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Bullwhip behavior in the Order-Up-To policy with ARIMA demand

Gerard Gaalman (*g.j.c.gaalman@rug.nl*)
Department of Operations, Faculty of Economics and Business, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands.

Stephen Disney
Logistics Systems Dynamics Group, Cardiff Business School, Cardiff University, Aberconway Building, Colum Drive, Cardiff, CF10 3EU, UK.

Abstract
This paper analyses the bullwhip effect produced by the Order-Up-To (OUT) policy for ARIMA demand processes. Areas in the parametrical space are identified where a bullwhip effect increases or decreases as function of the lead time. In remaining areas the bullwhip effect might be increasing, decreasing or fluctuating, depending upon the parameter values of the demand process.

Keywords: Bullwhip, Order-Up-To policy, ARMA(*p*,*q*) demand, Eigenvalue analysis

Introduction
The bullwhip effect produced by inventory replenishment policies in supply chains has been extensively studied. Usually a combination of a specific demand process, a forecasting method, and an inventory replenishment policy is chosen to evaluate the bullwhip effect by means of a metric. Some system parameters are varied within a suitable range and as a result a bullwhip effect may or may not exist. For example a common approach is that an autoregressive demand process of the first order (AR(1)), an Order-Up-To replenishment policy (OUT) policy and the simple exponential smoothing forecasting method is chosen. Next the AR parameter and the smoothing parameter are varied within a certain range to show how the bullwhip effect is affected. In some case the impact of the lead-time is also considered, see Chen et al., (2000), Zhang (2004).

This type of analysis is valuable for practice because insights are obtained that indicate what improvement measures can be taken and how successful they are likely to be. These improvements could include: more sophisticated forecasting; information sharing; or lead-time reduction. On the other hand the combination of the selected models might hide deeper insights into the cause of the bullwhip effect. Has the replenishment policy the strongest influence to the bullwhip effect? Or the forecasting method for a given demand process? Or is it just the combination of models? Or even the selected set of parameter values? We wonder, it is possible to get more generic, robust insights?

As a first step to answer this question we consider the bullwhip behavior of OUT policy for stationary autoregressive moving average (ARMA(*p*,*q*)) and non-stationary autoregressive integrated moving average (ARIMA(*p,d,q*)) demand process, Box et al., (2008). Since the notable work of Box and Jenkins modeling data characteristics by means of ARIMA processes has been given much attention in both the POM and the demand forecasting literature. The ARIMA(*p,d,q*) process can be modeled with an ARMA(*p,q*) process when the AR coefficients, $\left(\phi_i,\ i=1 \text{ to } \max\left(p,q\right)\right)$ are allowed to exist in the stationary region, including the border.

One of the advantages of the ARIMA approach is that when the demand process has been identified its forecast is easily determined as it is equal to its conditional expectation. This forecast is generally seen as the theoretically best, or optimal, forecast. The influence of non-optimal forecast methods on the bullwhip effect relative to the best forecast will be studied in a future paper. A relatively large amount of data is required to statistically identify an ARMA(*p,q*) process. This is frequently not available in short term demand forecasting scenarios, so ad-hoc methods like exponential forecasting smoothing methods (Holt (1957), Brown (1963), Brown and Meyer (1961), Gardner (1985)) are commonly used. However several authors have has shown that the forecasts of these methods are equal to the conditional expected forecasts of certain ARMA(*p,q*) processes (see for instance Muth (1960), McKenzie (1976) and Roberts, (1982)). This provides extra motivation for considering ARMA(*p,q*) processes.

The OUT inventory replenishment policy is selected because it is widely used in practice for high volume demand and its mathematical tractability. Using a state space approach the variance of the orders are expressed analytically as a sum of the demand variance, components dependent upon the $p+q$ ARMA process parameters, the lead time and the variance of the one period ahead forecast error. Both the demand and the order variances only exist for stationary ARMA(*p,q*) processes. From these variances however necessary conditions can be derived for which an increasing or decreasing bullwhip effect as function of the lead time exists. The variances do not exist for ARIMA processes. However their difference is finite for finite lead times and this indicates whether a bullwhip effect is present.

