Spectral convergence for high contrast elliptic periodic problems with a defect via homogenisation

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Abstract

We consider an eigenvalue problem for a divergence form elliptic operator A_ε with high contrast periodic coefficients with period ε in each coordinate, where ε is a small parameter. The coefficients are perturbed on a bounded domain of 'order one' size. The local perturbation of coefficients for such operator could result in emergence of localised waves - eigenfunctions with corresponding eigenvalues lying in the gaps of the Floquet-Bloch spectrum. We prove that, for the so-called double porosity type scaling, the eigenfunctions decay exponentially at infinity, uniformly in ε. Then, using the tools of two-scale convergence for high contrast homogenisation, we prove the strong two-scale compactness of the eigenfunctions of A_ε. This implies that the eigenfunctions converge in the sense of the strong two-scale convergence to the eigenfunctions of a two-scale limit homogenised operator A_0, consequently establishing 'asymptotic one-to-one correspondence' between the eigenvalues and the eigenfunctions of these two operators. We also prove by direct means the stability of the essential spectrum of the homogenised operator with respect to the local perturbation of its coefficients. That allows us to establish not only the strong two-scale resolvent convergence of A_ε to A_0 but also the Hausdorff convergence of the spectra of A_ε to the spectrum of A_0, preserving the multiplicity of the isolated eigenvalues.

Keywords: localised modes, elliptic operators, perturbed periodic operators, multiscale methods, two-scale convergence, high-contrast homogenisation

AMS Subject Classifications: 35B27, 35P99

1 Introduction

In this paper we consider a high contrast two-phase periodic medium with a small period and with a 'finite size' defect filled by a third phase, see Fig. 1. This physically represents, for instance, a simplified model of cross-section of a photonic crystal fiber. Mathematically, the problem relates to a compact perturbation of ε-periodic coefficients in a divergence form elliptic operator A_ε. The behaviour of A_ε and its spectral characteristics as $\varepsilon \to 0$ is of the main interest. A similar problem is considered in [16] using the method of asymptotic expansions, but the present study pursues different aims and approaches the problem from another direction, namely developing an appropriate version of the two-scale convergence technique [20, 2, 23]. As a result we obtain a complete description of the asymptotic (with respect to ε) behaviour of the localised modes and other spectral characteristics for

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the operator A_ε in terms of an explicitly described (two-scale) limit operator A_0. For other recent applications of the high contrast homogenisation techniques see also [22, 10, 8, 6, 11, 12, 4, 17].

In the absence of a defect, Zhikov considers in [24] a divergence form elliptic operator $\widehat{A}_\varepsilon$ (denoted in [24] by A_ε) with periodic coefficients corresponding to a double-porosity model [3, 9] (A_ε in our notation is obtained from $\widehat{A}_\varepsilon$ by a compact perturbation of its coefficients). Operators of such type have the Floquet-Bloch essential spectrum, displaying a band-gap structure. Zhikov proves that the spectra of $\widehat{A}_\varepsilon$ converge in the sense of Hausdorff to the spectrum of a certain two-scale homogenised operator $\widehat{A}_0$ with constant coefficients, see also [14, 23], and that $\widehat{A}_0$ is the limit of $\widehat{A}_\varepsilon$ in the sense of strong two-scale resolvent convergence. The spectrum of $\widehat{A}_0$ is purely essential and displays an explicit band-gap structure. It is well known, see e.g. [21, 13], that in the case of a compact perturbation of periodic coefficients in the elliptic operator $\widehat{A}_\varepsilon$ its essential spectrum remains unperturbed. The only extra spectrum that can emerge in the gaps due to the perturbation is a discrete one (isolated eigenvalues with finite multiplicity).[1] Such an extra spectrum does emerge at least under some assumptions, e.g. [13, 16]. This corresponds physically to localised modes emerging near the defect.

One of the main goals of the paper is to establish the strong two-scale convergence of the eigenfunctions of A_ε corresponding to the eigenvalues in the gaps. In order to obtain this we need the strong two-scale compactness of eigenfunctions. This requires in turn an exponential decay of the eigenfunctions *uniform in ε*.

The problem of wave localization (i.e. of the existence of eigenvalues with corresponding eigenfunctions decaying exponentially) in the gaps of the essential spectrum has been intensively investigated for a wide range of differential operators over the last decades. The results obtained up to date ensure the exponential decay of eigenfunctions of A_ε for a *fixed ε*, see e.g. [13]. However this is insufficient for establishing the required compactness. Moreover, the developed methods, e.g. [5] and [13] (the latter using the method of Agmon[1]), seem to be insufficient for the present purpose. The reason is that in order to obtain the *uniform* exponential decay one has to perform some kind of two-scale asymptotic analysis, investigating the behaviour of the eigenfunctions on small and large scales simultaneously. To achieve this we supplement the method of [1] by the related two-scale techniques, which play a crucial role. As a result, we obtain a uniform estimate with the decay exponent α (see (3.1) and (2.14) below) which ensures the compactness, but may also be of an independent interest. On one hand, it is sharp in a sense. On the other hand, it behaves qualitatively entirely differently compared to e.g. the one in [5]: while the one in [5] is proportional to the square root of the distance to the gap end, the decay exponent we derive becomes large on approaching the left end of the gap and small near the right end.

The structure of the paper is the following. We first define the problem in Section 2, describe the two-scale limit operator A_0 and state the main result. We then consider a subsequence of eigenvalues of A_ε converging to some point λ_0 lying in a gap of the spectrum of $\widehat{A}_0$. In Section 3 we prove (Theorem 3.1) the uniform exponential decay for the eigenfunctions of A_ε. Section 4 is devoted to the proof of a main auxiliary lemma that is employed in the previous section, which may also be of an independent interest. In Section 5 we list some properties of the two-scale convergence and several related statements which we use in the next section. Employing the uniform exponential decay, we establish in Section 6 (see Theorem 6.1) the strong two-scale compactness of (normalised) eigenfunctions of A_ε, see e.g. [23, 24]. This implies that, up to a subsequence, the eigenfunctions two-scale converge to a function, which is eventually proved to be an eigenfunction of the two-scale limit operator A_0 with a defect, which could be considered as a perturbation of $\widehat{A}_0$. Accordingly λ_0 is an eigenvalue of A_0. The two-scale convergence of the eigenfunctions together with the results of [16] on the existence of the eigenvalues in the gaps and related error bounds allow us to make a conclusion about the 'asymptotic one-to-one correspondence' between eigenfunctions and eigenvalues

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[1] We do not concern in this paper the issue of whether *embedded* eigenvalues can emerge on the bands as a result of the perturbation.
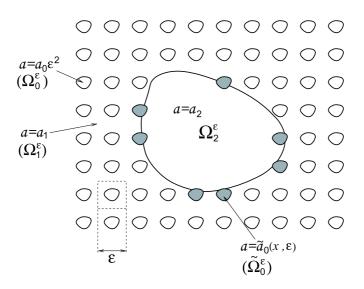
Figure 1: A defect in a rapidly oscillating high contrast periodic medium, cf. [16, Fig. 1].

of the operators A_ε and A_0 as $\varepsilon \to 0$. In the last section we prove by direct means (via the Weyl's sequences) the stability of the essential spectrum of $\widehat{A}_0$ with respect to the local perturbation of its coefficients (see Theorem 7.1). Thereby this establishes the convergence of the spectra of A_ε to the spectrum of A_0 in the sense of Hausdorff (Theorem 2.1).

2 Notation, problem formulation, limit operator and the main result

We will use the following notation for the geometric configuration visualised on Figure 1, cf. [16]. Consider a periodic set of unit cubes

$$\{Q : Q = [0,1)^n + \xi, \ \xi \in \mathbb{Z}^n\}. \tag{2.1}$$

Let F_0 be an open periodic set with period one in each coordinate such that $F_0 \cap Q \Subset Q$ is a connected domain with infinitely smooth boundary. We denote $F_0 \cap Q$ by Q_0 and $Q \backslash \overline{Q_0}$ by Q_1. Notice that the position of the particular set Q_0, Q_1 or Q depends on $\xi \in \mathbb{Z}^n$, however we will not reflect this in the notation to simplify what follows. Regularity assumptions on the boundary could be relaxed.[2] Let Ω_2 be a bounded domain with a sufficiently smooth boundary, containing the origin; its complement is denoted by Ω_1, $\Omega_1 = \mathbb{R}^n \backslash \overline{\Omega_2}$.

We define the 'inclusion phase' or the 'soft phase' Ω_0^ε as

$$\Omega_0^\varepsilon = \bigcup_{\varepsilon Q_0 \subset \Omega_1} \varepsilon Q_0,$$

where $\varepsilon > 0$ is a small parameter. The set of inclusions εQ_0 which intersect the boundary of Ω_2 is denoted by $\widetilde{\Omega}_0^\varepsilon$. The 'matrix phase', denoted by Ω_1^ε, is the complement to the inclusions in Ω_1, i.e. $\Omega_1^\varepsilon = \Omega_1 \backslash (\Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon)$. 'Defect domain' Ω_2^ε is defined by $\Omega_2 \backslash \widetilde{\Omega}_0^\varepsilon$. We also use the notation θ_Ω for the characteristic function of a set Ω and B_R for the open ball of radius R centered at the origin.

We consider an eigenvalue problem

$$A_\varepsilon u^\varepsilon = \lambda_\varepsilon u^\varepsilon \tag{2.2}$$

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[2]In particular, the results on the two-scale convergence stated in the paper remain valid at least under the assumption of Lipschitz regular boundaries. The $\varepsilon^{1/2}$-order bounds, as they were obtained in [16], require higher regularity.

for the point spectrum of an elliptic operator A_ε, self-adjoint in $L^2(\mathbb{R}^n)$,

$$A_\varepsilon u^\varepsilon := -\nabla \cdot \left(a(x,\varepsilon)\nabla u^\varepsilon(x) \right), \quad x \in \mathbb{R}^n, \tag{2.3}$$

with coefficient $a(x,\varepsilon)$ given by the formula

$$a(x,\varepsilon) = \begin{cases} a_0\varepsilon^2, & x \in \Omega_0^\varepsilon, \\ a_1, & x \in \Omega_1^\varepsilon, \\ a_2, & x \in \Omega_2^\varepsilon, \\ \widetilde{a}_0(x,\varepsilon), & x \in \widetilde{\Omega}_0^\varepsilon, \end{cases} \tag{2.4}$$

where measurable $\widetilde{a}_0(x,\varepsilon)$ is such that

$$\text{either } \widetilde{A}_0\,\varepsilon^{2-\theta} \le \widetilde{a}_0(x,\varepsilon) \le \sigma_0\,\varepsilon^{2-\theta} \text{ for all } \varepsilon, \text{ or } \widetilde{a}_0(x,\varepsilon) = a_0\,\varepsilon^2 \text{ for all } \varepsilon. \tag{2.5}$$

Here a_0, a_1, a_2, $\widetilde{A}_0$, $\widetilde{B}_0$ and θ are some positive constants independent of ε, $\theta \in (0,2]$. Notice that this includes as particular cases e.g. the case of 'removed' boundary inclusions, i.e. $a(x,\varepsilon) = a_1$ if $x \in \widetilde{\Omega}_0^\varepsilon \cap \Omega_1$, $a(x,\varepsilon) = a_2$ if $x \in \widetilde{\Omega}_0^\varepsilon \cap \Omega_2$, and the case of the 'full' inclusions, $\widetilde{a}_0(x,\varepsilon) = a_0\varepsilon^2$. The domain of A_ε is defined in a standard way via Friedrichs extension procedure with a bilinear form, see (2.6) below, defined on $H^1(\mathbb{R}^n)$.

For any $\varepsilon > 0$ the operator A_ε is an operator with ε-periodic coefficients, which are compactly perturbed (within bounded domain $\Omega_2^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon$). This implies (e.g. [21, 13]) that its essential spectrum coincides with the Floquet-Bloch spectrum of the associated 'unperturbed' operator $\widehat{A}_\varepsilon$, with only extra spectrum being hence the discrete spectrum in the gaps of $\widehat{A}_\varepsilon$.[3] Note that the spectrum of $\widehat{A}_\varepsilon$ contains gaps for small enough ε, cf. [14, 23, 24], and there is often an extra discrete spectrum in the gaps of $\sigma_{ess}(A_\varepsilon)$, e.g. [13, 16]. By definition, $u^\varepsilon \in H^1(\mathbb{R}^n)$, $u^\varepsilon \not\equiv 0$, is an eigenfunction of the eigenvalue problem (2.2) with an eigenvalue λ_ε if

$$\int_{\mathbb{R}^n} a(x,\varepsilon)\nabla u^\varepsilon \cdot \nabla w\, dx = \lambda_\varepsilon \int_{\mathbb{R}^n} u^\varepsilon w\, dx \tag{2.6}$$

for all $w \in H^1(\mathbb{R}^n)$.

