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Parameter identification for Maxwell's equations

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DECLARATION

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Abstract

In this work we present a variational algorithm to determine the parameters $\mu_r(x)$ and $\epsilon_r(x)$ in the Maxwell system

$$\nabla \times E + k\mu_r H = 0,$$

$$\nabla \times H - k\epsilon_r E = 0$$

in a body Ω from boundary measurements of electromagnetic pairs $(\boldsymbol{n} \times E_n|_{\partial\Omega}, \boldsymbol{n} \times H_n|_{\partial\Omega})$, $n = 1, 2, \dots$, where $\boldsymbol{n}$ is the outer unit normal. We show that this inverse problem can be solved by minimizing a positive functional $G(m, c)$ and using a conjugate gradient scheme. Apart from implementations with global boundary, we also consider the case of partial boundary, where we have only data available on a subset $\Gamma \subset \partial\Omega$. Further do we develop uniqueness results, to show that the given data $(\boldsymbol{n} \times E_n|_{\partial\Omega}, \boldsymbol{n} \times H_n|_{\partial\Omega})$, $n = 1, 2, \dots$, is a sufficient basis to solve the inverse problem. We investigate the uniqueness properties of the inverse problem in the case of global boundary data as well as in the case of partial boundary data. To show the effectivness and the stability of our approach we present various numerical results with noisy data. Finally we outline an alternative method, where one is only interested in recovering the support of the functions $\mu_r^{-1} - 1$ and $\epsilon_r - 1$.

Contents

1 Introduction

Since at least the time of the second world war many scientist have investigated the problem of land-mine detection. In search of a solution to this challenging problem a combination of differing technologies which include ground penetrating radar, infrared imaging and electromagnetic induction arrays have been used to try and produce a good detection system. These approaches have produced partially successful results, especially in the case of landmines that were made up of steel or similar materials. However in recent years landmines have become more sophisticated and are often made from synthetic materials which makes them harder to detect by the above technologies. We want to point out that apart from these engineering approaches methods from other fields of research have been pursued as well. For example at the University of Montana, Prof. Bromenshenk is training bees to find landmines. Another approach is the chemical demining of landmines, using the so-called Remote Explosive Scent Testing (REST), where the scent of the chemicals inside landmines is transferred to the surface and then detected by dogs or rats. Although these approaches are quite interesting, the main research is still focusing on engineering approaches and especially electromagnetic imaging, which is the topic of this work.

1.1 Landmine detection

Many groups have 'beamed' electromagnetic energy at mine targets in the ground and collected the scattered radiation. They have processed the data and detected mines. This has been done from airborne platforms, for example the American REMIDS programme and through MINE-SEEKER. It has also been done on ground based systems for example Portable Humanitarian Mine Detector (PHMD). However these systems still need further development. Mathematicians view this problem as an inverse problem, where one collects data on the surface of the earth which is then used to get an image of the ground below. Usually ground penetrating radar (GPR) is used to send electromagnetic waves into the ground. GPR works by transmitting pulses of ultra high frequency radio waves that is transmitted into the ground via a transducer or antenna. Most of these waves are absorbed by the ground (dissipated as heat energy), but a small proportion of the injected waves is reflected and the reflected waves can be measured at the surface with receiving antennas. Most GPR makes use only of measurements of the magnitude of the electric field strength at the receiving antennas. This is effective if one only needs a vague image of the subsurface, as in the detection of the position of known objects, like underground cables and the like. If one is to detect objects like landmines, the imaging has to be much better. These days it is possible for the receiving antennas to gather data on the electric field vector and the magnetic field vector or the magnetic flux density. The latter can for example be measured with an accurate magnetometer. With these measurements we will show that we can determine the electromagnetic properties of landmines and thus get an image of the subsurface. Electromagnetic imaging is not only used for landmine detection, but has many other applications like biomedical imaging, navigation, building restoration or airport security systems.

The mathematical model for electromagnetic waves is given by Maxwell's equations.

$$\nabla \cdot (\epsilon \mathcal{E}) = \rho, \tag{1.1.1}$$

$$\nabla \times \mathcal{E} + \frac{\partial(\mu \mathcal{H})}{\partial t} = 0, \tag{1.1.2}$$

$$\nabla \times \mathcal{H} - \frac{\partial(\epsilon \mathcal{E})}{\partial t} = \sigma \mathcal{E} + \mathcal{J}_a, \tag{1.1.3}$$

$$\nabla \cdot (\mu \mathcal{H}) = 0, \tag{1.1.4}$$

where $\mathcal{E}$ is the electric field, $\mathcal{H}$ the magnetic field, $\mathcal{J}_a$ is the electric current density and ρ is the charge density. To get an image of the subsurface we use boundary measurements to determine the functions $\epsilon(x)$, $\sigma(x)$ and $\mu(x)$, which characterize the Maxwell system above, where the permeability μ and the permittivity ϵ are strictly positive, bounded, scalar functions and the conductivity σ is bounded and non-negative. These three functions describe the electromagnetic properties of the subsurface and thus are key to electromagnetic imaging.

In this work we develop the necessary mathematical tools for a variational algorithm to recover these functions from boundary measurements. Further we present uniqueness and other theoretical results to show the well-posedness of the algorithm. Finally we present various numerical results that show the effectiveness of our approach, especially when done in a parallel environment. Before we go into more detail on these things, we give a short outline on inverse and ill-posed problems.

1.2 Inverse and ill-posed problems

We start our discussion of inverse problems by the following definition used by Keller and Kirsch (cf. [Kir96]).

Definition 1.1.

Two problems are inverse to each other if the formulation of each requires full or partial knowledge of the solution of the other.

Usually one problem has been studied in more detail or is easier to solve than the other. This one is then called the direct problem. A more rigorous distinction between the direct and the inverse problem can be made if one considers the following definition due to Hadamard.

Definition 1.2 (well-posed problem).

A well-posed problem has the following three properties

(i) *There exists a solution of the problem (existence).*

(ii) *There is at most one solution of the problem (uniqueness).*

(iii) *The solution depends continuously on the data (stability).*

If one of the three properties of existence, uniqueness and stability of the solution fails to hold, Hadamard called the problem ill-posed and of no physical interest. However these days many important and interesting problems in science lead to ill-posed inverse problems, where the direct problems are well-posed. Given an inverse problem it is often possible to establish existence by enlarging the solution set and uniqueness by reducing the solution set. However it is often very difficult to numerically compute the solution of an inverse problem and it often requires a priori knowledge about the solution of the inverse problem and a good numerical algorithm to establish stability. We outline some of the typical uniqueness and stability problems by considering one of the most famous inverse problems.

Example 1.3 (Electrical impedance tomography).

Let $\Omega \in \mathbb{R}^n$ *be a simply-connected bounded open set with a* $C^{1,1}$*-boundary and let* $p \in L^\infty(\Omega)$ *satisfy*

$$p(x) \geq \alpha > 0, \tag{1.2.1}$$

for some constant α. *We consider the elliptic equation*

$$-\nabla \cdot (p(x)\nabla u) = 0, \quad \forall x \in \Omega. \tag{1.2.2}$$

If we have a given $\Phi \in H^{1/2}(\partial\Omega)$ *the direct problem is to find a function* u *that satisfies* (1.2.2) *in the weak sense and the boundary condition*

$$u|_{\partial\Omega} = \Phi. \tag{1.2.3}$$

Here Φ *may be interpreted as a voltage. This direct Dirichlet boundary value problem has been studied in great detail and a very good mathematical theory for it exists. Before we introduce the inverse problem we first define the so called voltage-to-current map or Dirichlet Neumann map.*

For each p satisfying (1.2.1) *we can define*

$$\Lambda_p \Phi := p \frac{\partial u}{\partial n}|_{\partial\Omega}, \tag{1.2.4}$$

where $p\frac{\partial u}{\partial n}$ *is the so-called co-normal derivative of u on the boundary and is interpreted physically as a current. The inverse problem can now be stated as follows: Given* Λ_p *for some p, find p. As with many inverse problems the main focus is to restore uniqueness and find a stable recovery algorithm. The question of uniqueness for this inverse problem is more complicated. One of the first results was established by Kohn and Vogelius for analytic p [KV84]. In 1987 Sylvester and Uhlmann showed uniqueness for* $p \in C^{1,1}(\Omega)$ *in the case* $n \geq 3$ *[SU87] and Nachman later gave a uniqueness proof for* $n = 2$ *[Nac96]. Recently it was shown by Astala and Päivarinta that uniqueness holds even for* $p \in L^\infty(\Omega)$, *if* $n = 2$ *[AP06].*

Even more interesting than the question of uniqueness is the problem of stability. We give a short example by Alessandrini that shows the instability of the EIT problem. Consider Ω *a unit disk and set*

$$p(x) = \begin{cases} 1+\alpha, & \textit{if } |x| \leq r < 1, \\ 1, & \textit{otherwise.} \end{cases}$$

Given an arbitrary

$$\Phi(\theta) = \sum_{n=-\infty}^{\infty} \Phi_n e^{in\theta} \in H^{1/2}(\partial\Omega)$$

we get

$$(\Lambda_p\Phi)(\theta) = \sum_{n=-\infty}^{\infty} \frac{2+\alpha(1+r^{2|n|})}{2+\alpha(1-r^{2|n|})}\Phi_n e^{in\theta},$$
$$(\Lambda_1\Phi)(\theta) = \sum_{n=-\infty}^{\infty} |n|\Phi_n e^{in\theta}$$

and therefore

$$\lim_{r\to 0}(\Lambda_p - \Lambda_1)\Phi = 0, \quad \textit{but} \quad \lim_{r\to 0}\|p-1\|_{L^\infty} = \alpha.$$

This shows that the solution of the EIT problem does not continuously depend on the given data. Thus one has to develop recovery algorithms to overcome this instability. This can either be done by using a general regularization method (see for example [EHN96]) or by transforming the inverse problem to another problem, for example a minimization problem (see [Kno98]).

In general the class of inverse problems associated with partial differential equations can be described as in the following sketch. The direct problem is to find a function u, that satisfies the equation

$$Lu = f$$

in an open, simply-connected set $\Omega \in \mathbb{R}^n$ with a smooth boundary and a boundary condition on $\partial\Omega$. Here L is a differential operator and f is a given function. If L is a second-order partial differential operator with sufficiently smooth conditions and one chooses the correct space for f, the boundary conditions and the solution u, this problem is well-posed. The inverse problem is usually not to find the function f, but to find some properties, or all properties of the differential operator L, given f and full or at least partial knowledge of the solution u. These problems are usually ill-posed as is the EIT problem. This is exactly the case if we consider electromagnetic imaging as an inverse problem and as in the case of the EIT problem the two problems concerning us are uniqueness of the solution and a stable recovery.

1.3 Electromagnetic imaging as an inverse problem

We have seen that in mathematical terms the problem of electromagnetic imaging is equivalent to recover information about the coefficients μ, ϵ and σ in (1.1.1), (1.1.2), (1.1.3), (1.1.4). We give a short outline of two possible approaches on how to do this. The first approach relies on performing a Fourier transformation of the system to get a time-harmonic system. The given data consists of scattered waves from the buried object for different plane wave incident fields with varying orientation and polarization. Secondly we present an approach where we apply a Laplace transformation to Maxwell's equations to obtain a time-independent and coercive system. The data used in this approach is a collection of corresponding electric and magnetic boundary measurements and we will transform the inverse problem to a minimization problem. All the later results in this work are based on this approach.

We will use time-independent form in the following Paragraph. The derivations of these forms is given in detail in Chapter 2 [Section 2.5].

1.3.1 The inverse problem as an inverse scattering problem

All results of this paragraph can be found in [CP92] or [CK98]. The scattering approach to electromagnetic imaging is the most common approach these days. The idea is to send an incident field E_i into the ground, which induces a scattered field E_s. The asymptotic behaviour of E_s is characterized by the so-called far field pattern E_∞ which can be measured above the surface. Here we consider the so-called time-harmonic Maxwell system

$$\nabla \times E - ik\mu_r H = 0, \tag{1.3.1}$$

$$\nabla \times H + ik\epsilon_r E = \frac{1}{ik}F, \tag{1.3.2}$$

which we get via a Fourier transformation in time

$$\hat{u}(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-i\omega t} u(t)dt$$

or by analyzing electromagnetic waves at a single frequency ω. We set

$$E = \epsilon_0^{1/2}\hat{E}, \qquad H = \mu_0^{1/2}\hat{H},$$

where $\hat{E}$ is the Fourier transform of $\mathcal{E}$ and

$$\epsilon_r = \frac{1}{\epsilon_0}\left(\epsilon + \frac{i\sigma}{\omega}\right), \quad \mu_r = \frac{\mu}{\mu_0}.$$

The wavenumber k is given by

$$k = \omega\sqrt{\epsilon_0\mu_0}.$$

To simplify things a bit more we set $F = 0$ and assume $\mu = \mu_0$ in $\mathbb{R}^3$ and define the refractive index $n(x) = \epsilon_r(x)$ in accordance with the common notation in scattering theory. Thus we get the system

$$\nabla \times E - ikH = 0, \quad \nabla \times H + iknE = 0. \tag{1.3.3}$$

We assume that $m(x) := 1 - n(x)$ is compactly supported in $\mathbb{R}^3$. The direct scattering problem associated with (1.3.3) is to find solutions E, H of (1.3.3) in the form

$$E = E^i + E^s, \quad H = H^i + H^s,$$

where the incident fields E^i, H^i are given by

$$E^i(x) = (d \times p)e^{ikx \cdot d}, \quad H^i(x) = \frac{1}{ik}\nabla \times E^i(x), \tag{1.3.4}$$

with constant vectors $p \in \mathbb{R}^3$, $d \in \Omega = \{x : |x| = 1\}$ and the scattered fields satisfy the Silver-Müller radiation condition

$$H^s \times \hat{x} - E^s = \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty, \tag{1.3.5}$$

uniformly for all directions $\hat{x} = \frac{x}{|x|}$. This radiation condition is necessary to guarantee the well-posedness of the direct scattering problem, since the system (1.3.3) is not coercive and thus the behaviour of the electric and magnetic fields at infinity has to be controlled. The electric field satisfies the second order equation

$$\nabla \times \nabla \times E - k^2 nE = 0, \tag{1.3.6}$$

in $\mathbb{R}^3$ as well as the integral equation

$$\begin{aligned} E(x) =& E^i(x) - k^2 \int_D m(y)\Phi(x,y)E(y)dy \\ &+ \nabla \int_D \frac{1}{n(y)} E(y) \cdot \nabla n(y)\Phi(x,y)dy, \quad x \in \mathbb{R}^3, \end{aligned} \tag{1.3.7}$$

where

$$\Phi(x,y) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|} \tag{1.3.8}$$

and D is the support of m. Equation (1.3.7) has a unique solution, which also satisfies the equations (1.3.3), (1.3.4), (1.3.5) (see for example [CK98]). Of extreme importance to the inverse scattering problem is the asymptotic behaviour of the scattered field E^s,

$$E^s(x) = \frac{e^{ikr}}{r} E_\infty(\hat{x}; d, p) + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{1.3.9}$$

where $r = |x|$ and E_∞ is the so-called far field pattern. In scattering applications the given data for the inverse problem consists of the far field pattern $E_\infty(\hat{x}; d_j, p_i)$ for all directions d_j in Ω and three basis vectors p_i in $\mathbb{R}^3$. The far field pattern determines the index of refraction uniquely.

Theorem 1.4.

Let $m = 1 - n \in C_0^3(\mathbb{R}^3)$, where n is the refractive index, let p_i, $i = 1, 2, 3$ be three basis vectors in $\mathbb{R}^3$, and let Ω be the unit sphere in $\mathbb{R}^3$. Then n is uniquely determined by the electric far field pattern $E_\infty(\hat{x}; d, p_i)$ corresponding to the incident fields $E^i(x) = (d \times p_i)e^{ikx\cdot d}$ for a fixed wave number $k > 0$, $d, \hat{x} \in \Omega$ and $i = 1, 2, 3$.

Proof.

See [CP92][Theorem 4.1]. □

There are several methods available on how to determine information on the refractive index n from the knowledge of the far field pattern E_∞ in a stable way and there is still ongoing research in this area. We refer the interested reader to the works of Kress (e.g. [Kre04])) Colton (e.g. [CHP03a], [CCM04]) and Kirsch to get an overview over some of the methods available to solve the inverse scattering problem. Especially the factorization method developed by Kirsch et. al. (e.g. [Kir03], [Kir04], [GHK+05]) is interesting with respect to this work, since in Chapter 6 we develop a factorization method for the case of given near field data.

1.3.2 The inverse problem as a variational problem

In this work we want to present an alternative to the scattering approach outlined above. In many inverse problems for elliptic equations, variational algorithms are successfully used (see [EHN96], [Kno98] or [BJK05]). In this work we show that this can also be done in the case of Maxwell's equations. The main difference to the scattering approach is, that we work with a coercive system, which we get via a Laplace transformation and that we use near field data instead of far field data. As we will show in Chapter 2 (see (2.5.26), (2.5.27)) we can reduce the system (1.1.1), (1.1.2), (1.1.3), (1.1.4) by applying a Laplace transformation with frequency $\lambda \in \mathbb{R}_+$

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t) dt$$

to get the coercive system

$$\nabla \times E + k\mu_r H = 0, \tag{1.3.10}$$

$$\nabla \times H - k\epsilon_r E = -\frac{1}{k}F. \tag{1.3.11}$$

Here E denotes a scaled Laplace transform of $\mathcal{E}$, i.e.

$$E = \epsilon_0^{1/2}\hat{E}, \qquad H = \mu_0^{1/2}\hat{H},$$

where $\hat{E}$, $\hat{H}$ denote the Laplace transforms of $\mathcal{E}$, $\mathcal{H}$ and ϵ_0 and μ_0 are the constant values of ϵ and μ outside of our given domain. The term F includes the influence of $\mathcal{J}_a$ and ρ. The new coefficients are

$$\epsilon_r = \frac{1}{\epsilon_0}\left(\epsilon + \frac{\sigma}{\lambda}\right), \quad \mu_r = \frac{\mu}{\mu_0},$$

and the wavenumber k is given by

$$k = \lambda\sqrt{\epsilon_0\mu_0}.$$

As a result of the Laplace transformation, the time-independent system (1.3.10), (1.3.11) is a coercive system. This is one of the differences between applying a Fourier transformation and applying a Laplace transformation. In general coercive systems are better suited for a variational approach to inverse problems, than non-coercive systems.

Note that k depends on the chosen frequency λ of the Laplace transformation. Choosing a different frequency gives different electromagnetic fields and thus different corresponding electric and magnetic boundary measurements as data for the inverse problem. Mathematically the given data is represented by the so-called impedance map

$$Z_{\mu_r,\epsilon_r}(\boldsymbol{n} \times H|_{\partial\Omega}) = \boldsymbol{n} \times E|_{\partial\Omega}, \tag{1.3.12}$$

or the so-called admittance map

$$\Lambda_{\mu_r,\epsilon_r}(\boldsymbol{n} \times E|_{\partial\Omega}) = \boldsymbol{n} \times H|_{\partial\Omega}. \tag{1.3.13}$$

This brings us to the second difference to the scattering approach. The data of the inverse problem does not consist of far field data, but of near field data, given by Z_{μ_r,ϵ_r} or $\Lambda_{\mu_r,\epsilon_r}$. We will see later (Theorem 3.3) that these maps uniquely identify the coefficients μ_r and ϵ_r. Using the coercivity of the system (1.3.10), (1.3.11) we are able to define a non-negative functional $G(m,c)$ with a unique global minimum at $m = \mu_r^{-1}$ and $c = \epsilon_r$. Thus we reduce the inverse problem to a minimization problem.

1.4 Outline of this thesis

The main topic of this work is the minimization approach outlined in Subsection 1.3.2 to recover μ_r and ϵ_r. In Chapter 5 we show that it is not only possible to recover the functions μ_r and ϵ_r by minimizing a non-negative functional $G(m,c)$, but also that this can be done without any constraints on the smoothness or the spatial dependence of the these coefficients.

However before we outline the variational approach, we first give an overview of the mathematical tools needed for this analysis in Chapter 2. Then we present various uniqueness results for the inverse problem in Chapter 3. We show that the inverse problem has a unique solution in the case of global boundary data and also present a partial uniqueness result in the case of local boundary data. Finally we give a simple uniqueness result in the case of given interior data, which is the basis of Chapter 4, where we explain the main idea of the variational approach using interior data. We also present a range of numerical implementations for the case of global boundary data as well as local boundary data, where we show the effectiveness of our variational approach. Finally, Chapter 6, we discuss an alternative method to partially solve the inverse problem of recovering μ_r and ϵ_r by using a so-called factorization method.

2 Mathematical Preliminaries

In this section we present mathematical theorems, definitions and tools which we need for our analysis of Maxwell's equations. What we do not explain are the definitions of the standard Sobolev spaces $H^s(\Omega)$, $s \in \mathbb{R}$, the corresponding trace spaces $H^r(\partial\Omega)$, $r \in \mathbb{R}$ and the basic properties of elliptic partial differential equations. For definitions and explanations of these we recommend any introductory book to partial differential equations, for example [Wlo87] or [GT01]. Throughout this chapter we assume that Ω is a bounded domain in $\mathbb{R}^3$ with a $C^{1,1}$-boundary $\partial\Omega$, if not stated otherwise. We closely follow the books of Monk [Mon03][Chapter 3] and Cessenat [Ces96][Chapter 2] in many parts of this chapter. We will not consider functional analysis and sepctral theory of unbounded operators, since this is not neccessary for this work. Thus in the following symmetric will be equivalent to self-adjoint.

2.1 Fourier and Laplace transformations

When working with time-dependent partial differential equations two very important tools are the Fourier and the Laplace transformation. Before we give a definition of these transformations we first define the proper spaces in which the Fourier transformation acts.

Definition 2.1 (Schwartz space).

We define the Schwartz space

$$\mathcal{S}(\mathbb{R}^n) = \{f : f \in C^\infty(\mathbb{R}^n), \sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha f(x)| < \infty \text{ for all Multiindices } \alpha, \beta\}. \tag{2.1.1}$$

Definition 2.2 (Tempered Distributions).

We call a functional $T : \mathcal{S}(\mathbb{R}^n) \mapsto \mathbb{R}$ a tempered distribution if there is an $N \in \mathbb{N}$ and a $C > 0$ with

$$|T(\varphi)| \leq C \sum_{|\beta|,|\alpha|\leq N} \sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha \varphi(x)|, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{2.1.2}$$

The space of tempered Distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$

For every function in the Schwartz space we can make the following definition.

Definition 2.3 (The Fourier Transform).

For every $f \in \mathcal{S}(\mathbb{R}^n)$ we define

$$(\mathcal{F}f)(\zeta) := \hat{f}(\zeta) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\zeta} dx, \quad \forall \zeta \in \mathbb{R}^n, \tag{2.1.3}$$

the Fourier transform of f.

$\mathcal{F}$ is continuous linear and invertible. The inverse Fourier transform is given by

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\zeta) e^{ix\cdot\zeta} d\zeta, \quad \forall \zeta \in \mathbb{R}^n. \tag{2.1.4}$$

The Fourier transform $\mathcal{F}$: $\mathcal{S}(\mathbb{R}^n) \mapsto \mathcal{S}(\mathbb{R}^n)$ can be extended to a linear and continuous operator from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ by the definition

$$(\mathcal{F}T)(\varphi) = T(\mathcal{F}\varphi), \quad T \in \mathcal{S}'(\mathbb{R}^n), \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Definition 2.4 (The Laplace Transform).

