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Derived McKay correspondence via pure-sheaf transforms

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Abstract. In most cases where it has been shown to exist the derived McKay correspondence $D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^n)$ can be written as a Fourier-Mukai transform which sends point sheaves of the crepant resolution Y to pure sheaves in $D^G(\mathbb{C}^n)$. We give a sufficient condition for $E \in D^G(Y \times \mathbb{C}^n)$ to be the defining object of such a transform. We use it to construct the first example of the derived McKay correspondence for a non-projective crepant resolution of $\mathbb{C}^3/G$. Along the way we extract more geometrical meaning out of the Intersection Theorem and learn to compute θ-stable families of G-constellations and their direct transforms.

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1. Introduction

It was observed by McKay in [McK80] that the representation graph (better known now as the *McKay quiver*) of a finite subgroup G of $\mathrm{SL}_2(\mathbb{C})$ is the Coxeter graph of one of the affine Lie algebras of type ADE, while the configuration of irreducible exceptional divisors on the minimal resolution Y of $\mathbb{C}^2/G$ is dual to the Coxeter graph of the finite-dimensional Lie algebra of the same type. It followed that the subgraph of nontrivial irreducible representations coincided with the graph of irreducible exceptional divisors. This led Gonzales-Sprinberg and Verdier in [GSV83] to construct an isomorphism of the G-equivariant K-theory of $\mathbb{C}^2$ to the K-theory of Y, which induced naturally a choice of such bijection. This became known as *the (classical) McKay correspondence*.

In [Rei97] M.Reid proposed that the K-theory isomorphism might lift to the level of derived categories. It became known as *the derived McKay correspondence* conjecture:

Conjecture 1. Let G be a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$ and let Y be a crepant resolution of $\mathbb{C}^n/G$, if one exists. Then

$$D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^n) \tag{1.1}$$

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where $D(Y)$ and $D^G(\mathbb{C}^n)$ are bounded derived categories of coherent sheaves on Y and of G-equivariant coherent sheaves on $\mathbb{C}^n$, respectively.

To date and to the extent of our knowledge this conjecture has been settled for the following situations:

1. $G \subset \mathrm{SL}_{2,3}(\mathbb{C})$; Y the distinguished crepant resolution G-Hilb; ([KV98], Theorem 1.4; [BKR01], Theorem 1.1).
2. $G \subset \mathrm{SL}_3(\mathbb{C})$ abelian; Y any projective crepant resolution; ([CI04], Theorem 1.1).
3. $G \subset \mathrm{SL}_n(\mathbb{C})$ abelian; Y any projective crepant resolution; ([Kaw05], special case of Theorem 4.2).
4. $G \subset \mathrm{Sp}_{2n}(\mathbb{C})$; Y any symplectic (crepant) resolution; ([BK04], Theorem 1.1).

In the case 3 the construction is not direct and it isn't clear what form does the equivalence (1.1) take, but in each of the cases 1, 2 and 4, the equivalence (1.1) is constructed directly and we observe that the constructed functor sends point sheaves $\mathcal{O}_y$ of Y to pure sheaves (i.e. complexes with cohomologies concentrated in degree zero) in $D^G(\mathbb{C}^n)$. Another property (cf. though [Orl97], Theorem 2.18) that these functors share is that each can be written as a Fourier-Mukai transform $\Phi_E(- \otimes \rho_0)$ (see Def. 3) for some object $E \in D^G(Y \times \mathbb{C}^n)$.

A straightforward application (Prop. 3) of the established machinery of Fourier-Mukai transforms shows that if an equivalence (1.1) is a Fourier-Mukai transform $\Phi_E(- \otimes \rho_0)$ which sends point sheaves to pure sheaves, then its defining object E is itself a pure sheaf. Moreover, the fibers of E over Y have to be simple (G-$\mathrm{End}_{\mathbb{C}^n}(E_{|y}) = \mathbb{C}$ for all $y \in Y$), orthogonal in all degrees (G-$\mathrm{Ext}^i_{\mathbb{C}^n}(E_{|y_1}, E_{|y_2}) = 0$ if $y_1 \neq y_2$) and the Kodaira-Spencer maps have to be isomorphisms.

Let Y now be any irreducible separated scheme of finite type over $\mathbb{C}$. A $gnat$-family $\mathcal{F}$ on Y is a coherent G-sheaf on $Y \times \mathbb{C}^n$, flat over Y, such that for any $y \in Y$ the fiber $\mathcal{F}_{|y}$ of $\mathcal{F}$ is a G-*constellation* supported on a single G-orbit. That is, $\mathcal{F}_{|y}$ is a finite length coherent G-sheaf on $\mathbb{C}^n$ whose support is a single G-orbit and whose global sections have G-representation structure of the regular representation. Such family $\mathcal{F}$ has a well-defined Hilbert-Chow morphism $\pi_{\mathcal{F}} : Y \to \mathbb{C}^n/G$, it sends any $y \in Y$ to the G-orbit that $\mathcal{F}_{|y}$ is supported on (Prop. 2). Let Y and $\mathcal{F}$ be any such for which $\pi_{\mathcal{F}}$ is birational and proper. In this paper we give a sufficient condition for the functor $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ to be an equivalence (1.1). Notable, in the view of Prop. 3, is that this condition only asks for the non-orthogonality locus of $\mathcal{F}$ to be of high enough codimension. The simplicity of $\mathcal{F}$ and the Kodaira-Spencer maps being isomorphisms follow automatically:

Theorem 1. *Let G be a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$. Let Y be an irreducible separated scheme of finite type over $\mathbb{C}$ and $\mathcal{F}$ be a $gnat$-family on Y. Assume Y and $\mathcal{F}$ such that the Hilbert-Chow morphism $\pi_{\mathcal{F}}$ is birational and proper.*

If for every $0 \leq k < (n+1)/2$, the codimension of the subset

$$\mathrm{N}_k = \overline{\{(y_1, y_2) \in Y \times Y \setminus \Delta \;\mid\; G\text{-}\operatorname{Ext}^k_{\mathbb{C}^n}(\mathcal{F}_{|y_1}, \mathcal{F}_{|y_2}) \neq 0\}} \tag{1.2}$$

in $Y \times Y$ is at least $n + 1 - 2k$, then the functor $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is an equivalence of categories $D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^n)$.

Once $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is known to be an equivalence usual methods ([Rob98], Theorem 6.2.2 and [BKR01], Lemma 3.1) apply to show that Y is non-singular and $\pi_{\mathcal{F}}$ is crepant. The set N_k in (1.2) can be thought of as the locus of the degree k non-orthogonality in $\mathcal{F}$.

Our proof of Theorem 1 is based on the ideas introduced in [BO95] and [BKR01], particularly on the Intersection Theorem trick introduced in the latter. However, not wishing to restrict ourselves to just quasi-projective schemes necessitates more work in applying the Intersection Theorem. This is done in Section 2, which is a self-contained piece of abstract derived category theory for a locally noetherian scheme X. There we propose a generalisation of the concept of the *homological dimension* of $E \in D^b_{\text{coh}}(X)$ which we call Tor-*amplitude* , and use it to show that the inequality

$$\text{hom. dim. } E \geq \operatorname{codim}_X \operatorname{Supp} E$$

of [BM02], Corollary 5.5 refines to

$$\operatorname{Tor-amp} E \geq \operatorname{codim}_X \operatorname{Supp} E + \operatorname{coh-amp} E.$$

Other notable points of our proof of Theorem 1 are a different approach to Grothendieck duality when constructing the left adjoint to $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ and an application of [Log06], Prop. 1.5 which states that outside the exceptional set of Y any $gnat$-family has to be locally isomorphic to the universal family of G-clusters. The latter is everywhere simple and its Kodaira-Spencer maps are isomorphisms. Then the locus of points of Y where objects of $\mathcal{F}$ are not simple or the Kodaira-Spencer map isn't an isomorphism turns out to have too high a codimension to exist at all.

The question of an existence of a derived McKay correspondence which sends point sheaves to pure sheaves is thus reduced to that of an existence of a $gnat$-family satisfying the non-orthogonality condition of Theorem 1. This is particularly relevant whenever G is abelian, for then all the $gnat$-families on a given resolution $Y \to \mathbb{C}^n/G$ had been classified and their number was shown to be finite and non-zero ([Log06], Theorem 4.1).

When $n = 3$, Theorem 1 reduces to:

Corollary 1. *Let G be a finite subgroup of $\mathrm{SL}_3(\mathbb{C})$. Let Y, $\mathcal{F}$ and $\pi_{\mathcal{F}}$ be as in Theorem 1. Let $E_1, \dots, E_k$ be the irreducible exceptional surfaces of $\pi_{\mathcal{F}}$. Then if general points of any surface E_i are orthogonal in degree 0 in $\mathcal{F}$ to general points of any surface E_j (including case $j = i$) and of any curve $E_l \cap E_m$, then $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is an equivalence of categories.*

By a general point of an intersection of k exceptional surfaces we mean a point that doesn't lie on an intersection of any $k+1$ exceptional surfaces.

In Section 4 we show how to compute the degree 0 non-orthogonality locus of a $gnat$-family. We use this in Section 5 to give following application of Corollary 1: for G the abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$ known as $\frac{1}{6}(1,1,4) \oplus \frac{1}{2}(1,0,1)$ (see Section 5.1) and for Y a certain non-projective crepant resolution of $\mathbb{C}^3/G$ (see Section 5.2) we construct a $gnat$-family $\mathcal{F}$ on Y which satisfies the condition in Corollary 1. This gives the first example of the derived McKay correspondence for a non-projective crepant resolution of $\mathbb{C}^3/G$.

It also leads to an important observation: the properties that $\mathcal{F}$ must then possess in view of Proposition 3 imply that Y is a fine moduli space of G-constellations, representing the functor of all $gnat$-families whose members (fibres over closed points) are isomorphic to members of $\mathcal{F}$. At present the only moduli functors known for G-constellations come from the notion of θ-stability. Their fine moduli spaces M (cf. [CI04]) are constructed via the method introduced by King in [Kin94]. However, Y can't be one of M as these are all, due to the GIT nature of their construction in [Kin94], projective over $\mathbb{C}^n/G$. This raises the question as to whether there could exist a more general notion of 'stability', related perhaps to Bridgeland-Douglas stability [Bri02], which would allow for functors with non-projective moduli spaces.

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2. Cohomological and Tor amplitudes

We clarify terminology and introduce notation. By a point of a scheme we mean both a closed and non-closed point unless specifically mentioned otherwise. Given a point x on a scheme X we write $(\mathcal{O}_x, \mathfrak{m}_x)$ for the local ring of x, $\mathbf{k}(x)$ for the residue field $\mathcal{O}_x/\mathfrak{m}_x$ and ι_x for the point-scheme inclusion $\mathrm{Spec}\ \mathbf{k}(x) \hookrightarrow X$. Given an irreducible closed set $C \subset X$, we write x_C for the generic point of C and we sometimes write simply $(\mathcal{O}_C, \mathfrak{m}_C)$ for the local ring of x_C. All complexes are cochain complexes. Given a right (resp. left) exact functor F between two abelian categories $\mathcal{A}$ and $\mathcal{B}$, we denote by $\mathbf{L}\,F$ (resp. $\mathbf{R}\,F$) the left (resp. right) derived functor between the appropriate derived categories, if it exists, and by $\mathbf{L}^i\, F(\bullet)$ (resp. $\mathbf{R}^i\, F(\bullet)$) the $-i$-th cohomology of $\mathbf{L}\, F(\bullet)$ (resp. the i-th cohomology of $\mathbf{R}\, F(\bullet)$).

For X a smooth variety the results of Lemmas 1 and 2 below have appeared in the proof of Proposition 1.5 in [BO95]. We show them to hold in a more general setting of a locally noetherian scheme.

Lemma 1. *Let X be a locally noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X and C be an irreducible component of $\operatorname{Supp}_X \mathcal{F}$. Then for every point $x \in C$*

$$\mathbf{L}^i\, \iota_x^* \mathcal{F} \neq 0 \quad \textit{for } 0 \leq i \leq \operatorname{codim}_X(C). \tag{2.1}$$

Proof. Recall (cf. [Mat86], §19) that if a minimal free resolution $L_\bullet$ of a finitely generated module M for a local ring $(R, \mathfrak{m}, k)$ exists, then

$$\dim_k \operatorname{Tor}^i(M, k) = \operatorname{rk} L_i$$

Since X is locally noetherian minimal free resolutions of $\mathcal{F}$ exist in all local rings. Write F_C for the localisation of $\mathcal{F}$ to the local ring $\mathcal{O}_C$ of x_C. As $\mathbf{L}^i\, \iota_x^* \mathcal{F} = \operatorname{Tor}^i_{\mathcal{O}_C}(F_C, \mathbf{k}(x))$ it suffices to prove that the length of the minimal free resolution of F_C is at least $\operatorname{codim}_X(C)$.

Consider the standard filtration ([Ser00], I, §7, Theorem 1) of F_C by submodules $0 = M_0 \subset \cdots \subset M_n = F_C$ with each M_i/M_{i-1} isomorphic to $\mathcal{O}_C/\mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Supp}_{\mathcal{O}_C}(F_C)$. As the defining ideal of C is minimal in $\operatorname{Supp}_X(\mathcal{F})$, $\operatorname{Supp}_{\mathcal{O}_C}(F_C)$ consists of just $\mathfrak{m}_C$. So each M_i/M_{i-1} is isomorphic to k_C and hence F is a finite-length $\mathcal{O}_C$-module. Then by the New Intersection Theorem (e.g. [Rob98], Theorem 6.2.2) the length of the minimal resolution of F_C is at least $\dim \mathcal{O}_C$. As $\dim \mathcal{O}_C = \operatorname{codim}_X(C)$ the claim follows.

