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THE ORDER-UP-TO POLICY "SWEET SPOT" – USING PROPORTIONAL CONTROLLERS TO ELIMINATE THE BULLWHIP PROBLEM

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ABSTRACT

We develop a discrete control theory model of a stochastic demand pattern with both Auto Regressive and Moving Average (ARMA) components. We show that the bullwhip effect arises when the myopic Order-Up-To (OUT) policy is used. This policy is optimal when the ordering cost is linear. We then derive a set of z-transform transfer functions of a modified policy that allows us to avoid the bullwhip problem by incorporating a proportional controller into the inventory position feedback loop. With this technique, the order variation can be reduced to the same level as the demand variation. However, bullwhip-effect avoidance in our policy always comes at the costs of holding extra inventory. When the ordering cost is piece-wise linear and increasing, we compare the total cost per period under the two types of ordering policies: with and without bullwhip-effect reduction. Numerical examples reveal that the cost saving can be substantial if order variance is reduced using the proportional controller.

Keywords: Bullwhip effect, Inventory, Order-Up-To policy, Control theory

INTRODUCTION

The purpose of an ordering policy is to control production or distribution in such a way that supply is matched to demand, inventory levels are maintained within acceptable levels and capacity requirements are kept to a minimum. In doing so, however, the bullwhip effect may arise (Lee, Padmanabhan and Whang, 1997). It has been estimated that the economic consequences of the bullwhip effect can be as much as 30% of factory gate profits (Metters, 1997). Carlsson and Fullér (2000) have further summarised the negative impacts as follows;

- Excessive inventory investments throughout the supply chain to cope with the increased demand variability
- Reduced customer service due to the inertia of the production/distribution system
- Lost revenues due to shortages
- Reduced productivity of capital investment

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- Increased investment in capacity
- Inefficient use of transport capacity
- Production schedules missed more frequently

The Order Up To (OUT) policy is a standard ordering algorithm in many MRP systems (Gilbert, 2002) that is used to achieve the customer service, inventory and capacity trade-off. This policy is often set by the company to coordinate orders for multiple items from the same supplier, where setup costs may be reasonably ignored. Conceptually, the OUT policy is very easy to understand. Periodically, we review our inventory position and place an “**order**” to bring the inventory position “**-up-to**” a defined level. However, because of the lead-time between placing an order and receiving the goods into stock, we need to forecast demand. Common forecasting techniques to exploit here include moving average and exponential smoothing (Chen, Drezner, Ryan and Simchi-Levi, 2000; and Dejonckheere, Disney, Lambrecht and Towill, 2003). These techniques are useful for identifying environmental changes when the mean demand moves from one level to another. However, if demand possesses a linear trend or an explosive geometric growth, other forecasting techniques such as double or triple exponential smoothing may be more appropriate (Dejonckheere, Disney, Lambrecht and Towill, 2002).

In this contribution, we consider demand to be a weakly stationary stochastic variable that contains Auto Regressive and Moving Average (ARMA) components (Box and Jenkins, 1970). In the specific case of ARMA demand, it is well known that conditional expectation will provide the minimum mean squared error forecast, see Lee, So and Tang (2000). Indeed, it has long been noted that using conditional expectation, as the forecasting mechanism within the OUT policy will minimise the total inventory related cost over time (Johnson and Thomson, 1975). Recently, it has been noticed that using conditional expectation forecasting within the OUT policy will produce a policy that can actually avoid the bullwhip effect for certain instances of the ARMA demand pattern (Alwyn, 2001). However, in certain instances of the class of ARMA demands bullwhip still can not be avoided with this optimal forecasting technique.

Herein we present a z-transform model of the myopic OUT policy with conditional expectation forecasting. We exactly quantify the bullwhip produced by the system, together with closed form expressions of the resulting inventory variance over time. We have modified the classical OUT policy by using a simple control engineering principle, as also exploited by Magee (1956), to eliminate the bullwhip problem for **all** instances of the ARMA demand pattern. The principle exploited is to incorporate proportional controllers into the inventory position feedback loop in the classical myopic OUT policy.