If $u \in L^1(\mathbb{R}_+)$ then we define its Laplace transform to be

$$\hat{u}(s) = \int_0^\infty e^{-st} u(t) dt. \tag{2.1.5}$$

If the upper limit of the Integral in (2.1.5) is a finite number, we speak of a finite Laplace transform.

2.2 Functional Analysis and Spectral Theory

Besides to specific results for Maxwell's equations, we first need a few more general tools from functional analysis.

Theorem 2.5 (Riesz-Schauder spectral theorem).

Let H be a Hilbert space and $A : H \mapsto H$ a linear, self-adjoint, compact operator. Then

(i) *The spectrum $\sigma(A) \subset \{\lambda : |\lambda| \leq ||A||\}$ consists only of a finite number of eigenvalues or of a sequence of eigenvalues λ_n with $\lambda_n \to 0$ as $n \to \infty$.*

(ii) *Each eigenspace is finite dimensional.*

(iii) *$\sigma(A) \subset \mathbb{R}$. The eigenspaces $E(\lambda_n, A)$ corresponding to two distinct eigenvalues are mutually orthogonal.*

(iv) *For each $x \in H$ we have*

$$Ax = \sum_n \lambda_n \langle x, z_n \rangle z_n, \quad z_n \in E(\lambda_n, A).$$

A further concept that will be useful is the one of positivity.

Definition 2.6 (Positive operators in Hilbert spaces).

Let H be a Hilbert space and T a linear operator. We call T positive if

$$\langle Tx, x \rangle_H \geq 0, \quad \forall x \in H.$$

Remark: Positive, self-adjoint operators have only positive eigenvalues.

The properties shown so far, were restricted to operators defined on Hilbert spaces. However often we have operators defined on Banach spaces, which have similar properties to those defined on Hilbert spaces.

Definition 2.7 (Dual Pairing).

Let X and Y be real Banach spaces. We call $\langle \cdot, \cdot \rangle : X \times Y \mapsto \mathbb{R}$ a dual pairing between X and Y if it is bilinear and

$$\forall x \in X \backslash \{0\}, \ \exists y \in Y \quad \langle x, y \rangle \neq 0,$$

$$\forall y \in Y \backslash \{0\}, \ \exists x \in X \quad \langle x, y \rangle \neq 0.$$

Often the dual pairing of two Banach spaces can be identified with the scalar product in a Hilbert space. In this case extensions of operators in a Banach space to the Hilbert space prove to be useful.

Theorem 2.8.

Let X_1, X_2 be Banach spaces, and H_1, H_2 be Hilbert spaces with continuous and dense embeddings $X_i \subset H_i$, $i = 1, 2$. Let $T : X_1 \mapsto X_2$ and $T' : X_2 \mapsto X_1$ be two bounded linear operators such that T and T' are adjoint to each other with respect to the inner products of H_1 and H_2. Then T can be extended to a bounded operator from H_1 to H_2. Furthermore the following results hold.

(a) If λ belongs to the point spectrum of T over X_1, then λ belongs to the point spectrum of T over H_1.

(b) If T has a standard discrete spectrum over X_1, i.e. all points of the spectrum belong to the point spectrum with the possible exception of $\lambda = 0$, the eigenfunctions of T in X_1 span - in the sense of the norm of H_1 - the range of T.

(c) If T is compact over X_1, then T is compact over H.

Proof.

See [Lax54]. □

Corollary 2.9.

Let the conditions of the Theorem 2.8 hold. If $H = H_1 = H_2$, X_1, X_2 are reflexive Banach spaces and T is compact over X_1 and self-adjoint with respect to the inner product of H, then T allows a spectral decomposition

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle_H \phi_n,$$

where the λ_n are the eigenvalues of T and ϕ_n the eigenfunctions.

Proof.

Follows from (b) and (c) in Theorem 2.8. □

Another crucial result for weak solutions of partial differential equations is the Sobolev embedding theorem.

Theorem 2.10.

Let Ω be bounded with a $C^{k,\kappa}$-boundary and $s < r \leq k + \kappa$, $s, r \in \mathbb{R}_+$. Then the embedding

$$H^s(\Omega) \subset H^r(\Omega)$$

is compact. Now let M be a compact $C^{k,\kappa}$-manifold and $s < r \leq k + \kappa$, $s, r \in \mathbb{R}_+$. Then the embedding

$$H^s(M) \subset H^r(M)$$

is compact.

Proof.

See for example [Wlo87][Chapter 7]. □

2.3 Differential operators on the boundary

In our discussion of the inverse Maxwell problem, we have to work with differential operators related to tangential vector fields on the boundary. For this we define the space of surface tangential vector fields in $L^2(\partial\Omega)^3$ by

$$L_t^2(\partial\Omega) = \{u \in L^2(\partial\Omega)^3 |\ \boldsymbol{n} \cdot u = 0, \text{ a.e. on } \partial\Omega\}, \tag{2.3.1}$$

where here and thereafter we denote the unit outer normal by $\boldsymbol{n}$. $L_t^2(\partial\Omega)$ equipped with the standard L^2 norm is a Hilbert space. Next we define the so-called surface gradient, the surface divergence and the surface scalar curl.

Definition 2.11.

Let $p \in H^1(\partial\Omega)^3$ and let the parametric representation of $x \in \partial\Omega$ locally be given by

$$x = (x_1(u_1, u_2), x_2(u_1, u_2), x_3(u_1, u_2))^T.$$

Then the surface gradient is independet of the chosen parametric reprsentation and $\nabla_{\partial\Omega} p \in L_t^2(\partial\Omega)$ is given by

$$\nabla_{\partial\Omega} p = \sum_{i,j}^{2} g^{ij} \frac{\partial p}{\partial u_i} \frac{\partial x}{\partial u_j},$$

where g^{ij} is the inverse of the matrix

$$g_{ij} = \frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j}, \quad i, j = 1, 2.$$

An important property of $\nabla_{\partial\Omega} p$ is that for differentiable functions in Ω

$$(\nabla p)|_{\partial\Omega} = \nabla_{\partial\Omega} p + \frac{\partial p}{\partial \boldsymbol{n}}$$

as well as

$$(\boldsymbol{n} \times \nabla p) \times \boldsymbol{n} = \nabla_{\partial\Omega} p.$$

Having defined the surface gradient we can define the surface divergence $\nabla_{\partial\Omega} \cdot : L_t^2(\partial\Omega) \mapsto$

$H^{-1}(\partial\Omega)$ by the duality relation

$$\int_{\partial\Omega} \nabla_{\partial\Omega} \cdot v\, p\, dS = -\int_{\partial\Omega} v \cdot \nabla_{\partial\Omega}\, p\, dS, \quad \forall p \in H^1(\partial\Omega).$$

Finally the surface scalar curl $\nabla_{\partial\Omega}\times : L_t^2(\partial\Omega) \mapsto H^{-1}(\partial\Omega)$ is defined by using Stoke's theorem

$$\int_{\partial\Omega} \nabla_{\partial\Omega} \times v\, p\, dS = -\int_{\partial\Omega} v \cdot \boldsymbol{n} \times \nabla_{\partial\Omega}\, p\, dS, \quad \forall p \in H^1(\partial\Omega).$$

By using the above definitions we see that for $v \in L_t^2(\partial\Omega)$ we have the relations

$$\nabla_{\partial\Omega} \times v = -\nabla_{\partial\Omega} \cdot (\boldsymbol{n} \times v), \qquad \nabla_{\partial\Omega} \cdot v = \nabla_{\partial\Omega} \times (\boldsymbol{n} \times v). \tag{2.3.2}$$

2.4 The spaces $H(\mathrm{div};\Omega)$ and $H(\mathrm{curl};\Omega)$

In our analysis of Maxwell's equations we consider functions with a square-integrable divergence.

Definition 2.12.

The space of functions with square-integrable divergence is defined as

$$H(\mathrm{div};\Omega) := \{u \in L^2(\Omega)^3 |\ \nabla \cdot u \in L^2(\Omega)\}.$$

The associated graph norm is given by

$$\|u\|_{H(\mathrm{div},\Omega)} := \left(\|u\|^2_{L^2(\Omega)^3} + \|\nabla \cdot u\|^2_{L^2(\Omega)}\right)^{1/2}.$$

$H(\mathrm{div};\Omega)$ has the following properties.

Theorem 2.13.

$$H(\mathrm{div};\Omega) = \textit{closure of } C^\infty(\overline{\Omega})^3 \textit{ in the } H(\mathrm{div};\Omega) \textit{ norm.}$$

Further the normal trace mapping $\gamma_n(v) = v|_{\partial\Omega} \cdot \boldsymbol{n}$ on $C^\infty(\overline{\Omega})^3$ can be continuously extended to a mapping from $H(\text{div};\Omega)$ onto $H^{-1/2}(\partial\Omega)$ and the following integration by parts formula holds for all $v \in H(\text{div};\Omega)$ and $\phi \in H^1(\Omega)$

$$\int_\Omega v \cdot \nabla\phi \, dx = -\int_\Omega \nabla \cdot v \, \phi \, dx + \int_{\partial\Omega} \gamma_n(v)\phi \, dS.$$

Setting

$$H_0(\text{div};\Omega) := \textit{closure of } C_0^\infty(\Omega)^3 \textit{ in the } H(\text{div};\Omega) \textit{ norm,}$$

gives

$$H_0(\text{div};\Omega) = \{v \in H(\text{div};\Omega)| \ \gamma_n(v) = 0\}.$$

Proof.

See [Mon03][Theorem 3.22, Theorem 3.24 and Theorem 3.25]. □

Apart from $H(\text{div};\Omega)$ we also need the following space.

Definition 2.14.

We define the space of functions with curl in $L^2(\Omega)$ by

$$H(\text{curl};\Omega) := \{u| \ u \in L^2(\Omega)^3, \ \nabla \times u \in L^2(\Omega)^3\},$$

with the associated graph norm

$$\|u\|_{H(\text{curl};\Omega)} := \left(\|u\|_{L^2(\Omega)^3} + \|\nabla \times u\|_{L^2(\Omega)^3}\right)^{1/2}.$$

Further we define the space

$$H_0(\text{curl};\Omega) = \{v \in H(\text{curl};\Omega)|\ \boldsymbol{n} \times v|_{\partial\Omega} = 0\}.$$

Beside the space $H(\text{curl};\Omega)$ we also need spaces with more regularity.

Definition 2.15.

Let $s \in \mathbb{R}$. The spaces $H^s(\text{curl};\Omega)$ are defined as

$$H^s(\text{curl};\Omega) := \{u|\ u \in H^s(\Omega)^3,\ \nabla \times u \in H^s(\Omega)^3\}.$$

Before we present an integration by parts formula for functions in $H(\text{curl};\Omega)$ we must define the appropriate trace spaces.

Definition 2.16.

Let $s \in \mathbb{R}$. We define the spaces

$$H^s(\text{div};\partial\Omega) = \{u \in H^s(\partial\Omega)^3,\ \boldsymbol{n} \cdot u = 0,\ \nabla_{\partial\Omega} \cdot u \in H^s(\partial\Omega)\}, \tag{2.4.1}$$

$$H^s(\text{curl};\partial\Omega) = \{u \in H^s(\partial\Omega)^3,\ \boldsymbol{n} \cdot u = 0,\ \nabla_{\partial\Omega} \times u \in H^s(\partial\Omega)\}. \tag{2.4.2}$$

The spaces $H^{-1/2}(\text{div};\partial\Omega)$ and $H^{-1/2}(\text{curl};\partial\Omega)$ are linked by duality.

Theorem 2.17.

The space $H^{-1/2}(\text{div};\partial\Omega)$ is identified with the dual of $H^{-1/2}(\text{curl};\partial\Omega)$, when we use $L^2_t(\partial\Omega)$ as pivot space. The mapping $v \mapsto \boldsymbol{n} \times v$ is an isomorphism from $H^{-1/2}(\text{div};\partial\Omega)$ onto $H^{-1/2}(\text{curl};\partial\Omega)$, with the inverse $w \mapsto -\boldsymbol{n} \times w$.

Proof.

See [Ces96][Section 2.4, Corollary 2 and Proposition 3]. □

Now we can define traces for functions in $H(\text{curl};\Omega)$.

Theorem 2.18.

We define the trace operators $\gamma_t : H(\text{curl};\Omega) \mapsto H^{-1/2}(\text{div};\partial\Omega)$ *and* $\gamma_T : H(\text{curl};\Omega) \mapsto H^{-1/2}(\text{curl};\partial\Omega)$ *by*

$$\gamma_t(v) = \boldsymbol{n} \times v|_{\partial\Omega}, \qquad \gamma_T(v) = (\boldsymbol{n} \times v|_{\partial\Omega}) \times \boldsymbol{n}.$$

These trace operators are continuous and onto. They are also continuous as operators from $H^1(\text{curl};\Omega)$ *onto* $H^{1/2}(\text{div};\partial\Omega)$ *and* $H^{1/2}(\text{curl};\partial\Omega)$ *respectively.*

Proof.

See [Ces96][Page 35 Theorem 4 and Page 37 Remark 5]. □

With the help of these trace spaces we get an integration by parts theorem for $H(\text{curl};\Omega)$.

Theorem 2.19.

Let u *and* v *be elements of* $H(\text{curl};\Omega)$. *Then the following integration by parts formula holds:*

$$\int_\Omega \nabla \times v(x) \cdot u(x)\, dx = \int_\Omega v(x) \cdot \nabla \times u(x)\, dx + \int_{\partial\Omega} \gamma_t(v) \cdot \gamma_T(u)\, dS. \tag{2.4.3}$$

Proof.

See for example [Mon03][Theorem 3.31]. □

2.5 Maxwell's equations

In this section we review the direct problem for Maxwell's equations. Maxwell's equations in their time-dependent form are given by

$$\nabla \cdot \mathcal{D} = \rho, \tag{2.5.1}$$

$$\nabla \times \mathcal{E} + \frac{\partial \mathcal{B}}{\partial t} = 0, \tag{2.5.2}$$

$$\nabla \times \mathcal{H} - \frac{\partial \mathcal{D}}{\partial t} = \mathcal{J}, \tag{2.5.3}$$

$$\nabla \cdot \mathcal{B} = 0. \tag{2.5.4}$$

The vectors $\mathcal{E}$ and $\mathcal{H}$ are called the electric and magnetic field. $\mathcal{D}$ and $\mathcal{B}$ are the electric displacement and the magnetic induction (or magnetic flux density). $\mathcal{J}$ denotes the vector current density function and ρ stands for the charge density. Equation (2.5.1) is known as Gauss' law, equation (2.5.2) is Faraday's law, equation (2.5.3) is the modified Ampère's law and equation (2.5.4) expresses the fact the magnetic induction $\mathcal{B}$ is solenoidal.

These equations can be reduced to a system for $\mathcal{E}$ and $\mathcal{H}$ only using so-called constitutive equations. These laws depend on the properties of the interior of the domain Ω. We consider a setting where outside Ω we have a free space model and in the interior of Ω we have inhomogeneous but isotropic materials. In a vacuum or free space the relations

$$\mathcal{D}(x,t) = \epsilon_0 \mathcal{E}(x,t), \tag{2.5.5}$$

$$\mathcal{B}(x,t) = \mu_0 \mathcal{H}(x,t) \tag{2.5.6}$$

hold, where μ_0 and ϵ_0 are positive constants. μ is called the magnetic permeability and ϵ the electric permittivity or dielectric constant. In inhomogeneous isotropic media the relations

$$\mathcal{D}(x,t) = \epsilon(x)\mathcal{E}(x,t), \tag{2.5.7}$$

$$\mathcal{B}(x,t) = \mu(x)\mathcal{H}(x,t) \tag{2.5.8}$$

$$\mathcal{J}(x,t) = \sigma(x)\mathcal{E}(x,t) + \mathcal{J}_a(x,t) \tag{2.5.9}$$

hold, where the permeability μ and the permittivity ϵ are strictly positive, bounded and scalar functions and the conductivity σ is bounded as well, however with lower bound 0. $\mathcal{J}_a$ is a given applied current density. If not stated otherwise we make the following assumptions.

- $\epsilon|_{\partial\Omega} = \epsilon_0$, $\mu|_{\partial\Omega} = \mu_0$, $\sigma|_{\partial\Omega} = 0$, $\epsilon_0, \mu_0 \in \mathbb{R}_+$.
- $0 < \epsilon_m \leq \epsilon(x) \leq \epsilon_M$, $0 \leq \sigma(x) \leq \sigma_M$, $0 < \mu_m \leq \mu(x) \leq \mu_M$.
- ϵ, μ, $\sigma \in C^3(\overline{\Omega})$.

This leads to the following system

$$\nabla \cdot (\epsilon\mathcal{E}) = \rho, \tag{2.5.10}$$

$$\nabla \times \mathcal{E} + \frac{\partial(\mu\mathcal{H})}{\partial t} = 0, \tag{2.5.11}$$

$$\nabla \times \mathcal{H} - \frac{\partial(\epsilon\mathcal{E})}{\partial t} = \sigma\mathcal{E} + \mathcal{J}_a, \tag{2.5.12}$$

$$\nabla \cdot (\mu\mathcal{H}) = 0. \tag{2.5.13}$$

Using a Fourier transform in time,

$$\hat{u}(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-i\omega t} u(t) dt$$

or analyzing electromagnetic waves at a single frequency ω, we can reduce the time-dependent problem to a time-harmonic problem

$$\nabla \cdot (\epsilon \hat{E}) = \frac{1}{i\omega} \nabla \cdot (\sigma \hat{E} + \hat{J}_a), \tag{2.5.14}$$

$$\nabla \times \hat{E} - i\omega\mu\hat{H} = 0, \tag{2.5.15}$$

$$\nabla \times \hat{H} + i\omega\epsilon\hat{E} - \sigma\hat{E} = \hat{J}_a, \tag{2.5.16}$$

$$\nabla \cdot (\mu\hat{H}) = 0, \tag{2.5.17}$$

where $\hat{A}$ stands for the Fourier transform of the time-dependent vector field $\mathcal{A}$. In (2.5.14) we have used the relation $i\omega\hat{\rho} = \nabla \cdot \hat{J}$ which can be seen by taking the divergence of equation (2.5.16). Following an approach by Colton and Kress [CK98], we define

$$E = \epsilon_0^{1/2} \hat{E}, \qquad H = \mu_0^{1/2} \hat{H}$$

as well as the relative permittivity

$$\epsilon_r = \frac{1}{\epsilon_0} \left(\epsilon + \frac{i\sigma}{\omega} \right)$$

and the relative permeability

$$\mu_r = \frac{\mu}{\mu_0}.$$

Thus we get the following system

$$\nabla \times E - ik\mu_r H = 0, \tag{2.5.18}$$

$$\nabla \times H + ik\epsilon_r E = \frac{1}{ik}F, \tag{2.5.19}$$

where the wavenumber $k = \omega\sqrt{\epsilon_0\mu_0}$ and $F = ik\mu_0^{1/2}\hat{J}_a$. Note that we get the divergence conditions back, if we take the divergence of equations (2.5.18) and (2.5.19). Often one reduces the above first-order system to a single second-order equation for either E or H. By solving (2.5.18) for H and substituting this into (2.5.19) we get the equation

$$\nabla \times (\mu_r^{-1}\nabla \times E) - k^2\epsilon_r E = F, \quad \text{in } \Omega. \tag{2.5.20}$$

To get a unique solution for the above equation in $H(\text{curl};\Omega)$ we have to apply either an electric boundary condition

$$\boldsymbol{n} \times E|_{\partial\Omega} = g \in H^{-1/2}(\text{div};\partial\Omega)$$

or a magnetic boundary condition

$$\boldsymbol{n} \times H|_{\partial\Omega} = \frac{1}{ik}(\nabla \times E)|_{\partial\Omega} = h \in H^{-1/2}(\text{div};\partial\Omega).$$

As we will see later, one can easily find conditions for ϵ_r, μ_r and ω under which the above boundary value problems have a unique solution. The equation (2.5.20) has a major disadvantage for our development of a variational algorithm to solve the inverse Maxwell problem: since k^2 is strictly positive, we can easily see that equation (2.5.20) is not coercive. However the algorithms in Chapter 4 and Chapter 5 depend on the coercivity of the underlying equation. This problem can be removed by imposing the initial conditions

$$\mathcal{E}|_{t=0} = 0, \qquad \mathcal{H}|_{t=0} = 0 \tag{2.5.21}$$

on the time-dependent fields $\mathcal{E}$ and $\mathcal{H}$ in (2.5.10) - (2.5.13). Given these homogeneous initial conditions we can apply a Laplace transformation

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t) dt$$

with a real Laplace parameter $\lambda \in \mathbb{R}_+$ to (2.5.10) - (2.5.13). This yields

$$\nabla \cdot (\epsilon \hat{E}) = -\frac{1}{\lambda} \nabla \cdot (\sigma \hat{E} + \hat{J}_a), \tag{2.5.22}$$

$$\nabla \times \hat{E} + \lambda \mu \hat{H} = 0, \tag{2.5.23}$$

$$\nabla \times \hat{H} - \lambda \epsilon \hat{E} - \sigma \hat{E} = \hat{J}_a, \tag{2.5.24}$$

$$\nabla \cdot (\mu \hat{H}) = 0, \tag{2.5.25}$$

where $\hat{A}$ stands now for the Laplace transform of the time-dependent vector field $\mathcal{A}$. We set

$$E = \epsilon_0^{1/2} \hat{E}, \qquad H = \mu_0^{1/2} \hat{H}$$

as well as

$$\epsilon_r = \frac{1}{\epsilon_0} \left(\epsilon + \frac{\sigma}{\lambda} \right), \quad \mu_r = \frac{\mu}{\mu_0}.$$

Now we get the system

$$\nabla \times E + k \mu_r H = 0, \tag{2.5.26}$$

$$\nabla \times H - k \epsilon_r E = -\frac{1}{k} F, \tag{2.5.27}$$

where the wavenumber is defined as $k = \lambda\sqrt{\epsilon_0\mu_0}$ and the right hand side is $F = -k\mu_0^{1/2}\hat{J}_a$. If we reduce this system to a second order equation for E we get the desired coercive equation

$$\nabla \times (\mu_r^{-1}\nabla \times E) + k^2\epsilon_r E = F, \quad \text{in } \Omega. \tag{2.5.28}$$

Apart from the desired properties for a variational algorithm to solve the corresponding inverse problem, equation (2.5.28) also has the advantage that the forward problem has a unique solution for every frequency $\lambda \in \mathbb{R}_+$. Furthermore we want to point out that in (2.5.26), (2.5.27) all the quantities are real and thus given real boundary conditions, the solutions (E, H) of (2.5.26), (2.5.27) are real as well. This will be very important in the derivation of our variational approach to solve the inverse problem for (2.5.26), (2.5.27). Before we go into further details on this, we discuss the well-posedness of the forward problems for equations (2.5.28) and (2.5.20). For equation (2.5.28) this can be done by using the famous Lax-Milgram Theorem. First we have to give a rigorous definition of coercivity.

Definition 2.20.

Let H be a Hilbert space. A bilinear form $a : H \times H \mapsto \mathbb{R}$ is called bounded if there exists a constant c_1 such that

$$|a(u, v)| \leq c_1\|u\|\|v\|, \quad \forall u, v \in H.$$

It is called coercive if there exists a number $c_2 > 0$ such that

$$a(u, u) \geq c_2\|u\|^2, \quad \forall u \in H.$$

Using this definition we can state the Lax-Milgram Theorem.

Theorem 2.21 (Lax-Milgram).

Let a be a bounded, coercive bilinear form on a Hilbert space H. Then for every linear functional $F \in H'$, there exists a unique element u such that

$$a(u, v) = F(v), \quad \forall v \in H,$$

where H' is the dual space of H.