For real, distinct eigenvalues orderings are derived for which the bullwhip effect exists and is increasing or decreasing in the lead time. Thus the property applies as long as a particular eigenvalue ordering holds, regardless of the specific values of the eigenvalues. These regions provide only necessary conditions as a bullwhip effect can exist in the remaining regions. However in those regions bullwhip behavior also depends on the value of the eigenvalues and order variance can be an increasing or decreasing as a function of the lead time. It may also be a fluctuating function of the lead-time. This may mean that for some lead-times, there exists a bullwhip effect, increasing the lead-time results in no bullwhip, but increasing the lead-time even further causes the bullwhip to re-appear again, and so on.

The structure of our paper is as follows. First we provide basic expressions for the inventory, the ARMA demand process, and the OUT policy. Then we propose a bullwhip criterion and derive an analytical expression of the order variance to be used in this criterion. We use this criterion to identify propositions that specify when a bullwhip effect exists for all lead times. Finally we provide concluding remarks.

The Order-Up-Policy with ARMA(*p, q*) demand

For unit lead-time the inventory balance equation is given by

$$i_{t+1+k} = i_{t+k} + o_t - d_{t+1+k}, \tag{1}$$

where i_t is the inventory level at time t, o_t the replenishment orders at t, d_t the demand at time t and $k \in \aleph_0$ is the physical lead-time between placing an order and receiving it into inventory. Notice in (1) that there is a sequence of events delay. This means that when $k = 0$ the order placed in period t is received in period t but that information is not used in a replenishment decision until the next period. The ARMA(p,q) demand process, (Box et al, 2008) is defined as

$$\left.\begin{array}{c} d_{t+1} = \bar{d} + z_{t+1} \\ z_{t+1} - \phi_1 z_t \cdots - \phi_p z_{t+1-p} = \eta_{t+1} - \theta_1 \eta_t \cdots - \theta_q \eta_{t+1-q} \end{array}\right\}, \tag{2}$$

where η_t is a i.i.d, zero mean random variable drawn from a normal distribution.

State space representation of the demand process

Gaalman (2006) shows that defining an m-dimensional state y_t and applying certain transformations then we may obtain the following state space representation of the ARMA(p,q) demand process

$$d_{t+1} = \bar{d} + z_{t+1}, z_{t+1} = My_{t+1} + \eta_{t+1},\ y_{t+1} = Dy_t + K\eta_t, \tag{3}$$

with

$$\left.D = \begin{pmatrix} \phi_1 & 1 & 0 & 0 \\ \phi_2 & 0 & \ddots & 0 \\ \vdots & 0 & \ddots & 1 \\ \phi_m & 0 & 0 & 0 \end{pmatrix} = \Phi,\ K = \begin{pmatrix} \phi_1 - \theta_1 \\ \phi_2 - \theta_2 \\ \vdots \\ \phi_m - \theta_m \end{pmatrix},\ M = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}\right\}, \tag{4}$$

where $m = \max\left(p, q\right)$ and K is the gain of the recursion. Note: D contains only auto-regressive (AR) coefficients. The characteristic polynomial of the matrix D is given by

$$\det(D - \lambda I) = \lambda^m - \phi_1 \lambda^{m-1} \ldots - \phi_m = \prod\nolimits_{j=1}^{m} (\lambda - \lambda_j^{\phi}), \tag{5}$$

where λ_j^{ϕ} is the eigenvalue of the AR part of the ARMA process. The characteristic polynomial (3) of the MA part satisfies

$$\lambda^m - \theta_1 \lambda^{m-1} \ldots - \theta_m = \prod\nolimits_{j=1}^{m} (\lambda - \lambda_j^{\theta}), \tag{6}$$

where λ_j^{θ} is a moving average eigenvalue. The conditional expectation of demand can be found from the one period ahead forecast using the Kalman filter approach

$$\left.\begin{array}{l} \hat{d}_{t+1,t} = \bar{d} + \hat{z}_{t+1,t},\ \hat{z}_{t+1,t} = M\hat{y}_{t+1,t},\ \hat{y}_{t+1,t} = D\hat{y}_{t,t-1} + Ke_t, \\ e_t = (z_t - \hat{z}_{t,t-1}),\ \hat{d}_{t+k+1,t} = \bar{d} + \hat{z}_{t+k+1,t},\ \hat{z}_{t+k+1,t} = MD^k \hat{y}_{t+1,t}. \end{array}\right\} \tag{7}$$

Note: $e_t = \left(z_t - \hat{z}_{t,t-1}\right)$ is the one period ahead forecast error. If the demand process (7) is invertible (Box et al., 2008) then after an increasing the number of observations the forecast error e_t converges to η_t. By this $\hat{y}_{t+1,t}$ converges to y_{t+1}. As is common we assume the system is in steady state. To stress the forecast character in combination with the forecast of other variables we will use $\hat{y}_{t+1,t}$ instead of y_{t+1} for the forecasted value of *y* in period $t+1$ made at time *t*.

