INTER-LINKING MRP THEORY
AND
PRODUCTION AND INVENTORY CONTROL MODELS

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Abstract

We review the field of Materials Requirements Planning (MRP) and Production and Inventory Control (PIC) theory and highlight their similarities, differences and some conditions for one-to-one correspondence. The most important similarity is the use of transform techniques to model time delays in the systems. However MRP theory treats complete product structures, whereas PIC has usually been only considered with single item problems. Moreover, MRP theory is able to further breakdown lead-times to deal with explicit detail concerned with the precise assembly of internal lead-times. This is often missing in conventional PIC theory. However, MRP theory requires information about future demands, whereas there is not always such a requirement with PIC theory. Thus an MRP system typically suffers from system nervousness, an important problem that has yet to be solved. Whereas the analogous problem in PIC theory, bullwhip, is relatively well-understood. We explore these issues with a simple model interlinking MRP and PIC theory.

Keywords: Material requirements planning, production and inventory control, transfer functions, Laplace transform, input-output analysis.

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1. Introduction

A substantial amount of research has been carried out on production and inventory control models since the seminal paper of Herbert Simon in 1952. Overviews are given, for instance by Axsäter (1985), Riddalls, Bennett and Tipi (2000). Although there are exceptions, these systems have essentially dealt with single-item systems, and when two or more production levels are assumed, the systems have been coupled in tandem. Therefore, this approach is essentially one-dimensional. Attention has also been directed towards the transmission of stochastic properties of the order flow from final demand and upwards along the stream of production, for example see Dejonckheere, Disney, Lambrecht and Towill (2004). Some robust perturbations of delays have been studied in Bogataj and Bogataj (1992), using the previous results of Bogataj (1989). Optimal control of such hereditary inventory systems in particular optimisation of short-term conservation effects, have been studied in Bogataj (1994). Sensitivity of quadratic cost functions under stochastically perturbed controls in inventory systems with delays has been studied by Čibej and Bogataj (1994), based on previous results of hereditary systems in Banach spaces, Bogataj (1989a).

A parallel development has been MRP theory. This basic theory deals with multi-level production systems with multiple items taking place either within a discrete or continuous time framework. Lead times are given constants and either the Net Present Value Principle or the traditional average cost measure has been applied as the objective function. The Bill-of-Materials capturing component as well as capacity requirements, in volume as well as advanced timing due to lead time, has been described using a generalised input matrix involving Laplace transforms or z-transforms. Fundamental equations stating the time development of the system including production, external and internal demand, inventories of different kinds, backlogs and economic consequences have been set forth and analysed, and often optimised. By using matrix and vectorial representations of the system properties, production on different levels of the Bill of Materials can be treated simultaneously.

This paper aims at identifying the similarities and differences between these two lines of theoretical developments. As examples of differences, one may note the inclusion of forecast modules and transfer functions in the production and inventory control approach, as well as an update (re-planning) of the system state as time evolves, which is not explicitly taken care of in MRP Theory. Whereas MRP Theory is exact in the sense that it accurately represents real-world planning systems, production and inventory control systems are based on assumptions concerning certain simple decision rules and assume symmetry between inventory and backlog consequences, thus avoiding the effects of non-negativity of various system properties but enabling the use of transfer functions and linear analytical techniques.

We proceed as follows. In the next two sections we review the literature of MRP theory and Production and Inventory Control (PIC) theory. Then we discuss the role of transform techniques both approaches as a mean of highlighting the common ground. Section 5 considers the uniqueness of transform representations of production and inventory control models. In section 6 we present a simple example of how we may integrate MRP and PIC theory. Section 7 remarks upon the similarities and differences between the two approaches and concludes.

2. MRP theory: An overview and literature review

In order to describe the production opportunities in a multi-level production system, it is essential to describe the inter-relationship between items and processes, both in terms of volume and timing. To address this problem, MRP theory has been developed.

Grubbström and Lundquist (1977) discuss the relationships between Input-Output Analysis (Leontief, 1928), MRP (Orlicky, 1975) and production functions. They concluded that there is an obvious relationship between master production scheduling, material requirements planning, the bill of material, MRP, and a general linear production-economic system interpreted in terms of an input-output model. As a result, an Input-Output Analysis model presents the opportunity to transform one set of resources into another set using an efficient mathematical language.

The Laplace transform has been used for describing time developments and lags of the relevant production, demand and inventory variables in a compact way including effects of order flows and lead times. Secondly, the transform has functioned as a moment-generating function, and thirdly, the transform has been applied for assessing cash flows adopting the net present value principle. This has made the analysis compact and distinct (Grubbström and Molinder, 1994, Grubbström 1999). This methodology is presented here in section 3.

By way of an overview, MRP considers that a production system is made up of a finite number of processes, m, and it contains a finite number of products, n, inputs as well as outputs. A

process is run on a certain activity level which may be varied. The activity levels, one for each process, are collected into a m-dimensional column vector $\mathbf{x}$. The input volumes of each product to each process are described by a set of constant input coefficients, which are collected in $n \times m$ input matrix $\mathbf{H}$, its element h_{ij} being the quantity of item i required for the j-th process running on unit activity level. Similarly, all output volumes of each product from each process composes of an $n \times m$ output matrix $\mathbf{G}$. The net production $\mathbf{y}$ of the total system, the volumes that may leave the production system, is then determined by

$$\mathbf{y} = (\mathbf{G} - \mathbf{H})\mathbf{x} = \mathbf{T}\mathbf{x} \quad , \qquad\qquad 2.1$$

where $\mathbf{T} = \mathbf{G} - \mathbf{H}$ is the technology matrix. When each process produces a unique product, we obtain an elementary system $\mathbf{G} = \mathbf{I}$. Solving for the activity levels, we obtain

$$\mathbf{x} = (\mathbf{I} - \mathbf{H})^{-1}\mathbf{y} = \mathbf{H}^*\mathbf{y} , \qquad\qquad 2.2$$

where $\mathbf{H}^*$ is well-known as the Leontief inverse in Input-Output Analysis.

In terms of MRP terminology (Grubbström and Tang 2000a) the vector $\mathbf{x}$ corresponds to the gross requirements, which consists of external demand for final products and spare parts, as well as internal demand generated via the BOM explosion. The net production vector $\mathbf{y}$ is generated from the master production schedule. The input matrix $\mathbf{H}$ contains the quantity relationships of the product structure. If all product structures are hierarchical, meaning that there is no feedback of some higher-level item entering into a lower-level item and no product being part of itself, then the items may be numbered in such a way that items on lower levels have indices taking on higher values. This makes $\mathbf{H}$ triangular with zeros in its main diagonal and above (an assembly system). The triangular nature of the input matrix is valid for assembly and pure arborescent systems only (if there is any form of feedback this property fails).

If the number of levels in the product structure and the number of items are large, it is laborious to calculate the inverse of $(\mathbf{I} - \mathbf{H})$. Since all eigenvalues of triangular matrix $\mathbf{H}$ are zero, the following Neumann series for $\mathbf{H}^*$ converges

$$\mathbf{H}^* = (\mathbf{I} - \mathbf{H})^{-1} = \mathbf{I} + \mathbf{H} + \mathbf{H}^2 + \cdots \qquad\qquad 2.3$$

and this series expansion will provide an exact value of $\mathbf{H}^*$ after a finite number of steps.

We assume that in this context production on different levels takes place in batches of possibly different sizes at different points in time. Let there be n items in the system altogether. Demand $\mathbf{D}$, stock $\mathbf{S}$ and production $\mathbf{P}$ are presented by n-dimensional column vectors each being a function of time. These vectors are rates with the dimension units per time unit and they are turned into Laplace transforms denoted by $\pounds\{.\}$.

For the production (assembly) of one unit of item j, there is a need in the amount of h_{kj} of item k, and there is a lead time τ_j ahead of the completion of the production at which the components are needed. The h_{kj} are arranged into the square input matrix $\mathbf{H}$ describing the product structures of all relevant products. The lead times τ_i, $i = 1, \ldots, n$, are represented by a diagonal matrix $\tilde{\boldsymbol{\tau}}$, the lead time matrix, having lead-times represented by transform techniques in its j-th diagonal position. The matrix $\tilde{\mathbf{H}} = \mathbf{H}\tilde{\boldsymbol{\tau}}$ is the generalized input matrix.