Proof.

See for example [GT01][Theorem 5.8]. □

Now we show that Maxwell's equations with electric or magnetic boundary conditions have unique solutions in $H(\text{curl};\Omega)$. For simplicity we do this for a vanishing right-hand side $F = 0$.

Theorem 2.22.

Let $F = 0$. Equation (2.5.28) together with the electric boundary condition

$$\boldsymbol{n} \times E|_{\partial\Omega} = g \in H^{-1/2}(\text{div};\partial\Omega)$$

or a magnetic boundary condition

$$\boldsymbol{n} \times (\nabla \times E)|_{\partial\Omega} = h \in H^{-1/2}(\text{div};\partial\Omega) \tag{2.5.29}$$

has a unique weak solution $E \in H(\text{curl};\Omega)$.

Proof.

We show that the corresponding bilinear form $a(E, v)$ for the equation (2.5.28) in $H(\text{curl};\Omega)$ is given by

$$a(E, v) := \int_\Omega \mu_r^{-1} \langle \nabla \times E, \nabla \times v \rangle \, dx + k^2 \int_\Omega \epsilon_r \langle E, v \rangle dx, \quad E, v \in H(\mathrm{curl}; \Omega)$$

and that this form is coercive and bounded. In the case of a magnetic boundary condition (2.5.29) this can be seen as follows. Multiplying equation (2.5.28) and performing an integration by parts we get

$$\begin{aligned} 0 &= \int_\Omega \langle \nabla \times (\mu_r^{-1} \nabla \times E), v \rangle \, dx + k^2 \int_\Omega \epsilon_r \langle E, v \rangle dx \\ &= \int_\Omega \mu_r^{-1} \langle \nabla \times E, \nabla \times v \rangle \, dx + k^2 \int_\Omega \epsilon_r \langle E, v \rangle dx \\ &\quad + \int_{\partial\Omega} \langle \gamma_t(\nabla \times E), \gamma_T(v) \rangle dS, \quad \forall v \in H(\mathrm{curl}; \Omega). \end{aligned}$$

Therefore we have to solve

$$a(E, v) = - \int_{\partial\Omega} \langle h, \gamma_T(v) \rangle dS, \quad \forall v \in H(\mathrm{curl}; \Omega).$$

Since μ_r^{-1} and ϵ_r have absolute upper and lower bounds (see the assumptions at the beginning of this section) and k^2 is a positive number, $a(u, v)$ is coercive and bounded. Thus the magnetic boundary value problem has a unique solution due to the Lax-Milgram Theorem.

For the electric boundary value problem this can be seen as follows. We take a function T such that $\boldsymbol{n} \times T|_{\partial\Omega} = g$ and then set $S := E - T$. The function S then satisfies

$$\nabla \times (\mu_r^{-1} \nabla \times S) + k^2 \epsilon_r S = F, \quad \text{in } \Omega,$$

$$\boldsymbol{n} \times S = 0, \quad \text{on } \partial\Omega,$$

where $F := \nabla \times (\mu_r^{-1} \nabla \times T) + k^2 \epsilon_r T$. We can then solve

$$a(S, v) = \int_\Omega \langle F, v \rangle dx, \quad \forall v \in H_0(\text{curl}; \Omega),$$

which has a unique solution by the Lax-Milgram Theorem. □

Remark: Due to the symmetry of Maxwell's equations, we see that if E and H are solutions of (2.5.26), (2.5.27) in $H(\text{curl}; \Omega)$, that these functions lie even in the space

$$H(\text{curl}^2; \Omega) := \{u \in H(\text{curl}; \Omega) | \ \nabla \times u \in H(\text{curl}; \Omega)\}. \tag{2.5.30}$$

In contrast to (2.5.28) the boundary value problem associated with equation (2.5.20) is not coercive. Thus one needs to distinguish between the cases when ω is a so-called magnetic resonance and when not.

Definition 2.23 (Magnetic resonance).

Let $E \in H_0(\text{curl}; \Omega)$ and let $\Im(\epsilon_r) = 0$. Every κ for which

$$\langle \mu_r^{-1} \nabla \times E, \nabla \times \phi \rangle_{L^2} - \kappa^2 \langle \epsilon_r E, \phi \rangle_{L^2} = 0, \quad \forall \phi \in H_0(\text{curl}; \Omega) \tag{2.5.31}$$

does not have a unique solution is called a magnetic resonance or a cavity eigenvalue.

This definition characterizes exactly those values for which the boundary value problems associated with (2.5.20) do not have a unique solution.

Theorem 2.24.

The solutions of equation (2.5.32) have the following properties:

(i) *Corresponding to the eigenvalue $\kappa = 0$ there is an infinite family of eigenfunctions.*

(ii) *There is an infinite discrete set of magnetic resonances $\kappa_j > 0$, $j = 1, 2, \ldots$ and corresponding eigenfunctions E_j such that $0 < \kappa_1 \leq \kappa_2 \leq \cdots$, as well as $\lim_{j\to\infty} \kappa_j = \infty$ and E_j is orthogonal to E_l with respect to the L^2 inner product if $j \neq l$.*

Proof.

See [Mon03][Theorem 4.18]. □

Now we can state the conditions for unique solvability of equation (2.5.32), if κ is not a resonance.

Theorem 2.25.

Suppose $\Im(\epsilon_r) = 0$. Then if κ is not a magnetic resonance, then

$$\langle \mu_r^{-1} \nabla \times E, \nabla \times \phi \rangle_{L^2} - \kappa^2 \langle \epsilon_r E, \phi \rangle_{L^2} = \langle F, \phi \rangle_{L^2}, \quad \forall \phi \in H_0(\text{curl}; \Omega) \tag{2.5.32}$$

has a unique solution for every right-hand side $F \in L^2(\Omega)^3$.

Proof.

See [Mon03][Corollary 4.19]. □

So far we have only considered solutions of Maxwell's equations in the space $H(\text{curl}; \Omega)$. However sometimes we need solutions with more regularity. In particular, we want to consider solutions whose boundary values are in $H^{1/2}(\text{div}; \partial\Omega)$. For this purpose we need the following definition.

Definition 2.26.

We define the space of H^1 functions with regular divergence of the tangential component as

$$H^1_{\nabla_{\partial\Omega}\cdot}(\Omega) := \{u \in H^1(\Omega)^3 | \ \nabla_{\partial\Omega} \cdot (\boldsymbol{n} \times u|_{\partial\Omega}) \in H^{1/2}(\partial\Omega)\}.$$

The following result can be found in [Cos91] and is used in the analysis of inverse problems in [PLE92] and [PE96].

Theorem 2.27.

Let $\boldsymbol{n} \times E|_{\partial\Omega} \in H^{1/2}(\mathrm{div}; \partial\Omega)$ and let $\partial\Omega$ be a $C^{1,1}$-boundary. Then the solutions of equations (2.5.20) and (2.5.28) are elements of $H^1_{\nabla_{\partial\Omega}\cdot}(\Omega)$.

This yields the following corollary.

Corollary 2.28.

Let $\partial\Omega$ be a $C^{1,1}$-boundary. Then the impedance map $Z_{\mu_r,\epsilon_r} : H^{1/2}(\mathrm{div}; \partial\Omega) \mapsto H^{1/2}(\mathrm{div}; \partial\Omega)$

$$Z_{\mu_r,\epsilon_r}(\boldsymbol{n} \times E|_{\partial\Omega}) = \boldsymbol{n} \times H|_{\partial\Omega}$$

for the system (2.5.26), (2.5.27) is an isomorphism.

Proof.

The linearity of the map is obvious. The injectivity follows from the unique solvability of Maxwell's equaitons with electric boundary conditions in $H^{1/2}(\mathrm{div}; \partial\Omega)$ and the surjectivity from the unique solvability for every magnetic boundary condition in $H^{1/2}(\mathrm{div}; \partial\Omega)$. □

The regularity in Theorem 2.27 is not confined to solutions of Maxwell's equations, as can be seen from the following more general result.

Theorem 2.29.

Let $\partial\Omega$ be a $C^{1,1}$-boundary and let $u \in L^2(\Omega)^3$ satisfy

$$\nabla \times u \in L^2(\Omega)^3, \quad \nabla \cdot u \in L^2(\Omega), \quad \boldsymbol{n} \times u|_{\partial\Omega} \in H_t^{1/2}(\partial\Omega).$$

Then we have $u \in H^1(\Omega)^3$. The same result holds if we replace the condition $\boldsymbol{n} \times u|_{\partial\Omega} \in H_t^{1/2}(\partial\Omega)$ with $\boldsymbol{n} \cdot n|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$.

Proof.

See [Cos90] or [GR86]. □

Besides regularity results for boundary value problems, we also need a regularity result on transmission problems.

Theorem 2.30.

Suppose Ω_1 and Ω_2 are two non-overlapping Lipschitz domains meeting at a common surface Σ (with non-zero measure) such that $\overline{\Omega_1} \cap \overline{\Omega_2} = \Sigma$. Suppose that $u_1 \in H(\mathrm{curl};\Omega_1)$ and $u_2 \in H(\mathrm{curl};\Omega_2)$ and define $u \in L^2(\Omega_1 \cap \Omega_2 \cap \Sigma)^3$ by

$$u = \begin{cases} u_1, & \text{on } \Omega_1, \\ u_2, & \text{on } \Omega_2. \end{cases} \tag{2.5.33}$$

Then if $\boldsymbol{n} \times u_1|_\Sigma = \boldsymbol{n} \times u_2|_\Sigma$, we have $u \in H(\mathrm{curl};\Omega_1 \cap \Omega_2 \cap \Sigma)$.

Proof.

See [Mon03][Lemma 5.3]. □

Corollary 2.31.

Let the assumptions of Theorem 2.30 hold. Suppose further that for $\mu_r, \epsilon_r \in C^3(\Omega_1 \cup \Omega_2 \cup \Sigma)$, u_1 and u_2 satisfy the equation

$$\nabla \times (\mu_r^{-1} \nabla \times u_i) + k^2 \epsilon_r u_i = 0, \quad in\ \Omega_i,\ i = 1, 2.$$

Then if $\boldsymbol{n} \times (\nabla \times u_1)|_\Sigma = \boldsymbol{n} \times (\nabla \times u_2)|_\Sigma$, we have $u \in H(\mathrm{curl}; \Omega_1 \cup \Omega_2 \cup \Sigma)$ for

$$u = \begin{cases} u_1, & on\ \Omega_1, \\ u_2, & on\ \Omega_2. \end{cases} \tag{2.5.34}$$

Proof.

We apply Theorem 2.30 to $v_i := \nabla \times u_i$ which are elements of $H(\mathrm{curl}; \Omega_i)$ (see (2.5.30)) and set

$$v = \begin{cases} v_1, & \text{on } \Omega_1, \\ v_2, & \text{on } \Omega_2. \end{cases}$$

Thus we get $v \in H(\mathrm{curl}; \Omega_1 \cup \Omega_2 \cup \Sigma)$. Since

$$\nabla \times \nabla \times u_i = \mu_r (\nabla \mu_r^{-1} \times \nabla \times u_i + k^2 \epsilon_r u_i), \quad i = 1, 2$$

and $\nabla \times v = \nabla \times \nabla \times u$ we see that $u \in H(\mathrm{curl}; \Omega_1 \cup \Omega_2 \cup \Sigma)$. □

Besides these regularity results we also need a unique continuation result for Maxwell's equations.

Theorem 2.32.

Suppose Ω_0 is an open, connected subdomain of Ω. Suppose that u is a solution of

$$\nabla \times (\mu_r^{-1} \nabla \times u) + k^2 \epsilon_r u = 0, \quad \text{in } \Omega_0,$$

with ϵ_r and μ_r differentiable functions. If u vanishes on a ball of non-zero radius in Ω_0, then u vanishes identically in Ω_0.

Proof.

The proof is analogous to the proof of [Mon03][Theorem 4.13]. □

We also need the uniqueness of the Cauchy problem for Maxwell's equations.

Theorem 2.33.

Let Ω be a bounded domain in $\mathbb{R}^3$ with $C^{1,1}$-boundary and let u be a solution of

$$\nabla \times (\mu_r^{-1} \nabla \times u) + k^2 \epsilon_r u = 0, \ \text{in } \Omega,$$

$$\boldsymbol{n} \times u = \gamma_T(\nabla \times u) = 0, \ \text{on } \Gamma,$$

where $\Gamma \subset \partial\Omega$ is an open subset of $\partial\Omega$. Then $u = 0$ in Ω.

Proof.

This follows from a generalization of Holmgren's theorem and is an application of Theorem 2.32. □

2.6 Green's functions

Finally we devote a small section to the definition and properties of Green's functions associated with second-order differential operators. For proofs of the results stated below and a further insight into Green's function we refer the interested reader to any comprehensive reference on partial differential equations (for example [Wlo87]). As before we assume that $\Omega \subset \mathbb{R}^n$ is open, connected and has a $C^{1,1}$-boundary. Given a symmetric second order scalar differential operator

$$Lu(x) = -\nabla \cdot (A(x)\nabla u(x)) + c(x)u(x),$$

where $c(x)$ is a scalar function, $A(x)$ is a symmetric matrix for every $x \in \Omega$, we want to define the Green's function for the boundary value problem

$$Lu(x) = f(x), \quad x \in \Omega \tag{2.6.1}$$

$$Bu(x) = 0, \quad x \in \partial\Omega. \tag{2.6.2}$$

Here B describes a well-posed boundary condition. The corresponding Green's function is then defined as

$$-\nabla_y \cdot (A(y)\nabla_y G(x,y)) + c(y)G(x,y) = \delta_x(y), \quad x, y \in \Omega, \tag{2.6.3}$$

$$B_y G(x,y) = 0, \quad y \in \partial\Omega,\ x \in \Omega. \tag{2.6.4}$$

The function G then has the following properties.

- $G(x,y) = G(y,x)$.

- For $f \in L^2(\Omega)$, the function

$$u(x) = \int_\Omega G(x,y)f(y)dy$$

is a solution of the problem (2.6.1), (2.6.2).

In the case of a second order differential system we replace the operator L with

$$(Pu(x))_i = -\sum_{j=1}^{n} \nabla \cdot (C_{i,j}(x)\nabla u_j(x)) + \sum_{j=1}^{n} Q_{i,j}(x)u_j(x), \quad i = 1, \dots, n,$$

where $C(x)$ is now a symmetric fourth-order tensor and $Q(x)$ is a symmetric matrix for every $x \in \Omega$. Again we want to define the Green's function (or in this case the Green's matrix) for the boundary value problem

$$Pu(x) = f(x), \quad x \in \Omega, \tag{2.6.5}$$

$$Bu(x) = 0, \quad x \in \partial\Omega, \tag{2.6.6}$$

where again B describes a well-posed boundary condition. The corresponding Green's matrix is then defined as

$$-\sum_{j=1}^{n} \nabla_y \cdot (C_{i,j}(y)\nabla_y(G^j(x,y)) + \sum_{j=1}^{n} Q_{i,j}(y)G^j(x,y) = \delta_x(y)I^i, \quad x, y \in \Omega, \tag{2.6.7}$$

$$B_y G(x,y) = 0, \quad y \in \partial\Omega, \ x \in \Omega. \tag{2.6.8}$$

where G^j is the j-th column of $G(x,y)$. The function G then has the following properties.

- $G(x,y) = G(y,x)^T$.

- For $f \in L^2(\Omega)$, the function

$$u(x) = \int_\Omega G(x,y)^T f(y)dy$$

is a solution of the problem (2.6.5), (2.6.6).

Finally we want to consider the special case of Maxwell's equations and the boundary value problem

$$\nabla \times (\mu_r^{-1}(x)\nabla \times E(x)) + k^2\epsilon_r(x)E(x) = F, \quad x \in \Omega, \tag{2.6.9}$$

$$n(x) \times E(x) = 0, \quad x \in \partial\Omega. \tag{2.6.10}$$

The corresponding Green's matrix is then defined as the solution of

$$\nabla \times_y (\mu_r^{-1}(y)\nabla \times_y G(x,y)) + k^2\epsilon_r(y)G(x,y) = d_x(y)I, \quad x,y \in \Omega, \tag{2.6.11}$$

$$n(y) \times G(x,y) = 0, \quad y \in \partial\Omega, \ x \in \Omega. \tag{2.6.12}$$

This ends our overview of the mathematical preliminaries we need for our analysis of the inverse problem for Maxwell's equations. In the next chapter we present several uniqueness results for different kinds of given data.

3 Uniqueness results

In this chapter we present uniqueness results for the inverse problems associated with equations (2.5.20) and (2.5.28). However before we do this we give an outline of these inverse problems. Throughout this chapter we make the following assumptions.

- Ω is a bounded domain in $\mathbb{R}^3$ with a $C^{1,1}$-boundary $\partial\Omega$.
- $\epsilon|_{\partial\Omega} = \epsilon_0$, $\mu|_{\partial\Omega} = \mu_0$, $\sigma|_{\partial\Omega} = 0$, $\epsilon_0, \mu_0 \in \mathbb{R}_+$.
- $0 < \epsilon_m \leq \epsilon(x) \leq \epsilon_M$, $0 \leq \sigma(x) \leq \sigma_M$, $0 < \mu_m \leq \mu(x) \leq \mu_M$.
- ϵ, μ, $\sigma \in C^\infty(\overline{\Omega})$.

We make the last assumption out of convenience. For most results $\epsilon, \mu, \sigma \in C^3(\overline{\Omega})$ or even less smoothness would be sufficient.

3.1 The inverse problem

In this section we discuss the inverse problems for Maxwell's equations outlined in Chapter 1 in more detail. Except for the case of given interior data, we will investigate inverse problems for the equations (2.5.20) and (2.5.28), for the case of a vanishing right-hand side $F = 0$. Thus we consider either the system

$$\nabla \times E - ik\mu_r H = 0, \tag{3.1.1}$$

$$\nabla \times H + ik\epsilon_r E = 0, \tag{3.1.2}$$

with $\epsilon_r = \frac{1}{\epsilon_0}\left(\epsilon + \frac{i\sigma}{\omega}\right)$, $\mu_r = \frac{\mu}{\mu_0}$ or

$$\nabla \times E + k\mu_r H = 0, \tag{3.1.3}$$

$$\nabla \times H - k\epsilon_r E = 0, \tag{3.1.4}$$

with $\epsilon_r = \frac{1}{\epsilon_0}\left(\epsilon + \frac{\sigma}{\lambda}\right)$, $\mu_r = \frac{\mu}{\mu_0}$. The reason for this setting is, that in practical applications J_a and thus F is often vanishing or can be controlled. From a mathematical point of view this is convenient, since for a homogeneous right hand side the given data

$$Z_{\mu_r,\epsilon_r}(\boldsymbol{n} \times E|_{\partial\Omega}) = \boldsymbol{n} \times H|_{\partial\Omega}$$

for the inverse problem, can be characterized by a linear map. Sometimes it is more convenient to work with a single second order equation, so we often use the equations

$$\nabla \times (\mu_r^{-1}\nabla \times E) - k^2\epsilon_r E = 0, \quad \text{in } \Omega \tag{3.1.5}$$

or

$$\nabla \times (\mu_r^{-1}\nabla \times E) + k^2\epsilon_r E = 0, \quad \text{in } \Omega, \tag{3.1.6}$$

instead of the first order Maxwell systems. The standard inverse problems in the literature (see [Mon03][Chapter 14], [PLE92], [PE96]) are associated with the equation (3.1.5). One of these problems is the inverse scattering problem which we have shortly discussed in Section 1.3.1. If the given data consists of near field data, then one applies either an electric field or a magnetic field on the boundary of the medium and measures the other. Thus the given data can be characterized either by the so-called impedance map

$$Z_{\mu_r,\epsilon_r}(\boldsymbol{n} \times H|_{\partial\Omega}) = \boldsymbol{n} \times E|_{\partial\Omega} \tag{3.1.7}$$

or the so-called admittance map

$$\Lambda_{\mu_r,\epsilon_r}(\boldsymbol{n} \times E|_{\partial\Omega}) = \boldsymbol{n} \times H|_{\partial\Omega}. \tag{3.1.8}$$

Note that in the analogous case of elliptic equations, where the given data consists either of the Dirichlet Neumann map or the Neumann Dirichlet map, these maps in general have different smoothness properties. In the case of Maxwell's equations there is no difference in smoothness between Z_{μ_r,ϵ_r} and $\Lambda_{\mu_r,\epsilon_r}$.

Lemma 3.1.

Let k not be a magnetic resonance. Then the impedance map associated with equation (2.5.20) $Z_{\mu_r,\epsilon_r} : H^{1/2}(\text{div}; \partial\Omega) \mapsto H^{1/2}(\text{div}; \partial\Omega)$

$$Z_{\mu_r,\epsilon_r}(\boldsymbol{n} \times H|_{\partial\Omega}) = \boldsymbol{n} \times E|_{\partial\Omega}$$

is an isomorphism. Its inverse is the admittance map

$$\Lambda_{\mu_r,\epsilon_r}(\boldsymbol{n} \times E|_{\partial\Omega}) = \boldsymbol{n} \times H|_{\partial\Omega}.$$

Proof.

See for example [PLE92]. □

Now we formulate the first inverse problem that we consider in this work.

Problem 3.2.

Suppose k is not a magnetic resonance. Given the impedance map Z_{μ_r,ϵ_r} associated with equation (2.5.20), recover the parameters ϵ_r and μ_r in

$$\nabla \times (\mu_r^{-1} \nabla \times E) - k^2 \epsilon_r E = 0.$$

This problem has a unique solution.

Theorem 3.3.

Suppose k is not a magnetic resonance. Then the impedance map Z_{μ_r,ϵ_r} or $\Lambda_{\mu_r,\epsilon_r}$ uniquely determines μ_r and ϵ_r.

Proof.

See for example [PLE92] or [PE96]. □

Although this result is encouraging, it is not a sufficient basis for a solution to the inverse problem we consider in this work. First of all it requires global knowledge of the boundary data, which in general is not available. Another problem is the one we have mentioned in Chapter 2, that equation (3.1.5) does not induce a coercive bilinear form in $H(\text{curl}; \Omega)$ and therefore is not really suited for a variational approach to the inverse problem. Thus in the next section we consider the following problem.

Problem 3.4.

Given the impedance map Z_{μ_r,ϵ_r} associated with equation (2.5.28), recover the parameters ϵ_r and μ_r in

$$\nabla \times (\mu_r^{-1} \nabla \times E) + k^2 \epsilon_r E = 0.$$

Although the idea of the uniqueness proof for Problem 3.4 is the same as for Problem 3.2, there are a few differences and therefore we present it here for the sake of completeness.

3.2 Global boundary data

We consider the Maxwell system

$$\nabla \times E + \lambda \mu H = M \tag{3.2.1}$$

$$\nabla \times H - \lambda \epsilon_q E = J, \tag{3.2.2}$$

where $\lambda \in \mathbb{R}_+$, and here $\epsilon_q = \epsilon + \frac{\sigma}{\lambda}$. Further we demand $M, J \in C^\infty(\overline{\Omega})$ and that $\epsilon_q|_{\mathbb{R}^3 \setminus \overline{\Omega}} = \epsilon_0$ and $\mu|_{\mathbb{R}^3 \setminus \overline{\Omega}} = \mu_0$. We have made a slight change in notation for this subsection. This is mainly due to the fact, that we want to consider a non-normalized first order system, instead of the normalized second order system of the previous subsection. We do this to comply with the notation in [PE96]. We show the following result.

Theorem 3.5.

The impedance map

$$Z_{\mu,\epsilon_q}(\boldsymbol{n} \times H|_{\partial\Omega}) = \boldsymbol{n} \times E|_{\partial\Omega}$$

associated with the system (3.2.1), (3.2.2) uniquely determines the coefficients ϵ_q *and* μ.