The inventory balance equation can be written as an up-date filter by introducing $i_{t+1+k} = \hat{i}_{t+k+1,t+1} + \left(i_{t+1+k} - \hat{i}_{t+k+1,t+1}\right)$, with $\left(i_{t+1+k} - \hat{i}_{t+k+1,t+1}\right)$ the zero mean i.i.d. inventory forecast error (Gaalman and Disney, 2009)

$$\hat{i}_{t+k+1,t+1} = \hat{i}_{t+k,t} + o_t - \hat{d}_{t+1+k,t} - E\left(k\right)e_{t+1};\ e_{t+1} = \left(d_{t+1} - \hat{d}_{t+1,t}\right) = \left(z_{t+1} - \hat{z}_{t+1,t}\right) = \eta_{t+1}, \tag{8}$$

where

$$E\left(k\right) = 1 + \sum_{j=0}^{k-1} M\left(D^j\right)K,\ E\left(0\right) = 1. \tag{9}$$

$E\left(k\right)$ is an auxiliary variable and corresponds with the sum over the first k impulse responses $M\left(D^j\right)K$ of the demand ARMA(p,q) process as described in (3). Note: $MD^jK = E\left(j+1\right) - E\left(j\right)$.

The Order-Up-To replenishment policy

The OUT policy can be derived by setting the conditionally expected inventory $\hat{i}_{t+k+1,t}$ equal to the inventory norm $\bar{i}$. From (8) we can derive $\hat{i}_{t+k+1,t} = \bar{i} = \hat{i}_{t+k,t} + o_t - \hat{d}_{t+1+k,t}$ giving the order rate as function of the forecasted demand in the period after the lead-time from which is subtracted the difference between the conditionally expected inventory $\hat{i}_{t+k+1,t}$ to the inventory norm $\bar{i}$

$$o_t = \hat{d}_{t+1+k,t} - \left(\hat{i}_{t+k,t} - \bar{i}\right). \tag{10}$$

This policy is the linear periodic review OUT policy which has been extensively studied in literature (Silver et al., 1998; Zipkin, 2000). This policy is suitable for the high volume demand processes we consider here. Substituting in (8) into (10) shows

$$\hat{i}_{t+k+1,t+1} - \bar{i} = -E(k)\eta_{t+1} \text{ or } \hat{i}_{t+k,t} - \bar{i} = -E(k)\eta_t. \tag{11}$$

By this

$$o_t = \hat{d}_{t+1+k,t} + E(k)\eta_t. \tag{12}$$

The forecasted demand $\hat{d}_{t+k+1,t}$ is a function of $\hat{y}_{t+1,t}$ (see (7). Iteration of (7) and (8) reveals that $\hat{y}_{t+1,t}$ and $\hat{i}_{t+k,t}$ are linear functions of $\left\{\eta_t, \eta_{t-1}, \eta_{t-2} \ldots\right\}$. Thus a correlation exists between $\hat{d}_{t+1+k,t}$ and $\hat{i}_{t+k,t}$, the two components of the ordering policy. To eliminate this we substitute $\hat{d}_{t+k+1,t} = \hat{d}_{t+k+1,t-1} + MD^kK\eta_t$ to yield

$$o_t = \hat{d}_{t+k+1,t-1} + E(k+1)\eta_t. \tag{13}$$

The bullwhip criterion
A bullwhip effect is said to exist if

$$\left(\Sigma_{oo}/\Sigma_{dd}\right) > 1, \tag{14}$$

where Σ_{dd}, Σ_{oo} are respectively the variance of the demand and of the orders. For both variances to exist the demand process and the inventory balance equation must be stable. The ordering variance can be calculated from (13),

$$\Sigma_{oo} = \Sigma_{\hat{d}\hat{d}}\left(k+1\right) + E^2\left(k+1\right)\Sigma_{\eta\eta}, \tag{15}$$

with $\Sigma_{\hat{d}\hat{d}}\left(k+1\right) = \Sigma_{\hat{z}\hat{z}}\left(k+1\right) = M\Sigma_{\hat{y}\hat{y}}\left(k+1\right)M^T = \left(MD^{k+1}\right)\Sigma_{\hat{y}\hat{y}}\left(MD^{k+1}\right)^T$, $\Sigma_{\hat{y}\hat{y}}$ the variance of $\hat{y}_{t+1,t}$. Using (3) we can write $\Sigma_{\hat{z}\hat{z}}\left(k+1\right)$ as function of Σ_{zz} and $\Sigma_{\eta\eta}$

$$\Sigma_{zz} = \Sigma_{\hat{z}\hat{z}}\left(k+1\right) + \Sigma_{\eta\eta} + \sum_{j=0}^{k}\left(E\left(j+1\right) - E\left(j\right)\right)^2 \Sigma_{\eta\eta}. \tag{16}$$

