Disney, S.M., Warburton, R.W.H. and Zhong, Q-C., (2013), "Net Present Value of the Economic Production Quantity model", IMA Journal of Management Mathematics, Vol. 24, No. 4, pp423-435. DOI: 10.1093/imaman/dpt002.

Net Present Value Analysis of the Economic Production Quantity

Stephen M. Disney[1], Roger D. H. Warburton[2] & Qing Chang Zhong[3]

1. Logistics Systems Dynamics Group, Cardiff Business School, Cardiff University, Cardiff, CF10 3EU, United Kingdom. DisneySM@ardiff.ac.uk. Corresponding Author.
2. Department of Administrative Sciences, Metropolitan College, Boston University, MA 02215, United States of America. rwarb@bu.edu
3. Department of Automatic Control and Systems Engineering, University of Sheffield, Mappin Street, Sheffield, S1 3JD, United Kingdom. Q.Zhong@sheffield.ac.uk

Abstract

Using Laplace transforms we extend the Economic Production Quantity (EPQ) model by analysing cash flows from a Net Present Value (NPV) viewpoint. We obtain an exact expression for the present value of the cash flows in the EPQ problem. From this we are able to derive the optimal batch size. We obtain insights into the monotonicity and convexity of the present value of each of the cash flows, and show that there is a unique minimum in the present value of the sum of the cash flows in the extended EPQ model. We also obtain exact point solutions at several values in the parameter space. We compare the exact solution to a Maclaurin series expansion and show that serious errors exist with the first order approximation when the production rate is close to the demand rate. Finally we consider an alternative formulation of the EPQ model when the opportunity cost of the inventory investment is made explicit.

Key words: Economic Production Quantity, Net Present Value, Lambert W Function, Maclaurin Series Expansion.

1. Introduction

It is almost one hundred years since the introduction of the Economic Order Quantity formula by Ford Whitman Harris in 1913. Erlenkotter (1990) provides an excellent historical review of its early development. While some authors question the relevance of this approach in the current "lean" environment, (see for example Voss, 2010), it is our experience that the EOQ philosophy is still important today, especially in process industries where expensive production capacity is required to produce several similar products. Indeed, both American (see for example Blackburn and Scudder (2009) and Grubbström and Kingsman (2004)) and European (see, Disney and Warburton (2011), Beullens and Janssens (2011)) academic outlets are still regularly publishing papers on the subject. Furthermore it is our experience that industry still finds this a valuable managerial tool.

Shortly after Harris introduced the EOQ solution, Taft (1918) generalized the approach in what is now known as the Economic Production Quantity (EPQ) problem. The main difference between the EPQ model and EOQ model is that the EPQ model assumes that it takes time to produce the batch quantity, whereas the EOQ model assumes that the entire batch arrives instantaneously, all in one go.

There are many variations and extensions to both the EOQ and EPQ models in the literature. Many of these problems have exact explicit solutions but most of the more complicated variations require heuristic approaches or exploit approximations. It appears that the first paper to consider the time value of money in an EOQ / EPQ inventory model is Hadley (1964), where a numerical approach was taken. The contribution of Grubbström (1980) makes the link between NPV and the Laplace transform of the cash flows in the EPQ model. Here, an expression for the NPV of the cash flows in the EPQ model and its equivalent Annuity Stream is derived, but no attempt is made to identify the exact optimal batch quantity, rather a Maclaurin expansion is used to obtain an approximate solution. Grubbström and Kingsman (2004) consider the NPV of an EOQ decision when it is known that there will be a future price increase. An interesting feature of that problem is that the batch size is dynamic in time, with large orders placed in the final moment before the price increase.

Recently, Warburton (2009) noticed that some EOQ problems that were thought not to have exact, explicit solutions can be solved by employing the Lambert W function. Disney and Warburton (2012) integrated the Laplace transform and the Lambert W function in an investigation of two different EOQ problems: an EOQ problem with perishable inventory, and the NPV of an EOQ problem with yield loss. They are able to obtain exact, explicit solutions for the optimal batch size in both these problems, and it is this approach that we follow here to study the NPV of the cash flows in the EPQ problem.

In section 2 we define the EPQ problem from an average cost perspective. Section 3 considers the EPQ problem from the Net Present Value perspective, and Section 4 undertakes a numerical investigation to demonstrate the validity and practical utility of the theoretical model. Here we also compare the exact solution to an approximation based on the Maclaurin expansion. Section 5 provides some conclusions.

2. The Economic Production Quantity

We briefly review the classical EPQ model and its derivation. Traditionally, the total annual cost (*TC*) is to be minimised. The cost is minimised by selecting a production batch size, $Q \in \Re > 0$, where Q is the decision variable. The total cost is assumed to be made up of the cost of holding inventory (the cost of holding one unit of inventory for one year is $h \in \Re \geq 0$); the cost of a production set-up is $k \in \Re \geq 0$ (a change-over cost between one product and another); and the direct cost of production per unit is $c \in \Re > 0$ (not including the holding or the set-up cost).

The external, and hence uncontrollable (at least not easily), variables are the demand rate, $D \in \Re > 0$, and the production rate, $P \in \Re \geq D$. It is usual to consider the EPQ operating on an annual basis, so D is the demand per year, and P is the production per year that could be achieved if the product were manufactured continuously. $P \geq D$, as otherwise the production would never be able to keep up with demand. When $P > D$, the product is manufactured intermittently, and it is this situation that is typically considered in an EPQ analysis. When $P = D$ we produce continuously and never conduct a change-over. In the interval when we are not producing the product, we assume that the manufacturing equipment either lays idle or is used to manufacture another product. Access to production capacity is available instantly and at any time.

2.1. Time-based evolution of the costs

The direct production costs (c per unit) are incurred during the period when manufacturing product. As product is manufactured at a rate of P, direct costs are incurred at a rate of cP. After Q items have been made, the production is turned off (after Q/P units of time since production was started). During the period when production is running, as $P > D$, inventory has been building up (at a rate of $P-D$). When production ceases, the inventory level is at $Q(P-D)/P$ and the inventory thereafter is depleted at a rate of $-D$. At the instant that inventory falls to zero (at Q/D units of time after the last set-up was conducted), we assume production starts again and inventory builds up. The average inventory being held at any point in the year is $\frac{Q(P-D)}{2P}$. Every time the production is started up, a production set-up cost of k is incurred. There are D/Q setups per year. Figure 1 sketches the time evolution of the three components of the EPQ costs.

