DYNAMIC SUPPLY CHAIN DESIGN: SQUARE ROOT LAW FOR BULLWHIP

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ABSTRACT

We present a methodology to dynamically design supply chain networks taking into account both inventory and capacity (bullwhip) related costs. We assume i.i.d. demand and the "order-up-to" policy is used to place replenishment orders with a lead-time of one period. Our methodology is complete, analytical and exact. We illustrate its application via a "toy" numerical example that seems to suggest that a "square root law for bullwhip costs" exists. We provide a sketch of a proof that shows this is indeed the case.

KEYWORDS: Distribution network design, supply chain dynamics, inventory, capacity, bullwhip, square root law.

INTRODUCTION

Distribution Network Design (DND) is concerned with the placement of an arbitrary number of distribution centres (DC's) that act as stock holding facilities to enable the efficient flow of materials through a supply chain. The number of distribution centres is a decision variable; as is the size of the warehouse and their geographical position. These DND decisions are important as the location and number of DC's influence transportation costs and delivery / collection lead-times. The lead-time influences the amount of stock that must be held to provide a certain level of product availability. This in turn influences the capacity of the DC's that are required.

By assuming at demand is independently and identically distributed (i.i.d.) over time we may use recently derived results (Disney et al 2006a) to characterise the optimum safety stock required at each DC to minimise inventory holding and backlog costs. Alternatively, we may use a pseudo backlog cost to ensure an appropriate amount of availability. This safety stock requirement will drive the DC's level of capacity, which we can also characterise with results from Disney et al (2006a). This reference also provides a mechanism to minimise lost capacity and over-time costs associated with labour and transportation. Here we use these results to uniquely provide a mechanism to determine the optimum investment in warehousing and transportation facilities.

As we have asserted, the number of DC's and their locations will affect the delivery and collection lead-times. These lead-times also influence the amount of safety stock, and hence the size of distribution centres required. By assuming each DC determines replenishment orders with the order-up-to policy we use ztransforms and properties of random numbers to identify how the variance of demand is amplified in different DND's. These variances are then used inside probability density functions to identify optimum investments of inventory and capacity.

These decisions depend heavily on the costs specified. Thus we illustrate our methodology to dynamically design distribution networks with a simple numerical example. Although greatly simplified, the scenario is based upon the automotive supply chain detailed in Hammant et al (1999). At present our methodology is able to evaluate and compare a range of different logistical scenarios. This may be useful for companies who have limited range of options available for the structure of their distribution network. However, if our methodology is coupled with readily available commercial software the optimal placement of facilities for given consumption and supply can be found via "centre of gravity" modelling.

In this paper, we illustrate our methodology with i.i.d. demand. However it is relatively easy to extend the methodology to the ARIMA type of models (Box and Jenkins 1970). In fact, the necessary results can be found in Chen and Disney (2006), Disney et al (2006b) and Disney and Grubbström (2004). Further limitations of our model exist; most importantly, it is a single product scenario.

We believe our methodology to be unique in that it captures the link between the supply chain dynamics and the distribution network structure. At present these aspects do not appear to be incorporated into modern supply chain design software. Correctly accounting for these issues will be of much interest to industry companies with large distribution networks.

DYNAMIC SUPPLY CHAIN DESIGN: A SCENARIO

It seems prudent to illustrate our approach via example. Consider the following (six) scenarios. There are 12 customers, each with a particular realisation of a stochastic demand process. Each process is characterised by a normally distributed, uncorrelated, random variable, with a mean of five and a unit standard deviation. Thus

$$D_c = N(5,1) \tag{1}$$

where D_c is the demand for customer c, c is the index for the customer, here ranging from 1 to 12. We have picked 12 customers as 12 has "nice" divisors; 1, 2, 3, 4, 6 and 12. These are useful when specifying which DC's satisfies which customer demand as it saves us having to "split" up a demand process. For example, if there was 1 DC, then this single DC faces a normally distributed demand, with a mean of 60 and a standard deviation of $\sqrt{12}$, that is $N(60,\sqrt{12})$. If there are 2 DC's, we assume that each DC has 6 customers and each DC then faces a demand process of $N(30,\sqrt{6})$. Table 1 shows the how the DC demand process changes based on different numbers of DC's.

