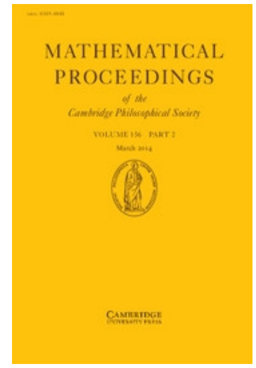

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Absolutely continuous spectrum of Dirac systems with potentials infinite at infinity

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Abstract

It is shown that the spectrum of a one-dimensional Dirac operator with a potential q tending to infinity at infinity, and such that the positive variation of $1/q$ is bounded, covers the whole real line and is purely absolutely continuous. An example is given to show that in general, pure absolute continuity is lost if the condition on the positive variation is dropped. The appendix contains a direct proof for the special case of subordinacy theory used.

1. *Introduction*

It is a well-known fact in non-relativistic quantum mechanics that the Schrödinger operator with a potential which tends to ∞ at $\pm\infty$ has a purely discrete spectrum. In contrast, the Dirac operator of relativistic quantum mechanics exhibits a very different behaviour in this situation. Specifically, for the one-dimensional Dirac operator

$$h = \sigma_2 p + \sigma_3 + q,$$

with a real-valued potential $q \in L^1_{\text{loc}}(\mathbb{R})$ (one has the limit point case at $\pm\infty$ and thus essential self-adjointness of the minimal operator) the spectrum is never purely discrete (cf. the appendix of [**10**]). If q tends to infinity at $+\infty$ and satisfies certain regularity conditions, the spectrum of h is purely absolutely continuous and comprises the whole real line. This has been observed by Plesset [**6**] in the case when q is a polynomial. Titchmarsh [**12**] has extended this result to functions $q \in C^1$ such that $q' \in AC_{\text{loc}}$ and

$$\int^\infty \frac{q'^2}{q^3} < \infty, \quad \int^\infty \frac{|q''|}{q^2} < \infty.$$

This regularity requirement has subsequently been weakened to essentially $q \in AC_{\text{loc}}$,

$$\int^\infty \frac{|q'|}{q^2} < \infty \tag{1}$$

by Erdélyi [**2**]. It should be mentioned that Rose and Newton [**8**], section III B, have found a purely continuous spectrum for all real values of the spectral parameter, assuming only that $\lim_{x\to\infty} q(x) = \infty$, and that q is monotonic near infinity; however, their reasoning is incomplete, assuming without justification that the σ_3 term of h does not significantly affect the asymptotics of solutions.

In the present paper we revisit the question of the quality of the spectrum for potentials infinite at infinity. Our main theorem (Theorem 1), which is proved in Section 2, states that the whole real line is purely absolutely continuous spectrum of h provided $\lim_{x\to\infty} q(x) = \infty$ and $1/q$ is of locally bounded variation, and of bounded positive variation near infinity. This extends Erdélyi's result not only because the local regularity of q is relaxed from absolute continuity to bounded variation, but because there is virtually no restriction on the growth of q; in particular any eventually non-decreasing function q is admissible (cf. Remarks 3, 4).

Our argument basically runs as follows. By the method of subordinacy, which was developed by Gilbert and Pearson [**3**], [**4**] for the Sturm–Liouville operator, and applied to the one-dimensional Dirac operator by Behncke [**1**], it suffices to prove that for every real λ, all solutions u of the eigenvalue equation

$$(\sigma_2 p + \sigma_3 + q - \lambda)\, u = 0 \tag{2}$$

are globally bounded. While the general subordinacy theory requires a subtle and quite circumstantial argument, one can give, based on an idea of [**11**], a rather direct proof for the special case we use, which is sketched in the appendix.