Total inventory $\widetilde{\mathbf{S}}(s)$ is the vector accounting for all items in the system being initial stock $\frac{\mathbf{S}(0)}{s}$ plus cumulative net production $\frac{(\mathbf{I}-\mathbf{H})\widetilde{\mathbf{P}}}{s}$ less cumulative deliveries from the system $\frac{\widetilde{\mathbf{F}}(s)}{s}$ (for details see Grubbström and Tang (1998)).

$$\widetilde{\mathbf{S}}(s) = \frac{\mathbf{S}(0) + (\mathbf{I}-\mathbf{H})\widetilde{\mathbf{P}}(s) - \widetilde{\mathbf{F}}(s)}{s} . \qquad\qquad 2.4$$

Available inventory $\widetilde{\mathbf{R}}(s)$ is initial available inventory $\frac{\mathbf{R}(0)}{s}$ plus cumulative production $\frac{\widetilde{\mathbf{P}}(s)}{s}$ less cumulative deliveries $\frac{\widetilde{\mathbf{F}}(s)}{s}$ and cumulative internal demand $\frac{\mathbf{H}\widetilde{\mathbf{P}}(s)}{s}$, which evaluated at the time the components are reserved is $\frac{\mathbf{H}\widetilde{\tau}\widetilde{\mathbf{P}}(s)}{s}$ and $\widetilde{\mathbf{R}}(s)$ must never be negative:

$$\widetilde{\mathbf{R}}(s) = \frac{\mathbf{R}(0) + (\mathbf{I}-\mathbf{H}\widetilde{\tau})\widetilde{\mathbf{P}}(s) - \widetilde{\mathbf{F}}(s)}{s} ; \qquad £^{-1}\{\widetilde{\mathbf{R}}(s)\} \geq 0 . \qquad 2.5$$

Internal demand must always be met, but if external demand cannot be met, we assume that this demand is either backlogged and satisfied at the time available inventory starts to become positive once again or that the sales are lost.

Allocated component stock $\widetilde{\mathbf{A}}(s)$ is defined as the difference between (total) inventory and available inventory:

$$\widetilde{\mathbf{A}}(s) = \widetilde{\mathbf{S}}(s) - \widetilde{\mathbf{R}}(s) = \frac{\mathbf{A}(0) + \mathbf{H}(\widetilde{\tau}-\mathbf{I})\widetilde{\mathbf{P}}(s)}{s} , \qquad\qquad 2.6$$

where $\mathbf{A}(0) = \mathbf{S}(0) - \mathbf{R}(0)$.

Backlogs $\widetilde{\mathbf{B}}(s)$, which are necessarily non-negative, only concern external demand and are given by initial backlogs plus cumulative external demand less cumulative deliveries:

$$\widetilde{\mathbf{B}}(s) = \frac{\mathbf{B}(0) + \widetilde{\mathbf{D}}(s) - \widetilde{\mathbf{F}}(s)}{s} . \qquad\qquad 2.7$$

In order to evaluate the system performance of a production and inventory system, an objective function is often required for further analysis. Along with the development of the MRP theory, the net present value approach is proposed as the objective function in the studies, taking advantage of the Laplace transform. This idea is first suggested in Grubbström (1967) and then followed up in economic analysis with the consideration of cash flow profiles.

Besides its advantage from an economic point of view, monitoring the cash flow in some circumstances is considered to be the only correct way towards obtaining an optimal evaluation compared with the traditional average cost approach (Grubbström 1980). A comprehensive survey of the application of the Laplace Transform to present value problems is given by Grubbström and Jiang (1990).

A production-inventory system usually faces uncertainty in its external demand. Uncertainty can be described by letting demand follow a renewal process, in which the demand is created by unit events separated by independent stochastic time intervals having the same but independent probability density functions. In several papers (Andersson and Grubbström 1994, Grubbström 1996 and Tang 2000, Bogataj and Bogataj 1998) efforts have been made to study the stochastic properties in terms of the Laplace transform.

The production and inventory systems are sometimes considered in a discrete time space in practice. Fundamental equations in the MRP theory can be found by Grubbström and Ovrin (1992), in which a z-transform is used instead of the Laplace transform to enable this.

Although capacity requirements are easily incorporated into the MRP theory, the solution procedure is rather difficult when constraints are considered. Some attempts have been made by Grubbström and Wang (1999 and 2003). The fundamental equations were further extended to include the considered constraints. Examples are given to illustrate the solution procedure in some simple cases with a discrete time.

The MRP theory has also been used to address the rescheduling and re-planning issues in production and inventory systems (Grubbström and Tang 2000b, Tang and Grubbström 2002a). In these studies, the authors first investigate the conditions for releasing a new schedule. Secondly, when rescheduling takes place cyclically, for instance with the same frequency as the basic time unit ("bucket") of the MRP schedule, they study how sensitive the former plan is to changes in conditions (such as demand realised etc). A closely related topic is the nervousness in a planning system. As it is difficult to estimate the cost that nervousness gives rise to, one line for analysing such consequences is to develop a control theory approach with the aim to find a system response analogy to a rescheduling event (Tang and Grubbström 2002b).

Other efforts have been made to create a theoretical background for practical operational problems using the MRP theory (Bogataj et al 1997). The framework introduced has been used and extended by other authors in different directions to interconnect with other fields, such as the volatility of demand in connection with spatial interactions (Bogataj 1999) and game theory, especially spatial games (Horvat and Bogataj 1996a, Horvat and Bogataj 1996b, Horvat and Bogataj 1999 and Bogataj and Bogataj 2001). The extension of the production part of a supply chain to the distribution part for the purpose of the compact presentation and analysis of a supply chain has been given by Bogataj and Bogataj (2004). This approach is especially useful for studying global supply chains (Bogataj, Bogataj and Vodopivec 2004).

Here, we have provided a brief review for the development of the MRP theory. For a comprehensive overview of the MRP theory, we refer to the study by Grubbström and Tang (2000a)

3. Production and inventory control: An overview and literature review

The field of Production and Inventory Control theory is vast. There are many different techniques that have been applied to the problem, for example, stochastic approaches, dynamic programming, simulation etc. One such technique is control theory, and possibly the first application of control the PIC problems was undertaken by Herbert Simon (1952) who considered a continuous time model with the Laplace transform.

Control theory models in PIC problems are usually concerned with the dynamic performance of a replenishment decision. This obviously depends on both the demand pattern and the structure of the replenishment decision and in the lead-time. Thus there is a wide range of scenarios to study. However common among all PIC models is the inventory balance equation, (see Table 3.1).

Representation of time	Time domain	Frequency domain
Continuous	$I_t = I_t + R_t - D_t$	$I\{s\} = \dfrac{R\{s\} - D\{s\}}{s}$
Discrete	$I_n = I_{n-1} + R_n - D_n$	$I\{z\} = \dfrac{R\{z\} - D\{z\}}{1 - z^{-1}}$

Table 3.1. The inventory balance equations

Key: I= inventory, R=receipts or completions, D=demand, t=continuous time index, n=discrete time index, s=Laplace operator, z= z-transform operator.

The role of the replenishment decision is the fundamental question the users of control theory are usually wishing to understand. Replenishment decisions usually consist of one of five sources of information. The first piece of information is the inventory level as replenishment decisions are mainly designed to maintain inventory or safety stock around a target level to provide product availability to customers. The second component of the replenishment order is a forecast of demand. This is usually based solely on previously observed realisations of demand and in this case we say that the system is physically realisable. The third component of a typical replenishment decision is a target safety stock. This may be either a constant or a function of the demand (or its forecast). The forth component of a typical replenishment decision is a feedback loop that keeps track of orders (replenishments) that have been placed, but due to the lead-time (the fifth component) have yet to be received.

Replenishment rules often contain parameters that may be "tuned" to produce a desirable dynamic response. For example forecasting constants, and feed-forward and feedback controllers may be use to regulate the dynamic response based on the demand. John, Naim and Towill (1994) coined the term APIOBPCS for an ordering policy with three controllers, a forecast, an inventory feedback loop and a WIP feedback loop. They studied various configurations of the system with the Laplace transform. Riddells and Bennett (2002) consider a continuous time APIOPBCS ordering policy with a pure time delay. They re-cast the APIOBPCS model as a Smith Predictor and derived some stability conditions using the Bellman and Cooke criterion. Warburton, Disney, Towill, and Hodgson (2004) studied the same system using the Lambert W function.