From the above description it is easy to obtain the following expression for the total annual cost.

$$TC = \frac{Dk}{Q} + \frac{Qh(P-D)}{2P} + cD \tag{1}$$

Taking the derivative with respect to Q yields:

$$\frac{\mathrm{d}TC}{\mathrm{d}Q} = \frac{Dk}{Q^2} + \frac{h(P-D)}{2P}. \tag{2}$$

Setting the derivative to zero and solving for the optimal batch quantity, Q^*, gives:

$$Q^* = \sqrt{\frac{2PDk}{h(P-D)}} = \sqrt{\frac{2Dk}{h}}\sqrt{\frac{P}{P-D}}. \tag{3}$$

It is easy to verify that Q^* in (3) is indeed a minimum by taking the second derivative $\left(\frac{\mathrm{d}^2TC}{\mathrm{d}Q^2} = \frac{2Dk}{Q^3}\right)$ and noting that it is always positive when $\{Q, D, k\} \in \Re > 0$. We notice in (3) that the Q^* given by the EPQ model is always bigger than the EOQ Q^* as $\sqrt{\frac{P}{P-D}} > 1$

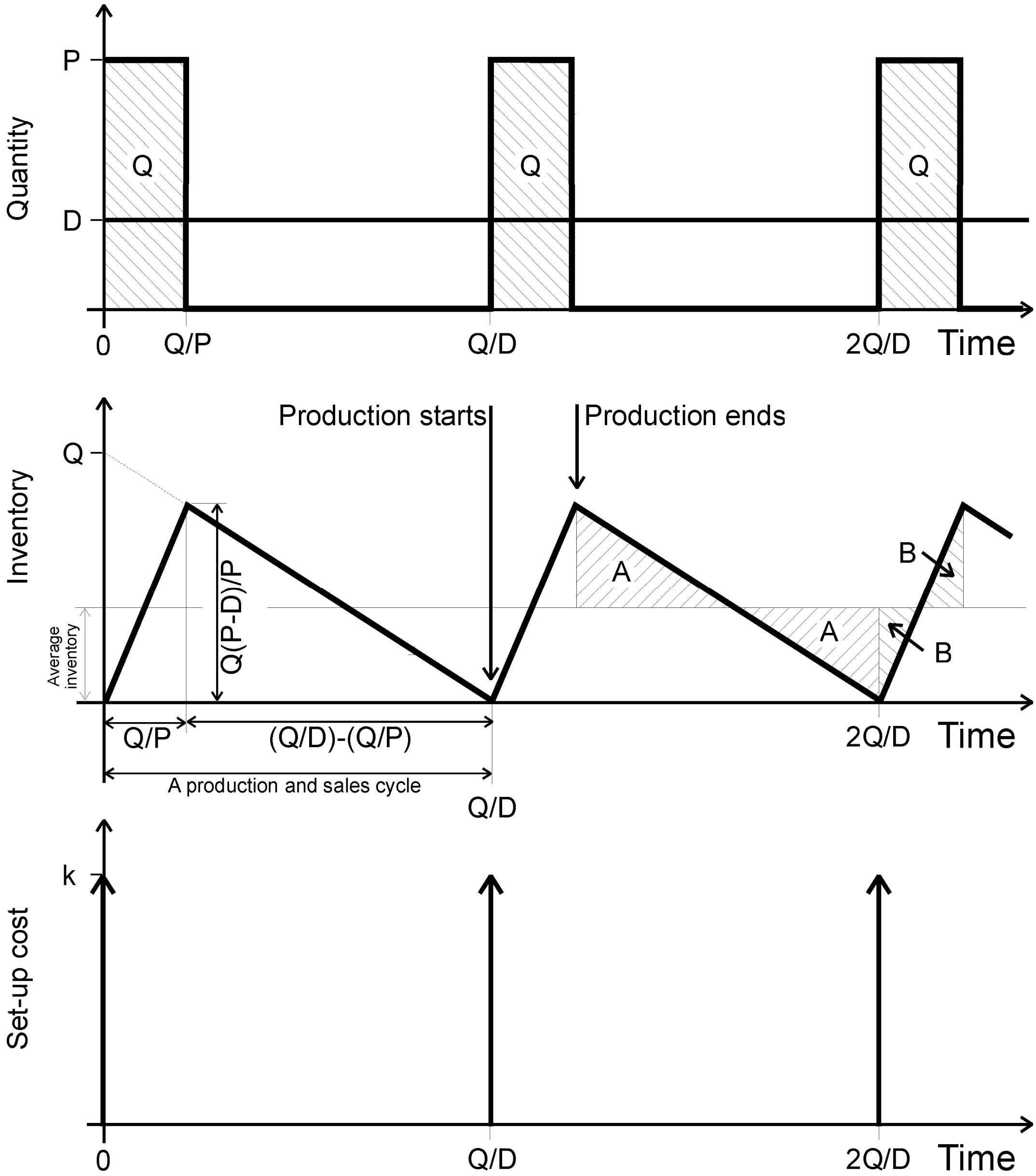

Figure 1. The production, inventory, and set-up costs over time in the EPQ problem

when $P > D$. Furthermore, increasing the production rate P results in a smaller Q^*, and indeed, when $P \to \infty$ we regain the EOQ result. As in the EOQ case, reducing the set-up cost, k, results in smaller optimal order quantities.

3. Net Present Value Analysis of the EPQ problem

Grubbström (1967) showed that if a Laplace transform is used to describe a cash flow over time and the Laplace operator, s, has been replaced by the continuous discount rate *r*, then the Laplace transform, *F*(*s*), of the cash flow, *f*(*t*), yields the present value of the cash flow. This fundamental relationship is formalized by:

$$PV = \left[F(s) = \int_0^\infty e^{-st} f(t)\, dt \right]_{s=r}. \tag{4}$$

Using some rather basic control engineering knowledge (see Nise (1995), or Buck and Hill (1971)) we may develop a block diagram to describe the cash flows in the EPQ system, see Figure 2. From this we will later obtain the Laplace transform of the cash flows in the EPQ model. Figure 3 illustrates the time based evolution of each of the signals in the EPQ problem showing how the cash flows are constructed.