Lemma 2. *Let X be a locally noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X of finite* Tor*-dimension. For any $p \in \mathbb{Z}$ define*

$$D_p = \{x \in X \mid \mathbf{L}^i\, \iota_x^* \mathcal{F} \neq 0 \textit{ for some } i \geq p\}. \tag{2.2}$$

Then each D_p is closed and $\operatorname{codim}_X(D_p) \geq p$.

Proof. It suffices to prove both claims for the case $X = \operatorname{Spec} R$ with R noetherian. Write F for $\Gamma(\mathcal{F})$. As $\mathbf{L}^p\, \iota_x^* \mathcal{F} = \operatorname{Tor}^p_R(F, \mathbf{k}(x))$ the first claim follows from the upper semicontinuity theorem ([GD63], *Théorème* **7.6.9**).

For the second claim let C be any irreducible component of D_p and let F_C be the localisation of F to the local ring $\mathcal{O}_C$. Then $\operatorname{Tor}^p_{\mathcal{O}_C}(F_C, \mathbf{k}(x_C)) \neq 0$ by the defining property of D_p. We have ([Mat86], §19, Lemma 1)

$$\operatorname{proj\,dim}_{\mathcal{O}_C} F_C = \sup\{i \in \mathbb{Z} \mid \operatorname{Tor}^i_{\mathcal{O}_C}(F_C, \mathbf{k}(x_C))\}$$

hence $\operatorname{proj\,dim}_{\mathcal{O}_C} F_C \geq p$. By the Auslander-Buchsbaum equality we have

$$\operatorname{depth}_{\mathcal{O}_C} \mathcal{O}_C = \operatorname{proj\,dim}_{\mathcal{O}_C} F_C + \operatorname{depth}_{\mathcal{O}_C} F_C$$

and thus $\operatorname{codim}_X C = \dim \mathcal{O}_C \geq \operatorname{depth}_{\mathcal{O}_C} \mathcal{O}_C \geq p$ as required.

The main idea behind the proof of the following proposition we owe to Bondal and Orlov in [BO95], Proposition 1.5.

Proposition 1. *Let X be a locally noetherian scheme and $F \in D^b_{coh}(X)$ an object of finite* Tor*-dimension. Denote by $\mathcal{H}^i$ the ith cohomology sheaf of F. Then for any point $x \in X$ we have*

$$-\sup\{i \in \mathbb{Z} \mid x \in \operatorname{Supp} \mathcal{H}^i\} = \inf\{j \in \mathbb{Z} \mid \mathbf{L}^j\, \iota_x^* F \neq 0\}. \tag{2.3}$$

Let C be an irreducible component of $\operatorname{Supp} \mathcal{H}^l$ for some l such that also $C \nsubseteq \operatorname{Supp} \mathcal{H}^m$ for any $m < l$. Then

$$\operatorname{codim}_X C - \inf\{i \in \mathbb{Z} \mid C \subseteq \operatorname{Supp} \mathcal{H}^i\} = \sup\{j \in \mathbb{Z} \mid \mathbf{L}^j\, \iota_{x_C}^* F \neq 0\}. \tag{2.4}$$

Proof. Fix a point $x \in X$. The main ingredient of the proof is the standard spectral sequence (eg. [GM03], Proposition III.7.10) associated to the filtration of $\mathbf{L}\, \iota_x^* F$ by the rows of the Cartan-Eilenberg resolution of F:

$$E_2^{-p,q} = \mathbf{L}^p\, \iota_x^*(\mathcal{H}^q) \Rightarrow E_\infty^{q-p} = \mathbf{L}^{p-q}\, \iota_x^*(F). \tag{2.5}$$

Denote by h the highest non-zero row of $E_2^{\bullet\bullet}$. As all rows above row h and all columns to the right of column 0 in $E_2^{\bullet\bullet}$ consist entirely of zeroes

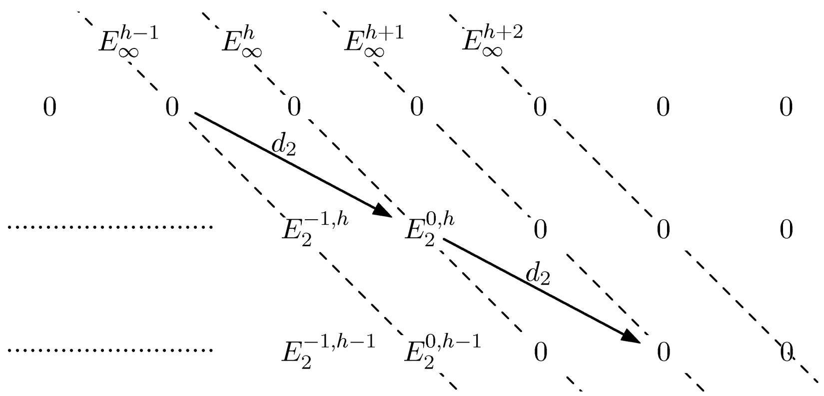


Figure 1

we conclude by inspection of the complex that $0 = E_\infty^n$ for all $n > h$ and $\mathcal{H}^h|_x = E_2^{0,h} = E_\infty^h = \mathbf{L}^{-h}(\iota_x^*(F))$. This gives (2.3).

To obtain (2.4) set x to be the generic point of C and define $E_{\bullet}^{\bullet\bullet}$ as above. For any $m < l$ we have $C \nsubseteq \operatorname{Supp} \mathcal{H}^m$ and hence $\mathbf{L}\, \iota_x^* \mathcal{H}^m = 0$. So all the rows of $E_2^{\bullet\bullet}$ below l consist of zeroes. On the other hand, C is an irreducible component of $\mathcal{H}^l$ and by Lemma 2 the set of points $y \in X$, such that there is a non-zero $\mathbf{L}^i\, \iota_y^*(\mathcal{H}^l)$ with $i > d$, is closed and of codimension at least $d+1$. Then this set can not contain x for the closure of x is C whose codimension is d. Hence all columns to the left of column $-d$ in $E_2^{\bullet\bullet}$ consist entirely of zeroes. We conclude that $E_\infty^n = 0$ for all $n > l-d$ and $\mathbf{L}^d\,\iota_x^*\mathcal{H}^l = E_2^{-d,l} = E_\infty^{l-d} = \mathbf{L}^{d-l}\,\iota_x^* F$. Thus, as $\mathbf{L}^d\,\iota_x^*\mathcal{H}^l \neq 0$ by Lemma 1, we obtain (2.4).

Definition 1. *Let* $\mathbf{A}$ *be an abelian category and* $E^\bullet$ *be a cochain complex of objects of* $\mathbf{A}$. *Define its cohomological amplitude,* denoted by $\operatorname{coh-amp} E^\bullet$, to be the length of the minimal interval in $\mathbb{Z}$ containing the set

$$\{i \in \mathbb{Z} \mid H^i(E^\bullet) \neq 0\}. \tag{2.6}$$

If no such interval exists we say that $\operatorname{coh-amp} E = \infty$.

Trivially $\operatorname{coh-amp} E^\bullet$ is the minimal length of a bounded complex quasi-isomorphic to $E^\bullet$, if any exist, and infinity, if none do.

Definition 2. *Let* R *be a ring or a sheaf of rings and* $E^\bullet$ *be a cochain complex of objects of* $\mathbf{Mod}\text{-}R$. *Define its* Tor*-amplitude,* denoted by $\operatorname{Tor-amp}_R E^\bullet$, to be the length of the minimal interval in $\mathbb{Z}$ containing the set

$$\{i \in \mathbb{Z} \mid \exists\, A \in \mathbf{Mod}\text{-}R \text{ such that } \operatorname{Tor}^i_R(E^\bullet, A) \neq 0\}. \tag{2.7}$$

If no such interval exists we say that $\operatorname{Tor-amp}_R E = \infty$.

Def. 2 can be seen to be equivalent to [Kuz05], Def. 2.20.

Let now X be any scheme. It follows from [Har66], Prop 4.2, that an object of $D^b(\mathbf{Mod}\text{-}X)$ has finite Tor-amplitude if and only if it is of finite Tor-dimension, i.e. quasi-isomorphic to a bounded complex of flat sheaves.

Lemma 3. *Let* X *be a locally noetherian scheme and* $E \in D^b_{coh}(X)$ *an object of finite* Tor*-dimension. Denote by* l *the length of the shortest complex of flat sheaves quasi-isomorphic to* E*, and by* k *the length of the smallest interval in* $\mathbb{Z}$ *containing the set*

$$\{i \in \mathbb{Z} \mid \exists\, x \in X \text{ such that } \mathbf{L}^i\,\iota_x^*(E) \neq 0\}. \tag{2.8}$$

Then $l = \operatorname{Tor-amp}_{\mathcal{O}_X} E = k$.

Proof. Implications $l \geq \operatorname{Tor-amp}_{\mathcal{O}_X} E$ and $\operatorname{Tor-amp}_{\mathcal{O}_X} E \geq k$ are trivial. We claim that $k \geq l$. Let $n, k \in \mathbb{Z}$ be such that the interval $[-n-k, -n]$ contains the set (2.8). Then (2.3) and (2.4) of Proposition 1 show that $\mathcal{H}^i(E) = 0$ unless $i \in [n, n+k]$. Since resolutions by flat modules exist on X, there exists a complex $F^\bullet$ of flat sheaves quasi-isomorphic to E and with $F_i = 0$ for all $i > n+k$. We claim that we can truncate $F^\bullet$ at degree n and keep it flat, i.e. that the sheaf $F^n / \operatorname{Im} F^{n-1}$ is flat. But as $\mathcal{H}^i(F^\bullet) = 0$ for $i < n$, the complex

$$\dots \to F^{n-2} \to F^{n-1} \to F^n \to 0 \to \dots$$

is a flat resolution of $F^n / \operatorname{Im} F^{n-1}$. Hence $\mathbf{L}^1\, \iota_x^*(F^n / \operatorname{Im} F^{n-1}) = \mathbf{L}^{-n+1}\, \iota_x^*(E)$ and so vanishes for all $x \in X$ by assumption. Thus we obtain a length k complex of flat-sheaves quasi-isomorphic to E, i.e. $k \geq l$.

Whenever X is a quasi-projective scheme, or any other scheme where there exist resolutions by locally-free sheaves, replacing the word 'flat' by the word 'locally-free' throughout Lemma 3 and its proof shows that for any $E \in D^b_{\text{coh}}(X)$ its Tor-amplitude is the length of the shortest complex of locally-free sheaves quasi-isomorphic to E. In other words, $\operatorname{Tor-amp}_{\mathcal{O}_X} E$ is the *homological dimension* of E introduced in [BM02]. The following can thus be compared to the inequality hom.dim.$E \geq \operatorname{codim} C$ of [BM02]:

Theorem 2. *Let X be a locally noetherian scheme and $E \in D^b_{coh}(X)$ an object of finite* Tor*-dimension. Then*

$$\operatorname{Tor-amp}_{\mathcal{O}_X} E \geq \operatorname{codim} \operatorname{Supp} E + \operatorname{coh-amp} E \tag{2.9}$$

and for any irreducible component C of $\operatorname{Supp} E$ *we have*

$$\operatorname{Tor-amp}_{\mathcal{O}_C} E_C = \operatorname{codim} C + \operatorname{coh-amp}_{\mathcal{O}_C} E_C. \tag{2.10}$$

Remark: To see that the inequality (2.9) can be strict, consider $X = \mathbb{A}^1$ and $E = \mathcal{O}_X \oplus \mathcal{O}_x$ for some closed point $x \in X$.

Proof. Denote by $\mathcal{H}^i$ the ith cohomology sheaf of E. Set

$$n = \inf_{x \in suppE} \{i \in \mathbb{Z} | x \in \operatorname{Supp} \mathcal{H}^i\} \qquad m = \sup_{x \in suppE} \{i \in \mathbb{Z} | x \in \operatorname{Supp} \mathcal{H}^i\}$$
$$l = \inf_{x \in suppE} \{i \in \mathbb{Z} |\, \mathbf{L}^i\, \iota_x^* E \neq 0\} \qquad h = \sup_{x \in suppE} \{i \in \mathbb{Z} |\, \mathbf{L}^i\, \iota_x^* E \neq 0\}$$

and observe that $m - n = \operatorname{coh-amp} E$ and $h - l = \operatorname{Tor-amp}_{\mathcal{O}_X} E$ (Lemma 3).

By (2.3) of Proposition 1 we have

$$-m = l. \tag{2.11}$$

Let D be any irreducible component of $\operatorname{Supp} \mathcal{H}^n$. We then have

$$\operatorname{codim} \operatorname{Supp} E - n \leq \operatorname{codim} D - n = \sup\{i \in \mathbb{Z} |\, \mathbf{L}^i\, \iota_{x_D}^* E \neq 0\} \leq h \tag{2.12}$$

with the middle equality due to (2.4) of Proposition 1 applied to D. Subtracting (2.11) from (2.12) we obtain $(m - n) + \operatorname{codim} \operatorname{Supp} E \leq (h - l)$. This shows (2.9).