No. of DC's	1	2	3	4	6	12
Demand process faced by each DC	$N\left(60,\sqrt{12}\right)$	$N\left(30,\sqrt{6}\right)$	$N\left(20,\sqrt{4}\right)$	$N\left(15,\sqrt{3}\right)$	$N\left(10,\sqrt{2}\right)$	$N(5,1)$

Table 1. The demand faced by DC's in each of the 6 scenarios

Now we need to specify a replenishment decision that the DC's use as this may introduce autocorrelation into the demand signal. For our purpose here it is appropriate to use the simplest replenishment rule we can think of. Let us consider the traditional "order-up-to" (OUT) policy with minimal mean squared error (MMSE) forecasting, Disney, Towill and van de Velde, (2004). The optimal, MMSE, forecast is simply the long term mean of the demand, as the demand is not auto-correlated. Under such conditions the classical OUT policy simply passes on demand as orders to the supplier or production system, Disney et al (2005). These replenishment orders are also uncorrelated.

We shall also presume that the DC's are replenished via a single factory in all scenarios. It turns out that the factory will face the same demand pattern in all six scenarios, characterised by $N(60,\sqrt{12})$. Figure 1 describes the six scenarios graphically.

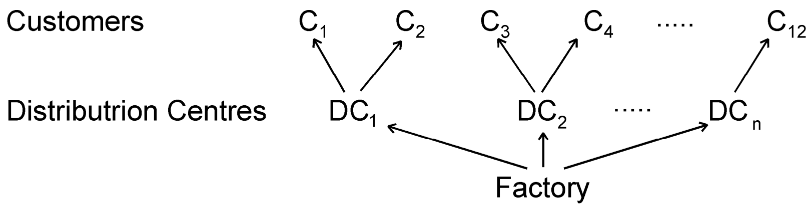

Figure 1. The six scenarios considered

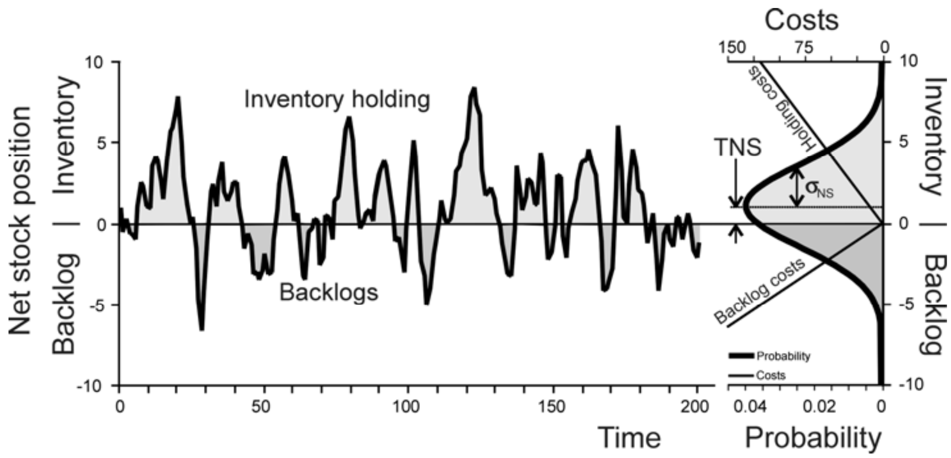

Figure 2. The inventory costs over time (Taken from Disney et al (2006a))

ASSIGNING COSTS TO A SUPPLY CHAIN NETWORK DESIGN: INVENTORY COSTS

We assume inventory related costs at each location in each period are incurred in the following manner

$$\text{Inventory cost in period } t = \begin{cases} H(NS_t), & \text{if } NS \geq 0 \\ B(-NS_t), & \text{if } NS < 0 \end{cases} \tag{2}$$

where NS_t is the Net Stock at each location in time period t, H is the cost of holding a single unit of inventory for one period and B is cost of a unit backlog in a single period. Graphically this illustrated in figure 2. Disney et al (2006a) have obtained a solution for the minimum inventory related costs at each location that we will exploit here. It is

$$I_{\underset{\min}{£}} = \text{Holding} + \text{Backlog costs} = \frac{\sigma_{NS}(B+H)e^{-erf^{-1}\left[\frac{2B}{B+H}-1\right]^2}}{\sqrt{2\pi}} \quad (3)$$

where the σ_{NS} is the standard deviation of the Net Stock levels over time and erf^{-1} is the inverse error function. The optimum amount of safety stock at each location is given by the Target Net Stock, TNS*,

$$TNS^* = \sigma_{NS}\sqrt{2}\left(erf^{-1}\left[\frac{B-H}{B+H}\right]\right). \quad (4)$$