To establish the boundedness of any real-valued solution u of (2), the key idea is to study not the behaviour of the norm of the solution $|u|$, but that of the function

$$R := \sqrt{|u|^2 + \frac{2}{q-\lambda-1}\, u_1^2} \tag{3}$$

on a right half-axis where $q-\lambda-1 > 0$. To motivate this choice, consider the case of a potential q which is piecewise constant. Then on each interval of constancy of q the general solution of (2) is found to be

$$u(t) = R \begin{pmatrix} \sqrt{\dfrac{q-\lambda-1}{q-\lambda+1}} \cos\left(\sqrt{(q-\lambda)^2-1}\, t - \phi\right) \\ \sin\left(\sqrt{(q-\lambda)^2-1}\, t - \phi\right) \end{pmatrix}, \quad \phi \in \mathbb{R},$$

$R > 0$ being constant in this interval. Thus the solution is moving on an ellipse of major radius R, which is more elongated the smaller $q-\lambda-1$ is, and which tends to a circle as $q \to \infty$. At a discontinuity of q, then, there is (by the absolute continuity of u) a continuous transition between two concentric ellipses of different eccentricities, which obviously requires that the major radius of the more eccentric ellipse is greater than or equal to that of the less eccentric one. Thus when q is shrinking, R grows or remains the same; when q is growing, R shrinks or remains the same, which establishes a close link between the qualitative behaviour of q and of R. In particular, R cannot grow at growth points of q. It turns out that in the general situation of a potential q of locally bounded variation, the growth of q can likewise be disregarded, while the growth of R as q decreases can be controlled, by means of a Gronwall type lemma (Lemma 2), in terms of the positive variation of $1/q$. In order to illustrate that $\lim_{x\to\infty} q(x) = \infty$ alone is not sufficient for the spectrum to be purely absolutely continuous, we give, in Section 3, a simple example of a potential with logarithmically divergent positive variation of $1/q$, such that the operator has an eigenvalue.

2. *Absolutely continuous spectrum*

We introduce the following notation. By $BV_{\text{loc}}(I)$ we denote the functions of bounded variation on each compact subinterval of I. Given a function $f \in BV_{\text{loc}}(I)$ we write Pf for the (indefinite) positive variation of f, the non-decreasing function determined up to a constant by

$$Pf(b) - Pf(a) = \sup \sum_{j=1}^{N} \max\{0, f(x_j) - f(x_{j-1})\}, \tag{4}$$

where the supremum is taken over all partitions

$$a = x_0 < x_1 < \ldots < x_N = b, \quad N \in \mathbb{N},$$

of the interval $[a, b] \subset I$. In an analogous way, the (non-decreasing) negative variation Nf and the total variation Tf are defined such that

$$Tf = Pf + Nf, \quad f = Pf - Nf.$$

By $\sigma_{ac}(h)$, $\sigma_s(h)$ we denote the absolutely continuous and the singular part of the spectrum of h, respectively.

THEOREM 1. *Let $q \in L^1_{\text{loc}}(\mathbb{R})$ be a real-valued function such that $\lim_{x\to\infty} q(x) = \infty$, $1/q \in BV_{\text{loc}}(c, \infty)$ for some $c \in \mathbb{R}$ and $P(1/q)(\infty) < \infty$; then $\sigma_{ac}(h) = \mathbb{R}$, $\sigma_s(h) = \emptyset$.*

Proof. By Lemma 1 below, it is sufficient to show that for every $\lambda \in \mathbb{R}$, any real-valued solution u of the eigenvalue equation (2), i.e. of

$$\left.\begin{aligned} u_1' &= -(q-\lambda-1)\, u_2, \\ u_2' &= (q-\lambda+1)\, u_1, \end{aligned}\right\} \tag{5}$$

is globally bounded on a right half-axis. Fix $\tilde{c} > c$ such that $\lambda + 1 \leqslant q/2$ on $[\tilde{c}, \infty)$. Then it follows from (4) that

$$P(1/(q-\lambda-1))(x) - P(1/(q-\lambda-1))(\tilde{c}) \leqslant 4(P(1/q)(x) - P(1/q)(\tilde{c})) \quad (x \geqslant \tilde{c}),$$

and similarly for the total variation; so that $1/(q-\lambda-1)$ is of locally bounded variation, and $P(1/(q-\lambda-1))(\infty) < \infty$. In particular,

