, the warping function f and the metric for the fibre. Hence, combining these three, for two points $\gamma(s_1), \gamma(s_2)$ on a curve $\gamma : [a, b] \to B \times X$, for $s_1, s_2 \in [a, b]$ and by denoting as m_{s_1,s_2} the minimum value of $f \circ \alpha$ in the interval $[s_1, s_2] \subseteq [a, b]$, we introduce the following expression:

$$\Psi_\gamma(\gamma(s_1), \gamma(s_2)) := \tau_b(\alpha(s_1), \alpha(s_2))^2 - m^2_{s_1,s_2} d^2_X(\beta(s_1), \beta(s_2)) \tag{2.1.2}$$

From the form of relation (2.1.2), we see that it serves as a generalization of (1.4.5), under the assumption that the base and the fibre need not be smooth spaces any more. However, in (1.4.5) (as it corresponds to the metric of Lorentzian length space) there is a certain causal structure that needs to be respected in defining a length measuring formula for the new topology given in Definition 2.1.1. To do so we use the following definition:

Definition 2.1.2. *For (X, d_X) a metric space, $(B, d_B, \ll, \leq, \tau_b)$ a Lorentzian length space, Y a space that has the product topology defined in 2.1.1 and a function $f : B \to (0, +\infty)$, a curve $\gamma : [a, b] \to Y$, s.t the projection of γ to B is a causal curve α and the projection to the fibre a curve β, there is a function P, which is defined to be the map:*

$$P : \gamma_{[s_1,s_2]} \to \mathbb{R} \tag{2.1.3}$$

which for two points $\gamma(s_1), \gamma(s_2)$, with $s_1, s_2 \in [a, b]$ and $\alpha(s_1) \leq \alpha(s_2)$, P is given by:

$$P(\gamma(s_1), \gamma(s_2)) = \begin{cases} \sqrt{\Psi_\gamma(\gamma(s_1), \gamma(s_2))} & \textit{if } \Psi_\gamma(\gamma(s_1), \gamma(s_2)) \geq 0 \\ \mathcal{C} & \textit{if } \Psi_\gamma(\gamma(s_1), \gamma(s_2)) < 0 \end{cases} \tag{2.1.4}$$

where $\mathcal{C} < 0$.

In what follows, whenever we refer to P we imply that P is specifically defined for a given curve γ in Y, as it is established by Definition 2.1.2. Later on, when we will have described the geometrical structure of our space and we will have constructed a generalised warped product space, equation (2.1.4) can be applied to any pair of points in Y, not necessarily on a causal curve.

In building a length-measuring formula for space Y, we have to reproduce the basic local properties of relation (1.4.5) for the Schwarzschild spacetime. A main property that we need to reckon with is that all points that are space-like related to each other have a negative value of (1.4.5) (in what follows these are termed as non-causal). The form of (2.1.4) gives us a good reason to perceive it as a tool similar to the local length for a curve γ in Y, as it seems to differentiate points with negative and positive values of it and it provides a direct analogy of (1.4.5) in the case Ψ_γ is differentiable and f is constant. However, in a case of non-differentiability, $P(\gamma(s_1), \gamma(s_2))$ would need to be calculated using an alternative way. Hence, the only method that we can naturally assume, in complete analogy to the metric case, is by considering a partition of $\gamma : [a, b] \to B \times X$ in smaller intervals and taking the infimum of P over all these possible partitions (hereafter any given partition of γ implies $\{s_0, s_1, s_2, ..., s_N; s_i \in [a, b], i = 0, 1, 2, ..., N\}$). Hence, using the tools that we have at our disposal, meaning the time separation function τ_b, the metric d_X, the function f and the Definition 2.1.4 that occurs from them, we get that the most general form for the length measuring formula could be:

Definition 2.1.3. *Let there be a metric space (X,d_X) and a Lorentzian length space $(B, d_B, \ll, \leq, \tau_b)$. Let there also be a space Y, with the product topology $B \times X$, as well as a continuous function $f : B \to I = (0, \infty)$. Then, for a curve $\gamma = (\alpha, \beta) : [a, b] \to Y$, such that for $s_i, s_{i+1} \in [a, b]$, $i \in [0, N-1] \subset \mathbb{N}$ and α a causal curve with $\alpha(s_i) \leq \alpha(s_{i+1})$, the variational length functional of γ is defined as:*

$$L_{var}(\gamma) = \inf\{ \sum_{i=0}^{N-1} P(\gamma(s_i), \gamma(s_{i+1})) \} \tag{2.1.5}$$

This formula is an analogue of the τ-length and the notion of rectifiability it entrains for the causal curves. However, a distinction needs be established at this point in that the rectifiability of our curve is wrt the

$P(\gamma(s_i), \gamma(s_{i+1}))$ expression in (2.1.5), which looks into the rectifiability of the projection of the curve into the base and the fibre separately. It is not a rectifiability wrt a time separation function in Y, which is the usual meaning of rectifiability for causal curves, as described in Definition 2.24, [KS17] and which is the direct and closest analogue of rectifiability of metric geometry. The only case where any notion of rectifiability wrt $P(\gamma(s_i), \gamma(s_{i+1}))$ is redundant is if α is a null curve, in which case though we know that the length of the curve is 0 if γ projects to a point in X and $\mathcal{C}$ if any section of γ projects to a curve is X.

One more reason that motivates the use of τ_b in the proposed measuring formula is that since the base is a Lorentzian space, we want a function that in case the curve γ is reduced to the base, the measuring tool adequately describes the causal relations. For this reason the time separation function is the only pre-existing tool and thus serves as an obvious candidate to be involved somewhere in the definition of a measuring tool. However, the question is whether to use the lengths (or τ-lengths) of curves that project to the base or just simply the time separation function. For simplicity, we consider the latter, however as we will show later (with the different equivalent length-measuring formulas) it doesn't affect the theory, as basically the additional extent of accuracy in measuring a length is rendered redundant by the nature of the formula. Therefore, we conclude that by taking the time separation function for the base and the metric distance for the fibre X, we can construct a measuring tool for the lengths of curves in a space Y. The exact structure and geometrical properties of this space will only become apparent later in this thesis, but the only way to fathom it for now, is that with this space Y (with the topology given in 2.1.1 and the formula (2.1.5), that calculates lengths in it) we intend to provide a generalization of the Schwarzschild spacetime to encapsulate the cases the base B and the fibre X, or one of them, need not be regular, or if the space has no Lorentzian metric.

We continue our analysis by checking some basic properties for $P(p, q; \gamma)$ in (2.1.4), that provide an interpretation of function $P(p, q)$ as the equivalent to the distance function of a metric space. In particular, we have the following lemma:

Lemma 2.1.4. *Let $\gamma = (\alpha, \beta)$ be a curve defined on the interval $[a, b] \to Y$, with Y having the topology of $B \times X$, $\alpha : [a, b] \to B$ and $\beta : [a, b] \to X$. Then for $s_1 < s_2 < s_3 \in [a, b]$ and for three points $\gamma(s_1), \gamma(s_2), \gamma(s_3)$, that* *satisfy* $\alpha(s_1) \leq \alpha(s_2) \leq \alpha(s_3)$*, we get that:*

$$P(\gamma(s_1), \gamma(s_3)) \geq P(\gamma(s_1), \gamma(s_2)) + P(\gamma(s_2), \gamma(s_3)) \tag{2.1.6}$$

Proof. Before we start proving the claim, in order to be assisted in the analysis below, we choose a more simplified notation. Specifically, we set $\gamma(s_1), \gamma(s_2), \gamma(s_3) = r, q, z$, respectively and $r = (\alpha(s_1) = \bar{r}, \beta(s_1) = \tilde{r}), q = (\alpha(s_2) = \bar{q}, \beta(s_2) = \tilde{q}), z = (\alpha(s_3) = \bar{z}, \beta(s_3) = \tilde{z})$.

From the Definition 2.1.2 above, we imply that (2.1.6) is considered for a given curve, on which we choose the points r, q, z. This fact facilitates us in handling the minimum of the warping function f along the interval $[s_1, s_3]$. If for this curve we have:

$$\tau_b(\bar{r}, \bar{z}) = \tau_b(\bar{r}, \bar{q}) + \tau_b(\bar{q}, \bar{z}) \tag{2.1.7}$$

then this is called a maximal curve in the base according to Definition 2.33, [KS17]. In this case the proof for the reverse triangle inequality is the same as in the one dimensional case Lemma 3.11 of [AGKS19] and which is a subcase of the method we use below.

For the general case we are not on a maximizer in B, we proceed in a different manner, similar to the corresponding case in Lemma 3.11, [AGKS19]. First, we consider the Minkowski spacetime (M^2), its metric:

$$-ds^2 = -dt^2 + dx^2 \tag{2.1.8}$$

and three points in it $\bar{R}, \bar{Q}, \bar{Z}$. Then we take a comparison triangle $\triangle_{\bar{R},\bar{Q},\bar{Z}}$ in M^2 for $\bar{r}, \bar{q}, \bar{z} \in B$, s.t $\tau_b(\bar{r}, \bar{q}) = \tau(\bar{R}, \bar{Q})$, $\tau_b(\bar{q}, \bar{z}) = \tau(\bar{Q}, \bar{Z})$, $\tau_b(\bar{r}, \bar{z}) = \tau(\bar{R}, \bar{Z})$, where we denote the time separation function on M (which coincides with the timelike part of the metric for a Minkowski spacetime) as τ. Additionally, we set that $t(\bar{r}, \bar{q}) = \tau_b(\bar{r}, \bar{z}) - \tau_b(\bar{q}, \bar{z})$, while also $c = \min(m_{\bar{r},\bar{q}}, m_{\bar{q},\bar{z}})$. We then take the 2-dimensional Euclidean space (R^2) and a comparison triangle $\triangle_{\tilde{R},\tilde{Q},\tilde{Z}}$ in it for $\tilde{r}, \tilde{q}, \tilde{z} \in X$, s.t $d_X(\tilde{r}, \tilde{q}) = d_{R^2}(\tilde{R}, \tilde{Q})$, $d_X(\tilde{q}, \tilde{z}) = d_{R^2}(\tilde{Q}, \tilde{Z})$, $d_X(\tilde{r}, \tilde{z}) = d_{R^2}(\tilde{R}, \tilde{Z})$, where $\tilde{R}, \tilde{Q}, \tilde{Z} \in R^2$. From the two comparison spaces, we can construct a new warped product space $M^2 \times_c R^2$. This space is an anisotropic FLRW-like spacetime, which is explained more in [MS11]. However, M^2 can be split into the space $I_1 \times I_2$ and hence the space $M^2 \times_c R^2$ is isomorphic to the total space $I_1 \times (I_2 \times_c R^2)$, which from Corollary 4.9 of [AGKS19] is a Lorentzian length space. The time separation function T for $I_1 \times (I_2 \times_c R^2)$

is given by the timelike part of the warped product metric (as the latter is introduced in equation (2.3) of [MS11]). Therefore, taking two points $\Pi \in Y$, $\Pi' \in I_1 \times (I_2 \times_c R^2)$ and considering that there is a mapping $\mathcal{H}$:

$$\mathcal{H} : \Pi = (\bar{\Pi}, \tilde{\Pi}) \to \Pi' = (t, \tilde{\Pi}') = (t, x, y, z) \tag{2.1.9}$$

with t, x, y, z having the usual meaning of the coordinates of each point in $I_1 \times (I_2 \times_c R^2)$, the time separation function in $I_1 \times (I_2 \times_c R^2)$ will be:

$$T(\Pi_i', \Pi_j') = \sqrt{\Delta t^2 - (\Delta x^2 + c^2(\Delta y^2 + \Delta z^2))} \tag{2.1.10}$$

where $\Pi_i', \Pi_j' = (\bar{R}', \tilde{R}'), (\bar{Q}', \tilde{Q}'), (\bar{Z}', \tilde{Z}') \in I_1 \times (I_2 \times_c R^2)$ and $(\Delta t, \Delta x, \Delta y, \Delta z) = \mathcal{H}(\Pi_j) - \mathcal{H}(\Pi_i)$.

Reckoning with the triangle inequality for $M^2 \times_c R^2$, we have that:

$$\begin{aligned}
&\sqrt{\tau_b(\bar{r}, \bar{q})^2 - m_{\bar{r},\bar{q}}^2 d_X^2(\tilde{r}, \tilde{q})} + \sqrt{\tau_b(\bar{q}, \bar{z})^2 - m_{\bar{q},\bar{z}}^2 d_X^2(\tilde{q}, \tilde{z})} \\
\leq &\sqrt{\tau_b(\bar{r}, \bar{q})^2 - m_{\bar{r},\bar{q}}^2 d_X^2(\tilde{r}, \tilde{q})} + \sqrt{t(\bar{q}, \bar{z})^2 - m_{\bar{q},\bar{z}}^2 d_X^2(\tilde{q}, \tilde{z})} \\
\leq &\sqrt{\tau_b(\bar{r}, \bar{q})^2 - c^2 d_X^2(\tilde{r}, \tilde{q})} + \sqrt{t(\bar{q}, \bar{z})^2 - c^2 d_X^2(\tilde{q}, \tilde{z})} \\
= &T(R', Q') + T(Q', Z') \leq T(R', Z') = \sqrt{\tau_b(\bar{r}, \bar{z})^2 - m_{\bar{r},\bar{z}}^2 d_X^2(\tilde{r}, \tilde{z})}
\end{aligned} \tag{2.1.11}$$

therefore we have proven (2.1.6).

□

In what we have described above as a measuring formula for the lengths of curves in Y, we required of course a given parametrization for the curve. Hence, before proceeding further, we could probe a bit more the extent of flexibility around the parametrization of the curve, so that our formula (2.1.5) remains invariant. Specifically, we see that:

Proposition 2.1.5. *The variational length L_{var} is reparametrization invariant, i.e., if we have a curve $\gamma = (\alpha, \beta) : S = [a, b] \to Y$ and in addition, $\phi : S' \to S$ is strictly monotonically increasing and such that ϕ and ϕ^{-1} are continuous, then $L_{var}(\gamma \circ \phi) = L_{var}(\gamma)$.*

Proof. Let $\gamma = (\alpha, \beta) : S = [a, b] \to Y$ and $\sigma : S' = [a', b'] \to Y$, with σ being the reparametrization of γ through $\phi : S' \to S$, i.e $\sigma = \gamma \circ \phi$. Then we have that the partition $\{s'_0, s'_1, s'_2, ..., s'_N; s'_i \in [a', b'], i = 0, 1, 2, ..., N-1\}$ will be mapped to $\{s_0, s_1, s_2, ..., s_N; s_i \in [a, b], i = 0, 1, 2, ..., N-1\}$. This gives:

$$\begin{aligned} L_{var}(\sigma) &\leq \sum_{i=0}^{N-1} P(\sigma(s'_i), \sigma(s'_{i+1})) \\ &= \sum_{i=0}^{N-1} P(\gamma(\phi(s'_i)), \gamma(\phi(s'_{i+1}))) \\ &= \sum_{i=0}^{N-1} P(\gamma(s_i), \gamma(s_{i+1})) \end{aligned} \tag{2.1.12}$$

If we take the infimum over both sides of the above equation we get that $L(\sigma) \leq L_{var}(\gamma)$. If we start with the length of γ instead, the same arguments as in the above equation give us that $L_{var}(\gamma) \leq L_{var}(\sigma)$, hence $L_{var}(\sigma) = L_{var}(\gamma)$. □

After having defined a formula for measuring the lengths of a curve and a given topology for our space, the next step is to check its compatibility with known special cases that it describes. Specifically, our proposed length-measuring formula, provides a general tool for spaces that are either regular (smooth) or non-regular. Hence, it is necessary for the validity of our measuring tool that it is compatible with the case of smooth base and fibre spaces, where there is a natural length-measuring formula from the metric occurring in a smooth space(time). Therefore, we proceed by considering (2.1.5) inside a smooth warped product space Y', comprised of a smooth Lorentzian base, B', a smooth metric space X' as fibre and a smooth warping function $f' : B' \to I \subseteq \mathbb{R}^+$, as $Y' = B' \times_{f'} X'$. We will check whether the variational length introduced in (2.1.5) equals the normal formula of integration for the length of an absolutely continuous curve in Y'. In these spaces the time separation function reduces to the usual notion of the length of a geodesic and the latter is differentiable. This fact furnishes the existence of a Lorentz structure function for a curve $\gamma' = (\alpha', \beta')$ as follows:

$$g = \dot{\tau_b}^2 - f' \circ \alpha'(s)^2 u_{\beta'}^2 \tag{2.1.13}$$

where $\dot{\tau_b} = \frac{d\tau_b(s)}{ds}$ and $u_{\beta'} = \frac{d_X(\beta'(s), \beta'(s+ds))}{ds}$, meaning the derivatives of the time separation function (τ_b) and the metric (d_X). In its turn, the structure function g gives rise to a formula for the length of γ' inside the space Y':

$$L = \int_{s_0}^{s_1} \sqrt{\dot{\tau_b}^2(s) - f'^2(s) u^2_{\beta'}(s)} ds \tag{2.1.14}$$

From equation (2.1.14), we can prove now the equivalence of the $L(\gamma)$ and $L_{var}(\gamma)$ in the following proposition:

Proposition 2.1.6. *Let $(B, d_B, \ll, \leq, \tau_b)$ be a smooth Lorentzian manifold and (X, d_X) a Riemannian manifold. Then the variational length L_{var} for an absolutely continuous causal curve $\gamma \in Y$ is equal to its length L.*