Proof.

For the proof we define the scalar fields

$$\Phi = \frac{1}{\lambda}\nabla \cdot (\epsilon_q E + \frac{1}{\lambda}J), \tag{3.2.3}$$

$$\Psi = \frac{1}{\lambda}\nabla \cdot (\mu H - \frac{1}{\lambda}M), \tag{3.2.4}$$

and work with the modified Maxwell equations

$$\nabla \times E - \frac{1}{\epsilon_q}\nabla\frac{1}{\mu}\Psi + \lambda\mu H = M, \tag{3.2.5}$$

$$\nabla \times H + \frac{1}{\mu}\nabla\frac{1}{\epsilon_q}\Phi - \lambda\epsilon_q E = J. \tag{3.2.6}$$

Later we will see that for large k that we the solutions of the modified Maxwell's equations are solutions of the original Maxwell's equations. This will be an essential feature of this proof. In what follows we need the rescaled fields

$$e = \epsilon_q^{1/2}E, \qquad h = \mu^{1/2}H, \tag{3.2.7}$$

$$\phi = \frac{1}{\epsilon_q\mu^{1/2}}\Phi, \qquad \psi = \frac{1}{\epsilon_q^{1/2}\mu}\Psi, \tag{3.2.8}$$

and we set $X = (\phi, e, h, \psi)$. We further define the 8×8 operator

$$P(\nabla) = \begin{pmatrix} 0 & \nabla\cdot & 0 & 0 \\ \nabla & 0 & \nabla\times & 0 \\ 0 & -\nabla\times & 0 & \nabla \\ 0 & 0 & \nabla\cdot & 0 \end{pmatrix}$$

and the matrix

$$V = (k-\kappa)1_8 + \frac{1}{2}\begin{pmatrix} 0 & \nabla \ln \epsilon_q\cdot & 0 & 0 \\ \nabla \ln \mu & 0 & -\nabla \ln \mu\times & 0 \\ 0 & \nabla \ln \epsilon_q\times & 0 & \nabla \ln \epsilon_q \\ 0 & 0 & \nabla \ln \mu\cdot & 0 \end{pmatrix},$$

where $k = \lambda(\epsilon_0\mu_0)^{1/2}$ and $\kappa = \lambda(\epsilon_q\mu)^{1/2}$. With this notation we get the following lemma.

Lemma 3.6. *Let X be a solution of the modified Maxwell's equations. Then*

$$(P(\nabla) - k + V)X = F, \tag{3.2.9}$$

where

$$F = \left(-\frac{1}{\lambda\epsilon_q^{1/2}}\nabla\cdot J, \mu^{1/2}J, -\epsilon_q^{1/2}M, \frac{1}{\lambda\mu^{1/2}}\nabla\cdot M\right).$$

Although the proof is standard we give it here for the convenience of the reader.

Proof.

We show the above relation for the first two components. The other components follow then from the symmetry of the Maxwell system. For the first component we get

$$\begin{aligned}
&\nabla \cdot e - \kappa\phi + \frac{1}{2}\nabla \ln \epsilon_q \cdot e \\
&= \nabla \cdot (\epsilon_q^{1/2} E) - \lambda \epsilon_q^{-1/2} \Phi + \frac{\nabla \epsilon_q}{2\epsilon_q}(\epsilon_q^{1/2} E) \\
&= \nabla(\epsilon_q^{1/2} \cdot E + \epsilon_q^{1/2} \nabla \cdot E - \epsilon_q^{-1/2} \nabla \cdot (\epsilon_q E + \frac{1}{\lambda} J) + \frac{\nabla \epsilon_q}{2\epsilon_q^{1/2}} E.
\end{aligned}$$

The result now follows from the fact that

$$\nabla \cdot (\epsilon_q E) = \nabla \cdot (\epsilon_q^{1/2} \epsilon_q^{1/2} E) = \nabla \epsilon_q^{1/2} \cdot (\epsilon_q^{1/2} E) + \epsilon_q \nabla \cdot E + \epsilon_q^{1/2} \nabla \epsilon_q^{1/2} \cdot E.$$

The formula for the second component is easily seen from the following simple calculation. Equation (3.2.6) is given by

$$\nabla \times H + \frac{1}{\mu} \nabla \frac{1}{\epsilon_q} \Phi - \lambda \epsilon_q E = J.$$

Substituting the rescaled fields gives

$$\nabla \times (\mu^{-1/2} h) + \frac{1}{\mu} \nabla \frac{1}{\epsilon_q} (\mu^{1/2} \phi) - \lambda \epsilon_q^{1/2} e = J,$$

which is the same as

$$\nabla \times h - \frac{1}{2} \nabla \ln \mu \times h + \nabla \phi + \frac{1}{2} \nabla \ln \mu \phi - \kappa e = \mu^{1/2} J. \tag{3.2.10}$$

□

A crucial point of this proof, is the following lemma.

Lemma 3.7.

The seemingly first-order operator

$$VP(\nabla) - P(\nabla)V^T$$

is in our case a zeroth-order operator.

Proof.

We show this for the first two components. The other components follow again from symmetry.

First we note that

$$P(\nabla)X = \begin{pmatrix} \nabla \cdot e \\ \nabla\phi + \nabla \times h \\ -\nabla \times e + \nabla\psi \\ \nabla \cdot \psi \end{pmatrix}$$

and that

$$V^T = (k-\kappa)1_8 + \frac{1}{2}\begin{pmatrix} 0 & \nabla \ln \mu \cdot & 0 & 0 \\ \nabla \ln \epsilon_q & 0 & -\nabla \ln \epsilon_q \times & 0 \\ 0 & \nabla \ln \mu \times & 0 & \nabla \ln \mu \\ 0 & 0 & \nabla \ln \epsilon_q \cdot & 0 \end{pmatrix}$$

and thus

$$V^T X = \begin{pmatrix} (k-\kappa)\phi + \frac{1}{2}\nabla \ln \mu \cdot e \\ (k-\kappa)e + \frac{1}{2}(\nabla \ln \epsilon_q \phi - \nabla \ln \epsilon_q \times h) \\ (k-\kappa)h + \frac{1}{2}(\nabla \ln \mu \times e + \nabla \ln \mu \psi) \\ (k-\kappa)\psi + \frac{1}{2}\nabla \ln \epsilon_q \cdot h \end{pmatrix}.$$

Therefore we get for the first component of $(VP(\nabla) - P(\nabla)V^T)X$

$$(k-\kappa)\nabla \cdot e + \frac{1}{2}\nabla \ln \epsilon_q \cdot \nabla\phi + \frac{1}{2}\nabla \ln \epsilon_q \cdot \nabla \times h - \nabla \cdot (k-\kappa)e - \frac{1}{2}\nabla \cdot (\nabla \ln \epsilon_q \phi - \nabla \ln \epsilon_q \nabla \times h)$$
$$= \nabla\kappa \cdot e - \frac{1}{2}\nabla \cdot \nabla \ln \epsilon_q \phi.$$

For the second component we get

$$\begin{aligned} &(k-\kappa)(\nabla\phi + \nabla \times h) + \frac{1}{2}\nabla \ln \mu \nabla \cdot e - \frac{1}{2}\nabla \ln \mu \times (-\nabla \times e + \nabla\psi) \\ &- \nabla((k-\kappa)\phi + \frac{1}{2}\nabla \ln \mu \cdot e) - \nabla \times ((k-\kappa)h + \frac{1}{2}(\nabla \ln \mu \times e + \nabla \ln \mu \psi)) \\ &= \nabla\kappa\phi + \nabla\kappa \times h + \frac{1}{2}\nabla \ln \mu \nabla \cdot e - \frac{1}{2}\nabla \ln \mu \times (-\nabla \times e + \nabla\psi) \\ &- \frac{1}{2}\nabla(\nabla \ln \mu \cdot e) - \frac{1}{2}\nabla \times (\nabla \ln \mu \times e + \nabla \ln \mu \psi). \end{aligned}$$

Notice that

$$\nabla(\nabla \ln \mu \cdot e) = -(\nabla \ln \mu \cdot \nabla)e - (e \cdot \nabla)\nabla \ln \mu - \nabla \ln \mu \times (\nabla \times e)$$

and

$$\nabla \times (\nabla \ln \mu \times e) = (e \cdot \nabla)\nabla \ln \mu - (\nabla \ln \mu \cdot \nabla)e + \nabla \ln \mu \nabla \cdot e + e\nabla \cdot \nabla \ln \mu$$

as well as

$$\nabla \times (\nabla \ln \mu\psi) = \nabla \ln \mu \times \nabla\psi$$

and the result easily follows. □

We conclude

$$-(P(\nabla) - k + V)(P(\nabla) + k - V^T) = -\Delta + k^2 + Q, \tag{3.2.11}$$

where

$$Q = -VP(\nabla) + P(\nabla)V^T - k(V + V^T) + VV^T. \tag{3.2.12}$$

Since we have already shown that Q is a zeroth-order operator, we can now define a so-called generalized Sommerfield potential Y (see [PE96]) by

$$X = -(P(\nabla) + k - V^T)Y. \tag{3.2.13}$$

It follows from (3.2.11) that Y satisfies

$$(-\Delta + k^2 + Q)Y = F. \tag{3.2.14}$$

In what follows we set $M = J = 0$ indicating $F = 0$. We construct exponentially growing solutions of (3.2.14) that will help us to show uniqueness for the inverse problem.

Let $\zeta \in \mathbb{C}^3$ with $\zeta \cdot \zeta = -k^2$. We set

$$G_\zeta(x) = e^{i\zeta\cdot x} g_\zeta(x), \tag{3.2.15}$$

where

$$g_\zeta(x) = \int_{\mathbb{R}^3} \frac{e^{i\xi\cdot x}}{|\xi|^2 + 2\xi\cdot\zeta} d\xi.$$

The fundamental solution G_ζ of the modified Helmholtz equation was introduced by L.D. Fadeev in 1960 [Fad66] and has some very useful properties. In what follows we need the weighted L^2 space

$$L^2_\delta = \{f \in L^2(\mathbb{R}^3) : \ \|f\|^2_\delta = \int_{\mathbb{R}^3} (1 + |x^2|)^\delta |f(x)|^2 dx < \infty\}. \tag{3.2.16}$$

The following result can be found in [Nac88].

Lemma 3.8.

Let $-1 < \delta < -\frac{1}{2}$. Then it holds for every $f \in L^2_\delta$ that

$$\|g_\zeta * f\|_\delta \leq \frac{C}{|\zeta|}\|f\|_\delta,$$

where C is independent of ζ.

We want to construct a solution Y_ζ of (3.2.14) with $F = 0$ of the form

$$Y_\zeta = Y_{0,\zeta} - G_\zeta * (QY_\zeta), \tag{3.2.17}$$

where $Y_{0,\zeta}$ is a solution of

$$(-\Delta + k^2)Y_{0,\zeta} = 0.$$

It follows from Lemma 3.8 that

$$\|Y_\zeta\|_\delta \leq \|Y_{0,\zeta}\|_\delta + \frac{C\|Q\|_\delta}{|\zeta|}\|Y_\zeta\|_\delta. \tag{3.2.18}$$

Thus (3.2.17) has a unique solution by the Neumann series if $|\zeta|$ is large enough. To be more specific on the form of $Y_{0,\zeta}$ we set

$$Y_{0,\zeta}(x) = e^{i\zeta\cdot x} y_{0,\zeta},$$

where $y_{0,\zeta}$ is bounded with respect to ζ, i.e.

$$\|y_{0,\zeta}\| \leq C\|\zeta\|$$

and constant in x. The choice of $Y_{0,\zeta}$ must be done in such a way that

$$X_\zeta = -(P(\nabla) + k - V^T)Y_\zeta$$

gives a solution of the original Maxwell's equations for large $|\zeta|$.

Lemma 3.9.

Assume that $y_{0,\zeta} \in \mathbb{C}^8$ is bounded with respect to ζ, and

$$((P(i\zeta) + k)y_{0,\zeta})_1 = ((P(i\zeta) + k)y_{0,\zeta})_8 = 0.$$

Then for large $|\zeta|$ we get

$$X_\zeta = (0, e_\zeta, h_\zeta, 0)^T.$$

Proof.

From equation (3.2.9) we know

$$\nabla\phi_\zeta + \nabla \times h_\zeta - \kappa e_\zeta + \frac{1}{2}\nabla \ln \mu\phi_\zeta - \frac{1}{2}\nabla \ln \mu \times h_\zeta = 0.$$

Applying a divergence on both sides we get

$$\Delta\phi_\zeta - \nabla \cdot (\kappa e_\zeta) + \frac{1}{2}\Delta \ln \mu\phi_\zeta + \frac{1}{2}\nabla \ln \mu \cdot \nabla\phi_\zeta + \frac{1}{2}\nabla \ln \mu \cdot \nabla \times h_\zeta. \qquad (3.2.19)$$

From (3.2.9) follows

$$\begin{aligned}\nabla\cdot(\kappa e_\zeta) &= \kappa\nabla\cdot e_\zeta + \nabla\kappa\cdot\epsilon_\zeta = \kappa^2\phi_\zeta - \kappa\frac{1}{2}\nabla\ln\epsilon_q\cdot e_\zeta + \nabla\kappa\cdot e_\zeta\\ &= \kappa^2\phi_\zeta - \frac{1}{2}\nabla\ln\epsilon_q\cdot(\kappa e_\zeta) + \frac{1}{2}\nabla\ln\epsilon_q\cdot(\kappa e_\zeta) + \frac{1}{2}\nabla\ln\mu\cdot(\kappa e_\zeta)\\ &= \kappa^2\phi_\zeta + \frac{1}{2}\nabla\ln\mu\cdot(\kappa e_\zeta).\end{aligned} \tag{3.2.20}$$

Now we use (3.2.10) to conclude

$$\frac{1}{2}\nabla\ln\mu\cdot\nabla\times h_\zeta = -\frac{1}{2}\nabla\ln\mu\cdot\nabla\phi_\zeta - \frac{1}{4}|\nabla\ln\mu|^2\phi_\zeta + \frac{1}{2}\nabla\ln\mu\cdot(\kappa e_\zeta).$$

Combining (3.2.19) and (3.2.20) yields

$$\Delta\phi_\zeta - \kappa^2\phi_\zeta + \frac{1}{2}\Delta\ln\mu\phi_\zeta - \frac{1}{4}|\nabla\ln\mu|^2\phi_\zeta = 0$$

or

$$-\Delta\phi_\zeta + k^2\phi_\zeta + q\phi_\zeta = 0, \tag{3.2.21}$$

where $q = (\kappa^2 - k^2) - \frac{1}{2}\Delta\ln\mu + \frac{1}{4}|\nabla\ln\mu|^2$. A similar equation can be derived for ψ_ζ. From the definition of Y_ζ we have

$$\begin{aligned}X_\zeta &= -(P(\nabla) + k - V^T)Y_\zeta\\ &= -e^{i\zeta\cdot x}((P(i\zeta)+k)y_{0,\zeta} - ((P(\nabla+i\zeta)+k)g_\zeta * (Qy_\zeta) - V^Ty_\zeta)).\end{aligned}$$

Now we set

$$x_{0,\zeta} = -(P(i\zeta)+k)y_{0,\zeta}, \quad x_{s,\zeta} = ((P(\nabla+i\zeta)+k)g_\zeta * (Qy_\zeta) - V^Ty_\zeta),$$

which yields

$$X_\zeta = -(P(\nabla) + k - V^T)Y_\zeta = e^{i\zeta\cdot x}(x_{0,\zeta} - x_{s,\zeta}).$$

Defining $w_{0,\zeta} = (x_{0,\zeta})_1$ and $w_{s,\zeta} = (x_{s,\zeta})_1$ we get

$$\phi_\zeta = e^{i\zeta\cdot x}(w_{0,\zeta} - w_{s,\zeta}).$$

From the definition of $w_\zeta = w_{0,\zeta} - w_{s,\zeta}$ and (3.2.21) we can easily see that w satisfies

$$-\Delta w_\zeta - 2i\zeta\cdot\nabla w_\zeta + qw_\zeta = 0.$$

(Let the reader be reminded that $\zeta\cdot\zeta = -k^2$). Thus we get the integral equation

$$w_\zeta = w_{0,\zeta} - g_\zeta * (qw_\zeta).$$

Since by our assumption $w_{0,\zeta} = -((P(i\zeta) + k)y_{0,\zeta})_1 = 0$ we get $w_\zeta = 0$ for $|\zeta|$ large enough. In the same way we can show that $\psi_\zeta = 0$ for $|\zeta|$ large enough. □

In light of the above lemma we take two constant vectors a and b and set

$$y_{0,\zeta} = \frac{1}{|\zeta|}(-i\zeta\cdot a, ka, kb, -i\zeta\cdot b)^T.$$

$y_{0,\zeta}$ obviously satisfies the conditions of Lemma 3.9. We further need an auxiliary complex vector ζ^* such that

$$\zeta^*\cdot\zeta = -k^2, \quad \zeta^* + \zeta = \xi$$

where $\xi \in \mathbb{R}^3$ is an arbitrary fixed vector.

Now we define

$$Y_{0,\zeta}^*(x) = e^{i\zeta^*\cdot x} y_{0,\zeta}^*,$$

where

$$y_{0,\zeta}^* = \frac{1}{|\zeta|}(P(i\zeta^*) - k)z, \quad z \in \mathbb{C}^8.$$

From this definition it follows that

$$(P(\nabla) + k)Y_{0,\zeta}^*(x) = 0.$$

The crucial point in the final stage of this proof is the following calculation. From Lemma 3.8 we derive

$$\begin{aligned}&\int_\Omega Y_{0,\zeta}^*(x) \cdot Q(x)Y_\zeta(x)dx \\ &= \int_\Omega e^{i\xi\cdot x} y_{0,\zeta}^* \cdot Q(x)y_{0,\zeta}dx + \mathcal{O}(\frac{1}{|\zeta|}),\end{aligned} \tag{3.2.22}$$

as well as

$$\begin{aligned}&\int_\Omega Y_{0,\zeta}^*(x) \cdot Q(x)Y_\zeta(x)dx \\ &= \int_\Omega Y_{0,\zeta}^*(x) \cdot (\Delta - k^2)Y_\zeta(x))dx \\ &= \int_\Omega Y_{0,\zeta}^*(x) \cdot (P(\nabla) - k)(P(\nabla) + k)Y_\zeta(x))dx.\end{aligned}$$

We set $\tilde{X}_\zeta = -(P(\nabla) + k)Y_\zeta = X_\zeta - V^T Y_\zeta$ integrate by parts and get

$$\begin{aligned}&- \int_\Omega Y_{0,\zeta}^*(x) \cdot (P(\nabla) - k)\tilde{X}_\zeta(x)dx \\ &= \int_\Omega (P(\nabla) + k)Y_{0,\zeta}^*(x) \cdot \tilde{X}_\zeta dx - \int_{\partial\Omega} Y_{0,\zeta}^*(x) \cdot P(\boldsymbol{n})\tilde{X}_\zeta(x)dS(x)\end{aligned}$$

$$= - \int_{\partial\Omega} Y^*_{0,\zeta}(x) \cdot P(\boldsymbol{n}) X_\zeta(x) dS(x), \tag{3.2.23}$$

since at the boundary we have $\tilde{X}_\zeta = X_\zeta$. Combining (3.2.22) and (3.2.23) and taking the limit $|\zeta| \to \infty$ gives

$$\hat{D}(\xi) := \lim_{|\zeta|\to\infty} \int_{\partial\Omega} Y^*_{0,\zeta}(x) \cdot P(\boldsymbol{n}) X_\zeta(x) dS(x) = - \int_\Omega e^{i\xi\cdot x} y^*_\infty \cdot Q(x) y_\infty dx, \tag{3.2.24}$$

where

$$y_\infty = \lim_{|\zeta|\to\infty} y_{0,\zeta}, \quad y^*_\infty = \lim_{|\zeta|\to\infty} y^*_{0,\zeta}.$$

The next step to show that the impedance map Z_{μ,ϵ_q} uniquely identifies μ and ϵ_q is to show that Z_{μ,ϵ_q} determines the boundary values of X_ζ and thus $\hat{D}(\xi)$ and Q. For this we note that for large $|\zeta|$ we get

$$P(\boldsymbol{n}) = (\boldsymbol{n} \cdot e_\zeta, \boldsymbol{n} \times h_\zeta, -\boldsymbol{n} \times e_\zeta, \boldsymbol{n} \cdot h_\zeta)^T.$$

To reconstruct the field components from Z_{μ,ϵ_q} we derive a boundary integral equation involving the operators

$$K_\zeta(\varphi)(x) = (vp)\ \boldsymbol{n} \times \int_{\partial\Omega} \nabla_x G_\zeta(x-y) \times \varphi(y) dS(y), \quad x \in \partial\Omega$$

and

$$D_\zeta(\varphi)(x) = (vp)\ \boldsymbol{n} \times \int_{\partial\Omega} \nabla_x G_\zeta(x-y) \nabla_{\partial\Omega} \cdot \varphi(y) - k^2 G_\zeta(x-y)\varphi(y) dS(y), \quad x \in \partial\Omega,$$

where $\nabla_{\partial\Omega}\cdot$ denotes the surface divergence. It is straightforward to extend these operators to continuous operators

$$K_\zeta,\ D_\zeta : H(\mathrm{div}; \partial\Omega)^{1/2} \mapsto H(\mathrm{div}; \partial\Omega)^{1/2}$$

(see [PLE92][Chapter 3]).

Theorem 3.10.

Let $Z_{\mu,\epsilon_q} : H(\mathrm{div}; \partial\Omega)^{1/2} \mapsto H(\mathrm{div}; \partial\Omega)^{1/2}$ *with* $Z_{\mu,\epsilon_q}(\boldsymbol{n} \times H) = \boldsymbol{n} \times E$ *is known. Then* $\boldsymbol{n} \times h_\zeta$ *satisfies the equation*

$$\frac{1}{2}\boldsymbol{n} \times h_\zeta = \boldsymbol{n} \times h_{0,\zeta} + \left(\frac{1}{\lambda\mu_0} D_\zeta Z_{\mu,\epsilon_q} - K_\zeta\right)(\boldsymbol{n} \times h_\zeta) \tag{3.2.25}$$

Proof.