Substituting in (15) gives

$$\Sigma_{oo} - \Sigma_{zz} = E^2\left(k+1\right)\Sigma_{\eta\eta} - \Sigma_{\eta\eta} - \sum_{j=0}^{k}\left(E\left(j+1\right) - E\left(j\right)\right)^2 \Sigma_{\eta\eta}. \tag{17}$$

The bullwhip criterion (14) is equivalent to $\Sigma_{oo} - \Sigma_{zz} > 0$. Let

$$CB(k) = -E^2\left(0\right) - \sum_{j=0}^{k}\left(E\left(j+1\right) - E\left(j\right)\right)^2 + E^2\left(k+1\right) = 2\sum_{j=0}^{k} E\left(j\right)\left(E\left(j+1\right) - E\left(j\right)\right), \tag{18}$$

and if $CB(k) > 0$ bullwhip is generated by the OUT policy. Another bullwhip criterion can be obtained that is closely related to $E\left(k\right)$. By simple inspection of the system's impulse response we can see that bullwhip will exist if

$$\sum_{t=0}^{k}\left(z_t\right)^2 < \left(\sum_{t=0}^{k+1} z_t\right)^2 \tag{19}$$

as $\sum_{t=0}^{k}\left(z_t\right)^2 = 1 + \sum_{j=0}^{k}\left(E\left(j+1\right) - E\left(j\right)\right)^2$ and $\left(\sum_{t=0}^{k+1} z_t\right)^2 = E^2\left(k+1\right)$. From (19), we see that bullwhip can only be avoided in the OUT policy if at least one z_t over the lead-time, *k*, is negative as $z_0 = 1$. Furthermore, we find that only the first *k* AR and MA terms of the ARMA(*p*,*q*) determine whether bullwhip exists or not. Several other bullwhip criteria exist. For example, Gilbert's (2005) criterion focuses on the uncertainty propagation over the supply chain and is equal to $E^2\left(k\right) > 0$.

Bullwhip effect properties
From $CB(k)$ necessary conditions can be derived that describe when the bullwhip effect exists and is an increasing function of the lead time. We first focus on real eigenvalues of the AR and MA part of the demand process. As before, the stationary situation is

considered and the AR eigenvalues satisfy $-1<\lambda_i^{\phi}<1,\ \forall i$. The invertibility conditions mean that the MA eigenvalues satisfy $-1<\lambda_i^{\theta}<1,\ \forall i$ and the following properties hold

$$\lim_{k\to\infty}\left(E\left(j+1\right)-E\left(j\right)\right)=0,\ E\left(\infty\right)=\lim_{k\to\infty}E\left(k\right)=\left(\frac{1-\theta_1-\ldots-\theta_m}{1-\phi_1-\ldots-\phi_m}\right)=\left(\frac{\prod_{j=1}^{m}(1-\lambda_j^{\theta})}{\prod_{j=1}^{m}(1-\lambda_j^{\phi})}\right)>0\Bigg\}. \tag{20}$$

This means that also the limit $CB\left(\infty\right)=\lim_{k\to\infty}CB\left(k\right)$ is finite when the demand process is stationary and invertible. Although the stationary conditions are satisfied for $-1<\lambda_i^{\phi}<1,\ \forall i$ we will now only consider the real AR eigenvalues in $0<\lambda_i^{\phi}<1,\ \forall i$ so as to avoid situations where the current demand is negatively correlated. A negative λ_l^{ϕ} causes alternating values of $\left(\lambda_l^{\phi}\right)^i$, which will result in a fluctuating $CB\left(k\right)$behavior .