ASSIGNING COSTS TO A SUPPLY CHAIN NETWORK DESIGN: CAPACITY COSTS

We assume that if an order is less then a certain capacity (of $S+\overline{D}$) then in each period a cost of N is incurred for each unit of unused capacity at each location. Otherwise, a cost of P is incurred for each unit that has to be produced in over-time capacity (or purchased from another source with the same lead-time). Thus,

$$\text{Bullwhip costs in period } t = \begin{cases} N\left(S+\overline{D}-O_t\right), & \text{if } O_t \le \left(S+\overline{D}\right) \\ P\left(O_t-\left(S+\overline{D}\right)\right), & \text{if } O_t > \left(S+\overline{D}\right) \end{cases} \quad (5)$$

where N is the cost per unit of not producing to capacity and P is the cost per unit of working over-time. $\overline{D}$ is the mean demand rate and S is the spare capacity above $\overline{D}$ that has been invested in by the company to improve responsiveness.

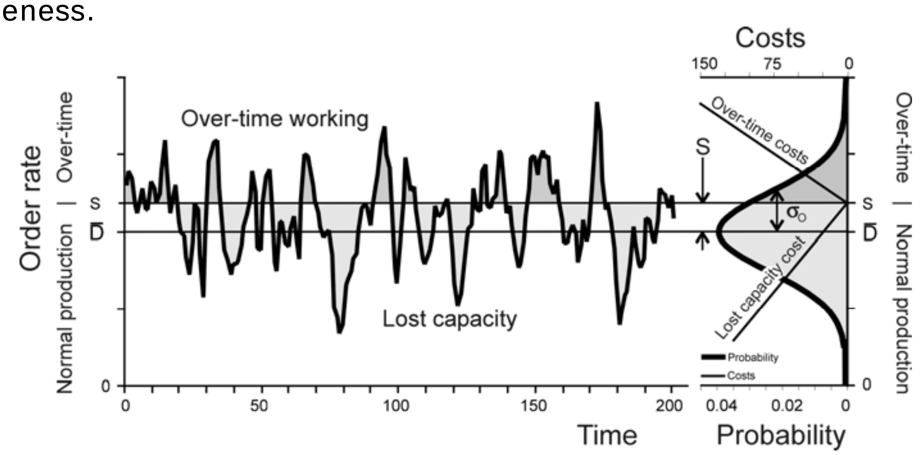

Figure 3. The capacity related costs over time (Taken from Disney et al (2006a))

Again exploiting Disney et al (2006a), the capacity or bullwhip related costs are given by

$$C_{\underset{\min}{£}} = \text{Lost capacity} + \text{Overtime costs} = \frac{\sigma_O(N+P)e^{-erf^{-1}\left[1-\frac{2N}{N+P}\right]^2}}{\sqrt{2\pi}} \quad (6)$$

where σ_O is the standard deviation of the order rate over time. Again there is an investment decision to be optimised; we need to determine the optimal amount of capacity to invest in (above / below average demand) at each location. Disney et al (2006a) have provided to necessary solution. It is given by

$$S^* = \sigma_O \sqrt{2}\left(erf^{-1}\left[\frac{P-N}{N+P} \right] \right). \qquad (7)$$

ENUMERATION OF THE SIX SCENARIOS

In this section we will enumerate the six scenarios and present the some numerical results. We will assume that the holding costs, H=9, the backlog costs, B=1, the lost capacity costs, N=4 and the over-time costs, P=6 at all locations, in all scenarios. We will also assume that the replenishment lead-times at all locations are of one period. Thus, with are assumptions that the OUT policy is used with MMSE forecasting the variance of the replenishment orders is the same as the variance of the demand and the variance of the Net Stock levels is twice the variance of the demand.

Scenario 1. A single DC

In the first scenario, we have 12 customers, one DC and one factory. Each customer places a demand of $N(5,1)$ onto the DC. The aggregate demand faced by the DC is thus $N\left(60,\sqrt{12}\right)$. The standard deviation of the DC orders is $\sigma_O = \sqrt{12}$ and the standard deviation of the DC net stock levels is $\sigma_{NS} = \sqrt{24}$.

Using (4) the optimal safety stock carried by the DC is TNS^*=6.2782 and the optimal slack capacity (above the average demand of 60) of the DC is S^*=0.87762 from (7). Exploiting (3) and (6) the minimum inventory related costs at the DC is $I_£$=8.5976 and the minimum capacity related costs at the DC is $C_£$=13.38. Thus the DC costs are equal to £21.9809.