As before let us denote $\tilde{X}_\zeta = -(P(\nabla) + k)Y_\zeta = X_\zeta - V^T Y_\zeta$. Integrating by parts we obtain

$$\begin{aligned} Y_{s,\zeta} &= \int_\Omega G_\zeta(x-y)Q(y)Y_\zeta dy \\ &= \int_\Omega G_\zeta(x-y)(\Delta - k^2)Y_\zeta dy \\ &= -\int_\Omega G_\zeta(x-y)(P(\nabla) - k)\tilde{X}_\zeta(y)dy \\ &= -(P(\nabla)_x - k)\int_\Omega G_\zeta(x-y)\tilde{X}_\zeta(y)dy - \int_{\partial\Omega} G_\zeta(x-y)P(n)\tilde{X}_\zeta(y)dS(y). \end{aligned} \tag{3.2.26}$$

Since on the boundary we have $X_\zeta = \tilde{X}_\zeta$ and

$$X_\zeta = -(P(\nabla) + k)Y_{0,\zeta} - (P(\nabla) + k)Y_{s,\zeta},$$

we get by applying $P(\nabla) + k$ to (3.2.26) for all $x \in \mathbb{R}^3 \backslash \overline{\Omega}$ (note that $(-\Delta + k^2)G_\zeta = 0$ for $x \neq y$)

$$X_\zeta = X_{0,\zeta} + (P(\nabla) + k)\int_{\partial\Omega} G_\zeta(x-y)P(n)\tilde{X}_\zeta(y)dS(y). \tag{3.2.27}$$

To get the desired integral equation we use the vector double layer potential

$$S_{\mathbb{R}^3\backslash\overline{\Omega},\zeta}(\varphi)(x) = \int_{\partial\Omega} G_\zeta(x-y)\varphi(y)dS(y).$$

Since for large $|\zeta|$ we have $X_\zeta = (0, e_\zeta, h_\zeta, 0)^T$ equation (3.2.27) gives

$$\begin{aligned}
\nabla \cdot S_{\mathbb{R}^3\backslash\overline{\Omega},\zeta}(\boldsymbol{n}\times h_\zeta) + kS_{\mathbb{R}^3\backslash\overline{\Omega},\zeta}(\boldsymbol{n}\cdot e_\zeta) &= 0,\\
e_\zeta = e_{0,\zeta} + \nabla S_{\mathbb{R}^3\backslash\overline{\Omega},\zeta}(\boldsymbol{n}\cdot e_\zeta) - \nabla\times S_{\mathbb{R}^3\backslash\overline{\Omega},\zeta}(\boldsymbol{n}\times e_\zeta) &+ k\boldsymbol{n}\times h_\zeta,\\
\nabla \cdot S_{\mathbb{R}^3\backslash\overline{\Omega},\zeta}(\boldsymbol{n}\times e_\zeta) - kS_{\mathbb{R}^3\backslash\overline{\Omega},\zeta}(\boldsymbol{n}\cdot h_\zeta) &= 0,\\
h_\zeta = h_{0,\zeta} + \nabla S_{\mathbb{R}^3\backslash\overline{\Omega},\zeta}(\boldsymbol{n}\cdot h_\zeta) - \nabla\times S_{\mathbb{R}^3\backslash\overline{\Omega},\zeta}(\boldsymbol{n}\times h_\zeta) &+ k\boldsymbol{n}\times e_\zeta.
\end{aligned} \tag{3.2.28}$$

Since e_ζ and h_ζ satisfy the scaled Maxwell's equations in $\mathbb{R}^3\backslash\overline{\Omega}$ we have

$$\boldsymbol{n}\cdot h_\zeta = -\frac{1}{k}\boldsymbol{n}\cdot\nabla\times e_\zeta = \frac{1}{k}\nabla_{\partial\Omega}\cdot(n\times e_\zeta).$$

Substituting this into (3.2.28) and using that on $\partial\Omega$ (see [CK98])

$$\boldsymbol{n}\times\nabla(S_{\mathbb{R}^3\backslash\overline{\Omega},\zeta}(\varphi)) = \boldsymbol{n}\times(\nabla S_{\mathbb{R}^3\backslash\overline{\Omega},\zeta}(\varphi)) - \frac{1}{2}\boldsymbol{n}\times\varphi,$$

we get

$$\frac{1}{2}\boldsymbol{n}\times h_\zeta = \boldsymbol{n}\times h_{0,\zeta} + \frac{1}{k}\boldsymbol{n}\times(\nabla S_{\mathbb{R}^3\backslash\overline{\Omega},\zeta}\nabla_{\partial\Omega}\cdot - k^2 S_{\mathbb{R}^3\backslash\overline{\Omega},\zeta})(\boldsymbol{n}\times e_\zeta) - \boldsymbol{n}\times(\nabla\times S_{\mathbb{R}^3\backslash\overline{\Omega},\zeta}(\boldsymbol{n}\times h_\zeta)). \tag{3.2.29}$$

The claim follows now from the definition of the impedance map Z_{μ,ϵ_q} and the definition of the operators D and K. □

Thus we get the term $\boldsymbol{n}\times h_\zeta$ by solving equation (3.2.25). By applying standard arguments one can check that the integral equation (3.2.25) has a unique solution (see [PLE92]). Having derived $\boldsymbol{n} \times h_\zeta$ we get the remaining components of $P(n)X_\zeta$ by

$$\boldsymbol{n} \times e_\zeta = \left(\frac{\epsilon_0}{\mu_0}\right)^{1/2} Z_{\mu,\epsilon_q}(\boldsymbol{n} \times h_\zeta),$$

$$\boldsymbol{n} \cdot e_\zeta = -\frac{1}{k}\nabla_{\partial\Omega} \cdot (\boldsymbol{n} \times h_\zeta),$$

$$\boldsymbol{n} \cdot h_\zeta = \frac{1}{\lambda\mu_0}\nabla_{\partial\Omega} \cdot Z_{\mu,\epsilon_q}(\boldsymbol{n} \times h_\zeta).$$

To identify ϵ_q and μ we have to make sure that our choices of the vectors ξ and ζ (and hence ζ^*) guarantee the existence of y_∞ and y^*_∞. For this we set

$$\xi = (|\xi|, 0, 0)^T, \quad \zeta = (i\xi/2, -(|\xi|^2/4 + R^2)^{1/2}, i(R^2 + k^2)^{1/2})^T,$$

where R is a real, positive parameter that controls the size of ζ. Letting R tend to infinity we get

$$(1/|\zeta|)\zeta \to \hat{\zeta} = \frac{1}{\sqrt{2}}(0, -1, i)^T$$

and thus the limits y_∞ and y^*_∞ exist and are given by

$$y_\infty = (-i\hat{\zeta} \cdot a, 0, 0, -i\hat{\zeta} \cdot b), \quad y^*_\infty = -iP(\hat{\zeta})z.$$

So far we have neither specified a, b nor z. We choose a and b such that $\hat{\zeta} \cdot a = i$, $\hat{\zeta} \cdot b = 0$ and we let $z = (0, a, 0, 0)$. Then

$$y^*_\infty = -i(\hat{\zeta} \cdot a, 0, -\hat{\zeta} \times a, 0)^T = (1, 0, i\hat{\zeta} \times a, 0)^T.$$

A calculation of the matrix Q shows that $Q_{1,j} = 0$, for $5 \leq j \leq 8$ and we find from (3.2.24)

$$\hat{D}(\xi) = \hat{Q}_{1,1}, \tag{3.2.30}$$

i.e. the given data determines the Fourier transform of the function $Q_{1,1}$ and thus the function itself. Analogously one shows that $\hat{D}(\xi)$ determines $Q_{8,8}$.

Since $Q = -VP(\nabla) + P(\nabla)V^T - k(V + V^T) + VV^T$ a simple calculation gives

$$Q_{1,1} = \frac{1}{2}\nabla \cdot \nabla \ln \mu + \frac{1}{4}|\nabla \ln \mu|^2 + (\kappa^2 - k^2)$$

and

$$Q_{8,8} = \frac{1}{2}\nabla \cdot \nabla \ln \epsilon_q + \frac{1}{4}|\nabla \ln \epsilon_q|^2 + (\kappa^2 - k^2).$$

Setting $u = \left(\frac{\mu}{\mu_0}\right)^{1/2}$ and $v = \left(\frac{\epsilon_q}{\epsilon_0}\right)^{1/2}$ we get the equations

$$\frac{1}{v}(\Delta v - k^2 v(1 - uv)) = Q_{1,1}, \tag{3.2.31}$$

$$\frac{1}{u}(\Delta u - k^2 u(1 - uv)) = Q_{8,8}, \tag{3.2.32}$$

which have unique solutions (see for example [PLE92]). This shows that the impedance map Z_{μ,ϵ_q} uniquely identifies μ and ϵ_q. □

The above result is not always sufficient to show uniqueness for inverse problems arising in real world problems. Often one has only partial boundary data available. If we are interested in applications like landmine detection we must consider the case of local boundary data, since we usually have only data available on the surface of the earth.

3.3 Local boundary data

After giving a uniqueness result for global boundary data, we want to present a uniqueness result for the inverse problem of the equation

$$\nabla \times (\mu_r^{-1} \nabla \times E) + k^2 \epsilon_r E = 0, \tag{3.3.1}$$

with only local boundary data. The result we show here is only a partial uniqueness result, since we only show that local boundary data determines one coefficient uniquely if the other is a known constant. In particular we will show the following. Let $\Gamma \subset \partial\Omega$ be an open subset of $\partial\Omega$ and let $\Gamma_c = \partial\Omega \backslash \overline{\Gamma}$. Then the set

$$C_{\epsilon_r} := \{(\boldsymbol{n} \times E|_{\partial\Omega}, \boldsymbol{n} \times (\nabla \times E|_{\Gamma}))|\ E \in H^1(\text{curl}; \Omega),\ \ E \text{ solves (3.3.1)},\ \ \boldsymbol{n} \times E|_{\Gamma_c} = 0\} \tag{3.3.2}$$

associated with equation (3.3.1) is sufficient to recover ϵ_r if $\mu_r = 1$ in Ω and $\epsilon_r = 1$ in a neighbourhood of the boundary. To be more precise we make the following assumptions.

(a1) $\mu_r = 1$ in Ω.

(a2) $\epsilon_r = 1$ in $\Omega \backslash \overline{\Omega'}$, where $\Omega' \subset\subset \Omega$ has a $C^{1,1}$-boundary and is connected.

The idea behind this proof goes back to Ammari and Uhlmann [AU04]. Note that in Theorem 2.27 we have shown that under the assumptions of this chapter, $E \in H^1(\text{curl}; \Omega)$ is satisfied. We show the following result in this section.

Theorem 3.11.

Let assumptions (a1) and (a2) hold. If we have two coefficients $\epsilon_{r,1}, \epsilon_{r,2}$ with $C_{\epsilon_{r,1}} = C_{\epsilon_{r,2}}$, then it must also hold that $\epsilon_{r,1} = \epsilon_{r,2}$ in Ω.

Proof.

The proof of the above theorem is in several steps. First we show the following auxiliary result.

Lemma 3.12.

Let the assumptions (a1) and (a2) hold. We set

$$\tilde{A}(\Omega) = \{E \in H^1(\mathrm{curl};\Omega)|\ \nabla \times \nabla \times E + k^2\epsilon_r E = 0,\ \boldsymbol{n} \times E = 0\ \text{on}\ \Gamma_c\}$$

and

$$A(\Omega) = \{E \in H^1(\mathrm{curl};\Omega)|\ \nabla \times \nabla \times E + k^2\epsilon_r E = 0\}.$$

Then $\tilde{A}(\Omega)$ is dense in $A(\Omega)$ with respect to the $L^2(\Omega')^3$ norm.

Proof.

Assume there is an element $u \in A(\Omega)$, such that

$$\int_{\Omega'} \langle u, w\rangle dx = 0 \quad \forall w \in \tilde{A}(\Omega).$$

We will show that u has to vanish. Let $G(x, y)$ be the Green matrix (see Section 2.6) for the equation (3.3.1) in Ω, i.e. $G(x, y)$ solves

$$\nabla_y \times \nabla_y \times G(x,y) + k^2\epsilon_r(y)G(x,y) = \delta_x(y), \quad x, y \in \Omega, \tag{3.3.3}$$

$$\boldsymbol{n}(y) \times G(x,y) = 0, \quad y \in \partial\Omega,\ x \in \Omega.$$

Given our regularity assumptions on ϵ_r and Ω that every solution of the equation (3.3.1) satisfies $\nabla \cdot E \in L^2(\Omega)$. Thus we define an Operator L on the space X_N given by

$$X_N := \{u \in H(\text{curl}; \Omega) \cap H(\text{div}; \Omega) |\ \boldsymbol{n} \times u|_{\partial\Omega} = 0\}$$

as $L : X_N \mapsto L^2(\Omega)^3$,

$$Lu := \nabla \times \nabla \times u + k^2 \epsilon_r u.$$

The bilinear form associated with the operator L is then

$$a(u, v) := \langle \nabla \times u, \nabla \times v \rangle_{L^2(\Omega)^3} + k^2 \langle \epsilon_r u, v \rangle_{L^2(\Omega)^3}.$$

The inverse $L^{-1} : L^2(\Omega)^3 \mapsto L^2(\Omega)^3$ then gives the solution of

$$a(u, v) = \langle f, v \rangle_{L^2(\Omega)^3}, \quad \forall v \in X_N$$

and thus L^{-1} is compact since X_N is compactly embedded in $L^2(\Omega)^3$ (see [Mon03][Theorem 3.49]). Therefore L^{-1} has discrete spectrum and the associated Green's function $G(x, y)$ can be written as

$$G(x, y) = \sum_{n=1}^{\infty} \frac{1}{\mu_n} \Psi_n(y) \Psi_n(x)^T, \tag{3.3.4}$$

where μ_n and Ψ_n are the eigenvalues and eigenfunctions of the operator L. To see how the differential operators and traces act on the components of the Green's matrix G we look at the addends in (3.3.4) and apply the operators to the generic matrix

$$g(x,y) = \begin{pmatrix} \Phi_1(y)\Phi_1(x) & \Phi_1(y)\Phi_2(x) & \Phi_1(y)\Phi_3(x) \\ \Phi_2(y)\Phi_1(x) & \Phi_2(y)\Phi_2(x) & \Phi_2(y)\Phi_3(x) \\ \Phi_3(y)\Phi_1(x) & \Phi_3(y)\Phi_2(x) & \Phi_3(y)\Phi_3(x) \end{pmatrix}. \tag{3.3.5}$$

We will denote the rows of $g(x,y)$ by $g_1(x,y)$, $g_2(x,y)$ and $g_3(x,y)$ and the columns by $g^1(x,y)$, $g^2(x,y)$ and $g^3(x,y)$. After transposing and then multiplying (3.3.3) with a $w(y) \in \tilde{A}(\Omega)$ on each side, integrating over Ω and replacing $G^i(x,y)$ with $g^i(x,y)$ we get

$$w_i(x) = \int_\Omega \langle \nabla_y \times \nabla_y \times g^i(x,y), w(y)\rangle + \epsilon_r(y)\langle g^i(x,y), w(y)\rangle dy, \quad i = 1,2,3. \tag{3.3.6}$$

For this analysis we will use the standard integration by parts formula

$$\int_\Omega \langle \nabla \times u, v\rangle dx = \int_\Omega \langle u, \nabla \times v\rangle dx + \int_{\partial\Omega} \langle \boldsymbol{n} \times u, v\rangle dS \tag{3.3.7}$$

instead of the more detailed formula (2.4.3). Thus if we apply an integration by parts to (3.3.6) we get

$$\begin{aligned} w_i(x) &= \int_\Omega \langle \nabla_y \times \nabla_y \times g^i(x,y), w(y)\rangle + \epsilon_r(y)\langle g^i(x,y), w(y)\rangle dy \\ &= \int_\Omega \langle \nabla_y \times g^i(x,y), \nabla_y \times w(y)\rangle + \epsilon_r(y)\langle g^i(x,y), w(y)\rangle dy \\ &\quad + \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla_y \times g^i(x,y)), w(y)\rangle dS(y) \\ &= \int_\Omega \underbrace{\langle g^i(x,y), \nabla_y \times \nabla_y \times w(y)\rangle + \epsilon_r(y)\langle g^i(x,y), w(y)\rangle}_{=0} dy \\ &\quad + \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla_y \times g^i(x,y)), w(y)\rangle dS(y) \end{aligned}$$

$$+\int_\Gamma \langle \underbrace{\boldsymbol{n}(y) \times g^i(x,y)}_{=0}, \nabla_y \times w(y)\rangle dS(y).$$

This yields

$$w_i(x) = \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla_y \times g^i(x,y)), w(y)\rangle dS(y), \quad i = 1, 2, 3. \tag{3.3.8}$$

We define

$$\tilde{H}^{1/2}(\text{curl}; \Gamma) = \{p \in H^{1/2}(\text{curl}; \partial\Omega),\ p = 0 \text{ on } \Gamma_c\}$$

and show that for any $p \in \tilde{H}^{1/2}(\text{curl}; \Gamma)$ the function v defined by

$$v_i(x) = \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla_y \times g^i(x,y)), p(y)\rangle dS(y), \quad i = 1, 2, 3, \tag{3.3.9}$$

is an element of $\tilde{A}(\Omega)$. Thus we define v by (3.3.9) and show that this gives $\boldsymbol{n} \times v|_{\Gamma^c} = 0$. To do this we first show that $\boldsymbol{n}(y) \times \nabla_y \times$ 'almost' commutes with $\boldsymbol{n}(x) \times$. The components of $\boldsymbol{n}(x) \times w(x)$, $x \in \Gamma_c$ satisfy

$$(\boldsymbol{n}(x) \times v(x))_1 =$$

$$\boldsymbol{n}_2(x) \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla_y \times g^3(x,y)), p(y)\rangle dS(y) - \boldsymbol{n}_3(x) \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla_y \times g^2(x,y)), p(y)\rangle dS(y),$$

$$(\boldsymbol{n}(x) \times v(x))_2 =$$

$$\boldsymbol{n}_3(x) \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla_y \times g^1(x,y)), p(y)\rangle dS(y) - \boldsymbol{n}_1(x) \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla_y \times g^3(x,y)), p(y)\rangle dS(y),$$

$$(\boldsymbol{n}(x) \times v(x))_3 =$$

$$\boldsymbol{n}_1(x) \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla_y \times g^2(x,y)), p(y)\rangle dS(y) - \boldsymbol{n}_2(x) \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla_y \times g^1(x,y)), p(y)\rangle dS(y).$$

However (3.3.5) yields

$$\boldsymbol{n}(y) \times (\nabla_y \times g^i(x,y)) = \Phi_i(x)\boldsymbol{n}(y) \times (\nabla_y \times \Phi(y)), \quad x \neq y$$

and therefore

$$(\boldsymbol{n}(x) \times v(x))_1 =$$

$$\boldsymbol{n}_2(x)\Phi_3(x) \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla \times \Phi(y)), p(y)\rangle dS(y) - \boldsymbol{n}_3(x)\Phi_2(x) \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla \times \Phi(y)), p(y)\rangle dS(y),$$

$$(\boldsymbol{n}(x) \times v(x))_2 =$$

$$\boldsymbol{n}_3(x)\Phi_1(x) \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla \times \Phi(y)), p(y)\rangle dS(y) - \boldsymbol{n}_1(x)\Phi_3(x) \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla \times \Phi(y)), p(y)\rangle dS(y),$$

$$(\boldsymbol{n}(x) \times v(x))_3 =$$

$$\boldsymbol{n}_1(x)\Phi_2(x) \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla \times \Phi(y)), p(y)\rangle dS(y) - \boldsymbol{n}_2(x)\Phi_1(x) \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla \times \Phi(y)), p(y)\rangle dS(y).$$

Thus after setting

$$F = \int_\Gamma \langle \boldsymbol{n}(y) \times (\nabla \times \Phi(y)), p(y)\rangle dS(y)$$

we get

$$n(x) \times v(x) = (n(x) \times \Phi(x))F, \quad x \in \Gamma_c.$$

Since Φ is an eigenfunction of L it satisfies

$$n(x) \times \Phi(x)|_{\partial\Omega} = 0$$

yielding

$$n(x) \times v(x) = 0, \quad x \in \Gamma_c.$$

Therefore any $w \in \tilde{A}(\Omega)$ can be represented as

$$w(x) = \int_\Gamma n(y) \times (\nabla_y \times G(x,y)^T)p(y)dS(y), \quad x \in \Omega,$$

for some $p \in \tilde{H}^{1/2}(\text{curl};\Gamma)$. Now using our assumption

$$\int_{\Omega'} \langle u, w \rangle dx = 0, \quad \forall w \in \tilde{A}(\Omega),$$

Fubini's theorem and the symmetry of G $(G(y,x)^T = G(x,y))$ we can conclude that

$$\int_\Gamma \langle \int_{\Omega'} n(y) \times (\nabla_y \times G(y,x)^T)u(x)dx, p(y)\rangle dy = 0, \quad \forall p \in \tilde{H}^{1/2}(\text{curl};\Gamma)$$

and therefore

$$\int_{\Omega'} n(x) \times (\nabla_x \times G(x,y)^T)u(y)dy = 0, \quad \forall x \in \Gamma. \tag{3.3.10}$$

Now we set

$$v(x) = \int_{\Omega'} G(x,y)^T u(y)dy.$$

Then $v \in H^1(\text{curl};\Omega)$ satisfies $n \times v|_{\partial\Omega} = 0$ and because of (3.3.10) also $n \times (\nabla \times v|_\Gamma) = 0$.

Further v satisfies

$$\nabla \times \nabla \times v + k^2 \epsilon_r v = \begin{cases} u & \text{if } x \in \Omega' \\ 0 & \text{if } x \in \Omega \backslash \bar{\Omega}' \end{cases}.$$

By the unique continuation principle for Maxwell's equations (Theorem 2.32) and the unique solvability of the Cauchy problem (Theorem 2.33) we get $v \equiv 0$ in $\Omega \backslash \bar{\Omega}'$ and thus

$$\boldsymbol{n} \times v|_{\partial\Omega'} = \boldsymbol{n} \times (\nabla \times v|_{\partial\Omega'}) = 0.$$

Now we take the scalar product of both sides of

$$\nabla \times \nabla v + k^2 \epsilon_r v = u$$

with u, integrate by parts and get

$$\int_{\Omega'} \langle u, u \rangle = 0.$$

Thus $\|u\|_{L^2(\Omega')} = 0$ and by the unique continuation principle we get $u = 0$ in Ω. □

Before we show the main result we need another auxiliary result.

Lemma 3.13.

Let $\Omega' \subset\subset \Omega$, Ω' contain the support of $\epsilon_{r,1} - \epsilon_{r,2}$. Let u_i satisfy

$$\nabla \times \nabla \times u_i + k^2 \epsilon_{r,i} u_i = 0, \quad \text{in } \Omega, \tag{3.3.11}$$

$$\boldsymbol{n} \times u_i|_{\Gamma_c} = 0, \quad i = 1, 2.$$

Further assume $\epsilon_{r,1} = \epsilon_{r,2} = 1$ *in* $\Omega \backslash \overline{\Omega'}$ *and* $C_{\epsilon_{r,1}} = C_{\epsilon_{r,2}}$ *(see (3.3.2)). Then*

$$\int_{\Omega'} k^2(\epsilon_{r,1} - \epsilon_{r,2})\langle u_1, u_2 \rangle dx = 0. \tag{3.3.12}$$

Proof.