Proposition 1: If the impulse response $\left(E\left(k+1\right)-E\left(k\right)\right)>0,\ \forall k$ then $E\left(k\right)>0$ and is increasing. As a consequence $CB\left(k\right)>0\ \ \forall k$ and is increasing. Moreover if the impulse response $\left(E\left(k+1\right)-E\left(k\right)\right)<0\ \ \forall k$ then $E\left(k\right)$ is decreasing and $E\left(k\right)>0$. This implies that $CB\left(k\right)<0$ and decreasing $\forall k$. In other words there is no bullwhip for all lead-times.
Proof 1: Due to its simplicity the proof is left for the reader's enjoyment.

These are only necessary conditions. In order to get further insights for which parameter AR and MA values $CB\left(k\right)$ will be increasing or decreasing we use the eigenvalue representation of the system. The impulse response can be written as

$$M\left(D^j\right)G=\left(E\left(k+1\right)-E\left(k\right)\right)=\sum_{l=1}^{m}r_l\left(\lambda_l^{\phi}\right)^j,\ r_l=\left(\frac{\Pi_{j=1}^{m}\left(\lambda_l^{\phi}-\lambda_j^{\theta}\right)}{\Pi_{\substack{j=1\\ j\neq l}}^{m}\left(\lambda_l^{\phi}-\lambda_j^{\phi}\right)}\right). \tag{21}$$

(due to space restrictions the derivation is omitted). Thus

$$E(k)=1+\sum_{l=1}^{m}r_l g_l\left(\lambda_l^{\phi},k\right),\ E(0)=1,\ g_l(\lambda_l^{\phi},k)=\sum_{j=0}^{k-1}\left(\lambda_l^{\phi}\right)^j=\tfrac{1-\left(\lambda_l^{\phi}\right)^k}{1-\lambda_l^{\phi}}\quad \forall k\geq 1. \tag{22}$$

For simplicity reasons we assumed here that all eigenvalues are distinct and real. However the real, but non-distinct, case will be equivalent. Using (21) and (22) in (18) gives,

$$CB(k)=2\sum_{j=0}^{k}\left(1+\sum_{l=1}^{m}\sum_{i=0}^{j-1}r_l\left(\lambda_l^{\phi}\right)^i\right)\left(\sum_{l=1}^{m}r_l\left(\lambda_l^{\phi}\right)^j\right) \tag{23}$$

This expression can be further investigated to reveal insight on $CB\left(k\right)$'s behavior as a function of the lead time. However because of the complexity of this expression we concentrate on the increasing / decreasing conditions of the impulse response and $E\left(k\right)$.

Since $0<\lambda_l^{\phi}<1$ then $\left(\lambda_l^{\phi}\right)^j,\ j\in\aleph_0$ is a decreasing, positive and convex function in k. Moreover $g_l(\lambda_l^{\phi},k)$ is a increasing, positive and concave in k with minimum at $k=1$ and maximum $\frac{1}{\left(1-\lambda_l^{\phi}\right)}>1$ for $k\to\infty$. Then, depending of the sign of r_l, the impulse response consists partly of decreasing, positive, convex functions and partly of increasing, negative, concave functions. The basic property behind proposition 1 is that the r_l weighted sum of impulse responses is either positive or negative. If, for example

$r_l > 0,\ \forall l$ then the impulse response $\left(E\left(k+1\right)-E\left(k\right)\right) > 0,\ \forall k$ and $E(k)$ is an increasing (concave) function in *k*. By this $CB\left(k\right) > 0$ and is increasing for all lead times. The sign of r_l depends on the ordering of the AR- and MA-eigenvalues. The condition $r_l > 0,\ \forall l$ is satisfied i.f.f.

$$\lambda_1^\theta < \lambda_1^\phi \ldots \lambda_{m-1}^\theta < \lambda_{m-1}^\phi < \lambda_m^\theta < \lambda_m^\phi, \tag{24}$$

where the eigenvalues are ordered in an increasing sequence. Note: the value of λ_1^θ might be negative. This ordering of the eigenvalues determines areas in the parametrical space for which $CB\left(k\right) > 0$ and is increasing in *k*. This raises the question whether other eigenvalue orderings can be found for which a bullwhip effect exists for all lead times.

Proposition 2: The impulse response is positive for all k and $E(k)$ is increasing if for each AR eigenvalue λ_l^ϕ the number of MA eigenvalues smaller than λ_l^ϕ is larger than the number of AR eigenvalues smaller then λ_l^ϕ. As a result $CB(k) > 0$ and is increasing in *k*.
Proof 2: The cumbersome proof is omitted here for space reasons.