The aim of this work is to establish that as $\varepsilon \to 0$ the operator A_ε converges in the appropriate sense (namely, in the sense of two-scale convergence, see Section 5) to a 'two-scale' limit operator A_0, which we describe next. For the rest of the present section we assume that $Q = [0,1)^n$, considering all functions of two variables (x,y) to be 1-periodic in each coordinate with respect to y. The 'two-scale' limit operator A_0 is analogous to the one introduced in the defect free setting by Zhikov [23, 24] and acts in a Hilbert space

$$\mathcal{H}_0 := \left\{ u(x,y) \in L^2(\mathbb{R}^n \times Q) \,\Big|\, u(x,y) = u_0(x) + v(x,y), u_0 \in L^2(\mathbb{R}^n), v \in L^2\left(\Omega_1; L^2(Q_0)\right) \right\}, \tag{2.7}$$

with the natural inner product inherited from $L^2(\mathbb{R}^n \times Q)$ and $\mathcal{H}_0$ being its closed subspace, cf. [24]. It is implied that v is extended by zero for $y \in Q_1$ or $x \in \Omega_2$. The operator A_0 is defined as generated by a (closed) symmetric and bounded from below bilinear form $B_0(u,w)$ acting in a dense subspace

$$\mathcal{V} = H^1(\mathbb{R}^n) + L^2\left(\Omega_1, H_0^1(Q_0)\right) \tag{2.8}$$

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[3]This does not rule out possible emergence of embedded eigenvalues on the bands, not considered in this paper.

of $\mathcal{H}_0 = L^2(\mathbb{R}^n) + L^2(\Omega_1, L^2(Q_0))$, which is defined as follows: for $u = u_0 + v, w = w_0 + z \in \mathcal{V}$,

$$B_0(u,w) = a_2 \int\limits_{\Omega_2} \nabla u_0 \cdot \nabla w_0 \, dx + \int\limits_{\Omega_1} A^{\mathrm{hom}} \nabla u_0 \cdot \nabla w_0 \, dx + a_0 \int\limits_{\Omega_1} \int\limits_{Q_0} \nabla_y v \cdot \nabla_y z \, dy \, dx. \qquad (2.9)$$

Here $A^{\mathrm{hom}} = \left(A^{\mathrm{hom}}_{ij} \right)$ is the standard "porous" homogenised (symmetric, positive-definite) matrix for the periodic medium as described above but when no defect is present and with $a_0 = 0$, see e.g. [15, §3.1]:

$$A^{\mathrm{hom}}_{ij} \xi_i \xi_j = \inf_{w \in C^\infty_{\mathrm{per}}(Q)} \int\limits_{Q_1} a_1 |\xi + \nabla w|^2 \, dy \quad (\xi \in \mathbb{R}^n). \qquad (2.10)$$

Here $C^\infty_{\mathrm{per}}(Q)$ stands for the set of infinitely smooth functions with periodic boundary conditions. Then one can see (cf. [24]) that the form is indeed bounded from below, densely defined and closed. Hence, according to the standard Friedrichs extension procedure, e.g. [21], A_0 can be defined as a self-adjoint operator with a domain $\mathcal{D}(A_0) \subset \mathcal{V}$. A function $u^0(x,y) = u_0(x) + v(x,y) \in \mathcal{V}$, $u^0(x,y) \not\equiv 0$, is an eigenfunction of the limit operator A_0 corresponding to an eigenvalue λ_0 if and only if

$$B_0(u^0, w) = \lambda_0 \int\limits_{\mathbb{R}^n} \int\limits_Q (u_0 + v)(w_0 + z) \, dy \, dx. \qquad (2.11)$$

for any $w = w_0 + z \in \mathcal{V}$ (we assume where it is possible that a function defined on a smaller domain is extended by zero on a larger domain).[4]

The 'unperturbed' operators $\widehat{A}_\varepsilon$ and $\widehat{A}_0$ could be defined analogously to A_ε and A_0 formally setting above $\Omega_2 = \emptyset$ and $\Omega_1 = \mathbb{R}^n$. (See also [23, 24], where these operators are denoted by A_ε and A respectively.)

We next describe a function $\beta(\lambda)$ which was introduced by Zhikov [23, 24] (cf. also [8]) and plays an important role in our considerations. Let λ_j and φ_j, $j = 1, 2, \ldots$, be eigenvalues and corresponding orthonormalised eigenfunctions of operator T defined as

$$Tf := -a_0 \Delta f, \quad f \in H^1_0(Q_0) \cap H^2(Q_0). \qquad (2.12)$$

Note that the eigenvalues of T belong to the spectrum of $\widehat{A}_0$, see [23]. For $\lambda \neq \lambda_j$, $j \geq 1$, denote by b the solution to

$$Tb - \lambda b = -a_0 \Delta b - \lambda b = 1, \ b \in H^1_0(Q_0). \qquad (2.13)$$

The function $\beta(\lambda)$ is defined by

$$\beta(\lambda) := \lambda \left(1 + \lambda \langle b \rangle_y \right) = \lambda + \lambda^2 \sum_{j=1}^\infty \frac{\langle \varphi_j \rangle_y^2}{\lambda_j - \lambda} \qquad (2.14)$$

where $\langle f \rangle_y := \int\limits_Q f(y) \, dy$. It is well-defined for any λ except $\lambda = \lambda_j$ with $\langle \varphi_j \rangle_y \neq 0$, monotonically increasing between such points, see Figure 2. This function describes the structure of $\sigma(\widehat{A}_0)$, see [23]. Namely, the intervals where $\beta(\lambda) \geq 0$ correspond to the bands of the spectrum of $\widehat{A}_0$. Isolated points of the spectrum of $\widehat{A}_0$, i.e. λ_j such that $\langle \varphi_j \rangle_y = 0$ and $\beta(\lambda_j) < 0$, can also be regarded as degenerate bands. The intervals on which $\beta(\lambda) < 0$ (excluding λ_j) are gaps.

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[4]Explicit examples in [16, §5] ensure the existence of isolated eigenvalues of A_0 of finite multiplicity in the gaps of $\sigma(\widehat{A}_0)$ in particular situations.
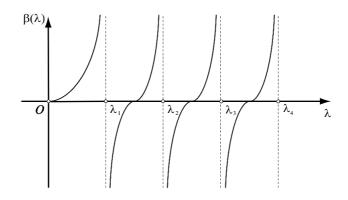
Figure 2: $\beta(\lambda)$, cf. [24].

It was shown in [24] (see also [14, 23]) that $\sigma(\widehat{A}_\varepsilon)$ converges in the sense of Hausdorff to $\sigma(\widehat{A}_0)$, while $\widehat{A}_\varepsilon$ converges to $\widehat{A}_0$ in the sense of the strong two-scale resolvent convergence (cf. Sections 5 and 6 below) implying the convergence of spectral projectors, etc.

We aim at showing that similar as well as some further results hold for the perturbed operators. Namely, our main result is the following

Theorem 2.1. *The operator A_ε converges to A_0 in the sense of the strong two-scale resolvent convergence. Hence the spectral projectors also strongly two-scale converge away from the point spectrum of A_0. The spectrum of A_ε converges in the sense of Hausdorff to the spectrum of A_0. Let λ_0 be an isolated eigenvalue of multiplicity m of the operator A_0 in the gap of its essential spectrum. Then, for small enough ε, there exist exactly m eigenvalues $\lambda_{\varepsilon,i}$ of A_ε (counted with their multiplicities) such that*

$$|\lambda_{\varepsilon,i} - \lambda_0| \leq C\varepsilon^{1/2}, \, i = 1, \ldots, m, \tag{2.15}$$

with a constant C independent of ε.[5] If for some sequence $\varepsilon_k \to 0$ a sequence of eigenvalues λ_{ε_k} of A_ε converges to λ_0 which is in the gap of the essential spectrum of A_0, then λ_0 is an isolated eigenvalue of A_0 of a finite multiplicity m and for large enough k, $\lambda_{\varepsilon_k} \in \{\lambda_{\varepsilon_k,i}, \, i = 1, \ldots, m\}$.

A key part in establishing the latter is in controlling the behaviour at infinity of the eigenfunctions corresponding to the extra point spectrum which may appear in the spectral gaps of the unperturbed operator. A central property providing this is a uniform exponential decay of the eigenfunctions which we prove next.

3 Uniform exponential decay of the eigenfunctions of A_ε

Let λ_0 be a point in a gap of $\sigma(\widehat{A}_0)$, i.e. such that $\beta(\lambda_0) < 0$ and $\lambda_0 \neq \lambda_j$ for all j. Assume λ_0 is an accumulation point of the point spectra of A_ε, i.e. for some subsequence $\varepsilon_k \to 0$ there exist eigenvalues λ_{ε_k} of A_ε such that $\lambda_{\varepsilon_k} \to \lambda_0$ as $k \to \infty$. (Notice that the results of [13, 16] ensure in particular that such sequences do exist.) We formulate the main result of this section (and also one of the principal results of the paper) in the following.

Theorem 3.1. *Let λ_{ε_k} and u^{ε_k} be sequences of eigenvalues of the operator A_ε and corresponding eigenfunctions normalised in $L^2(\mathbb{R}^n)$, where ε_k is some positive sequence converging to zero as $k \to \infty$. Let λ_0 be such that $\beta(\lambda_0)$ is negative and λ_0 is not an eigenvalue of the operator T given by (2.12).*

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[5]The error bound (2.15) employs the results of [16] requiring, as stated, higher regularity of ∂Q_0. The rest of the statement of the theorem applies potentially to less regular boundaries.

Suppose that λ_{ε_k} converges to λ_0. Then for small enough ε_k eigenfunctions u^{ε_k} decay uniformly exponentially at infinity, namely, for

$$0 < \alpha < \sqrt{-\beta(\lambda_0)/a_1} \tag{3.1}$$

the following holds:

$$\|e^{\alpha|x|}u^{\varepsilon_k}\|_{L^2(\mathbb{R}^n)} \leq C,$$

uniformly in ε_k, i.e. for any $0 < \varepsilon_k < \varepsilon(\alpha)$, with $C = C(\alpha)$ independent of ε.

Proof. We drop the index k in ε_k for the sake of simplification of notation. So, when we say, for instance, 'sequence λ_ε' we actually mean 'subsequence λ_{ε_k}'.

The plan of the proof is the following. We first derive 'elementary' a priori estimates for the eigenfunction u^ε outside the set of inclusions $\Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon$. Next we study the structure of the eigenfunction at the small scale and deduce some vital inequalities for $\varepsilon\nabla u^\varepsilon$ inside the inclusions. As a central technical step, we then employ in the integral identity (2.6) a test function with exponentially growing weight $g^2(|x|)$, see (3.12)–(3.13) below, and perform some delicate two-scale uniform estimates to achieve the result. The main auxiliary technical results are proven in Lemma 3.3 and Proposition 4.1.

Step 1. Setting $w = u^\varepsilon$ in (2.6) we have

$$\varepsilon^2 a_0\|\nabla u^\varepsilon\|_{L^2(\Omega_0^\varepsilon)}^2 + a_1\|\nabla u^\varepsilon\|_{L^2(\Omega_1^\varepsilon)}^2 + a_2\|\nabla u^\varepsilon\|_{L^2(\Omega_2^\varepsilon)}^2 + \|\widetilde{a}_0^{1/2}(x,\varepsilon)\,\nabla u^\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)}^2 = \lambda_\varepsilon\|u^\varepsilon\|_{L^2(\mathbb{R}^n)}^2 = \lambda_\varepsilon.$$

Therefore

$$\|u^\varepsilon\|_{H^1(\mathbb{R}^n\setminus(\Omega_0^\varepsilon\cup\widetilde{\Omega}_0^\varepsilon))} \leq C \tag{3.2}$$

uniformly in ε. From now on C denotes a generic constant whose precise value is insignificant and can change from line to line.

Step 2. Let us consider the function u^ε in a cell εQ corresponding to such $\xi = \xi(\varepsilon) \in \mathbb{Z}^n$, see 2.1, that the corresponding 'inclusion' εQ_0 has a nonempty intersection with Ω_1. There exists an extension $\widetilde{u}^\varepsilon$ of $u^\varepsilon|_{\varepsilon Q_1}$ to the whole cell εQ such that

$$\|\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)} \leq C\|u^\varepsilon\|_{L^2(\varepsilon Q_1)}, \quad \|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)} \leq C\|\nabla u^\varepsilon\|_{L^2(\varepsilon Q_1)}, \tag{3.3}$$

where C does not depend on ε or ξ, see e. g. [19, Ch. 3, §4, Th. 1], which is a version of the so-called 'extension lemma', see also e.g. [15, §3.1, L. 3.2]. In particular, we can choose the following extension:

$$\widetilde{u}^\varepsilon \equiv u^\varepsilon, \qquad x \in \Omega_1^\varepsilon \cup \Omega_2^\varepsilon,$$
$$-\nabla \cdot \big(a(x,\varepsilon)\nabla\widetilde{u}^\varepsilon(x)\big) = 0, \quad x \in \Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon,$$

which minimises $\|a^{1/2}(x,\varepsilon)\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)}$ subject to the prescribed boundary conditions, with (2.4) and (2.5) ensuring that (3.3) still holds. From (3.2) and (3.3) we conclude that

$$\|\widetilde{u}^\varepsilon\|_{H^1(\mathbb{R}^n)} \leq C. \tag{3.4}$$

We represent u^ε in the form

$$u^\varepsilon(x) = \widetilde{u}^\varepsilon(x) + v^\varepsilon(x) \tag{3.5}$$

and consider the function $v^\varepsilon \in H_0^1(\Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon)$.[6] In each inclusion $\varepsilon Q_0 \subset \Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon$ we have the following boundary value problem for $v^\varepsilon(x)$:

$$-\nabla \cdot (a(x,\varepsilon)\nabla v^\varepsilon) - \lambda_\varepsilon v^\varepsilon = \lambda_\varepsilon\widetilde{u}^\varepsilon, \ x \in \varepsilon Q_0; \quad v^\varepsilon(x) = 0, \ x \in \partial(\varepsilon Q_0). \tag{3.6}$$

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[6] In a sense, (3.5) decomposes u^ε into a slowly varying part $\widetilde{u}^\varepsilon$ and rapidly varying v^ε. The two are coupled and subsequently analyzed simultaneously, which is the essence of two-scale asymptotic analysis.