To obtain (2.10) we observe that on $\operatorname{Spec}\ \mathcal{O}_C$ the support of the localisation E_C consists of a single point x_C. Therefore applying the above argument to $X' = \operatorname{Spec}\ \mathcal{O}_C$ and $E' = E_C$ we have $D = x_C = \operatorname{Supp} E'$ which makes both the inequalities in (2.12) into equalities.

3. Derived McKay correspondence

Given a scheme S denote by $D_{qc}(S)$ (resp. $D(S)$) the full subcategory of the derived category of $\mathcal{O}_S\text{-}\mathbf{Mod}$ consisting of complexes with quasi-coherent (resp. bounded and coherent) cohomology. For S a scheme of finite type over $\mathbb{C}$ and H a finite group acting on S on the left by automorphisms an *H-sheaf* is a sheaf $\mathcal{E}$ of $\mathcal{O}_S$-modules equipped with a lift of the H-action to $\mathcal{E}$. For the technical details see [BKR01], Section 4. Denote by $\mathcal{O}_S\text{-}\mathbf{Mod}^H$ (resp. $\mathrm{QCoh}^H S$, $\mathrm{Coh}^H S$) the abelian category of H-sheaves (resp. quasi-coherent, coherent H-sheaves) on S and by $D^H_{qc}(S)$ (resp. $D^H(S)$) the full subcategory of the derived category of $\mathcal{O}_S\text{-}\mathbf{Mod}^H$ consisting of complexes with quasi-coherent (resp. bounded and coherent) cohomology.

3.1. *Integral transforms*

Let N and M be schemes of finite type over $\mathbb{C}$. Denote by π_N and π_M the projections $N \times M \to N$ and $N \times M \to M$.

Definition 3. *Let E be an object of $D_{qc}(N \times M)$ of finite* Tor*-dimension. An integral transform Φ_E* is a functor $D_{qc}(N) \to D_{qc}(M)$ defined by

$$\Phi_E(-) = \mathbf{R}\,\pi_{M*}(E \overset{\mathbf{L}}{\otimes} \pi_N^*(-)). \tag{3.1}$$

The object E is called *the kernel* of the transform. If Φ_E is an equivalence of categories it is further called *a Fourier-Mukai transform*.

If a group G acts on N and M then, for any $E \in D^G_{qc}(N \times M)$ of finite Tor-dimension, (3.1) defines an integral transform $D^G_{qc}(N) \to D^G_{qc}(M)$. If the group action on N is trivial denote by $(-\otimes\rho_0)$ the functor $D_{qc}(N) \to D^G_{qc}(N)$ which gives a sheaf the trivial G-equivariant structure. It is exact and has an exact left and right adjoint $(-)^G$, the functor of taking the G-invariant part ([BKR01], Section 4.2). We also use the terms *integral* and *Fourier-Mukai transform* for the functors $D_{qc}(N) \to D^G_{qc}(M)$ of the form $\Phi_E(- \otimes \rho_0)$ where Φ_E is some integral transform $D^G_{qc}(N) \to D^G_{qc}(M)$.

When N and M are smooth and proper varieties it is well known that Φ_E has a left adjoint $\Phi_{E^\vee \otimes \pi_M^*(\omega_M)[\dim M]}$ ([BO95], Lemma 1.2). The lemma below allows to generalise this to certain integral transforms between non-proper schemes. We use methods of Verdier-Deligne as per the exposition in [Del66] to which we refer the reader for all the necessary definitions.

Lemma 4. *Let N and M be separable schemes of finite type over $\mathbb{C}$ with M smooth of dimension n. Let $E \in D(N \times M)$ be of finite homological dimension with* $\mathrm{Supp}(E)$ *proper over N. Then the functor*

$$\pi_N^*(-) \overset{\mathbf{L}}{\otimes} E : \quad D(N) \to D(N \times M)$$

has a left adjoint

$$\mathbf{R}\,\pi_{N*}(- \overset{\mathbf{L}}{\otimes} E^{\vee} \otimes \pi_M^*(\omega_M))[n] : \quad D(N \times M) \to D(N). \tag{3.2}$$

Proof. First we compactify M: choose an open immersion $M \hookrightarrow \bar{M}$ with $\bar{M}$ smooth and proper [Nag]. Then π_N decomposes as an open immersion $\iota : N \times M \hookrightarrow N \times \bar{M}$ followed by the projection $\bar{\pi}_N : N \times \bar{M} \to N$. As $\bar{\pi}_N$ is smooth and proper Grothendieck-Serre duality for smooth and proper morphisms (e.g. [Har66], VII4.3) implies that $\bar{\pi}_N^* : D(N) \to D(N \times \bar{M})$ has a left adjoint

$$\mathbf{R}\,\bar{\pi}_{N*}(-) \otimes \bar{\pi}_M^* \omega_{\bar{M}}[n]$$

where $\bar{\pi}_M : \; N \times \bar{M} \to \bar{M}$ is the projection onto the second component.

By the duality for open immersions ([Del66], Prop. 4) the left adjoint to the (exact) functor $\iota^*(-)$ exists as an (exact) functor $\iota_!$ from $\mathrm{Coh}(N \times M)$ to the category $\text{pro-}\mathrm{Coh}(N \times \bar{M})$. For the definition of $\text{pro-}\mathrm{Coh}(N \times \bar{M})$ and the generalities on pro-objects see [Del66], n° 1. The functor $\iota_!$ may be calculated as follows: given $\mathcal{A} \in \mathrm{Coh}(N \times M)$ take any $\bar{\mathcal{A}} \in \mathrm{Coh}(N \times \bar{M})$ which restricts to $\mathcal{A}$ on $N \times M$. Then

$$\iota_!(\mathcal{A}) = \varinjlim \mathrm{Hom}(\mathcal{I}^n \bar{\mathcal{A}}, -) \tag{3.3}$$

where $\mathcal{I}$ is the ideal sheaf defining the complement $N \times (\bar{M} \setminus M)$.

Finally, as E is of finite homological dimension, the left adjoint of $(-) \overset{\mathbf{L}}{\otimes} E$ is $(-) \overset{\mathbf{L}}{\otimes} E^{\vee}$ where $E^{\vee}$ is $\mathbf{R}\,\mathcal{H}om(E, \mathcal{O}_{N \times M})$.

Therefore the left adjoint of $\pi_N^*(-) \overset{\mathbf{L}}{\otimes} E$ exists as the functor

$$\mathbf{R}\,\bar{\pi}_{N*}(\iota_!(- \overset{\mathbf{L}}{\otimes} E^{\vee}) \otimes \bar{\pi}_M^*(\omega_M))[n] \tag{3.4}$$

from $\text{pro-}D(N \times M)$ to $\text{pro-}D(N)$. To finish the proof it suffices now to show that $\iota_!(- \overset{\mathbf{L}}{\otimes} E^{\vee}) = \iota_*(- \overset{\mathbf{L}}{\otimes} E^{\vee})$. Then applying the projection formula to $\iota_*(- \overset{\mathbf{L}}{\otimes} E^{\vee}) \otimes \bar{\pi}_M^*(\omega_M)$ in (3.4) and observing that $\iota \circ \bar{\pi}_M = \pi_M$ and $\iota \circ \bar{\pi}_N = \pi_N$ yields (3.2).

We have $\mathrm{Id} = \iota^* \iota_*$ on $\mathrm{QCoh}(N \times M)$ ([GD60], *Prop. 9.4.2*). It induces by the adjunction of [Del66], Prop. 4 natural transformations $\Upsilon : \; \iota_! \to \iota_*$ of functors $\mathrm{Coh}(N \times M) \to \text{pro-}\mathrm{QCoh}(N \times \bar{M})$ and $\Upsilon' : \; \iota_!(- \overset{\mathbf{L}}{\otimes} E^{\vee}) \to \iota_*(- \overset{\mathbf{L}}{\otimes} E^{\vee})$ of functors $D(N \times M) \to \text{pro-}D(N \times \bar{M})$. By [Del66], Prop. 3 and the exactness of $\iota_!$ and ι_*, to show Υ' to be an isomorphism of functors it suffices to show that Υ is an isomorphism on the cohomology sheaves of

$- \overset{\mathbf{L}}{\otimes} E^\vee$. The support of these is proper over N by the assumption on E. For any $\mathcal{A} \in \mathrm{Coh}(N \times M)$ we have

$$\mathrm{Hom}(\iota_!(\mathcal{A}), \iota_*(\mathcal{A})) = \varinjlim \mathrm{Hom}_{N \times \bar{M}}(\mathcal{I}^k \bar{\mathcal{A}}, \iota_*(\mathcal{A})) \tag{3.5}$$

using the notation of (3.3). From the construction of the adjunction in [Del66], Prop. 4 it is immediate that $\Upsilon(\mathcal{A})$ is the unique element of RHS of (3.5) which restricts to $N \times M$ as $\mathrm{Id} \in \mathrm{Hom}_{N \times M}(\mathcal{A}, \mathcal{A})$. If $\mathrm{Supp}(\mathcal{A})$ is proper over N, we can take $\bar{\mathcal{A}} = \iota_* \mathcal{A}$ in (3.3). Moreover, $\mathcal{I}^k \iota_*(\mathcal{A}) = \iota_*(\mathcal{A})$ for all k. Therefore (3.3) yields $\iota_!(\mathcal{A}) = \iota_*(\mathcal{A})$ and moreover the RHS of (3.5) is just $\mathrm{Hom}(\iota_* \mathcal{A}, \iota_* \mathcal{A})$. It is then clear that $\Upsilon(\mathcal{A}) = \mathrm{Id}$, as required.

3.2. G-constellations and gnat-families

Definition 4. *Let G be a finite subgroup of* $\mathrm{GL}_n(\mathbb{C})$. *A G-constellation* is a coherent G-sheaf $\mathcal{V}$ on $\mathbb{C}^n$ whose global sections $\Gamma(\mathcal{V})$ have the G-representation structure of the regular representation V_{reg}.

Two G-constellations $\mathcal{V}, \mathcal{W}$ are *orthogonal in degree k* if $G\text{-}\mathrm{Ext}^k_{\mathbb{C}^n}(\mathcal{V}, \mathcal{W}) = G\text{-}\mathrm{Ext}^k_{\mathbb{C}^n}(\mathcal{W}, \mathcal{V}) = 0$.

Let now Y be a scheme of finite type over $\mathbb{C}$. We endow Y with the trivial G-action, thus we can speak of G-sheaves on Y and on $Y \times \mathbb{C}^n$.

Definition 5. *A gnat-family on Y* (short for *G-natural* or *geometrically natural*) is an object $\mathcal{F}$ of $\mathrm{Coh}^G(Y \times \mathbb{C}^n)$, flat over Y, such that for every closed $y \in Y$ the fiber $\mathcal{F}_{|y}$ is a G-constellation supported on a single G-orbit. The *Hilbert-Chow map* $\pi_{\mathcal{F}}$ of $\mathcal{F}$ is the map $Y \to \mathbb{C}^n/G$ defined by $y \mapsto \mathrm{Supp}_{\mathbb{C}^n} \mathcal{F}_{|y}$. A *gnat*-family on a fixed morphism $Y \xrightarrow{\pi} \mathbb{C}^n/G$ is a *gnat*-family on Y whose Hilbert-Chow map coincides with π.

Two subsets C and C' of Y are *orthogonal in degree k in $\mathcal{F}$* if for every $y \in C$ and $y' \in C'$ the fibers $\mathcal{F}_{|y}$ and $\mathcal{F}_{|y'}$ are orthogonal in degree k. The family $\mathcal{F}$ is *orthogonal in degree k* if Y is orthogonal to Y in degree k in $\mathcal{F}$.

Proposition 2. *For any gnat-family $\mathcal{F}$ its Hilbert-Chow map $\pi_{\mathcal{F}}$ is a morphism.*

Proof. Denote by R the ring $\mathbb{C}[x_1, \dots, x_n]$. For any G-constellation $\mathcal{V}$, the action of R on $H^0(\mathcal{V})$ restricts to the action of R^G on $H^0(\mathcal{V})^G$. Clearly

$$(\mathrm{Ann}_R H^0(\mathcal{V}))^G \subseteq \mathrm{Ann}_{R^G} H^0(\mathcal{V})^G. \tag{3.6}$$

The LHS of (3.6) is the image of $\mathrm{Supp}_{\mathbb{C}^n} \mathcal{V}$ in $\mathbb{C}^n/G$. If this support is a single G-orbit, then $(\mathrm{Ann}_R H^0(\mathcal{V}))^G$ is maximal in R^G and (3.6) is an equality. Therefore it suffices to construct a morphism $Y \to \mathbb{C}^n/G$ which sends each $y \in Y$ to $\mathrm{Ann}_{R^G} H^0(\mathcal{F}_{|y})^G$. We construct it thus: the invariant part of $\pi_{Y*}(\mathcal{F})$

is a line bundle on Y, which has a R^G-module structure induced from $\mathcal{F}$. This structure defines a homomorphism $R^G \rightarrow \mathcal{O}_Y$. The corresponding morphism $Y \rightarrow \mathbb{C}^n/G$ is easily seen to send each $y \in Y$ to $\operatorname{Ann}_{R^G} H^0(\mathcal{F}_{|y})^G$.