Control theory may also been used to study discrete time realisations of replenishment policies. This was quickly achieved by Vassian (1954) after Simon's work by exploiting the new discovered z-transform. However, Vassian sadly died before publication of the work. This was finished by Magee, who also published results (but not the mathematical detail), concerning the variance of the orders and inventory level is a simple production system, Magee (1958). Brown (1963) appears to be the first book containing a z-transform analysis of a PIC system. Interestingly Vassian, Magee and Brown, all worked for the Boston based consultancy firm, Arthur, J. Little. Howard (1963) also provides an illuminating introduction to the use of discrete linear control theory in PIC problems. The United States Naval Postgraduate School also made important contributions to inventory models using control theory, for example see Reilly (1965), DeWinter (1966), Bessler and Zehna (1968).

Adelson (1966) studies different exponential smoothing forecasting mechanism using the z-transform, as did Wikner (1994). In fact Adelson (1966), seems to be one of the first to study the bullwhip problem analytically. Burns, and Sivazlian (1978), consider a four-echelon serial supply chain using z-transforms and signal flow diagrams. Deziel and Eilon (1967) study a ordering policy that is guaranteed to be stable using simulation and z-transforms. They consider numerically the variance of the orders and inventory level via Tsypkin's relation (1964).

Recently work has been directed towards the discrete time, Order-Up-To policy. Disney and Grubbström (2003), study the case of a stochastic demand with Auto Regressive components and characterise the economic consequences resulting from stock holding, backlog and over-time working. Chen and Disney, (2003) study the economic consequences of the optimal myopic order-up-to policy in response to Auto Regressive and Moving Average (ARMA) stochastic demand patterns. Disney and Towill (2003), study the bullwhip and inventory variance problem in a single echelon of a supply chain when demand is a stationary i.i.d. random variable. Dejonckheere, Disney, Farasyn, Janssen, Lambrecht, Towill and Van de Velde (2002) study the customer service implications (in terms of the fill-rate) when demand is i.i.d. Disney, Farasyn, Lambrecht, Towill and Van de Velde, (2003) extend this to the case of ARMA demands. Disney and Towill (2002) studied the discrete time stability in VMI (and traditional) supply chains via the Routh-Hurwitz array. Dejonckheere, Disney, Lambrecht and Towill, (2004) consider a supply chain with shared end consumer demand information and different forecasting techniques.

4. Transforms: The common ground

4.1 The time lags

Time lags always exist in the production-inventory system, for instance setup times, processing times, queuing times, etc. These properties can be captured by the Laplace transform due to algebraic operation corresponding to the time shift operation (Aseltine, 1958, see also Figure 4.1)

$$\pounds\{f_2(t)\} = \pounds\{f_1(t+\tau)\} = e^{s\tau}\pounds\{f_1(t)\} \tag{4.1}$$

where $\pounds$ refers the Laplace transform operation and τ is the time shift. The lead times are then represented in the form of $e^{s\tau}$, where τ is the length of lead-time that is usually assumed to be a constant (Grubbström and Molinder 1994).
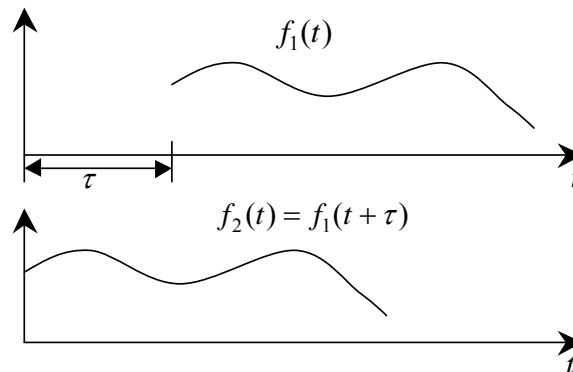

Figure 4.1. An original function and its corresponding function subject to a time lag

The above method can be easily incorporated into the input matrix of a multi-level system to capture amounts as well as their advance timing of assembly. For instance, in an assembly system, in order to assemble one unit of item j, there is a need in the amount of h_{kj} of item k, and there is a *lead time* τ_j ahead of the completion of the production at which the components are needed. The h_{kj} are arranged into the square *input matrix* **H** describing the product structures of all relevant products. The lead times $\tau_1, \tau_2, ..., \tau_N$, are represented by a diagonal matrix $\widetilde{\tau}$, the *lead time matrix,* having $e^{s\tau_j}$ in its jth diagonal position.

$$\widetilde{\tau} = \begin{bmatrix} e^{14s} & 0 & 0 & 0 & 0 \\ 0 & e^{12s} & 0 & 0 & 0 \\ 0 & 0 & e^{10s} & 0 & 0 \\ 0 & 0 & 0 & e^{8s} & 0 \\ 0 & 0 & 0 & 0 & e^{6s} \end{bmatrix}. \qquad 4.2$$

The matrix $\widetilde{\mathbf{H}} = \mathbf{H}\widetilde{\tau}$ is the *generalised input matrix* and it captures component requirements together with their required advance timing

$$\widetilde{\mathbf{H}} = \mathbf{H}\widetilde{\tau} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4e^{14s} & e^{12s} & 0 & 0 & 0 \\ 2e^{14s} & 0 & e^{10s} & 0 & 0 \\ 0 & 3e^{12s} & 2e^{10s} & 2e^{8s} & 0 \end{bmatrix}. \qquad 4.3$$

4.2 Describing time intervals using transform methods

In developing the MRP theory, the transform method is also used to describe the lead time interval in a renewal process, which is often stochastic. Along with this framework, studies have been conducted in Andersson and Grubbström (1994), Grubbström (1996), among others, to investigate the stochastic properties of the production-inventory systems.

A process is defined to be a renewal process if its demand is created by unit events separated by independent stochastic time intervals which have the same but independent probability density functions $f(t)$, $t \geq 0$. This time interval can also been viewed as the lead time to generate a unit of output in a production inventory system, especially at the operational level where queues are unavoidable.

In previous papers (Andersson and Grubbström 1994, Grubbström 1996), efforts have been made to study the stochastic properties of the system in terms of the Laplace transform. Two most important theorems are present in the following.

Theorem 1
If the planning horizon is infinite, expected demand $\mathrm{E}(D(t))$ has the transform

$$E(D(t)) = \pounds^{-1}\left\{E(\widetilde{D}(s))\right\} = \pounds^{-1}\left\{\frac{\widetilde{f}(s)}{1 - \widetilde{f}(s)}\right\}$$

4.4

Theorem 2

If cumulative production at time t is $\overline{P}$ (including initial stockout), then, during the time that cumulative production remains at $\overline{P}$, the level of expected stockout $E(B(t))$ for the item will be part of the function

$$E(B(t)) = \pounds^{-1}\left\{E(\widetilde{B}(s))\right\} = \pounds^{-1}\left\{\frac{\widetilde{f}^{\,\overline{P}+1}(s)}{s(1 - \widetilde{f}(s))}\right\}$$

4.5

in this interval.

For instance the simplest and most commonly used distribution for time intervals is an exponential distribution

$$f(t) = \lambda e^{-\lambda t},$$

4.6

that leads to a well-known Poisson process. According to the transform approach, we first have

$$\widetilde{f}(s) = \frac{\lambda}{\lambda + s}.$$

4.7

Then expected demand will be

$$E(D(t)) = \pounds^{-1}\left\{E(\widetilde{D}(s))\right\} = \pounds^{-1}\left\{\frac{\lambda}{s}\right\} = \lambda$$

4.8

Hence expected demand is the constant λ which is also its time average. This gives the first property of the Poisson process, namely that the demand rate is independent of time.

Besides capturing the stochastic properties of demand, the transform also makes explicit the expression for the stockout function, which is not easily handled in any simple method in the time domain. The inverse transform of $E(\widetilde{B}(s))$ can be evaluated by Cauchy's Residue Theorem (Churchill, 1958). After inserting Equation 4.7 into 4.5 and taking the inverse transform, we have

$$E(B(t)) = \lambda t - \overline{P} + e^{-\lambda t} \sum_{j=0}^{\overline{P}-1} \frac{(\lambda t)^j}{j!}(\overline{P} - j)$$

4.9

It has been shown in this section that the stochastic properties of a renewal process can be captured through the transform method. The above theorems can be applied for other kinds of probability functions, such as the Gamma-distributed time interval (Grubbström and Tang 1997, Tang 2000).