Proof. Take an absolutely continuous curve $\gamma' = (\alpha', \beta') \in Y$ and its length for $S' = [a', b']$ being given from (2.1.14). Since, as shown in Corollary 3.7 of [AGKS19] this formula for the length, as well as the formula from (2.1.5) are reparametrization invariant, we can consider an appropriate map $\phi : S' \to S = [a, b]$, as described in Proposition 3.34 of [KS17] such that $\dot{\tau_b} = 1$ (this curve, with $\dot{\tau_b} > 0$, for the purposes of this proof - without loss of generality - will be called future directed causal curve) and that $\gamma' = \gamma \circ \phi$. By utilising the reasoning of the proof for Proposition 3.14 of [AGKS19] we get the following inequality:

$$\begin{aligned} L(\gamma) &= \int_{a'}^{b'} \sqrt{\dot{\tau_b}^2 - (f \circ \alpha')^2 u^2_{\beta'}}\, ds' = \int_a^b \sqrt{1 - (f \circ \alpha)^2 u^2_\beta}\, ds \\ &= \sum_{i=0}^{N-1} \int_{s_i}^{s_{i+1}} \sqrt{\dot{\tau_b}^2 - f^2 u^2_\beta}\, ds \\ &\leq \sum_{i=0}^{N-1} \sqrt{\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - m^2_{s_i, s_{i+1}} d_X(\beta(s_i), \beta(s_{i+1}))^2} \end{aligned} \tag{2.1.15}$$

where we assumed above that $s_0 = a, s_N = b$. By taking the infimum over both the last line and the $L(\gamma)$, we have that: $L(\gamma) \leq L_{var}$. Next, we write the derivatives inside the integral into a variational form, in order to take advantage locally of the inequality written above between $L(\gamma)$ and L_{var}. Hence, we use the fact that $L_{var} \leq \sum_{i=0}^{N-1} P_\gamma(\gamma(s_i), \gamma(s_{i+1}))$. In addition, we consider that $\exists \epsilon$ s.t $0 < \epsilon < b - a$ and we set $b' = b - \epsilon > a$, $h = \frac{b' - a}{N}$, with $h \leq \epsilon$ and $s_i = a + ih$, for $i = 0, 1, 2, ..., N$. Then for a value $s \in [a, b']$,

$\sqrt{h^2 - m^2_{s,s+h} d_X(\beta(s), \beta(s+h))^2} \geq 0$. Consequently:

$$\int_{s_0}^{s_N} \sqrt{\dot{\tau_b}^2 - f^2 u_\beta^2}\, ds = \lim_{h\to 0} \frac{1}{h} \int_{s_0}^{s_N} \sqrt{h^2 - m^2_{s,s+h} d_X(\beta(s), \beta(s+h))^2}\, ds$$
$$= \lim_{h\to 0} \frac{1}{h} \sum_{i=0}^{N-1} \int_{s_i}^{s_{i+1}} \sqrt{h^2 - m^2_{s,s+h} d_X(\beta(s), \beta(s+h))^2}\, ds \tag{2.1.16}$$

Now, we make the substitution $s \to s_i + s'$, with $s' \in [0, h]$ and thus:

$$\begin{aligned}
\int_{s_0}^{s_N} \sqrt{\dot{\tau_b}^2 - f^2 u_\beta^2}\, ds &= \lim_{h\to 0} \frac{1}{h} \sum_{i=0}^{N-1} \int_{s_i}^{s_{i+1}} \sqrt{h^2 - m^2_{s,s+h} d_X(\beta(s), \beta(s+h))^2}\, ds \\
&= \lim_{h\to 0} \frac{1}{h} \int_0^h \sum_{i=0}^{N-1} [h^2 - m^2_{s_i+s, s_{i+1}+s} \\
&\qquad \times d_X(\beta(s_i+s), \beta(s_{i+1}+s))^2]^{\frac{1}{2}}\, ds \\
&\geq \lim_{h\to 0} \frac{1}{h} \int_0^h L_{var}(\gamma_{[\alpha+s, b'+s]}) ds \\
&= \lim_{h\to 0} \frac{1}{h} \int_0^h L_{var}(\gamma) - L_{var}(\gamma_{[a,a+s]}) - L_{var}(\gamma_{[b'+s,b]}) ds \\
&\geq \lim_{h\to 0} \frac{1}{h} \int_0^h L_{var}(\gamma) - L_{var}(\gamma_{[a,a+h]}) - L_{var}(\gamma_{[b-h,b]}) ds \\
&\geq L_{var}(\gamma) - \sqrt{\epsilon^2 - m^2_{b-\epsilon,b} d_X(\beta(b-\epsilon), \beta(b))^2}
\end{aligned} \tag{2.1.17}$$

Now, by taking the limit of $\epsilon \to 0$ we get that $L(\gamma) \geq L_{var}$. Therefore, since $L(\gamma) \geq L_{var} \geq L(\gamma)$, we have that $L_{var} = L(\gamma)$ for our suggested length measuring formula (2.1.5). □

2.2 Causal curves

Having defined a measuring tool for the lengths of curves in Y, the next step is to use it in order to define the causal curves in it, along the lines of the smooth Lorentzian geometry. In the smooth setting, where τ_b is given by the metric of the smooth spacetime that constitutes the base, which metric is also differentiable with derivative $\dot{\tau_b}$ and where the metric of the fibre is differentiable, with derivative v_β, we have from Definition 2.1.3 and the formula (2.1.5), that a causal curve is:

$$\begin{cases} \textit{timelike} \\ \textit{null} \\ \textit{causal} \end{cases} \quad \text{if} \quad -\dot{\tau_b}^2 + (f \circ \alpha)^2 v_\beta^2 \quad \begin{cases} < 0 \\ = 0 \\ \leq 0 \end{cases} \tag{2.2.1}$$

Therefore, as a first step towards defining the causal curves in the space Y, we need to be able to match the picture from (2.2.1) in the non-smooth (length space analogue) setting. To this end, we assume that if there is a curve $\gamma : [a, b] \to B \times X$, then by using the formula for $L_{var}(\gamma)$, the following definition is given:

Definition 2.2.1. *Let $(B, d_B, \ll, \leq, \tau_b)$ be a Lorentzian length space, (X, d_X) a metric space, $f : B \to I$ a continuous function in B and $\gamma : [a, b] \to B \times X$ be a curve, such that its length is given by (2.1.5). Then for all $s \in [a, b]$ and for $\epsilon \in \mathbb{R}^+$, so that $b - s > \epsilon$, if we denote the section of the curve from $\gamma(s)$ to $\gamma(s + \epsilon)$ as $\gamma_{[s,s+\epsilon]}$, we classify γ as:*

$$\begin{cases} \textit{timelike} \\ \textit{null} \\ \textit{causal} \end{cases} \quad \textit{if} \quad L_{var}(\gamma_{[s,s+\epsilon]}) \quad \begin{cases} > 0 \\ = 0 \\ \geq 0 \end{cases} \tag{2.2.2}$$

In what follows when we refer to a causal curve, we imply the causality given by the Definition 2.2.1. This definition just indicates that on a given curve γ, as defined above, any two "neighbouring" points have to be connected with a segment of the curve that has length greater than zero if the curve is timelike, or greater or equal to zero if the curve is causal. This classification of causal curves allows us to connect it with the local behaviour of the expression for P (and its equivalents as shown below) and then reach to some clear connections between causal curves and ordering relations of points in Y.

Moreover, from the definition above we can try to make more detailed connections of causal curves with the length measuring formula for our space Y. Hence, the following lemma ensues:

Corollary 2.2.2. *Given a Lorentzian length space* $(B, d_B, \ll, \leq, \tau_b)$*, a metric space* (X, d_X)*, a space* $Y := B \times X$ *with length in it given by (2.1.5) and a curve* $\gamma : [a, b] \to Y$*, then* γ *is causal if and only if* $P(\gamma(c), \gamma(d)) \geq 0$ $\forall c, d \in [a, b]$*, such that* $a \leq c < d \leq b$ *and* $\alpha(a) \leq \alpha(c) \leq \alpha(d) \leq \alpha(d)$*.*

Proof. We start by considering γ to be a causal curve. In order to be assisted in the analysis of our arguments here, we denote $\gamma(c) = p = (\bar{p}, \tilde{p})$, $\gamma(d) = q = (\bar{q}, \tilde{q})$ and for the section of the causal curve γ from p to q, we write $\gamma_{[p,q]}$. By definition, $L_{var}(\gamma_{[p,q]})$ is greater or equal to zero. Having determined the notation, we proceed by considering a partition $\{t_i : t_i \in [c, d], i = 1, ..., N; t_1 = c, t_N = d\}$ of γ. If the $\gamma_{[p,q]}$ section of the γ partition gives that $P(p, q) < 0$, because of the relation:

$$P(p, q) = \inf\{P(p, q)\} \geq \inf\{P(p, z) + P(z, q); \forall z \in \gamma_{[p,q]}\} \tag{2.2.3}$$

which for the partition $\{t_i : t_i \in [c, d], i = 1, ..., N; t_1 = c, t_N = d\}$ of γ means that:

$$P(p, q) \geq \inf\{\sum_{i=0}^{N-1} P(\gamma(t_i), \gamma(t_{i+1}))\} \tag{2.2.4}$$

we get that $L(\gamma_{[p,q]}) < 0$ as the infimum of the partition, which is a contradiction to the causal character of γ. Note here that if there are any two points that give a negative value for $P(p, q)$, then the correspondence with $L(\gamma)$ in the smooth setting is lost. $\square$

Remark 2.2.3. *From the Corollary 2.2.2 we see that we can capture all causal curves with* $P(p, q) \geq 0$*, for* p, q *on a causal curve* γ*. Therefore, in order to simplify our analysis of causal curves wrt Definition 2.2.1, we can go back to Definition 2.1.2 and equation (2.1.4) and set* $\mathcal{C} = -\infty$ *when* $\Psi_\gamma(\gamma(s_1), \gamma(s_2)) < 0$*.*

From the Remark 2.2.3 and the Corollary 2.2.2 we see that we can capture some connection between the causal character of a curve and the local variational behaviour of the length-measuring formula in (2.1.5) (i.e $P(p, q)$). This connection though is attained through this given formula (2.1.5), the choice of which is assumed based on some reasonable assumptions that we motivated above and moreover, in order to get $P(p, q)$ we use the minimum of $f \circ \alpha$. Therefore, the use of $P(p, q)$ in order to derive any further information for the local causal character of the curve is not possible. However, as we show later, the causal curves classification wrt

Definition 2.2.1 agrees with the causal curves wrt to Definition 1.3.5. This implies that although we build a length-measuring formula following natural some assumptions (i.e the choice of the particular form for $P(p,q)$), we end up with a formula that accurately captures causal relations in our space.

Null curves

As we will see later, there is a second route we can take to define null curves, but for now we focus on the one given in our Definition 2.2.1. According to it, null curves are those that have an L_{var} equal to zero. Before we proceed to check some basic properties of null curves, we give the following definition:

Definition 2.2.4. *Let $(B, d_B, \ll, \leq, \tau_b)$ be a Lorentzian length space, (X, d_X) a length space, $Y := B \times X$, $f : B \to (0, +\infty)$ a continuous function in B and $\gamma = (\alpha, \beta) : [a,b] \to Y$ a causal curve, with L being the d_X-arclength of β, $L(\beta_{[t_1,t_2]}) > 0\ \forall t_1 \leq t_2 \in [a,b]$ and $t_1 = t'_0$, $t_2 = t'_M$. Then, there is a function $\tau_\gamma : \gamma_{[t_1,t_2]} \to R_{\geq 0}$, which is given by:*

$$\tau_\gamma(\gamma; t_1, t_2) = \sup\{\sum_{k=0}^{M-1} m_{t'_k, t'_{k+1}} d_X(\beta(t'_k), \beta(t'_{k+1}))\} \tag{2.2.5}$$

In what follows, unless there is a need to specify, we adopt the notation $\tau_\gamma(t_1, t_2)$, meaning $\tau_\gamma(\gamma; t_1, t_2)$.

Corollary 2.2.5. *$\tau_\gamma(t_1, t_2)$ is additive, meaning that for $t_1 < t_2 < t_3 \in [a,b]$, we have:*

$$\tau_\gamma(t_1, t_3) = \tau_\gamma(t_1, t_2) + \tau_\gamma(t_2, t_3) \tag{2.2.6}$$

Proof. Take a $c \in (a,b)$. Then we know from Definition 2.2.4 that there is one set of partitions for $[a,c]$, that we call $W_{[a,c]}$ and another for $[c,b]$, that we call $W_{[c,b]}$ the union of which is a subset or equal to the set of partitions in $[a,b]$, which we denote as $W_{[a,b]}$. Hence, for an arbitrarily small $\epsilon > 0$ and

$t_n = c$:

$$\begin{aligned}\tau_\gamma(a,b)+\epsilon \geq & \sum_{i=0}^{n} m_{t_i,t_{i+1}} d_X(\beta(t_i),\beta(t_{i+1})) + \frac{\epsilon}{2} \\ & + \sum_{i=n}^{N-1} m_{t_i,t_{i+1}} d_X(\beta(t_i),\beta(t_{i+1})) + \frac{\epsilon}{2} \\ & > \tau_\gamma(a,c)+\tau_\gamma(c,b) \end{aligned} \tag{2.2.7}$$

Moreover

$$\sum_{i=0}^{N-1} m_{t_i,t_{i+1}} d_X(\beta(t_i),\beta(t_{i+1})) \leq \tau_\gamma(a,c)+\tau_\gamma(c,b) \tag{2.2.8}$$

which if we take the supremum of both sides in (2.2.8) becomes:

$$\tau_\gamma(a,b)-\epsilon < \tau_\gamma(a,c)+\tau_\gamma(c,b) \tag{2.2.9}$$

and therefore:

$$\tau_\gamma(a,b)-\epsilon < \tau_\gamma(a,c)+\tau_\gamma(c,b) < \tau_\gamma(a,b)+\epsilon \tag{2.2.10}$$

which gives that:

$$\tau_\gamma(a,b) = \tau_\gamma(a,c)+\tau_\gamma(c,b) \tag{2.2.11}$$

since ϵ is arbitrarily small. □

An additional step that is needed before the discussion about null curves is the following two equivalent formulas for (2.1.5):

Corollary 2.2.6. *For (X, d_X) a length space, $(B, d_B, \ll, \leq, \tau_b)$ a Lorentzian length space, $Y := B \times X$, in which lengths of curves are given by (2.1.5), a continuous function $f : B \to (0, +\infty)$ and $\Psi_\gamma \geq 0$, the length of a causal curve $\gamma : [a, b] \to Y$ is also equal to:*

$$L(\gamma) = \inf\{\sum_{i=0}^{N-1} \sqrt{L_{\tau_b}(\alpha_{[t_i,t_{i+1}]})^2 - m^2_{t_i,t_{i+1}} d_X(\beta(t_i),\beta(t_{i+1}))^2}\} \tag{2.2.12}$$

where $N \in \mathbb{N}$, $i \in [0, N-1]$, $t_i, t_{i+1} \in \{a = t_0 < t_1 < \cdots < t_N = b\}$.

Proof. We take the length formula given in (2.1.5). Moreover, we take a partition ζ of every interval $[t_i, t_{i+1}]$:
$\zeta = \{t'_{i,k} : t_i = t_{i,0} < t_{i,1} < \cdots < t_{i,M} = t_{i+1}; M \in \mathbb{N}, k \in [0, M-1]\}$. We know from the Definition 1.3.6 (see also 2.24 in [KS17]) that there is always a partition with given M, N for which we can have a value $\epsilon_i > 0$, arbitrarily small, s.t $L_{\tau_b}(\alpha_{[t_i,t_{i+1}]}) + \epsilon_i > \tau_b(\alpha(t_i), \alpha(t_{i+1})) > \sum_{k=0}^{M-1} \tau_b(\alpha(t'_{i,k}), \alpha(t'_{i,k+1}))$. Therefore, taking also into account that $P(\gamma(t_i), \gamma(t_{i+1})) > 0$, we get:

$$\begin{aligned}
L(\gamma) =& \inf\{\sum_{i=0}^{N-1} \sqrt{\tau_b(\alpha(t_i), \alpha(t_{i+1}))^2 - m^2_{t_i,t_{i+1}} d_X(\beta(t_i), \beta(_{i+1}))^2}\} \\
\geq& \inf\{\sum_{i=0}^{N-1} [\ \sum_{k=0}^{M-1} \tau_b(\alpha(t'_{i,k}), \alpha(t'_{i,k+1}))\ ]^2 - m^2_{t_i,t_{i+1}} \\
& \qquad \times d_X(\beta(t_i), \beta(t_{i+1}))^2\ ]^{\frac{1}{2}}\} \\
=& L(\gamma)_L
\end{aligned} \tag{2.2.13}$$

Moreover:

$$\begin{aligned}
L(\gamma) =& \inf\{\sum_{i=0}^{N-1} \sqrt{\tau_b(\alpha(t_i), \alpha(t_{i+1}))^2 - m^2_{t_i,t_{i+1}} d_X(\beta(t_i), \beta(_{i+1}))^2}\} \\
\leq& \inf\{\sum_{i=0}^{N-1} \sqrt{(L_{\tau_b}(\alpha_{[t_i,t_{i+1}]}) + \epsilon_i)^2 - [m_{t_i,t_{i+1}} d_X(\beta(t_i), \beta(t_i))]^2}\} \\
=& L(\gamma)_U
\end{aligned} \tag{2.2.14}$$

However:

$$\begin{aligned}
L(\gamma)_U =& \inf\{\sum_{i=0}^{N-1} \sqrt{(L_{\tau_b}(\alpha_{[t_i,t_{i+1}]}) + \epsilon_i)^2 - [m_{t_i,t_{i+1}} d_X(\beta(t_i), \beta(t_i))]^2}\} \\
=& \inf\{\sum_{i=0}^{N-1} [\ \sum_{k=0}^{M-1} \tau_b(\alpha(t'_{i,k}), \alpha(t'_{i,k+1}))\ ]^2 - m^2_{t_i,t_{i+1}} \\
& \qquad \times d_X(\beta(t_i), \beta(t_{i+1}))^2\ ]^{\frac{1}{2}}\} \\
=& L(\gamma)_L
\end{aligned} \tag{2.2.15}$$

Consequently, since $L(\gamma)_U = L(\gamma)_L$:

$$L(\gamma) = \inf\{\sum_{i=0}^{N-1} \sqrt{\ L_{\tau_b}(\alpha_{[t_i,t_{i+1}]})^2 - m^2_{t_i,t_{i+1}} d_X(\beta(t_i), \beta(t_{i+1}))^2}\} \tag{2.2.16}$$

and thus we have proven the claim. $\square$

As an immediate consequence of Corollary 2.2.6, we get the following formula that is also equivalent to (2.1.5):