When $a(x, \varepsilon) = a_0 \varepsilon^2$, i.e. everywhere in Ω_0^ε and also in $\widetilde{\Omega}_0^\varepsilon$ in the case $\widetilde{a}_0(x, \varepsilon) = a_0 \varepsilon^2$, after changing the variables $x \to y = x/\varepsilon$ we obtain

$$-a_0 \Delta_y v^\varepsilon(\varepsilon y) - \lambda_\varepsilon v^\varepsilon(\varepsilon y) = \lambda_\varepsilon \widetilde{u}^\varepsilon(\varepsilon y), \ y \in Q_0, \quad v^\varepsilon(\varepsilon y) = 0, \ y \in \partial Q_0. \tag{3.7}$$

Since $\lambda_0 \neq \lambda_j$ by the assumptions of the theorem, λ_ε is separated uniformly from the spectrum of operator (2.12) for small enough ε. Hence the resolvent of T at λ_ε is bounded uniformly in ε and (3.7) implies

$$\|v^\varepsilon(\varepsilon y)\|_{H^1(Q_0)} \leq C \|\widetilde{u}^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}. \tag{3.8}$$

In the case when $\widetilde{A}_0 \, \varepsilon^{2-\theta} \leq \widetilde{a}_0(x, \varepsilon) \leq \widetilde{B}_0 \, \varepsilon^{2-\theta}$, $\theta \in (0, 2]$, we multiply equation (3.6) by v^ε and integrate by parts to obtain after rescaling

$$\varepsilon^{-2} \int\limits_{\varepsilon Q_0} \widetilde{a}_0(\varepsilon y, \varepsilon) |\nabla_y v^\varepsilon(\varepsilon y)|^2 dx - \lambda_\varepsilon \int\limits_{\varepsilon Q_0} \left(v^\varepsilon(\varepsilon y) \right)^2 dx = \lambda_\varepsilon \int\limits_{\varepsilon Q_0} \widetilde{u}^\varepsilon(\varepsilon y) v^\varepsilon(\varepsilon y) \, dx. \tag{3.9}$$

Notice that $\varepsilon^{-2} \widetilde{a}_0(\varepsilon y, \varepsilon) \geq \widetilde{A}_0 \varepsilon^{-\theta} \to \infty$ as $\varepsilon \to 0$. Then using Poincaré inequality one easily derives

$$\varepsilon^{-2} \|\widetilde{a}_0^{1/2} \nabla_y v^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}^2 + \|v^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}^2 \leq C \|\widetilde{u}^\varepsilon(\varepsilon y)\|_{L^2(Q_0)}^2, \tag{3.10}$$

for small enough ε. Returning in (3.8) and (3.10) to the variable x we arrive at the following inequality that describes the behaviour of v^ε and its gradient in $\Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon$,

$$\|a^{1/2} \nabla v^\varepsilon(x)\|_{L^2(\varepsilon Q_0)}^2 + \|v^\varepsilon(x)\|_{L^2(\varepsilon Q_0)}^2 \leq C \|\widetilde{u}^\varepsilon(x)\|_{L^2(\varepsilon Q_0)}^2, \tag{3.11}$$

with an ε-independent constant C.

Step 3. In order to get the uniform exponential decay of the eigenfunctions we next substitute in (2.6) a test function of a special form:

$$w = g^2(|x|) \widetilde{u}^\varepsilon(x). \tag{3.12}$$

Here we define function g as follows

$$g(t) = \begin{cases} e^{\alpha t}, & t \in [0, R], \\ e^{\alpha R}, & t \in (R, +\infty), \end{cases} \tag{3.13}$$

where R is some arbitrary positive number. The exponent α will be chosen later. This method was employed e.g. by Agmon, see [1], but in the present case its realization is not straightforward. Namely, to obtain the desired estimates we have to implement the approach of [1] in the context of the two-scale analysis. We will show that $g(|x|) \widetilde{u}^\varepsilon(x)$, and consequently $g(|x|) u^\varepsilon(x)$, are bounded in $L^2(\mathbb{R}^n)$ uniformly with respect to R and ε. Then we will show via passing to the limit as $R \to \infty$ that we can replace $g(|x|)$ by $e^{\alpha|x|}$.

Remark 3.2. We cannot use $e^{2\alpha|x|} \widetilde{u}^\varepsilon(x)$ as a test function directly, since it is not known at this stage that this function is square integrable.

The following identity holds by direct inspection

$$\nabla \widetilde{u}^\varepsilon \nabla(g^2 \widetilde{u}^\varepsilon) = |\nabla(g \widetilde{u}^\varepsilon)|^2 - |\nabla g|^2 (\widetilde{u}^\varepsilon)^2. \tag{3.14}$$

Notice that the absolute value of ∇g is bounded by g with α (uniformly in R):

$$\left| \nabla g(|x|) \right| \leq \alpha g(|x|). \tag{3.15}$$

After the substitution of (3.12) into (2.6) we have, via (3.5) and (3.14),

$$
\varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla u^\varepsilon \cdot \nabla (g^2 \widetilde{u}^\varepsilon)\, dx + \int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 \nabla v^\varepsilon \cdot \nabla (g^2 \widetilde{u}^\varepsilon)\, dx + \int\limits_{\mathbb{R}^n \setminus \Omega_0^\varepsilon} a(x,\varepsilon) |\nabla (g \widetilde{u}^\varepsilon)|^2\, dx -
$$

$$
- a_1 \int\limits_{\Omega_1^\varepsilon} |\nabla g|^2 (\widetilde{u}^\varepsilon)^2\, dx - \lambda_\varepsilon \int\limits_{\Omega_0^\varepsilon \cup \Omega_1^\varepsilon} g^2 (\widetilde{u}^\varepsilon)^2\, dx - \lambda_\varepsilon \int\limits_{\Omega_0^\varepsilon} g^2 v^\varepsilon \widetilde{u}^\varepsilon\, dx = \tag{3.16}
$$

$$
= \lambda_\varepsilon \int\limits_{\widetilde{\Omega}_0^\varepsilon} g^2 u^\varepsilon \widetilde{u}^\varepsilon\, dx + \lambda_\varepsilon \int\limits_{\Omega_2^\varepsilon} g^2 (\widetilde{u}^\varepsilon)^2 dx + \int\limits_{\Omega_2^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon} a(x,\varepsilon) |\nabla g|^2 (\widetilde{u}^\varepsilon)^2\, dx.
$$

Notice that the right hand side is bounded by some constant C independent of ε and R due to (3.2), (3.4), (3.11) and the boundedness of the domains of integration.

We employ (3.4), (3.11) and the boundedness of $\widetilde{a}_0$ to conclude that the second term on the left hand side of (3.16) tends to zero (uniformly in R):

$$
\left| \int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 \nabla v^\varepsilon \cdot \nabla (g^2 \widetilde{u}^\varepsilon)\, dx \right| \le C \|\widetilde{u}^\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \to 0, \tag{3.17}
$$

as follows. Let us take an arbitrary subsequence $\widetilde{u}^\varepsilon$. Since $\|\widetilde{u}^\varepsilon\|_{H^1(\mathbb{R}^n)}$ is bounded uniformly in ε, see (3.4), the set of functions $\widetilde{u}^\varepsilon$ is weakly compact in $H^1(B_R)$, hence strongly compact in $L^2(B_R)$ for any R; we take R large enough so that $\Omega_2 \subset\subset B_R$. Then there exists further subsequence $\widetilde{u}^\varepsilon$ that converges to some function u_0 strongly in $L^2(B_R)$. Then

$$
\|\widetilde{u}^\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \le \|u_0\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} + \|\widetilde{u}^\varepsilon - u_0\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \to 0
$$

as Lebesgue measure of the set $\widetilde{\Omega}_0^\varepsilon$ tends to zero. Since we have chosen in the beginning an arbitrary subsequence $\widetilde{u}^\varepsilon$, (3.17) follows. From (3.11) and (3.17) we also obtain

$$
\|v^\varepsilon\|_{L^2(\widetilde{\Omega}_0^\varepsilon)} \to 0. \tag{3.18}
$$

Step 4. The following Lemma approximates and bounds the last and the first terms (both, in a sense, of a 'two-scale' nature) on the left hand side of (3.16).

Lemma 3.3. *There exists $\varepsilon_0 > 0$ such that for all positive $\varepsilon < \varepsilon_0$ the following estimates are valid*

$$
\left| \lambda_\varepsilon \int\limits_{\Omega_0^\varepsilon} g^2 v^\varepsilon \widetilde{u}^\varepsilon\, dx - (\beta(\lambda_\varepsilon) - \lambda_\varepsilon) \int\limits_{\Omega_0^\varepsilon \cup \Omega_1^\varepsilon} g^2 (\widetilde{u}^\varepsilon)^2\, dx \right| \le \tag{3.19}
$$

$$
\le C \varepsilon \left(\|\nabla (g \widetilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)}^2 + \|g \widetilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}^2 \right) + C,
$$

and

$$
\left| \varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla u^\varepsilon \nabla (g^2 \widetilde{u}^\varepsilon)\, dx \right| \le C \varepsilon \left(\|\nabla (g \widetilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)}^2 + \|g \widetilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}^2 + C \right), \tag{3.20}
$$

where C does not depend on ε and R.

The proof of this lemma is quite technical and we give it in the next section. We make use of Lemma 3.3 and convergence (3.17) to transform identity (3.16) into the following inequality, valid for small enough ε:

$$a_1\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\Omega_1^\varepsilon)} - a_1\|(\nabla g)\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_1^\varepsilon)} - \beta(\lambda_\varepsilon)\|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)} -$$
$$- 2C\varepsilon\left(\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\Omega_1^\varepsilon)} + \|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}\right) \le C,$$

where C is independent of ε and R. Notice that $\beta(\lambda_\varepsilon)$ is negative and uniformly bounded away from zero as $\lambda_\varepsilon \to \lambda_0$. Applying (3.15) to the second term on the left hand side we arrive at

$$(a_1 - 2C\varepsilon)\|\nabla(g\widetilde{u}^\varepsilon)\|^2_{L^2(\Omega_1^\varepsilon)} + \left(-\beta(\lambda_\varepsilon) - \alpha^2 a_1 - 2C\varepsilon\right)\|g\widetilde{u}^\varepsilon\|^2_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)} \le C. \tag{3.21}$$

Hence we should choose α such that $-\beta(\lambda_0) - \alpha^2 a_1$ is positive, i.e.

$$\alpha < \sqrt{-\beta(\lambda_0)/a_1}.$$

Since $g(|x|)$ coincides with $e^{\alpha|x|}$ on the ball B_R, restricting the L^2-norms to B_R we obtain that

$$\left\|e^{\alpha|x|}\widetilde{u}^\varepsilon\right\|_{L^2(B_R)} \le C$$

for small enough ε, where C does not depend on ε and R. Then passing to the limit as $R \to \infty$ we obtain

$$\left\|e^{\alpha|x|}\widetilde{u}^\varepsilon\right\|_{L^2(\mathbb{R}^n)} \le C. \tag{3.22}$$

Step 5. Now we easily get the same estimate for the function u^ε:

$$\left\|e^{\alpha|x|}u^\varepsilon\right\|_{L^2(\mathbb{R}^n)} \le \left\|e^{\alpha|x|}\widetilde{u}^\varepsilon\right\|_{L^2(\mathbb{R}^n)} + \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon \cup \widetilde{\Omega}_0^\varepsilon}\left\|e^{\alpha|x|}v^\varepsilon\right\|_{L^2(\varepsilon Q_0)}.$$

In each cell we use inequality (3.11) and

$$\sup_{x' \in \varepsilon Q} e^{\alpha|x'|} \le e^{\alpha\sqrt{n}\varepsilon}e^{\alpha|x|}, \qquad \forall x \in \varepsilon Q,$$

to obtain

$$\left\|e^{\alpha|x|}v^\varepsilon\right\|_{L^2(\varepsilon Q_0)} \le Ce^{\alpha\sqrt{n}\varepsilon}\left\|e^{\alpha|x|}\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q_0)} \le C\left\|e^{\alpha|x|}\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q_0)},$$

and hence, finally,

$$\left\|e^{\alpha|x|}u^\varepsilon\right\|_{L^2(\mathbb{R}^n)} \le C$$

uniformly in ε. $\qquad\square$

Remark 3.4. From (3.3), (3.15) and (3.21) it also follows that the gradient of $\widetilde{u}^\varepsilon$ decays exponentially at infinity,

$$\left\|e^{\alpha|x|}\nabla\widetilde{u}^\varepsilon\right\|_{L^2(\mathbb{R}^n)} \le C \tag{3.23}$$

uniformly in ε.

Remark 3.5. Estimate (3.1) is sharp in a sense. As we will show later, u_ε strongly two-scale converges to u_0, for which $\sqrt{-\beta(\lambda_0)/a_1}$ is the optimal estimate for its decay exponent, cf. (7.15).