We start by introducing the classical ARMA demand model, then proceed to discuss the OUT policy and then move on to derive expressions for bullwhip and inventory variance. We will note that conditional expectation forecasting is very good at removing bullwhip for some demand patterns, but fails to remove bullwhip in all cases of the ARMA demand. We incorporate a proportional controller into the OUT policy to satisfy the smoothing objective, but note that with conditional expectation forecasting the OUT policy can only remove bullwhip in all cases of the ARMA demand pattern by holding extra inventory. We then demonstrate that our bullwhip reduction method results in lower total costs than the simple OUT policy without bullwhip reduction. For more details, the reader may refer to the longer version of the paper (Chen and Disney, 2003).

ARMA DEMAND AND CONDITIONAL EXPECTATION FORECASTING

We have chosen the ARMA demand pattern for our analysis as it is sufficiently general to represent real demand patterns, but it is still mathematically tractable. We have elected to use the mean centred ARMA demand pattern without lose of generality. It is commonly expressed as a difference equation (1) as follows:

$$\left.\begin{aligned} D_0^{ARMA} &= \varepsilon_t + \mu \\ D_t^{ARMA} &= \rho(D_{t=1}^{ARMA} - \mu) - \theta\varepsilon_{t-1} + \varepsilon_t + \mu \end{aligned}\right\} \quad (1)$$

where, D_t = the ARMA demand at time, *t*; μ = unconditional mean of the ARMA demand sequence; ρ = the Auto Regressive constant; θ = the Moving Average constant; ε_t = a white noise process, that is, a normally distributed independently and identically distributed stochastic variable with mean zero and unity variance, $\sigma^2 = 1$. Let $\hat{D}_t$ be the conditional forecast for period *t* on based on information in the previous period (D_{t-1} and ε_{t-1}). We assume μ to be at least four times the variance of the ARMA demand so that the probability of negative demand is small as in Johnson and Thompson (1975). Note that the forecast error $\varepsilon_t = D_t - \hat{D}_t$, hence the variance of the forecast error is obviously unity.

We may express Eq 1 as a block diagram using standard techniques from discrete linear control theory as shown below in Figure 1. For a general introduction of control theory we refer readers to Nise (1995).

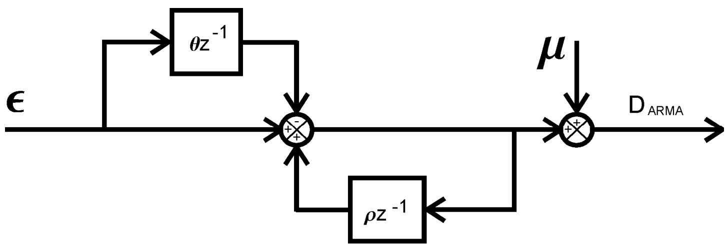

Figure 1. Block diagram of the ARMA demand generator

Re-arranging the block diagram, using common techniques, yields the ARMA demand transfer function (2),

$$\frac{D(z)}{\varepsilon(z)} = \frac{z-\theta}{z-\rho} \quad (2)$$

where z is the z-transform operator, $F(z)=\sum_{t=0}^{\infty} f(t)z^{-t}$. The variance of the ARMA demand is given by, $\sigma_D^2 = \dfrac{1+\theta^2-2\theta\rho}{1-\rho^2}$.

THE ORDER-UP-TO POLICY

The sequence of events in any period is: inventory level is reviewed and ordering decision is made at the beginning of the period, then the customers order is received and demand is realized and fulfilled at the end of the period. Thus, it takes one period to receive the order placed. Unmet demand in a period is backordered. Two costs are considered at the end of the each period, inventory holding and stock-out. Both are proportional functions with cost parameters constant over time, *h* and *s*, respectively. Piece-wise ordering costs will be considered later. In this section, we assume only a linear ordering cost. The objective is to minimize the long-run average total cost per period.