Lemma 5. *If $\mathcal{F}$ is a gnat-family on Y and $\pi_{\mathcal{F}} : Y \rightarrow \mathbb{C}^n/G$ is proper, then $\mathcal{F}$ is of finite homological dimension in $D^G(Y \times \mathbb{C}^n)$ and the integral transform $\Phi_{\mathcal{F}} : D^G_{qc}(Y) \rightarrow D^G_{qc}(\mathbb{C}^n)$ restricts to $D^G(Y) \rightarrow D^G(\mathbb{C}^n)$.*

Proof. Let ι be the open immersion $Y \times \mathbb{C}^n \rightarrow Y \times \mathbb{P}^n$. As $\operatorname{Supp} \mathcal{F}$ is proper over Y, $\iota_* \mathcal{F}$ is coherent. Quite generally, given any coherent sheaf $\mathcal{A}$ on $Y \times \mathbb{P}^n$ flat over Y, consider the adjunction co-unit $\xi : \pi_Y^* \pi_{Y*} \mathcal{A} \rightarrow \mathcal{A}$. As π_Y is proper and $\mathcal{A}$ is flat over Y, $\pi_Y^* \pi_{Y*} \mathcal{A}$ is lffr (locally free of finite rank). Twisting by some power of $\pi_{\mathbb{P}^n}^* \mathcal{O}(1)$ we can make ξ surjective. But then $\ker \xi$ is again coherent and flat over Y. We set initially $\mathcal{A} = \iota_* \mathcal{F}$ and repeat this construction until $\ker \xi$ becomes lffr. This has to happen eventually as $\iota_* \mathcal{F}$ is flat over Y and $\mathbb{P}^n$ is smooth. Thus we obtain an lffr resolution of $\iota_* \mathcal{F}$ of finite length. Restricting it to $Y \times \mathbb{C}^n$ demonstrates the first claim.

For the second claim: since π_Y is flat, the pullback $\pi_Y^*(- \otimes \rho_0)$ is exact and takes $D(Y)$ to $D^G(Y \times \mathbb{C}^n)$. Since $\mathcal{F}$ is of finite homological dimension, $\mathcal{F} \overset{\mathbf{L}}{\otimes} -$ takes $D^G(Y \times \mathbb{C}^n)$ to $D^G(Y \times \mathbb{C}^n)$. Moreover the image $\operatorname{Im}(\mathcal{F} \overset{\mathbf{L}}{\otimes} -)$ lies in the full subcategory of $D^G(Y \times \mathbb{C}^n)$ consisting of the objects with support in $\operatorname{Supp} \mathcal{F}$. Finally, $\pi_{\mathcal{F}}$ being proper implies that $\operatorname{Supp} \mathcal{F}$ is proper over $\mathbb{C}^n$, hence $\mathbf{R}\, \pi_{\mathbb{C}^n *}$ takes $\operatorname{Im}(\mathcal{F} \overset{\mathbf{L}}{\otimes} -)$ to $D^G(\mathbb{C}^n)$ ([GD61], *Corollaire 3.2.4*).

The following demonstrates a certain relevance of gnat-families:

Proposition 3. *Let G be a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$, Y a variety and $E \in D^G(Y \times \mathbb{C}^n)$ an object such that $\Phi_E(- \otimes \rho_0)$ is an equivalence $D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^n)$ which sends point sheaves on Y to pure sheaves. Then E is a gnat-family over Y and its Hilbert-Chow map π_E is a crepant resolution of $\mathbb{C}^n/G$. Moreover*

$$G\text{-}\operatorname{Ext}^i(E_{|y_1}, E_{|y_2}) = \begin{cases} \mathbb{C} & \textit{if } y_1 = y_2, i = 0 \\ 0 & \textit{if } y_1 \neq y_2 \end{cases} \tag{3.7}$$

and for any $y \in Y$ the (Kodaira-Spencer) map $\operatorname{Ext}^1(\mathcal{O}_y, \mathcal{O}_y) \rightarrow G\text{-}\operatorname{Ext}^1(E_{|y}, E_{|y})$ is an isomorphism.

Proof. By [Huy06], Example 5.1(vi), $E_{|y} = \Phi_E(\mathcal{O}_y \otimes \rho_0)$, whence the assertion (3.7) and the Kodaira-Spencer maps being isomorphisms. By [Bri99], Lemma 4.3, it follows that E is a pure sheaf flat over Y. Then by Lemma 4 the inverse of $\Phi_E(- \otimes \rho_0)$ is $\Phi_{E^\vee[n]}(-)^G$. It maps $\mathcal{O}_{\mathbb{C}^n}$ to $(\pi_{Y*} E^\vee[n])^G$, so the cohomology sheaves of $(\pi_{Y*} E^\vee[n])^G$ are coherent $\mathcal{O}_Y$-modules. Since π_{Y*} is affine, the support of $E^\vee[n]$ is finite over Y. As $\operatorname{Supp}(E^\vee[n]) = \operatorname{Supp} E$, we conclude that for each $y \in Y$ the support of $E_{|y}$ is a finite union of G-orbits. The simplicity of $E_{|y}$ further implies that it has to be a single G-orbit. To show that $\Gamma(E_{|y})$ has G-representation structure of V_{reg} it suffices, by flatness of E, to show it for any single $y \in Y$. As the set $\{E_{|y}\}_{y\in Y}$ is an image of a spanning class of $D(Y)$ under $\Phi(- \otimes \rho_0)$, it is a spanning class for $D^G(\mathbb{C}^n)$. Hence for every G-orbit Z in $\mathbb{C}^n$ there exists $y \in Y$ such that $E_{|y}$ is supported at Z. Choose Z to be any free orbit. The only simple G-sheaf supported on a free orbit is its structure sheaf, therefore $\Gamma(E_{|y}) \simeq V_{\text{reg}}$. We conclude that E is a gnat-family and that π_E is surjective and an isomorphism outside the singularities of $\mathbb{C}^n/G$. By [Rob98], Theorem 6.2.2 and [BKR01], Lemma 3.1, Y is smooth and π_E is crepant. It remains to show that π_E is proper, which is equivalent to $\operatorname{Supp}_{Y\times\mathbb{C}^n} E$ being proper over $\mathbb{C}^n$ and that follows, e.g., from $\pi_{\mathbb{C}^n *}E$ having to be coherent, as it is a cohomology sheaf of the complex $\Phi_E(\mathcal{O}_Y \otimes \rho_0)$.

3.3. *Main results*

We now give the proof of Theorem 1. Its general framework follows those of [BO95], Theorem 1.1 and of [BKR01], Theorem 1.1. We note two principal differences: [BO95] works with smooth varieties, while we assume Y to be a not necessarily smooth scheme (whence the content of Section 2); [BKR01] adopts a two-step strategy to establish the left adjoint of $\Phi_{\mathcal{F}}(- \otimes \rho_0)$, whereas our Lemma 4 achieves this directly.

Proof (Proof of Theorem 1).

We divide the proof into five steps:

Step 1: We claim that $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ has a left adjoint $(\Psi_{\mathcal{F}})^G$, where $\Psi_{\mathcal{F}}$ is a certain integral transform $D^G(\mathbb{C}^n) \to D^G(Y)$.

Recall that $\Phi_{\mathcal{F}} = \mathbf{R}\,\pi_{\mathbb{C}^n *}(\mathcal{F} \overset{\mathbf{L}}{\otimes} \pi_Y^*(-))$. The issue here is the left adjoint of $\pi_Y^*(-)$ as π_Y, though smooth, is manifestly non-proper. But the support of $\mathcal{F}$ is proper, so by Lemma 4 the functor $\mathbf{R}\,\pi_{Y*}(- \overset{\mathbf{L}}{\otimes} \mathcal{F}^\vee[n])$ is the left adjoint to $\pi_Y^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{F}$. The claim now follows, for $\pi_{\mathbb{C}^n}^*$ is the left adjoint to $\mathbf{R}\,\pi_{\mathbb{C}^n *}$ and $(-)^G$ is the left (and right) adjoint of $- \otimes \rho_0$.

Step 2: We claim that the composition $(\Psi_{\mathcal{F}})^G \circ \Phi_{\mathcal{F}}(- \otimes \rho_0)$ is an integral transform Φ_Q for some $Q \in D(Y \times Y)$ and that for any closed point (y_1, y_2) in $Y \times Y$ and any $k \in \mathbb{Z}$ we have

$$\mathbf{L}^k\,\iota_{y_1,y_2}^* Q = G\text{-}\operatorname{Ext}^k(\mathcal{F}_{|y_1}, \mathcal{F}_{|y_2})^\vee. \tag{3.8}$$

The first assertion is a standard result due to Mukai in [Muk81], Proposition 1.3. The second assertion follows from the formula (5) of [BKR01], Sec. 6, Step 2 by the adjunction of $\mathbf{L}\,\iota_{y_1,y_2}^*$ and $\iota_{y_1,y_2 *}$.

Step 3: We claim that Q is a pure sheaf and that its support lies within the diagonal $Y \xrightarrow{\Delta} Y \times Y$.

First note that since $Y \times Y$ is of finite type over $\mathbb{C}$, it is certainly Jacobson (see [GD66], §10.3) and so any closed set of $Y \times Y$ is uniquely identified by its set of closed points. We implicitly use this property at several points of the argument below.

Recall the closed set N_k of (1.2). As the support of any G-constellation is proper and as $\omega_{\mathbb{C}^n} = \mathcal{O}_{\mathbb{C}^n} \otimes \rho_0$ as a G-sheaf since $G \subseteq \mathrm{SL}_n(\mathbb{C})$, Serre duality applies to yield

$$G\text{-}\operatorname{Ext}^k_{\mathbb{C}^n}(\mathcal{F}_{|y_1}, \mathcal{F}_{|y_2}) = G\text{-}\operatorname{Ext}^{n-k}_{\mathbb{C}^n}(\mathcal{F}_{|y_2}, \mathcal{F}_{|y_1})^\vee.$$

It follows that $\operatorname{codim} N_k = \operatorname{codim} N_{n-k}$ for all k.

Let C be an irreducible component of $\operatorname{Supp} Q$. Denote by y_C its generic point, by $\mathcal{O}_C$ the local ring of y_C and by Q_C the localisation of Q to $\mathcal{O}_C$. For any k denote by M_k the set $\{y \in Y \times Y \mid \mathbf{L}^k \iota_y^* Q \neq 0\}$ and let l and m be the infimum and the supremum of the set $\{k \in \mathbb{Z} \mid y_C \in M_k\}$, hence $\operatorname{Tor-amp}_{\mathcal{O}_C} Q_C = m - l$ (Lemma 3). By (3.8) the closure of $M_l \setminus \Delta$ is N_l, so $y_C \in M_l$ implies $y_C \in \Delta$ or $y_C \in N_l$. Similarly for N_m. Thus either $y_C \in \Delta$ or $y_C \in N_l \cap N_m$. The latter would imply that

$$\operatorname{codim} C \geq \operatorname{codim} N_l \geq n - 2l + 1$$
$$\operatorname{codim} C \geq \operatorname{codim} N_m = \operatorname{codim} N_{n-m} \geq 2m - n + 1$$

and therefore that $\operatorname{codim} C \geq m - l + 1$. But then $\operatorname{codim} C$ would be strictly greater than $\operatorname{Tor-amp}_{\mathcal{O}_C} Q_C$, which contradicts Theorem 2. Thus y_C lies within Δ and, since Y is separated, so does all of C.

We have now shown that $\operatorname{Supp} Q \subseteq \Delta$, so $\operatorname{codim} \operatorname{Supp} Q \geq n$. But as $\mathbb{C}^n$ is smooth and n-dimensional, (3.8) implies

$$\mathbf{L}^k \iota_y^* Q = 0 \qquad \forall y \in Y,\ k \notin 0, \ldots, n \tag{3.9}$$

so $\operatorname{Tor-amp} Q \leq n$. By Theorem 2 $\operatorname{Tor-amp} Q = n$ and $\operatorname{coh-amp} Q = 0$. Together with (3.9) this implies that Q is a pure sheaf.

Step 4: We claim that Q is the structure sheaf $\mathcal{O}_\Delta$ of the diagonal Δ and therefore $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is fully faithful.

The adjunction co-unit $\Phi_Q \to \mathrm{Id}_{D(Y)}$ induces a surjective $\mathcal{O}_{Y\times Y}$-module morphism $Q \xrightarrow{\epsilon} \mathcal{O}_\Delta$. Let K be its kernel, we then have a short exact sequence

$$0 \to K \to Q \xrightarrow{\epsilon} \mathcal{O}_\Delta \to 0. \tag{3.10}$$

Choosing some closed point $(y, y) \in \Delta$ and applying functor $\mathbf{L}\,\iota^*_{y,y}(-)$ to (3.10) we obtain a long exact sequence of $\mathbb{C}$-modules

$$\cdots \to G\text{-}\operatorname{Ext}^1_{\mathbb{C}^n}(\mathcal{F}_{|y}, \mathcal{F}_{|y})^* \xrightarrow{\alpha_y} \Omega^1_{Y,y} \to K_{y,y} \to G\text{-}\operatorname{End}_{\mathbb{C}^n}(\mathcal{F}_{|y})^* \xrightarrow{\epsilon_y} \mathbb{C} \to 0 \to \ldots.$$

The map ϵ_y is surjective due to any G-constellation having automorphisms consisting of scalar multiplication. It is an isomorphism whenever $\mathcal{F}_{|y}$ is simple, i.e. when the scalar multiplication automorphisms are all we get. The map α_y is the dual of the Kodaira-Spencer map of $\mathcal{F}$ at $y \in Y$, which takes a tangent vector at y to the infinitesimal deformation in that direction in the family $\mathcal{F}$. Hence for any $y \in Y$, such that $\mathcal{F}_{|y}$ is simple and such that the Kodaira-Spencer map of $\mathcal{F}$ is injective at y, the long exact sequence above shows that $K|_{y,y} = 0$.