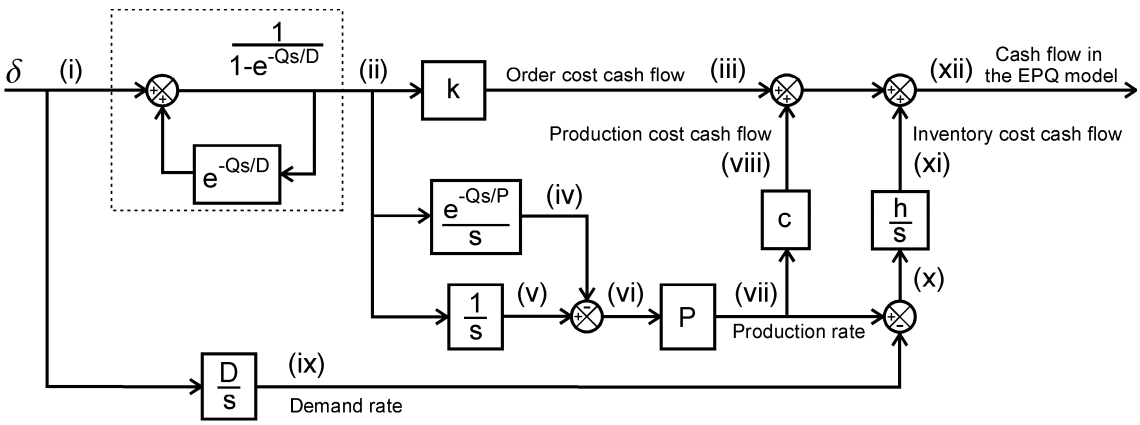

Figure 2. Block diagram of the cash flows in the EPQ problem

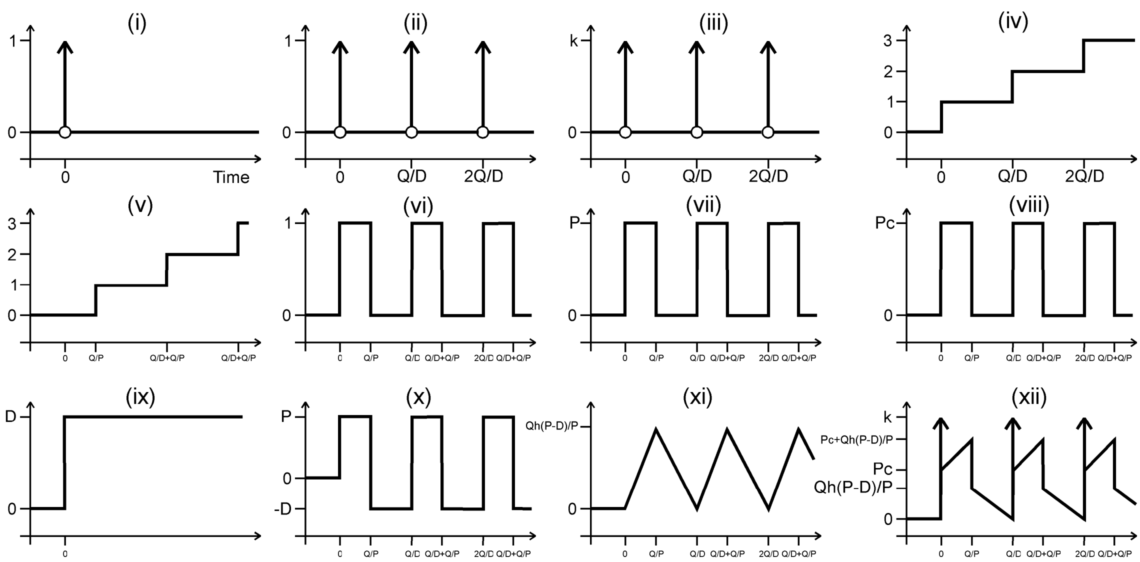

Figure 3. Time evolution of the signals that generate the cash flows in the EPQ problem

Grubbström (1980) argued that the inventory holding costs are unnecessary in the EPQ case, as they have already been accounted for in the production cost cash flow. Indeed Harris (1913) defined inventory holding costs as an opportunity cost related to the production cost. We have elected to include them as we note that, besides the capital inventory cost, there may be other out-of-pocket expenses such as storage, spoilage, shrinkage and insurance to be accounted for. However, if these out-of-pocket costs can indeed be ignored as advocated by Grubbström (1980), this can easily be modeled by setting $h = 0$.

The block diagram in Figure 2 may be manipulated to obtain the following Laplace transform transfer function that also describes the Present Value (PV) of the cash flows in the EPQ decision.

$$PV_{\text{Costs}} = K(Q) + C(Q) + H(Q) \tag{5}$$

where

$$K(Q) = \frac{k}{1-e^{-Qs/D}};\ C(Q) = \frac{cP\left(1-e^{-Qs/P}\right)}{s\left(1-e^{-Qs/D}\right)};\ H(Q) = \frac{hP\left(1-e^{-Qs/P}\right)}{s^2\left(1-e^{-Qs/D}\right)} - \frac{hD}{s^2}. \tag{6}$$

(5) can be reduced to

$$PV_{\text{Costs}} = \frac{e^{-Qs/P}\left(e^{Qs/P}\left(Dh + e^{Qs/D}\left(h(P-D) + s(cP+ks)\right)\right) - e^{Qs/D}P(h+cs)\right)}{\left(e^{Qs/D}-1\right)s^2} \tag{7}$$

We first study the present value of each of the costs individually. The present value of the set-up costs, $K(Q)$, is shown in the first term in (5) and they are:

- Monotonically decreasing in Q as the first derivative, $\frac{\mathrm{d}K(Q)}{\mathrm{d}Q} = -\frac{e^{-Qs/D}ks}{D\left(1-e^{-Qs/D}\right)^2} \le 0$ $\forall\{s,k,Q,D\} > 0$;

- Strictly convex in Q as $\frac{\mathrm{d}^2K(Q)}{\mathrm{d}Q^2}=\frac{e^{Qs/D}\left(1+e^{Qs/D}\right)ks^2}{D^2\left(e^{Qs/D}-1\right)^3}>0\ \forall\{s,k,Q,D\}>0$;
- Infinite when $Q=0$; and
- k when $Q\to\infty$.