Using integration by parts (2.4.3) we get

$$\int_{\Omega'} k^2(\epsilon_{r,1} - \epsilon_{r,2})\langle u_1, u_2 \rangle dx = \int_{\Gamma} \langle \gamma_t(\nabla \times u_1), \gamma_T(u_2) \rangle - \langle \gamma_t(\nabla \times u_2), \gamma_T(u_1) \rangle dS.$$

Now let $v \in H^1(\text{curl}; \Omega)$ be the unique solution of

$$\nabla \times \nabla \times v + k^2 \epsilon_{r,1} v = 0, \quad \text{in } \Omega,$$

$$\boldsymbol{n} \times v = \boldsymbol{n} \times u_2, \quad \text{on } \partial\Omega.$$

From $C_{\epsilon_{r,1}} = C_{\epsilon_{r,2}}$ we conclude that

$$\boldsymbol{n} \times v|_{\Gamma_c} = 0, \quad \boldsymbol{n} \times v|_{\Gamma} = \boldsymbol{n} \times u_2|_{\Gamma} \quad \rightarrow \quad \boldsymbol{n} \times (\nabla \times v)|_{\Gamma} = \boldsymbol{n} \times (\nabla \times u_2)|_{\Gamma}.$$

Another integration by parts then gives

$$0 = \int_{\Omega'} k^2(\epsilon_{r,1} - \epsilon_{r,1})\langle u_1, v \rangle dx = \int_{\Gamma} \langle \gamma_t(\nabla \times u_1), \gamma_T(v) \rangle - \langle \gamma_t(\nabla \times v), \gamma_T(u_1) \rangle dS$$

which proves the result. □

Now we extend ϵ_r with 1 to $\mathbb{R}^3 \backslash \bar{\Omega}$ and construct solutions of

$$\nabla \times \nabla \times v + k^2 \epsilon_r v = 0$$

in $\mathbb{R}^3$ of the form

$$v_i = e^{\langle x,\zeta_i\rangle}(\epsilon_{r,i}^{-1/2}\nu_i + R_{\epsilon_{r,i}}(x,\zeta_i)),\ i = 1,2, \tag{3.3.13}$$

for $|\zeta_i|$ sufficiently large with $R(\cdot,\zeta_i) \in L^2_\delta(\mathbb{R}^3)$, $-1 < \delta < -1/2$. See (3.2.16) for the definition of $L^2_\delta(\mathbb{R}^n)$. These solutions satisfy

$$\|R_{\epsilon_{r,i}}(\cdot,\zeta_i)\|_{L^2_\delta(\mathbb{R}^3)} \leq \frac{C}{\|\zeta_i\|} \tag{3.3.14}$$

and $\nu_1 \cdot \nu_2 \neq 0$. The existence of such solutions is for example shown in [PLE92][Lemma 2.4, Theorem 2.5] and can also be shown using our calculations from Section 3.2 (see the solutions X_ζ in Section 3.2). We choose

$$\zeta_1 = \frac{\eta}{2} + i\frac{j+l}{2},$$
$$\zeta_2 = -\frac{\eta}{2} + i\frac{j-l}{2},$$

with $\eta \cdot j = 0$, $\eta \cdot l = 0$, $j \cdot l = 0$ and $\|\eta\|^2 - \|j\|^2 + \|l\|^2 = k^2$. Using Lemma 3.12 we can approximate any $v_i \in A(\Omega)$ by elements of $\tilde{A}(\Omega)$ and thus

$$\int_{\Omega'} (\epsilon_{r,1} - \epsilon_{r,2})\langle v_1, v_2\rangle dx = 0 \tag{3.3.15}$$

$\forall v_i \in H^1(\mathrm{curl};\Omega)$ with $\nabla \times \nabla \times v_i + k^2\epsilon_r v_i = 0$. We substitute (3.3.13) into (3.3.15) and get

$$\int_{\Omega'} (\epsilon_{r,1} - \epsilon_{r,2})\langle e^{\langle x,\zeta_1\rangle}(\epsilon_{r,1}^{-1/2}\nu_1 + R_{\epsilon_{r_1}}(x,\zeta_1)), e^{\langle x,\zeta_2\rangle}(\epsilon_{r,2}^{-1/2}\nu_2 + R_{\epsilon_{r,2}}(x,\zeta_2))\rangle dx = 0.$$

Letting $\|l\|$ be big enough we get, due to (3.3.14),

$$\int_{\Omega'} (\epsilon_{r,1} - \epsilon_{r,2})\langle e^{\langle x,\zeta_1\rangle}\epsilon_{r,1}^{-1/2}\nu_1, e^{\langle x,\zeta_2\rangle}\epsilon_{r,2}^{-1/2}\nu_2\rangle dx = 0$$

and thus

$$\langle \nu_1, \nu_2 \rangle \int_{\Omega'} (\epsilon_{r,1} - \epsilon_{r,2}) e^{\langle x, \zeta_1 \rangle} e^{\langle x, \zeta_2 \rangle} \epsilon_{r,1}^{-1/2} \epsilon_{r,2}^{-1/2} dx = 0$$

and therefore (because of $\langle \nu_1, \nu_2 \rangle \neq 0$) we get

$$\widehat{(\epsilon_{r,1} - \epsilon_{r,2}) \epsilon_{r,2}^{-1/2} \epsilon_{r,2}^{-1/2}}(j) = 0, \quad \forall j \in \mathbb{R}^3,$$

which means $(\epsilon_{r,1} - \epsilon_{r,2}) \epsilon_{r,1}^{-1/2} \epsilon_{r,2}^{-1/2}(x) = 0$ ($\widehat{f}$ stands for the Fourier transform of f). Since $\epsilon_{r,1}^{-1/2} \epsilon_{r,2}^{-1/2} > 0$ we can conclude that

$$\epsilon_{r,1} = \epsilon_{r,2}.$$

This concludes the proof. □

The idea of the above proof can unfortunately not be used to show that local boundary data determines two coefficients μ_r and ϵ_r uniquely. Nevertheless the result is encouraging and provides a sufficient theoretical basis to justify a variational algorithm using only local boundary data. Since in the next chapter we want to outline the idea of our variational algorithm by using interior data, we give a short discussion on the uniqueness of the inverse problem given interior data.

3.4 Interior data

Finally we present two results that show also interior data, i.e. one or more solutions of the boundary value problem

$$\nabla \times (\mu_r^{-1} \nabla \times E) + k^2 \epsilon_r E = F, \quad \text{in } \Omega, \tag{3.4.1}$$

$$n \times E = 0, \quad \text{on } \partial\Omega \tag{3.4.2}$$

can uniquely determine one or both of the coefficients μ_r and ϵ_r. Since the main focus of this thesis is not on interior data we do not state any comprehensive results, but just present two results to show that the right kind of interior data identifies the coefficients uniquely. There are not too many uniqueness results for parameter identification in the literature for given interior data. Some results for elliptic equations can be found in [Ale86], [VK93] and [KY02].

Here we consider the weak formulation of (3.4.1), (3.4.2), i.e.

$$\int_\Omega \mu_r^{-1} \langle \nabla \times E, \nabla \times \Phi \rangle \, dx + \int_\Omega k^2 \epsilon_r \langle E, \Phi \rangle \, dx = \int_\Omega \langle F, \Phi \rangle \, dx, \quad \forall \Phi \in H(\text{curl}; \Omega). \tag{3.4.3}$$

If we assume that besides F also μ_r is known and that the solution E is given, we see that the unique identifiability of ϵ_r corresponds to the uniqueness of the solution of the integral equation

$$T(\epsilon_r) := \int_\Omega k^2 \epsilon_r \langle E, \Phi \rangle \, dx = 0, \quad \forall \Phi \in H(\text{curl}; \Omega). \tag{3.4.4}$$

Since $H(\text{curl}; \Omega)$ is dense in $L^2(\Omega)^3$ we see immediately that (3.4.4) can only be true if $k^2 \epsilon_r E = 0$. Thus the unique solvability of (3.4.3) depends only on the properties of E. A sufficient condition for uniqueness of the inverse problem is then given by the following theorem.

Theorem 3.14.

Let μ_r and F be known. Then if the solution $E \in H(\text{curl}; \Omega)$ has only positive entries, it uniquely *identifies* ϵ_r.

Proof.

Since we have to consider the equation $k^2\epsilon_r E = 0$, we see that due to the positivity of E and k^2, this can only be true if $\epsilon_r = 0$ and thus (3.4.4) has a unique solution. □

This result is of course very simplistic, however we have shown that given sufficient assumptions on E, interior data uniquely identifies ϵ_r. Now we want to show, that in general one does not have to impose positivity on E and that given several solutions E_i we can expect to recover more than one coefficient. For this we consider an anisotropic system. Let λ_i, $i = 1, 2, 3$ be given and let E^i be given solutions of the equations

$$\nabla \times (\mu_r^{-1}(x)\nabla \times E^i(x)) + k_i^2\epsilon_r(x)E^i(x) = F, \quad i = 1, 2, 3, \tag{3.4.5}$$

$$\nabla \times (\tilde{\mu}_r^{-1}(x)\nabla \times E^i(x)) + k_i^2\epsilon_r(x)E^i(x) = F, \quad i = 1, 2, 3, \tag{3.4.6}$$

where $k_i = \lambda_i\sqrt{\mu_0\epsilon_0}$ and let μ_r^{-1} and $\tilde{\mu}_r^{-1}$ be two diagonal matrices

$$\mu_r^{-1}(x) = \begin{pmatrix} \mu_{11}(x) & 0 & 0 \\ 0 & \mu_{22}(x) & 0 \\ 0 & 0 & \mu_{33}(x) \end{pmatrix}, \quad \tilde{\mu}_r^{-1}(x) = \begin{pmatrix} \tilde{\mu}_{11}(x) & 0 & 0 \\ 0 & \tilde{\mu}_{22}(x) & 0 \\ 0 & 0 & \tilde{\mu}_{33}(x) \end{pmatrix}.$$

From (3.4.5), (3.4.6) we conclude

$$\nabla \times ((\mu_r^{-1}(x) - \tilde{\mu}_r^{-1}(x))\nabla \times E^i(x)) = 0, \quad i = 1, 2, 3.$$

Setting $r = \mu_{11} - \tilde{\mu}_{11}$, $s = \mu_{22} - \tilde{\mu}_{22}$ and $t = \mu_{33} - \tilde{\mu}_{33}$ gives

$$\nabla \times \begin{pmatrix} r(\partial_2 E_3^i - \partial_3 E_2^i) \\ s(\partial_3 E_1^i - \partial_1 E_3^i) \\ t(\partial_1 E_2^i - \partial_2 E_1^i). \end{pmatrix} = 0, \quad i = 1, 2, 3.$$

We define

$$\hat{\mu} = \begin{pmatrix} r \\ s \\ t \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & \partial_3 E_1^1 - \partial_1 E_3^1 & \partial_2 E_1^1 - \partial_1 E_2^1 \\ 0 & \partial_3 E_1^2 - \partial_1 E_3^2 & \partial_2 E_1^2 - \partial_1 E_2^2 \\ 0 & \partial_3 E_1^3 - \partial_1 E_3^3 & \partial_2 E_1^3 - \partial_1 E_2^3 \end{pmatrix},$$

$$B = \begin{pmatrix} \partial_3 E_2^1 - \partial_2 E_3^1 & 0 & \partial_1 E_2^1 - \partial_2 E_1^1 \\ \partial_3 E_2^2 - \partial_2 E_3^2 & 0 & \partial_1 E_2^2 - \partial_2 E_1^2 \\ \partial_3 E_2^3 - \partial_2 E_3^3 & 0 & \partial_1 E_2^3 - \partial_2 E_1^3 \end{pmatrix},$$

$$C = \begin{pmatrix} \partial_2 E_3^1 - \partial_3 E_2^1 & \partial_1 E_3^1 - \partial_3 E_1^1 & 0 \\ \partial_2 E_3^2 - \partial_3 E_2^2 & \partial_1 E_3^2 - \partial_3 E_1^2 & 0 \\ \partial_2 E_3^3 - \partial_3 E_2^3 & \partial_1 E_3^3 - \partial_3 E_1^3 & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} (\partial_3 - \partial_2)(\partial_2 E_3^1 - \partial_3 E_2^1) & (\partial_1 - \partial_3)(\partial_3 E_1^1 - \partial_1 E_3^1) & (\partial_2 - \partial_1)(\partial_1 E_2^1 - \partial_2 E_1^1) \\ (\partial_3 - \partial_2)(\partial_2 E_3^2 - \partial_3 E_2^2) & (\partial_1 - \partial_3)(\partial_3 E_1^2 - \partial_1 E_3^2) & (\partial_2 - \partial_1)(\partial_1 E_2^2 - \partial_2 E_1^2) \\ (\partial_3 - \partial_2)(\partial_2 E_3^3 - \partial_3 E_2^3) & (\partial_1 - \partial_3)(\partial_3 E_1^3 - \partial_1 E_3^3) & (\partial_2 - \partial_1)(\partial_1 E_2^3 - \partial_2 E_1^3) \end{pmatrix}$$

to get

$$A\hat{\mu}_{x_1} + B\hat{\mu}_{x_2} + C\hat{\mu}_{x_3} + D\hat{\mu} = 0, \quad \text{in } \Omega, \tag{3.4.7}$$

$$\hat{\mu}|_{\partial\Omega} = 0. \tag{3.4.8}$$

Thus we have transformed the problem of the unique identifiability of the coefficient μ_r in (3.4.5) to the unique solvability of the initial value problem (3.4.7), (3.4.8). This problem has of course a unique solution if and only if $\partial\Omega$ is not a characteristic. This yields the following theorem.

Theorem 3.15.

Let ϵ_r be known and assume we have three solutions E_i corresponding to three different frequencies λ_i, $i = 1, 2, 3$ given. Then if the $\partial\Omega$ is not a characteristic for the system (3.4.7), (3.4.8), the coefficient μ_r is uniquely determined by the data E_i, $i = 1, 2, 3$.

Remark: In two dimensions one can easily derive explicit conditions under which $\partial\Omega$ is not a characteristic for systems of the form (3.4.7), (3.4.8) (see for example [CH89][Page 171]).

This ends our discussion of uniqueness results for the inverse problem for Maxwell's equations. Equipped with these results we will develop a variational algorithm to recover the coefficients μ_r and ϵ_r from global and local boundary data. However we first outline the idea behind this algorithm for given interior data.

4 A variational algorithm using interior data

In this chapter we outline the basic idea for our variational algorithm to recover the functions μ_r and ϵ_r. We do this using interior data, i.e. solutions E_i of the boundary value problem

$$\nabla \times (\mu_r^{-1} \nabla \times E_i) + k_i^2 \epsilon_r E_i = F, \quad \text{in } \Omega, \tag{4.0.1}$$

$$\boldsymbol{n} \times E_i = 0, \quad \text{on } \partial\Omega, \quad i = 1, 2, \dots, M. \tag{4.0.2}$$

In the case of given boundary data we will use a similar approach, however the theoretical framework will be more complex than in the case of interior data. Again we set $k_i = \lambda_i \sqrt{\mu_0 \epsilon_0}$, M is a positive integer and $\lambda_i \in \mathbb{R}_+$ for $i = 1, 2, \dots, M$. We assume the coefficients μ_r and ϵ_r satisfy the following conditions.

$$0 < \mu_m \leq \mu_r^{-1}(x) \leq \mu_M, \quad 0 < \epsilon_m \leq \epsilon_r(x) \leq \epsilon_M, \quad x \in \Omega. \tag{4.0.3}$$

We present a variational algorithm to recover μ_r and ϵ_r using a convex functional. Although this setting is inadequate in the case of given boundary data, it shows the basic idea of our intended approach. It further helps to point out the arising difficulties in the case of boundary data.

4.1 The algorithm

Any reliable variational algorithm to recover the coefficients in Maxwell's equations relies on the unique identifiability of the desired parameters. We have shown in Theorem 3.14 and Theorem 3.15 that interior data in some cases identifies the coefficients uniquely. We assume throughout this chapter that our given solutions identify the coefficients uniquely. Thus given M solutions E_i of (4.0.1) and (4.0.2) we define a functional H on the domain

$$D_H = \{(m,c) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) |\ m|_{\partial\Omega} = c|_{\partial\Omega} = 1,\ m(x) \text{ and } c(x) \text{ satisfy } (4.0.3)\}.$$

The condition $(m,c) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ is imposed to ensure that m and c have a well-defined trace. We set

$$G(m,c,\lambda_i) = \int_\Omega m\|\nabla \times (E_i - E_i^{m,c})\|^2 + k_i^2 c\|(E_i - E_i^{m,c})\|^2 dx, \tag{4.1.1}$$

where $E_i^{m,c}$ solves (4.0.1), (4.0.2) with $\mu_r^{-1} = m$, $\epsilon_r = c$ and $k_i = \lambda_i\sqrt{\epsilon_0\mu_0}$, for $i = 1,2,\ldots,M$.

To incorporate all the given data into a convex functional we set

$$H(m,c) = \sum_{i=1}^{M} G(m,c,\lambda_i). \tag{4.1.2}$$

The functional $H(m,c)$ has two very important properties.

Theorem 4.1.

For all $m,c \in C^1(\overline{\Omega})$ we have $H(m,c) \geq 0$. Given our assumption that the solutions E_i, $i = 1,2,\ldots,M$ uniquely identify μ_r and ϵ_r it also holds that

$$H(m,c) = 0 \iff (m,c) = (\mu_r^{-1}, \epsilon_r).$$

Proof.

The first property is obvious, since we have $G(m, c, \lambda_i) \geq 0,\ 1 \leq i \leq M$. The second property can be shown as follows. If $H(m, c) = 0$ we have $G(m, c, \lambda_i) = 0$, for $i = 1, 2, \ldots, M$. Thus we have $E_i = E_i^{m,c}$ for all $i = 1, 2, \ldots, M$. However since the solutions E_i uniquely identify the coefficients μ_r and ϵ_r we get $(m, c) = (\mu_r^{-1}, \epsilon_r)$. □

The last theorem shows that the functional H is positive and has a unique global minimum. To minimize H we have to make sure that we can calculate a proper descent direction and that there are no other local minima than the global minimum. The first of these problems is addressed in the next theorem.

Theorem 4.2.

The Gâteaux derivative of $H(m, c)$ in the direction (h_m, h_c), with $h_m|_{\partial\Omega} = h_c|_{\partial\Omega} = 0$, is given by

$$H'(m, c)[h_m, h_c] = \sum_{i=1}^{M} \int_\Omega h_m(\|\nabla \times E_i\|^2 - \|\nabla \times E_i^{m,c}\|^2) + k_i^2 h_c(\|E_i\|^2 - \|E_i^{m,c}\|^2)dx.$$

Proof.

We show the above formula for a single term $G(m, c, \lambda)$ for a fixed Laplace parameter λ. We simply write $G(m, c)$ instead of $G(m, c, \lambda)$. Thus we consider

$$
\begin{aligned}
&G(m+\delta h_m, c+\delta h_c) - G(m,c) = \\
&\int_\Omega m[(\nabla\times(E-E^{m+\delta h_m,c+\delta h_c}))\cdot(\nabla\times(E-E^{m+\delta h_m,c+\delta h_c})) \\
&-(\nabla\times(E-E^{m,c}))\cdot(\nabla\times(E-E^{m,c}))] \\
&+k^2c((E-E^{m+\delta h_m,c+\delta h_c})\cdot(E-E^{m+\delta h_m,c+\delta h_c})-(E-E^{m,c})\cdot(E-E^{m,c})) \\
&+\delta h_m(\nabla\times(E-E^{m+\delta h_m,c+\delta h_c}))\cdot(\nabla\times(E-E^{m+\delta h_m,c+\delta h_c})) \\
&+\delta k^2 h_c(E-E^{m+\delta h_m,c+\delta h_c})\cdot(E-E^{m+\delta h_m,c+\delta h_c})\,dx.
\end{aligned}
$$

Using the formula $|a|^2 - |b|^2 = (a-b)\cdot(a+b)$ we can rewrite the above expression as

$$
\begin{aligned}
&G(m+\delta h_m, c+\delta h_c) - G(m,c) = \\
&\int_\Omega m\nabla\times(E^{m+\delta h_m,c+\delta h_c}-E^{m,c})\cdot\nabla\times(E^{m+\delta h_m,c+\delta h_c}-E^{m,c}) \\
&-2m\nabla\times E\cdot\nabla\times E^{m+\delta h_m,c+\delta h_c}+2m\nabla\times E\cdot\nabla\times E^{m,c} \\
&+k^2c(E^{m+\delta h_m,c+\delta h_c}-E^{m,c})\cdot(E^{m+\delta h_m,c+\delta h_c}-E^{m,c}) \\
&-2k^2cE\cdot E^{m+\delta h_m,c+\delta h_c}+2k^2cE\cdot E^{m,c} \\
&+\delta h_m(\nabla\times(E-E^{m+\delta h_m,c+\delta h_c}))\cdot(\nabla\times(E-E^{m+\delta h_m,c+\delta h_c})) \\
&+\delta k^2 h_c(E-E^{m+\delta h_m,c+\delta h_c})\cdot(E-E^{m+\delta h_m,c+\delta h_c})\,dx.
\end{aligned}
$$

An integration by parts (Theorem 2.19) yields

$$
\begin{aligned}
&G(m+\delta h_m, c+\delta h_c) - G(m,c) = \\
&\quad \int_\Omega (E^{m+\delta h_m, c+\delta h_c} - E^{m,c}) \cdot (\nabla \times (m \nabla \times (E^{m+\delta h_m, c+\delta h_c} - E^{m,c})) \\
&\quad + k^2 c(E^{m+\delta h_m, c+\delta h_c} - E^{m,c}))) \\
&\quad - 2E \cdot (\nabla \times (m \nabla \times (E^{m+\delta h_m, c+\delta h_c} - E^{m,c}) + k^2 c(E^{m+\delta h_m, c+\delta h_c} - E^{m,c}))) \\
&\quad + \delta h_m (\nabla \times (E - E^{m+\delta h_m, c+\delta h_c})) \cdot (\nabla \times (E - E^{m+\delta h_m, c+\delta h_c})) \\
&\quad + \delta k^2 h_c (E - E^{m+\delta h_m, c+\delta h_c}) \cdot (E - E^{m+\delta h_m, c+\delta h_c}) \, dx.
\end{aligned}
$$

Now we make use of the fact that $E^{m+\delta h_m, c+\delta h_c}$ and $E^{m,c}$ satisfy equation (4.0.1) with μ_r^{-1} and ϵ_r replaced by $m + \delta h_m$ and $c + \delta h_c$ or m and c resp. and get

$$
\begin{aligned}
&G(m+\delta h_m, c+\delta h_c) - G(m,c) = \\
&\quad \int_\Omega (E^{m+\delta h_m, c+\delta h_c} - E^{m,c}) \cdot (-\nabla \times (\delta h_m \nabla \times E^{m+\delta h_m, c+\delta h_c}) - \delta k^2 h_c E^{m+\delta h_m, c+\delta h_c}) \\
&\quad + 2E \cdot (\nabla \times (\delta h_m \nabla \times E^{m+\delta h_m, c+\delta h_c}) + \delta k^2 h_c E^{m+\delta h_m, c+\delta h_c}) \\
&\quad + \delta h_m (\nabla \times (E - E^{m+\delta h_m, c+\delta h_c})) \cdot (\nabla \times (E - E^{m+\delta h_m, c+\delta h_c})) \\
&\quad + \delta k^2 h_c (E - E^{m+\delta h_m, c+\delta h_c}) \cdot (E - E^{m+\delta h_m, c+\delta h_c}) \, dx.
\end{aligned}
$$

Dividing by δ and taking the limit $\delta \to 0$ yields,

$$
\begin{aligned}
G'(m,c)[h_m, h_c] = &\int_\Omega 2E \cdot (\nabla \times (h_m \nabla \times E^{m,h_c}) + k^2 h_c E^{m,c}) \\
&+ h_m (\nabla \times (E - E^{m,c})) \cdot (\nabla \times (E - E^{m,c})) + k^2 h_c (E - E^{m,c}) \cdot (E - E^{m,c}) \, dx.
\end{aligned}
$$

Another integration by parts gives

$$G'(m,c)[h_m,h_c] = \int_\Omega 2h_m \nabla \times E \cdot \nabla \times E^{m,h_c} + 2k^2 h_c E \cdot E^{m,c})$$

$$+ h_m(\nabla \times (E - E^{m,c})) \cdot (\nabla \times (E - E^{m,c})) + k^2 h_c (E - E^{m,c}) \cdot (E - E^{m,c})\, dx.$$

Simplifying this expression yields

$$G'(m,c)[h_m,h_c] = \int_\Omega h_m(\|\nabla \times E\|^2 - \|\nabla \times E^{m,c}\|^2) + k^2 h_c(\|E\|^2 - \|E^{m,c}\|^2) dx.$$

This also gives the desired expression for $H(m,c)$. □

To show that H has a single local minimum we calculate the second Gâteaux derivative of H.

Theorem 4.3.