Corollary 2.2.7. *For (X, d_X) a length space, $(B, d_B, \ll, \leq, \tau_b)$ a Lorentzian length space, $Y := B \times X$, in which lengths of curves are given by (2.1.5), a continuous function $f : B \to (0, +\infty)$ and $\Psi_\gamma \geq 0$, the length of a causal curve $\gamma : [a, b] \to Y$ is given by:*

$$L(\gamma) = \inf\{\sum_{i=0}^{N-1} \sqrt{L_{\tau_b}(\alpha_{[t_i,t_{i+1}]})^2 - \sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta(t'_{i,k}), \beta(t'_{i,k+1}))\}^2}\ \} \tag{2.2.17}$$

where

$$\begin{aligned} &N, M \in \mathbb{N}, i \in [0, N-1], k \in [0, M-1], \\ &t_i, t_{i+1} \in \{a = t_0 < t_1 < \cdots < t_N = b\}, \\ &t'_{i,k}, t'_{i,k+1} \in \{t_i = t_{i,0} < t_{i,1} < \cdots < t_{i,M} = t_{i+1}\} \end{aligned}$$

Proof. The proof is exactly the same as in Corollary 2.2.6, so here we will give the outline of it. Specifically, we know that there is always a partition with given M, N for which we can have a value $\epsilon_i > 0$, arbitrarily small, s.t $\sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta(t'_{i,k}), \beta(t'_{i,k+1}))\} - \epsilon_i < m_{t_i,t_{i+1}} d_X(\beta(t_i), \beta(t_{i+1})) < \sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta(t'_{i,k}), \beta(t'_{i,k+1}))$. From this point onwards we proceed as in Corollary 2.2.6 $\square$

If we take a null curve $n : [a, b] \to Y$, then from general relativity for warped product spaces, like the FRWL spacetime, or a generalisation of it in the case of generalized cones presented in [AGKS19], we know that the time it takes (the time separation function for one dimensional base) to traverse a curve in the fibre is bounded from below, meaning that we can never take less time than light to go between two points in the fibre, respecting thus the fundamental principle of causality. By following this fundamental axiom in our generalised case of having a base that is not one dimensional, we can provide a bound on the time separation function between points in the base. This is done through the following Proposition:

Proposition 2.2.8. *Let $(B, d_B, \ll, \leq, \tau_b)$ be a Lorentzian length space, (X, d_X) a length space, $f : B \to I$ a continuous function in B and $\gamma = (\alpha, \beta) : [a, b] \to Y$ a causal curve, with $L(\beta_{[t_1,t_2]}) \neq 0\ \forall t_1 < t_2 \in [a, b]$. Then, for $t_1 = t'_{1,0}$, $t_2 = t'_{1,M}$, γ is a null curve if and only if:*

$$L_{\tau_b}(\alpha_{[t_1,t_2]}) = \sup\{\sum_{k=1}^{M-1} m_{t'_{1,k},t'_{1,k+1}} d_X(\beta(t'_{1,k}), \beta(t'_{1,k+1}))\} \tag{2.2.18}$$

Proof. We consider a partition of γ as $\{t_i \in [a, b] : i \in [0, N-1] \subset \mathbb{N}\}$ and we take the case that $L_{\tau_b} = \sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta(t'_{i,k}), \beta(t'_{i,k+1}))\}$. In this case, from Corollary 2.2.7, we see that $L(\gamma_{[t_1,t_2]}) = 0\ \forall t_1, t_2 \in [a, b]$ and hence γ is a null curve.

In contrast we want to show that if (2.2.18) doesn't hold, then γ can't be null. We assume instead, that $\gamma \in Y$ is null and it doesn't comply with (2.2.18). We have:

$$L_{\tau_b}(\alpha_{[t_i,t_{i+1}]}) = \sup\{\sum_{k=1}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta'(t'_{i,k}), \beta'(t'_{i,k+1}))\} + \delta_i \tag{2.2.19}$$

where $\delta_i > 0$, $L(\gamma) = 0$ and for $N = 1$ in the partition $\{t_i\}$ of γ, $\delta_0 = \delta$. Now, we can use the additivity of L_{τ_b} and τ_γ from Definition 2.2.4, together with three points $t_1 < t_2 < t_3 \in [a, b]$, in order to get:

$$L_{\tau_b}(\alpha_{[t_1,t_3]}) = L_{\tau_b}(\alpha_{[t_1,t_2]}) + L_{\tau_b}(\alpha_{[t_2,t_3]}) \tag{2.2.20}$$

and

$$\begin{aligned} &\sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta'(t'_{i,k}), \beta'(t'_{i,k+1}))\}_{[t_1,t_3]} = \\ &\qquad \sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta'(t'_{i,k}), \beta'(t'_{i,k+1}))\}_{[t_1,t_2]} \\ &\qquad + \sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta'(t'_{i,k}), \beta'(t'_{i,k+1}))\}_{[t_2,t_3]} \end{aligned} \tag{2.2.21}$$

and thus from (2.2.19), (2.2.20), (2.2.21) we get:

$$\begin{aligned}
\sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta'(t'_{i,k}),\beta'(t'_{i,k+1}))\}_{[t_1,t_2]} + \delta_1 = \\
\sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta'(t'_{i,k}),\beta'(t'_{i,k+1}))\}_{[t_1,t_2]} + \delta_2 \\
+\sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta'(t'_{i,k}),\beta'(t'_{i,k+1}))\}_{[t_2,t_3]} + \delta_3 \qquad (2.2.22)
\end{aligned}$$

which gives

$$\delta_1 = \delta_2 + \delta_3 \qquad (2.2.23)$$

Hence, by using (2.2.19) in (2.2.17), together with (2.2.23), we get that:

$$\begin{aligned}
L(\gamma) &= \\
&= \inf\{\sum_{i=0}^{N-1} \sqrt{L_{\tau_b}(\alpha_{[t_i,t_{i+1}]})^2 - \sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta'(t'_{i,k}),\beta'(t'_{i,k+1}))\}^2}\ \} \\
&= \inf\{\sum_{i=0}^{N-1} [(\sup\{\sum_{k=1}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta'(t'_{i,k}),\beta'(t'_{i,k+1}))\} + \delta_i)^2 \\
&\qquad - \sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta'(t'_{i,k}),\beta'(t'_{i,k+1}))\}^2]^{\frac{1}{2}}\ \} \\
&= \inf\{\sum_{i=0}^{N-1} \sqrt{2\delta_i \sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} d_X(\beta'(t'_{i,k}),\beta'(t'_{i,k+1}))\} + \delta_i^2}\ \} \\
&\geq \inf\{\sum_{i=0}^{N-1} \sqrt{\delta_i^2}\ \} \\
&= \inf\{\sum_{i=0}^{N-1} \delta_i\ \} \qquad (2.2.24)
\end{aligned}$$

From (2.2.23) $L(\gamma) \geq \inf\{\sum_{i=0}^{N-1} \delta_i\} = \delta > 0$ and hence γ is not null, which is a contradiction. For $\delta_i < 0$, we would get a complex number in the subroot for any t_i, t_{i+1} and hence it would be not a causal curve. □

Remark 2.2.9. *If we choose a reparametrisation of γ, s.t for a continuous and monotonically increasing map $\Phi : [a, b] \to [c, d]$, $\gamma = n \circ \Phi(\cdot)$, then from Proposition 2.1.5, if n is a null curve, so is γ.*

Null curves, and the limits that they pose in the causal relations between points in Y will be a focal point later in our work, when we try to negotiate the relations between points in the space Y.

2.3 Speed bounds for causal curves

With the definition of the causal curves in hand, we need to delve a bit more into the matter of continuity for the curves in Y, whose lengths are given by (2.1.5). In particular, from the smooth Lorentzian geometric spaces we know that the speed on a given curve can never be greater than the speed of light. This restriction translates to the condition of Definition 2.18 in [KS17] for causal curves being locally Lipschitz continuous with respect to the background metric d. In the construction of Y though, we have chosen a Lorentzian length space (B) as the base and from Definition 2.1.2 we know that the projection of any curve to B will also be causal and hence from Definition 2.18, in [KS17] will always be Lipschitz continuous with respect to the background metric of the base (d_B). Therefore, we are left to ask whether there is such a parametrisation that the corresponding projection to the fibre is also Lipschitz continuous.

In what follows, we need to distinguish between three cases. Specifically, we might have causal curves that move on both the space and the fibre and causal curves that have a non-constant projection only to the base. The additional case of a curve that has a curve projection on the fibre, but projects only to a point in the base is not a causal case, as we can see from Proposition 2.2.8. Therefore, we can treat the first two cases separately, as in the case that a curve constitutes a combination of the two we can choose different reparametrisations for each one of them. Hence, for any continuous curve (wrt d), we get the following proposition:

Proposition 2.3.1. *Let there be a metric space (X, d_X) and $(B, d_B, \ll, \leq, \tau_b)$ a Lorentzian length space, while also $Y := B \times X$. In addition, $f : B \to I$ is a continuous function. Then, for a future-directed causal curve α, every curve $\gamma = (\alpha, \beta) : [a, b] \to Y$ has a reparametrization that is locally Lipschitz continuous.*

Proof. We want to show that a causal curve $\gamma \in Y$ (i.e for γ s.t $L_{var}(\gamma) \geq 0$) is locally Lipschitz continuous (i.e in a compact interval $[a, b]$), for at least one parametrisation. Thus, we need to show that if $s \in [a, b]$ is the parameter in the original parametrisation, then there is at least one $\Phi : [a, b] \to [c, d]$ that is continuous and monotonically increasing (according to Definition 2.26, in [KS17]), s.t for $s < t$:

$$\frac{d(\gamma(t), \gamma(s))}{\Phi(t) - \Phi(s)} \leq K \tag{2.3.1}$$

where $d(\gamma(t), \gamma(s)) = \sqrt{d_B^2(\alpha(s), \alpha(t)) + d_X^2(\beta(s), \beta(t))}$ and $K \geq 0$. For $s < t \in [a, b]$ we know that:

$$d_X(\beta(s), \beta(t)) \leq L_{d_X}(\beta_{[s,t]}) \tag{2.3.2}$$

$$d_B(\alpha(s), \alpha(t)) \leq L_{d_B}(\alpha_{[s,t]}) \tag{2.3.3}$$

where $L_{d_X}(\beta_{[s,t]}) = L_{d_X}(\beta_{[a,t]}) - L_{d_X}(\beta_{[a,s]})$ and $L_{d_B}(\alpha_{[s,t]}) = L_{d_B}(\alpha_{[a,t]}) - L_{d_B}(\alpha_{[a,s]})$ are the lengths of the sections of β, α respectively from s to t and thus $L_{d_X}(\beta_{[a,x]}), L_{d_B}(\alpha_{[a,x]})$ are both continuous wrt $x \in [a, b]$ (additivity). Because of (2.3.2) and (2.3.3) we get that α, β are of course Lipschitz continuous wrt the d_B, d_X-arclength respectively.

Now, we consider a reparametrisation of γ as $\gamma = \gamma' \circ \Phi$ (along the lines of Definition 2.26, in [KS17]) s.t $\Phi : [a, b] \to [a', b']$ and $\gamma' = (\lambda, \sigma)$. The reparametrisation is chosen to be the following:

$$\Phi(s) = L_{d_X}(\beta_{[a,s]}) + L_{d_B}(\alpha_{[a,s]}) + s \tag{2.3.4}$$

which is continuous and monotonically increasing. Thus, from the additivity of $L_{d_B}(\alpha_{[a,s]})$, $L_{d_X}(\beta_{[a,s]})$:

$$\begin{aligned}\frac{d_X(\sigma(\Phi(s)), \sigma(\Phi(t)))}{(\Phi(t) - \Phi(s))} =& \frac{d_X(\sigma(\Phi(s)), \sigma(\Phi(t)))}{L_{d_B}(\alpha_{[s,t]}) + L_{d_X}(\beta_{[s,t]}) + s} \\ \leq& \frac{d_X(\sigma(\Phi(s)), \sigma(\Phi(t)))}{L_{d_X}(\beta_{[s,t]})} \\ \leq& C_{a,b}\end{aligned} \tag{2.3.5}$$

with $C_{a,b} = \max(\{\limsup_{t \to s} \frac{d_X(\sigma(s), \sigma(t))}{L_{d_X}(\beta_{[s,t]})}; \forall\ s \in [a, b]\}) \in (0, 1]$. In the same context:

$$\begin{aligned}\frac{d_B(\lambda(\Phi(s)),\lambda(\Phi(t)))}{(\Phi(t)-\Phi(s))} =& \frac{d_B(\lambda(\Phi(s)),\lambda(\Phi(t)))}{L_{d_B}(\alpha_{[s,t]})+L_{d_X}(\beta_{[s,t]})+s} \\ \leq& \frac{d_B(\lambda(\Phi(s)),\lambda(\Phi(t)))}{L_{d_B}(\alpha_{[s,t]})} \\ \leq& C'_{a,b} \end{aligned} \tag{2.3.6}$$

with $C'_{a,b} = \max(\{\limsup_{t\to s} \frac{d_B(\lambda(s),\lambda(t))}{L_{d_B}(\beta_{[s,t]})}; \forall\ s \in [s_i, s_{i+1}]\}) \in (0,1]$. Consequently, (w.l.o.g) for $s, t \in [a,b]$, from (2.3.5) and (2.3.6) we get:

$$\begin{aligned} d(\gamma'(\Phi(u)),\gamma'(\Phi(s))) =& \sqrt{d_B^2(\lambda(\Phi(s)),\lambda(\Phi(u))) + d_X^2(\sigma(\Phi(s)),\sigma(\Phi(u)))} \\ \leq& \max(C_{a,b}, C'_{a,b}, 1)(\,\Phi(u)-\Phi(s)\,) \end{aligned} \tag{2.3.7}$$

and thus we have shown that with our chosen reparametrisation γ' is locally Lipschitz continuous. □

So far we have given the topology of a space Y and we have proposed a formula for measuring lengths in it, with the proof that this formula is equivalent with the lengths of curves in smooth spaces and that it satisfies some of the basic properties of Lorentzian pre-length spaces (section 2.3, [KS17]). Our aim going forward, is to show that there is a Lorentzian pre-length space lying behind the defined topology and the designated length formula in (2.1.5). Before we proceed to further establishing the geometrical picture for our space, we need to check whether our length-measuring formula is upper semicontinuous, as it is a requirement for Lorentzian pre-length spaces, analogously to the length-measuring formulas for Riemannian length spaces.

2.4 *Upper semicontinuity of length formula*

An important element of metric geometry is that given a metric d_X for a length space X, the length functional of a d_X-rectifiable curve β is a lower semicontinuous function. The nature of this kind of semicontinuity is to be ascribed to the use of sup for the d_X-functional that provides the length of β and the triangle inequality. However, due to the use of inf in the length functional for causal curves and the inverse triangle inequality for points on them, we want our particular length functional to be an upper semicontinuous function. Before we proceed to this proof, we define $C(I,Y)$ to be the space of causal curves in Y, with Y having the topology, length formula and properties as described so far.

Proposition 2.4.1. *If $(B, d_B, \ll, \leq, \tau_b)$ is a Lorentzian length space, (X, d_X) is a metric space and $\gamma_n, \gamma \in C(I,Y)$, s.t $\gamma_n \to \gamma$ pointwise, then:*

$$\limsup_n L(\gamma_n) \leq L(\gamma) \tag{2.4.1}$$

Proof. We have that $\gamma_n = (\alpha_n, \beta_n) : [a,b] \to Y$ and $\gamma = (\alpha, \beta) : [a,b] \to Y$, for which we have a projection $\pi = (\pi_B, \pi_X)$ such that $\alpha = \pi_B \circ \gamma \in B$ and $\beta = \pi_X \circ \gamma \in X$. In addition, f is taken to be an everywhere continuous function in its domain. We are going to show here that the equivalent length formulas we have derived in (2.1.5), (2.2.12) and (2.2.17) are both upper semicontinuous. We first show the upper semicontinuity for (2.1.5), where in general, the time separation function for the base is a lower semicontinuous function. However, the space B is Lorentzian and hence we have that there is always a neighbourhood U, where the time separation function is locally continuous ([KS17], Definition 3.16), meaning that the existence of a local maximal length for any curve between two points in a neighbourhood U, gives rise to a different local time separation function, which is given by a mapping $\tau_{b,U} : U \times U \mapsto R$ and is in general locally continuous. Therefore, we consider again the causal curve $\gamma = (\alpha, \beta) \in C(I,Y)$ and we take a partition $\sigma = \{t_i : t_i \in [a,b], i = 1, ..., N\}$. Then we consider that since B is Lorentzian, there is always a neighbourhood V_i such that any two points on α, namely $\alpha(t_i), \alpha(t_{i+1})$ will be connected via a locally maximal curve $\lambda_{[t_i, t_{i+1}]}$, which has a length equal to $\tau_{b,V_i}(\alpha(t_i), \alpha(t_{i+1}))$, since, as mentioned above, there is always a locally continuous function $\tau_{b,V_i} : (V_i, V_i) \to R$, defined in V_i with the property $\tau_{b,V_i}(\alpha(t_i), \alpha(t_{i+1})) \leq \tau_b(\alpha(t_i), \alpha(t_{i+1}))$. Due to that last property, if we substitute τ_{b,V_i} in (2.1.4) and set:

$$P_{V_i}(\gamma(t_i), \gamma(t_{i+1})) = \sqrt{\tau_{b,V_i}(\alpha(t_i), \alpha(t_{i+1}))^2 - m^2_{t_i,t_{i+1}} d^2_X(\beta(t_i), \beta(t_{i+1}))},$$

we have that:

$$\begin{aligned} A &= \sum_{i=1}^{N-1} P(\gamma(t_i), \gamma(t_{i+1})) \\ &\geq \sum_{i=1}^{N-1} P_{V_i}(\gamma(t_i), \gamma(t_{i+1})) \end{aligned} \tag{2.4.2}$$

Next, we consider a sequence of causal curves $(\gamma_n)_n$ that is pointwise convergent to the causal curve γ. From the fact that τ_{b,V_i} is continuous in V_i, we can find a value n_0 for n such that $\tilde{n} \in \{\forall n \geq n_0\}$ and a given partition $\{t_i : \text{ for } i \in [1, N] \subset \mathbb{N}\}$ of $[a, b]$, every pair of points $(\gamma_{\tilde{n}}(t_i), \gamma_{\tilde{n}}(t_{i+1}))$ will be inside V_i and the same is true for γ. In addition:

$$|P_{V_i}(\gamma(t_i), \gamma(t_{i+1})) - P_{V_i}(\gamma_{\tilde{n}}(t_i), \gamma_{\tilde{n}}(t_{i+1}))| < \epsilon_i = \frac{\epsilon}{N} \tag{2.4.3}$$

for all i and $\epsilon > 0$. Therefore, since $P_{V_i}(\gamma_{\tilde{n}}(t_i))$ is continuous, $\exists N$ s.t:

$$\sum_{i=1}^{N-1} P_{V_i}(\gamma(t_i), \gamma(t_{i+1})) \geq \sum_{i=1}^{N-1} P_{V_i}(\gamma_{\tilde{n}}(t_i), \gamma_{\tilde{n}}(t_{i+1})) - \epsilon_i \tag{2.4.4}$$

and because of (2.4.2):

$$\sum_{i=1}^{N-1} P(\gamma(t_i), \gamma(t_{i+1})) \geq \sum_{i=1}^{N-1} P_{V_i}(\gamma_{\tilde{n}}(t_i), \gamma_{\tilde{n}}(t_{i+1})) - \epsilon_i \tag{2.4.5}$$

By taking the infimum over both sides of (2.4.5), we get that:

$$L(\gamma) \geq L(\gamma_{\tilde{n}}) - \epsilon \tag{2.4.6}$$

which shows that $L(\gamma)$ is an upper semicontinuous function.