$$R^2(t) := |u|^2(t) + \frac{2}{q(t)-\lambda-1}\, u_1^2(t) \quad (t \geqslant \tilde{c})$$

is of locally bounded variation, and by the rule of integration by parts for Stieltjes integrals (cf. [**7**] §54) we have for $t_2 \geqslant t_1 \geqslant \tilde{c}$:

$$R^2(t_2) - R^2(t_1) = \int_{t_1}^{t_2} (|u|^2)' + \int_{t_1}^{t_2} \frac{2}{q(t)-\lambda-1}\, d(u_1^2(t)) + \int_{t_1}^{t_2} 2u_1^2(t)\, d\left(\frac{1}{q-\lambda-1}\right).$$

As u_1^2 is locally absolutely continuous, the first two integrals can be taken together as Lebesgue integrals:

$$\int_{t_1}^{t_2} \left((|u|^2)' + \frac{2}{q-\lambda-1}\,(u_1^2)'\right) = \int_{t_1}^{t_2} \left(2u_1' u_1 + 2u_2' u_2 + \frac{4}{q-\lambda-1}\, u_1' u_1\right) = 0,$$

since the integrand vanishes by (5). Hence

$$R^2(t_2)-R^2(t_1) = 2\int_{t_1}^{t_2} u_1^2(t)\,dP\left(\frac{1}{q-\lambda-1}\right)(t)-2\int_{t_1}^{t_2} u_1^2(t)\,dN\left(\frac{1}{q-\lambda-1}\right)(t)$$

$$\leqslant 2\sup_{t\in[t_1,t_2]} R^2(t)\int_{t_1}^{t_2} dP\left(\frac{1}{q-\lambda-1}\right),$$

since $0\leqslant u_1^2(t)\leqslant R^2(t)$.

By virtue of Lemma 2 below, this implies the boundedness of R.

Lemma 1. *Let $q\in L^1_{\mathrm{loc}}(\mathbb{R})$ be real-valued. Assume that for each $\lambda\in\mathbb{R}$ there exist $c\in\mathbb{R}$ and $C>0$ such that for each solution u of (2) $|u(x)|<C|u(c)|$ $(x\in[c,\infty))$. Then $\sigma_{ac}(h)=\mathbb{R}$, $\sigma_s(h)=\emptyset$.*

This is a consequence of [**1**], theorem 1. For a direct proof, cf. the appendix below.

Lemma 2. *Let $c\in\mathbb{R}$, $f\colon[c,\infty)\to[0,\infty)$ locally bounded, and $\alpha\colon[c,\infty)\to\mathbb{R}$ non-decreasing and bounded. Assume*

$$f(t_2)-f(t_1)\leqslant \sup_{t\in[t_1,t_2]} f(t)\cdot(\alpha(t_2)-\alpha(t_1)) \quad (t_2\geqslant t_1\geqslant c).$$

Then there is a constant $C>0$ such that $f\leqslant C$.

Proof. Choose $t_1\geqslant c$ so large that $\alpha(\infty)-\alpha(t_1)<1$. Then for every $t_2\geqslant t_1$ we have

$$\sup_{t\in[t_1,t_2]} f(t)\leqslant f(t_1)+\sup_{t\in[t_1,t_2]}\Big(\sup_{s\in[t_1,t]} f(s)(\alpha(t)-\alpha(t_1))\Big)$$

$$\leqslant f(t_1)+\sup_{t\in[t_1,t_2]} f(t)(\alpha(\infty)-\alpha(t_1)),$$

and thus

$$\sup_{t\in[t_1,t_2]} f(t)\leqslant\frac{f(t_1)}{1-\alpha(\infty)+\alpha(t_1)}<\infty.$$

Remark 1. For reasons of symmetry it is clear that the assertion of Theorem 1 holds as well if q tends to $-\infty$, and $1/q$ is of locally bounded variation and bounded negative variation near infinity.