For such a problem, Johnson and Thompson (1975) have shown that the simple order-up-to (OUT) level policy is optimal. The OUT level is updated every period according to

$$S_t = \hat{D}_t + k\sigma_D \tag{3}$$

where $\hat{D}_t$ is an estimate of mean demand in period t, σ_D is the standard deviation of demand, and k is the safety factor, $k=F^{-1}(s/(s+h))$, where F is the standard normal cumulative distribution. This is the so-called myopic OUT policy.

The classical order-up-to policy definition is completed as follows; inventory position equals net stock (*NS*). We then successively obtain:

$$O_t = \hat{D}_t + k\sigma_D - NS_t \tag{4}$$

Notice that, in this system, $O_t \geq 0$ (Johnson and Thompson, 1975). Now we propose to make a modification to the classical OUT policy to provide more freedom in shaping its dynamic response. Our change is that we are going to use a proportional controller in the inventory position feedback loop. Specifically, we introduce a proportional controller, ($1/Ti$), as follows:

$$O_t = \hat{D}_t + \frac{1}{Ti}\left(k\sigma - NS_t\right) \tag{5}$$

The new policy will be called the ***modified OUT policy,*** where (6) completes the definition.

$$NS_t = NS_{t-1} + O_{t-1} - D_{t-1} \tag{6}$$

The conditional expectation forecast of the demand in the current period (remember our order of events) is described by the following transfer function

$$\frac{\hat{D}(z)}{\varepsilon(z)} = \frac{\rho(\rho-\theta)}{(z-\rho)} - \theta \tag{7}$$

and the standard deviation of the forecast error, $\sigma_D = 1$.

From Eqs 2, 5, 6, and 7, and from our description we may now develop a block diagram of the ARMA demand and the modified OUT policy as shown in Figure 2. In the block diagram, $k\sigma_D$ is a time-invariant constant that reflects the standard deviation of the forecast error over the lead-time and the safety factor, k, required.

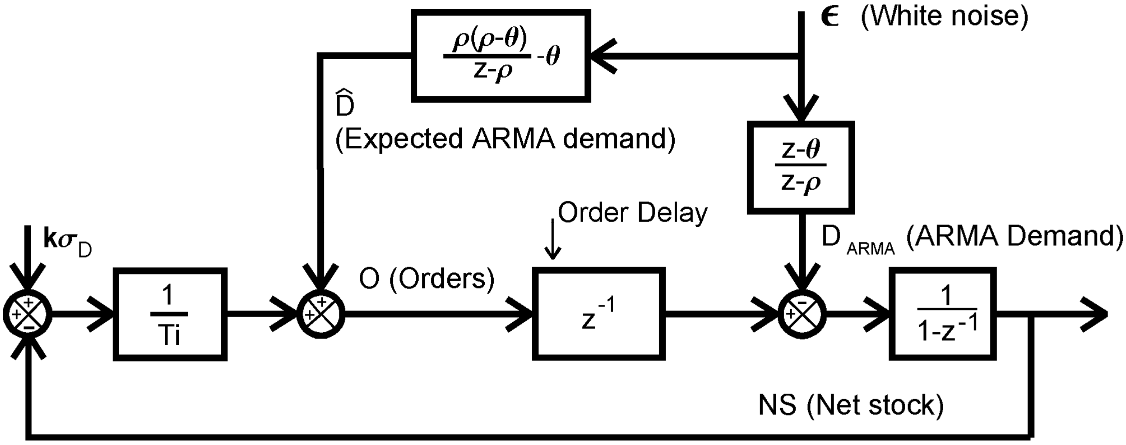

Figure 2. Block diagram of our modified OUT policy with ARMA demand and conditional expectation forecasting

Rearranging Figure 2 for the transfer function that describes the relationship between orders and the white noise process that drives the ARMA demand we have:

$$\frac{O(z)}{\varepsilon(z)} = \frac{z^2 - z\theta + Tiz\theta - Tiz^2\theta - Tiz\rho + Tiz^2\rho}{(1 - Ti + Tiz)(z-\rho)} \tag{8}$$

BULLWHIP IN THE MODIFIED MYOPIC OUT POLICY

Disney and Towill (2003) have recently investigated Tsypkin's relation (Tsypkin, 1964). Applying this relation to Eq 7 and dividing by the variance of the ARMA demand process, will yield a closed form expression for the bullwhip produced by the OUT policy with unconditional expectation forecasting in response to ARMA demand. We have plotted the bullwhip produced by the classical OUT policy (so Ti=1) in Figure 3. Here we can see that the classical OUT policy with conditional expectation forecasting is able to remove bullwhip when $\theta > \rho$.