In this scenario the factory faces the same demand process as the DC, thus the same settings for TNS^* and S^* exist and the same costs ($I_£$=8.5976 and $C_£$=13.38) are incurred. Thus the factory incurs a total expected cost per period of £21.9809. In fact, the factory always faces the same demand process and incurs the same costs in all scenarios.

Scenario 2. Two DC’s

In this scenario we have 12 customers, 2 DC’s and a single factory. Each DC replenishes 6 customers and faces a demand characterised by $N\left(30,\sqrt{6}\right)$. The optimal safety stock, TNS^*, at both DC’s is 4.4394 and the optimal slack capacity in both DC’s (above the average demand of 30) is S^*=0.62057. The inventory costs $I_£$=6.074 at each of the two DC’s and $C_£$=9.4634 at each of the two DC’s. Thus the total cost of the DC echelon is 2*(6.074+9.4634)=2*15.5428=£31.0857.

Scenario 3. Three DC’s

In this scenario we have 12 customers, 3 DC’s and a single factory. Each DC replenishes 4 customers and faces a demand characterised by $N\left(20,\sqrt{4}\right)$. The optimal safety stock, TNS^*, at all DC's is 3.62477 and the optimal slack capacity in all DC's (above the average demand of 20) is $S^*=0.50669$. The inventory costs $I_£=4.9638$ at each of the three DC's and $C_£=7.7268$ at each of the three DC's. Thus the total cost of the DC echelon is 3*(4.9638+7.7268) =3*12.69069= £38.072.

Scenario 4. Four DC's

In this scenario we have 12 customers, 4 DC's and a single factory. Each DC replenishes 3 customers and faces a demand characterised by $N\left(15,\sqrt{3}\right)$. The optimal safety stock, TNS^*, at all DC's is 3.14915 and the optimal slack capacity in all DC's (above the average demand of 15) is $S^*=0.43881$. The inventory costs $I_£=4.298811$ at each of the four DC's and $C_£=6.69164$ at each of the four DC's. Thus the total cost of the DC echelon is 4*(4.298811+6.69164)= 4*10.99046=£43.96184.

Scenario 5. Six DC's

In this scenario we have 12 customers, 6 DC's and a single factory. Each DC replenishes 2 customers and faces a demand characterised by $N\left(10,\sqrt{2}\right)$. The optimal safety stock, TNS^*, at all DC's is 2.5631 and the optimal slack capacity in all DC's (above the average demand of 10) is $S^*=0.35828$. The inventory costs $I_£=3.50996$ at each of the six DC's and $C_£=5.4637$ at each of the six DC's. Thus the total cost of the DC echelon is 6*(3.50996+5.4637)= 6*8.9736=£53.8420.

Scenario 6. Twelve DC's

In this scenario we have 12 customers, 12 DC's and a single factory. Each DC's replenishes a single customer and faces a demand characterised by $N(5,1)$. The optimal safety stock, TNS^*, at all DC's is 1.812388 and the optimal slack capacity in all DC's (above the average demand of 15) is $S^*=0.25334$. The inventory costs $I_£=2.48192$ at each of the twelve DC's and $C_£=3.8634$ at each of the twelve DC's. Thus the total cost of the DC echelon is 12*(2.48192+3.8634)= 12*6.345345=£76.1441.

Summary of the six scenarios

Table 2 below summaries the different scenarios considered above. We find it interesting to note that adding in the bullwhip costs into the distribution network design actually has the same consequences as the inventory related costs.

	Number of DC's					
	1	2	3	4	6	12
Safety stock at each DC	6.28	4.44	3.62	3.14	2.56	1.81
Capacity at each DC	60.88	30.62	20.51	14.44	10.36	5.25
Inventory cost at DC echelon	£8.59	£12.15	£14.89	£17.20	£21.06	£29.78
Capacity cost at DC echelon	£13.38	£18.93	£23.18	£26.77	£32.78	£46.36
Safety stock at factory	6.28					
Capacity at factory	60.87					
Inventory cost at factory	£8.59					
Capacity cost at factory	£13.38					
Total costs	£43.96	£53.07	£60.05	£65.94	£75.84	£98.13

Table 2. Summary expected costs per period

THE SQUARE ROOT LAW FOR INVENTORIES

Maister (1976) introduced the "Square Root Law" for inventory costs when consolidation occurs in a distribution network. Quoting directly from Maister, "If the inventories of a single product (or stock keeping unit) are originally maintained at a number (n) of field locations (refereed to as the decentralised system) but are then consolidated into one central inventory then the ratio $\frac{\text{decentralised system inventory}}{\text{centralised system inventory}} = \sqrt{n}$ exists", Maister, (1976).