Having proved that $\operatorname{Supp} Q \subseteq \Delta$ we have proved by (3.8) that any two G-constellations in $\mathcal{F}$ are orthogonal. Denoting by q the quotient map $\mathbb{C}^n \to \mathbb{C}^n/G$ we claim that for any closed point $x \in \mathbb{C}^n/G$, such that $q^{-1}(x)$ is a free orbit of G, the fiber $\pi_{\mathcal{F}}^{-1}(x)$ consists of at most a single point. This is because, by definition of $\pi_{\mathcal{F}}$, all the G-constellations parametrised by $\pi_{\mathcal{F}}^{-1}(x)$ are supported on $q^{-1}(x)$ - and any two G-constellations supported at the same free orbit are easily seen to be isomorphic. Thus $\pi_{\mathcal{F}}$ is an isomorphism on the smooth locus X_0 of $\mathbb{C}^n/G$. By [Log06], Proposition 1.5 the family $\mathcal{F}$ on X_0 (identified with an open subset of Y via $\pi_{\mathcal{F}}$) is locally isomorphic to the canonical G-cluster family $q_*\mathcal{O}_{\mathbb{C}^n}|_{X_0}$. As any G-cluster is simple and as the Kodaira-Spencer map of $q_*\mathcal{O}_{\mathbb{C}^n}|_{X_0}$ is trivially injective $K|_{y,y} = 0$ for any $y \in X_0$. Therefore $\operatorname{codim}_{Y\times Y} \operatorname{Supp} K \geq n+1$, as X_0 is open in Δ.

On the other hand, since $\operatorname{Tor-amp} Q = \operatorname{Tor-amp} \mathcal{O}_\Delta = n$, the short exact sequence (3.10) implies that $\operatorname{Tor-amp} K \leq n$. As that is smaller than the codimension of its support, $K = 0$ by Theorem 2. Thus $Q \simeq \mathcal{O}_\Delta$, the adjunction co-unit is an isomorphism and $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is fully faithful.

Step 5: We claim that $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is an equivalence of categories.

As $D(Y)$ is fully faithfully embedded in $D^G(\mathbb{C}^n)$ the trivial Serre functor of the latter induces a trivial Serre functor on the former. Therefore the left adjoint to $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is also its right adjoint. Then $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is an equivalence of categories by [Bri99], Theorem 3.3.

Proof (Proof of Corollary 1).

It suffices to demonstrate that $\mathcal{F}$ satisfies the condition of Theorem 1. Thus we have to show that $\operatorname{codim} N_0 \geq 4$ and $\operatorname{codim} N_1 \geq 2$. But, as seen in the proof of Theorem 1, N_k lies within the fibre product $Y \times_{\mathbb{C}^3/G} Y$ for all k. As $\pi_{\mathcal{F}}$ is birational its fibres are at most divisors and so the codimension of $Y \times_{\mathbb{C}^3/G} Y$ is at least 2.

It remains to show that $N_0 \geq 4$. The assumptions of the Corollary ensure that N_0 is contained in the union of all sets of form $(E_i \cap E_j) \times (E_k \cap E_l)$ or $E_i \times (E_i \cap E_j \cap E_k)$, and the codimension of each of these sets is 4.

4. Orthogonality in degree zero

Throughout this section we denote by G a finite abelian subgroup of $\mathrm{SL}_n(\mathbb{C})$, by Y a smooth scheme of finite type over $\mathbb{C}$ and by $\mathcal{F}$ a $gnat$-family on Y. We assume that the Hilbert-Chow morphism $\pi_{\mathcal{F}}$ associated to $\mathcal{F}$ is birational and proper. The main purpose of this section is to show how, given any pair of closed points of Y, one checks whether the corresponding pair of G-constellations are orthogonal in degree 0.

We denote by V_{giv} the representation of G given by its inclusion into $\mathrm{SL}_n(\mathbb{C})$. The (left) action of G on V_{giv} induces a right action of G on ${V_{\text{giv}}}^{\vee}$ which we make into a left action by setting:

$$g \cdot f(v) = f(g^{-1} \cdot v) \qquad \text{for all } v \in V_{\text{giv}},\ f \in {V_{\text{giv}}}^{\vee},\ g \in G. \tag{4.1}$$

We denote by $x_1, \dots, x_n$ the common eigenvectors of the action of G on ${V_{\text{giv}}}^{\vee}$. We denote by R the symmetric algebra $S({V_{\text{giv}}}^{\vee})$ with the induced left action of G. Then $R = \mathbb{C}[x_1, \dots, x_n]$ and as an affine G-scheme $\mathbb{C}^n$ is $\operatorname{Spec} R$. We denote by $G^{\vee}$ the character group $\operatorname{Hom}(G, \mathbb{C}^*)$ of G. A rational function $f \in K(\mathbb{C}^n)$ is said to be G-homogeneous of weight $\chi \in G^{\vee}$ if we have $f(g.v) = \chi(g)\ f(v)$ for all $v \in \mathbb{C}^n$ where f is defined. We denote by $\rho(f)$ the weight χ of such f. It follows from (4.1) that G acts on f by $\rho(f)^{-1}$.

From here on we employ freely the terminology and the results of [Log06].

4.1. The McKay quiver of G

By a *quiver* we mean a vertex set Q_0, an arrow set Q_1 and a pair of maps $h\colon Q_1 \to Q_0$ and $t\colon Q_1 \to Q_0$ giving the head $hq \in Q_0$ and the tail $tq \in Q_0$ of each arrow $q \in Q_1$. By a *representation of a quiver* we mean a graded vector space $\bigoplus_{i \in Q_0} V_i$ and a collection of linear maps $\{\alpha_q\colon V_{tq} \to V_{hq}\}_{q \in Q_1}$.

Definition 6. *The McKay quiver of G* is the quiver whose vertex set Q_0 are the irreducible representations ρ of G and whose arrow set Q_1 has $\dim \operatorname{Hom}_G(\rho_i, \rho_j \otimes V_{\text{giv}})$ arrows going from the vertex ρ_i to the vertex ρ_j.

We have ${V_{\text{giv}}}^{\vee} = \bigoplus \mathbb{C} x_i$, as G-representations. Denote by U_χ the 1-dimensional representation on which G acts by $\chi \in G^{\vee}$. By Schur's lemma

$$G\text{-}\operatorname{Hom}(U_{\chi_i} \otimes {V_{\text{giv}}}^{\vee}, U_{\chi_j}) = \begin{cases} \mathbb{C} & \text{if } \chi_j = \chi_i \rho(x_k)^{-1} \quad k \in \{1, \dots, n\} \\ 0 & \text{otherwise} \end{cases} .$$

Thus each vertex χ of the McKay quiver of G has n arrows emerging from it and going to vertices $\chi\rho(x_k)^{-1}$ for $k = 1, \dots, n$. We denote the arrow from χ to $\chi\rho(x_k)^{-1}$ by (χ, x_k). Let now A be a G-constellation viewed as an $R \rtimes G$-module ([Log06], Section 1.1) and let $\oplus A_\chi$ be its decomposition into irreducible representations of G. Then the $R \rtimes G$-module structure on A defines a representation of the McKay quiver into the graded vector space $\oplus A_\chi$, where the map α_{χ, x_k} is just the multiplication by x_k, i.e.

$$\alpha_{\chi, x_k} :\ A_\chi \to A_{\chi\rho(x_k)^{-1}},\ v \mapsto x_k \cdot v. \tag{4.2}$$

4.2. *Degree* 0 *orthogonality of G-constellations*

Let A and A' be two G-constellations and ϕ be an $R \rtimes G$-module morphism $A \to A'$. Let $\bigoplus_{G^\vee} A_\chi$ and $\bigoplus_{G^\vee} A'_\chi$ be decompositions of A and A' into one-dimensional representations of G. By G-equivariance ϕ decomposes into linear maps $\phi_\chi : \; A_\chi \to A'_\chi$.

Let $\{\alpha_q\}$ and $\{\alpha'_q\}$ be the corresponding representations of the McKay quiver into graded vector spaces $\oplus A_\chi$ and $\oplus A'_\chi$, as per (4.2). Each α_q is a linear map between one-dimensional vector spaces A_{tq} and A_{hq} and so is either a zero-map or an isomorphism, similarly for the maps α'_q. So for each arrow of the McKay quiver we distinguish the following four possibilities:

Definition 7. *Let q be an arrow of McKay quiver of G. With the notation above we say that with respect to an ordered pair (A, A') of G-constellations the arrow q is:*

1. *a type $[1,1]$ arrow, if both α_q and α'_q are isomorphisms.*
2. *a type $[1,0]$ arrow, if α_q is an isomorphism and α'_q is a zero map.*
3. *a type $[0,1]$ arrow, if α_q is a zero map and α'_q is an isomorphism.*
4. *a type $[0,0]$ arrow, if both α_q and α'_q are zero maps.*

Proposition 4. *Let q and (A, A') be as in Definition 7 and let ϕ be any $R \rtimes G$-module morphism $A \to A'$. Then:*

1. *If q is a $[1,0]$ arrow, then $A_{hq} \subseteq \ker \phi$.*
2. *If q is a $[0,1]$ arrow, then $A_{tq} \subseteq \ker \phi$.*
3. *If q is a $[1,1]$ arrow, A_{tq} and A_{hq} either both lie in $\ker \phi$ or both don't.*

Proof. Write $q = (\chi, i)$ where $\chi \in G^\vee$ and $i \in \{1, \dots, n\}$. Recall that α_q is the map $A_{tq} \to A_{hq}$ corresponding to the action of x_i on A_{tq}. Then R-equivariance of the morphism ϕ implies a commutative square

$$\begin{array}{ccc} A_{hq} & \xrightarrow{\phi_{hq}} & A'_{hq} \\ {\scriptstyle \alpha_q}\uparrow & & \uparrow{\scriptstyle \alpha'_q} \\ A_{tq} & \xrightarrow[\phi_{tq}]{} & A'_{tq} \end{array}$$

from which all three claims immediately follow.

Corollary 2. *Let (A, A') be an ordered pair of G-constellations. If every component of the McKay quiver path-connected by $[1,1]$-arrows has either a $[0,1]$-arrow emerging from it or a $[1,0]$-arrow entering it, then*

$$\operatorname{Hom}_{R \rtimes G}(A, A') = 0.$$

If, also, every component has either a $[0,1]$-arrow entering it or a $[1,0]$-arrow emerging from it, then we further have

$$\mathrm{Hom}_{R\rtimes G}(A', A) = 0$$

and therefore A and A' are orthogonal in degree 0.

4.3. *Divisors of zeroes*

The Hilbert-Chow morphism $\pi_{\mathcal{F}} : Y \to \mathbb{C}^n/G$ is birational, thus it defines a notion of G-Cartier and G-Weil divisors on Y ([Log06]), Section 2). The family $\mathcal{F}$, in a sense of a sheaf of $\mathcal{O}_Y \otimes (R \rtimes G)$-modules on Y, can be written as $\bigoplus_{\chi \in G^\vee} \mathcal{L}(-D_\chi)$, where D_χ are G-Weil divisors. For any other such expression $\bigoplus \mathcal{L}(-D'_\chi)$ of $\mathcal{F}$ there exist $f \in K(Y)$ such that $D'_\chi = D_\chi + (f)$ for all $\chi \in G^\vee$ ([Log06], Section 3.1).

Definition 8. *Let $q = (\chi, x_k)$ be an arrow in the McKay quiver of G. We define the divisor of zeroes B_q* of q in $\mathcal{F}$ to be the Weil divisor

$$D_{\chi^{-1}} + (x_i) - D_{\chi^{-1}\rho(x_i)}. \tag{4.3}$$

Note that B_q is always an ordinary, integral Weil divisor on Y.

Proposition 5. *Let (χ, x_k) be an arrow in the McKay quiver of G and B_{χ,x_k} be its divisor of zeroes in $\mathcal{F}$. Let y be a closed point of Y and A be the G-constellation $\mathcal{F}_{|y}$. Then in the corresponding representation $\{\alpha_q\}_{q\in Q_1}$ of the McKay quiver the map α_{χ,x_k} is a zero map if and only if $y \in B_{\chi,x_k}$.*

Proof. The map $\alpha_{\chi,x_k} : A_\chi \to A_{\chi\rho(x_k)^{-1}}$ is the action of x_k on A_χ. This map is the restriction to the point y of the global section β of the $\mathcal{O}_Y$-module

$$Hom_{G,\mathcal{O}_Y}(\mathcal{O}_Y x_k \otimes \mathcal{F}_\chi, \mathcal{F}_{\chi\rho^{-1}(x_k)}) \tag{4.4}$$

defined by $x_k \otimes s \mapsto x_k \cdot s$ for any section s of the χ-eigensheaf $\mathcal{F}_\chi$.

As G acts on a monomial of weight χ by χ^{-1} the χ-eigensheaf of $\mathcal{F}$ is $\mathcal{L}(-D_{\chi^{-1}})$. Hence (4.4) is canonically isomorphic to the following sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^n)$:

$$\mathcal{L}(D_{\chi^{-1}} + (x_k) - D_{\chi^{-1}\rho(x_k)}) \tag{4.5}$$

and the isomorphism maps β to the global section $1 \in K(\mathbb{C}^n)$ of (4.5). Which vanishes precisely on the Weil divisor $B_{\chi,x_k} = D_{\chi^{-1}} + (x_k) - D_{\chi^{-1}\rho(x_k)}$.