4 Proof of Lemma 3.3.

Proof. **Step 1.** First we decompose the function v^ε in Ω_0^ε into the sum of two functions:

$$v^\varepsilon = \widetilde{v}^\varepsilon + \widehat{v}^\varepsilon, \tag{4.1}$$

solving the following equations (cf. (3.7)):

$$-a_0\Delta_y\widetilde{v}^\varepsilon(\varepsilon y) - \lambda_\varepsilon\widetilde{v}^\varepsilon(\varepsilon y) = \lambda_\varepsilon\langle\widetilde{u}^\varepsilon(\varepsilon y)\rangle_y, \ y \in Q_0, \quad \widetilde{v}^\varepsilon(\varepsilon y) = 0, \ y \in \partial Q_0, \tag{4.2}$$

$$-a_0\Delta_y\widehat{v}^\varepsilon(\varepsilon y) - \lambda_\varepsilon\widehat{v}^\varepsilon(\varepsilon y) = \lambda_\varepsilon\left(\widetilde{u}^\varepsilon(\varepsilon y) - \langle\widetilde{u}^\varepsilon(\varepsilon y)\rangle_y\right), \ y \in Q_0, \quad \widehat{v}^\varepsilon(\varepsilon y) = 0, \ y \in \partial Q_0. \tag{4.3}$$

The solution of (4.2) could by presented in the form

$$\widetilde{v}^\varepsilon(\varepsilon y) = \lambda_\varepsilon\langle\widetilde{u}^\varepsilon\rangle_y b_\varepsilon(y), \tag{4.4}$$

where b_ε is a solution of (2.13) with $\lambda = \lambda_\varepsilon$. Due to the uniform (with respect to ε) boundedness of the resolvent of the operator T in the neighborhood of λ_0, the solution of (4.3) is bounded as follows,

$$\|\widehat{v}^\varepsilon(\varepsilon y)\|_{H^1(Q_0)} \leq C\,\|\widetilde{u}^\varepsilon(\varepsilon y) - \langle\widetilde{u}^\varepsilon\rangle_y\|_{L^2(Q_0)} \leq C\,\|\nabla_y\widetilde{u}^\varepsilon\|_{L^2(Q_0)},$$

here we also employed the Poincaré inequality. In particular

$$\|\widehat{v}^\varepsilon(x)\|_{L^2(\varepsilon Q_0)} \leq \varepsilon\,C\|\nabla\widetilde{u}^\varepsilon(x)\|_{L^2(\varepsilon Q)}, \tag{4.5}$$

where C in the inequality does not depend on ε or $\xi \in \mathbb{Z}^n$.

Step 2. At this stage we will need several inequalities which follow from the properties of g and $\widetilde{u}^\varepsilon$.

Proposition 4.1. *The following estimates are valid for small enough ε with constants independent of ε and the choice of particular εQ:*

$$\left\|g^2\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q)}\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \leq C\left(\|\nabla(g\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_1)}^2 + \|g\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}^2\right), \tag{4.6}$$

$$\|\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\left\|\nabla(g^2\widetilde{u}^\varepsilon)\right\|_{L^2(\varepsilon Q)} \leq C\left(\|\nabla(g\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_1)}^2 + \|g\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}^2\right), \tag{4.7}$$

$$\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\left\|\nabla(g^2\widetilde{u}^\varepsilon)\right\|_{L^2(\varepsilon Q)} \leq C\left(\|\nabla(g\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_1)}^2 + \|g\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}^2\right). \tag{4.8}$$

Proof. Notice that

$$\sup_{x'\in\varepsilon Q} g(x') \leq e^{\alpha\sqrt{n}\varepsilon}g(x), \qquad x \in \varepsilon Q. \tag{4.9}$$

We apply (3.3), (3.15) and (4.9) to get (4.6):

$$\left\|g^2\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q)}\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)} \leq C\left\|g^2\widetilde{u}^\varepsilon\right\|_{L^2(\varepsilon Q)}\|\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_1)} \leq C\|g\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\|g\nabla\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_1)} =$$

$$= C\|g\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\|\nabla(g\widetilde{u}^\varepsilon) - (\nabla g)\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_1)} \leq C\left(\|g\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}\|\nabla(g\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_1)} + \|g\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}^2\right) \leq$$

$$\leq C\left(\|\nabla(g\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_1)}^2 + \|g\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}^2\right).$$

The proof of (4.7) and (4.8) is analogous. $\qquad\square$

Let us show that the entity $\int_{\Omega_0^\varepsilon} g^2 \widehat{v}^\varepsilon \widetilde{u}^\varepsilon \, dx$ is relatively small (compared to the first term on the right hand side of (3.19)). Indeed, applying inequalities (4.5) and (4.6) in each cell we obtain

$$\int_{\Omega_0^\varepsilon} g^2 \widehat{v}^\varepsilon \widetilde{u}^\varepsilon \, dx \leq \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \|g^2 \widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q_0)} \|\widehat{v}^\varepsilon\|_{L^2(\varepsilon Q_0)} \leq \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon \, C \left(\|\nabla(g\widetilde{u}^\varepsilon)\|_{L^2(\varepsilon Q_1)}^2 + \|g\widetilde{u}^\varepsilon\|_{L^2(\varepsilon Q)}^2 \right).$$

$$(4.10)$$

Considering sets

$$\bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q \qquad \text{and} \qquad \bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q_1,$$

one can notice that they are "nearly" equal to

$$\Omega_0^\varepsilon \cup \Omega_1^\varepsilon \qquad \text{and} \qquad \Omega_1^\varepsilon,$$

respectively. Namely,

$$\Omega_0^\varepsilon \cup \Omega_1^\varepsilon = \left(\bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q \right) \cup \Omega_{1,+}^\varepsilon \setminus \Omega_{1,-}^\varepsilon,$$

$$\Omega_1^\varepsilon = \left(\bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q_1 \right) \cup \Omega_{1,+}^\varepsilon \setminus \Omega_{1,-}^\varepsilon,$$

where

$$\Omega_{1,-}^\varepsilon = \bigcup_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \varepsilon Q \cap \Omega_2,$$

$$\Omega_{1,+}^\varepsilon = \bigcup_{\varepsilon Q_0 \cap \Omega_2 \neq \emptyset} \varepsilon Q \cap \Omega_1^\varepsilon.$$

We introduce two 'correctors'

$$r^\varepsilon = \|\nabla(g\widetilde{u}^\varepsilon)\|_{L^2(\Omega_{1,-}^\varepsilon)}^2 + \|g\widetilde{u}^\varepsilon\|_{L^2(\Omega_{1,-}^\varepsilon)}^2,$$

and

$$r_1^\varepsilon = \|g\widetilde{u}^\varepsilon\|_{L^2(\Omega_{1,+}^\varepsilon \cup \Omega_{1,-}^\varepsilon)}^2.$$

Then inequality (4.10) transforms into

$$\int_{\Omega_0^\varepsilon} g^2 \widehat{v}^\varepsilon \widetilde{u}^\varepsilon \, dx \leq \varepsilon C \left(\|\nabla(g\widetilde{u}^\varepsilon)\|_{L^2(\Omega_1^\varepsilon)}^2 + \|g\widetilde{u}^\varepsilon\|_{L^2(\Omega_0^\varepsilon \cup \Omega_1^\varepsilon)}^2 + r^\varepsilon \right). \qquad (4.11)$$

Step 3. Now we consider the term $\int_{\Omega_0^\varepsilon} g^2 \widetilde{v}^\varepsilon \widetilde{u}^\varepsilon \, dx$ (cf. (3.19)) using also (4.4) and (2.14):

$$\left| \lambda_\varepsilon \int_{\Omega_0^\varepsilon} g^2 \widetilde{v}^\varepsilon \widetilde{u}^\varepsilon \, dx - (\beta(\lambda_\varepsilon) - \lambda_\varepsilon) \int_{\Omega_0^\varepsilon \cup \Omega_1^\varepsilon} g^2 (\widetilde{u}^\varepsilon)^2 \, dx \right| \leq$$

$$\leq C\varepsilon^n \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \left| \int_Q g^2 \widetilde{u}^\varepsilon(\varepsilon y) b_\varepsilon(y) \langle \widetilde{u}^\varepsilon \rangle_y \, dy - \langle b_\varepsilon \rangle_y \int_Q g^2(\varepsilon y)(\widetilde{u}^\varepsilon(\varepsilon y))^2 \, dy \right| + C \, r_1^\varepsilon \leq \qquad (4.12)$$

$$\leq C\varepsilon^n \sum_{\varepsilon Q_0 \subset \Omega_0^\varepsilon} \left| \langle \widetilde{u}^\varepsilon \rangle_y \int_Q \left(g^2 \widetilde{u}^\varepsilon - \langle g^2 \widetilde{u}^\varepsilon \rangle_y \right) b_\varepsilon \, dy \right| + \left| \langle b_\varepsilon \rangle_y \int_Q \left(g^2 \widetilde{u}^\varepsilon - \langle g^2 \widetilde{u}^\varepsilon \rangle_y \right) \widetilde{u}^\varepsilon \, dy \right| + C \, r_1^\varepsilon.$$

Notice that the mean value of $\widetilde{u}^{\varepsilon}$ is bounded by its norm in L^2

$$|\langle\widetilde{u}^{\varepsilon}(\varepsilon y)\rangle_y| = \left|\int\limits_Q \widetilde{u}^{\varepsilon}\,dy\right| \le \|\widetilde{u}^{\varepsilon}(x)\|_{L^2(Q)}. \tag{4.13}$$

Similarly,

$$\langle b_{\varepsilon}\rangle_y \le \|b_{\varepsilon}\|_{L^2(Q_0)} \le C, \tag{4.14}$$

where C does not depend on ε due to the uniform boundedness of $(T-\lambda)^{-1}$ in the neighborhood of λ_0. Via the Poincaré inequality we derive

$$\left|\int\limits_Q \left(g^2\widetilde{u}^{\varepsilon} - \langle g^2\widetilde{u}^{\varepsilon}\rangle_y\right)\widetilde{u}^{\varepsilon}\,dy\right| \le \left\|g^2\widetilde{u}^{\varepsilon} - \langle g^2\widetilde{u}^{\varepsilon}\rangle_y\right\|_{L^2(Q)}\|\widetilde{u}^{\varepsilon}\|_{L^2(Q)} \le C\left\|\nabla_y\left(g^2\widetilde{u}^{\varepsilon}\right)\right\|_{L^2(Q)}\|\widetilde{u}^{\varepsilon}\|_{L^2(Q)}, \tag{4.15}$$

and

$$\left|\int\limits_Q \left(g^2\widetilde{u}^{\varepsilon} - \langle g^2\widetilde{u}^{\varepsilon}\rangle_y\right)b_{\varepsilon}\,dy\right| \le C\left\|\nabla_y\left(g^2\widetilde{u}^{\varepsilon}\right)\right\|_{L^2(Q)}, \tag{4.16}$$

with constants independent of ε and ξ. Applying inequalities (4.13)–(4.16) and then (4.7) to (4.12) we arrive at

$$\left|\lambda_{\varepsilon}\int\limits_{\Omega_0^{\varepsilon}} g^2\widetilde{v}^{\varepsilon}\widetilde{u}^{\varepsilon}\,dx + (\lambda_{\varepsilon} - \beta(\lambda_{\varepsilon}))\int\limits_{\Omega_0^{\varepsilon}\cup\Omega_1^{\varepsilon}} g^2(\widetilde{u}^{\varepsilon})^2\,dx\right| \le$$
$$\le \varepsilon\,C\left(\|\nabla(g\widetilde{u}^{\varepsilon})\|^2_{L^2(\Omega_1^{\varepsilon})} + \|g\widetilde{u}^{\varepsilon}\|^2_{L^2(\Omega_0^{\varepsilon}\cup\Omega_1^{\varepsilon})} + r^{\varepsilon}\right) + C\,r_1^{\varepsilon}, \tag{4.17}$$

where C is ε-independent. Since the correctors r^{ε}, r_1^{ε} are uniformly bounded, inequalities (4.11) and (4.17) together imply the validity of (3.19).