Bullwhip is given by;

$$Bullwhip = \frac{\sigma_O^2}{\sigma_D^2} = \frac{\begin{pmatrix} 2\theta - 2Ti^2(\theta-\rho)^2(\rho-1) - (1+\theta^2)\rho + \\ Ti(1 - \theta(4+\theta) + 3\rho + \theta(2+3\theta)\rho - 2(1+\theta)\rho^2) \end{pmatrix}}{((2Ti-1)(Ti(\rho-1)-\rho)(2\theta\rho - \theta^2 - 1))} \tag{9}$$

Our modification to the OUT policy ($1/Ti$) does however allow us to remove bullwhip for all instances of the ARMA demand pattern. Letting the bullwhip equal 1 yields the minimum value of Ti that is required to avoid the bullwhip problem. This expression is shown in (10) below, and a high enough Ti will always eliminate the bullwhip problem. We have however plotted the case of Ti=5 in Figure 4. This clearly shows the bullwhip reduction properties.

$$TiMin = \frac{1-2\theta+\sqrt{1+4\theta(\theta-\rho)}}{2-2\rho} \tag{10}$$

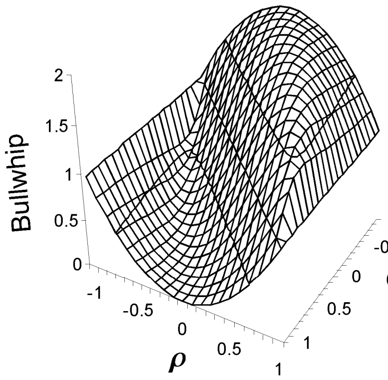

Figure 3. Bullwhip generated by the myopic OUT policy with ARMA demand pattern

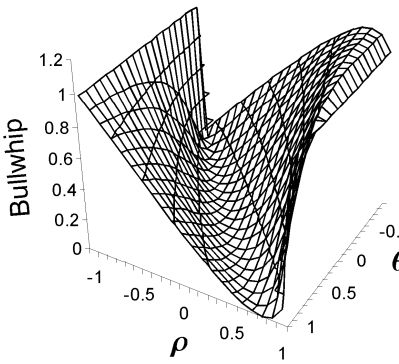

Figure 4. Bullwhip when Tp=1 and Ti=5 in modified OUT policy with CE forecasting

INVENTORY VARIANCE IN THE MODIFIED OUT POLICY

The transfer function of the inventory level can be found from the block diagram shown in Figure 2. Amazingly, the inventory variance is independent of the demand properties and only depends on the proportional controller, Ti, see (11).

$$\frac{NS(z)}{\varepsilon(z)} = \frac{Tiz}{Ti-Tiz-1} \tag{11}$$

The inventory variance is given by $\sigma_{NS}^2 = \dfrac{Ti^2}{2Ti-1}$, which we have plotted in Figure 5 below. We note that in the inventory variance when Ti=1 is unity and minimum for all of the class of ARMA demand patterns.