The present value of the production costs, $C(Q)$:

- Are monotonically increasing in Q as the first derivative, $\frac{\mathrm{d}C(Q)}{\mathrm{d}Q}=\frac{ce^{Qs\left(\frac{1}{D}-\frac{1}{P}\right)}\left(D\left(e^{Qs/D}-1\right)+P\left(1-e^{Qs/P}\right)\right)}{D\left(e^{Qs/D}-1\right)^2}\geq 0\quad\forall\{s,c,Q,D,P\}>0$, a relationship that is more obvious to determine from $C(Q)=\frac{cP}{s}\left(\frac{1-e^{-Qs/P}}{1-e^{-Qs/D}}\right)$, as both the numerator and denominator of the bracketed term are non-decreasing functions of Q in the range (0,1);
- Are not concave in Q as the second derivative $\left.\frac{\mathrm{d}^2C(Q)}{\mathrm{d}Q^2}\right|_{Q=0}=\frac{cs\left(D(2D-3P)+P^2\right)}{6DP^2}$ is positive when $D\leq P\leq 2D$. When $P>2D$ the present value of the production costs appear to be concave in Q;
- $C(Q)=\frac{cD}{s}$ and $\left.\frac{\mathrm{d}C(Q)}{\mathrm{d}Q}\right|_{Q=0}=\frac{c(P-D)}{2P}$ when $Q=0$;
- $C(Q)=\frac{cP}{s}$ and $\left.\frac{\mathrm{d}C(Q)}{\mathrm{d}Q}\right|_{Q\to\infty}=\left.\frac{\mathrm{d}^2C(Q)}{\mathrm{d}Q^2}\right|_{Q\to\infty}=0$ when $Q\to\infty$.

The present value of the inventory costs, $H(Q)$, (the third component of (5)):

- Are monotonically increasing in Q as the first derivative $\frac{\mathrm{d}H(Q)}{\mathrm{d}Q}=\frac{he^{Qs\left(\frac{1}{D}-\frac{1}{P}\right)}\left(D\left(e^{Qs/D}-1\right)+P\left(1-e^{Qs/P}\right)\right)}{Ds\left(e^{Qs/D}-1\right)^2}\geq 0\quad\forall\{s,h,Q,D,P\}>0$. Again this relationship is more obvious to determine from $H(Q)=\frac{hP}{s^2}\left(\frac{1-e^{-Qs/P}}{1-e^{-Qs/D}}\right)-\frac{hD}{s_2}$ as both numerator and denominator of the bracketed term are non-decreasing functions of Q in the range (0,1);
- Are not concave in Q as the second derivative $\left.\frac{\mathrm{d}^2H(Q)}{\mathrm{d}Q^2}\right|_{Q=0}=\frac{h\left(D(2D-3P)+P^2\right)}{6DP^2}$ is positive when $D\leq P\leq 2D$. When $P>2D$ the present value inventory costs appear to be concave in Q;

- $H(Q)=0$ and $\left.\frac{\mathrm{d}H(Q)}{\mathrm{d}Q}\right|_{Q=0}=\frac{h(P-D)}{2Ps}$ when $Q=0$;
- $H(Q)=\frac{h(P-D)}{s^2}$ and $\left.\frac{\mathrm{d}H(Q)}{\mathrm{d}Q}\right|_{Q\to\infty}=\left.\frac{\mathrm{d}^2H(Q)}{\mathrm{d}Q^2}\right|_{Q\to\infty}=0$ when $Q\to\infty$.

As both the present value of the production and inventory costs are monotonically increasing functions of *Q*, their sum is also a monotonically increasing function of *Q*. The present value of the set-up costs are monotonically decreasing in *Q*. It then follows that the present value of the sum of all three costs in the EPQ model has a unique minimum in *Q*.

Taking the derivative of (5) with respect to *Q* yields,

$$\frac{\mathrm{d}PV_{\text{Costs}}}{\mathrm{d}Q}=\frac{e^{\left(\frac{1}{D}-\frac{1}{P}\right)Qs}\left(\left(D\left(e^{Qs/D}-1\right)+P\right)(h+cs)-e^{Qs/P}\left(hP+s\left(cP+ks\right)\right)\right)}{Ds\left(e^{Qs/D}-1\right)^2}, \tag{8}$$

from which the following characteristic equation can be obtained that describes the optimal batch size Q_{NPV}^*:

$$\left(D\left(e^{Q_{NPV}^* s/D}-1\right)+P\right)(h+cs)-e^{Q_{NPV}^* s/P}\left(hP+s\left(cP+ks\right)\right)=0. \tag{9}$$

(9) can be rearranged into the following form $Ae^{aQ_{NPV}^*}+Be^{bQ_{NPV}^*}+C=0$ where

$$\begin{gathered} a=\tfrac{s}{D};\ A=D(h+cs);\ b=\tfrac{s}{P};\\ B=P(cs+h)+ks^2;\ C=P(cs+h)-D(h+cs). \end{gathered} \tag{10}$$

In (9), while all the variables are real, there is no known general solution to this equation. However, we are able to obtain solutions at specific points in the parameter space. When the production rate, *P*, is less than (or equal to) the demand rate *D*, then it is best to produce continuously, forever (as the production rate can not keep up with demand), so,

$$Q^*\Big|_{P\le D} = \infty \tag{11}$$

holds. This can also be verified by letting $P \to D$ in (5) and simplifying to yield $PV_{Costs} = k + \frac{k}{e^{Qs/D}-1} + \frac{cD}{s}$, which is clearly minimized when $Q \to \infty$.