Proposition 5 together with Corollary 2 show that the data of the divisors of zeroes of $\mathcal{F}$ is all that is necessary to determine whether any given pair of closed points of Y are orthogonal in degree 0 in $\mathcal{F}$.

4.4. *Direct transforms*

Let Y' and Y'' be two crepant resolutions of $\mathbb{C}^n/G$ isomorphic outside of a closed set of codimension ≥ 2. E.g. the case $n = 3$ where all crepant resolutions are related by a chain of flops ([Kol89]). We fix a birational isomorphism and use it to identify Y' and Y'' along the isomorphism locus U. Since the complement of U is of codimension ≥ 2 in Y' (resp. Y'') any line bundle or divisor on U extends uniquely to a line bundle or a divisor on Y' (resp. Y''). The same is true of a family of G-constellations as for G abelian any such family is a direct sum of line bundles. For any family $\mathcal{V}'$ of G-constellations on Y' we define its *direct transform* $\mathcal{V}''$ to Y'' to be the unique extension to Y'' of the restriction of $\mathcal{V}'$ to U. Observe that if $\mathcal{V}'$ is of form $\bigoplus_\chi \mathcal{L}(-D'_\chi)$ for some G-Weil divisors D'_χ on Y' then $\mathcal{V}''$ is the family $\bigoplus \mathcal{L}(-D''_\chi)$ where each D''_χ is the direct transform of D'_χ.

If $\mathcal{F}$ can be shown to be a direct transform of some everywhere orthogonal in degree 0 family $\mathcal{F}'$ on some Y', it greatly reduces the number of calculations necessary to determine the degree 0 non-orthogonality locus of $\mathcal{F}$. Let U be as above. As $\mathcal{F}$ is the direct transform of F', the restriction of $\mathcal{F}$ to $U \subset Y$ is isomorphic to the restriction of F' to $U \subset Y'$. So the calculations only have to be carried out for points in $Y \times Y \setminus U \times U$.

4.5. *Theta stability and $gnat$-families*

We recall basic facts about θ-stability for G-constellations, cf. [CI04], Section 2.1. Let $\mathbb{Z}(G) = \bigoplus_{\chi \in G^\vee} \mathbb{Z}\chi$ be the representation ring of G and set

$$\Theta = \{\theta \in \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}(G), \mathbb{Q}) \mid \theta(V_{\text{reg}}) = 0\}$$

For any $\theta \in \Theta$, a G-constellation A is θ-stable (resp. θ-semistable) if for every sub-$R \rtimes G$-module B of A we have $\theta(B) > 0$ (resp. $\theta(B) \geq 0$). We say that θ is generic if every θ-semistable G-constellation is θ-stable. This is equivalent to θ being non-zero on any proper subrepresentation of V_{reg}.

Let π be any proper birational morphism $Y \to \mathbb{C}^n/G$. A $gnat$-family $\mathcal{V}$ on $Y \xrightarrow{\pi} \mathbb{C}^n/G$ is *normalized* if $\mathcal{V}^G \simeq \mathcal{O}_Y$. Such $\mathcal{V}$ can be written uniquely as $\bigoplus_{\chi \in G^\vee} \mathcal{L}(-D_\chi)$ for some G-Weil divisors D_χ with $D_{\chi_0} = 0$ ([Log06], Cor. 3.5). Denote by $\mathfrak{E}$ the set of all prime Weil divisors on Y whose image in $\mathbb{C}^n/G$ is either a point or a coordinate hyperplane $x_i^{|G|} = 0$. As G is abelian, any branch divisor of $\mathbb{C}^n \to \mathbb{C}^n/G$, if it exists, is one of the hyperplanes $x_i^{|G|} = 0$. Hence, by [Log06], Prop. 3.14 and 3.15, each D_χ is of form $\sum_{E \in \mathfrak{E}} q_{\chi,E} E$. Denote by U the open subset of Y consisting of points lying on at most one divisor in $\mathfrak{E}$.

Definition 9. *Let θ be an element of Θ. We define a map*

$$w_\theta : \quad \left\{ \textit{normalized gnat-families on } Y \xrightarrow{\pi} \mathbb{C}^n/G \right\} \to \mathbb{Q}$$

by

$$w_\theta(\mathcal{V}) = \sum_{E \in \mathfrak{E}} \sum_{\chi \in G^\vee} \theta(\chi) q_{\chi,E}. \tag{4.6}$$

The domain of definition of w_θ is finite ([Log06], Corollary 3.16), so for any $\theta \in \Theta$ there is at least one normalized *gnat*-family maximizing w_θ.

Proposition 6. *Let $\mathcal{M}$ be any family which maximizes $w_\theta(\mathcal{M})$. Then for any point $y \in U$ the fiber of $\mathcal{M}$ at y is a θ-semistable G-constellation. If, moreover, θ is generic, then such family $\mathcal{M}$ is unique.*

Proof. Write $\mathcal{M}$ as $\bigoplus \mathcal{L}(-M_\chi)$. Suppose that the fiber of $\mathcal{M}$ is not θ-semistable at some $y \in U$. Denote this fiber by A, its decomposition into irreducible representations by $\bigoplus_{\chi \in G^\vee} A_\chi$ and the corresponding representation of the McKay quiver by $\{\alpha_q\}$. As A isn't θ-semistable there exists a non-empty proper subset I of $G^\vee$ such that $A' = \bigoplus_{\chi \in I} A_\chi$ is a sub-$R \rtimes G$-module of A and $\theta(A') < 0$. Denote by J the complement $G^\vee \setminus I$. Denote by $Q_{I \to J}$ the subset $\{q \in Q_1 \mid tq \in I, hq \in J\}$ of the arrow set Q_1 of the McKay quiver and similarly for $Q_{J \to I}$, $Q_{I \to I}$, $Q_{J \to J}$. Then A' being closed under the action of R implies that for any $q \in Q_{I \to J}$ the map α_q is a zero map. Which by Proposition 5 implies $y \in B_q$.

The support of each M_χ consists only of the prime divisors in $\mathfrak{E}$ ([Log06], Prop. 3.14 and 3.15). The same is true of the principal divisors (x_i) for their images in $\mathbb{C}^n/G$ are the coordinate hyperplanes $x_i^{|G|} = 0$. Therefore, by their defining equation (4.3), the support of each of the divisors of zeroes B_q of $\mathcal{M}$ consists also only of the prime divisors in $\mathfrak{E}$. As y lies on all B_q with $q \in Q_{I \to J}$, y must lie on at least one divisor in $\mathfrak{E}$. But, as $y \in U$, y also lies on at most one divisor in $\mathfrak{E}$. Denote this unique divisor by E, then

$$q \in Q_{I \to J} \Rightarrow E \subset B_q. \tag{4.7}$$

Define a new G-Weil divisor set $\{M'_\chi\}$ by setting M'_χ to be M_χ if $\chi \in I$ and $M_\chi + E$ if $\chi \in J$. Then divisors $\{B'_q\}$ defined from $\{M'_\chi\}$ by equations (4.3) can be expressed as

$$B'_q = \begin{cases} B_q & \text{if } q \in Q_{I \to I}, Q_{J \to J} \\ B_q + E & \text{if } q \in Q_{J \to I} \\ B_q - E & \text{if } q \in Q_{I \to J} \end{cases}. \tag{4.8}$$

Since $\{B_q\}$ are all effective (4.8) and (4.7) imply that $\{B'_q\}$ are also all effective. Therefore $\bigoplus \mathcal{L}(-M'_\chi)$ is a normalized $gnat$-family. But

$$w_\theta(\mathcal{M}') = w_\theta(\mathcal{M}) + \sum_{\chi \in J} \theta(\chi) \tag{4.9}$$

which contradicts the maximality of $w_\theta(\mathcal{M})$ since $\sum_{\chi \in J} \theta(\chi) = -\theta(A') > 0$.

For the second claim let $\mathcal{N} = \bigoplus \mathcal{L}(-N_\chi)$ be another normalized family θ-semistable over U. Let B'_q be divisors of zeroes of $\mathcal{N}$. Then

$$B_q - B'_q = (M_{tq} - N_{tq}) - (M_{hq} - N_{hq}). \tag{4.10}$$

Take any $E' \in \mathfrak{E}$ such that the sets $\{m_{\chi,E'}\}$ and $\{n_{\chi,E'}\}$ of the coefficients of E' in $\{M_\chi\}$ and $\{N_\chi\}$ are distinct. Then $J' = \{\chi \in G^\vee \mid n_{\chi,E'} > m_{\chi,E'}\}$ is a non-empty proper subset of $G^\vee$. Denote by I' its complement. For any $q \in Q_{I' \to J'}$ the coefficient of E' in the RHS of (4.10) is strictly positive. As B'_q is effective we conclude that $q \in Q_{I' \to J'}$ implies $E' \subset B_q$. So for any $y \in E'$ the restriction $(\bigoplus_{\chi \in I'} \mathcal{L}(M_\chi))|_y$ is a sub-$R \rtimes G$-module of $\mathcal{M}_{|y}$. But as $\mathcal{M}$ is θ-semistable on U and as $U \cap E' \neq \emptyset$ we must have $\sum_{\chi \in I'} \theta(\chi) \geq 0$. Similarly if $q \in Q_{J' \to I'}$, then the RHS of (4.10) is strictly negative, so $E' \subset B'_q$ and θ-semistability of $\mathcal{N}$ implies $\sum_{\chi \in J'} \theta(\chi) = -\sum_{\chi \in I'} \theta(\chi) \geq 0$. Therefore $\sum_{\chi \in I'} \theta(\chi) = 0$ and θ is not generic.

The fine moduli space M_θ of θ-stable G-constellations can be constructed via GIT theory, together with the universal family $\mathcal{M}_\theta$. The Hilbert-Chow morphism π_θ of $\mathcal{M}_\theta$ is projective. As the universal family is defined up to an equivalence of families, that is up to a twist by a line bundle, we can assume $\mathcal{M}_\theta$ to be normalised.

Assume for the rest of this section that $n = 3$. If θ is generic, then M_θ is a projective crepant resolution of $\mathbb{C}^3/G$ and $\mathcal{M}_\theta$ is everywhere orthogonal in all degrees. As any two crepant resolutions of a canonical treefold are connected by a chain of flops, M_θ and Y are isomorphic outside of a codimension 2 subset. The maps $Y \xrightarrow{\pi} \mathbb{C}^3/G$ and $M_\theta \xrightarrow{\pi_\theta} \mathbb{C}^3/G$ fix a choice of a birational isomorphism between Y and M_θ. This, as described in Section 4.4, defines a notion of direct transforms between Y and M_θ.

Corollary 3. *Let $\theta \in \Theta$ be generic. Let $\mathcal{M}$ be the unique normalized $gnat$-family on Y which maximizes the map w_θ. Then $\mathcal{M}$ is isomorphic to the direct transform of $\mathcal{M}_\theta$ from M_θ to Y.*

Proof. By the first claim of Proposition 6, $\mathcal{M}$ is θ-stable on U. So, by its definition, is the direct transform of $\mathcal{M}_\theta$ to Y. Hence, by the second claim of Proposition 6, $\mathcal{M}$ and the direct transform of $\mathcal{M}_\theta$ must be isomorphic.

5. Non-projective example

In this section we give an application of the Theorem 1 whereby we construct explicitly a derived McKay correspondence for a choice of an abelian $G \subset \mathrm{SL}_3(\mathbb{C})$ and of a non-projective crepant resolution Y of $\mathbb{C}^3/G$.

5.1. The group

We set the group G to be $\frac{1}{6}(1,1,4) \oplus \frac{1}{2}(1,0,1)$. That is, the image in $\mathrm{SL}_3(\mathbb{C})$ of the product $\mu_6 \times \mu_2$ of groups of 6th and 2nd roots of unity, respectively, under the embedding:

$$(\xi_1, \xi_2) \mapsto \begin{pmatrix} \xi_1\xi_2 & & \\ & \xi_1 & \\ & & \xi_1^4\xi_2 \end{pmatrix}. \tag{5.1}$$

We denote by $\chi_{i,j}$ the character of G induced by $(\xi_1, \xi_2) \mapsto \xi_1^i \xi_2^j$. Calculating the McKay quiver of G (cf. Section 4.1), we obtain:

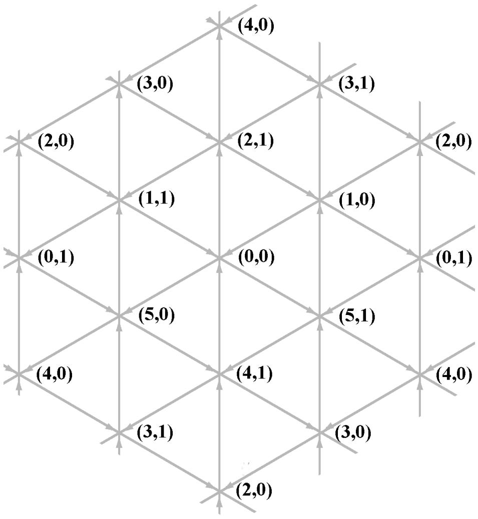


Figure 2

The way we've chosen to depict the McKay quiver reflects the fact that it has a universal cover quiver naturally embedded into $\mathbb{R}^2$. This point of view will not be essential for our argument but a curious reader should consult [CI04], Section 10.2 and [Log04], Section 6.4.