Remark 2. A Hartman–Wintner type argument shows that the central fact of our proof, namely, that for each $\lambda\in\mathbb{R}$ all solutions of (2) are bounded, is invariant under the addition of an L^1 perturbation. Thus Theorem 1 remains valid if $q=q_1+q_2$ with q_1 satisfying our hypotheses and $q_2\in L^1(c,\infty)$, and also if an angular momentum term is added. For this, cf. [**9**], proof of proposition 1.

Remark 3. If $q\in AC_{\mathrm{loc}}(c,\infty)$,

$$P(1/q)(\infty)=const+\int^{\infty}\frac{(q')_-}{q^2},$$

where $(q')_-:=\max\{0,-q'\}$. Thus Theorem 1 imposes no condition on the positive part of q' and in this way (in conjunction with Remark 2) generalizes the result of Erdélyi (cf. (1)).

Remark 4. If q is any non-decreasing function, it is of locally bounded variation, and the positive variation of $1/q$ vanishes; thus Theorem 1 applies.

3. *A counterexample*

In this section we construct a potential q such that $\lim_{x\to\infty} q(x) = \infty$, but 0 is an eigenvalue of h. This shows that infinite growth of q alone does not guarantee that the spectrum is purely absolutely continuous.

For $m \in \{3, 4, \ldots\}$ set $q_{2m} := m/2$, $q_{2m+1} := m$. We consider the potential q assuming the value q_j on the interval $I_j := [\sum_{k=6}^{j-1} l_k, \sum_{k=6}^{j} l_k)$, where

$$l_j := \frac{\pi(1+4q_j)}{2\sqrt{q_j^2-1}} \quad (j \in \{6, 7, \ldots\})$$

(this defines q on $[0, \infty)$, since $l_j = 2\pi + o(1)$ as $j \to \infty$), and extended to $\mathbb{R}$ by $q(-x) := q(x)\,(x \in [0, \infty))$.

Now consider the solution u of (2) for $\lambda = 0$, with $u(0) = \binom{0}{1}$; it is sufficient to study the solution in $[0, \infty)$ as the solution in $(-\infty, 0]$ is an exact mirror image. The function R defined in (3) has the constant value R_j in the interval I_j. The positions of the points of discontinuity of q are chosen in such a way (note that $2q_j$ is an integer) that the decrease from q_{2m-1} to q_{2m} takes place at a zero of u_1, so that $R_{2m} = R_{2m-1}$. On the other hand, the increase from q_{2m} to q_{2m+1} takes place at a zero of u_2 and thus leads to a decrease of R according to

$$R_{2m+1} = R_{2m}\sqrt{\frac{q_{2m+1}+1}{q_{2m+1}-1}\frac{q_{2m}-1}{q_{2m}+1}} = R_{2m}\sqrt{\frac{m+1}{m-1}\frac{m-2}{m+2}}.$$

By induction

$$R_{2m}^2 = R_{2m-1}^2 = \left(\prod_{j=3}^{m-1}\frac{j+1}{(j+1)+1}\right)\left(\prod_{j=3}^{m-1}\frac{j-2}{(j+1)-2}\right) = \frac{4}{m^2-m-2},$$

since the products telescope. Consequently,

$$\|u\|^2 \leqslant 2\sum_{m=3}^{\infty} 2(2\pi+o(1))R_{2m}^2 < \infty.$$

We observe that the points of increase of $1/q$ are asymptotically evenly spaced and the transition from $1/q_{2m-1}$ to $1/q_{2m}$ contributes $2/m - 1/(m-1) \sim 1/m$, so that the positive variation of $1/q$ is logarithmically divergent.

Appendix

Lemma 1 above, i.e. the statement that global boundedness of all solutions of (2) for all $\lambda \in \mathbb{R}$ (or, more generally, for a λ interval) implies purely absolutely continuous spectrum in $\mathbb{R}$ (resp. in the interior of that interval), can be viewed as a special case of the general subordinacy method developed by Gilbert and Pearson, cf. [**1**]. As this special case proves particularly useful in applications, it is interesting to note that Simon [**11**] has recently given a fairly direct proof for the analogous statement about Sturm–Liouville operators; we now sketch a proof of Lemma 1 along these lines.