Let $L^{-1}_{m,c,i}$ be the inverse of the operator $L_{m,c,i}u : \nabla \times (m\nabla \times u) + k_i^2 cu$ with the boundary condition $n \times u|_{\partial\Omega} = 0$. Then if $h_m|_{\partial\Omega} = h_c|_{\partial\Omega} = 0$ and $l_m|_{\partial\Omega} = l_c|_{\partial\Omega} = 0$ the second Gâteaux derivative of H is given by

$$H''(m,c)[(l_m,l_c),(h_m,h_c)] = \sum_{i=1}^{M} \int_\Omega 2L^{-1}_{m,c,i} d_i(l_m,l_c) \cdot d_i(h_m,h_c) dx$$

where

$$d_i(l_m,l_c) = \nabla \times (l_m \nabla \times E_i^{m,c}) + k_i^2 l_c E_i^{m,c}.$$

$d_i(h_m,h_c)$ is defined analogously.

Proof.

Let $\delta > 0$. We use the fact that

$$-L_{m,c,i}E_i^{m,c} + L_{m+\delta l_m,c+\delta l_c,i}E_i^{m+\delta l_m,c+\delta l_c} = 0$$

to get

$$L_{m,c,i}(E_i^{m+\delta l_m,c+\delta l_c} - E_i^{m,c}) = -\nabla \times (\delta l_m \nabla \times E_i^{m+\delta l_m,c+\delta l_c}) - k_i^2 l_c E_i^{m+\delta l_m,c+\delta l_c}, \qquad (4.1.3)$$

and conclude

$$E_i^{m+\delta l_m,c+\delta l_c} - E_i^{m,c} = -L_{m,c,i}^{-1}(\nabla \times (\delta l_m \nabla \times E_i^{m+\delta l_m,c+\delta l_c}) + k_i^2 \delta l_c E_i^{m+\delta l_m,c+\delta l_c}). \qquad (4.1.4)$$

Thus we get for the functional G

$$\begin{aligned}
&G'(m+\delta l_m, c+\delta l_c)(h_m,h_c) - G'(m,c)(h_m,h_c) = \\
&\int_\Omega h_m(\|\nabla \times E\|^2 - \|\nabla \times E^{m+\delta l_m,c+\delta l_c}\|^2) + k^2 h_c(\|E\|^2 - \|E^{m+\delta l_m,c+\delta l_c}\|^2) \\
&-h_m(\|\nabla \times E\|^2 - \|\nabla \times E^{m,c}\|^2) - k^2 h_c(\|E\|^2 - \|E^{m,c}\|^2)dx = \\
&\int_\Omega h_m(\nabla \times E^{m,c} \cdot (\nabla \times E^{m,c} - \nabla \times E^{m+\delta l_m,c+\delta l_c}) \\
&+\nabla \times E^{m+\delta l_m,c+\delta l_c} \cdot (\nabla \times E^{m,c} - \nabla \times E^{m+\delta l_m,c+\delta l_c})) \\
&+k^2 h_c(\|E^{m,c}\|^2 - \|E^{m+\delta l_m,c+\delta l_c}\|^2)\, dx.
\end{aligned}$$

Now we integrate by parts and use the formula $a^2 - b^2 = (a-b)(a+b)$ to get

$$\begin{aligned}
&G'(m+\delta l_m, c+\delta l_c)(h_m,h_c) - G'(m,c)(h_m,h_c) \\
&\qquad \int_\Omega (E^{m,c} - E^{m+\delta l_m,c+\delta l_c}) \cdot (\nabla \times (h_m \nabla \times E^{m,c} + \nabla \times E^{m+\delta l_m,c+\delta l_c})) \\
&\qquad +(E^{m,c} - E^{m+\delta l_m,c+\delta l_c}) \cdot (k^2 h_c(E^{m,c} + E^{m+\delta l_m,c+\delta l_c}))\, dx.
\end{aligned}$$

Applying equation (4.1.4) yields

$$G'(m + \delta l_m, c + \delta l_c)(h_m, h_c) - G'(m, c)(h_m, h_c) =$$

$$\delta \int_\Omega L^{-1}_{m,c}(\nabla \times (l_m \nabla \times E^{m+\delta l_m, c+\delta l_c}) + k^2 l_c E^{m+\delta l_m, c+\delta l_c}) \cdot$$

$$\cdot (\nabla \times (h_m \nabla \times (E^{m+\delta l_m, c+\delta l_c} + E^{m,c})) + k^2 h_c (E^{m+\delta l_m, c+\delta l_c} + E^{m,c}))\, dx.$$

If we now take the limit $\delta \longrightarrow 0$ we get the desired result. □

Since equation (4.0.1) is coercive, the operators $L^{-1}_{m,c,i}$ are strictly positive. Thus the above theorem shows, that the functional H does not only have a unique global minimum, but is actually convex. Therefore we can apply a conjugate gradient method to minimize H and do not have to worry about getting trapped in any local minimum.

4.2 Numerical implementations of the inverse problem for given interior data

In this section we present a few numerical implementations for given interior data, to show the effectiveness of the approach outlined in the previous section. To minimize the functional H we use a conjugate gradient scheme.

4.2.1 The Polak-Ribiere scheme

We present an abstract formulation of the descent algorithm on which we build our minimization procedure. Given a function $f : V \mapsto \mathbb{R}$ where V is some normed vector space the simplest descent method is the following.

- Start at a point P_0.
- Update the point P_i by a new point P_{i+1} as many times as needed, by minimizing the function $H(\alpha) = f(P_i - \alpha\nabla f(P_i))$ and then set $P_{i+1} = P_i - \alpha_{min}\nabla f(P_i)$.

However this simple approach has some major drawbacks. If $V = \mathbb{R}^n$, then we can expand any function f by its Taylor series

$$f(x) = f(0) + \sum_i \frac{\partial f}{\partial x_i} f + \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} + O(|x|^3)$$

and therefore

$$f(x) \approx \frac{1}{2}\langle Ax, x\rangle_{\mathbb{R}^n} - \langle b, x\rangle_{\mathbb{R}^n} + c \qquad (4.2.1)$$

with $c = f(0)$, $b = -\nabla f(x)|_{x=0}$, $A_{i,j} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}|_{x=0}$. The gradient of the approximation is then $\nabla f = Ax - b$. Now suppose we have started at a point x_0 and moved along a minimization direction $v_0 = \nabla f(x_0)$ to a minimum, say at $x_1 = x_0 + \alpha_{min}\nabla f(x_0)$. Since α_{min} minimizes the expression $H(\alpha) = f(x_0 + \alpha\nabla f(x_0))$ we can conclude

$$0 = \frac{\partial}{\partial\alpha} f(x_0 + \alpha\nabla f(x_0))|_{\alpha=\alpha_{min}} = (\nabla f(x_1))^T \frac{\partial}{\partial\alpha}(x_0 + \alpha\nabla f(x_0))|_{\alpha=\alpha_{min}} = (\nabla f(x_1))^T v_0.$$

This shows that the new descent direction $v_1 = f'(x_1)$ is perpendicular to v_0. Thus we get

$$0 = v_1 \cdot \delta(\nabla f) = v_1 \cdot Av_0. \qquad (4.2.2)$$

In order not to ruin the descent from the direction v_0, we would like the new descent direction v_2 to be perpendicular to v_1 and v_0. A good minimization procedure thus ensures that (4.2.2) holds pairwise for the set of produced descent directions v_i, $i = 0, 1, 2, \ldots$. If the relation (4.2.2) holds pairwise for a set of vectors, the set is said to be conjugate. However this is not the case for the steepest descent method. This problem leads to the so-called conjugate gradient methods. The two most famous of these are the Fletcher-Reeves and the Polak-Ribiere methods. The Polak-Ribiere scheme (as well as the Fletcher-Reeves method) is based on the results of the following theorem (see [PTVF92][chapter 10]).

Theorem 4.4.

Let A be a symmetric positive definite matrix. Let g_0 be an arbitrary vector, set $h_0 = g_0$. For $i = 1, 2, \ldots$ define the two sequences

$$g_{i+1} = g_i - \alpha_i A h_i, \qquad h_{i+1} = g_{i+1} + \gamma_i h_i,$$

where γ_i and α_i are defined as

$$\alpha_i = \frac{g_i \cdot h_i}{h \cdot A h_i}, \qquad \gamma_i = \frac{(g_{i+1} - g_i) \cdot g_{i+1}}{g_i \cdot g_i}.$$

Then for all $i \neq j$,

$$g_i \cdot g_j = 0, \qquad h_i \cdot A h_j = 0.$$

The calculation of the matrix A, can be avoided.

Theorem 4.5.

Let g_i and h_i be defined as in the above theorem. Suppose we have $g_i = -\nabla f(P_i)$ for some point P_i where f is of the form (4.2.1). If we proceed from P_i along the direction h_i to the minimum point P_{i+1} and set $g_{i+1} = -\nabla f(P_{i+1})$, then g_{i+1} is the same as would have been constructed by *the procedure in the above theorem.*

4.2.2 The Neuberger gradient

To apply the Polak-Ribiere scheme to H, we must calculate the gradient of H. From the formula of the Gâteaux derivative we see that the L^2-gradient for the functional H is given by,

$$\nabla H(m,c) = \begin{pmatrix} \sum_{i=1}^{M} \|\nabla \times E_i\|^2 - \|\nabla \times E_i^{m,c}\|^2 \\ \sum_{i=1}^{M} k_i^2(\|E_i\|^2 - \|E_i^{m,c}\|^2) \end{pmatrix}.$$

One of the major error sources in steepest descent methods, is that the updated function after a descent step, does not continue to lie in the domain D_G anymore. In our case this presents a major problem. The update direction of (m,c) must vanish on the boundary of Ω, since the coefficients $(m,c) \in D_G$ have to satisfy the conditions $m|_{\partial\Omega} = 1$, $c|_{\partial\Omega} = 1$. However the terms

$$\sum_{i=1}^{M} \left(\|\nabla \times E_i\|^2 - \|\nabla \times E_i^{m,c}\|^2\right), \quad \sum_{i=1}^{M} k_i^2 \left(\|E_i\|^2 - \|E_i^{m,c}\|^2\right)$$

do not vanish on $\partial\Omega$ in general. We overcome this problem by using a H_0^1 gradient ϑ (or Sobolev gradient) (see [Neu97]) for the update direction of (m,c). We use the following definition of the H_0^1 gradient.

Lemma 4.6.

The H_0^1 gradient $\vartheta = (\vartheta_m, \vartheta_c)$ is the solution of

$$-\Delta\vartheta_m + \vartheta_m = \sum_{i=1}^{M} \|\nabla \times E_i\|^2 - \|\nabla \times E_i^{m,c}\|^2, \quad \text{in } \Omega \tag{4.2.3}$$

$$-\Delta\vartheta_c + \vartheta_c = \sum_{i=1}^{M} k_i^2(\|E_i\|^2 - \|E_i^{m,c}\|^2), \quad \text{in } \Omega$$

$$\vartheta_m|_{\partial\Omega} = \vartheta_c|_{\partial\Omega} = 0.$$

Proof.

We have to show that the H_0^1 gradient ϑ satisfies

$$H'(m,c)(h_m,h_c) = \langle \vartheta, (h_m,h_c)\rangle_{H_0^1(\Omega)^3}.$$

An integration by parts gives

$$\begin{aligned}\langle \vartheta, (h_m,h_c)\rangle_{H_0^1(\Omega)^3} &= \int_\Omega \vartheta_m h_m + \vartheta_c h_c \, dx + \int_\Omega \nabla\vartheta_m \cdot \nabla h_m + \nabla\vartheta_c \cdot \nabla h_c \, dx \\ &= \int_\Omega \vartheta_m h_m + \vartheta_c h_c \, dx - \int_\Omega \Delta\vartheta_m h_m + \vartheta_c h_c \, dx \\ &= H'(m,c)(h_m,h_c).\end{aligned}$$

□

Therefore we can see, that the H_0^1 gradient ϑ is a proper descent direction.

Remark: ϑ behaves like a preconditioned version of $\nabla H(m,c)$, as can be seen from its definition. Thus the entries of ϑ belong to H_0^1. This makes it easier on the one hand to recover smooth functions with the H_0^1 gradient than with the L^2-gradient. However it could be a slight disadvantage to use the H_0^1 gradient to recover discontinuous functions. In the one-dimensional case of the inverse spectral problem for the Sturm-Liouville equation, this is certainly the case (see [BSKM03]). However in the case of Maxwell's equations we do not have any experimental evidence to support this.

4.2.3 Regularization

To guarantee a stable recovery of the coefficients μ_r and ϵ_r under noise, a regularization of some kind is usually necessary. We do not implement any classical regularization schemes like Tikhonov regularization or similar methods. However we regularize the update of m and c after each descent step, to ensure m and c lie still in D_G. In particular this means that the updated values of m and c have to stay positive. This is one characteristic of the ill-posedness of our problem, since if m or c is not positive definite anymore, we loose the coercivity in our Maxwell system (4.0.1). This would not only affect the numerical stability of our solver, but also ruin the basis of our variational algorithm itself. We control this problem by cutting off the values of m and c after each iteration, if they are below a certain cut-off value. This is justified on physical grounds by the usual presence of earlier measurements of data which allows one to establish a minimum for coefficients to be recovered. We are thus getting a better condition for our algorithm and making it well-posed. The slight disadvantage of introducing a cut-off value is that our algorithm is not a real descent algorithm anymore, but an iterative algorithm and we are not descending that fast. However this is a small price to pay, if we get a stable minimization procedure. In our calculations we always chose a cut-off value between 0.4 and 0.5.

4.2.4 The stopping criterion

An important issue is that of a stopping criterion for our algorithm. Since $H(m, c)$ tends to zero, one might suggest to stop the algorithm if the value of $H(m, c)$ is small enough. However this is not a sufficient criterion since we have no guarantee that if $H(m,c)$ is below a certain value, then the recovered coefficients must be good approximations. Especially in the presence of noise, the minimal value of $H(m,c)$ need not be zero any longer and therefore the above criterion would certainly fail. Another criterion is to measure the norm of the L^2-gradient and if it is small enough, to abort the algorithm since the function $H(m,c)$ has only one local minima. In general one cannot be sure that the gradient does not have a small norm away from the local minima. Nevertheless in this case we know that if $H(m,c)$ tends to zero, the corresponding solutions $E_{m,c,i}$ tend to the given data E_i, $i = 1 \leq i \leq M$. Since the solutions $E_{m,c,i}$ depend continuously on m and c, we can expect satisfactory results, if the difference

$$\sum_i \|E_{m,c,i} - E_i\|^2 \tag{4.2.4}$$

is sufficiently small. Since this is basically the second entry of the L^2-gradient, a small norm of the L^2-gradient also guarantees that the difference of (4.2.4) is small and thus is a good stopping criterion.

4.2.5 The implementation

The data for the implementation of the inverse problem with interior data, consists of M solutions E_i, $1 \leq i \leq M$, of the equation (4.0.1). We construct the data by solving M boundary value problems with the real coefficients μ_r and ϵ_r. All the implementations are done on the cube $\Omega = [-1,1]^3$. We also did an implementation were we constructed the data by solving a time-dependent system and applying M finite Laplace transformations. This added about 1% of extra noise to the data. All our computations are done on single PC with a 3.6 GHz processor and 2GB Ram or on parallel Linux system with 7 2.8 GHz processors and 1 GB Ram. A parallel implementation is the logical way forward since there is natural parallelism in the algorithm. We use a simple remote login setup to connect within the network and solved the arising forward problems in each iteration in parallel. We then save the solutions of these problems to files and load them on a single PC to process them further. Due to the remote login, the saving and loading of the solutions proves to be the most time-expensive part of the parallel implementation and we have to admit that with a more sophisticated parallelization than ours, this could probably be done in a much more efficient way. Nevertheless our implementation is enough to show the effectiveness and the quality of our approach.

Parallel implementation

If one has M solutions E_i, $1 \leq i \leq M$ given, then the natural choice is to use a system with M processors. However since our resources are a bit limited we use 21 solutions E_i and implement the algorithm on a network with 7 processors. We use Comsol Multiphysics (http://www.comsol.com) to solve the various direct problems for Maxwell's equations to obtain the solutions $E_i^{m,c}$ in every step of our descent algorithm. Comsol Multiphysics allows a $3D$ implementation of our Maxwell systems with magnetic or electric boundary conditions. An advantage of Comsol Multiphysics is the easy and straightforward scripting language Comsol Script, which is similar to Matlab and very easy to implement. Comsol Multiphysics also allows the user to solve a time-dependent system, so that we can create our data by solving a time-dependent system and then just do M

Laplace transformations to obtain the solutions E_i, $1 \leq i \leq M$.

The frequency

We choose our Laplace parameters λ_i, such that the wavenumber $k_i = \lambda_i\sqrt{\mu_0\epsilon_0}$ is in the interval $0.8 \leq k_i \leq 3$. The choice of the frequency is quite important, especially in the case of given boundary data, discussed in Chapter 5. A higher frequency usually gives a better resolution of the recovered image. The problem is that if it is too high, the direct problem of Maxwell's equations becomes numerically unstable. Remember that $k = \lambda\sqrt{\mu_o\epsilon_0}$ and consider an application (for example in land-mine detection) in which the background medium is soil. Then the values for the permeability and the permittivity in the background medium are

$$\mu_0 = 1.26 * 10^{-6}\frac{Vs}{Am}, \quad \epsilon_0 = 8.85 * 10^{-12}\frac{As}{Vm}.$$

This yields $k = \frac{\lambda}{3*10^{-8}}$. Thus the frequencies λ_i chosen for our computations were approximately $240 - 900$ MHz, which is realistic if for example ground penetrating radar is used.

Finite elements

In the case of interior data we use a finite element mesh with 21624 tetrahedra. In this case it is sufficient to use linear elements on the tetrahedra. However as we will see later, in the case of given boundary data it is better to use a mesh with quadratic elements. The numerical derivatives arising to get $\nabla \times E_i^{m,c}$ are computed with a finite difference scheme in Comsol Multiphysics which uses central differences. In the case of given boundary data, especially with noise, this is rather problematic, since differentiating itself is an inverse problem and then simple numerical differentiation techniques are often not sufficient. In this case one has to make sure that the algorithm is stable under noise or/and use better differentiation techniques (see for example [KW95]). All integrations are done by Simpson's quadrature rule.

The descent

The line minimization in each descent step should be done by a proper line minimization function like 'Brent' in [PTVF92][Chapter 10]. However this proved to be not efficient in our case. The problem here is, that for each line minimization we would need about $8-10$ iterations and thus we have to compute the gradient $\nabla H(m,c)$ $8-10$ times. This is not feasible on a single computer. Even on a small parallel network this proves inefficient and we developed a heuristic choice of the length α of each descent step. For this we used our experience from the implementation of the inverse problem for elasticity systems ([BJK05]). Here we have seen that length of the descent steps does not change significantly anymore after a certain amount of iterations. Thus we start with a 'good' guess for the length α and reduce it if we do not minimize the functional anymore or increase it if the minimization is not fast enough anymore. In the next section we show a few implementations using interior data.

4.2.6 Results given interior data

In this section we present some numerical results for the recovered Maxwell coefficients μ_r and ϵ_r using interior data. We present implementations for the case of smooth and discontinuous functions. In all implementations we used 21 given solutions E_i. We rescaled our system for these implementations such that $\mu_r|_{\partial\Omega} = \epsilon_r|_{\partial\Omega} = 0.5$ instead of 1. In our first implementation we consider the case of a known $\mu_r = 0.5$ and try to recover ϵ_r. We apply a white noise of 3% to the data. The function we try to recover is

$$\epsilon_{r,1}(x) = (x_1^2 - 1)(x_2^2 - 1)(x_3^2 - 1)^2 + 0.5.$$

The pictures in Figure 4.1 and Figure 4.2 show the true and recovered functions at $x_3 = 0$. We see that the recovery procedure is quite effective and the form as well as the amplitude of the coefficient are approximated quite well.

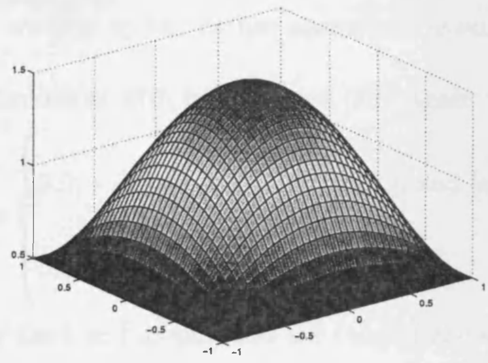

Figure 4.1: True $\epsilon_{r,1}$ at $x_3 = 0$

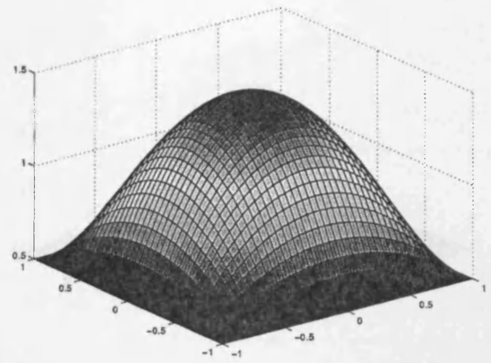

Figure 4.2: Computed $\epsilon_{r,1}$ at $x_3 = 0$, 100 iterations, L^2-error $= 0.041$

In our next implementation we give up the rather unrealistic assumption of smooth coefficients and try to recover a discontinuous ϵ_r with known $\mu_r = 0.5$. Again we apply a noise of 3%.

$$\epsilon_{r,2}(x) = \begin{cases} 2.0, & \text{if } |x_1| < 0.5,\ |x_2| < 0.5 \text{ and } |x_3| < 0.5, \\ 0.5, & \text{otherwise.} \end{cases}$$

The Figures 4.3 and 4.4 show the true function and the recovered coefficient at $x_3 = 0$. Again we

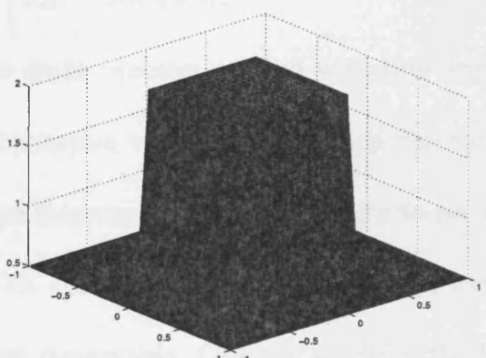

Figure 4.3: True $\epsilon_{r,2}$ at $x_3 = 0$

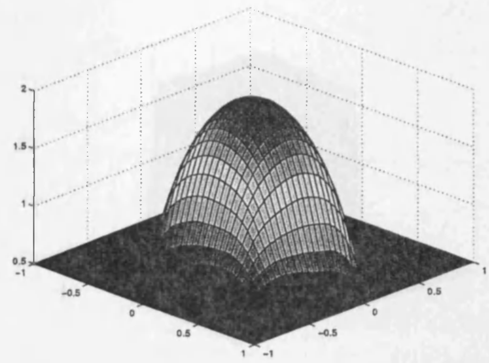

Figure 4.4: Computed $\epsilon_{r,2}$ at $x_3 = 0$, 100 iterations, L^2-error $= 0.167$

can see that not only the support of the coefficient but also the general shape are approximated quite well. The higher L^2-error is due to the lack of smoothness in the coefficient and thus to be expected. Finally we look into the recovery of the coefficient μ_r as well. Again we recover the function

$$\mu_{r,1}(x) = \begin{cases} 2.0, & \text{if } |x_1| < 0.5,\ |x_2| < 0.5 \text{ and } |x_3| < 0.5, \\ 0.5, & \text{otherwise.} \end{cases}$$

We added a noise of 5%. The slight problem we had with this implementation was, that with linear finite elements, the computation of $\nabla \times E_i^{m,c}$ in each descent step introduced additional noise in each step. Nevertheless this was a good opportunity to see if our method still produces satisfying results, even with this amount of noise. As before we look at the recovered function at $x_3 = 0$. The results are quite remarkable. One can see from the Figures 4.5 and 4.6 that the recovery is still very convincing, even with this level of noise in the data.