This proposition results in many possibilities between two "extreme" cases. One is indicated in (24) and the other is

$$\lambda_1^\theta < \ldots \lambda_{m-1}^\theta < \lambda_m^\theta < \lambda_1^\phi \ldots < \lambda_{m-1}^\phi < \lambda_m^\phi \tag{25}$$

where the r_l's alternate in sign.

Similar eigenvalue orderings can be derived for which $CB\left(k\right) < 0$ and decreasing. Again these are only necessary conditions. This means that for other $\phi_i,\ \theta_j$ then, as proposition 2 shows, either $CB\left(k\right) > 0$ and increases with lead-time or $CB\left(k\right) < 0$ and decreases with lead-time can happen. Depending on the AR and MA parameter values these areas may also contain fluctuating $CB\left(k\right)$'s.

Bullwhip effect properties for ARMA(2,2) demand

Recall, the bullwhip criterion, given by (18) is $CB\left(k\right) = 2\sum_{j=0}^{k} E\left(j\right)\left(E\left(j+1\right)-E\left(j\right)\right)$. For general ARMA(*p,q*) demand a z-transform of $CB\left(k\right)$ can be obtained via the z-transform multiplication theorem, Oppenheim and Schafer (1975).

$$Z\left\{CB\left(k\right)\right\} = \frac{2z}{z-1}\left(\frac{1}{2\pi\sqrt{-1}}\oint_C \frac{z^2\left(-1+\sum_{i=1}^{q} z^{-i}\theta_i\right)\left(\sum_{i=1}^{q}\left(\frac{z}{\upsilon}\right)^{-i}\theta_i - \sum_{j=1}^{p}\left(\frac{z}{\upsilon}\right)^{-j}\phi_j\right)}{\upsilon^2\left(z-1\right)\left(-1+\sum_{j=1}^{p} z^{-j}\phi_j\right)\left(-1+\sum_{j=1}^{p}\left(\frac{z}{\upsilon}\right)^{-j}\phi_j\right)}\,\mathrm{d}\upsilon\right) \tag{26}$$

We will now investigate the $CB\left(k\right)$ for ARMA(2,2) demand. For ARMA(2,2) (26) is

$$Z\left\{CB(k)\right\} = \frac{\left(2z^2\left(\begin{array}{l} z^2\left(z^2\phi_1+\theta_2^2\phi_1-\theta_2\left(z+\phi_1\left(z+\left(-1+\phi_1\right)\phi_1\right)\right)\right)+\theta_2\phi+z\theta_1^2\left(z^2\phi_1+\left(z-\phi_2\right)\phi_2\right) \\ -z\theta_1\left(z-\theta_2\right)\left(z\left(z+\phi_1^2\right)+2\phi_1\phi_2-\phi_2^2\right)+z\phi_2\left(z^2+\theta_2\left(z+\theta_2-\phi_1\left(2z+\phi_1\right)\right)\right)-\phi_2^2\left(z^2+\theta_2\left(z+\theta_2-z\phi_1\right)\right)_2^3\end{array}\right)\right)}{\left(\left(z-1\right)\left(\left(z-\phi_2\right)^2-z\phi_1^2\right)\left(z^2-z\phi_1-\phi_2\right)\left(z+\phi_2\right)\right)} \tag{27}$$

ARMA(2,2) demand (and $CB\left(k\right)$ after ignoring a root at 1) is stable if

$$\phi_2 > -1 \wedge \phi_1 < 1-\phi_2 \wedge \phi_1 > \phi_2 - 1. \quad (28)$$

There are six poles to (tf),at $z=\left\{1,-\phi_2,\tfrac{1}{2}\left(\phi_1 \pm \sqrt{\phi_1^2+4\phi_2}\right),\tfrac{1}{2}\left(\phi_1^2+2\phi_2 \pm \sqrt{\phi_1^4+4\phi_1^2\phi_2}\right)\right\}$.The last four poles are complex if $\phi_2 \le 0 \wedge \left(-2\sqrt{-\phi_2} < \phi_1 < 0 \vee 0 < \phi_1 < 2\sqrt{-\phi_2}\right)$, and real otherwise. This is important as complex roots will mean that $CB(k)$ will be oscillating with a period greater than 2. If this oscillation repeatedly crosses the origin, than for some lead-times there will be a bullwhip effect (when $CB(k) > 0$) and for other lead-times there will not be a bullwhip effect (when $CB(k) < 0$). Note $CB(k)$ can also fluctuate with real roots. Applying the final value theorem, $\lim_{z\to 1}(z-1)X(z) = \lim_{k\to\infty} CB(k)$, to (27) we may obtain an expression that details the value of $CB(\infty)$,

$$CB(\infty)=\left(\frac{(1+\theta_2)^2+(\theta_1-\phi_1)(\theta_1+\theta_2\phi_1)}{(\phi_1^2-4)(1+\phi_2)}+\frac{1}{4}\left(\frac{(1+\theta_1-\theta_2)^2}{(1+\phi_1)(\phi_2-\phi_1-1)}+\frac{(\theta_1+\theta_2-1)^2}{(\phi_1+\phi_2-1)}\left(2+\frac{1}{(2-\phi_1)}\right)\right)\right). \quad (29)$$