In case of (2.2.12), L_{τ_b} is upper semicontinuous and so if we consider $C(I, Y)$ in Y and take a given partition $\sigma = \{t_i : t_i \in [a, b], i = 1, ..., N\}$ of γ the mapping $T : C(I, Y) \to R$, that is provided by the following expression:

$$T(\gamma) = \sum_{i=0}^{N-1} \sqrt{\tau_{b,U}(\alpha(s_i), \alpha(s_{i+1}))^2 - m^2_{s_i, s_{i+1}} d^2_X(\beta(s_i), \beta(s_{i+1}))} \tag{2.4.7}$$

will also be upper semicontinuous with respect to pointwise convergence. Therefore, the infimum of this function is an upper semicontinuous function, as the infimum of a sequence of upper semicontinuous functions (cf., e.g., [AB06], Lem. 2.41).

Finally, the case of (2.2.17) is a combination of the above two cases and can be easily shown to be upper semicontinuous. □

With the proof for upper semicontinuity, we have established so far that should (2.1.5) be used to define a tool for measuring lengths, inside the space Y, and for the causal curve as designated in Definition 2.2.1, it reproduces basic properties of a length functional for a Lorentzian pre-length space. However, in order to infer that Y is itself actually a Lorentzian pre-length space, we need to have a time separation function, that explicitly depicts the causal structure.

Causal character with respect to pointwise convergence

An important question that we want to answer in order to be used in what follows is related to what kind of causal behaviour we expect to see wrt pointwise convergence. In particular, we consider that we have a sequence of causal curves (wrt 2.2.1) $\gamma_n = (\alpha_n, \beta_n) : [a, b] \to Y$, for $n \in \mathbb{N}$. Then, if we consider a curve γ and allow the sequence γ_n to converge pointwise to γ, in the sense that for $s \in [a, b]$ $\lim_{n\to\infty} \gamma_n(s) = \gamma(s)$, we can deduce the causal character of γ from the following proposition:

Proposition 2.4.2. *Let $(B, d_B, \ll, \leq, \tau_b)$ be a Lorentzian length space, (X, d_X) a length space, $f : B \to (0, \infty)$ a continuous function in B and $\gamma_n = (\alpha_n, \beta_n) : [a, b] \to Y$ a future directed causal curve, with $n \in \mathbb{N}$. Then, if $\lim_{n\to\infty} \gamma_n(s) = \gamma(s)$, for $s \in [a, b]$, γ is also a future directed causal curve in $[a, b]$. The same applies for past-directed causal curves.*

Proof. Take $a \leq s < t \leq b$. Thus, from (2.4.1) we have that if $\gamma_n(s) \to \gamma(s)$ when $n \to \infty$ and γ_n are causal then:

$$L(\gamma_{[s,t]}) \geq 0 \ , \ \forall s < t \in [a, b] \tag{2.4.8}$$

Consequently, γ will also be a causal curve. □

The limit curve causal character is an important result that we will invoke several times in what follows. Most notably, is a result that we will rely upon in order to prove the validity of our definition of causal character for curves, for which (up until now) all we know about is that it is dependent on the length-measuring formula for our space.

2.5 The time separation function

In equation (2.1.5), we have defined the variational formula for the length of a space following the topology in 2.1.1. In Proposition 2.1.6, we have also shown the equivalence of the variational length of (2.1.5) with the length of an absolute continuous curve in a smooth spacetime. Therefore, using (2.1.5), we can define the time separation function as:

Definition 2.5.1. *Let γ be a causal curve wrt Definition 2.2.1. Then, if:*

$$\Gamma = \{L(\gamma) : \gamma \textit{ future directed causal curve from } p \textit{ to } q\} \tag{2.5.1}$$

the time separation function $\tau : Y \times Y \to [0, \infty) \cup \{\infty\}$ is defined as:

$$\tau(p,q) := \begin{cases} \sup \Gamma & \textit{if } \Gamma \textit{ is non-empty} \\ 0 & \textit{if } \Gamma \textit{ is empty} \end{cases} \tag{2.5.2}$$

The time separation function constitutes a fundamental part for a Lorentzian pre-Length space, according to Definition 2.8 in [KS17].

2.6 Causal relations between points

The final step before providing a Lorentzian pre-length space is to ordain the causal relations for the space Y. In order to do it, we proceed analogously to (Definition 3.18, [AGKS19]):

Definition 2.6.1. *Let $p, q \in Y$, then p and q are chronologically related, denoted by $p \ll q$, if there exists a future directed timelike curve from p to q. Moreover, p and q are causally related, denoted by $p \leq q$ if there exists a future directed causal curve from p to q or $p = q$. Moreover, we define the chronological and causal future and past of a point as:*

$$I^+(p) = \{q \in Y : p \ll q\} \ \ , \ \ I^-(p) = \{q \in Y : q \ll p\} \tag{2.6.1}$$

$$J^+(p) = \{q \in Y : p \leq q\} \ \ , \ \ J^-(p) = \{q \in Y : q \leq p\} \tag{2.6.2}$$

From this definition and the Definition 2.2.1 we can also get another important set of properties with regard to the use of the time separation function for defining the causal relations between points in Y:

$$\tau(\bar{p},\bar{q}) > 0 \qquad \text{if } \bar{p} \ll \bar{q} \tag{2.6.3}$$
$$\tau(\bar{p},\bar{q}) = 0 \qquad \text{if } \bar{p} \not\geq \bar{q} \tag{2.6.4}$$
$$\tau(\bar{p},\bar{q}) = 0 \qquad \text{if } (\bar{p},\bar{q}) \text{ on a null curve} \tag{2.6.5}$$

Lemma 2.6.2. *The relations $\ll$ and $\leq$ are transitive and $\leq$ is reflexive too, with $\ll \subseteq \leq$.*

Proof. Along the lines of Lemma 3.19 in [AGKS19], the proof for reflexivity and the relations hierarchy is given by Definition 2.6.1, while the transitivity is easily shown by concatenating curves. □

With the definition of causal relations in place, we can revisit the time separation function to show one very fundamental property, that of the reverse triangle inequality.

Lemma 2.6.3. *Let $(B, d_B, \ll, \leq, \tau_b)$ be a Lorentzian length space and (X, d_X) a metric space, while also $Y := B \times X$ and $f : B \to (0, +\infty)$ a continuous function in B. Moreover, curves in Y have their lengths given by (2.1.5) and the time separation function for Y is given by (2.5.2). Then for three points $p \leq q \leq z$ on a causal curve γ in Y, the time separation function τ satisfies:*

$$\tau(p,z) \geq \tau(p,q) + \tau(q,z) \tag{2.6.6}$$

Proof. Reverse triangle inequality for τ follows naturally from its definition as the supremum over a set of lengths. First we consider that the points p, q, z belong on a segment of the causal curve γ, namely $\gamma_{p,z}$. Then, we assume that $\gamma_{p,q}, \gamma_{q,z}$ connect p with q and q with z respectively. Using these curve segments and comparing them with τ, since the latter is the supremum of the set in (2.5.2), there is always an $\epsilon > 0$ such that $\tau(p,q) - \frac{\epsilon}{2} < L(\gamma_{p,q})$ and $\tau(q,z) - \frac{\epsilon}{2} < L(\gamma_{q,z})$. Therefore, from the additivity of lengths for the segments of a curve we get that:

$$\tau(p,q) + \tau(q,z) < L(\gamma_{p,q}) + L(\gamma_{q,z}) + \epsilon = L(\gamma_{p,z}) + \epsilon \leq \tau(p,z) + \epsilon \tag{2.6.7}$$

which proves the case, when we have a non-empty Γ. If an empty Γ was to be considered, then $\tau(p,z) = \tau(p,q) = \tau(q,z) = 0$ and hence the equality is trivially satisfied. □

In Definition 2.2.1 we have given the notion of a causal curve, which is later used to define the causal relations between points in Y, as the latter are given in Definition 2.6.1. Now, we try to delve into the set of points possessing chronological and causal relations in Y and see the kind of structure that emerges from them. In order to populate the sets of $I^{\pm}$(timelike causally related points) and $J^{\pm}$(causally related points) we will use null curves to obtain values for the L_{τ_b} between the points they connect. In order to do that, we define a function χ_α, which gives an upper bound for the d_X-arclength, when traversing a potential geodesic in X.

Definition 2.6.4. *For B a Lorentzian length space, a pair of points $p, q \in B$, an integrable function $f \circ \alpha(u)$ and a curve α from p to q, the mapping χ is given by:*

$$\chi(\alpha; \rho(z), \delta) = \int_{\delta}^{\delta+\rho} f \circ \alpha(u) \dot{\rho} du \tag{2.6.8}$$

where $\rho : [0, 1] \to \mathbb{R}_{>0}$ monotonically increasing and $\delta > 0$ determine the integration interval.

For simplicity, from now on, whenever χ is used, we write $\chi_\alpha(\rho; \delta)$, with the appropriate subscript indicating the curve used. Additionally, wherever $\delta = 0$, we write $\chi_\alpha(\rho)$. Making use of the Definition 2.6.4, we additionally define the following set:

Definition 2.6.5. *Let $(B, d_B, \ll, \leq, \tau_b)$ be a Lorentzian length space, (X, d_X) be a length space and $p = (\bar{p}, \tilde{p}) \in Y$. Let also $I_B^+(\bar{p})$ be the set of future timelike-related points with $\bar{p}$. Then, the following set for p is defined:*

$$\begin{aligned} \mathcal{G}^+(p) := \ \{ & q = (\bar{q}, \tilde{q}) \in Y : \bar{q} \in I_B^+(\bar{p});\ \exists\, \alpha \text{ from } \bar{p} \text{ to } \bar{q} \text{ s.t } L_{\tau_b}(\alpha) < b\,, \\ & b_{\tilde{p}} = \chi_\alpha^{-1}(b)\,,\ d_X(\tilde{p}, \tilde{q}) < b_{\tilde{p}}\,;\ \forall\, \bar{z}, \bar{w} \in \alpha\,,\ \exists\, \tilde{z}, \tilde{w} \in X, \\ & L_{\tau_b}(\alpha_{[\bar{z},\bar{w}]}) > \chi_\alpha(\, d_X(\tilde{z}, \tilde{w})\,) \} \end{aligned} \tag{2.6.9}$$

where $l = d_X(\tilde{p}, \tilde{q}) + \epsilon$. $\mathcal{G}^-(p)$, corresponding to timelike points in the past of p, is defined analogously.

Making use of Definition 2.6.5, we associate the timelike future of a point p in Y with the length functional through the following Lemma:

Lemma 2.6.6. *Let $p = (\bar{p}, \tilde{p}), q = (\bar{q}, \tilde{q}) \in Y$. Then $p \ll q$ if and only if there exists a future directed causal curve from p to q of positive length. Moreover, $I^+(p)$ is given by:*

$$I^+(p) = \mathcal{G}^+(p) \tag{2.6.10}$$

Same for $I^-(p)$.

Proof. Firstly we show that if $q \in \mathcal{G}^+(p)$, then it is timelike connected to p. We start the negotiation of the proof by considering any of the curves $\alpha(u) : [0, L] \to B$ in $\mathcal{G}^+(p)$, that connect p to q. Upon this, we seek to find if there exists a curve in X from $\tilde{p}$ to $\tilde{q}$, which if passed in τ_γ, the latter will be less than the $L_{\tau_b}(a)$, so that from Proposition 2.2.8 we get the condition of Definition 2.2.1 for a timelike curve between p, q. This is done through an almost minimizing curve unit speed curve $\lambda_{\beta,am} : [0, d_X(\tilde{p}, \tilde{q}) + \epsilon] \to X$, where $L_{d_X}((\lambda_{\beta,am})_{[\tilde{p},\tilde{q}]}) = d(\tilde{p}, \tilde{q}) + \epsilon$. Next, we take two values s, s' and a constant $c > 0$, s.t:

1. $r, z \in [0, 1]$
2. $w = zd(\tilde{p}, \tilde{q})$
3. $s = z(d(\tilde{p}, \tilde{q}) + \epsilon)$
4. $s' = \chi_\alpha^{-1}(\, r\chi_\alpha(d_X(\tilde{p}, \tilde{q}) + \epsilon + c)\,)$
5. $b_{\tilde{p}} > d_X(\tilde{p}, \tilde{q}) + \epsilon + c$

Moreover, we use that $f \circ \alpha$ is continuous and hence integrable, while in addition that for $\eta_s > 0$, $d_X(\lambda_{\beta,am}(s), \lambda_{\beta,am}(s + ds)) = L_{d_X}(\lambda_{\beta,am})dz - \eta_s dz$ and thus:

$$\begin{aligned}
\chi_\alpha(s) =& \sup\{\sum_{k=0}^{M-1} m_{s_k,s_{k+1}} d_X(\lambda_{\beta,am}(s_k), \lambda_{\beta,am}(s_{k+1}))\} \\
&\geq \inf_{\lambda_{\beta,am}} \{\, \sup\{\sum_{k=0}^{M-1} m_{s_k,s_{k+1}} d_X(\lambda_{\beta,am}(s_k), \lambda_{\beta,am}(s_{k+1}))\}\,\} \\
&= \chi_\alpha(w)
\end{aligned} \tag{2.6.11}$$

where in the last line of (2.6.11) we used the fact the infimum is obtained for $\eta_s \to 0$. From (2.6.11) we see that there is always an ϵ and hence an appropriately chosen $\lambda_{\beta,am}$, which gives a τ_γ value arbitrarily close to $\chi_\alpha(w)$, unless X is geodesic, where there is a curve $\lambda_{\beta,am}$, that realises the infimum. In order to take advantage of the inequalities in (2.6.11) we need to set (without loss of generality) $\bar{p} = \alpha(0)$ and that:

$$L_{\tau_b}(\alpha) = \chi_\alpha(d_X(\tilde{p},\tilde{q}) + \epsilon + c) \tag{2.6.12}$$

Taking advantage of (2.6.12) and by requiring that $L_{\tau_b}(\alpha_{[0,s']}) \geq rL_{\tau_b}(\alpha)$, in order to relate $\chi_\alpha(s')$ to $L_{\tau_b}(s')$, we set $s' = \phi(s) = s + \frac{\delta(s)}{d_X(\tilde{p},\tilde{q})+\epsilon}$ and $\delta(s) > 0$, $\delta(d_X(\tilde{p},\tilde{q}) + \epsilon) = c(d_X(\tilde{p},\tilde{q}) + \epsilon)$ to get:

$$L_{\tau_b}(\alpha_{[0,s']}) \geq \chi_\alpha(s') \tag{2.6.13}$$

Thus, we have created a curve $\gamma = (\alpha \circ \phi, \lambda_{\beta,am})$, which from Corollary 2.2.7 gives:

$$\begin{aligned} L(\gamma) = \inf\{\sum_{i=0}^{N-1} [L_{\tau_b}(\alpha_{[\phi(s_i),\phi(s_{i+1})]})^2 - \sup\{\sum_{k=0}^{M-1} m_{s_{i,k},s_{i,k+1}} \\ \times d_X(\lambda_{\beta,am}(s_{i,k}), \lambda_{\beta,am}(s_{i,k+1}))\}^2]^{\frac{1}{2}}\ \} \end{aligned} \tag{2.6.14}$$

In order to simplify the notation, we write:

$$n_\alpha = \alpha \circ \phi \tag{2.6.15}$$

and we also use the fact from Definition 2.6.8, that:

$$\chi(\alpha; s', 0) = \chi(n_\alpha; \phi(s), 0) \tag{2.6.16}$$

to get $\chi_\alpha(s') = \chi_{n_\alpha}(\phi(s))$. Therefore, from (2.6.11) and (2.6.13), as well as from reparametrisation invariance of L_{τ_b}:

$$\begin{aligned} L_{\tau_b}(\alpha_{[s',s'+\Delta s']}) &= L_{\tau_b}((n_\alpha)_{[s,s+\Delta s]}) \\ &\geq \chi_{n_\alpha}(\,\Delta\phi(s); \phi(s)\,) \\ &> \chi_{n_\alpha}(\,\Delta s; s\,) \\ &= \tau_\gamma(s, s+\Delta s) \end{aligned} \tag{2.6.17}$$

Thus, equation (2.6.17) together with (2.6.11), (2.6.16) and the definition of ϕ give:

$$\begin{aligned} L_{\tau_b}(\alpha_{[s',s'+\Delta s']}) &\geq \chi_\alpha(\Delta s'; s') \\ &> \chi_\alpha(\Delta s; s) \\ &= \tau_\gamma(\gamma; s, s+\Delta s) \\ &\geq \chi_\alpha(\Delta w; w) \end{aligned} \tag{2.6.18}$$

Consequently, for every curve connecting p, q (satisfying the above conditions), there is always an ϵ' and hence a $\lambda_{\beta,am}$, s.t $\gamma = (n_\alpha, \lambda_{\beta,am})$ is a timelike curve connecting p, q:

$$\begin{aligned} L(\gamma) =& \inf\{\sum_{i=0}^{N-1}[\tau_b(n_\alpha(s_i), n_\alpha(s_{i+1}))^2 - m^2_{s_i,s_{i+1}} \\ &\qquad \times d_X(\lambda_{\beta,am}(s_i), \lambda_{\beta,am}(s_{i+1}))^2]^{\frac{1}{2}}\} \\ =& \inf\{\sum_{i=0}^{N-1}[L_{\tau_b}((n_\alpha)_{[s_i,s_{i+1}]})^2 - m^2_{s_i,s_{i+1}} \\ &\qquad \times d_X(\lambda_{\beta,am}(s_i), \lambda_{\beta,am}(s_{i+1}))^2]^{\frac{1}{2}}\} \\ >& \inf\{\sum_{i=0}^{N-1}[\tau_\gamma(\gamma; s_i, s_{i+1})^2 - m^2_{s_i,s_{i+1}} \\ &\qquad \times d_X(\lambda_{\beta,am}(s_i), \lambda_{\beta,am}(s_{i+1}))^2]^{\frac{1}{2}}\} \\ =& 0 \end{aligned} \tag{2.6.19}$$

where in the final line the equality comes from the second line of (2.6.11), together with the equivalence of (2.2.12) and (2.2.17). Hence, from Definitions 2.2.1 and 2.6.1, q will be part of the set of points that are $<<$-connected with p, i.e $\mathcal{G}(p) \subseteq I^+(p)$.