Although (9) has no general solution, point solutions can be obtained when $\frac{P}{D} = \{1,\ 2,\ 3...\}$. For example, the solution at $P = 2D$ is:

$$Q^*_{NPV}\Big|_{P=2D} = \frac{P}{s}\,\mathrm{Log}\left[\frac{hP + cPs + ks^2 + \sqrt{4D(D-P)(h+cs)^2 + (hP + s(cP+ks))^2}}{2D(h+cs)}\right], \tag{12}$$

and the solution at $P = 3D$ is:

$$Q^*_{NPV}\Big|_{P=3D} = \frac{P}{s} Log\left[\frac{2\,3^{1/3} D(h+cs)(hP+s(cP+ks)) + \sqrt[3]{2}\sqrt[3]{\left[9D^3(h+cs)^3 - 9D^2P(h+cs)^3 + \sqrt{3}\sqrt{D^3(h+cs)^3\left(27D(D-P)^2(h+cs)^3 - 4(hP+s(cP+ks))^3\right)}\right]^2}}{\sqrt[3]{6^2}\, D(h+cs)\sqrt[3]{9D^3(h+cs)^3 - 9D^2P(h+cs)^3 + \sqrt{3}\sqrt{D^3(h+cs)^3\left(27D(D-P)^2(h+cs)^3 - 4(hP+s(cP+ks))^3\right)}}}\right]. \tag{13}$$

This approach can be exploited further and solutions found at $P = 4D$ etc., but the equations become rather lengthy, so we will not present them. When the production rate, *P,* is infinite, the batch is delivered instantaneously, all at once. The solution in the limit where $P \to \infty$ is:

$$Q^*_{NPV}\Big|_{P\uparrow\infty} = \frac{-ks}{h+cs} - \frac{D}{s}\left(1 + W_{-1}\left[-\exp\left[-1 - \frac{ks^2}{D(h+cs)}\right]\right]\right), \tag{14}$$

where $W_{-1}[x]$ is the Lambert *W* function, evaluated on the alternative branch. We note that this is the solution for the EOQ problem given by Warburton (2009). Disney and Warburton (2012) provide some pedagogical insights on how to use the Lambert W function for EOQ problems in classroom settings.

4. Numerical investigations

It is interesting to numerically investigate the impact of P on the optimal batch size Q^*_{NPV}. Consider the industrially relevant case detailed in Disney and Warburton (2012) of annual demand D = 18, discount rate s = 0.2, order placement cost k = 27, direct cost c = 10, and inventory holding cost h = 4.

This is illustrated in Figure 4, where we have plotted $1/Q^*_{NPV}$ for convenience. Here as P becomes much greater than D, Q^*_{NPV} approaches the EOQ solution asymptotically. Furthermore when P is only just greater than D, Q^*_{NPV} is rather large.

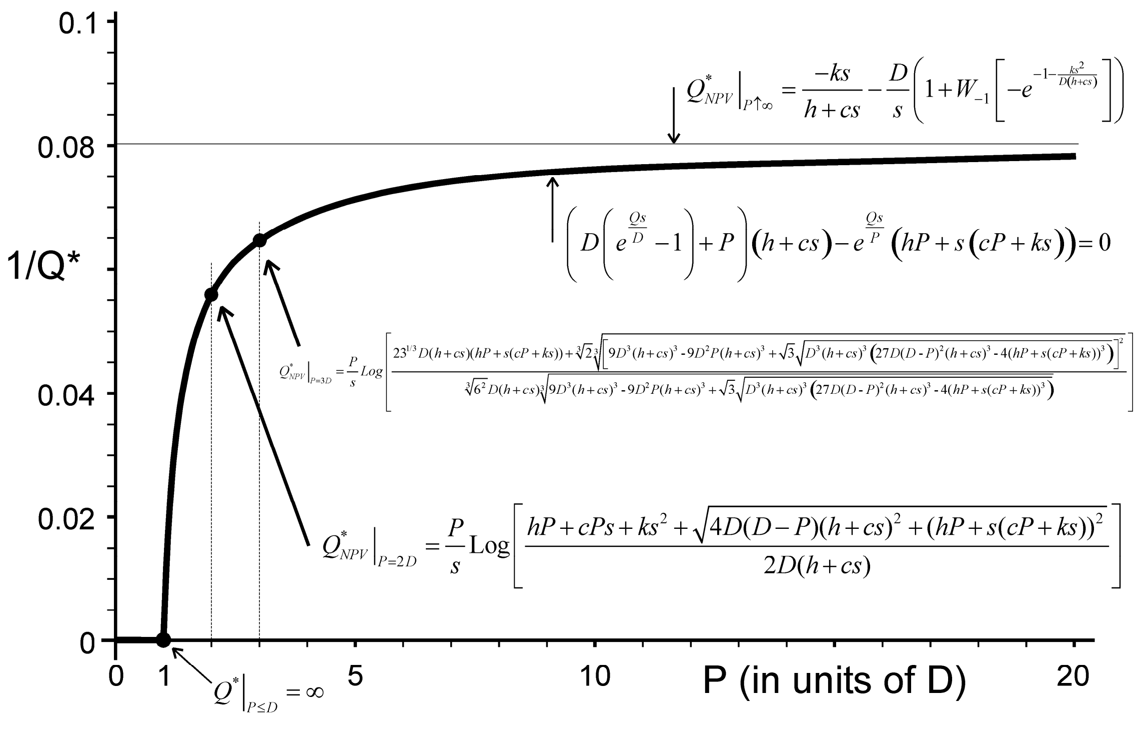

Figure 4. The optimal order quantity in the NPV EPQ problem

Figure 5 shows the present value of the costs when $Q = Q^*_{NPV}$. We can see that in our numerical example the PV ranges from 917 to 1300 and is increasing in P. If the NPV EOQ Q^* is used (14) instead of Q^*_{NPV} then the percentage increase in the present value of the costs falls quite rapidly from 28.7% at P = D to 9% when P = 2D, 5.7% at P =

$3D$, 3.3% at $P = 5D$, 2% at $P = 8D$ and less than 1% when $P > 16D$. So althought the error is significant in practical situation when P is close to D, (14) provides a useful near optimal solution when $P >> D$ when there is no appetite to calculate Q^*_{NPV} from (9). However, we note that (9) is quite easily determined with the help of a good scientific calculator or with the Microsoft Excel Solver function.