A stable $CB(k)$ is required to ensure that a finite final value of $CB(\infty)$ exists. Applying the initial value theorem, $\lim_{z\to\infty} X(z) = CB(0)$, to (27) we may obtain $CB(0)$,

$$CB(0)=\phi_1-\theta_1. \quad (30)$$

Interestingly $CB(0)=\phi_1-\theta_1$ for all cases of ARMA(*p*,*q*) demand. As the stability of $CB(k)$ is influence by the AR parameters, we now investigate when bullwhip exists for AR(2) demand as shown in Figure 1 where the triangles denotes the stability area. Each graph represents a different lead-time *k*. Note that we are able to determine when bullwhip will be generated or not outside of the stability region as a positive $CB(k)$ indicates when $\Sigma_{oo}-\Sigma_{dd} > 0$, even when both $\{\Sigma_{oo},\Sigma_{dd}\}=\infty$. Thus this AR(2) map is also relevant for ARIMA(1,1,0) demand. It is clear to see that some complex lead-time bullwhip behavior is present. We have highlighted several cases of $CB(k)$ in Figure 2.

Bullwhip effect properties for non-stationary ARIMA demand

Several forecasting methods exist to cope with (non-seasonal) polynomial components in the demand, such as a constant level, a trend or a quadratic term. Examples include General Exponential Smoothing (Brown & Meyer, 1961), Holts method (Holt, 1957), Exponentially Weighted Regression polynomial models (D'Esopo, 1961) and ARIMA (Box et al., 2008). The forecasts of theses approaches can be represented by an ARIMA process for which the conditional expected value is "optimal". Recently Damped Trend, a variant of Holts Method, Gardner (1985), has received attention because of high scores in the M-3 competition (Makridakis and Hibon, 2000) .

All these approaches have one or more AR eigenvalues at one. Theoretically, the demand variance does not exist as well as the variance of the orders. Also the limits of $E(k)$ and $CB(k)$ at $k=\infty$ do not exist. However for finite lead times (*k*) they do. So the difference between the ordering and demand variance has some value. Put another way, the identification of AR, I and MA parameters of an ARIMA processes based on a limit set of data is never 100% accurate. Assuming the polynomial elements simplifies the identification process. However, given the limited data one might also assume that some AR eigenvalues are smaller but nearby to one. As a consequence the above analysis may apply if the condition that the largest MA eigenvalue is not larger than the AR eigenvalue near one.

Brown's general exponential approach for an m^{th} degree polynomial has m AR eigenvalues equal to one and m MA eigenvalues equal to β, $0<\beta<1$. Thus $CB(k)>0$ and is increasing in *k*. Holt's approach has two degrees of freedom, one more than second order exponential smoothing. The two AR eigenvalues are one and because the two real MA eigenvalues are smaller than one also $CB(k)>0$ and increasing. Damped Trend has AR eigenvalues $\lambda_1^{\phi}=\phi$ and $\lambda_2^{\phi}=1$, where generally $0<\phi<1$ is advised. Three eigenvalues orderings are possible, where the ordering $\lambda_1^{\phi}=\phi<\lambda_1^{\theta}<\lambda_2^{\theta}<\lambda_2^{\phi}=1$ could lead to decreasing behavior of $CB(k)$.

Concluding remarks

Using eigenvalues of the ARMA demand process areas of the parameter space are identified for which the bullwhip effect is increasing / decreasing as function of the lead time. Within these areas, regardless of the specific parameter values, this property holds, giving a robust result. Popular smoothing models also have this property. We are not sure of the bullwhip behavior if ARIMA models are identified by using demand data. In general, seasonal models show fluctuating bullwhip behavior and the OUT policy may not perform well in these cases.