For some fixed $\alpha\in\mathbb{R}$, let $\theta(\cdot,\lambda)$, $\phi(\cdot,\lambda)$ be the solutions of (2) for $\lambda\in\mathbb{C}$ such that

$$\theta(0,\lambda)=\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix},\quad \phi(0,\lambda)=\begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix}.$$

As the singular endpoints $\pm\infty$ are in the limit point case, for $\lambda\in\mathbb{C}\backslash\mathbb{R}$ one has $m(\lambda)$, $n(\lambda)\in\mathbb{C}$ such that

$$\left.\begin{aligned}\psi(\cdot,\lambda)&=\theta(\cdot,\lambda)+m(\lambda)\,\phi(\cdot,\lambda)\in L^2(0,\infty)\\ \chi(\cdot,\lambda)&=\theta(\cdot,\lambda)+n(\lambda)\,\phi(\cdot,\lambda)\in L^2(-\infty,0)\end{aligned}\right\}. \tag{6}$$

From the Weyl–Titchmarsh theory (cf. [**5**], [**11**]) it is known that m,n are regular functions in the upper half plane and that $\operatorname{Im} m(\lambda)>0$, $\operatorname{Im} n(\lambda)<0$ $(\operatorname{Im}\lambda>0)$.

Furthermore,

$$\operatorname{Im} m(\lambda)=\operatorname{Im}\lambda\int_0^\infty|\psi(\cdot,\lambda)|^2,$$

and
$$\rho_\alpha(\lambda)=\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_0^\lambda \operatorname{Im} m(\mu+i\epsilon)\,d\mu\quad(\lambda\in\mathbb{R})$$

is the spectral function for the operator h_α given as $\sigma_2 p+\sigma_3+q$ on the half-axis $[0,\infty)$ with boundary condition $(\cos\alpha,\sin\alpha)\cdot u(0)=0$. Thus its singular part $\rho_{\alpha,s}$ is supported on the set

$$\{\lambda\in\mathbb{R}\,|\,\lim_{\epsilon\downarrow 0}\operatorname{Im} m(\lambda+i\epsilon)=\infty\},$$

and its absolutely continuous part $\rho_{\alpha,ac}$ has derivative

$$\rho'_{\alpha,ac}(\lambda)=\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\operatorname{Im} m(\lambda+i\epsilon).$$

Similarly, with the matrix

$$M(\lambda)=\frac{1}{n(\lambda)-m(\lambda)}\begin{pmatrix}1 & \frac12(n(\lambda)+m(\lambda))\\ \frac12(n(\lambda)+m(\lambda)) & n(\lambda)\,m(\lambda)\end{pmatrix},$$

the spectral matrix ρ for the operator h is given as

$$\rho(\lambda)=\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_0^\lambda \operatorname{Im} M(\mu+i\epsilon)\,d\mu\quad(\lambda\in\mathbb{R}),$$

and its singular part is supported on

$$\{\lambda\in\mathbb{R}\,|\sup_{j,k\in\{1,2\}}\lim_{\epsilon\downarrow 0}\operatorname{Im} M_{jk}(\lambda+i\epsilon)=\infty\}.$$

To prove Lemma 1, it is sufficient to show that this support is empty and thus $\sigma_s(h)=\emptyset$, and that $\rho'_{\alpha,ac}(\lambda)>0$ for all real λ, since then by Glazman's decomposition principle (cf. [**13**] p. 165 seq.) $\sigma(h)\supset\sigma_e(h_\alpha)\supset\sigma_{ac}(h_\alpha)=\mathbb{R}$. With the above general information, these two properties are guaranteed by:

LEMMA 3. *Let $\lambda\in\mathbb{R}$. Assume all solutions of* (2) *are bounded in* $[0,\infty)$. *Then*

$$\liminf_{\epsilon\downarrow 0}\operatorname{Im} m(\lambda+i\epsilon)>0,\quad \limsup_{\epsilon\downarrow 0}|m(\lambda+i\epsilon)|<\infty.$$

Furthermore,
$$\limsup_{\epsilon\downarrow 0} |M_{jk}(\lambda+i\epsilon)| < \infty \quad (j,k\in\{1,2\}).$$

Proof. For $\mu\in\mathbb{C}$, let $T(\cdot,\mu)$ be the matrix valued solution of (2) (with eigenvalue parameter μ) on $[0,\infty)$ with $T(0,\mu)=1$. As $\det T\equiv 1$, $|T^{-1}|=|T|$, where $|\cdot|$ denotes the operator norm in $\mathbb{C}^2$. By hypothesis, there is a constant $C>0$ such that $|T(x,\lambda)| \leqslant C (x\in[0,\infty))$.