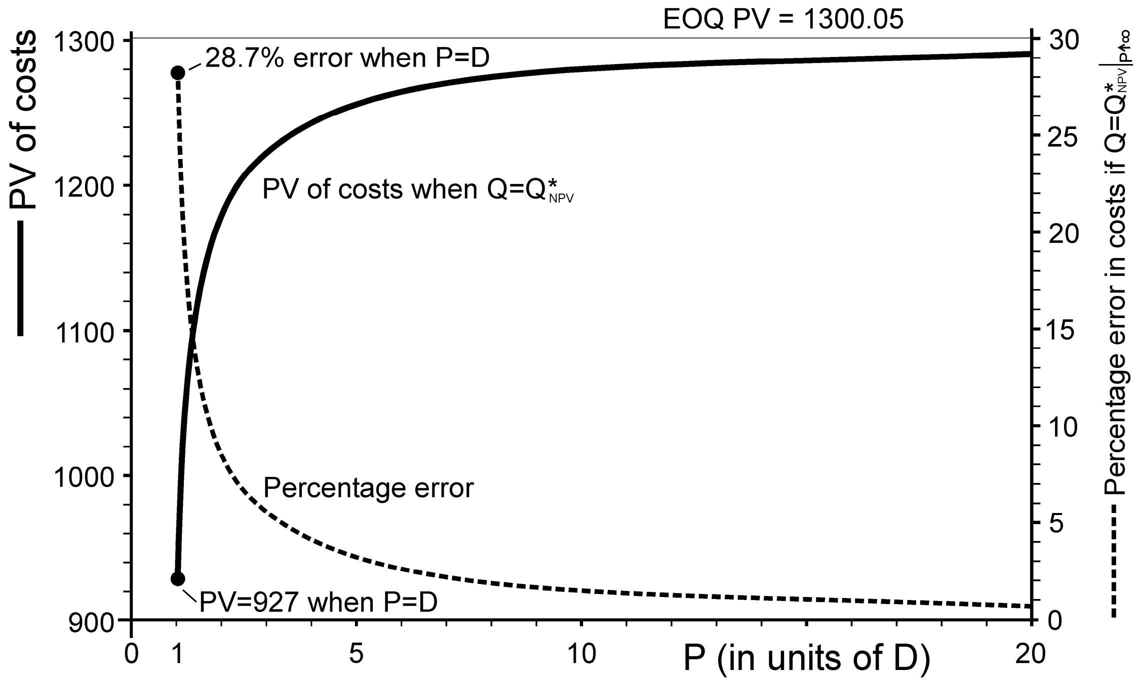

Figure 5. Cost performance of the NPV EPQ model and the NPV EOQ solution

4.1. Approximations to the optimal batch size

The first order Maclaurin expansion of (7) yields the following power series for the present value of the costs

$$PV_{\text{Costs}} \approx \frac{k}{2} + \frac{D(k+cQ)}{sQ} + \frac{kPs^2 + 6D(h+cs)(PQ-D)}{12sDP} + O[Q]^2 \tag{15}$$

Taking the derivative of (15) with respect to Q and solving for the first order conditions yields Q^*_2, an approximate value of Q^*_{NPV}, the optimal batch quantity when the NPV of the cash flow is accounted for,

$$Q_2^* = \frac{2D\sqrt{3kP}}{\sqrt{kPs^2 - 6D^2(h+cs) + 6DP(h+cs)}}. \tag{16}$$

Figure 6 shows the percentage error between the Maclaurin expansion and the optimal batch quantity for minimizing the of the costs in the EPQ decision, Q_{NPV}^*. The numerical example chosen in this figure is D = 10, P = 25, k = 20, c = 10, h = 4. Here we have plotted the errors for the first, Q_2^*, second, Q_3^*, and third order, Q_4^*, Maclaurin expansions. The second and third order Maclaurin expansions for the present value of the costs are

$$\begin{aligned} PV_{\text{Costs}} \approx{}& \frac{k}{2} + \frac{D(k+cQ)}{sQ} + \frac{kPs^2 + 6D(h+cs)(PQ-D)}{12sDP} + \\ & \frac{Q^2(h+cs)(D-P)(2D-P)}{12DP^2} + O[Q]^3 \end{aligned} \tag{17}$$

and

$$\begin{aligned} PV_{\text{Costs}} \approx{}& \frac{k}{2} + \frac{D(k+cQ)}{sQ} + \frac{kPs^2 + 6D(h+cs)(PQ-D)}{12sDP} + \\ & \frac{Q^2(h+cs)(D-P)(2D-P)}{12DP^2} - \frac{\left(sQ^3\left(30D^2(D-P)^2(h+cs) + kP^3s^2\right)\right)}{720D^3P^3} + O[Q]^4, \end{aligned} \tag{18}$$

but we have not re-arranged them for Q_3^* and Q_4^* as the results are very lengthy. These higher order expansions do indeed lead to a more accurate approximations for Q_{NPV}^*, as shown in Figure 6.