### 4.3 Modelling lead times using control theory

The lead time models in the study of control theory have been discussed by Wikner (1994). He states that there are three common approaches in modelling lead times, namely, pure delay, first order delay and third order delay.

The pure delay model is the same as the deterministic lead time in the MRP theory, in which case the lead time is shifted for a fixed time lag τ. The input $u(t)$ and the response $y(t)$ are connected in the frequency domain as

$$y(s) = e^{-s\tau} u(s). \tag{4.10}$$

Here the response y represents the completion of the job whereas the input u is a function of the orders released. However, this pure delay model is seldom used in the study of production-inventory systems using control theory. Whereas the most common approach is the first order delay approach to describe the dynamics of lead times, for instance in Towill (1982), John, Naim and Towill (1994) and Tang and Naim (2003) as this results in rational transfer functions. However, some progress has been made in the continuous time pure time delay case. For example, see Warburton, Disney, Towill and Hodgson (2004) and Riddells and Bennett (2002).

Using a block diagram, the relation between the input and response is illustrated. In an APIOBPCS model (John, Naim and Towill, 1994, Figure 2.1), the ordering rate (*ORATE*) and completion rate (*COMRATE*) is connected through a transfer function, $\dfrac{1}{1 + sT_M}$, where T_M is the manufacturing lead time.
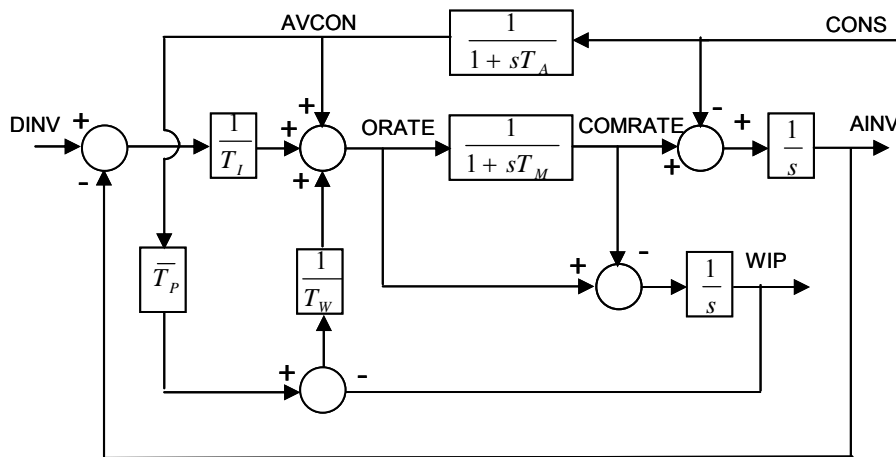

Figure 4.2. Block diagram of *APIOBPCS* model, John, Naim and Towill (1994)

The lead time is interpreted in the following diagram with a feedback (Figure 4.3). First the ordering rate is adjusted by a feedback completion rate. It is then accumulated through a transform $1/s$ to obtain a measure of feed-in work-in-process (WIP). According to Wikner's interpretation, the factor $1/T_M$ is explained as the fraction of *WIP* which is finished each time unit. Or alternatively T_M is the average lead time of the production unit. From Figure 4.3, we have the following balance condition

$$(ORATE - COMRATE)\frac{1}{s}\frac{1}{T_M} = COMRATE . \tag{4.11}$$

After rearranging the terms in the above equation, we finally obtain

$$COMRATE = \frac{1}{1 + sT_M} ORATE .$$ 4.12
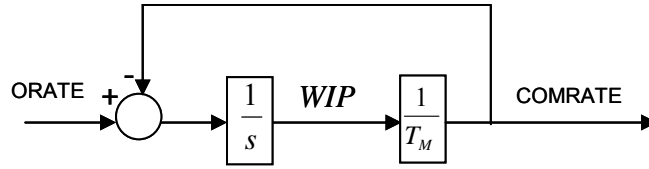
Figure 4.3. Interpretations of the lead times in the block diagram

The average lead time T_M is also interpreted as an inverse of the smoothing factor in exponential smoothing forecasting. It is well know that in a discrete time space, exponential smoothing is expressed as

$$F(k+1) = F(k) + \alpha(A(k) - F(k)),$$ 4.13

where $F(k)$ is the forecast in period k and $A(k)$ is the actual demand in period k. After rewriting the above difference function, we obtain a differential function in a continuous time space as

$$\frac{dF(t)}{dt} = \alpha(A(t) - F(t)).$$ 4.14

We take the transformation of the above function to obtain $sF(s) = \alpha A(s) - \alpha F(s)$, which leads to

$$F(s) = \frac{1}{1 + s/\alpha} A(s).$$ 4.15

Comparing Equations 4.12 and 4.15, we conclude that the lead time in this first order delay model is also interpreted as the smoothing parameter $T_M = 1/\alpha$. This has actually been illustrated in the *APIOBPCS* model (Figure 2.2), where the actual customer consumption (*CONS*) is added into the system as a part of ordering rate (*ORATE*) through exponential smoothing forecasting with a factor $1/T_A$.

Since $ORATE = e^{sT_M} COMRATE = (1 + sT_M + \frac{(sT_M)^2}{2!} + ...)COMRATE$, it is also easy to conclude that the first order delay can be viewed as the first order approximation of the pure delay model.

The third order delay model appears in Forrester (1961). This approach is identical to cascaded first order delays where the overall lead time T_M is evenly divided into three parts. Therefore response and input are written as

$$y(s) = \frac{1}{(1 + T_M s/3)^3} u(s)$$ 4.16

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The lead time in this case can also be interpreted as following a Gamma/Erlang distribution, with a mean value T_M and variance $T_M^2/3$. In principle, the delay could be described with any order n and the transfer function is $\dfrac{1}{(1+T_M s/n)^n}$. The mean lead-time is again T_M, whereas the variance becomes T_M^2/n. In one extreme case when $n=1$, we obtain a first order delay and the lead time follows an exponential distribution. In another extreme case when n approaches infinite, the transfer function turn out to be e^{sTM}. Its lead-time is then deterministic since the variance reduces to zero.

According to Forrester (1961), the third order delay model has been indicated as an appropriated compromise between complexity and accuracy. A comparison study of the delay model with different orders has been made by Wikner (1994), where generic lead-time model is presented and a simulation study is used to capture the system dynamics when lead time is stochastic.

4.4 The discrete time case

Pure time delays are much easier to handle in discrete time, due to the definition in the z-transform. The corresponding time shift theorem is simply,

$$Z\{f_2(t)\} = Z\{f_1(t+\tau)\} = z^{-\tau} Z\{f_1(t)\} \qquad\qquad 4.17$$

The discrete time analogue of a general order lags is given by,

$$\frac{COMRATE}{ORATE} = \left(\frac{\alpha}{1 - z^{-1}(1-\alpha)}\right)^n \qquad\qquad 4.18$$

This is important because it yields discrete time case of the expected dynamic behaviour of the stochastic lead-times of Erlang k distribution of order $k=n$ (Wikner 1994). Setting $\alpha = \dfrac{1}{1+Ta}$, as it simplifies the exposition, the variance of this general order lag is given by

$$\sigma_\tau^2 = n^{2n}(n+Ta)^{-2n} {}_2F_1\left[n,n,1,\frac{Ta^2}{(n+Ta)^2}\right] \qquad\qquad 4.19$$

where ${}_2F_1$ is the Gauss hyper-geometric function and n is the order of the lag and Ta is the expected value of the lag.