Step 4. Finally, it is not difficult to obtain similarly (3.20) via (3.11), (4.7) and (4.8):

$$\left|\varepsilon^2 a_0\int\limits_{\Omega_0^{\varepsilon}}\nabla u^{\varepsilon}\nabla(g^2\widetilde{u}^{\varepsilon})\,dx\right| \le \varepsilon^2 C\sum_{\varepsilon Q_0\subset\Omega_0^{\varepsilon}}\|\nabla u^{\varepsilon}\|_{L^2(\varepsilon Q_0)}\left\|\nabla(g^2\widetilde{u}^{\varepsilon})\right\|_{L^2(\varepsilon Q_0)} \le$$

$$\le\varepsilon C\sum_{\varepsilon Q_0\subset\Omega_0^{\varepsilon}}\left(\|\varepsilon\nabla v^{\varepsilon}\|_{L^2(\varepsilon Q_0)} + \varepsilon\|\nabla\widetilde{u}^{\varepsilon}\|_{L^2(\varepsilon Q_0)}\right)\left\|\nabla(g^2\widetilde{u}^{\varepsilon})\right\|_{L^2(\varepsilon Q_0)} \le$$

$$\le\varepsilon C\sum_{\varepsilon Q_0\subset\Omega_0^{\varepsilon}}\left(\|\widetilde{u}^{\varepsilon}\|_{L^2(\varepsilon Q_0)}\left\|\nabla(g^2\widetilde{u}^{\varepsilon})\right\|_{L^2(\varepsilon Q_0)} + \varepsilon\|\nabla\widetilde{u}^{\varepsilon}\|_{L^2(\varepsilon Q_0)}\left\|\nabla(g^2\widetilde{u}^{\varepsilon})\right\|_{L^2(\varepsilon Q_0)}\right) \le$$

$$\le\varepsilon C\left(\|\nabla(g\widetilde{u}^{\varepsilon})\|^2_{L^2(\Omega_1^{\varepsilon})} + \|g\widetilde{u}^{\varepsilon}\|^2_{L^2(\Omega_0^{\varepsilon}\cup\Omega_1^{\varepsilon})} + r^{\varepsilon}\right) \le \varepsilon C\left(\|\nabla(g\widetilde{u}^{\varepsilon})\|^2_{L^2(\Omega_1^{\varepsilon})} + \|g\widetilde{u}^{\varepsilon}\|^2_{L^2(\Omega_0^{\varepsilon}\cup\Omega_1^{\varepsilon})} + C\right)$$

for small enough ε.

Notice that all the estimates obtained in this section are independent of R. $\qquad\square$

5 Some properties of two-scale convergence

In this section we list the definitions and some properties of the two-scale convergence, see [2, 20, 23, 24]. We also formulate several statements (analogous to those in [23]) which are necessary for obtaining the two-scale convergence of the eigenfunctions of A_{ε} and derivation of the limit equation.

Let Ω be an arbitrary region in $\mathbb{R}^n$, in particular $\Omega = \mathbb{R}^n$. Denote by $\square$ the unit cube $[0,1)^n$. We consider all functions of the form $u(x,y)$ to be 1-periodic in y in each coordinate.

Definition 5.1. *We say that a bounded in $L^2(\Omega)$ sequence v_ε is weakly two-scale convergent to a function $v \in L^2(\Omega \times \square)$, $v_\varepsilon(x) \overset{2}{\rightharpoonup} v(x,y)$, if*

$$\lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon(x)\varphi(x)b\left(\frac{x}{\varepsilon}\right) dx = \int_\Omega \int v(x,y)\varphi(x)b(y)\,dydx$$

for all $\varphi \in C_0^\infty(\Omega)$ and all $b \in C_{\mathrm{per}}^\infty(\square)$ (where $C_{\mathrm{per}}^\infty(\square)$ is the set of 1-periodic functions from $C^\infty(\mathbb{R}^n)$).

Definition 5.2. *We say that a bounded in $L^2(\Omega)$ sequence u_ε is strongly two-scale convergent to a function $u \in L^2(\Omega \times \square)$, $u_\varepsilon(x) \overset{2}{\to} u(x,y)$, if*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)v_\varepsilon(x)\,dx = \int_\Omega \int u(x,y)v(x,y)\,dy\,dx$$

for all $v_\varepsilon(x) \overset{2}{\rightharpoonup} v(x,y)$.

Proposition 5.3. *(Properties of the two-scale convergence.)*
(i) If $u_\varepsilon(x) \overset{2}{\rightharpoonup} u(x,y)$ and $a \in L_{\mathrm{per}}^\infty(\square)$ then

$$a(x/\varepsilon)u_\varepsilon(x) \overset{2}{\rightharpoonup} a(y)u(x,y).$$

(ii) $v_\varepsilon(x) \overset{2}{\to} v(x,y)$ if and only if $v_\varepsilon(x) \overset{2}{\rightharpoonup} v(x,y)$ and

$$\lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon^2\,dx = \int_\Omega \int v^2\,dy\,dx.$$

(iii) If $f_\varepsilon(x) \to f(x)$ in $L^2(\Omega)$, then $f_\varepsilon(x) \overset{2}{\to} f(x)$.

Proposition 5.4. *(The mean value property of periodic functions.) Let $\Phi(y) \in L_{\mathrm{per}}^1(\square)$. Then for each $\phi(x) \in C_0^\infty(\mathbb{R}^n)$ we have*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \phi(x)\Phi(x/\varepsilon)dx = \langle\Phi\rangle_y \int_{\mathbb{R}^n} \phi(x)dx.$$

Potential vector space V_{pot} is defined as a closure of the set $\{\nabla\varphi : \varphi \in C_{\mathrm{per}}^\infty(\square)\}$ in $L^2(\square)^n$. We say that a vector $b \in L^2(\square)^n$ is solenoidal ($b \in V_{\mathrm{sol}}$) if it is orthogonal to all potential vectors. Thus,

$$L^2(\square)^n = V_{\mathrm{pot}} \oplus V_{\mathrm{sol}},$$

and

$$L^2(\Omega \times \square)^n = L^2(\Omega, V_{\mathrm{pot}}) \oplus L^2(\Omega, V_{\mathrm{sol}}).$$

Lemma 5.5. *Let u_ε and $\varepsilon\nabla u_\varepsilon$ be bounded in $L^2(\mathbb{R}^n)$. Then (up to a subsequence)*

$$u_\varepsilon(x) \overset{2}{\rightharpoonup} u(x,y) \in L^2(\mathbb{R}^n, H_{\mathrm{per}}^1),$$
$$\varepsilon\nabla u_\varepsilon(x) \overset{2}{\rightharpoonup} \nabla_y u(x,y),$$

where $H_{\mathrm{per}}^1 = H_{\mathrm{per}}^1(\square)$ is the Sobolev space of periodic functions.

Lemma 5.6. *Let $u_\varepsilon \in H^1(\mathbb{R}^n)$,*

$$u_\varepsilon(x) \overset{2}{\rightharpoonup} u(x) \in H^1(\mathbb{R}^n), \tag{5.1}$$

and ∇u_ε is bounded in $L^2(\mathbb{R}^n)$. Then, up to a subsequence,

$$\nabla u_\varepsilon(x) \overset{2}{\rightharpoonup} \nabla u(x) + v(x,y), \;\; where \; v \in L^2(\mathbb{R}^n, V_{\mathrm{pot}}). \tag{5.2}$$

Lemma 5.7. *Let (5.1) and (5.2) be valid. Let also*

$$\lim_{\varepsilon \to 0} \int_{\Omega_1^\varepsilon} a_1 \nabla u_\varepsilon(x) \cdot \nabla_y w(\varepsilon^{-1}x)\varphi(x)\, dx = 0 \tag{5.3}$$

for any $\varphi \in C_0^\infty(\Omega_1)$ and $w \in C_{\mathrm{per}}^\infty(\square)$. Then the following weak convergence of the flows takes place:

$$a_1 \theta_{Q_1}(\varepsilon^{-1}x)\nabla u_\varepsilon(x) \rightharpoonup A^{\mathrm{hom}}\nabla u(x) \; in \; \Omega_1,$$

where homogenised matrix A^{hom} is defined by (2.10).

The proofs of the listed statements repeat the proofs of the corresponding assertions in [23] with no or only small alterations, and are not given here.

Definition 5.8. *Let A_ε, $\varepsilon > 0$, and A_0 be non-negative self-adjoint operators in $L^2(\mathbb{R}^n)$ and $\mathcal{H}_0 \subset L^2(\mathbb{R}^n \times Q)$, see (2.7), respectively. We say that $A_\varepsilon \overset{2}{\rightarrow} A_0$ in the sense of the strong two-scale resolvent convergence if $(A_\varepsilon + I)^{-1} f_\varepsilon \overset{2}{\rightarrow} (A_0 + I)^{-1} f_0$ as long as $f_\varepsilon \overset{2}{\rightarrow} f_0$.*

6 Strong two-scale convergence of the eigenfunctions and multiplicity of the eigenvalues of A_ε

In this section we will show that the normalised eigenfunctions u_ε are compact in the sense of strong two-scale convergence. Namely, provided $\lambda_\varepsilon \to \lambda_0$, a sequence of normalised eigenfunctions u_ε of the operator A_ε strongly two-scale converges, up to a subsequence, to a function $u^0(x,y)$. This implies that $u^0(x,y)$ is an eigenfunction corresponding to the eigenvalue λ_0 of the limit operator A_0. This, together with results of [16], establishes an 'asymptotic one-to-one correspondence' between isolated eigenvalues and corresponding eigenfunctions of the operators A_ε and A_0.

Theorem 6.1. *Under the assumptions of Theorem 3.1 λ_0 is an eigenvalue of the operator A_0. Moreover, there exists a subsequence ε such that eigenfunctions u^ε of the operator A_ε strongly two-scale converge to an eigenfunction $u^0(x,y)$ of A_0 corresponding to the eigenvalue λ_0.*

Proof. **Step 1.** In order to establish strong two-scale convergence of the eigenfunctions $u^\varepsilon = \widetilde{u}^\varepsilon + v^\varepsilon$ we prove it for each of its components separately. From (3.22) and (3.23) it follows that

$$\|\widetilde{u}^\varepsilon\|_{H^1(\mathbb{R}^n \setminus B_R)} \le C e^{-\alpha R} \tag{6.1}$$

with C independent of ε and R. From this one can easily conclude that $\widetilde{u}^\varepsilon$ is weakly compact in $H^1(\mathbb{R}^n)$ and strongly compact in $L^2(\mathbb{R}^n)$. Indeed, since $\widetilde{u}^\varepsilon$ are bounded in $H^1(\mathbb{R}^n)$ uniformly in ε,

$$\widetilde{u}^\varepsilon \rightharpoonup u_0 \; in \; H^1(\mathbb{R}^n), \tag{6.2}$$

up to a subsequence. For any fixed R function $\widetilde{u}^\varepsilon$ converges to u_0 weakly in $H^1(B_R)$ and, hence, strongly in $L^2(B_R)$ up to a subsequence. Considering a sequence of balls B_R, $R \in \mathbb{N}$, one can use the method of extracting a diagonal subsequence such that

$$\widetilde{u}^\varepsilon \to u_0 \; in \; L^2(B_R) \tag{6.3}$$

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for any $R > 0$.

For any $\delta > 0$ we can choose R such that $\|u_0\|_{L^2(\mathbb{R}^n \setminus B_R)} < \delta/3$ and $\|\widetilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n \setminus B_R)} < \delta/3$ for sufficiently small ε (the latter follows from (6.1)). From (6.3) it follows that $\|u_0 - \widetilde{u}^\varepsilon\|_{L^2(B_R)} < \delta/3$ for sufficiently small ε. Then, up to a subsequence,

$$\|u_0 - \widetilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n)} \leq \|u_0 - \widetilde{u}^\varepsilon\|_{L^2(B_R)} + \|u_0\|_{L^2(\mathbb{R}^n \setminus B_R)} + \|\widetilde{u}^\varepsilon\|_{L^2(\mathbb{R}^n \setminus B_R)} < \delta$$

for small enough ε. Hence, up to a subsequence, we have

$$\widetilde{u}^\varepsilon \to u_0 \text{ in } L^2(\mathbb{R}^n).$$

Then from properties of the two-scale convergence we conclude that

$$\widetilde{u}^\varepsilon \xrightarrow{2} u_0. \tag{6.4}$$

Step 2. Now let us consider v^ε. We denote by v_1^ε and v_2^ε its restrictions $v_\varepsilon|_{\Omega_0^\varepsilon}$ and $v_\varepsilon|_{\widetilde{\Omega}_0^\varepsilon}$ respectively, extended by zero to the rest of $\mathbb{R}^n$.

Lemma 6.2. *The following convergence properties are valid for v_1^ε (up to a subsequence):*

$$v_1^\varepsilon(x) \xrightarrow{2} v(x,y) \in L^2(\Omega_1, H_0^1(Q_0)),$$

$$\varepsilon \nabla v_1^\varepsilon(x) \xrightarrow{2} \nabla_y v(x,y),$$

where $v(x,y)$ is a solution to the following problem:

$$- a_0 \Delta_y v - \lambda_0 v = \lambda_0 u_0, \quad y \in Q_0. \tag{6.5}$$

Here u_0 is a function from (6.4).

Proof. Function $v_1^\varepsilon \in H^1(\Omega_0^\varepsilon)$ satisfies the following differential equation:

$$- \varepsilon^2 a_0 \Delta v_1^\varepsilon - \lambda_\varepsilon v_1^\varepsilon = \lambda_\varepsilon \widetilde{u}^\varepsilon \text{ in } \Omega_0^\varepsilon. \tag{6.6}$$

The right hand side of this equation is of the form $\lambda_\varepsilon \theta_{\Omega_0^\varepsilon} \widetilde{u}^\varepsilon$. By (6.4) and the properties of the two-scale convergence we have

$$\lambda_\varepsilon \theta_{\Omega_0^\varepsilon}(x) \widetilde{u}^\varepsilon(x) \xrightarrow{2} \lambda_0 \theta_{Q_0}(y) \theta_{\Omega_1}(x) u_0(x). \tag{6.7}$$

Following [23] we consider more general problem

$$z_\varepsilon \in H^1(\Omega_0^\varepsilon), \quad -\varepsilon^2 a_0 \Delta z_\varepsilon - \lambda_\varepsilon z_\varepsilon = f_\varepsilon, \quad f_\varepsilon \in L^2(\Omega_0^\varepsilon). \tag{6.8}$$

(It is implicit that $f_\varepsilon = z_\varepsilon = 0$ in $\mathbb{R}^n \setminus \Omega_0^\varepsilon$.)