5.2. The resolution

We define the crepant resolution Y of $\mathbb{C}^3/G$ using methods of toric geometry. For the specifics related to G-constellations see [Log03], Section 3.

We define the relevant notation. The embedding (5.1) defines a surjection of torii

$$0 \longrightarrow G \longrightarrow (\mathbb{C}^*)^3 \longrightarrow T \longrightarrow 0\,. \tag{5.2}$$

Applying $\mathrm{Hom}(\bullet, \mathbb{C}^*)$ to (5.2) we obtain the character lattices of the torii:

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^3 \xrightarrow{\rho} G^\vee \longrightarrow 0\,. \tag{5.3}$$

Given any character $m = (k_1, k_2, k_3) \in \mathbb{Z}^3$ of $(\mathbb{C}^*)^3$ we denote by x^m the Laurent monomial $x_1^{k_1} x_2^{k_2} x_3^{k_3}$ in R. Applying $\mathrm{Hom}(\bullet, \mathbb{Z})$ to (5.3) we obtain the dual lattices

$$0 \longrightarrow (\mathbb{Z}^3)^\vee \longrightarrow N \longrightarrow \mathrm{Ext}^1(G^\vee, \mathbb{Z}) \longrightarrow 0\,.$$

Let e_1, e_2, e_3 be the basis of $(\mathbb{Z}^3)^\vee$ dual to x_1, x_2, x_3. The dual lattice N is generated over $(\mathbb{Z}^3)^\vee$ by $\frac{1}{6}(1,1,4)$ and $\frac{1}{2}(1,0,1)$. The quotient space $\mathbb{C}^3/G$ is the toric variety given by a single cone $\sigma_{\geq 0} = \sum \mathbb{R}_{\geq 0} e_i$ in N. Let Y be the toric variety whose fan $\mathfrak{F}$ in N is the subdivision of $\sigma_{\geq 0}$ which triangulates the junior simplex $\Delta = \{(k_1, k_2, k_3) \in \sigma_{\geq 0} \mid \sum k_i = 1\}$ as depicted below

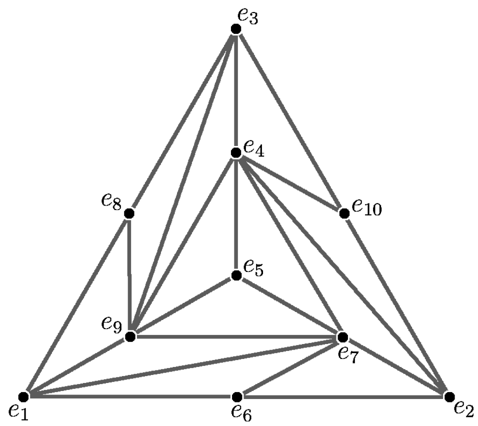


Figure 3

where by e_i we denote the following elements of N

$$\begin{array}{lll} e_1 = (1,0,0) & e_2 = (0,1,0) & e_3 = (0,0,1) \\ e_4 = \frac{1}{6}(1,1,4) & e_5 = \frac{1}{3}(1,1,1) & e_6 = \frac{1}{2}(1,1,0) \\ e_7 = \frac{1}{6}(1,4,1) & e_8 = \frac{1}{2}(1,0,1) & e_9 = \frac{1}{6}(4,1,1) \\ e_{10} = \frac{1}{2}(0,1,1). & & \end{array} \tag{5.4}$$

Denote by π the map $Y \to \mathbb{C}^3/G$ defined by the inclusion of $\mathfrak{F}$ into $\sigma_{\geq 0}$. All the maximal cones of $\mathfrak{F}$ are basic in N, so Y is smooth. The generators e_i of the rays of $\mathfrak{F}$ lie in Δ, so the map π is crepant([Rei87], Prop. 4.8). Finally, the argument of [KKMSD73], Chapter III, §2E, Example 2 shows that π is non-projective.

The quotient torus T acts on Y and to each k-dimensional cone σ in $\mathfrak{F}$ corresponds a $(3-k)$-dimensional orbit of T. We denote it by S_σ and denote by E_σ the closure of S_σ, it is the union of all orbits $S_{\sigma'}$ with $\sigma \subseteq \sigma'$. For each cone $\langle e_i \rangle$ in the fan $\mathfrak{F}$, we denote by S_i the codimension 1 orbit $S_{\langle e_i \rangle}$ and by E_i the divisor $E_{\langle e_i \rangle}$. Similarly we use $S_{i,j}$ and $E_{i,j}$ for the codimension 2 orbit $S_{\langle e_i, e_j \rangle}$ and the surface $E_{\langle e_i, e_j \rangle}$ and we use $E_{i,j,k}$ for the toric fixed point $E_{\langle e_i, e_j, e_k \rangle}$.

5.3. *The family*

The map $Y \xrightarrow{\pi} \mathbb{C}^3/G$ defines the notion of G-Weil divisors on Y. Any normalized $gnat$-family on $Y \xrightarrow{\pi} \mathbb{C}^3/G$ is of the form $\bigoplus_{\chi \in G^\vee} \mathcal{L}(-D_\chi)$ for some G-Weil divisors D_χ with $D_{\chi_{0,0}} = 0$. Moreover, as explained in [Log06], Section 3.5, there exists the *maximal shift family* $\oplus \mathcal{L}(-M_\chi)$ such that for any other normalized $gnat$-family $\oplus \mathcal{L}(-D_\chi)$ we have

$$M_\chi \geq D_\chi \tag{5.5}$$

for all $\chi \in G^\vee$. We denote this family by $\mathcal{F}$ and shall prove it to satisfy the assumptions of Corollary 1.

In the notation of Section 5.2 each divisor M_χ is of form $\sum q_{\chi,i} E_i$. The coefficients $q_{\chi,i}$ can be calculated via formula

$$q_{\chi,i} = \inf\{e_i(m) \mid m \in \sigma^\vee_{\geq 0} \cap \rho^{-1}(\chi)\}. \tag{5.6}$$

A detailed example of such calculation can be seen in [Log03], Example 4.21. In our case, we obtain $q_{\chi,i}$ to be:

$\chi \setminus i$	4	5	6	7	8	9	10	$\chi \setminus i$	4	5	6	7	8	9	10
$\chi_{0,0}$	0	0	0	0	0	0	0	$\chi_{2,0}$	$\frac{2}{6}$	$\frac{4}{6}$	0	$\frac{2}{6}$	0	$\frac{2}{6}$	0
$\chi_{4,0}$	$\frac{4}{6}$	$1\frac{2}{6}$	0	$\frac{4}{6}$	0	$\frac{4}{6}$	0	$\chi_{1,1}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	0
$\chi_{1,0}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	0	$\frac{1}{6}$	$\frac{3}{6}$	$\chi_{4,1}$	$\frac{4}{6}$	$\frac{2}{6}$	0	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{1}{6}$	$\frac{3}{6}$
$\chi_{3,1}$	$\frac{3}{6}$	1	$\frac{3}{6}$	$\frac{3}{6}$	$\frac{3}{6}$	1	0	$\chi_{3,0}$	$\frac{3}{6}$	1	$\frac{3}{6}$	1	0	$\frac{3}{6}$	$\frac{3}{6}$
$\chi_{0,1}$	1	1	0	$\frac{3}{6}$	$\frac{3}{6}$	$\frac{3}{6}$	$\frac{3}{6}$	$\chi_{5,1}$	$\frac{5}{6}$	$\frac{4}{6}$	$\frac{3}{6}$	$\frac{5}{6}$	$\frac{3}{6}$	$\frac{2}{6}$	0
$\chi_{5,0}$	$\frac{5}{6}$	$\frac{4}{6}$	$\frac{3}{6}$	$\frac{2}{6}$	0	$\frac{5}{6}$	$\frac{3}{6}$	$\chi_{2,1}$	$\frac{2}{6}$	$\frac{4}{6}$	0	$\frac{5}{6}$	$\frac{3}{6}$	$\frac{5}{6}$	$\frac{3}{6}$

(5.7)

The principal G-Weil divisors (x_k) can be calculated with a formula

$$(x_i) = \frac{1}{12} \sum_{j=1}^{10} e_j(x_i^{12}) E_j, \tag{5.8}$$

cf. [Log03], Prop. 3.2. In our case we obtain:

$$\begin{aligned}
(x_1) &= E_1 + \frac{1}{6}E_4 + \frac{1}{3}E_5 + \frac{1}{2}E_6 + \frac{1}{6}E_7 + \frac{1}{2}E_8 + \frac{4}{6}E_9 \\
(x_2) &= E_2 + \frac{1}{6}E_4 + \frac{1}{3}E_5 + \frac{1}{2}E_6 + \frac{4}{6}E_7 + \frac{1}{6}E_9 + \frac{1}{2}E_{10} \\
(x_3) &= E_3 + \frac{4}{6}E_4 + \frac{1}{3}E_5 + \frac{1}{6}E_7 + \frac{1}{2}E_8 + \frac{1}{6}E_9 + \frac{1}{2}E_{10}
\end{aligned} \tag{5.9}$$

Substituting the data of (5.9) and (5.7) into the formula (4.3) we calculate for every arrow of the McKay quiver its divisor of zeroes in $\mathcal{F}$:

$$\begin{aligned}
&B_{\chi_{0,0},1} = E_1 && B_{\chi_{1,1},1} = E_1 + E_4 + E_5 + E_6 + E_7 + E_8 + E_9 \\
&B_{\chi_{0,0},2} = E_2 && B_{\chi_{1,1},2} = E_2 + E_6 + E_7 \\
&B_{\chi_{0,0},3} = E_3 && B_{\chi_{1,1},3} = E_3 + E_4 + E_8 \\
&B_{\chi_{4,0},1} = E_1 && B_{\chi_{1,0},1} = E_1 + E_6 + E_9 \\
&B_{\chi_{4,0},2} = E_2 && B_{\chi_{1,0},2} = E_2 + E_4 + E_5 + E_6 + E_7 + E_9 + E_{10} \\
&B_{\chi_{4,0},3} = E_3 && B_{\chi_{1,0},3} = E_3 + E_4 + E_{10} \\
&B_{\chi_{2,0},1} = E_1 + E_5 + E_9 && B_{\chi_{4,1},1} = E_1 + E_8 + E_9 \\
&B_{\chi_{2,0},2} = E_2 + E_5 + E_7 && B_{\chi_{4,1},2} = E_2 + E_7 + E_{10} \\
&B_{\chi_{2,0},3} = E_3 + E_4 + E_5 && B_{\chi_{4,1},3} = E_3 + E_4 + E_5 + E_7 + E_8 + E_9 + E_{10} \\
&B_{\chi_{5,1},1} = E_1 + E_6 + E_8 + E_9 && B_{\chi_{3,1},1} = E_1 + E_6 + E_8 + E_9 \\
&B_{\chi_{5,1},2} = E_2 + E_6 && B_{\chi_{3,1},2} = E_2 + E_5 + E_6 + E_7 + E_9 \\
&B_{\chi_{5,1},3} = E_3 + E_8 && B_{\chi_{3,1},3} = E_3 + E_4 + E_5 + E_8 + E_9 \\
&B_{\chi_{5,0},1} = E_1 + E_6 && B_{\chi_{3,0},1} = E_1 + E_5 + E_6 + E_7 + E_9 \\
&B_{\chi_{5,0},2} = E_2 + E_6 + E_7 + E_{10} && B_{\chi_{3,0},2} = E_2 + E_6 + E_7 + E_{10} \\
&B_{\chi_{5,0},3} = E_3 + E_{10} && B_{\chi_{3,0},3} = E_3 + E_4 + E_5 + E_7 + E_{10} \\
&B_{\chi_{2,1},1} = E_1 + E_8 && B_{\chi_{0,1},1} = E_1 + E_4 + E_5 + E_8 + E_9 \\
&B_{\chi_{2,1},2} = E_2 + E_{10} && B_{\chi_{0,1},2} = E_2 + E_4 + E_5 + E_7 + E_{10} \\
&B_{\chi_{2,1},3} = E_3 + E_4 + E_8 + E_{10} && B_{\chi_{0,1},3} = E_3 + E_4 + E_8 + E_{10}.
\end{aligned} \tag{5.10}$$

5.4. A sample calculation

Corollary 2 together with the table (5.10) are all that we need to check any two G-constellations in $\mathcal{F}$ for the degree 0 orthogonality. Below we give an example of a calculation which verifies that any point on the torus orbit S_8 and any point on the torus orbit $S_{1,7}$ are orthogonal in degree 0 in $\mathcal{F}$.

Let a be any point of S_8. Then a lies on no divisor E_i other than E_8. Hence $a \in B_q$ if and only if $E_8 \subset B_q$. Let A be the fiber of $\mathcal{F}$ at a and $\{\alpha_q\}$ be the corresponding representation of the McKay quiver. By Proposition 5 for any arrow q the map α_q is a zero map if and only if $E_8 \in B_q$. On Figure 4 we use the table (5.10) and mark all the zero-maps in $\{\alpha_q\}$ by drawing a line through the corresponding arrow of the McKay quiver. Similarly if b is a point of $S_{1,7}$ then b lies on no E_i other than E_1 and E_7. Let B be the fiber of $\mathcal{F}$ at b and $\{\beta_q\}$ be the corresponding representation. As above β_q is a zero-map if and only if either E_1 or E_7 belongs to B_q. On Figure 5 we mark all the zero-maps $\{\beta_q\}$.