5. The problem of one – to –one correspondence between MRP and production and inventory control model

Linear production systems without delay are usually studied in the form:

$$\frac{d\mathbf{x}(t)}{dt} = \Pi \mathbf{x}(t) + \Theta \mathbf{u}(t) \qquad\qquad 5.1$$

with the initial conditions

$$\mathbf{x}(0) = \mathbf{x}_0 \,, \mathbf{x}(t) \in \Re^n \,, \mathbf{u}(t) \in \Re^m \,, \Pi \in L(\Re^n, \Re^n) \,, \Theta \in L(\Re^m, \Re^n)$$

and output relation

$$\mathbf{y}(t) = \Gamma \mathbf{x}(t) \,, \quad \mathbf{y}(t) \in \Re^r \,, \Gamma \in L(\Re^n, \Re^r) \,, \tag{5.2}$$

where $L(\Re^m, \Re^n)$ denotes the real Banach space of all continuous linear maps $\Lambda : \Re^m \to \Re^n$. The unique solution of equation (5.1) is

$$\mathbf{x}(t) = \varphi(t,0)\mathbf{x}_0 + \int_0^t \varphi(t,\tau)\Theta\mathbf{u}(\tau)d\tau \,, \tag{5.3}$$

where the transition matrix $\varphi(t,\tau)$ can be immediately determined as

$$\varphi(t,\tau) = e^{\Pi(t-\tau)} \,. \tag{5.4}$$

The matrix exponential is defined as

$$e^{\Pi t} = \mathbf{I} + \Pi t + \frac{\Pi^2 t^2}{2!} + \frac{\Pi^3 t^3}{3!} + \cdots \tag{5.5}$$

and this series converges uniformly and absolutely on any finite interval. We can write the state of the system as

$$\mathbf{x}(t) = e^{\Pi t}\mathbf{x}_0 + \int_0^t e^{\Pi(t-\tau)}\Theta\mathbf{u}(\tau)d\tau \,. \tag{5.6}$$

Before we use Laplace transform of (5.1) – (5.2), let us consider the procedures above in $L_2([0,\infty), \, \Re^n)$ space which is the proper state space for Laplace transformed production functions which take place in fundamental equations. In L_2 space all properties which follow from (5.1) – (5.11) are still valid, if we assume:

$$\mathbf{x} \in \Re^n \,, \mathbf{x}(.) \in L_2([0,\infty), \, \Re^n), \mathbf{u} \in \Re^m \,,$$
$$\mathbf{u}(.) \in L_2([0,\infty), \, \Re^m), \Pi \in L(\Re^n, \Re^n) \,, \Theta \in L(\Re^m, \Re^n) \tag{5.7}$$

and output relation is

$$\mathbf{y}(t) = \Gamma\mathbf{x}(t) \,, \mathbf{y} \in \Re^r \,, \mathbf{y}(.) \in L_2([0,\infty), \, \Re^r), \Gamma \in L(\Re^n, \Re^r) \,, \tag{5.8}$$

where $L(\Re^m, \Re^n)$ denote the real Banach space of all continuous linear maps $\Lambda : \Re^m \to \Re^n$.

From the basic properties of L_2 functions it follows that Laplace transform of L_2 functions has one-to-one relationship with its inverse, Bogataj (1989). Specifically, if the functions in fundamental equations are L_2 functions, uniqueness is assured and the mapping $f \mapsto £\{f\}$ is one-to-one all along the procedure. In this case the appropriate integrals are taken in the Lebesque – Riemann sense.

Such an approach has several advantages over the traditional work in sequentially continuous functions, which had been considered by Churchill (1958). In real time the fundamental equations are composed of sequentially continuous functions having derivatives almost everywhere. These derivatives present also the finite changes of inventory values in infinitesimally small time intervals. In general it means that there also finite number of finite value jumps take place on the set of measure zero. This is the reason why we have to work in L_2 space and we can not use Churchill's results in general. **L_2** space is rich enough for all necessary generalizations of inventory systems. In the state space of sequentially continuous functions there is a great limitation because the modern tools of functional analysis well developed in abstract Hilbert spaces can not be used for investigating:

- structural properties,
- optimization procedures,
- sensitivity,
- stabilizability
- and other important properties of the models,

when one-to-one correspondence should be assures in each step of parallel working in real time and frequency space. When we are doing these procedures in required spaces, the theory of MRP Input – Output analysis and Laplace transforms, developed by Linköping School and others, is well founded for further investigation of structural properties, optimizations and other properties of delayed production / inventory systems.

Using Laplace transform of functions in L_2 space, we obtain from (5.1) and (5.2):

$$£\{\dot{\mathbf{x}}(t)\} = \Pi\,£\{\mathbf{x}(t)\} + \Theta\,£\{\mathbf{u}(t)\}\,, \qquad\qquad 5.9$$

$$£\{\mathbf{y}(t)\} = \Gamma\,£\{\mathbf{x}(t)\}\,. \qquad\qquad 5.10$$

Considering $£\{\dot{\mathbf{x}}(t)\} = -\mathbf{x}(0) + s\,£\{\mathbf{x}(t)\}$ and notation $£\{\mathbf{x}(t)\} = \widetilde{\mathbf{x}}(s)$, we obtain

$$\widetilde{\mathbf{x}}(s) = (s\mathbf{I}-\Pi)^{-1}\mathbf{x}(0) + (s\mathbf{I}-\Pi)^{-1}\Theta\widetilde{\mathbf{u}}(s)\,,$$
$$\widetilde{\mathbf{y}}(s) = \Gamma(s\mathbf{I}-\Pi)^{-1}\mathbf{x}(0) + \Gamma(s\mathbf{I}-\Pi)^{-1}\Theta\widetilde{\mathbf{u}}(s)\,.$$

If we use Laplace transform for equation (5.9), we obtain

$$\widetilde{\mathbf{x}}(s) = £\{e^{\Pi t}\}\mathbf{x}_0 + £\{e^{\Pi t}\}\Theta\,\widetilde{\mathbf{u}}(s)\,, \qquad\qquad 5.11$$
$$\widetilde{\mathbf{y}}(s) = \Gamma\,£\{e^{\Pi t}\}\mathbf{x}_0 + \Gamma\,£\{e^{\Pi t}\}\Theta\,\widetilde{\mathbf{u}}(s)\,, \qquad\qquad 5.12$$

where
$$£\{f_1 * f_2\} = £\{\textstyle\int_0^t f_1(\tau)f_2(t-\tau)d\tau\} = £\{f_1(t)\}£\{f_2(t)\}\,.$$

Thus, the Laplace transform of the state transition matrix $e^{\Pi t}$ is seen to be the matrix $(s\mathbf{I}-\Pi)^{-1}$. We can prove this, if we consider the expression (5.5) for the matrix exponential

$$£\{e^{\Pi t}\} = £\left\{\sum_{k=0}^{\infty}\frac{(\Pi t)^k}{k!}\right\} = \sum_{k=0}^{\infty}\frac{\Pi^k}{k!}\,£\left\{t^k\right\} = \sum_{k=0}^{\infty}\frac{\Pi^k}{k!}\cdot\frac{k!}{s^{k+1}}$$

$$= \frac{1}{s}\sum_{k=0}^{\infty}\left(s^{-1}\Pi\right)^k = s^{-1}(\mathbf{I} - s^{-1}\Pi)^{-1} = (s\mathbf{I} - \Pi)^{-1}.$$

The approach suggested above is especially necessary when time delays in the time domain take place. In such cases of functional – differential equations the system can be modelled by the differential delay equation. Using Laplace transform of functions in product space M_2 x L_2, we obtain from (5.1) and (5.2):

$$\pounds\{\dot{\mathbf{x}}(t)\} = \Pi\,\pounds\{\mathbf{x}(t,\tau^1)\} + \Theta\,\pounds\{\mathbf{u}(t,\tau^2)\}\,,$$ 5.13

$$\mathbf{x}(t)\in\Re^n,\ \mathbf{x}(t+\tau)\in L_2\left(\left[-\tau,\infty\right),\ \Re^n\right),$$
$$\mathbf{u}(t+\eta)\in L_2\left(\left[-\eta,\infty\right),\ \Re^m\right),\Pi\in L(\Re^n,\Re^n),\ \Theta\in L(\Re^m,\Re^n)$$ 5.14

and output relation is
$$\mathbf{y}(t)=\Gamma\mathbf{x}(t)\,,$$
$$\mathbf{y}(t)\in L_2\left(\left[0,\infty\right),\ \Re^r\right),\ \Gamma\in L(\Re^n,\Re^r)$$

In this general setting differential delay equation (5.13) can be written as an abstract evolution equation where the infinitesimal generator of a strongly continuous semigroup of operators can be stated and the solution derived using the semigroup approach. For details see Hale and Verduyn (1993). Using analogue procedures as Churchill (1958) uniqueness and therefore one-to-one correspondence all along the parallel procedures in Laplace transformed space is also assured when we introduce in the procedure the derivatives of sequentially continuous functions, with values almost everywhere, which is the case in production/inventory control study of one-to-one correspondence in the time and frequency domain. In this case we overcome many inconveniences, which appear using the properties of the state space of continuous functions. Therefore the theory of MRP, Input – Output analysis and Laplace transforms developed up to now, can be mathematically well founded.