Similarly, we proceed to show that if $q \notin \mathcal{G}^+(p)$, then we don't have a causal curve of positive length connecting the two points p, q. We proceed by considering that such a curve exists and we call it $\gamma'_\epsilon = (\alpha'_\epsilon, \beta'_\epsilon) : [0, d(\bar{p}, \bar{q}) + \epsilon] \to Y$. From this point there are two different cases that lead a point outside of $\mathcal{G}^+(p)$ and a third one that is a combination of the other two. Hence, it suffices to show the first two cases only. On the first one of these cases, we consider a point q such that for $\alpha'_\epsilon(0) = \bar{p}$ and $\alpha'_\epsilon(t) = \bar{q}$, $L_{\tau_b}(\alpha'_\epsilon) \leq \chi_\alpha(d(\tilde{p}, \tilde{q}))$. Since χ_α is a continuous and monotonically increasing function in its domain, we can consider an appropriately chosen curve $n_b = \lambda_{\beta,am} \circ \chi_\alpha^{-1}(u) : [0, L_{\tau_b}((\alpha'_\epsilon)_{[a,b]})] \to X$, s.t $n = (\bar{\alpha}_\epsilon, n_b) \in Y$ is a null curve, where $\lambda_{\beta,am}$ goes again from $\tilde{p}$ to $\tilde{q}$. Additionally, we need:

$$L(\gamma'_\epsilon) > 0 \tag{2.6.20}$$

because γ_ϵ is set to be causal with positive length. Then:

$$\begin{aligned} L(\gamma'_\epsilon) =& \inf\{\sum_{i=0}^{N-1}[\,L_{\tau_b}((\alpha'_\epsilon)_{[t_i,t_{i+1}]})^2 - \sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} \\ &\times d_X(\beta'_\epsilon(t'_{i,k}), \beta'_\epsilon(t'_{i,k+1}))\}^2\,]^{\frac{1}{2}}\ \} \\ \leq& \inf\{\sum_{i=0}^{N-1}[\chi_\alpha(t_{i+1}-t_i;t_i)^2 - \sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} \\ &\times d_X(\beta'_\epsilon(t'_{i,k}), \beta'_\epsilon(t'_{i,k+1}))\}^2\,]^{\frac{1}{2}}\ \} \end{aligned} \tag{2.6.21}$$

Thus:

$$\begin{aligned} L(\gamma'_\epsilon) \leq& \inf\{\sum_{i=0}^{N-1}[\,\chi_\alpha(t_{i+1}-t_i;t_i)^2 - \sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} \\ &\times d_X(\beta'_\epsilon(t'_{i,k}), \beta'_\epsilon(t'_{i,k+1}))\}^2\,]^{\frac{1}{2}}\ \} \\ \leq& \inf\{\sum_{i=0}^{N-1}[\,\chi_\alpha(t_{i+1}-t_i;t_i)^2 - \sup\{\sum_{k=0}^{M-1} m_{t'_{i,k},t'_{i,k+1}} \\ &\times d_X(n_b(t'_{i,k}), n_b(t'_{i,k+1}))\}^2\,]^{\frac{1}{2}}\ \} \\ =&0 \end{aligned} \tag{2.6.22}$$

and hence from (2.6.20) and (2.6.21):

$$L(\gamma'_\epsilon) = 0 \tag{2.6.23}$$

meaning that $L(\gamma'_\epsilon)$ isn't a timilike curve for the case $L_{\tau_b}(\alpha'_\epsilon) \leq \chi_\alpha(d(\tilde{p},\tilde{q}))$ and hence it doesn't belong to $I^+(p)$.

The other case that we need to prove is that $d_X(\tilde{p},\tilde{q}) \geq b_{\tilde{p}}$. As we did for the previous case, we assume that $\gamma = (\alpha, \beta) \in Y$ is a causal curve connecting the two points $(p, q) \in Y$. Since $L_{\tau_b}(\alpha) < b$ and

$\lim_{s \to b_{\bar{p}}} \chi_\alpha(s) \to b$ we can choose $\epsilon > 0$ and a point $x = (\bar{x}, \tilde{x})$ on γ, as well as $\gamma'_\epsilon = \gamma_{[p,x]}$ connecting p, x, such that $L_{\tau_b}(\alpha'_\epsilon) \leq \chi_\alpha(b_{\bar{p}} - \epsilon) = \chi_\alpha(d(\tilde{p}, \tilde{x}))$ and thus $d_X(\tilde{p}, \tilde{x}) < b_{\tilde{p}}$. Therefore, from this point onwards the proof proceeds as in the case above, where $L_{\tau_b}(\alpha'_\epsilon) \leq \chi_\alpha(d(\tilde{p}, \tilde{q}))$, through which we get that p, x are not timelike related and hence the curve with points p, x, q is not timelike, according to Definition 2.2.1. □

Remark 2.6.7. *Lemma 2.6.6 implies that $I^{\pm}(p)$ has the push-up property, Definition 1.3.10.*

Now, in order to transition to the case of $J^{\pm}$, we want to add to the picture above for $I^{\pm}$ those points $p = (\bar{p}, \tilde{p}), q = (\bar{q}, \tilde{q})$, which are connected via curves $\gamma = (\alpha, \beta)$ that have $L_{\tau_b}(\alpha_{[s_1,s_2]}) = \tau_\gamma(s_1, s_2)$, i.e null curves. Hence, in order to obtain the complete causal picture we need to add the points of the null curves in those of $I^{\pm}$. To do so, we first define that:

Definition 2.6.8. *Let $(B, d_B, \ll, \leq, \tau_b)$ be a Lorentzian length space and (X, d_X) be a length space. If $p = (\bar{p}, \tilde{p}) \in Y$, it is defined that:*

$$\begin{aligned} \mathcal{J}^+(p) := \mathcal{G}^+(p) \cup \{q = (\bar{q}, \tilde{q}) \in Y : \exists \text{ a causal curve } \alpha \text{ from } \bar{p} \text{ to } \bar{q} \\ \text{and a minimizing curve } \beta \text{ from } \tilde{p} \text{ to } \tilde{q}, \text{ s.t } d_X(\tilde{p}, \tilde{q}) < b_{\tilde{p}}, \\ \forall\, \bar{z}, \bar{w} \in \alpha\,,\, \exists\, \tilde{z}, \tilde{w} \in \beta,\ L_{\tau_b}(\alpha_{[\bar{z},\bar{w}]}) = \chi_\alpha(\, d_X(\tilde{z}, \tilde{w})\,)\,\} \end{aligned} \tag{2.6.24}$$

Analogously for $\mathcal{J}^-(p)$.

Making use of Definition 2.6.8, we can give the following Corollary:

Proposition 2.6.9. *If (X, d_X) is a length space, $(B, d_B, \ll, \leq, \tau_b)$ is a Lorentzian length space and $p = (\bar{p}, \tilde{p}) \in Y$, then:*

$$J^+(p) = \mathcal{J}^+(p) \tag{2.6.25}$$

$J^-(p)$ is defined analogously.

Proof. First we show that if a point belongs in $\mathcal{J}^+(p)$, then it belongs to $J^+(p)$. In the case that a point belongs to $I^+(p)$, it is timelike connected to p, as we have shown in Lemma 2.6.6. Moreover, if a point belongs to the set:

$$\begin{aligned} \mathcal{R}^+(p) = \{q = (\bar{q}, \tilde{q}) \in Y : \exists \text{ a causal curve } \alpha \text{ from } \bar{p} \text{ to } \bar{q} \\ \text{and a minimizing curve } \beta \text{ from } \tilde{p} \text{ to } \tilde{q}, \text{ s.t } d_X(\tilde{p}, \tilde{q}) < b_{\tilde{p}}, \\ L_{\tau_b}(\alpha) = \chi_\alpha(d_X(\tilde{p}, \tilde{q}))\} \end{aligned} \tag{2.6.26}$$

then for $\gamma = (\alpha, \beta)$, $\tau_\gamma(\gamma; 0, kd_X(\tilde{p}, \tilde{q})) = \chi_\alpha(kd_X(\tilde{p}, \tilde{q}))$, with $k \in [0, 1]$ and thus from Proposition 2.2.8 we know that it will be null-connected to p, hence $\mathcal{J}^+(p) \subseteq J^+(p)$.

Next is to show that the curve β needs to be minimizing. The reason is that if a point q belongs to $\mathcal{J}^+(p)$ it is causally related to p as we have shown above, but if it doesn't belong in $I^+(p)$, then it must be null related to p. Therefore, the causal curve γ from p to q must have $L(\gamma) = 0$ and γ must be a maximizer. If $\beta : [0, L] \to X$, parametrised by its arclength is not a minimizer, then there is a curve $\beta' : [0, L] \to X$, parametrised proportianally to arclength, with arclength strictly less than β and so there will be s_1, s_2 in the common parametrisation which give a curve $\gamma' = (\alpha, \beta')$ connecting p and q s.t:

$$L_{\tau_b}(\alpha_{[s_1,s_2]}) = \chi_\alpha(s_2 - s_1; s_1) = \tau_\gamma(\gamma; s_1, s_2) > \tau_\gamma(\gamma'; s_1, s_2) \tag{2.6.27}$$

and therefore γ' has positive length. However, no pair of null-related points can have a non-negative length for their connecting curve. Hence, β is minimizing.

Finally, we can use the same arguments as above in proving that a point which doesn't belong to $\mathcal{G}^+(p)$ can't be connected to a causal curve of positive length with p, to show that when a point doesn't belong to the set $\mathcal{J}^+(p)$ then it won't be causally related to p. Hence, $\mathcal{J}^+(p)$ is equal to $J^+(p)$. □

Corollary 2.6.10. *If $(B, d_B, \ll, \leq, \tau_b)$ is a globally hyperbolic Lorentzian length space and (X, d_X) is a geodesic length space then $J(p)$ in Proposition 2.6.9 is closed.*

Proof. In order to show that $J(p)$ is closed, we need to show that any sequence of points $\{q_n\}$ inside it converges to a point q that is also inside $J(q)$. This is true for any point apart from the ones for which $d_X(\bar{p}, \bar{q}) = b_{\tilde{p}}$. However, in this case, for a_{q_n} being the curve connecting the points p, q_n, we get that $L_{\tau_b}(a_{q_n}) = \chi_\alpha(d_X(\bar{p}, \bar{q_n}))$ and for $q_n \to q$, we get $\lim_{\bar{q}_n \to \bar{q}} \chi_\alpha(d_X(\bar{p}, \bar{q}_n)) = \chi_\alpha(d_X(\bar{p}, \bar{q})) = b$, but we have chosen $L_{\tau_b}(a) < b$ and hence it is a contradiction. □

2.7 The time separation function - revisited

Having introduced the time separation function above, we have also shown that our Definition 2.5.1 yields the required reverse triangle property, that any time separation needs to obey as a constituent of a Lorentzian pre-length space. However, as we mentioned therein, we need the property of lower semicontinuity for our formula (2.5.2) to be considered as a time separation function in a Lorentzian pre-length space. To accomplish this, we use the proof from [AGKS19], where the same arguments apply here, since we have the openness of I^+ and the reverse triangle inequality.

Lemma 2.7.1. *Let $(B, d_B, \ll, \leq, \tau_b)$ be a Lorentzian length space, (X, d_X) be a length space and $f : B \to (0, \infty)$ a continuous function in B. Moreover, there is a space Y that admits the product topology as in Definition 2.1.1 and in which lengths are given via (2.1.5). Then the time separation function τ is lower semi-continuous (with respect to the metric d).*

Proof. The proof proceeds similarly to Lemma 3.25, in [AGKS19], but for simplicity we reproduce here. Let $p = (\bar{p}, \tilde{p}), q = (\bar{q}, \tilde{q})$ be two points in Y and τ such that $\tau(p, q) > \epsilon > 0$. By definition of τ there is a curve γ, such that $L(\gamma) \geq \tau(p, q) - \frac{\epsilon}{2}$. We choose two points $z_1 = (\bar{z_1}, \tilde{z_1}), z_2 = (\bar{z_2}, \tilde{z_2})$ on γ, such that, if $\gamma : [a, b] \to Y$ and $a < t_1 < t_2 < b$, $\gamma(t_1) = z_1$ and $\gamma(t_2) = z_2$. Moreover, we set $0 \leq L(\gamma_{[a,t_1]}) < \frac{\epsilon}{4}$, $0 \leq L(\gamma_{[t_2,b]}) < \frac{\epsilon}{4}$. Due to $\tau(p, q) > 0$, γ is timelike and thus $\tau(p, z_1) \geq L(\gamma_{[a,t_1]}) > 0$, $\tau(z_2, q) \geq L(\gamma_{[t_2,b]}) > 0$. Therefore, if $U := I^-(z_1)$ and $V := I^+(z_2)$, $p \in U$ and $q \in V$. Now if $U, V \subset Y$ and $r \in U$, $r' \in V$, then due to the reverse triangle inequality of τ and the fact that there is a causal curve connecting p, z_1, z_2, q in pairs, it occurs:

$$\begin{aligned} \tau(r, r') &\geq \tau(r, z_1) + \tau(z_1, z_2) + \tau(z_2, r') \\ &\geq \tau(z_1, z_2) \\ &\geq L(\gamma_{[t_1,t_2]}) \\ &= L(\gamma) - L(\gamma_{[a,t_1]}) - L(\gamma_{[t_2,b]}) \\ &\geq \tau(p, q) - \frac{\epsilon}{2} - \frac{\epsilon}{4} - \frac{\epsilon}{4} \end{aligned} \tag{2.7.1}$$

$$\Rightarrow \tau(r, r') \geq \tau(p, q) - \epsilon \tag{2.7.2}$$

The above constructions also covers the case that $\tau(p, q) = \infty$. □

Therefore, having shown the lower semicontinuity for our formula in (2.5.2), we have a time separation function for the space we introduced in Definition 2.1.1, for which we have given a length-measuring formula in (2.1.5).

2.8 Warped product space as a Lorentzian pre-length space

Therefore, taking into account the properties for our variational length and its relevance in describing the causal relations of the spacetime it pertains to, as well as the time separation function we introduced in (2.5.2), including the Definition 2.6.1, the following definition can be given:

Definition 2.8.1. *Let $(B, \ll, \leq, d, \tau_b, d_B)$ be a Lorentzian length space, (X, d_X) a length space and $f : B \to (0, \infty)$ be continuous. Then the space Y with the product topology $B \times X$, the time separation function τ from Definition 2.5.1 and $\ll, \leq$ from Definition 2.6.1 is defined to be the warped product space of B with X and it is denoted as $B \times_f X$. B is called the base of the warped product space, f is called the warping function and X is called the fibre.*

In practical terms, the way to fathom the meaning of Definition 2.8.1 is in the context of the example of the Schwarzschild metric, given in (1.4.1). In particular, in the context of the (almost) smooth manifold described by the Schwarzschild metric of 2.8.1, our base B is the space $R \times (0, \infty)$ and X is the S^2.

Proposition 2.8.2. *Under the assumptions of 2.8.1, the warped product space $Y = B \times_f X$ is a Lorentzian pre-length space.*

Proof From the formula in (2.1.5) we get a measuring tool for the lengths in the space Y, which leads to a definition of causal curves in Definition 2.2.1 and then in the Definition 2.6.1 that renders Y a causal space. Moreover, from (2.5.2) we get the time separation function τ, which is proven in Lemma 2.6.3 that satisfies the triangle inequality and from Lemma 2.7.1 it is proven to be lower semicontinuous. In addition, from (2.6.3) and (2.6.4) we see that the time separation function satisfies the required properties for the causal relations between points in Y and by taking into account 2.6.9, we see that Y is a Lorentzian pre-length space.

2.9 Fibre independence

So far we have worked with fibres that are length spaces. On this particular case, the fibre doesn't need to have a minimal length-realising curve. However, from the point of view of the base, the equivalent notion of minimisers (in the case of LLS it becomes a maximal curve) is always guaranteed at least locally. In what concerns Y, the existence of maximal curves (i.e curves with maximum length) will play a pivotal role in what follows to lift Y into a LLS, which are spaces of much greater importance, especially for Physics.

In the case that the space Y does have maximal curves connecting two points and if X is geodesic, then we get an important result regarding the connection between the geometry of the fibre and that of the warped product space. Specifically, one of the latent properties in our warped product space is the independence of our structure from the fibre. The following proposition analyses this very topic.

Proposition 2.9.1. *Consider $(B, d_B, \ll, \leq, \tau_b)$ to be a Lorentzian length space, (X, d_X) a geodesic length space, $f : B \to (0, \infty)$ a continuous function in B, $Y = B \times_f X$ and $\gamma = (\alpha, \beta) : [a, b] \to Y$ a future directed causal and unique maximal curve in $[a, b]$. Then:*

1. *The fibre component β is minimizing*
2. *the base component α is independent of β. Specifically, let $(X', d_{X'})$ be another geodesic length space, $\beta' : [a, b] \to X'$ minimizing in X' with $L_{d_{X'}}(\beta') = L_{d_X}(\beta)$ and for any $s_i = s'_{i,1}, s_{i+1} = s'_{i,M} \in [a, b]$, $\sup\{\sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\} =$ $\sup\{\sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} d_{X'}(\beta'(s'_{i,k}), \beta'(s'_{i,k+1}))\}$. Then, $\gamma = (\alpha, \beta')$ is a future directed maximal causal curve in $Y' = B \times_f X'$.*

Proof.