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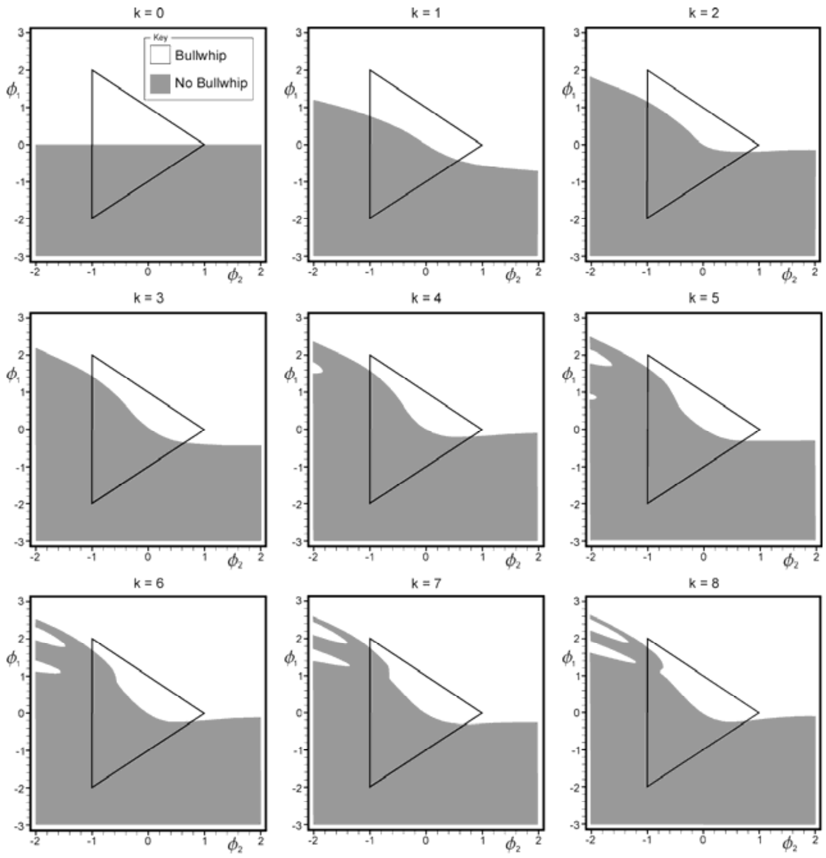

Figure 1. Regions of the AR(2) parameter space where a bullwhip exists in the OUT policy

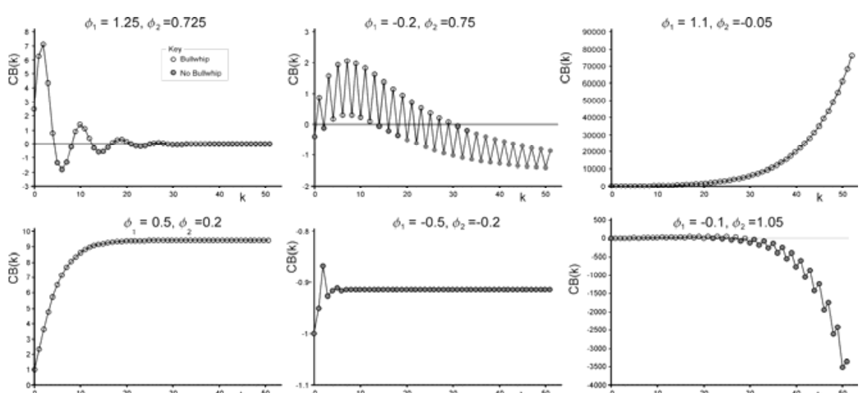

Figure 2. CB(k) for various AR(2) demands

Case	Ordering of eigenvalues	ϕ_1	ϕ_2	θ_1	θ_2	Eigenvalues
A	0-o-o-x-x-1	1.15	-0.33	0.21	-0.01	0.0729, 0.137, 0.55, 0.6
B	0-o-x-o-x-1	1.15	-0.2	0.6	-0.02	0.035,0.214, 0.565, 0.936
C	0-o-x-x-o-1	1.15	-0.2	1	-0.02	0.02, 0.214, 0.936, 0.979
D	0-x-o-o-x-1	1.18	-0.19	1	-0.02	0.02, 0.214, 0.936, 0.979
E	0-x-x-o-o-1	0.3	-0.02	1.15	-0.33	0.1, 0.2, 0.55, 0.6
F	0-x-o-x-o-1	0.6	-0.02	1.18	-0.19	0.035, 0.214, 0.564, 0.936