6. Integrating MRP and production and inventory control models: An example

As a way of investigating the link between MRP and PIC let us consider a simple industrial scenario where it is easy to interlink MRP and PIC theory. As we do this we will discuss the issues that arise in exploiting the full potential of the available theories.

Consider that a world class manufacturing company (WCM) has a product range of two items, A and B. Figure 5.1 graphically describes their joint Bill of Materials (BOM). BOM's such as these are completely described by MRP theory via the input matrix. We could generalize the input matrix and shift the requirements for sub-assemblies (items C, D and E) forward in time, via conventional MRP logic. However, this would require knowledge of future events, something that is not usually done in PIC theory. If we were to not undertake this time shifting, we may model correctly a make to stock environment. That is WCM company, maintains an inventory of finished goods of product A and B, and from this inventory, satisfies consumer demand. WCM also maintains an inventory of components items (C, D and E). This raw material inventory is used to satisfy production requirements of item A and B.
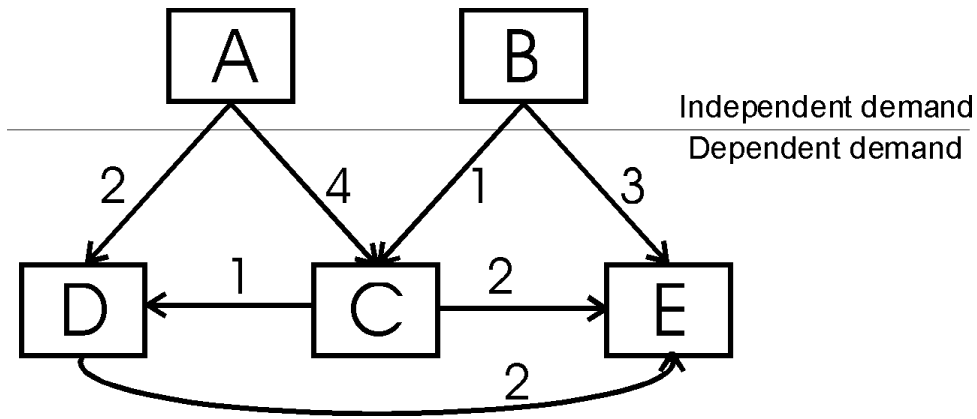
Figure 6.1. Bill of materials

We may convert this BOM into an MRP input matrix as follows. Here row 1 (column 1) represents item A, row 2 (column 2) represents item B, row 3 (column 3) represents item C, row 4 (column 4) represents item D and row 5 (column 5) represents item E. A column in the input matrix is read follows, item A (column), is made up of no A's, no B's, four C's, two D's and no E's.

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & 2 & 0 \end{bmatrix} \qquad\qquad 6.1$$

From (6.1) we take the Leontief inverse of **H**, to yield the total requirements for all components as follows,

$$(\mathbf{I} - \mathbf{H})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & 0 \\ 6 & 1 & 1 & 1 & 0 \\ 20 & 7 & 4 & 2 & 1 \end{bmatrix} \qquad\qquad 6.2$$

The next step is to determine the influence of demand on the product structure. We consider the independent demand to be two Heaviside step functions; for product A the step occurs at time $n=0$, for product B the step occurs at time, $t=5$. The Laplace transform representations of demands may be collected into a demand vector as follows. Notice that we could have also considered that demands can be placed on the dependent products in the demand column vector (6.3). This might represent, for example, a demand of spare parts or consumables in the product structure. However we have set this to zero here.

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$$\widetilde{\mathbf{D}} = \begin{bmatrix} 1/s \\ e^{-5s}/s \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad\qquad 6.3$$

The independent demand is satisfied from finish goods inventory and any unmet independent demand is fully backlogged. Using APIOPBCS model, as a means to generating the production targets for the independent items, yields the following requirements vector. $\widetilde{\mathbf{R}}$ is simply the product of the transfer function of the replenishment rule and the demand vector.

$$\widetilde{\mathbf{R}} = \left[\frac{1 + s(Ta + Ti + Tp)}{(1 + Tas)(1 + Tis)s} \quad \frac{1 + s(Ta + Ti + Tp)}{(1 + Tas)(1 + Tis)s}e^{-5s} \quad 0 \quad 0 \quad 0 \right]^{\mathbf{T}} \qquad 6.4$$

(6.4) has incorporated a typical PIC policy into our MRP model (this model is the APIOBPCS model of John, Naim and Towill (1994), with *Tw=Ti*). Thus, rather than exploiting traditional MRP theory to schedule production we are exploiting the APIOBPCS replenishment rule for this task.

The first two rows of (6.4) describe the time evolution of the production requirements for items A and B. Combining this with the Leontief inverse reveals the dynamic production of the complete BOM for the two products;

$$\widetilde{\mathbf{R}} = (\mathbf{I} - \mathbf{H})^{-1}\widetilde{\mathbf{D}} = \begin{bmatrix} \dfrac{1 + s(Ta + Ti + Tp)}{(1 + Tas)(1 + Tis)s} \\[1em] \dfrac{1 + s(Ta + Ti + Tp)}{(1 + Tas)(1 + Tis)s}e^{-5s} \\[1em] \dfrac{4(1 + s(Ta + Ti + Tp))}{(1 + Tas)(1 + Tis)s} + \dfrac{1 + s(Ta + Ti + Tp)}{(1 + Tas)(1 + Tis)s}e^{-5s} \\[1em] \dfrac{6(1 + s(Ta + Ti + Tp))}{(1 + Tas)(1 + Tis)s} + \dfrac{1 + s(Ta + Ti + Tp)}{(1 + Tas)(1 + Tis)s}e^{-5s} \\[1em] \dfrac{20(1 + s(Ta + Ti + Tp))}{(1 + Tas)(1 + Tis)s} + \dfrac{1 + s(Ta + Ti + Tp)}{(1 + Tas)(1 + Tis)s}7e^{-5s} \end{bmatrix}^{\mathbf{T}} \qquad 6.5$$

Taking the inverse Laplace transform of (6.5) yields the time domain response of the complete production of the BOM. We have plotted this in Figure 6.2. It is clear to see the time varying production of the various items in the BOM.
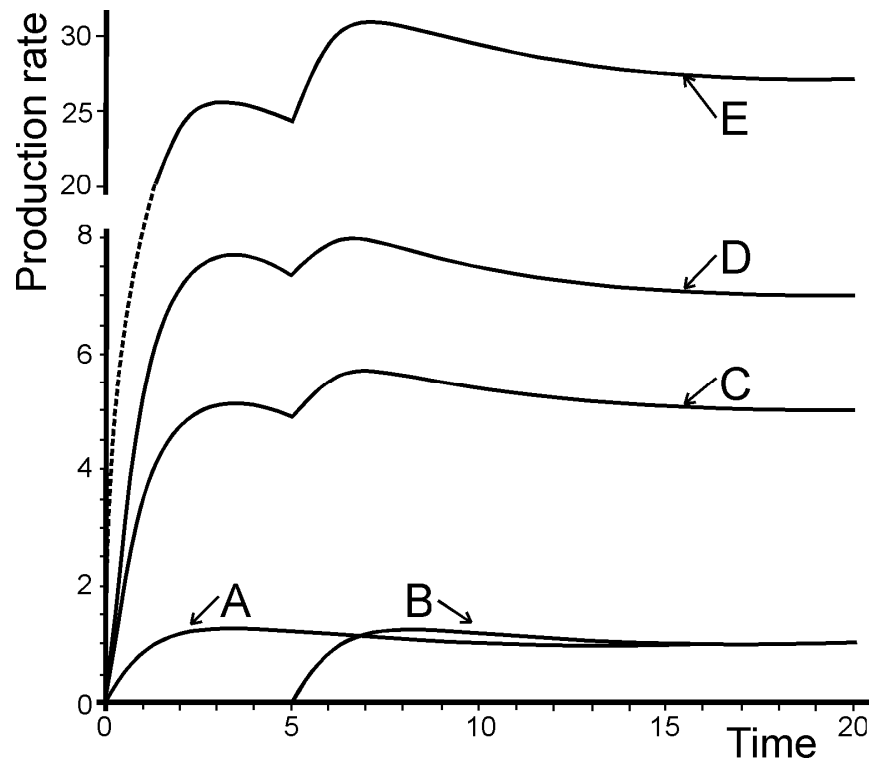
Figure 6**.2. Total production rates for the complete bill of materials in our example**

This simple example has shown that is possible to combine MRP and PIC theory. However we, have yet to exploit the full potential of the MRP theory. In particular, the time phasing of component items during the lead-time for finished items, and the non-negativity of inventory levels. However the MRP fundamental equations and simple control theory have allowed us to investigate the dynamic behaviour of a product range and their component items in a neat and compact manner. Of course, there is a lot more work to do to integrate and fully exploit both fields of endeavour, and this will be the considered in future research. Now we will conclude some final remarks on the two approaches.