1. We assume that β is not minimizing and that there is such a curve β', for which we get a curve $\gamma' = (\alpha, \beta') : [a, b] \to Y$. Hence, from Corollary A.4.1, we have that:

$$\begin{aligned} L(\gamma') &= \inf\{ \sum_{i=0}^{N-1} \sqrt{\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - m^2_{s_i, s_{i+1}} d^2_X(\beta'(s_i), \beta'(s_{i+1}))} \} \\ &= \inf\{ \sum_{i=0}^{N-1} (\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - \\ &\qquad \sup\{ \sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2)^{\frac{1}{2}} \} \end{aligned} \tag{2.9.1}$$

Now, we take an $\epsilon_i > 0$, s.t $\sum_{i=0}^{N-1} \epsilon_i = \epsilon$ and we can write $L_{d_X}(\beta'_{[s_i,s_{i+1}]}) = L_{d_X}(\beta_{[s_i,s_{i+1}]}) + \epsilon_i = d_X(\beta(s_i), \beta(s_{i+1})) + \epsilon_i$. Thus, we have:

$$\begin{aligned}
L(\gamma') =& \inf\{ \sum_{i=0}^{N-1} (\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 \\
& - \sup\{ \sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} d_X(\beta'(s'_{i,k}), \beta'(s'_{i,k+1}))\}^2)^{\frac{1}{2}} \} \\
=& \inf\{ \sum_{i=0}^{N-1} [\ \tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 \\
& - \sup\{ \sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} (d_X(\beta(s_i), \beta(s_{i+1})) + \epsilon_i)\}^2 \ \ ]^{\frac{1}{2}} \} \\
=& \inf\{ \sum_{i=0}^{N-1} [\ \tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - \sup\{ \sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} \\
& \times d_X(\beta(s_i), \beta(s_{i+1})) + \sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} \epsilon_i\}^2 \ \ ]^{\frac{1}{2}} \}
\end{aligned} \tag{2.9.2}$$

Now, if we consider that $\sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} \epsilon_i > m_{a,b} \sum_{k=1}^{M-1} \epsilon_i = m_{a,b}\epsilon$ and we Taylor-expand, we get:

$$\begin{aligned}
L(\gamma') =& \inf\{ \sum_{i=0}^{N-1} [\ \tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - \sup\{ \sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} \\
& \times d_X(\beta(s_i), \beta(s_{i+1})) + \sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} \epsilon_i\}^2 \ \ ]^{\frac{1}{2}} \} \\
\leq& \inf\{ \sum_{i=0}^{N-1} [\ \tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - (\sup\{ \sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} \\
& \times d_X(\beta(s_i), \beta(s_{i+1}))\} + m_{a,b}\epsilon)^2 \ \ ]^{\frac{1}{2}} \} \\
<& \inf\{ \sum_{i=0}^{N-1} \sqrt{\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - m^2_{s_i,s_{i+1}} d_X^2(\beta(s_i), \beta(s_i))} \} \\
=& L(\gamma)
\end{aligned} \tag{2.9.3}$$

and thus $L(\gamma') < L(\gamma)$, which is a contradiction to the maximality of γ' and hence β is minimizing.

2. We consider a curve β' in the space $Y' = B \times_f X'$, defined on $[a, b]$, such that $L_{d_{X'}}(\beta') = L_{d_X}(\beta)$ and for any $s_i = s'_{i,1}, s_{i+1} = s'_{i,M} \in [a, b]$, $\sup\{\sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}$. Set $\gamma' = (\alpha, \beta') : [a, b] \to Y'$. Then γ' is future directed causal and we have that:

$$\begin{aligned} L(\gamma') =& \inf\{ \sum_{i=0}^{N-1} (\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - \\ & \sup\{ \sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} d_{X'}(\beta'(s'_{i,k}), \beta'(s'_{i,k+1}))\}^2)^{\frac{1}{2}} \} \\ =& \inf\{ \sum_{i=0}^{N-1} (\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - \\ & \sup\{ \sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2)^{\frac{1}{2}} \} \\ =& L(\gamma) \end{aligned} \tag{2.9.4}$$

Moreover, γ' will be maximal. If it wasn't, there would exist a curve $\tilde{\gamma} = (\tilde{\alpha}, \tilde{\beta})$ that would have a length greater than γ' in Y'. However, in that case $L_{d_X}(\tilde{\beta}_{[a,b]}) > L_{d_{X'}}(\beta'_{[a,b]})$, from $\tilde{\beta}(a) = \beta'(a)$ to $\tilde{\beta}(b) = \beta'(b)$. Therefore, by denoting $\bar{\gamma} = (\tilde{\alpha}, \beta')$, it occurs:

$$\begin{aligned} L(\bar{\gamma}) =& \inf\{ \sum_{i=0}^{N-1} (\tau_b(\tilde{\alpha}(s_i), \tilde{\alpha}(s_{i+1}))^2 - \\ & \sup\{ \sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} d_{X'}(\beta'(s'_{i,k}), \beta'(s'_{i,k+1}))\}^2)^{\frac{1}{2}} \} \\ \geq& \inf\{ \sum_{i=0}^{N-1} (\tau_b(\tilde{\alpha}(s_i), \tilde{\alpha}(s_{i+1}))^2 - \\ & \sup\{ \sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} d_{X'}(\tilde{\beta}(s'_{i,k}), \tilde{\beta}(s'_{i,k+1}))\}^2)^{\frac{1}{2}} \} \\ =& L(\tilde{\gamma}) \end{aligned} \tag{2.9.5}$$

and because:

$$\sup\{\sum_{k=1}^{M-1} m_{s'_{i,k},s'_{i,k+1}} d_{X'}(\beta'(s'_{i,k}), \beta'(s'_{i,k+1}))\} = \tag{2.9.6}$$

$$\sup\{\sum_{k=1}^{M-1} m_{s'_{i,k},s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\} \tag{2.9.7}$$

$L((\tilde{\alpha}, \beta)) = L((\tilde{\alpha}, \beta')) = L(\bar{\gamma})$. Hence, we get that $L((\tilde{\alpha}, \beta)) = L(\bar{\gamma}) \geq L(\tilde{\gamma}) \geq L(\gamma') \geq L(\gamma)$, which means that there is a curve $(\tilde{\alpha}, \beta) \in Y$ that contradicts either the maximality or the uniqueness of γ.

$\square$

3

Warped product spaces as Lorentzian length spaces

So far we have shown that if we start with two length spaces (one of which is Lorentzian) and construct a space following the product topology, followed by a formula for the measurement of lengths in this space, i.e equation (2.1.5), we can construct a Lorentzian pre-length space. However, Lorentzian pre-length spaces are not of as great importance as Lorentzian length space, since the latter have a greater extent of applicability, and their existence is imperative for Physics. Hence, in the section that follows we try to go a step further and decide about the conditions our structure needs to obey in order for it to yield a Lorentzian length space.

The direction we need to follow in order to decide about the conditions our structure needs to obey for it to be rendered a Lorentzian length space is given by Definition 3.22, [KS17]. The first step in this direction concerns our topological properties and in particular the question regarding the causal closure of neighbourhoods in our space, as described in Definition 3.4, [KS17]. These neighbourhoods come in the form of what is called the causal diamond, which is defined to be $J(p,q) = J^+(p) \cap J^-(q)$, for two points $p \leq q$ in the warped product space Y.

Before we proceed in the rest of the analysis in this section, we introduce a couple of tools that will be useful in what follows. First and foremost, we introduce $B^d_\delta(z) = \{w \in S : 0 \leq d(z,w) \leq \delta\}$ to be the closed ball of radius δ, for a space S with metric d and two points $z, w \in S$. Moreover, as a continuation of the statement 2. in Definition 1.3.12, we set ω_b to be a local time separation function for neighbourhoods Ω_B of B, with the properties of localisability mentioned in Definition 1.3.12.

3.1 Causal diamond

Therefore, having defined the causal diamond, we give the following proposition that provides a closed boundary for any causal diamond in our space:

Proposition 3.1.1. *If $(B, d_B, \ll, \leq, \tau_b)$ is a Lorentzian length space, (X, d_X) is a length space, $f : B \to (c, +\infty)$ is a continuous function, with $c \in \mathbb{R}^+$ and $p = (\bar{p}, \tilde{p}), q = (\bar{q}, \tilde{q}) \in Y$, $J_B(\bar{p}, \bar{q}) = J_B^+(\bar{p}) \cap J_B^-(\bar{q})$, $M = \inf(\{f : \bar{x} \in J_B(\bar{p}, \bar{q})\})$ and $\mathcal{A}$ is a set given by:*

$$\begin{aligned}\mathcal{A}(p,q) = \{r = (\bar{r}, \tilde{r}) \in Y : &\bar{r} \in J_B(\bar{p}, \bar{q}) \subseteq \Omega_B, \\ &\tilde{r} \in B^{d_X}_{\frac{\omega_b(\bar{p},\bar{r})}{M}}(\tilde{p}) \cap B^{d_X}_{\frac{\omega_b(\bar{r},\bar{q})}{M}}(\tilde{q})\}\end{aligned} \tag{3.1.1}$$

then for the causal diamond $J(p, q)$ we get that:

$$J(p,q) \subseteq \mathcal{A}(p,q) \tag{3.1.2}$$

Proof. To begin with, the projection of $J(p, q)$ will belong to $J_B(\bar{p}, \bar{q})$. Moreover, since B is a LLS (for brevity, we refer to Lorentzian length spaces as LLS), from Definition 1.3.12, there is always a neighbourhood Ω_B, s.t renders $J_B(\bar{p}, \bar{q})$ bounded, while there is also a local time separation ω_b valued in $[0, \infty)$. In addition, from Proposition 2.2.8 we have $L_{\tau_b}(\alpha_{[\bar{p},\bar{r}]}) \geq m_{\bar{p},\bar{r}} d_X(\tilde{p}, \tilde{r})$ for any causal curve $\alpha_{[\bar{p},\bar{r}]} \subset J_B(\bar{p}, \bar{q}) \subset \Omega_B$ and from Definition 1.3.12 $\omega_b(\bar{p}, \bar{r}) \geq L_{\tau_b}(\alpha_{[\bar{p},\bar{r}]})$. Hence we get that:

$$\omega_b(\bar{p}, \bar{r}) \geq m_{\bar{p},\bar{r}} d_X(\tilde{p}, \tilde{r}) \tag{3.1.3}$$

Moreover, for any pair of point $r \in J^+(p) \cap J^-(q)$, we have from Proposition 2.6.9 that $p \leq r \leq q$ and in addition there is a causal curve

$\gamma = (\alpha, \beta) : [a, b] \to Y$ connecting them in pairs, s.t from Corollary 2.2.6:

$$\begin{aligned} L(\gamma_{[p,r]}) =& \inf\{\sum_{i=0}^{N-1} \sqrt{L_{\tau_b}(\alpha_{[s_i,s_{i+1}]})^2 - m^2_{s_i,s_{i+1}} d_X(\beta(s_i), \beta(s_{i+1}))^2}\ \} \\ \leq& \sum_{i=0}^{N-1} \sqrt{L_{\tau_b}(\alpha_{[s_i,s_{i+1}]})^2 - m^2_{s_i,s_{i+1}} d_X(\beta(s_i), \beta(s_{i+1}))^2} \\ \leq& \sum_{i=0}^{N-1} \sqrt{L_{\tau_b}(\alpha_{[s_i,s_{i+1}]})^2 - M^2 d_X(\beta(s_i), \beta(s_{i+1}))^2} \\ \leq& \sqrt{L_{\tau_b}(\alpha_{[\bar{p},\bar{r}]})^2 - M^2 d_X(\beta_{[\tilde{p},\tilde{r}]})^2} \\ \leq& \sqrt{\omega_b(\bar{p},\bar{r})^2 - M^2 d_X(\tilde{p},\tilde{r})^2} \end{aligned} \tag{3.1.4}$$

From the last line in (3.1.4) we get an upper limit for the length of any causal curve connecting p, r. Hence, it occurs that the biggest neighbourhood in the fibre that can lift the pairs of points $\bar{p}, \bar{r}$ to pairs of causally related points in Y will be: $B^{d_X}_{\frac{\omega_b(\bar{p},\bar{r})}{M}}$. Therefore, by making the same consideration for the past causal curve from q, we end up with the neighbourhood $B^{d_X}_{\frac{\omega_b(\bar{r},\bar{q})}{M}}$ and thus, by taking their intersection and by taking the neighbourhood $\Omega_B \times B^{d_X}_{\frac{\omega_b(\bar{p},\bar{r})}{M}}(\tilde{p}) \cap B^{d_X}_{\frac{\omega_b(\bar{r},\bar{q})}{M}}(\tilde{q})$, we prove the claim. □

Remark 3.1.2. *Proposition 3.1.1 also holds if B is a globally hyperbolic Lorentzian length space and $f : B \to (0, +\infty)$, because in this case $J_B(\bar{p}, \bar{q})$ is closed and compact and thus we can set $M = \min(\{f : \bar{x} \in J_B(\bar{p}, \bar{q})\})$, which is always positive.*

From the causal picture above, we get the following lemma describing the convexity of causal neighbourhoods:

Lemma 3.1.3. *If $(B, d_B, \ll, \leq, \tau_b)$ is a strongly causal Lorentzian length space, (X, d_X) is a length space and $f : B \to (c, +\infty)$ is a continuous function, with $c \in \mathbb{R}^+$, then any $p = (\bar{p}, \tilde{p}) \in Y$ has a basis of open, causally convex neighbourhoods, i.e., neighbourhoods such that any causal curve with endpoints in that neighbourhood is contained in it.*

Proof. Let a triplet of points $p = (\bar{p}, \tilde{p})$, $p_{-\epsilon} = (\bar{p}_{-\epsilon}, \tilde{p}_{-\epsilon}), p_{+\epsilon} = (\bar{p}_{+\epsilon}, \tilde{p}_{+\epsilon})$, such that $p_{-\epsilon} \leq p_{+\epsilon}$ are on a causal curve $\gamma = (\alpha, \beta) : [a, b] \to U \subset Y$ and in addition $p, p_{-\epsilon}, p_{+\epsilon} \in \mathcal{F}(p_{-\epsilon}, p_{+\epsilon})$, where:

$$\begin{aligned}\mathcal{F}(p_{-\epsilon}, p_{+\epsilon}) = \{r = (\bar{r}, \tilde{r}) :& \bar{r} \in J_B(\bar{p}, \bar{q}), \\ & \tilde{r} \in \bar{B}^{d_X}_{\frac{\omega_b(\bar{p}_{+\epsilon}, \bar{p})}{M}}(\tilde{p}_{+\epsilon}) \cap \bar{B}^{d_X}_{\frac{\omega_b(\bar{p}, \bar{p}_{-\epsilon})}{M}}(\tilde{p}_{-\epsilon})\}_{\epsilon>0}\end{aligned} \tag{3.1.5}$$

Then, if $\gamma(s_1) = p_{-\epsilon}$, $\gamma(s_2) = p_{+\epsilon}$ and $\gamma(s) = p$, from the fact that γ is a causal curve, $p \in J(p_{-\epsilon}, p_{+\epsilon}) = J^+(p_{-\epsilon}) \cap J^-(p_{+\epsilon})$ and we have that $\omega_b(\alpha(s_1), \alpha(s)) \geq m_{s_1,s_2} d_X(\beta(s_1), \beta(s))$ and $\omega_b(\alpha(s), \alpha(s_2)) \geq m_{s_1,s_2} d_X(\beta(s), \beta(s_2))$. Therefore, for a neighbourhood $\mathcal{U}_X$ in space X, given by the set of points $\mathcal{U}_X = B^{d_X}_{\frac{\omega_b(\alpha(s_1),\alpha(s))}{m_{s_1,s_2}}}(p_{-\epsilon}) \cap B^{d_X}_{\frac{\omega_b(\alpha(s),\alpha(s_2))}{m_{s_1,s_2}}}(p_{+\epsilon})$ and for a neighbourhood $\mathcal{U}_B$, s.t $J_B(\bar{p}, \bar{q}) \subseteq \mathcal{U}_B$, we get a d-neighbourhood U in Y, such that $p \in U = \mathcal{U}_B \times \mathcal{U}_X$. Thus we have that any point on the causal curve connecting $p_{-\epsilon}, p_{+\epsilon}$ will be inside the set $\mathcal{A}(p_{-\epsilon}, p_{+\epsilon})$ from Proposition 3.1.1 and hence inside U, which is a subset of $\mathcal{F}(p_{-\epsilon}, p_{+\epsilon})$ in (3.1.5) and hence the neighbourhood $\mathcal{F}(p_{-\epsilon}, p_{+\epsilon})$ is convex.

For the proof now that the neighbourhoods $\mathcal{F}(p_{-\epsilon}, p_{+\epsilon})$ constitute a basis, we have to observe that if we use as $m_{s_1,s}$ above a minimum corresponding to a given causal curve α, connecting points $\bar{p}_{-\epsilon}, \bar{p}_{+\epsilon}$, then the neighbourhoods U, as described above, would be dependant on the curve α. Hence, if we consider that f is continuous and finite over the set of admissible causal curves from $\bar{p}_{-\epsilon}, \bar{p}_{+\epsilon}$, valued in $(c, +\infty)$, with $c \in \mathbb{R}^+$, then we understand that f obtains a finite value for its infimum in $\mathcal{U}_B$, which we call M. Thus, if we substitute m_{s_1,s_2} with M in $\mathcal{U}_X$ above, the set $\mathcal{F}(p_{-\epsilon}, p_{-\epsilon})$ will always contain $J(p_{-\epsilon}, p_{-\epsilon})$ and any set of $\mathcal{F}(p_{-\epsilon}, p_{-\epsilon})$ can be written as a union of sets belonging to a subfamily, hence proving the claim. $\square$

From the proof of the Lemma 3.1.3 we can also get another important property of causality. Specifically, we see that the two different topologies (according to Definition 2.4, [KS17]) match. Hence, this shows that:

Lemma 3.1.4. *If $Y = B \times_f X$ is a Lorentzian pre-length space, with $(B, d_B, \ll, \leq, \tau_b)$ a strongly causal Lorentzian length space and (X, d_X) a length space, then Y is strongly causal.*

Proof. As in the proof above, if we have a triplet of points $p = (\bar{p}, \tilde{p})$, $p_{-\epsilon} = (\bar{p}_{-\epsilon}, \tilde{p}_{-\epsilon}), p_{+\epsilon} = (\bar{p}_{+\epsilon}, \tilde{p}_{+\epsilon})$, such that $p_{-\epsilon} \ll p_{+\epsilon}$ are on a causal curve

$\gamma = (\alpha, \beta) : [a, b] \to U \subset Y$ and that $p \in \mathcal{I}(p_{-\epsilon}, p_{+\epsilon}) = I^+(p_{-\epsilon}) \cap I^-(p_{+\epsilon})$, then from the push-up property of Lemma 2.6.9 we get that any causal (timelike in specific) curve connecting $p_{-\epsilon}, p_{+\epsilon}$ will be entirely in $\in \mathcal{I}(p_{-\epsilon}, p_{+\epsilon})$, as $p_{+\epsilon} \in I^+(p_{-\epsilon})$ and $p_{-\epsilon} \in I^-(p_{+\epsilon})$. Therefore, $\mathcal{I}(p_{-\epsilon}, p_{+\epsilon})$ agrees with $\mathcal{F}(p_{-\epsilon}, p_{+\epsilon})$ and therefore could be used as a family of open sets that could constitute a basis for the topology, meaning that the warped product Lorentzian pre-length space $Y = B \times_f X$ is strongly causal. □

3.2 *Causal curves - revisited*

The notion of the causal curves for a general warped product space (as the latter is introduced in Definition 2.8.1) are given in 2.2.1. From that definition and the Lemma 2.6.6 we can get a connection between the timelike curves and all the timelike related points in the neighbourhood of a point in Y. However, the restriction imposed to the reasoning that has been followed in order to obtain this particular result is that it is all dependent on L_{var} and therefore dependent on relation (2.1.5).