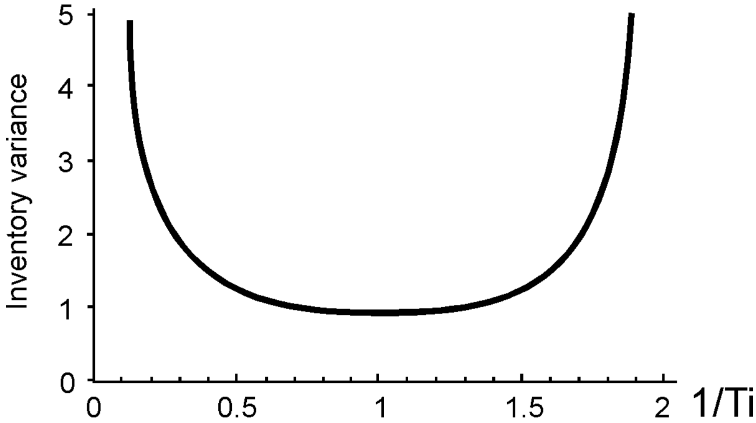

Figure 5. Inventory variance for the modified OUT policy reacting to ARMA demand

EXPECTED COSTS PER PERIOD: COMPARISON BETWEEN THE TWO OUT POLICIES

Now we assume the ordering cost to be piece-wise linear: within the normal production capacity K, the unit cost is c, while it is $c_0>c$ if the order quantity is greater than K. Think of this as incurring an overtime premium. We assume the overtime capacity is practically infinite (think of this as having infinite sub-contracting capacity at the same cost as in-house overtime working).

For such a model with a piece-wise linear ordering cost structure, Karlin (1960) shows that when the distributions of demands over time are time-independent, the generalized OUT is optimal. Under such a policy, the OUT level in a period depends on the beginning inventory level if it is below a certain level, and if it is above this level, then no order is needed. Though this policy structure can be shown to remain optimal for our problem with ARMA demands, it is difficult to implement this policy and obtain it numerically. Thus, we take a practical approach by adopting a sub-optimal policy, i.e., the modified OUT policy.

A numerical example

With our OUT policy the order quantity in a period is a normal random variable; if it is greater than K, then ordering cost c_0 is charged instead of c. We may find the expected amount of ordering costs by studying the probability density function of order levels over time (details omitted for brevity). Holding and shortage costs are calculated similarly. The sum of these costs yields the total expected cost per period. As the probability density function of the normal distribution is essentially non-algebraic, analytic results are difficult to obtain. Hence, we will consider the following numerical scenario.

The average ARMA demand is 10 units per period, the capacity limit, K, is 12 units per period, the cost to produce a unit in normal production is \$100, and in overtime production the unit ordering (production) cost is \$200. The inventory holding cost is \$10 per unit per period and the shortage cost is \$50 per unit per period. We set the inventory safety factor $k\sigma_D = 0.2\mu$ for simplicity (although, after investigation, this does not seem to make a great deal of qualitative difference). We consider four ARMA demand patterns at combinations of $\theta = \pm 0.5$ and $\rho = \pm 0.5$. We note that all solutions when $\theta = \rho$ result in identical curves. This symmetry comes from the fact the ARMA demand is a stochastic i.i.d process under this condition.