The second order error appears to be increasing in as the discount rate, s, increases for this numerical setting. The first and second order errors are around 1% for $0 < s < 0.8$. The third order error is rather small. We also note that when P is close to D then the second and third order Maclaurin expansions are numerically difficult to evaluate. As

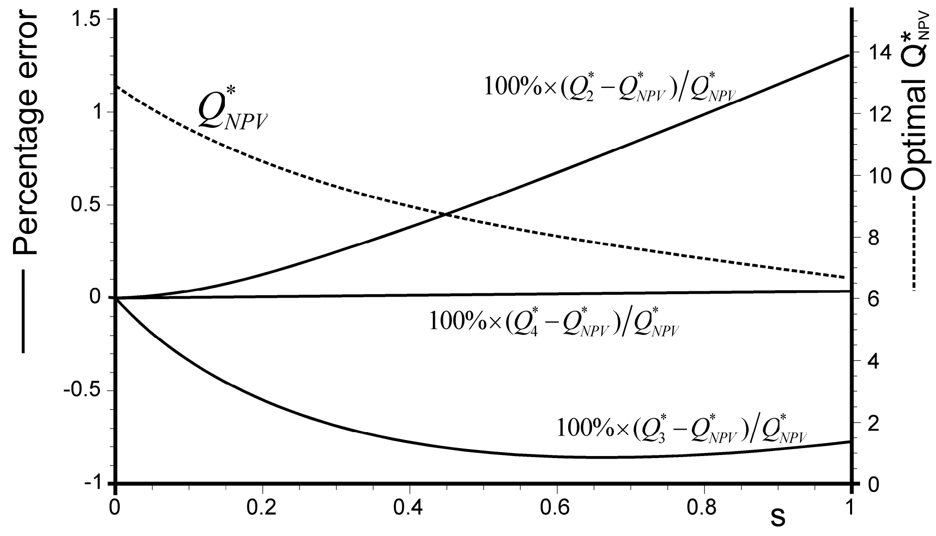

Figure 6. Accuracy of the Maclaurin expansion for determining the optimal batch quantity

$$Q_2^*\Big|_{s\to 0} = Q_{NPV}^*\Big|_{s\to 0} = Q^* = \sqrt{\frac{2PDk}{h(P-D)}}\,,\ \frac{\mathrm{d}Q_2^*}{\mathrm{d}s} = \frac{2D\sqrt{3kP}\big(3cD(D-P)-kPs\big)}{\big(kPs^2+6D(P-D)(h+cs)\big)^{3/2}} < 0\ \forall s, \tag{19}$$

and $P \geq D$, then $Q_2^* < Q^*$ when $s > 0$ and Q_2^* is strictly decreasing in s.

From this fact we might propose that $Q_{NPV}^* < Q^*$ when $s > 0$. However this is erroneous reasoning as numerical investigations, which are illustrated in Figure 7, reveal that when P is close to D, Q_{NPV}^* is actually an increasing function in s, see plots a) and b). This demonstrates a fundamental structural difference between the behavior of the first order Maclaurin expansion and the true behavior. Plot c) show that Q_{NPV}^* is initially a decreasing function in s, but then becomes an increasing function is s near s = 0.298 (it then becomes a decreasing function again near s = 1.224, but this is not shown). Furthermore we can see from Figure 7 that sometimes the Maclaurin series expansion under-estimates Q_{NPV}^* (see plots a) to i)), and at other times it is an over-estimate (see plots j) to k)). In Figure 7 we have also highlighted the value of the classic Economic Production Quantity, Q^*, as well at the case when the Production rate, P, becomes infinite (the EOQ) case, plot l).

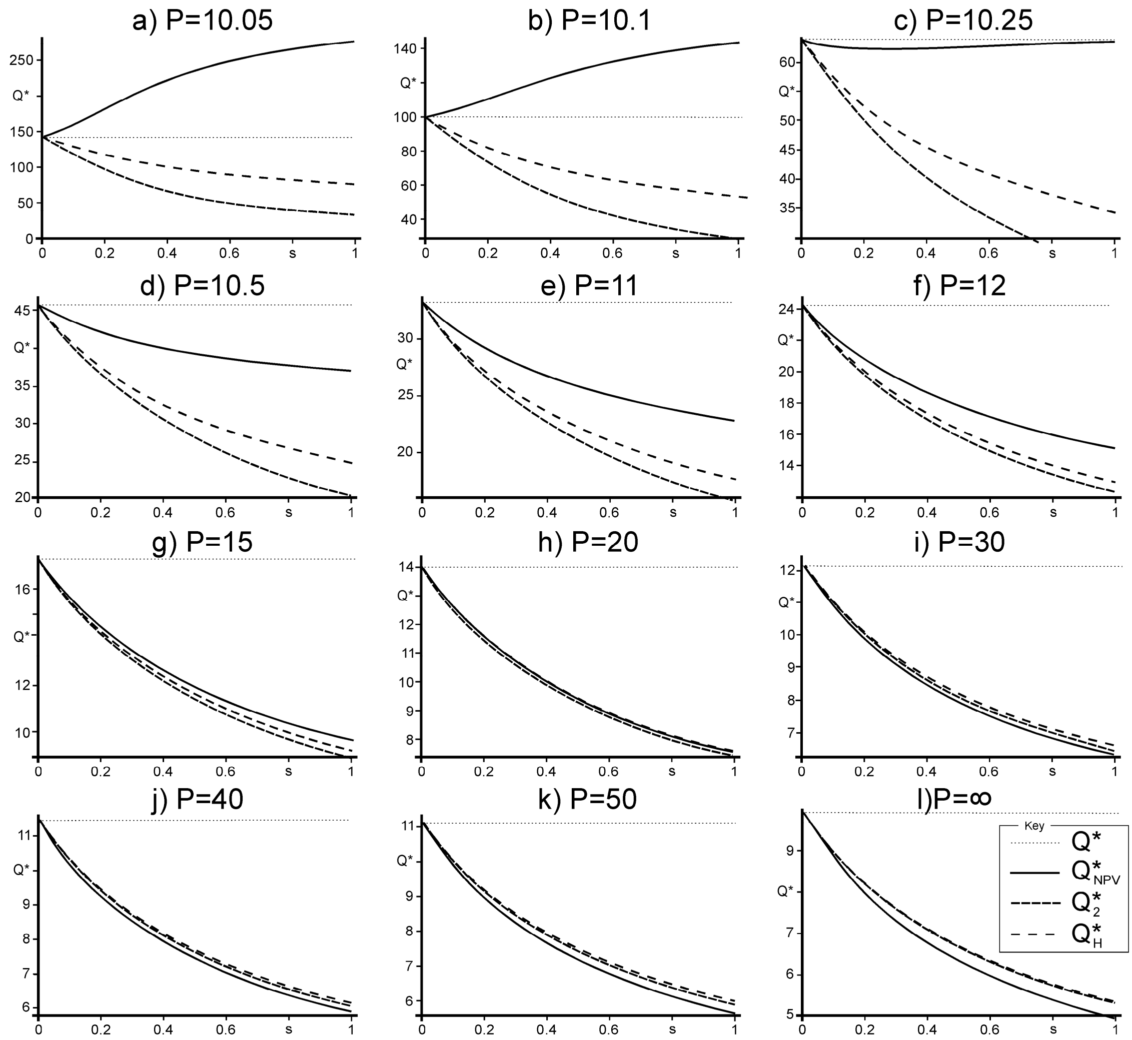

Figure 7. The effect of P on Q^* when D = 10, k = 20, c = 10 and h = 4

Grubbström (1980) argues for an alternative formulation of the EPQ model. In our situation it amounts to replacing h, the unit inventory holding cost with $\left(h' + cs\right)$ where h' is the out-of-pocket inventory costs and cs is the opportunity cost of the inventory investment. This leads to the following annual Total Cost function,