Proposition 6.3. *Let*

$$f^\varepsilon(x) \xrightarrow{2} f(x,y). \tag{6.9}$$

Then

$$z^\varepsilon(x) \xrightarrow{2} z(x,y) \in L^2(\Omega_1, H_0^1(Q_0)),$$

$$\varepsilon \nabla z^\varepsilon(x) \xrightarrow{2} \nabla_y z(x,y),$$

where function $z(x,y)$ solves the following equation:

$$- a_0 \Delta_y z - \lambda_0 z = f, \quad y \in Q_0. \tag{6.10}$$

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Proof. One can easily derive an estimate for z^ε analogous to (3.11), applying to (6.8) a reasoning similar to those for the solution of equation (3.6). This gives us the weak two-scale convergence of z^ε and $\varepsilon \nabla z^\varepsilon$ via Lemma 5.5. The result follows by a straightforward passing to the limit in the integral identity corresponding to (6.8) with appropriately chosen test function. The full proof could be found in [23] and applies to the present situation with no alteration. $\square$

The above proposition together with (6.7) establishes a "weak" form of the statement of the lemma, i.e. weak two-scale convergence of v_1^ε. We now prove that the convergence is actually strong, following again [23]. Multiply (6.6) and (6.8) by z^ε and v_1^ε respectively and integrate by parts. The left hand sides of the resulting equalities are identical. So, equating the right hand sides, we obtain the following identity

$$\int\limits_{\Omega_1} f^\varepsilon v^\varepsilon \, dx = \lambda_\varepsilon \int\limits_{\Omega_1} \widetilde{u}^\varepsilon z^\varepsilon \, dx.$$

By the definition of the strong two-scale convergence we have

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon \int\limits_{\Omega_1} \widetilde{u}^\varepsilon z^\varepsilon \, dx = \lambda_0 \int\limits_{\Omega_1} \int\limits_{Q_0} u_0(x) z(x,y) \, dy \, dx.$$

Multiplying (6.5) and (6.10) by z and v respectively and integrating by parts it is easy to see that

$$\lambda_0 \int\limits_{\Omega_1} \int\limits_{Q_0} u_0(x) z(x,y) \, dy dx = \int\limits_{\Omega_1} \int\limits_{Q_0} f(x,y) v(x,y) \, dy \, dx.$$

Thus, we have a convergence of the integrals:

$$\lim_{\varepsilon \to 0} \int\limits_{\Omega_1} f^\varepsilon v_1^\varepsilon \, dx = \int\limits_{\Omega_1} \int\limits_{Q_0} f(x,y) v(x,y) \, dy \, dx$$

for any weakly two-scale convergent sequence f^ε. Hence, by the definition,

$$v_1^\varepsilon(x) \xrightarrow{2} v(x,y).$$

$\square$

Lemma 6.4. *Sequence of functions v_2^ε converges to zero in the sense of strong two-scale convergence:*

$$v_2^\varepsilon \xrightarrow{2} 0 \; as \; \varepsilon \to 0.$$

Proof. Straightforward from (3.18) and Proposition 5.3 *(iii)*. $\square$

Combining (6.4) with Lemmas 6.2 and 6.4, we arrive at

$$u^\varepsilon(x) \xrightarrow{2} u^0(x,y) = u_0(x) + v(x,y), \tag{6.11}$$

where $u_0 \in H^1(\mathbb{R}^n)$, $v \in L^2(\Omega_1, H_0^1(Q_0))$.

Step 3. Now it remains to show that $u^0(x,y)$ is an eigenfunction and λ_0 is the corresponding eigenvalue of the limit operator A_0, i.e. that $u^0(x,y)$ satisfies (2.11). In order to do that we need to choose appropriate test-function ψ^ε and pass to the limit in the integral identity

$$\varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx + a_1 \int\limits_{\Omega_1^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx + \int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx +$$

$$+ a_2 \int\limits_{\Omega_2^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx = \lambda_\varepsilon \int\limits_{\mathbb{R}^n} u^\varepsilon \psi^\varepsilon \, dx \tag{6.12}$$

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corresponding to the original eigenvalue problem (2.2)–(2.3). Let us take

$$
\begin{aligned}
&\psi^\varepsilon(x) = \psi_0(x) + \varphi(x)b(\varepsilon^{-1}x), \\
&\psi_0 \in C_0^\infty(\mathbb{R}^n),\ \varphi \in C_0^\infty(\Omega_1),\ b(y) \in C_0^\infty(Q_0),
\end{aligned}
\tag{6.13}
$$

and consider each term of (6.12) separately. Let us expand the first term:

$$
\varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla u^\varepsilon \nabla \psi^\varepsilon\, dx = \varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla \widetilde{u}^\varepsilon \nabla \psi^\varepsilon\, dx +
$$

$$
+\varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla v^\varepsilon \left(\nabla \psi_0 + b(\varepsilon^{-1}x)\nabla\varphi \right) dx + a_0 \int\limits_{\Omega_0^\varepsilon} \varepsilon \nabla v^\varepsilon \varphi \nabla_y b(\varepsilon^{-1}x)\, dx.
$$

As $\nabla \widetilde{u}^\varepsilon$ is bounded in L^2-norm and $|\nabla \psi^\varepsilon| \le C\varepsilon^{-1}$ then the first term on the right hand side tends to zero. From (3.11) and the boundedness of $\nabla \psi_0 + b\nabla\varphi$ we conclude that the second term also converges to zero. Since by Lemma 6.2 $\varepsilon \nabla v^\varepsilon$ converges two-scale weakly, from the definition of the weak two-scale convergence we obtain

$$
\lim_{\varepsilon\to 0} \varepsilon^2 a_0 \int\limits_{\Omega_0^\varepsilon} \nabla u^\varepsilon \nabla \psi^\varepsilon\, dx = a_0 \int\limits_{\Omega_1} \int\limits_{Q_0} \nabla_y v(x,y)\varphi(x)\nabla_y b(y)\, dy\, dx.
\tag{6.14}
$$

Let us show that convergence property (5.3) holds for u^ε. To this end we substitute into (6.12) a test function of the form $\varepsilon\, w(\varepsilon^{-1}x)\varphi(x)$, $\varphi \in C_0^\infty(\Omega_1)$, $w \in C_{\mathrm{per}}^\infty(\square)$, cf. [23]. Then all the terms except, possibly,

$$
\int\limits_{\Omega_1^\varepsilon} a_1 \nabla u^\varepsilon(x) \cdot \nabla_y w(\varepsilon^{-1}x)\varphi(x)\, dx
$$

converge to zero. As a result, the above term also converges to zero. We then apply Lemma 5.6 for u_ε replaced by $\widetilde{u}^\varepsilon$. Since $\widetilde{u}^\varepsilon$ coincides with u^ε on Ω_1^ε, by Lemma 5.7 applied to the second term on the left hand side of (6.12) with ψ^ε as in (6.13) we obtain

$$
\lim_{\varepsilon\to 0} a_1 \int\limits_{\Omega_1^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon\, dx = \lim_{\varepsilon\to 0} a_1 \int\limits_{\Omega_1^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi_0\, dx = \int\limits_{\Omega_1} A^{\mathrm{hom}} \nabla u_0 \cdot \nabla \psi_0\, dx.
\tag{6.15}
$$

For small enough ε the function ψ^ε is equal to ψ_0 in $\widetilde{\Omega}_0^\varepsilon$, so $\nabla \psi^\varepsilon$ is bounded in $\widetilde{\Omega}_0^\varepsilon$. Since $\int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 |\nabla u^\varepsilon|^2\, dx$ is bounded uniformly in ε and $\left|\widetilde{\Omega}_0^\varepsilon\right| \to 0$ as $\varepsilon \to 0$, we have

$$
\left| \int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 \nabla u^\varepsilon \nabla \psi^\varepsilon\, dx \right| \le C \int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 |\nabla u^\varepsilon|\, dx \le C \left|\widetilde{\Omega}_0^\varepsilon\right|^{1/2} \widetilde{a}_0^{1/2} \left(\int\limits_{\widetilde{\Omega}_0^\varepsilon} \widetilde{a}_0 |\nabla u^\varepsilon|^2\, dx \right)^{1/2} \to 0.
\tag{6.16}
$$

The function u^ε coincides with $\widetilde{u}^\varepsilon$ on Ω_2^ε. Then, via (6.2) we have convergence of the last term on the left hand side of (6.12):

$$
\lim_{\varepsilon\to 0} a_2 \int\limits_{\Omega_2^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon\, dx = \lim_{\varepsilon\to 0} \left[a_2 \int\limits_{\Omega_2} \nabla \widetilde{u}^\varepsilon \cdot \nabla \psi_0\, dx - a_2 \int\limits_{\widetilde{\Omega}_0^\varepsilon \cap \Omega_2} \nabla \widetilde{u}^\varepsilon \cdot \nabla \psi_0\, dx \right] = a_2 \int\limits_{\Omega_2} \nabla u_0 \cdot \nabla \psi_0\, dx.
$$

$$
\tag{6.17}
$$

Thus, passing to the limit as $\varepsilon \to 0$ on the left hand side of (6.12) via (6.14)–(6.17), and on the right hand side via (6.11), we arrive at

$$a_0 \int\limits_{\Omega_1} \int\limits_{Q_0} \nabla_y v \cdot \varphi \nabla_y b \, dy \, dx + \int\limits_{\Omega_1} A^{\mathrm{hom}} \nabla u_0 \cdot \nabla \psi_0 \, dx + a_2 \int\limits_{\Omega_2} \nabla u_0 \cdot \nabla \psi_0 \, dx = \lambda_0 \int\limits_{\mathbb{R}^n} \int\limits_{Q} (u_0 + v)(\psi_0 + \varphi \, b) \, dy \, dx.$$

Since the space of functions from (6.13) is dense in $\mathcal{V}$ (see (2.8)), the latter is equivalent to (2.11). It follows from (6.11), Proposition 5.3 (ii) and the normalization of u^ε that $u^0(x, y) \not\equiv 0$. Thus we have proved that λ_0 and $u^0(x, y)$ are respectively an eigenvalue and an eigenfunction of the operator A_0, completing the proof of the theorem. $\qquad\square$

Remark 6.5. Theorem 6.1 combined with [13, Theorem 2] implies the existence of eigenvalues of A_0 in the gaps of its essential spectrum, provided Ω_2 is large enough and/or a_2 is small enough.

Remark 6.6. It is not hard to show that there holds the strong two-scale resolvent convergence $A_\varepsilon \xrightarrow{2} A_0$, see Definition 5.8. Namely, considering the resolvent equation

$$A_\varepsilon w^\varepsilon + w^\varepsilon = f^\varepsilon,$$

where $f^\varepsilon \xrightarrow{2} f^0$, and employing essentially the same arguments as above (cf. also [23, Theorem 5.1]), one can pass to the limit as $\varepsilon \to 0$ in the weak form of the resolvent equation choosing appropriate test functions, cf. (6.12)–(6.17), to obtain that $w^\varepsilon \xrightarrow{2} w^0$, with

$$A_0 w^0 + w^0 = f^0.$$

Further, arguing as in [23, §4.3], cf. also proof of Lemma 6.2 above, one can show that the above weak two-scale convergence implies the strong one, i.e. $w^\varepsilon \xrightarrow{2} w^0$ as long as $f^\varepsilon \xrightarrow{2} f^0$, which means the strong two-scale resolvent convergence by the definition. The latter implies in particular the strong two-scale convergence of spectral projectors $(P_\varepsilon(\lambda) \xrightarrow{2} P_0(\lambda)$ if λ is not an eigenvalue of A_0), see [21, 24], and has other nice properties, however it does not imply in its own the convergence of the spectra. The latter requires an additional (two-scale) compactness property to hold, which Theorem 6.1 provides.

Remark 6.7. The function $v(x, y)$ could be represented as a product of $u_0(x)\big|_{\Omega_1}$ and $\lambda_0 b(y)$, where $b(y)$ solves (2.13) with $\lambda = \lambda_0$. Then $v(x, \varepsilon^{-1}x)$ strongly two-scale converges to $v(x, y)$ by the mean value property and the properties of two-scale convergence. Then

$$u^{\mathrm{appr}}(x, \varepsilon) := \begin{cases} u_0(x) + v(x, x/\varepsilon), & x \in \Omega_0^\varepsilon, \\ u_0(x), & x \in \mathbb{R}^n \backslash \Omega_0^\varepsilon, \end{cases} \tag{6.18}$$

also strongly two-scale converges to $u^0(x, y)$. Hence it approximates the eigenfunction $u^\varepsilon(x)$:

$$\|u^{\mathrm{appr}} - u^\varepsilon\|_{L_2(\mathbb{R}^n)}^2 \to 0. \tag{6.19}$$

Now, using the result of Theorem 6.1 we will discuss the multiplicity properties of the eigenvalues λ_ε and λ_0. Let us assume that the multiplicity of the eigenvalue λ_0 of A_0 is m. Suppose that for a subsequence $\varepsilon_k \to 0$ there exist l (accounting for multiplicities) eigenvalues of A_ε, $\lambda_{\varepsilon_k, 1} \leq \lambda_{\varepsilon_k, 2}, \ldots \leq \lambda_{\varepsilon_k, l}$, such that $\lambda_{\varepsilon_k, i} \to \lambda_0$, $i = 1, \ldots, l$. Let $u_i^{\varepsilon_k}$ be the corresponding eigenfunctions orthonormalised in $L^2(\mathbb{R}^n)$. It follows from Theorem 6.1 that there exists a subsequence k_m such that

$$u_i^{\varepsilon_{k_m}} \xrightarrow{2} u_i^0, \ i = 1, \ldots, l,$$

where u_i^0 are eigenfunctions of A_0 corresponding to λ_0. In particular, due to the strong two-scale convergence, we have convergence of the inner products as a consequence of the convergence of norms:

$$(u_i^{\varepsilon_{k_m}}, u_j^{\varepsilon_{k_m}})_{L^2(\mathbb{R}^n)} \to (u_i^0, u_j^0)_{\mathcal{H}_0}.$$

However $(u_i^{\varepsilon_{k_m}}, u_j^{\varepsilon_{k_m}})_{L^2(\mathbb{R}^n)} = \delta_{ij}$. Then u_i^0, $i = 1, \ldots, l$ are also orthonormal (in $\mathcal{H}_0$), i.e. there exist at least l linearly independent eigenfunctions of A_0 corresponding to λ_0. Thus, $l \le m$.