From Definition 2.18 of [KS17] though, we have the general definition of a causal curve, that is not dependent on any particular length-measuring formula and hence it serves as a more general definition. To the best of our knowledge, at the moment of conducting this research, this is the most general definition of a causal curve in the (published) literature. Nevertheless, we are in position to know that there are attempts for this definition to be amended. Hence, we need to show that our definition of causal curves, emerging from the length-measuring formula in (2.1.5), is compliant with the more general Definition 2.18 of [KS17]. This is achieved in our setting via the following Lemma:

Lemma 3.2.1. *The notion of causal curves wrt to the causal relation $\leq$, as given in Definition 2.18 in [KS17] agrees with the notion of causal curves for a warped product space from Definition 2.2.1, for B being a strongly causal Lorentzian length space.*

Proof. The proof is along the lines of Lemma 4.4, [AGKS19]. Let's take a curve γ, such that $\gamma = (\alpha, \beta) : [a, b] \to Y = B \times_f X$. From the causal relations given in (2.6.1) and from the structure of $J^+(p)$ in 2.6.9, we have that if γ is future-directed causal in the warped product space $Y = B \times_f X$, it has $L(\gamma_{[s,s+\epsilon]}) \geq 0$ and it is contained in $J^+(p)$. Since all the $\leq$-related points are in $J^+(p)$, then we have that all causal curves wrt Definition 2.2.1 agree with the definition wrt $\leq$ relation. The same logic applies to past-directed causal curves leading to the same equivalence.

For the converse case, if γ is a future directed causal curve wrt $\leq$ causal relations (the same arguments apply to a past directed causal curve) then $\forall a \leq s < t \leq b$, $\gamma(s) \leq \gamma(t)$. Since, $\gamma(s), \gamma(t)$ are $\leq$-related, from Definition 2.6.1 there is a curve γ' and a section $\gamma'_{s,t}$ of it, with the property that $L(\gamma'_{s,t}) \geq 0$, which in general does not need necessarily to coincide with γ. This coincidence is what we will try to show below.

Therefore, utilizing this fact, we partition the interval $[a, b]$ into smaller subintervals $[t_i, t_{i+1}]$, for which $[a, b] = \bigcup_i [t_i, t_{i+1}]$, with $t_0 = a$ and $t_N = b$. For each $[t_i, t_{i+1}]$ there is a causal curve, as described by Definition 2.2.1, connecting $\gamma(t_i), \gamma(t_{i+1})$. Now we want to take the concatenation of these locally causal curves and see that they converge pointwise to γ. Indeed by taking a partition of $[a, b]$, corresponding to N causal segments $\gamma'_{t_i,t_{i+1}}$, we get $\gamma_N = \gamma'_{[a,t_1]} * \gamma'_{[t_1,t_2]} \cdots * \gamma'_{[t_{N-1},b]}$. Let σ_N be a sequence of such partitions, whose norms tend to zero as $N \to \infty$. Now, we consider a neighbourhood U of a point γ_{t_i}, that can be convex from the Lemma 3.1.3. This means that any causal curve with endpoints on a segment of γ and inside U will have points that remain in U. Moreover, from Proposition 2.3.1, we know that γ will have a reparametrization that renders it locally Lipschitz continuous and hence all segments of the curve in the given partition inside U will be uniformly Lipschitz continuous too. By allowing $N \to \infty$ and thus shrinking these U neighbourhoods across γ, we get a pointwise convergence of γ_N to γ and thus, from Proposition 2.4.2, the sequence converges to a causal curve that is causal wrt $\leq$ causal relations (i.e Definition 2.18 of [KS17]). □

From the Lemma 3.2.1 we also get as immediate consequence of it and Definition 3.1 in [KS17] that:

Lemma 3.2.2. *Let $(B, d_B, \ll, \leq, \tau)$ a Lorentzian length space and (X, d_X) be a length space. Then the warped product space $Y = B \times_f X$ is a causally path connected Lorentzian pre-length space.*

Proof. The proof is straightforward as for any pair of points $p, q \in Y$ that are $\leq$-related, there is a causal curve (in the sense of the Definition 2.2.1) that is inside $J(p, q)$. □

3.3 Warped product space as Lorentzian length space

Before proceeding to the discussion of Lorentzian length space structure for Y, another property that is found in our given structure so far, is that our space is locally d-compatible.

Lemma 3.3.1. *Every warped product space $Y = B \times_f X$, with $f : B \to (c, +\infty)$ and $c \in \mathbb{R}^+$ has the property that for every point $x \in Y$ there is a neighbourhood U of x and a constant $C > 0$ such that the d-arclength of every causal curve which is contained in U is bounded by C, i.e., $L_d(\gamma) \leq C$.*

Proof. As we did in the proof for Proposition 3.1.1, from Proposition 2.2.8, Definition 1.3.12 and (3.1.4), we have that d_X is bounded for every pair of points $p, q \in Y$. If we substitute M, in (2.2.17) and because $\omega_b(\bar{p}, \bar{q}) \geq L_{\tau_b}(\alpha_{[\bar{p},\bar{q}]})$, we get that:

$$\frac{\omega_b(\bar{p}, \bar{q})}{M} \geq \sup\{\sum_{k=0}^{M-1} d_X(\beta(t_k), \beta(t_{k+1}))\} = L_{d_X}(\beta_{[\tilde{p},\tilde{q}]}) \tag{3.3.1}$$

and hence the d_X-arclength between $\tilde{p}, \tilde{q}$ will be bounded too, by a constant C_X. Moreover, since B is a LLS, we get that any curve connecting $\bar{p}, \bar{q}$ will have d_B-arclength that is bounded by a constant C_B. Therefore, for any curve $\gamma : [a, b] \to Y$, with $s_1, s_2 \in [a, b]$, we get:

$$\begin{aligned} d(\gamma(s_1), \gamma(s_2)) =& \sqrt{d_B^2(\alpha(s_1), \alpha(s_2)) + d_X^2(\beta(s_1), \beta(s_2))} \\ \leq& \sqrt{L_{d_B}^2(\alpha_{[s_1,s_2]}) + L_{d_X}^2(\beta_{[s_1,s_2]})} \\ \leq& L_{d_B}(\alpha_{[s_1,s_2]}) + L_{d_X}(\beta_{[s_1,s_2]}) \end{aligned} \tag{3.3.2}$$

which leads to:

$$\begin{aligned} L_d(\gamma_{[p,q]}) =& \sup\{\sum_{i=0}^{N-1} d(\gamma(s_i), \gamma(s_{i+1}))\} \\ \leq& \sup\{\sum_{i=0}^{N-1} L_{d_B}(\alpha_{[s_i,s_{i+1}]}) + L_{d_X}(\beta_{[s_i,s_{i+1}]})\} \\ =& L_{d_B}(\alpha_{[\bar{p},\bar{q}]}) + L_{d_X}(\beta_{[\tilde{p},\tilde{q}]}) \\ \leq& C_B + C_X \end{aligned} \tag{3.3.3}$$

and hence the d-arclength of any curve γ will also be bounded from above by a constant. $\square$

The arguments of the proof above can be used to handle another basic property of the warped product spaces; the existence of maximal causal curves.

Proposition 3.3.2. *Let $(B, d_B, \ll, \leq, \tau_b)$ be a globally hyperbolic Lorentzian length space and (X, d_X) be a locally compact metric space. Then every point in $Y = I \times_f X$ has a neighborhood U such that any two causally related points in U can be connected by a maximal causal curve.*

Proof. First thing is to create neighbourhoods of points. For that, we need to focus on the base, as the overall neighbourhood is defined as in the one dimensional case. Therefore, considered a neighbourhood $U = V \times W$ in Y, such that $V \subset B$ and $W \subset X$, we get a neighbourhood U' s.t $U' \subseteq U$, as well as causally closed and convex. So for the base, there is the metric, which can be used to parametrize a curve α, using the d_B-length. Since, the curve is causal it will be Lipschitz continuous, i.e its d_B-length will be bounded by a constant C. Therefore for any point $p = (\bar{p}, \tilde{p})$, with $\bar{p} \in V$, any other point $q = (\bar{q}, \tilde{q})$ that belongs in U' is causally connected with p through a causal curve, which abides by the convexity lemma and thus lives inside U'. Take also $\gamma_n : [a, b] \to Y$ to be a sequence of future directed causal curves from p to q, such that $L(\gamma_n) \to \tau(p, q)$. Since γ_n are causal curves connecting p, q they live inside U'. From reparametrization invariance in Proposition 2.1.5, we see that we can use $\Phi(s)$ from Proposition 2.3.1 to reparametrize γ_n as $\gamma_n = \lambda_n \circ \Phi$, without changes to their lengths and causal character and thus obtain a reparametrization that is Lipschitz continuous. Thus from the theorem of Arzela-Ascoli, obtained is a subsequence $\tilde{\gamma}_{n'}$ of $\tilde{\gamma}_n$ that convergences uniformly to a Lipschitz curve γ from p to q. Since the latter is a curve that is non-constant, connecting the causally related points p, q, it must be causal and inside U'. Therefore, we get that:

$$L(\gamma) \leq \tau(p, q) = \lim_{n \to \infty} L(\gamma_{n'}) \leq L(\gamma) \tag{3.3.4}$$

and therefore there exists a maximal curve in U'. $\square$

The above reasoning also indicates that Y is locally causally closed (as local closeness is defined in Definition 1.3.9):

Lemma 3.3.3. *Let $Y = B \times_f X$ be a warped product space, with $(B, d_B, \ll \leq, \tau_b)$ a globally hyperbolic Lorentzian length space. Moreover, (X, d) is a locally compact length space and $f : (c, +\infty) \to \mathbb{R}^+$. Then, every point in Y has a neighbourhood U such that for any y_n, $z_n \in Y$ with $y_n \to y \in U$, $z_n \to z \in U$ and $y_n \leq z_n$ for all $n \in N$, it follows that $y \leq z$.*

Proof. The proof makes use of the same pointwise convergence arguments that we used above, in Proposition 3.3.2, to prove the existence of maximal curves. □

Next thing to establish is the relation between the length of a curve and the τ-length (calculated via the time separation function). From Proposition 4.7 of [AGKS19] we get the identification of the two and so for our case of warped product spaces:

Proposition 3.3.4. *Let $(B, d_B, \ll, \leq, \tau_b)$ is a Lorentzian length space and (X, d_X) is a metric space. If $\gamma : [a, b] \to Y$ is a future directed causal curve, then $L(\gamma) = L_\tau(\gamma)$:*

$$L(\gamma) = L_\tau = \inf\{\sum_{i=0}^{N-1} \tau(\alpha(s_i), \alpha(s_{i+1})) \colon \ a = t_0 < t_1 < ... < t_n = b\} \quad (3.3.5)$$

From this proposition, together with (Lemma 3.2.1) and by further probing the localisability of neighbourhoods U of Y, we can argue about the classification of our warped product space as a Lorentzian length space. Starting with the latter we get:

Lemma 3.3.5. *Any warped product space $Y = B \times_f X$, with $(B, d_B, \ll, \leq, \tau_b)$ a globally hyperbolic Lorentzian length space and (X, d_X) a locally compact length space, is localisable in the sense that every point p has a neighbourhood U such that:*

- *$L_d(\gamma) \leq C$ for some $C > 0$ and all causal curves γ contained in U.*
- *there is a continuous $\omega : U \times U \to [0, \infty)$ such that $(U, d_{U\times U}, \ll_{U\times U} \leq_{U\times U}, \omega)$ is a Lorentzian pre-length space with $I^\pm(q) \cap U \neq \oslash$, for all $q \in U$*
- *for all pairs of points $q \leq q'$ in U, there is a causal curve γ from q to q' that is maximal and for which $L(\gamma) = \omega(q, q') \leq \tau(q, q')$.*

Proof. The first point concerning bounds of L_d is proven in Lemma 3.3.1.

For the second point, the specific case of warped product space in the form of the generalized cone with the one dimensional base in [AGKS19] offers knowledge regarding the localisability of open neighbourhoods U of Y that is also transferable to our case of warped product space. Hence, following a similar series of arguments, we can consider a neighbourhood U, s.t besides the requirement for $L_d(\gamma) \leq C$ above, it also belongs to the causally convex neighbourhoods of 3.1.3, while it is also set to be a neighbourhood where all points in it can be connected with maximal curves. In addition, we define $\omega =: \tau|_{U\times U}$, which $\tau|_{U\times U}$ is given by Definition 2.5.1, for curves in U and thus it is finite inside it. Moreover, $\tau|_{U\times U}$ is lower semicontinuous and therefore so is ω. In addition, from Proposition 3.3.2, we know that inside U there are always maximal causal curves between any pair of points and so there is a causal curve in U with length that realises $\tau|_{U\times U}$. But since the length is shown in Proposition 2.4.1 to be upper semicontinuous, then $\tau|_{U\times U}$ has to be too. Consequently, ω must be continuous inside U and reckoning with the local existence of maximal causal curves, our space Y is shown to be localisable. □

Having proven localisability of Y, we are ready to argue about the latter being classified as Lorentzian length space:

Proposition 3.3.6. *Any warped product space $Y = B \times_f X$, with $(B, d_B, \ll \leq, \tau_b)$ a globally hyperbolic Lorentzian length space and (X, d_X) a locally compact length space, is a Lorentzian length space.*

Proof. By Proposition 2.8.2 and Lemma 3.3.3 Y is a locally causally closed Lorentzian pre-length space. Moreover, from Lemma 3.2.2, it is also causally path connected.

From equation (1.3.1), Proposition 3.3.4 and the equivalence in the notion of causal curves from Lemma 3.2.1 we directly obtain that $\tau = \mathcal{T}$. Finally, from Lemma 3.3.5 we have that Y is localisable and hence it is proven to be a Lorentzian length space. □

Remark 3.3.7. *Any Lorentzian warped product space $Y = B \times_f X$ in the sense of Proposition 3.3.6 automatically implies Y being strongly causal, from the fact that any globally hyperbolic LLS is strongly causal (Theorem 3.26-iv in [KS17]), as well as the convexity of Lemma 3.1.3, Definition 2.35-iv, and Theorem 3.26-iv in [KS17].*

Having given the conditions of closure of causal diamond $J^{\pm}(p,q)$ for $p,q \in Y$ in 2.6.9 and for a globally hyperbolic LLS B, we infer that:

Proposition 3.3.8. *If $Y = B \times_f X$ is a warped product space with $(B, d_B, \ll, \leq, \tau_b)$ a globally hyperbolic Lorentzian length space and (X, d_X) a geodesic length space that is also proper metric space, then Y is a globally hyperbolic space.*

Proof. From Proposition 3.3.6 we know that Y is a strongly causal Lorentzian length space and hence non-totally imprisoning by Theorem. 3.26-iii, [KS17]. Therefore, from Proposition 2.6.9, where we see that $J^{\pm}(p)$ is closed and from 3.1.1 that the causal diamond $J(p,q)$ is contained within a closed set. Therefore, since B is globally hyperbolic, every $J_B(p,q)$ is compact and so $J(p,q)$ is compact too. Consequently, from Definition 2.35-v, [KS17] it is globally hyperbolic. $\square$

4 Discussion

Our purpose in what has preceded was to establish the geometrical foundations for a generalisation of warped product spaces and generalised cones (see [AGKS19]). In doing so, we showed that by choosing a formula for the lengths of causal curves in a space of product topology $Y := B \times X$, our formula satisfies the properties required for a length-measuring formula. In addition, in Proposition 2.1.6, we showed that this formula is equivalent to the length measuring formula in the smooth setting.

Moreover, in corollaries 2.2.7 and 2.2.6, we showed that there are alternative formulas to (2.1.5), that are also useful in providing a more local picture of the causal character of curves in the absence of curves, as proven in Proposition 2.2.8.

In addition, we showed in Proposition 2.8.2, that the geometrical structure we have chosen, with the given length-measuring formulas, the time separation function and the set of causally related points, constitute a Lorentzian pre-length space and the properties shown to be present in the case of the generalised cones of [AGKS19] and those known for smooth spacetimes, do generalise in the case of higher dimension for the base B. Finally, in section 3, we showed that our structure further satisfies the conditions of Definition 1.3.14, rendering it thus a Lorentzian length space.

Having provided the casual picture for a generalised warped product space, we see that the structure of these spaces lacks general pathologies, frequently encountered in the causal structure of spacetimes, like causal bubbling (as an example see Example 1.11 of [CG11]).