Let $\epsilon>0$ and define $S_\epsilon := T^{-1}(\cdot,\lambda)\,T(\cdot,\lambda+i\epsilon)$; then S_ϵ is the solution of the initial value problem

$$S_\epsilon'(x) = \epsilon T^{-1}(x,\lambda)(-\sigma_2)\,T(x,\lambda)\,S_\epsilon(x) \quad (x\in[0,\infty)), \quad S_\epsilon(0)=1.$$

A Gronwall argument yields the bound

$$|T(x,\lambda+i\epsilon)| \leqslant |T(x,\lambda)|\,|S_\epsilon(x)| \leqslant C\exp(\epsilon C^2 x) \quad (x\in[0,\infty)).$$

As T operates as a transfer matrix for solutions, we have in particular

$$\psi(0,\lambda+i\epsilon) = T(x,\lambda+i\epsilon)^{-1}\,\psi(x,\lambda+i\epsilon).$$

By a straightforward calculation based on (6), $|\psi(0,\lambda+i\epsilon)| = \sqrt{1+|m(\lambda+i\epsilon)|^2}$, and thus

$$|\psi(x,\lambda+i\epsilon)| \geqslant \frac{1}{C}\exp(-\epsilon C^2 x)\sqrt{1+|m(\lambda+i\epsilon)|^2}.$$

Hence
$$\operatorname{Im} m(\lambda+i\epsilon) = \epsilon\int_0^\infty |\psi(x,\lambda+i\epsilon)|^2 \geqslant \frac{1+|m(\lambda+i\epsilon)|^2}{2C^4} \geqslant \frac{1}{2C^4} > 0,$$

and
$$|m(\lambda+i\epsilon)| \geqslant \operatorname{Im} m(\lambda+i\epsilon) \geqslant \frac{|m(\lambda+i\epsilon)|^2}{2C^4},$$

i.e. $|m(\lambda+i\epsilon)| \leqslant 2C^4 < \infty$.

To prove the statement about the matrix M, we observe that since $\operatorname{Im} n(\lambda+i\epsilon) < 0$,

$$|n(\lambda+i\epsilon)-m(\lambda+i\epsilon)| \geqslant \operatorname{Im} m(\lambda+i\epsilon) - \operatorname{Im} n(\lambda+i\epsilon) \geqslant \frac{1}{2C^4},$$

so $|M_{11}(\lambda+i\epsilon)| \leqslant 2C^4 < \infty$. On the other hand, $\operatorname{Im}(n(\lambda+i\epsilon)^{-1}) > 0$ and

$$\operatorname{Im}(m(\lambda+i\epsilon)^{-1}) = -\frac{\operatorname{Im} m(\lambda+i\epsilon)}{|m(\lambda+i\epsilon)|^2} \leqslant -\frac{1}{8C^{12}},$$

so
$$|M_{22}(\lambda+i\epsilon)| = |m(\lambda+i\epsilon)^{-1} - n(\lambda+i\epsilon)^{-1}|^{-1} \leqslant 8C^{12} < \infty.$$

Finally,

$$\begin{aligned}|M_{12}(\lambda+i\epsilon)| &= \left|\frac{1}{2}\left(\frac{1}{m(\lambda+i\epsilon)}M_{22}(\lambda+i\epsilon) + m(\lambda+i\epsilon)M_{11}(\lambda+i\epsilon)\right)\right| \\ &\leqslant 8C^{16} + 2C^8 < \infty.\end{aligned}$$

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