The results presented in [16] remain also valid for the setting of the problem in the present paper, i.e. when the coefficients of the divergence form operator A_ε are of the form (2.4). By Theorem 4.1 of [16], if λ_0 is an eigenvalue of the limit operator A_0 lying in a gap of its essential spectrum, then for small enough ε, there exist eigenvalues (or at least one eigenvalue) of A_ε such that

$$|\lambda_{\varepsilon,i} - \lambda_0| \le C\varepsilon^{1/2}, \ i = 1, \ldots, l(\varepsilon).$$

Moreover, again by [16, Thm 4.1], for any eigenfunction u_i^0 of A_0 corresponding to λ_0 the related u_i^{appr}, see (6.18), can be approximated by a linear combination of the eigenfunctions of A_ε corresponding to $\lambda_{\varepsilon,i}$, $i = 1, \ldots, l(\varepsilon)$. Since, by the above, $(u_i^{\mathrm{appr}}, u_j^{\mathrm{appr}})_{L^2(\mathbb{R}^n)} \to \delta_{ij}$, as $\varepsilon \to 0$, $i, j = 1, \ldots, m$, it is not hard to show that $l(\varepsilon) \ge m$. Hence we conclude that there exist exactly m eigenvalues (counted with their multiplicities) of A_ε such that

$$|\lambda_{\varepsilon,i} - \lambda_0| \le C\varepsilon^{1/2}, \ i = 1, \ldots, m,$$

where m is a multiplicity of λ_0. In other words there is an "asymptotic one-to-one correspondence" between isolated eigenvalues and eigenfunctions of the operators A_ε and A_0.

7 Identity of the essential spectra of $\widehat{A}_0$ and A_0, convergence of the spectra of A_ε in the sense of Hausdorff

By definition, the Hausdorff convergence of spectra, $\sigma(A_\varepsilon) \overset{H}{\to} \sigma(A_0)$ as $\varepsilon \to 0$, means that

- for all $\lambda \in \sigma(A_0)$ there are $\lambda_\varepsilon \in \sigma(A_\varepsilon)$ such that $\lambda_\varepsilon \to \lambda$;

- if $\lambda_\varepsilon \in \sigma(A_\varepsilon)$ and $\lambda_\varepsilon \to \lambda$, then $\lambda \in \sigma(A_0)$.

We remind that $\widehat{A}_\varepsilon$ and $\widehat{A}_0$ denote the 'unperturbed' operators corresponding to A_ε and A_0, see Section 2. It was shown in [24] that $\sigma(\widehat{A}_\varepsilon) \overset{H}{\to} \sigma(\widehat{A}_0)$ (the spectra of both $\widehat{A}_\varepsilon$ and $\widehat{A}_0$ are purely essential). In [13] it is proved that the essential spectrum of a divergence form operator $-\nabla \cdot a(x)\nabla$ (where $a(x) \ge \delta > 0$ is a scalar function) remains unperturbed with respect to the local perturbation of the coefficient $a(x)$. Applying this assertion to the operator $\widehat{A}_\varepsilon$ and its perturbation A_ε we conclude that $\sigma(\widehat{A}_\varepsilon) = \sigma_{\mathrm{ess}}(A_\varepsilon) \overset{H}{\to} \sigma(\widehat{A}_0)$. Let us assume that $\sigma(\widehat{A}_0) = \sigma_{\mathrm{ess}}(A_0)$. Then $\sigma_{\mathrm{ess}}(A_\varepsilon) \overset{H}{\to} \sigma_{\mathrm{ess}}(A_0)$. In this case Theorem 6.1 together with the results of [16] imply the convergence of the discrete spectra in the gaps $(\sigma_{\mathrm{disc}}(A_\varepsilon) \overset{H}{\to} \sigma_{\mathrm{disc}}(A_0))$ and, consequently, we would have $\sigma(A_\varepsilon) \overset{H}{\to} \sigma(A_0)$. However, we cannot apply the result of [13] as it is stated to the case of the two-scale operators $\widehat{A}_0$ and A_0. In this section we prove the stability of the essential spectrum of $\widehat{A}_0$ with respect to the local perturbation of its coefficients, establishing thereby the missing part of the reasoning. We do this by direct means using the Weyl's criterium for the essential spectrum of an operator, see e.g. [7].

Theorem 7.1. *The essential spectra of the operators $\widehat{A}_0$ and A_0 coincide.*

Proof. **Step 1.** First we describe the domains of $\widehat{A}_0$ and A_0. According to the Friedrichs extension procedure, see e.g. [21], a function u belongs to $\mathcal{D}(A_0)$ if and only if $u = u_0(x) + v(x, y) \in \mathcal{V}$ and there exists $h \in \mathcal{H}_0$ such that

$$B_0(u, w) = (h, w)_{\mathcal{H}_0}$$

for all $w \in \mathcal{V}$, see (2.7)–(2.9). If $u = u_0 + v \in \mathcal{D}(A_0)$ then $u_0, v \in \mathcal{D}(A_0)$. Due to the regularity properties of solutions of elliptic equations, $u_0 \in H^2_{\text{loc}}$ everywhere away from the boundary of Ω_2.

Operator $\widehat{A}_0$ acting in the Hilbert space $\widehat{\mathcal{H}}_0$ was described in [24] and is generated by a (closed) symmetric and bounded from below bilinear form $\widehat{B}_0(u, w)$ on a dense subspace $\widehat{\mathcal{V}}$ of $\widehat{\mathcal{H}}_0$, where $\widehat{\mathcal{H}}_0$, $\widehat{\mathcal{V}}$ and $\widehat{B}_0(u, w)$ are defined by (2.7)–(2.9) with $\Omega_2 = \emptyset$ and $\Omega_1 = \mathbb{R}^n$. A function u belongs to domain $\mathcal{D}(\widehat{A}_0)$ if and only if $u = u_0(x) + v(x, y) \in \widehat{\mathcal{V}}$ and there exists $h \in \widehat{\mathcal{H}}_0$ such that

$$\widehat{B}_0(u, w) = (h, w)_{\widehat{\mathcal{H}}_0}$$

for all $w \in \widehat{\mathcal{V}}$. If $u = u_0 + v \in \mathcal{D}(\widehat{A}_0)$ then $u_0, v \in \mathcal{D}(\widehat{A}_0)$, $u_0 \in H^2(\mathbb{R}^n)$.

Let A be a self-adjoint operator with domain $\mathcal{D}(A)$ acting in a Hilbert space H. By the Weyl's criterium, see e.g. [7], condition $\lambda \in \sigma_{\text{ess}}(A)$ is equivalent to the existence of a singular sequence $u^{(k)} \in \mathcal{D}(A)$, i.e. such that

$$0 < C_1 \leq \|u^{(k)}\|_H \leq C_2, \tag{7.1}$$

$$u^{(k)} \rightharpoonup 0 \text{ weakly in } H, \tag{7.2}$$

$$(A - \lambda)u^{(k)} \to 0 \text{ strongly in } H. \tag{7.3}$$

Step 2. Let $\lambda \in \sigma_{\text{ess}}(\widehat{A}_0)$ and $u^{(k)} = u_0^{(k)}(x) + v^{(k)}(x, y)$ be the corresponding singular sequence in $\mathcal{D}(\widehat{A}_0) \subset \widehat{\mathcal{H}}_0$. We want to construct on its basis a singular sequence for the operator A_0, i.e. in $\mathcal{D}(A_0) \subset \mathcal{H}_0$ and satisfying properties (7.1)–(7.3). First notice that the gradient of $u_0^{(k)}$ is bounded in $L^2(\mathbb{R}^n)$. Indeed, from (2.9) and (7.3) we have

$$\|\nabla u_0^{(k)}\|^2_{L^2(\mathbb{R}^n)} \leq C\widehat{B}_0(u^{(k)}, u^{(k)}) = C\lambda(u^{(k)}, u^{(k)})_{\widehat{\mathcal{H}}_0} + o(1) \leq C. \tag{7.4}$$

Let us define a cut-off function

$$\eta_{k,R}(x) = \eta\left(\frac{1}{k}(|x| - R)\right),$$

where $\eta \in C^2(\mathbb{R})$ is such that

$$\eta(t) = \begin{cases} 1, & t \leq 0, \\ 0, & t \geq 1. \end{cases}$$

Consider the following sequence, $u^{(k)}\eta_{k,R_k} \in \mathcal{D}(\widehat{A}_0)$, where R_k is chosen large enough so that $\|u^{(k)}(1 - \eta_{k,R_k})\|_{\widehat{\mathcal{H}}_0} \leq \frac{1}{k}$. This sequence obviously satisfies (7.1) regarding the operator $\widehat{A}_0$.

Let us check property (7.3). The operator $\widehat{A}_0$ acts on a function $u \in H^2(\mathbb{R}^n) \subset \widehat{\mathcal{H}}_0$ as follows[7], cf. [24]. Let

$$-\nabla \cdot A^{\text{hom}}\nabla u(x) = f(x) \in L^2(\mathbb{R}^n).$$

Then, by the definition of $\widehat{A}_0$, we have

$$\widehat{A}_0 u(x) = |Q_1|^{-1}\theta_{Q_1}(y)f(x) \in \widehat{\mathcal{H}}_0.$$

Note that

$$\|\widehat{A}_0 u\|_{\widehat{\mathcal{H}}_0} = |Q_1|^{-1/2}\|f\|_{L^2(\mathbb{R}^n)}.$$

---

[7]If $u = u_0(x) + v(x, y)$ then $\widehat{A}_0 u = h \in \widehat{\mathcal{H}}_0$ implies $-\nabla \cdot A^{\text{hom}}\nabla u_0 = \langle h \rangle_y$ and $-a_0\Delta_y v = h(x, y)$, $y \in Q_0$.

For $u^{(k)}\eta_{k,R_k}$ we derive

$$\widehat{A}_0\left(u^{(k)}\eta_{k,R_k}\right) = \eta_{k,R_k}\widehat{A}_0 u^{(k)} - |Q_1|^{-1}\theta_{Q_1}(y)\left(2\nabla\eta_{k,R_k}\cdot A^{\text{hom}}\nabla u_0^{(k)} + u_0^{(k)}\nabla\cdot A^{\text{hom}}\nabla\eta_{k,R_k}\right).$$

Thus we arrive at

$$\left\|(\widehat{A}_0 - \lambda)(u^{(k)}\eta_{k,R_k})\right\|_{\widehat{\mathcal{H}}_0} \le \left\|\eta_{k,R_k}(\widehat{A}_0 - \lambda)u^{(k)}\right\|_{\widehat{\mathcal{H}}_0} + |Q_1|^{-1/2}\left(2\left\|\nabla\eta_{k,R_k}\cdot A^{\text{hom}}\nabla u_0^{(k)}\right\|_{L^2(\mathbb{R}^n)} + \right.$$

$$\left. + \left\|u_0^{(k)}\nabla\cdot A^{\text{hom}}\nabla\eta_{k,R_k}\right\|_{L^2(\mathbb{R}^n)}\right) = o(1) + \frac{1}{k}O\left(\left\|\nabla u_0^{(k)}\right\|_{L^2(\mathbb{R}^n)}\right) + \frac{1}{k^2}O\left(\left\|u_0^{(k)}\right\|_{L^2(\mathbb{R}^n)}\right). \tag{7.5}$$

Due to (7.1) and (7.4) the latter converges to 0 as $k \to \infty$. Hence (7.3) holds regarding $\widehat{A}_0$.

Now notice that if $\operatorname{supp} u \cap \overline{\Omega}_2 = \emptyset$, then $u \in \mathcal{D}(\widehat{A}_0)$ if and only if $u \in \mathcal{D}(A_0)$; besides $\widehat{A}_0 u = A_0 u$. We hence next shift the supports of the elements of the sequence away from Ω_2 ensuring also that the new sequence is weakly convergent to maintain (7.2). Since $\operatorname{supp}\eta_{k,R_k}$ is a closed ball of radius $R_k + k$ centered at the origin, the shift of x by $\xi_k := (R_k + 2k + \operatorname{diam}(\Omega_2))\xi$ for every k, where ξ is an arbitrary unit vector from $\mathbb{R}^n$, will do the job. Hence, for the given λ we have constructed a singular sequence

$$w^{(k)}(x,y) = u^{(k)}(x + \xi_k, y)\,\eta_{k,R_k}(x + \xi_k),$$

satisfying all the properties (7.1)–(7.3) for the operator A_0. Namely, the translational invariance of $\widehat{A}_0$ in x ensures that (7.1) and (7.3) are satisfied. Finally, (7.2) follows from the pointwise convergence of $w^{(k)}$ to zero as $k \to \infty$ (since for any fixed x, $w^{(k)}(x,y) = 0$ for large enough k). Thus $\lambda \in \sigma_{\text{ess}}(A_0)$.