If we look at the DC's inventory costs in Table 3, we can see, numerically at least, that the square root law also holds in our case, even though we have not used the same replenishment rule as Maister (1976).

	Number of DC's, n					
	1	2	3	4	6	12
Inventory cost	£8.59	£12.15	£14.89	£17.20	£21.06	£29.78
$\frac{\text{Inventory cost}}{\sqrt{n}}$	£8.59	£8.59	£8.60	£8.60	£8.60	£8.60

Table 3. The square root law for inventories

THE SQUARE ROOT LAW FOR BULLWHIP COSTS

Amazingly, the square root law also exists for bullwhip or capacity costs. Consider again capacity related costs at the DC echelon. In figure 4 we have again scaled the cost by the inverse of the square root of the number of DC's and we can see that the bullwhip costs also follow the square root law. This should not surprise us as the bullwhip cost structure is exactly the same as inventory cost structure. Furthermore, as the square root law is just a ratio, it will also work for the sum of the inventory and capacity costs.

	Number of DC's, n					
	1	2	3	4	6	12
Capacity cost	£13.38	£18.93	£23.18	£26.77	£32.78	£46.36
$\frac{\text{Capacity cost}}{\sqrt{n}}$	£13.38	£13.39	£13.38	£13.39	£13.38	£13.38

Table 4. The square root law for bullwhip (capacity costs)

PROOF OF THE SQUARE ROOT LAW FOR BULLWHIP

Equation (7) shows us that if bullwhip or capacity costs are given by

$$C_{£} = \frac{\sigma_O (N+P) e^{-erf^{-1}\left[1-\frac{2N}{N+P}\right]^2}}{\sqrt{2\pi}} = \sigma_O Y \; ; \; Y = \frac{(N+P) e^{-erf^{-1}\left[1-\frac{2N}{N+P}\right]^2}}{\sqrt{2\pi}} . \tag{8}$$

Y is a constant determined by the lost capacity and overtime costs. It is easy too prove the square root law for bullwhip exists by considering that in the decentralised supply chain the standard deviation of the orders is $\sigma_O = n\sqrt{\sigma_c^2}$, and in the centralised supply chain the standard deviation of the orders is $\sigma_O = \sqrt{n\sigma_c^2}$. Thus,

$$\frac{\text{decentralised bullwhip costs}}{\text{centralised bullwhip costs}} = \frac{\left(n\sqrt{\sigma_c^2}\right)Y}{\left(\sqrt{n\sigma_c^2}\right)Y} = \sqrt{n} \, , \tag{9}$$

which is the "Square Root Law for Bullwhip".

CONCLUSIONS

We have presented a methodology to dynamically design a distribution network based on inventory and capacity costs. At each point in the network the optimum safety stock has been set to minimise the sum of the inventory holding and backlog costs in the face of the stochastic demand pattern that each location faces. We have also defined the optimum capacity at each point in the network in order to minimise the sum of the lost capacity and over-time working at each location. The maths that we have exploited to achieve this is based on a linear system, thus inventory has been backlogged rather then lost when a negative inventory position has occurred, and production over the capacity limit has been made up in over-time (or provided via a sub-contractor with the same lead-time). All lead-times in the distribution network have been assumed to be of one period.

We have shown via a numerical example that the addition of the bullwhip costs into a dynamic distribution network design methodology actually behaves in exactly the same way as the inventory related costs. This surprised us, as intuitively, we expected it to have the opposite impact. Our result suggests that reason to consolidate distribution networks are actually a lot stronger than previously thought of based solely on inventory costs. The likely impact of this is to force companies to consolidate even further than they have in the past, increasing the amount of traffic on the road. Thus, internalising the external costs transportation causes is now even more important.

For further work, we acknowledge that there is a long to go with this short conference paper. We have not proved a rigorous proof of the "square root law for bullwhip", only provided a numerical example and a simple proof based on rather restrictive assumptions. These could be easily extended to consider more complex replenishment rules, lead-times and demand patterns. These matters will be considered in the future.

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