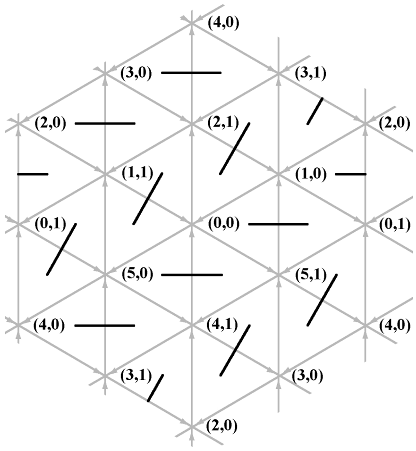


Figure 4

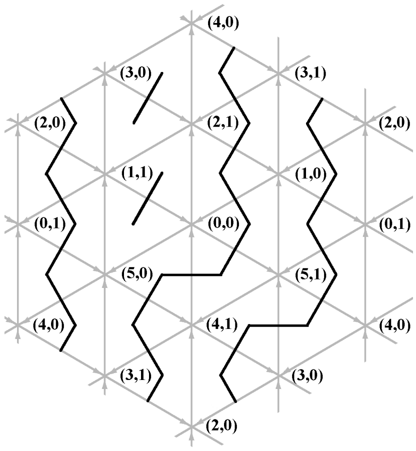


Figure 5

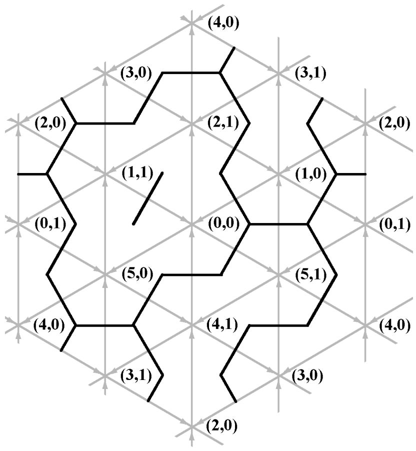


Figure 6

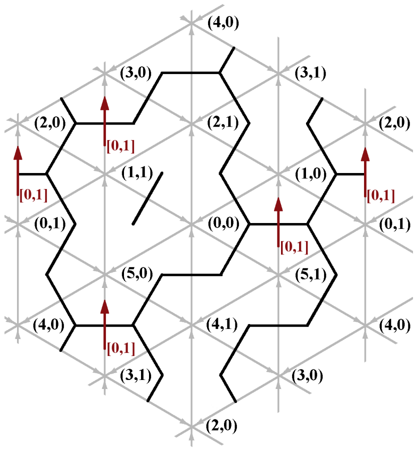


Figure 7

On Figure 6 we combine the markings of Figures 4 and 5. The arrows left unmarked are the arrows of type $[1,1]$ with respect to the pair A, B (Def. 7). It is clear that the components path-connected by $[1,1]$-arrows are: $\{\chi_{0,0}, \chi_{2,1}, \chi_{5,0}, \chi_{1,1}\}$, $\{\chi_{5,1}, \chi_{4,1}, \chi_{2,0}\}$, $\{\chi_{1,0}, \chi_{3,1}\}$ and $\{\chi_{0,1}, \chi_{4,0}, \chi_{3,0}\}$. Now, with Cor. 2 in mind, we search the borders of these four regions for the $[1,0]$ and $[0,1]$-arrows. The $[1,0]$-arrows are the ones unmarked on Figure 4 but marked on Figure 5 and vice versa for $[0,1]$. On Figure 7 we've marked on the border of each region an incoming and an outgoing $[0,1]$-arrow. By Cor. 2 we see that A and B are orthogonal in degree 0.

5.5. *Final calculations*

We now claim that $\mathcal{F}$ is the direct transform of the universal family of G-clusters on $G\text{-Hilb}(\mathbb{C}^3)$. In the notation of Section 4.5 define $\theta_+ \in \Theta$ by $\theta_+(\chi_{0,0}) = 1 - |G|$ and $\theta_+(\chi) = 1$ for $\chi \neq \chi_{0,0}$. Evidently θ_+ is generic. It follows from the original observation by Ito and Nakajima in [IN00], §3, that G-clusters can be identified with θ_+-stable G-constellations, thus identifying $G\text{-Hilb}(\mathbb{C}^3)$ with the fine moduli space M_{θ_+}. On the other hand, inequalities (5.5) imply that $\mathcal{F}$ maximizes ω_{θ_+} on $Y \xrightarrow{\pi} \mathbb{C}^3/G$. Hence, by Corollary 3, $\mathcal{F}$ is the direct transform of $\mathcal{M}_{\theta_+}$ from $G\text{-Hilb}(\mathbb{C}^3)$ to Y.

For a detailed description of an algorithm which allows one to calculate the toric fan of $G\text{-Hilb}(\mathbb{C}^3)$ see in [CR02]. For our group G we obtain:

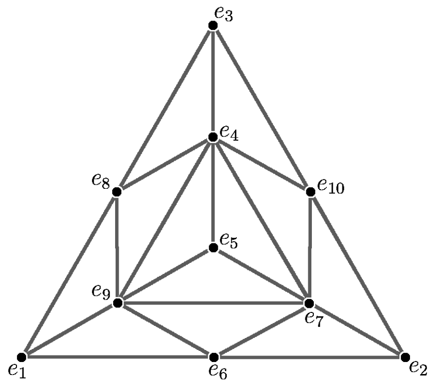


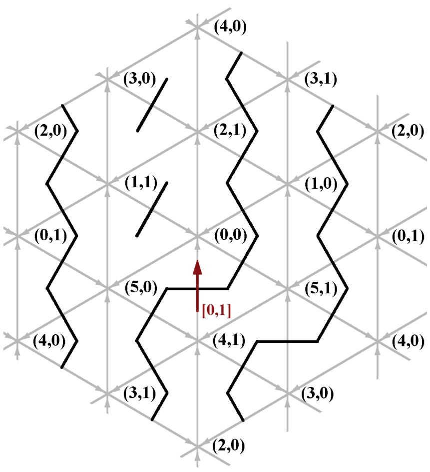
Figure 8

The general points of an exceptional surface E_i, as per the statement of Corollary 1, are precisely the codimension 1 torus orbit S_i. Similarly, the general points of an exceptional curve $E_i \cap E_j$ are precisely the codimension 2 torus orbit $S_{i,j}$. Comparing Figure 8 with the fan of Y on Figure 3 we see that the only codimension 1 or 2 torus orbits in Y whose corresponding cones aren't also contained in the fan of $G\text{-Hilb}(\mathbb{C}^3)$ are $S_{1,7}$, $S_{2,4}$ and $S_{3,9}$. The argument in Section 4.4 reduces verifying that $\mathcal{F}$ satisfies the conditions of Corollary 1, to checking that each of these three orbits is orthogonal in degree 0 in $\mathcal{F}$ to every codimension 1 orbit S_i.

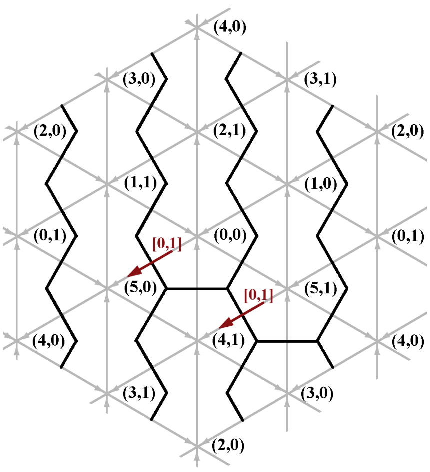
We claim that, in fact, it suffices to check it for just one of these orbits. Let ϕ be the rotation of the fan of Y around the ray e_5 which rotates Figure 2 clockwise by $2\pi/3$. Let ψ be the rotation of the plane containing the McKay quiver on the Figure 3 anti-clockwise by $2\pi/3$ with center at $\chi_{0,0}$. Observe that the permutation of the divisors E_i defined by ϕ and the permutation of the arrows of the McKay quiver defined by ψ leave the numerical data (5.10) of divisors of zeroes of $\mathcal{F}$ invariant[1]. It follows that the orthogonality calculation of Section 5.4 for any pair of torus orbits S, S' and the same calculation for $\phi(S), \phi(S')$ differ on Figures 4-7 only by a rotation by ψ. The claim now follows as the cones of $S_{1,7}$, $S_{2,4}$ and $S_{3,9}$ are permuted by ϕ.

We choose to treat $S_{1,7}$. We repeat the calculation of Section 5.4 for $S_{1,7}$ and every other orbit S_i and list below the analogues of Figure 7. From them, as elaborated in Section 5.4, the reader could readily ascertain the orthogonality in $\mathcal{F}$ of the torus orbits involved.

We conclude, by Corollary 1, that the integral transform $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ is an equivalence of categories $D(Y) \to D^G(\mathbb{C}^3)$ and that *a posteriori* the family $\mathcal{F}$ is everywhere orthogonal in all degrees.



$(S_1, S_{1,7})$ and $(S_7, S_{1,7})$



$(S_2, S_{1,7})$

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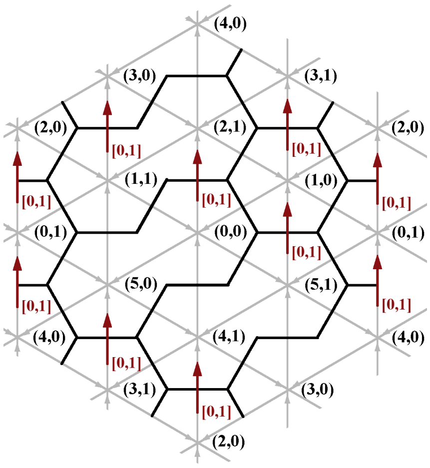
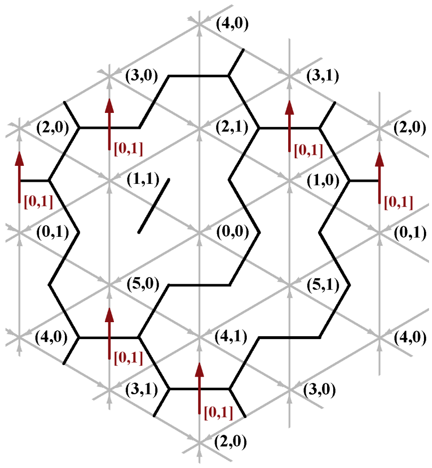
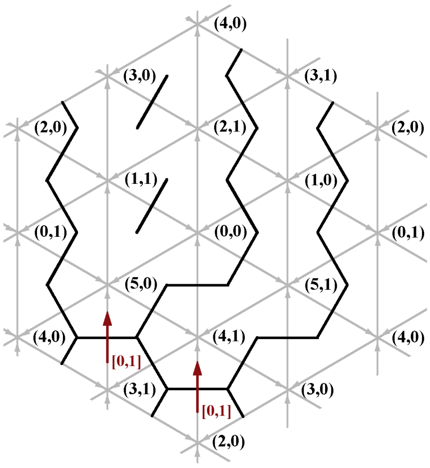
[1] This invariance is a consequence of the fan of Y being symmetric and of $\mathcal{F}$ being intrinsically defined as the maximal shift family.



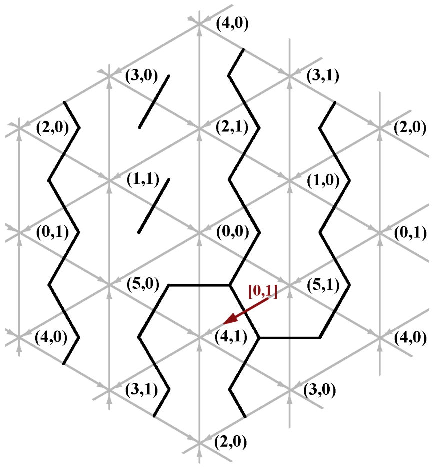
$(S_3, S_{1,7})$



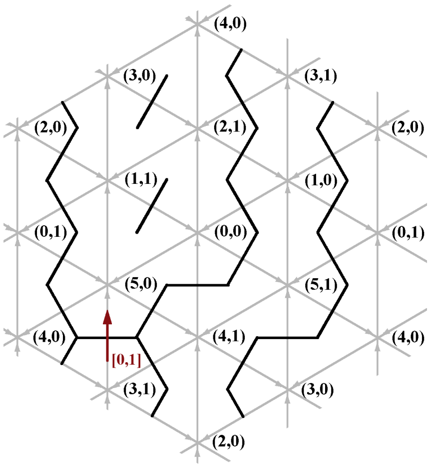
$(S_4, S_{1,7})$



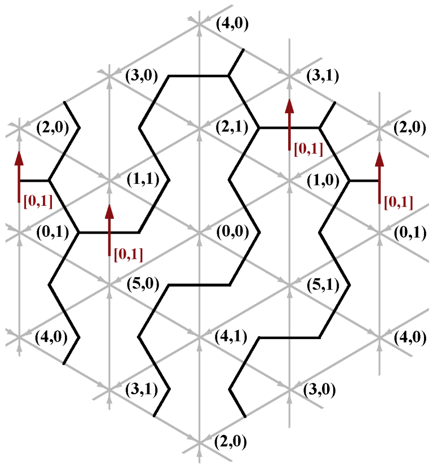
$(S_5, S_{1,7})$



$(S_6, S_{1,7})$



$(S_9, S_{1,7})$



$(S_{10}, S_{1,7})$

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