Step 3. Suppose now that $\lambda \in \sigma_{\text{ess}}(A_0)$ and $u^{(k)} = u_0^{(k)}(x) + v^{(k)}(x,y)$ is the corresponding singular sequence. Let R be such that $\overline{\Omega}_2 \subset B_R$. There are only two alternative possibilities[8]:

- There exists a sequence $\delta_i \to 0$ such that for any $i \in \mathbb{N}$

$$\|u^{(k)}(1 - \theta_{B_{R+i}})\|_{\mathcal{H}_0} \le \delta_i \tag{7.6}$$

  for all k.

- There exist a constant $M > 0$ and subsequences $k(j) \to \infty$, $i(j) \to \infty$ as $j \to \infty$ such that

$$\|u^{(k(j))}(1 - \theta_{B_{R+i(j)}})\|_{\mathcal{H}_0} \ge M \tag{7.7}$$

  for all j.

Let (7.6) take place. The sequence $\nabla u_0^{(k)}$ is bounded in $L^2(\mathbb{R}^n)$, cf. (7.4). From (7.6) and

$$\|f\|_{L^2(\mathbb{R}^n)} = \|f\|_{\mathcal{H}_0}, \text{ for all } f \in L^2(\mathbb{R}^n) \subset \mathcal{H}_0, \tag{7.8}$$

it follows that

$$u_0^{(k)} \to u(x) \text{ in } L^2(\mathbb{R}^n),$$

up to a subsequence. The reasoning leading to this assertion is essentially identical to the one in (6.1)–(6.4) and is not reproduced. From (7.2) and the latter we conclude that

$$v^{(k)}(x,y) \rightharpoonup -u(x) \text{ weakly in } \mathcal{H}_0.$$

---

[8]Let $A_{ki} := \|u^{(k)}(1 - \theta_{B_{R+i}})\|_{\mathcal{H}_0}$ and let $\delta_i := \sup_k A_{ki}$. Then either $\delta_i \to 0$ giving (7.6) or $\delta_i \nrightarrow 0$ yielding (7.7).

Hence, on one hand, we have

$$\left(u, v^{(k)}\right)_{\mathcal{H}_0} \to -(u, u)_{\mathcal{H}_0} = -\int_{\mathbb{R}^n} u^2 \, dx.$$

On the other hand,

$$\left(u, v^{(k)}\right)_{\mathcal{H}_0} = \int_{\mathbb{R}^n} \int_{Q_0} u \, v^{(k)} \, dy \, dx = \left(u \, \theta_{Q_0}(y), v^{(k)}\right)_{\mathcal{H}_0} \to -(u \, \theta_{Q_0}(y), u)_{\mathcal{H}_0} = -|Q_0| \int_{\mathbb{R}^n} u^2 \, dx.$$

Comparing the last two formulas, conclude that at $u \equiv 0$, i.e.

$$u_0^{(k)} \to 0 \text{ in } L^2(\mathbb{R}^n). \tag{7.9}$$

Denote $A_0 u^{(k)}$ by $g^{(k)}(x, y) = g_0^{(k)}(x) + h^{(k)}(x, y) \in \mathcal{H}_0$. From (7.3) and (7.8) we get the following convergence:

$$\|g_0^{(k)} - \lambda u_0^{(k)}\|_{L^2(\mathbb{R}^n)} \to 0,$$
$$\|h^{(k)} - \lambda v^{(k)}\|_{\mathcal{H}_0} \to 0. \tag{7.10}$$

Then from (7.9) we have

$$g_0^{(k)} \to 0 \text{ in } L^2(\mathbb{R}^n). \tag{7.11}$$

Analogously to [23] we define a self-adjoint operator A_y acting in $L^2(\Omega_1 \times Q_0)$ by

$$A_y v = -a_0 \Delta_y v = p, \quad p \in L^2(\Omega_1 \times Q_0).$$

The domain of the operator, $\mathcal{D}(A_y) \subset L^2(\Omega_1, H_0^1(Q_0))$, is the set of all the solution of this equation. Similarly we define a self-adjoint operator $\widehat{A}_y$ acting in $L^2(\mathbb{R}^n \times Q_0)$, which corresponds to the defect-free setting,

$$\widehat{A}_y v = -a_0 \Delta_y v = p, \quad p \in L^2(\mathbb{R}^n \times Q_0).$$

One can easily check the following properties: $\mathcal{D}(A_y) \subset \mathcal{D}(A_0)$, $\mathcal{D}(\widehat{A}_y) \subset \mathcal{D}(\widehat{A}_0)$, $\sigma(A_y) \subset \sigma_{\mathrm{ess}}(A_0)$ and, in particular,

$$\sigma(\widehat{A}_y) \subset \sigma_{\mathrm{ess}}(\widehat{A}_0), \quad \sigma(A_y) = \sigma(\widehat{A}_y). \tag{7.12}$$

It is not difficult to see (by analyzing (2.9), see also [23]) that

$$A_y v^{(k)} = g_0^{(k)} \theta_{\Omega_1}(x) \theta_{Q_0}(y) + h^{(k)}. \tag{7.13}$$

(Note that $A_y v^{(k)} \neq A_0 v^{(k)}$.)

Combining (7.10), (7.11) and (7.13) we arrive at

$$\|(A_y - \lambda) v^{(k)}\|_{L^2(\Omega_1 \times Q_0)} = \|g_0^{(k)} \theta_{Q_0}(y) + h^{(k)} - \lambda v^{(k)}\|_{L^2(\Omega_1 \times Q_0)} \to 0.$$

This implies that λ belongs to the spectrum of A_y (notice that (7.1) holds for $v^{(k)}$ via (7.9)). Hence $\lambda \in \sigma_{\mathrm{ess}}(\widehat{A}_0)$, see (7.12).

Now let (7.7) hold. Consider a sequence $w^{(j)} = u^{(k(j))}(1 - \eta_{i(j),R}) \in \mathcal{D}(\widehat{A}_0)$ (we remind that R is large enough to ensure $\Omega_2 \subset\subset B_R$). Then

$$\|w^{(j)}\|_{\widehat{\mathcal{H}}_0} \geq \|u^{(k(j))}(1 - \theta_{B_{R+i(j)}})\|_{\mathcal{H}_0} \geq M,$$

i.e. (7.1) is satisfied for $w^{(j)}$. Since the sequence $1 - \eta_{i(j),R}$ tends to 0 pointwise, (7.2) is valid. Analogously to (7.5) we derive

$$\|(\widehat{A}_0 - \lambda) w^{(j)}\|_{\widehat{\mathcal{H}}_0} = \|(A_0 - \lambda) w^{(j)}\|_{\mathcal{H}_0} \to 0, \tag{7.14}$$

yielding (7.3). Thus, we conclude that $\lambda \in \sigma_{\mathrm{ess}}(\widehat{A}_0)$, completing the proof of the theorem. $\qquad \square$

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Remark 7.2. Theorem 7.1 combined with [24] implies that $\sigma_{\mathrm{ess}}(A_0) = \{\lambda : \beta(\lambda) \geq 0\} \cup \sigma(A_y)$. Using the methods of [24] it is not hard to show further that $\sigma_{\mathrm{ess}}(A_0)$ contains no point spectrum (in particular, no embedded eigenvalues) except if λ is an eigenvalue of A_y corresponding to an eigenfunction with zero mean. It is natural to conjecture (cf. [24]) that, outside these eigenvalues, the spectrum is absolutely continuous and the "eigenfunctions of the continuous spectrum" are $u(x,y,\lambda) = u_0(x,\lambda)(1 + \lambda b(y,\lambda))$, where $u_0(x,\lambda)$ are solutions of the appropriate scattering problems:

$$
\begin{aligned}
\nabla \cdot A^{\mathrm{hom}} \nabla u_0 + \beta(\lambda) u_0 = 0, \ x \in \mathbb{R}^n \backslash \overline{\Omega}_2, \\
a_2 \Delta u_0 + \lambda u_0 = 0, \ x \in \Omega_2
\end{aligned}
\tag{7.15}
$$

with the appropriate matching condition at $\partial \Omega_2$ and radiation condition at infinity. A detailed study of this as well as of the convergence of the related generalised eigenfunctions (cf. [24] for the defect-free case) is beyond the scope of the present paper.

Summarizing the main results of the present paper we conclude that Theorems 6.1 and 7.1 together with the results of [13, 16] (see the discussions at the end of Section 6 and in the beginning of the present section) establish the validity of Theorem 2.1.

References

[1] Agmon S., 1982, Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators. *Math. Notes,* **29**. Princeton University Press.

[2] Allaire G., 1992, Homogenization and two-scale convergence. *SIAM J. Math. Anal.,* **23**, 1482-1518.

[3] Arbogast T., Douglas J. Jr. and Hornung U., 1990, Derivation of the double porosity model of single phase flow via homogenization theory. *SIAM J. Math. Anal.,* **21**, no. 4, 823-836.

[4] Babych N. O., Kamotski I. V., Smyshlyaev V. P., 2008, Homogenization of spectral problems in bounded domains with doubly high contrasts. *Netw. Heterog. Media,* **3**, no. 3, 413–436.

[5] Barbaroux J. M., Combes J. M., and Hislop P. D., 1997, Localization near band edges for random Schrödinger operators. *Helv. Phys. Acta,* **70**, 16–43

[6] Bellieud M., 2005, Homogenization of evolution problems for a composite medium with very small and heavy inclusions. *ESAIM Control Optim. Calc. Var.,* **11**, no. 2, 266–284.

[7] Birman M.S., Solomyak M.Z., 1987, Spectral Theory of Self-Adjoint Operators in Hilbert Space. *D. Reidel Publishing Company.*

[8] Bouchitté, G., Felbacq, D., 2004, Homogenization near resonances and artificial magnetism from dielectrics. *C. R. Math. Acad. Sci. Paris,* **339**, no. 5, 377–382.

[9] Bourgeat A., Mikelic A. and Piatnitski A., 2003, On the double porosity model of a single phase flow in random media. *Asymptot. Anal.,* **34**, no. 3–4, 311-332.

[10] Briane M., 2003, Homogenization of the Stokes equations with high-contrast viscosity. *J. Math. Pures Appl.,* **82**, no. 7, 843–876.

[11] Cherednichenko K.D., Smyshlyaev V.P. and Zhikov V.V., 2006, Non-local homogenised limits for composite media with highly anisotropic periodic fibres. *Proc. Roy. Soc. Edinb. A,* **136**, no. 1, 87-114.

[12] Cherednichenko K.D., 2006, Two-scale asymptotics for non-local effects in composites with highly anisotropic fibres. *Asymptot. Anal.,* **49**, no. 1-2, 39-59.

[13] Figotin A., Klein A., 1997, Localised classical waves created by defects. *J. Statist. Phys.,* **86**, no. 1-2, 165-177.

[14] Hempell R., Lienau J., 2000, Spectral properties of periodic media in the large coupling limit. *Comm. Partial Diff. Equations*, **25**, 1445-1470.

[15] Jikov V.V., Kozlov S.M., Oleinik O.A., 1994, Homogenization of differential operators and integral functionals. Springer-Verlag, Berlin.

[16] Kamotski I.V., Smyshlyaev V.P., 2006, Localised modes due to defects in high contrast periodic media via homogenization. BICS preprint 3/06. Available online at: www.bath.ac.uk/math-sci/preprints/BICS06_3.pdf.

[17] Kohn R.V., Shipman S.P., 2008, Magnetism and Homogenization of Microresonators. *Multiscale Modeling & Simulation, SIAM,* **7**, no. 1, 62–92.

[18] Kuchment P., 2001, The mathematics of photonic crystals, in Mathematical Modeling in Optical Science. *Frontiers in Applied Mathematics, SIAM, Philadelphia,* **22**, 207–272.

[19] Mikhailov V.P., 1978, Partial differential equations. Mir, Moscow. Translated from Russian.

[20] Nguetseng G., 1989, A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.,* **20**, 608-623.

[21] Reed M., Simon B., 1978, Methods of modern mathematical physics. Academic Press.

[22] Sandrakov G.V., 1999, Homogenization of elasticity equations with contrasting coefficients. *SB MATH*, **190**, no. 12, 1749–1806.

[23] Zhikov V.V., 2000, On an extension of the method of two-scale convergence and its applications. *(Russian) Mat. Sb.,* **191**, no. 7, 31–72; *translation in Sb. Math.,* **191**, no. 7-8, 973–1014.

[24] Zhikov V.V., 2004, Gaps in the spectrum of some elliptic operators in divergent form with periodic coefficients. *(Russian) Algebra i Analiz,* **16**, no. 5, 34–58; 2005, *translation in St. Petersburg Math. J.,* **16**, no. 5, 773–790.