7. Remarks on the two approaches

It is of interest to note that in both approaches, the Laplace transform plays an important role. In the MRP theory approach, the transform is used to capture the time lags using a time shift operation. Stochastic properties are captured through the probability density function of the time intervals (lead times). The lead-time can be further incorporated into a production-inventory system, for instance the fundamental equations in Section 2. The production (ordering) decisions in this case can be either continuous or discrete.

However, in the control theory approach, the control variables are rather rate-based. For instance it is easy in the *APIOBPCS* systems to describe the dynamics of order and completion rate if these two rates are continuous variables. However, if due to the technical constraints, we need to model the ordering releasing as a discrete volume, (i.e. order one batch and after a certain time order the second batch) it becomes rather difficult to implement the control theory approach.

It is also indicated that in an MRP approach, we can handle multi-level/multi-item systems. The demand relationship between items at different levels can be explicitly described in a generalized input matrix. Using control theory to study a production-inventory model, we often need to exploit the systems dynamics from the top level down to a lower one. It applies in a demand pull system as well as a push type MRP system (Bonney, Popplewell and Matoug, 1994). In the latter case, the transfer function mainly consists of a summation of feed-forward and a summation of feedback (Popplewell and Bonney, 1987).

According to the literature, modelling the setup time, which is often a major ingredient of lead-time, has not been thoroughly investigated in either approach. Even though, since setup times are often a constant regardless the size of batch, it is straightforward to add the set-up time to the generalised input matrix in the MRP approach. Whereas in the control theory approach, it becomes rather unclear how to express this part of lead time: first a setup adds nonlinearity into the system and secondly a constant time lag often needs a pure delay model which is more difficult manage in a continuous time control theory approach.

It is not obvious how to cope with the non-negativity in inventory levels in the PIC approach. Convolution may be used in specific cases to yield analytical solutions, but it is not possible to say anything of a general nature with this approach. Some progress has been made with this aspect with MRP theory however. The fine-tuning of the time phasing of individual requirements in the assembly structure is treated properly with MRP theory; there is no such consideration in PIC theory.

The major difference between the two approaches seems to be how they treat knowledge of demand information. The MRP theory is relevant for scenarios where future demands are known. If the realisation of demand is different from that previous thought, a re-scheduling problem exists that is closely related to the problem of system nervousness. However, PIC theory does not make any prediction about the future, a PIC model is completely determined by the known inputs and previous actions. Thus, in a sense, a PIC system re-schedules continuously and the system nervousness problem does not exist. Although, the bullwhip problem, could be considered to be analogous to system nervousness at the instant an production order is made.

8. References

Adelson, R.M. (1966), The dynamic behaviour of linear forecasting and scheduling rules", *Operational Research Quarterly*, Vol. 17, No. 4, pp. 447-462

Andersson, L. E. and Grubbström, R. W. (1994), *Asymptotic Behaviour of a Stochastic Multi-Period Inventory Process with Planned Production,* Working Paper WP-210, Department of Production Economics, Linköping Institute of Technology.

Aseltine, J. A. (1958), *Transform Method in Linear System Analysis*, McGraw-Hill, New York

Axsäter, S. (1985), Control theory concepts in production and inventory control, *International Journal of Systems Science*, Vol. 16, No 2, pp. 161-169.

Bessler, S.A. and Zehna, P.W. (1968), An application of servomechanisms to inventory, *Naval Research Logistics Quarterly*, Vol. 15, June, pp. 157-168.

Bogataj L. and Bogataj M. (1992), Perturbed inventory systems with delays, *International Journal of Production Economics*, Vol. 26, No. 1-3, pp. 277-28.

Bogataj L. and Čibej, J.A. (1994), Perturbations in living stock and similar biological inventory systems, *International Journal of Production Economics, Volume 35, Issues 1-3, pp.233-239*

Bogataj L. and Horvat L. (1996) Stochastic considerations of Grubbström—Molinder model of MRP, input-output and multi-echelon inventory systems[*1], *International Journal of Production Economics*, Vol. 45, No. 1-3, pp. 329-336.

Bogataj, L. (1989), The adjoint operator of the infinitesimal generator for the linear differential-difference and in the control. *Glas. mat.*, 24, 2-3, pp. 349-354.

Bogataj, L. (1989a), Sensitivity of linear-quadratic systems with delay in the state and in the control for perturbation of the systems matrices. *Glas. mat.*, 24, 2-3, pp. 355-360.

Bogataj, L. and Bogataj, M. (1998). *Input-output analysis applied to MRP models with compound distribution of total demand,* www.inform.umd.edu/EdRes/Topic/Economics/ EconData/WorkPaper/IOPAP/journali.pdf

Bogataj, L., Ferbar, L. and Bogataj, M. (2003), Controllability of supply chains, controlled in Laplace - transformed space, New results in Mathematical Economics, Operational Research and Logistics, WP 2 , KMOR, EF University of Ljubljana.

Bogataj, L., Grubbström, R. W. and Bogataj, M. (1997), Eds, *Input-Output Analysis and Laplace Transforms in Material Requirements Planning*, Storlien, Sweden

Bogataj, M. (1994), Optimal control of hereditary inventory systems with short-time conservation effects, *International Journal of Production Economics*, Vol. 35, No. 1-3, pp. 241-244.

Bogataj, M. (1999), Inventory Allocation and Customer Traveling Problem in Spatial Duopoly, *International Journal of Production Economics*, Vol. 59, pp. 271–279

Bogataj, M. , Bogataj, L. Vodopivec, R. (2004), Stability of Perishable Goods in Cold Logistic Chains, *International Journal of Production Economics*, (to appear).

Bogataj, M. and Bogataj, L. (2004), On the compact presentation of the lead times perturbations in distribution networks, *International Journal of Production Economics,* Vol. 88, No. 2, pp. 145-155.

Bogataj, M. and Bogataj. L. (2001), Supply chain coordination in spatial games, *International Journal of Production Economics*, Vol. 71, No 1-3, pp. 277-28.

Bonney, M, Popplewell, K. P. and Matoug, M. (1989), Effect of Errors and Delays in Inventory Reporting on Production System Performance, *International Journal of Production Economics*, Vol. 35, pp. 93-105.

Bonney, M. C. and Popplewell, K. (1987), The Application of Discrete Linear Control Theory to the Analysis and Simulation of Multi-product, Multi-level Production Control Systems, *International Journal of Production Research,* Vol. 25, No 1 pp. 45–56

Brown, R.G., 1963, "Smoothing, forecasting and prediction of discrete time series", Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Burns, J.F. and Sivazlian, B.D. (1978), Dynamic analysis of multi-echelon supply systems, *Computer and Industrial Engineering*, Vol. 2, pp. 181-193

Chen, Y.F. and Disney, S.M., (2003) "The order-up-to policy "sweet spot" - Using proportional controllers to eliminate the bullwhip problem", EUROMA POMS Conference, Como Lake, Italy, 16-18 June, Vol. 2, pp. 551-560

Churchill, R. V. (1958), *Operational Mathematics*, McGraw-Hill, New York

Čibej J.A. and Bogataj L. (1994) Sensitivity of quadratic cost functionals under stochastically perturbed controls in inventory systems with delays, *International Journal of Production Economics*, Vol 35, No. 1-3, pp.265-270.

Clark, A. J. and Scarf, H. (1960), Optimal Policies for a Multi-Echelon Inventory Problem, *Management Science*, Vol. 6, pp. 475–490

Dejonckheere, J., Disney, S.M., Farasyn, I., Janssen, F., Lambrecht, M., Towill, D.R. and Van de Velde, W. (2002), Production and Inventory Control: The variability trade-off, Proceedings of the 9th EUROMA Conference, June 2-4, Copenhagen, Denmark.

Dejonckheere, J., Disney, S.M., Lambrecht, M.R. and Towill, D.R., (2004), The impact of information enrichment on the bullwhip effect in supply chains: A control engineering perspective, *European Journal of Operational Research*, Vol. 152, pp. 727-750.