As an immediate next step in the work done here, would be the study of curvature bounds. Due to the further complexity of the generalised warped product spaces studied here, in comparison to the generalised cones (product spaces with one dimensional base) the study of curvature bounds, requires some further treatment that is still not properly structured. For example, if someone was to choose the route of [AB04] in showing the curvature bounds of a structure like the one shown here, the absence of an equivalent of the gluing theorem for metric spaces as well as the meagre knowledge on hyperbolic angles and spaces of directions would be the main obstacles of such an endeavour. Therefore, an attempt to bridge these gaps should help propel the knowledge established on this thesis on warped product spaces and help to further advance the wider scope of understanding on Lorentzian geometry.

APPENDIX

A [redacted]

Appendix

A.1 Non-constant f and connections to P

We assume that we have a Lorentzian length space $(B, d_B, \ll, \leq, \tau_b)$, a metric space (X, d_X), a space Y the topology of which is given by Definition 2.1.1 and a curve $\gamma : [a, b] \to Y$, along with two points on it, $\gamma(c), \gamma(d)$, such that $a \leq c < d \leq b$ and $\alpha(a) \leq \alpha(c) \leq \alpha(d) \leq \alpha(d)$. From Definition 2.1.2 and Remark 2.2.3 we have that:

$$P(\gamma(s_1), \gamma(s_2)) = \sqrt{\Psi_\gamma(\gamma(s_1), \gamma(s_2))} \tag{A.1.1}$$

when $\Psi_\gamma(\gamma(s_1), \gamma(s_2)) \geq 0$. Moreover, from Corollary 2.2.2, we have that $P(\gamma(s_1), \gamma(s_2)) \geq 0$. An implicit assumption of this condition is that the time separation function in B, τ_b, is actually bounded for a curve to be causal, as it would be naturally expected. In order to take the actual bounds of this function, we need to consider what happens in the case that $P(\gamma(s_1), \gamma(s_2)) = 0$ and hence $\tau_b(\alpha(s_1), \alpha(s_2)) = m_{s_1,s_2} d_X(\beta(s_1), \beta(s_2))$ $\forall s_i \in [a, b]$.

In the case that f is a constant function in $[a, b]$ on γ, then the bounds on τ_b are set indeed by taking the case that $P(\gamma(s_1), \gamma(s_2)) = 0$. Same reasoning also applies in the case that τ_b, d_X are differentiable. However, in the case that f is not constant in $[a, b]$ we need to follow a different reasoning. Hence, in what follows we assume that $m_{s_1,s_2} < m_{s_2,s_3}$ and we consider the following cases, with any other possible case pertaining to one of those listed below:

$\underline{P(\gamma(s_i),\gamma(s_j)) = 0\ ,\ \forall s_i, s_j \in [a,b],\ i,j \in \mathbb{N}}$:

$$\begin{aligned}\tau_b(\alpha(s_1),\alpha(s_2)) =& m_{s_1,s_2} d_X(\beta(s_1),\beta(s_2))\\ \tau_b(\alpha(s_2),\alpha(s_3)) =& m_{s_2,s_3} d_X(\beta(s_2),\beta(s_3)),\\ \tau_b(\alpha(s_1),\alpha(s_3)) =& m_{s_1,s_3} d_X(\beta(s_1),\beta(s_3))\end{aligned} \tag{A.1.2}$$

for $s_1, s_2, s_3 \in [a,b]$. Hence, we get:

$$d_X(\beta(s_1),\beta(s_2)) + d_X(\beta(s_2),\beta(s_3)) \geq d_X(\beta(s_1),\beta(s_3)) \tag{A.1.3}$$

$$\begin{aligned}\Rightarrow\quad & m_{s_1,s_3} d_X(\beta(s_1),\beta(s_2)) + m_{s_1,s_3} d_X(\beta(s_2),\beta(s_3)) = m_{s_1,s_2} d_X(\beta(s_1),\beta(s_2))\\ & + m_{s_1,s_3} d_X(\beta(s_2),\beta(s_3)) \geq m_{s_1,s_3} d_X(\beta(s_1),\beta(s_3))\end{aligned} \tag{A.1.4}$$

where in the last step we used (w.l.o.g) that $m_{s_1,s_2} = \min\{m_{s_1,s_2}, m_{s_2,s_3}\}$ and that $m_{s_1,s_2} = m_{s_1,s_3}$. Consequently:

$$\begin{aligned}& m_{s_1,s_2} d_X(\beta(s_1),\beta(s_2)) + m_{s_1,s_3} d_X(\beta(s_2),\beta(s_3)) \geq m_{s_1,s_3} d_X(\beta(s_1),\beta(s_3))\\ \Rightarrow\quad & m_{s_1,s_2} d_X(\beta(s_1),\beta(s_2)) + m_{s_2,s_3} d_X(\beta(s_2),\beta(s_3)) > m_{s_1,s_3} d_X(\beta(s_1),\beta(s_3))\\ \Rightarrow\quad & \tau_b(\alpha(s_1),\alpha(s_2)) + \tau_b(\alpha(s_2),\alpha(s_3)) > \tau_b(\alpha(s_1),\alpha(s_3))\end{aligned} \tag{A.1.5}$$

which is a contradiction.

$\underline{P(\gamma(s_1),\gamma(s_3)) = 0\ ,\ P(\gamma(s_1),\gamma(s_2)) > 0\ ,\ P(\gamma(s_2),\gamma(s_3)) > 0}$:

From (A.1.3), (A.1.4) and the reasoning of (A.1.5) we are led to another contradiction.

$\underline{P(\gamma(s_1),\gamma(s_3))\ ,\ P(\gamma(s_1),\gamma(s_2))\ ,\ P(\gamma(s_2),\gamma(s_3)) > 0}$:

This is the only case in which we don't end up back to (A.1.5) and hence it is admissible.

Therefore, we end up with the fact that $P(\gamma(s),\gamma(t)) > 0$, $\forall s,t \in [a,b]$, for the case that f is a monotonically increasing in $[a,b]$. With similar arguments we get the same result when f is monotonically decreasing in $[a,b]$.

A.2 Alternative formula for Corollary 2.2.6

Corollary A.2.1. *For (X, d_X) a length space, $(B, d_B, \ll, \leq, \tau_b)$ a Lorentzian length space, Y a space that has the product topology defined in 2.1.1, a continuous function $f : B \to (0, +\infty)$ and $\Psi_\gamma \geq 0$, the length of a causal curve $\gamma : [a, b] \to Y$ is given by:*

$$L(\gamma) = \inf\{\sum_{i=0}^{N-1}[\ \tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - \sup\{\sum_{k=0}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} \times d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2\ ]^{\frac{1}{2}}\} \tag{A.2.1}$$

where $M \in \mathbb{N}$, $i \in [0, N-1]$, $k \in [0, M-1]$, $s_{i+1} \in (a = s_0, b = s_N]$ and $s'_{i,k+1} \in (s_i = s'_{i,0}, s_{i+1} = s'_{i,M}]$.

Proof. We take the length formula given in (2.1.5). We know that there is always a partition with given M, N for which we can have a value $\epsilon_i > 0$, arbitrarily small, s.t
$\sup\{\sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\} - \epsilon_i < m_{s_i, s_{i+1}} d_X(\beta(s_i), \beta(s_{i+1}))$.
Therefore, taking also into account that $P(\gamma(s_i), \gamma(s_{i+1})) > 0$, we get:

$$\begin{aligned}
L(\gamma) &= \inf\{\sum_{i=0}^{N-1} \sqrt{\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - m^2_{s_i, s_{i+1}} d_X(\beta(s_i), \beta(s_{i+1}))^2}\} \\
&= \inf\{\sum_{i=0}^{N-1} [\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - \sup\{\sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} \\
&\qquad \times d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2 - \epsilon_i]^{\frac{1}{2}}\} \\
&= \inf\{\sum_{i=0}^{N-1} [\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - \sup\{\sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} \\
&\qquad \times d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2]^{\frac{1}{2}} \\
&\quad \times [\ 1 - \frac{\epsilon_i}{\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - \sup\{\sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2}\ ]^{\frac{1}{2}}\} \\
&= \inf\{\sum_{i=0}^{N-1} [\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - \sup\{\sum_{k=1}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} \\
&\qquad \times d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2] \\
&\quad \times \sqrt{1 - \epsilon'_i}\}
\end{aligned} \tag{A.2.2}$$

where we have set $\epsilon'_i = \frac{\epsilon_i}{\tau_b(\alpha(s_i),\alpha(s_{i+1}))^2 - \sup\{\sum_{k=1}^{M-1} m_{s'_{i,k},s'_{i,k+1}} d_X(\beta(s'_{i,k}),\beta(s'_{i,k+1}))\}^2}$.
Next, we consider the Taylor expansion of the second factor in the last line of (A.2.2) and we get:

$$
\begin{aligned}
L(\gamma) = &\inf\{\sum_{i=0}^{N-1}[\tau_b(\alpha(s_i),\alpha(s_{i+1}))^2 - \sup\{\sum_{k=1}^{M-1} m_{s'_{i,k},s'_{i,k+1}} \\
&\times d_X(\beta(s'_{i,k}),\beta(s'_{i,k+1}))\}^2]^{\frac{1}{2}}\sqrt{1-\epsilon'_i}\} \\
= &\inf\{\sum_{i=0}^{N-1}[\tau_b(\alpha(s_i),\alpha(s_{i+1}))^2 - \sup\{\sum_{k=1}^{M-1} m_{s'_{i,k},s'_{i,k+1}} \\
&\times d_X(\beta(s'_{i,k}),\beta(s'_{i,k+1}))\}^2]^{\frac{1}{2}}(1-\frac{1}{2}\epsilon'_i + O(\epsilon'_i))\}
\end{aligned} \tag{A.2.3}
$$

As we make the partition denser in the subroot expression of (A.2.3), we allow $\epsilon' \to 0$. Consequently:

$$
\begin{aligned}
\lim_{\epsilon'\to 0} L(\gamma) = &\inf\{\sum_{i=0}^{N-1}(\tau_b(\alpha(s_i),\alpha(s_{i+1}))^2 - \sup\{\sum_{k=1}^{M-1} m_{s'_{i,k},s'_{i,k+1}} \\
&\times d_X(\beta(s'_{i,k}),\beta(s'_{i,k+1}))\}^2)^{\frac{1}{2}} \times (1-\frac{1}{2}\epsilon'_i + O(\epsilon'_i))\} \\
= &\inf\{\sum_{i=0}^{N-1}(\tau_b(\alpha(s_i),\alpha(s_{i+1}))^2 - \sup\{\sum_{k=1}^{M-1} m_{s'_{i,k},s'_{i,k+1}} \\
&\times d_X(\beta(s'_{i,k}),\beta(s'_{i,k+1}))\}^2)^{\frac{1}{2}}\}
\end{aligned} \tag{A.2.4}
$$

and thus we have proven the claim. □

A.3 Alternative proof for Corollary 2.2.6

Corollary A.3.1. *For (X, d_X) a length space, $(B, d_B, \ll, \leq, \tau_b)$ a Lorentzian length space, Y a space that has the product topology defined in 2.1.1, a continuous function $f : B \to (0, +\infty)$ and $\Psi_\gamma \geq 0$, the length of a causal curve $\gamma : [a, b] \to Y$ is given by:*

$$L(\gamma) = \inf\{\sum_{i=0}^{N-1} \sqrt{L_{\tau_b}(\alpha_{[s_i,s_{i+1}]})^2 - \sup\{\sum_{k=0}^{M-1} m_{s'_{i,k},s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2}\ \} \tag{A.3.1}$$

where $M \in \mathbb{N}$, $i \in [0, N-1]$, $k \in [0, M-1]$, $s_{i+1} \in (a = s_0, b = s_N]$ and $s'_{i,k+1} \in (s_i, s_{i+1}]$, with $s_i = s'_{i,0}, s_{i+1} = s'_{i,M} = s'_{i+1,0}$.

Proof. The proof is exactly the same as in Corollary A.2.1, so here we will give the outline of it. Specifically, we know that there is always a partition with given M, N for which we can have a value $\epsilon_i > 0$, arbitrarily small, s.t $\sup\{\sum_{k=0}^{M-1} m_{s'_{i,k},s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\} - \epsilon_i < m_{s_i,s_i} d_X(\beta(s_i), \beta(s_i))$. Therefore:

$$\begin{aligned}
&\inf\{\sum_{i=0}^{N-1}[\ [\inf\{\sum_{k=0}^{M-1} \tau_b(\alpha(s'_{i,k}), \alpha(s'_{i,k+1}))\} + \epsilon_i\ ]^2 - \sup\{\sum_{k=0}^{M-1} m_{s'_{i,k},s'_{i,k+1}} \\
&\quad \times d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2\ ]^{\frac{1}{2}}\} \\
&= \inf\{\sum_{i=0}^{N-1} \sqrt{L_{\tau_b}(\alpha_{[s_i,s_{i+1}]})^2 + \epsilon'_i - \sup\{\sum_{k=0}^{M-1} m_{s'_{i,k},s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2}\ \} \\
&= \inf\{\sum_{i=0}^{N-1} \sqrt{L_{\tau_b}(\alpha_{[s_i,s_{i+1}]})^2 - \sup\{\sum_{k=0}^{M-1} m_{s'_{i,k},s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2} \\
&\quad \times \sqrt{1 + \epsilon''_i}\ \}
\end{aligned} \tag{A.3.2}$$

where we set

$$\epsilon'_i = \frac{2\epsilon_i}{L_{\tau_b}(\alpha_{[s_i,s_{i+1}]})} + \frac{\epsilon_i^2}{L_{\tau_b}(\alpha_{[s_i,s_{i+1}]})^2} << 1$$

and also made the substitution

$$\epsilon''_i = \frac{\epsilon'_i}{\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - \sup\{\sum_{k=1}^{M-1} m_{s'_{i,k},s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2} << 1$$

Hence, from here the proof proceeds as in Corollary A.2.1 and we get:

$$\begin{aligned} L(\gamma) = \inf\{\sum_{i=0}^{N-1} [\, L_{\tau_b}(\alpha_{[s_i,s_{i+1}]})^2 - \sup\{\sum_{k=0}^{M-1} m_{s'_{i,k},s'_{i,k+1}} \\ \times d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2 \,]^{\frac{1}{2}}\} \end{aligned} \tag{A.3.3}$$

□

A.4 Alternative formula for L_{var}

Corollary A.4.1. *For (X, d_X) a length space, $(B, d_B, \ll, \leq, \tau_b)$ a Lorentzian length space, Y a space that has the product topology defined in 2.1.1, a continuous function $f : B \to (0, +\infty)$ and $\Psi_\gamma \geq 0$, the length of a causal curve $\gamma : [a, b] \to Y$ is given by:*

$$L(\gamma) = \inf\{\sum_{i=0}^{N-1}[\ \tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - \sup\{\sum_{k=0}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} \times d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\}^2\ ]^{\frac{1}{2}}\} \tag{A.4.1}$$

where $M \in \mathbb{N}$, $i \in [0, N-1]$, $k \in [0, M-1]$, $s_{i+1} \in (a = s_0, b = s_N]$ and $s'_{i,k+1} \in (s_i, s_{i+1}]$, with $s_i = s'_{i,0}, s_{i+1} = s'_{i,M} = s'_{i+1,0}$.

Proof. We take the length formula given in (2.1.5). We know that there is always a partition with given M, N for which we can have a value $\epsilon_i > 0$, arbitrarily small, s.t $\sup\{\sum_{k=0}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\} - \epsilon_i < m_{s_i, s_{i+1}} d_X(\beta(s_i), \beta(s_{i+1})) < \sum_{k=0}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))$. Therefore, taking also into account that $P(\gamma(s_i), \gamma(s_{i+1})) > 0$, we get:

$$\begin{aligned}
L(\gamma) =& \inf\{\sum_{i=0}^{N-1} \sqrt{\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - m^2_{s_i, s_{i+1}} d_X(\beta(s_i), \beta(s_{i+1}))^2}\} \\
\leq& \inf\{\sum_{i=0}^{N-1}[\ \ \tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - [\ \ \sup\{\sum_{k=0}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} \\
&\times d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\} - \epsilon_i\ \ ]^2\ \ ]^{\frac{1}{2}}\} \\
=& L(\gamma)_U
\end{aligned} \tag{A.4.2}$$

Moreover:

$$\begin{aligned}
L(\gamma) =& \inf\{\sum_{i=0}^{N-1} \sqrt{\tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - m^2_{s_i, s_{i+1}} d_X(\beta(s_i), \beta(s_{i+1}))^2}\} \\
\geq& \inf\{\sum_{i=0}^{N-1}[\ \ \tau_b(\alpha(s_i), \alpha(s_{i+1}))^2 - [\ \ \sum_{k=0}^{M-1} m_{s'_{i,k}, s'_{i,k+1}} \\
&\times d_X(\beta(s'_{i,k}), \beta(s'_{i,k+1}))\ \ ]^2\ \ ]^{\frac{1}{2}}\} \\
=& L(\gamma)_L
\end{aligned} \tag{A.4.3}$$

However:

$$
\begin{aligned}
L(\gamma)_U =& \inf\{\sum_{i=0}^{N-1}[\ \ \tau_b(\alpha(s_i),\alpha(s_{i+1}))^2 - [\ \ \sum_{k=0}^{M-1} m_{s'_{i,k},s'_{i,k+1}} \\
& \qquad\qquad \times d_X(\beta(s'_{i,k}),\beta(s'_{i,k+1}))\ \ ]^2\ \ ]^{\frac{1}{2}}\} \\
=& \inf\{\sum_{i=0}^{N-1}[\ \ \tau_b(\alpha(s_i),\alpha(s_{i+1}))^2 - [\ \ \sup\{\sum_{k=0}^{M-1} m_{s'_{i,k},s'_{i,k+1}} \\
& \qquad\qquad \times d_X(\beta(s'_{i,k}),\beta(s'_{i,k+1}))\} - \epsilon_i\ \ ]^2\ \ ]^{\frac{1}{2}}\} \\
=& L(\gamma)_L
\end{aligned}
\tag{A.4.4}
$$

Consequently, since $L(\gamma)_U = L(\gamma)_L$:

$$
\begin{aligned}
L(\gamma) = \inf\{\sum_{i=0}^{N-1}[\ \tau_b(\alpha(s_i),\alpha(s_{i+1}))^2 - \sup\{\sum_{k=0}^{M-1} m_{s'_{i,k},s'_{i,k+1}} \\
\times d_X(\beta(s'_{i,k}),\beta(s'_{i,k+1}))\}^2\ ]^{\frac{1}{2}}\}
\end{aligned}
\tag{A.4.5}
$$

and thus we have proven the claim. $\square$

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