$$TC_H = cD + \frac{Dk}{Q} + \frac{(P-D)Q\left(h' + cs\right)}{2P} \tag{20}$$

which has the following, first and second order derivatives w.r.t. Q

$$\frac{\mathrm{d}TC_H}{\mathrm{d}Q}=\frac{(P-D)(h'+cs)}{2P}-\frac{Dk}{Q^2},\ \frac{\mathrm{d}^2TC_H}{\mathrm{d}Q^2}=\frac{2Dk}{Q^3} \tag{21}$$

from which we may obtain the following expression for Q_H^*, an optimal production quantity when the opportunity costs from the inventory investment have been explicitly linked to the discount rate *s*,

$$Q_H^*=\sqrt{\frac{2kDP}{(h'+cs)(P-D)}}. \tag{22}$$

We have also plotted Q_H^* in Figure 7. We can see that although it has the same structural deficiencies as the first order Maclaurin Expansion approximation, it is more accurate when *P* is relatively small (in plots a) to h)). Furthermore, it is only marginally less accurate than the first order Maclaurin Expansion approximation when *P* is large (see plots i) to l)). We propose therefore, that the simple expression (22) is at least as useful as the approximation given first order Maclarurin Series Expansion. Indeed it may be very useful when the *P* is reasonably larger than *Q*.

5. Concluding remarks

We have enhanced the Economic Production Quantity model by including the present value of the cash flows. To capture the present values we exploited the Laplace Transform. We focused on identifying the influence on the optimal batch size, Q_{NPV}^*, of the present value of the costs. We have obtained important managerial insights into the monotonicity and convexity of each of the costs in the EPQ model, and have shown that there is a unique batch quantity that minimizes the present value of the cash flows.

We were able to obtain an exact expression for the NPV of the EPQ problem in terms of a characteristic equation for the optimal batch size Q_{NPV}^*. We were also able to obtain exact point solutions in the parameter space, and showed that the Lambert W function plays an important role in the case where the production rate *P* is large compared to the demand rate *D*. We were unable to obtain a complete explicit solution to the equation for Q^*_{NPV}. However, numerical solutions to the characteristic equation given in (9) can easily be obtained using either a scientific calculator or the Excel Solver.

We have compared our results to a Maclaurin Series expansion of the NPV of the cash flows and found that the first order series expansion results in structurally erroneous insights as Q^*_{NPV} can become greater than the Q^*. However, this appears (only) to happen only when *P* is very close to (but still greater than) *D*. We have also investigated an alternative formulation of the EPQ model when the opportunity cost of the inventory investment is linked to the discount rate *s*. While this EPQ formulation suffers from the same limitation of the first order Maclaurin approximation, it appears to be more accurate when *P* is not too large. When *P* is sufficiently greater (and numerical investigation seem to suggest that sufficiently greater is not that much greater) than *D*, then $Q^*_{NPV} < Q^*$. This may help explain why the EPQ / EOQ approach is often ignored by the Lean Production community who frequently advocate that the production batch quantity, *Q*, should be a small as practically possible.

6. Acknowledgements

We thank the referees and editors for some useful suggestions that have significantly improved this paper.

7. Bibliography

Beullens P and Janssens GK, (2011) Holding costs under push or pull conditions - The impact of the Anchor Point. *European Journal of Operational Research* **215**: 115-125.

Blackburn J and Scudder G, (2009) Supply chain strategies for perishable products: The case of fresh produce. *Production & Operations Management*, **18** (2): 127-137.

Buck JR and Hill TW, (1971) Laplace transforms for the economic analysis of deterministic problems in engineering. *The Engineering Economist* 16 (4): 247-263.

Disney SM and Warburton RDH, (2012) On the Lambert W function: Economic Order Quantity applications and pedagogical considerations. *International Journal of Production Economics* **40** (2), 756-764.

Erlenkotter D, (1990) Ford Whitman Harris and the Economic Order Quantity Model. *Operations Research* **38** (6): 937-946.

Grubbström RW and Kingsman BG, (2004) Ordering and Inventory Policies for Step Changes in the Unit Item Cost: A Discounted Cash Flow Approach. *Management Science*, **50** (2): 253-267.

Grubbström RW, (1967) On the application of the Laplace transform to certain economic problems. *Management Science* **13** (7): 558-567.

Grubbström RW, (1980) A Principle for Determining the Correct Capital Costs of Work-in-progress and Inventory. *International Journal of Production Research* **18** (2): 259-271.

Hadley G, (1964) A comparison of order quantities computed using the average annual cost and the discounted cost. *Management Science* **10** (3): 472-476.

Harris FW, (1913) How many parts to make at once. *Factory, The Magazine of Management* 10 (2): 135-136, 152. Reprinted in *Operations Research*, 1990, **38** (6): 947-950.

Nise NS, (1995) *Control Systems Engineering*. Benjamin Cummings, Redwood, CA.

Taft EW, (1918) The most economical production lot. *The Iron Age* **101**: 1410-1412.

Voss C, (2010) Thoughts on the state of OM. http://om.aomonline.org/dyn/news/Editorial_Voss.pdf . Verified 11/6/11.

Warburton RDH, (2009) EOQ extensions exploiting the Lambert W function. *European Journal of Industrial Engineering* **3** (1): 45-69.