DeWinter, R.E., (1966), Inventory applications of servo-mechanism models, MSc Thesis, United States Naval Postgraduate School.

Deziel, D.P. and Eilon, S., (1967), A linear production-inventory control rule, *The Production Engineer*, Vol. 43, pp. 93-104.

Disney, S.M. and Grubbström, R.W. (2003) "The economic consequences of a production and inventory control policy", 17th International conference on Production Research, Virginia, USA, 3-7 August, ISBN 0-9721257-3-6.

Disney, S.M. and Towill, D.R., (2002), A discrete linear control theory model to determine the dynamic stability of Vendor Managed Inventory supply chains, *International Journal of Production Research*, Vol. 40, No. 1, pp. 179-204

Disney, S.M. and Towill, D.R., (2003) On the bullwhip and inventory variance produced by an ordering policy, *OMEGA: The International Journal of Management Science,* Vol. 31, No. 3, pp. 157-167

Disney, S.M., Farasyn, I., Lambrecht, M., Towill, D.R. and Van de Velde, W., (2003), "Creating win-win scenarios from the bullwhip problem - By design not accident", EUROMA POMS Conference, Como Lake, Italy, 16-18 June, Vol. 2, pp. 561-570

Forrester, J. W. (1961), Industrial Dynamics, MIT Press, Cambridge, MA

Grubbström, R. W. (1967), On the Application of the Laplace Transform to Certain Economic Problems, *Management Science*, Vol. 13, pp. 558–567

Grubbström, R. W. (1996), Stochastic Properties of a Production-Inventory Process with Planned Production Using Transform Methodology, *International Journal of Production Economics,* Vol. 45, pp. 407–419

Grubbström, R. W. (1998a), A Net Present Value Approach to Safety Stocks in a Planned Production, *International Journal of Production Economics*, Vol. 56–57, pp. 213–229

Grubbström, R. W. (1999), *The Fundamental Equations of MRP Theory for a Simple Feed-Back System with Lead Times*, Working Paper WP-264, Department of Production Economics, Linköping Institute of Technology

Grubbström, R. W. and Jiang, Y. (1990), A Survey and Analysis of the Application of the Laplace Transform to Present Value Problems, *Revista di matematica per le scienze economiche e sociale*, Vol. 12, pp. 43–62

Grubbström, R. W. and Lundquist, J. (1977), The Axsäter Integrated Production-Inventory System Interpreted in Terms of the Theory of Relatively Closed Systems, *Journal of Cybernetics*, Vol. 7, pp. 49–67

Grubbström, R. W. and Molinder, A. (1994), Further Theoretical Considerations on the Relationship between MRP, Input-Output Analysis and Multi-Echelon Inventory System, *International Journal of Production Economics*, Vol. 35, pp. 299–311

Grubbström, R. W. and Ovrin, P. (1992), Intertemporal Generalization of the Relationship between Material Requirements Planning and Input-Output Analysis, *International Journal of Production Economics*, Vol. 26, pp. 311–318

Grubbström, R. W. and Tang, O. (1998) An Overview of Input-Output Analysis Applied to Production − Inventory Systems, Paper presented at *The Twelfth International Conference on Input-Output Techniques,* New York.

Grubbström, R. W. and Tang, O. (1997), *Stochastic Properties of a Demand Renewal Process with a Gamma-Distributed Time Interval*, Working Paper WP-245, Department of Production Economics, Linköping Institute of Technology, Sweden

Grubbström, R. W. and Tang, O. (2000a), An Overview of Input-Output Analysis Applied to Production-Inventory Systems, *Economic Systems Research*, Vol. 12, No 1, pp. 3-25

Grubbström, R. W. and Tang, O. (2000b), Modelling Rescheduling Activities in a Multi-Period Production-Inventory System, *International Journal of Production Economics*, Vol. 68, No 2, pp. 123-135.

Grubbström, R. W. and Wang, Z. (1999), Introducing Capacity Limitations into Multi-Level, Multi-Stage Production-Inventory Systems applying the Input-Output/Laplace Transform Approach. In Hillery, M. T. and Lewis, H. J., (Eds.), *The 15th International Conference on Production Research*, Limerick, Ireland, August 9-12, pp. 435–438

Grubbström, R. W. and Wang, Z. (2003), A Stochastic Model of Multi-Level/Multi-Stage Capacity-constrained Production-Inventory Systems, *International of production Economics,* Vol. 81-82, pp 483-494.

Grubbström, R. W., 1996, Stochastic Properties of a Production-Inventory Process with Planned Production Using Transform Methodology, *International Journal of Production Economics,* Vol. 45, pp. 407-419.

Hale, J.K. and Verduyn, L.S., (1993), *Introduction to Functional Differential Equations*, Springer Verlag Gmbh.

Horvat, L. and Bogataj, L. (1996a), MRP, Input-Output Analysis and Multi-Echelon Inventory Systems with Exponentially Distributed External Demand, *Proceedings of GLOCOSM Conference*, Bangalore, India, pp. 149-154

Horvat, L. and Bogataj, L. (1996b), Game Theory Aspects of MRP, Input-Output and Multi-Echelon Inventory Systems, *Pre-Prints, Ninth International Working Seminar on Production Economics*, Vol. 3, Igls/Innsbruck, pp. 1-12

Horvat, L. and Bogataj, L. (1999), A Market Game with the Characteristic Function According to the MRP and Input-Output Analysis Model, *International Journal of Production Economics*, Vol. 59, pp. 281-288

Howard, R.A. (1963), System analysis of linear models, in Multistage inventory models and techniques, Edited by Scarf, H.E., Gilford, D. and Shelly, M., Stanford University Press, Stanford, CA, pp. 143-184

John, S., Naim, M.M. and Towill, D. R. (1994), Dynamic Analysis of a WIP Compensated Decision Support System, *International Journal of Manufacturing System Design*, Vol. 1, No. 4, pp 283-297

Leontief, W. W. (1928), Die Wirtschaft als Kreislauf, *Archiv für Sozialwissenschaft und Sozialpolitik*, Vol. 60, No 3, pp. 577–623

Magee, J.F. (1958), Production planning and inventory control, McGraw-Hill Book Company, New York

Orlicky, J. (1975), *Material Requirements Planning*, McGraw-Hill, New York

Reilly, J.V. (1965), A dynamic inventory model using exponential smoothing", Unpublished MSc Thesis, US Naval Post Graduate School.

Riddalls, C.E. and Bennett, S. (2002) The stability of supply chains, *International Journal of Production Research*, Vol. 40, No. 2 pp 459-475.

Riddalls, C.E., Bennett, S. and Tipi, N.S. (2000) Modelling the dynamics of supply chains, *International Journal of Systems Science* Vol. 31 No. 8 pp 969-976.

Segerstedt, A. (1995), *Multi-Level Production and Inventory Control Problems–Related to MRP and Cover–Time Planning*, PROFIL 13, Production-Economic Research in Linköping, Linköping

Simon, H.A., (1952) On the application of servomechanism theory to the study of production control, *Econometrica* Vol. 20 pp 247-268.

Tang, O. (2000), Modelling Stochastic Lead Times in a Production-Inventory System Based on the Laplace Transform Method, *International Journal of Production Research*, Vol. 38, No 17, pp 4217-4226.

Towill, D. R. (1982), Dynamic Analysis of an Inventory and Order Based Production Control System, *International Journal of Production Research*, Vol. 20, No 6, pp 671-687.

Grubbström, R.W., Bogataj, L., Bonney, M.C., Disney, S.M. and Tang, O., (2004) "Inter-linking MRP theory and production and inventory control models", Proceedings of the 13th International Working Conference on Production Economics, Igls, Austria, February, Vol. 2, pp259-278, ISSN 0925 5273.

Tsypkin, Y.Z. (1964), Sampling systems theory and its application. Pergamon Press, Oxford.

Vassian H.J., (1955) Application of discrete variable servo theory to inventory control, *Journal of the Operations Research Society of America,* Vol. 3, No. 3, pp. 272-282.

Warburton, R.D.H., Disney, S.M. Towill, D.R., and Hodgson, J.P.E. (2004), Further insights into "The stability of supply chains", *International Journal of Production Research*, Vol. 42, No. 3, pp. 639-649.

Wikner J. (1994), *Dynamic Modelling and Analysis of Information Flows in Production-Inventory and Supply Chain System*, PROFIL 12, Production-Economic Research in Linköping, Linköping.