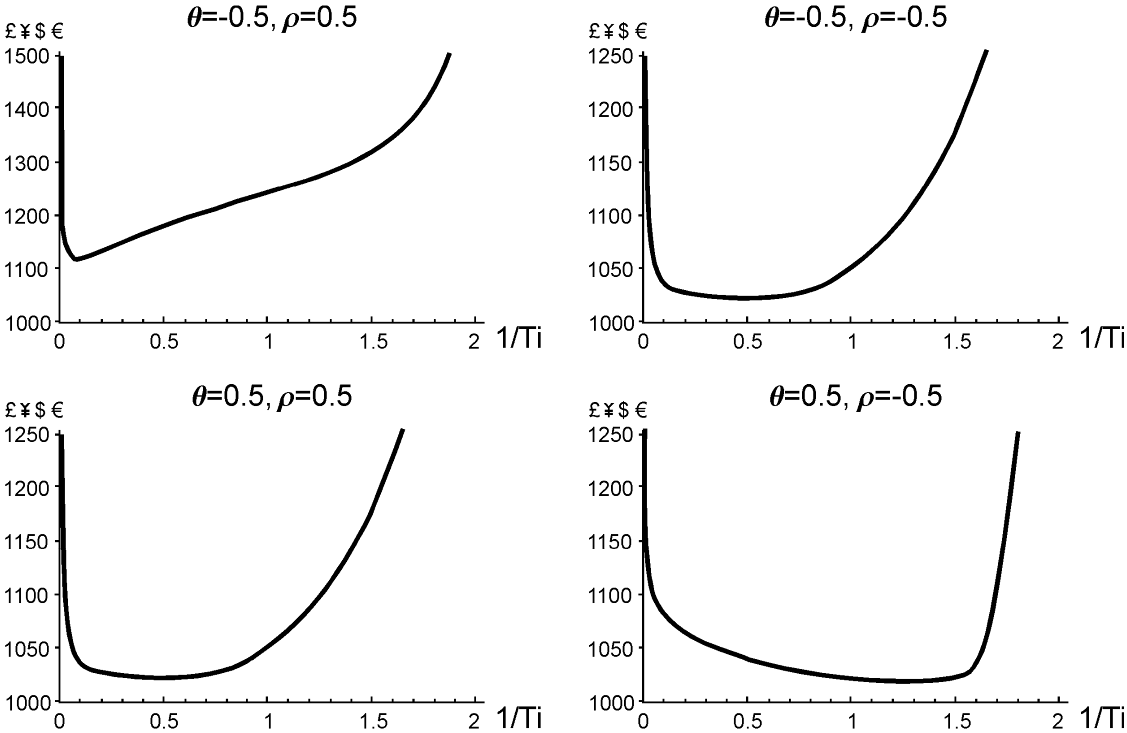

Figure 6. Expected total costs per period by the modified myopic OUT policy in response to ARMA demand

We can see that our introduction to the myopic OUT policy, the proportional controller – *Ti*, is clearly capable of reducing total expected costs per period, when compared to the classical myopic policy (when *Ti*=1). We note that the relationship is complex and sometimes, we should use a value of *Ti*>1 in order to minimise total expected costs and at other times we should use *Ti*<1. Our results are completely analytic and exact, although we have yet to fully understand the complete expected costs "solution space" as there are many dimensions to the problem. In general, the controller is much more effective when $\rho > 0$.

It is interesting to highlight the economic impact of our modification. It is by no means insignificant. For our example, average demand is 10 units per period. In a perfect world all products would be solely manufactured in normal capacity and the there would be no inventory or backlog costs. Therefore we expect at least $1000 of unavoidable costs per period. The avoidable costs (that is the over-capacity, inventory and backlog costs) for the classical myopic OUT policy (when Ti=1) and our modified OUT policy (where *Ti*>0.5) are shown in Table 1. We can see that we are able to avoid up to 50% of the avoidable costs by "tuning" *Ti* in the ordering policy to the demand pattern.

θ	ρ	Classical OUT		Modified OUT		Reduction of avoidable costs
		Ti	**Avoidable costs**	***Ti***	**Avoidable costs**	
$\theta = \rho$		1	\$48.65	2.0618	\$21.48	55.8%
0.5	-0.5	1	\$20.83	0.8928	\$20.55	1.3%
-0.5	0.5	1	\$241.19	12.987	\$115.42	52.1%

Table 1. Sample economic impact of the modification to the myopic OUT policy

CONCLUSIONS

Using the z-transform and the normal distribution probability density function, we have studied the myopic OUT policy reacting to the stochastic ARMA demand pattern. We have achieved this in an environment where there are piece-wise linear ordering costs and piece-wise linear inventory costs are present. Exploiting basic control engineering principles we have made a slight modification to the modified myopic OUT policy, the proportional controller in the inventory feedback loop, to allow us to better exploit the structure of our defined cost function. The myopic OUT policy with our bullwhip effect reduction technique, *Ti*, outperforms the classical myopic OUT policy when the convex ordering cost is considered. In some cases, the savings can be quite substantial. However, the comparison is based on heuristic inventory replenishment policies. As an ongoing research project, we are currently refining the optimal policies and developing procedures to quantify them.

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