With these results we end our discussion of variational methods using interior data. We have

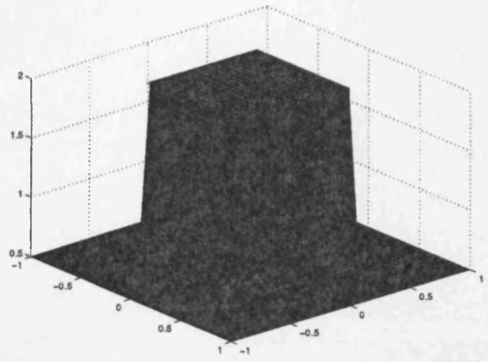

Figure 4.5: True $\mu_{r,1}$ at $x_3 = 0$

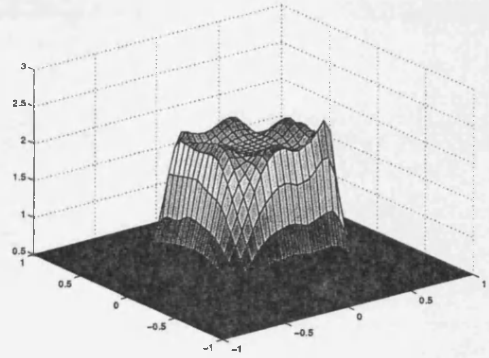

Figure 4.6: Computed $\mu_{r,1}$ at $x_3 = 0$, 80 iterations, L^2-error $= 0.126$

seen that we can recover the coefficients in Maxwell's equations using interior data. These results are still satisfying, even under a high level of white noise. In the next chapter we will present a variational formulation to solve the inverse problem for Maxwell's equations using boundary data.

5 Variational algorithms for given boundary data

After showing the idea of a variational algorithm for given interior data in the previous section, we present a procedure when only boundary data is available.

5.1 The variational algorithm for given boundary data

In this section we outline the variational algorithm to solve the inverse Maxwell problem. Again we consider the system

$$\nabla \times E + k\mu_r H = 0, \tag{5.1.1}$$

$$\nabla \times H - k\epsilon_r E = 0 \tag{5.1.2}$$

and the corresponding second order equation

$$\nabla \times (\mu_r^{-1} \nabla \times E) + k^2 \epsilon_r E = 0, \quad \text{in } \Omega. \tag{5.1.3}$$

The coefficients μ_r and ϵ_r satisfy the conditions

$$0 < \mu_m \leq \mu_r^{-1}(x) \leq \mu_M, \quad 0 < \epsilon_m \leq \epsilon_r(x) \leq \epsilon_M, \quad x \in \Omega \tag{5.1.4}$$

and $k = \lambda\sqrt{\mu_0\epsilon_0}$. Our aim is to recover the parameters μ_r and ϵ_r in (5.1.3) from the knowledge of the impedance map

$$Z_{\mu_r,\epsilon_r}(\boldsymbol{n} \times H|_{\partial\Omega}) = \boldsymbol{n} \times E|_{\partial\Omega}, \tag{5.1.5}$$

or the admittance map

$$\Lambda_{\mu_r,\epsilon_r}(\boldsymbol{n} \times E|_{\partial\Omega}) = \boldsymbol{n} \times H|_{\partial\Omega}. \tag{5.1.6}$$

In what follows we will use the equivalent maps

$$Z_{\mu_r,\epsilon_r}(\boldsymbol{n} \times H|_{\partial\Omega}) = (\boldsymbol{n} \times (\nabla \times H)|_{\partial\Omega}) \times \boldsymbol{n} = \gamma_T(\nabla \times H), \tag{5.1.7}$$

or the admittance map

$$\Lambda_{\mu_r,\epsilon_r}(\boldsymbol{n} \times E|_{\partial\Omega}) = (\boldsymbol{n} \times (\nabla \times E)|_{\partial\Omega}) \times \boldsymbol{n} = \gamma_T(\nabla \times E). \tag{5.1.8}$$

We already know from Theorem 3.5 that Z_{μ,ϵ_r} (and thus $\Lambda_{\mu_r,\epsilon_r}$) uniquely identifies μ_r and ϵ_r. Given the map $\Lambda_{\mu_r,\epsilon_r}$ we now define a functional G on the domain

$$D_G = \{(m,c) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})|\ m|_{\partial\Omega} = c|_{\partial\Omega} = 1,\ m(x) \text{ and } c(x) \text{ satisfy (5.1.4)}\}. \tag{5.1.9}$$

Let $\phi_1, \phi_2, \ldots$ be a basis of $H^{-1/2}(\text{div}; \partial\Omega)$. We set

$$G(m,c) = \sum_{n=1}^{\infty} \theta_n \int_\Omega m\|\nabla \times (E_n^{m,c} - \tilde{E}_n^{m,c})\|^2 + k^2 c\|(E_n^{m,c} - \tilde{E}_n^{m,c})\|^2 dx, \tag{5.1.10}$$

where the $\theta_n > 0$ are chosen such that the series converges. $E_n^{m,c}$ solves (5.1.3) with $\mu_r^{-1} = m$, $\epsilon_r = c$ and $\boldsymbol{n} \times E_n^{m,c}|_{\partial\Omega} = \phi_n$. $\tilde{E}_n^{m,c}$ is the solution of (5.1.3) with $\mu_r^{-1} = m$ and $\epsilon_r = c$ and $\gamma_T(\nabla \times \tilde{E}_n^{m,c})|_{\partial\Omega} = \Lambda_{\mu_r,\epsilon_r}\phi_n$. As in the previous chapter the functional $G(m,c)$ is non-negative.

Theorem 5.1.

For all $m, c \in D_G$ *we have* $G(m,c) \geq 0$ *as well as*

$$G(m,c) = 0 \iff (m,c) = (\mu_r^{-1}, \epsilon_r).$$

Proof.

The first property is obvious. If $G(m,c) = 0$ we have $E_n^{m,c} = \tilde{E}_n^{m,c}$, for all $n \in \mathbb{N}$ and they satisfy the same boundary conditions. Thus we have

$$\Lambda_{m^{-1},c}(\boldsymbol{n} \times E_n^{m,c}) = \gamma_T(\nabla \times E_n^{m,c}) = \gamma_T(\nabla \times \tilde{E}_n^{m,c}) = \Lambda_{\mu_r,\epsilon_r}\phi_n = \Lambda_{\mu_r,\epsilon_r}(\boldsymbol{n} \times E_n^{m,c}),$$

for all $n \in \mathbb{N}$ and thus

$$\Lambda_{m^{-1},c} = \Lambda_{\mu_r,\epsilon_r}.$$

Due to the uniqueness property of Theorem 3.5 we get $(m,c) = (\mu_r^{-1}, \epsilon_r)$. □

As in the case of interior data, we can find a closed form expression for the Gâteaux derivative of G.

Theorem 5.2.

The Gâteaux derivative of $G(m,c)$ *in the direction* (h_m, h_c), *with* $h_m|_{\partial\Omega} = h_c|_{\partial\Omega} = 0$, *is given by*

$$G'(m,c)[h_m,h_c] = \sum_{n=1}^{\infty} \theta_n \int_\Omega h_m(\|\nabla \times E_n^{m,c}\|^2 - \|\nabla \times \tilde{E}_n^{m,c}\|^2) + k^2 h_c(\|E_n^{m,c}\|^2 - \|\tilde{E}_n^{m,c}\|^2)dx.$$

Proof.

We consider

$$
\begin{aligned}
&G(m+\delta h_m, c+\delta h_c) - G(m,c) = \\
&\sum_{n=1}^{\infty} \theta_n \int_\Omega m[(\nabla \times (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c})) \cdot (\nabla \times (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c})) \\
&- (\nabla \times (E_n^{m,c} - \tilde{E}_n^{m,c})) \cdot (\nabla \times (E_n^{m,c} - \tilde{E}_n^{m,c}))] \\
&+ k^2 c((E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c}) \cdot (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c}) \\
&- (E_n^{m,c} - \tilde{E}_n^{m,c}) \cdot (E_n^{m,c} - \tilde{E}_n^{m,c})) \\
&+ \delta h_m (\nabla \times (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c})) \cdot (\nabla \times (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c})) \\
&+ \delta k^2 h_c (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c}) \cdot (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c})\, dx.
\end{aligned}
$$

Using the formula $|a|^2 - |b|^2 = (a-b)\cdot(a+b)$ we can rewrite the above expression as

$$
\begin{aligned}
&G(m+\delta h_m, c+\delta h_c) - G(m,c) = \\
&\sum_{n=1}^{\infty} \theta_n \int_\Omega (\nabla \times ((E_n^{m+\delta h_m, c+\delta h_c} - E_n^{m,c}) - (\tilde{E}_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m,c})))\cdot \\
&\cdot m(\nabla \times ((E_n^{m+\delta h_m, c+\delta h_c} + E_n^{m,c}) - (\tilde{E}_n^{m+\delta h_m, c+\delta h_c} + \tilde{E}_n^{m,c}))) \\
&+ ck^2((E_n^{m+\delta h_m, c+\delta h_c} - E_n^{m,c}) - (\tilde{E}_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m,c}))\cdot \\
&\cdot ((E_n^{m+\delta h_m, c+\delta h_c} + E_n^{m,c}) - (\tilde{E}_n^{m+\delta h_m, c+\delta h_c} + \tilde{E}_n^{m,c}))) \\
&+ \delta h_m (\nabla \times (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c})) \cdot (\nabla \times (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c})) \\
&+ \delta k^2 h_c (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c}) \cdot (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c})\, dx.
\end{aligned}
$$

Now we apply the integration by parts formula (2.19) and get

$$
\begin{aligned}
&G(m+\delta h_m, c+\delta h_c) - G(m,c) = \\
&\sum_{n=1}^{\infty} \theta_n \int_\Omega (E_n^{m+\delta h_m,c+\delta h_c} - E_n^{m,c}) \cdot (\nabla \times (m\nabla \times (E_n^{m+\delta h_m,c+\delta h_c} + E_n^{m,c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m,c}))) \\
&- (E_n^{m+\delta h_m,c+\delta h_c} + E_n^{m,c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m,c}) \cdot (\nabla \times (m\nabla \times (\tilde{E}_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m,c})))dx \\
&+ \int_{\partial\Omega} \underbrace{\gamma_t((E_n^{m+\delta h_m,c+\delta h_c} - E_n^{m,c})}_{=0} \cdot (\gamma_T(m\nabla \times (E_n^{m+\delta h_m,c+\delta h_c} + E_n^{m,c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m,c}))dS \\
&+ \int_{\partial\Omega} \gamma_t(E_n^{m+\delta h_m,c+\delta h_c} + E_n^{m,c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m,c}) \cdot \underbrace{\gamma_T(m\nabla \times (\tilde{E}_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m,c}))}_{=0} dS \\
&+ \int_\Omega k^2 c((E_n^{m+\delta h_m,c+\delta h_c} - E_n^{m,c}) - (\tilde{E}_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m,c})) \cdot \\
&\cdot ((E_n^{m+\delta h_m,c+\delta h_c} + E_n^{m,c}) - (\tilde{E}_n^{m+\delta h_m,c+\delta h_c} + \tilde{E}_n^{m,c}))) \\
&+ \delta h_m(\nabla \times (E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c})) \cdot (\nabla \times (E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c})) \\
&+ \delta k^2 h_c(E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c}) \cdot (E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c})\, dx.
\end{aligned}
$$

We make use of the fact that $E_n^{m+\delta h_m,c+\delta h_c}, \tilde{E}_n^{m+\delta h_m,c+\delta h_c}$ and $E_n^{m,c}, \tilde{E}_n^{m,c}$ satisfy equation (5.1.3) with μ_r^{-1} and ϵ_r replaced by $m + \delta h_m$ and $c + \delta h_c$ or m and c resp. and get

$$
\begin{aligned}
&G(m+\delta h_m, c+\delta h_c) - G(m,c) = \\
&\sum_{n=1}^{\infty} \theta_n \int_\Omega (E_n^{m+\delta h_m,c+\delta h_c} - E_n^{m,c}) \cdot (-\nabla \times (\delta h_m \nabla \times (E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c})) \\
&- k^2(c+\delta h_c)(E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c})) \\
&+ (E_n^{m+\delta h_m,c+\delta h_c} - E_n^{m,c}) \cdot (-c(E_n^{m,c} - \tilde{E}_n^{m,c}))
\end{aligned}
$$

$$- (E_n^{m+\delta h_m,c+\delta h_c} + E_n^{m,c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m,c}) \cdot (-\nabla \times (\delta h_m \nabla \times \tilde{E}_n^{m+\delta h_m,c+\delta h_c})$$

$$- k^2(c + \delta h_c)\tilde{E}_n^{m+\delta h_m,c+\delta h_c})dx$$

$$- (E_n^{m+\delta h_m,c+\delta h_c} + E_n^{m,c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m,c}) \cdot (c\tilde{E}_n^{m,c})$$

$$+ k^2 c((E_n^{m+\delta h_m,c+\delta h_c} - E_n^{m,c}) - (\tilde{E}_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m,c}))\cdot$$

$$\cdot ((E_n^{m+\delta h_m,c+\delta h_c} + E_n^{m,c}) - (\tilde{E}_n^{m+\delta h_m,c+\delta h_c} + \tilde{E}_n^{m,c})))$$

$$+ \delta h_m(\nabla \times (E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c})) \cdot (\nabla \times (E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c}))$$

$$+ \delta k^2 h_c(E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c}) \cdot (E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c})\, dx.$$

As the sum over all terms with the factor c add up to 0 we get

$$G(m + \delta h_m, c + \delta h_c) - G(m, c) =$$

$$\sum_{n=1}^{\infty} \theta_n \int_{\Omega} (E_n^{m+\delta h_m,c+\delta h_c} - E_n^{m,c}) \cdot (-\nabla \times (\delta h_m \nabla \times (E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c}))$$

$$- \delta k^2 h_c(E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c}))$$

$$- (E_n^{m+\delta h_m,c+\delta h_c} + E_n^{m,c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m,c}) \cdot (-\nabla \times (\delta h_m \nabla \times \tilde{E}_n^{m+\delta h_m,c+\delta h_c})$$

$$- \delta k^2 h_c \tilde{E}_n^{m+\delta h_m,c+\delta h_c})$$

$$+ \delta h_m(\nabla \times (E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c})) \cdot (\nabla \times (E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c}))$$

$$+ \delta k^2 h_c(E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c}) \cdot (E_n^{m+\delta h_m,c+\delta h_c} - \tilde{E}_n^{m+\delta h_m,c+\delta h_c})\, dx.$$

After another integration by parts we get

$$
\begin{aligned}
&G(m+\delta h_m, c+\delta h_c) - G(m,c) = \\
&\quad \delta \sum_{n=1}^{\infty} \theta_n \int_\Omega (\nabla \times (E_n^{m+\delta h_m, c+\delta h_c} - E_n^{m,c})) \cdot (-\nabla \times (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c}))) \\
&\quad - (E_n^{m+\delta h_m, c+\delta h_c} - E_n^{m,c})) \cdot (k^2 h_c (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c})) \\
&\quad - (\nabla \times (E_n^{m+\delta h_m, c+\delta h_c} + E_n^{m,c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m,c})) \cdot (-h_m \nabla \times \tilde{E}_n^{m+\delta h_m, c+\delta h_c}) \\
&\quad - (E_n^{m+\delta h_m, c+\delta h_c} + E_n^{m,c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m,c})) \cdot (-k^2 h_c \tilde{E}_n^{m+\delta h_m, c+\delta h_c})) \\
&\quad + h_m (\nabla \times (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c})) \cdot (\nabla \times (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c})) \\
&\quad + k^2 h_c (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c}) \cdot (E_n^{m+\delta h_m, c+\delta h_c} - \tilde{E}_n^{m+\delta h_m, c+\delta h_c})\, dx.
\end{aligned}
$$

We divide the term above by δ and let $\delta \to 0$ to get

$$
\begin{aligned}
&\lim_{\delta \to 0} \frac{G(m+\delta h_m, c+\delta h_c) - G(m,c)}{\delta} \\
&= \sum_{n=1}^{\infty} \theta_n \int_\Omega \underbrace{(\nabla \times (E_n^{m,c} - E_n^{m,c})) \cdot (-\nabla \times (E_n^{m,c} - \tilde{E}_n^{m,c})))}_{=0} \\
&\quad - \underbrace{k^2 h_c (E_n^{m,c} - E_n^{m,c}))(E_n^{m,c} - \tilde{E}_n^{m,c}))}_{=0} \\
&\quad - (\nabla \times (2E_n^{m,c} - 2\tilde{E}_n^{m,c})) \cdot (-h_m \nabla \times \tilde{E}_n) \\
&\quad - (2E_n^{m,c} - 2\tilde{E}_n^{m,c})) \cdot (-k^2 h_c \tilde{E}_n^{m,c}))dx \\
&\quad + h_m (\nabla \times (E_n^{m,c} - \tilde{E}_n^{m,c})) \cdot (\nabla \times (E_n^{m,c} - \tilde{E}_n^{m,c})) \\
&\quad + k^2 h_c (E_n^{m,c} - \tilde{E}_n^{m,c}) \cdot (E_n^{m,c} - \tilde{E}_n^{m,c})\, dx.
\end{aligned}
$$

After reordering the terms we get

$$\lim_{\delta\to 0}\frac{G(m+\delta h_m, c+\delta h_c)-G(m,c)}{\delta}$$
$$=\sum_{n=1}^{\infty}\theta_n\int_\Omega h_m(|\nabla\times E_n^{m,c}|^2-|\nabla\times\tilde{E}_n^{m,c}|^2)+k^2h_c(|E_n^{m,c}|^2-|\tilde{E}_n^{m,c}|^2)\,dx.$$

□

The above theorem enables us to calculate the L^2-gradient of $G(m,c)$ in a closed form. We can also calculate the second Gâteaux derivative of G.

Theorem 5.3 (Second Gâteaux derivative of G).

Let $L_{m,c}^{-1}$ be the inverse of the operator

$$Lu:\nabla\times(m\nabla\times u)+k^2cu,$$

with the boundary condition $\boldsymbol{n}\times u|_{\partial\Omega}=0$ and $\tilde{L}_{m,c}^{-1}$ the inverse of L given $\gamma_T(\nabla\times u)|_{\partial\Omega}=0$. Then if $h_m|_{\partial\Omega}=h_c|_{\partial\Omega}=0$ and $l_m|_{\partial\Omega}=l_c|_{\partial\Omega}=0$ the second Gâteaux derivative of G is given by

$$G''(m,c)[(l_m,l_c),(h_m,h_c)]=\sum_{n=1}^{\infty}\theta_n\int_\Omega 2(\langle(\tilde{L}_{m,c}^{-1}\tilde{d}_n(l_m,l_c)),\tilde{d}_n(h_m,h_c)\rangle$$
$$-\langle(L_{m,c}^{-1}d_n(l_m,l_c)),d_n(h_m,h_c)\rangle)\,dx,$$

where

$$d_n(l_m,l_c)=\nabla\times(l_m\nabla\times E_n^{m,c})+k^2l_cE_n^{m,c},$$

and

$$\tilde{d}_n(l_m,l_c)=\nabla\times(l_m\nabla\times\tilde{E}_n^{m,c})+k^2l_c\tilde{E}_n^{m,c}.$$

$d_n(h_m, h_c)$, and $\tilde{d}_n(h_m, h_c)$ are defined analogously.

Proof.

Let $\delta > 0$. We use the fact that

$$-L_{m,c}E_n^{m,c} + L_{m+\delta l_m, c+\delta l_c}E^{m+\delta l_m, c+\delta l_c} = 0$$

to get

$$L_{m,c}(E^{m+\delta l_m,c+\delta l_c} - E_n^{m,c}) = -\nabla \times (\delta l_m \nabla \times E^{m+\delta l_m,c+\delta l_c}) - k^2 l_c E^{m+\delta l_m,c+\delta l_c}. \qquad (5.1.11)$$

We conclude

$$E^{m+\delta l_m,c+\delta l_c} - E_n^{m,c} = -L_{m,c}^{-1}(\nabla \times (\delta l_m \nabla \times E^{m+\delta l_m,c+\delta l_c}) + k^2 \delta l_c E^{m+\delta l_m,c+\delta l_c}). \qquad (5.1.12)$$

Now

$$G'(m + \delta l_m, c + \delta l_c)(h_m, h_c) - G'(m, c)(h_m, h_c)$$

$$= \sum_{n=1}^{\infty} \theta_n \int_\Omega h_m(\|\nabla \times E_n^{m+\delta l_m,c+\delta l_c}\|^2 - \|\nabla \times \tilde{E}_n^{m+\delta l_m,c+\delta l_c}\|^2)$$

$$+ k^2 h_c(\|E_n^{m+\delta l_m,c+\delta l_c}\|^2 - \|\tilde{E}_n^{m+\delta l_m,c+\delta l_c}\|^2)$$

$$- h_m(\|\nabla \times E_n^{m,c}\|^2 - \|\nabla \times \tilde{E}_n^{m,c}\|^2) - k^2 h_c(\|E_n^{m,c}\|^2 - \|\tilde{E}_n^{m,c}\|^2) dx$$

$$= \sum_{n=1}^{\infty} \int_\Omega h_m(\nabla \times E_n^{m,c} \cdot (\nabla \times E_n^{m+\delta l_m,c+\delta l_c} - \nabla \times E_n^{m,c})$$

$$+ \nabla \times E_n^{m+\delta l_m,c+\delta l_c} \cdot (\nabla \times E_n^{m+\delta l_m,c+\delta l_c} - \nabla \times E_n^{m,c})$$

$$- \nabla \times \tilde{E}_n^{m,c} \cdot (\nabla \times \tilde{E}_n^{m+\delta l_m,c+\delta l_c} - \nabla \times \tilde{E}_n^{m,c})$$

$$+ \nabla \times \tilde{E}_n^{m+\delta l_m, c+\delta l_c} \cdot (\nabla \times \tilde{E}_n^{m+\delta l_m, c+\delta l_c} - \nabla \times \tilde{E}_n^{m,c}))$$

$$+ k^2 h_c((\|E_n^{m+\delta l_m, c+\delta l_c}\|^2 - \|\tilde{E}_n^{m+\delta l_m, c+\delta l_c}\|^2 - \|E_n^{m,c}\|^2 + \|\tilde{E}_n^{m,c}\|^2)\, dx.$$

We integrate by parts and use the formula $a^2 - b^2 = (a-b)(a+b)$ to get

$$G'(m + \delta l_m, c + \delta l_c)(h_m, h_c) - G'(m,c)(h_m, h_c)$$

$$= \sum_{n=1}^{\infty} \int_\Omega (E_n^{m+\delta l_m, c+\delta l_c} - E_n^{m,c}) \cdot (\nabla \times (h_m \nabla \times (E_n^{m+\delta l_m, c+\delta l_c} + E_n^{m,c})))$$

$$- (\tilde{E}_n^{m+\delta l_m, c+\delta l_c} - \tilde{E}_n^{m,c}) \cdot (\nabla \times (h_m \nabla \times (\tilde{E}_n^{m+\delta l_m, c+\delta l_c} + \tilde{E}_n^{m,c})))$$

$$+ (E_n^{m+\delta l_m, c+\delta l_c} - E_n^{m,c}) \cdot (k^2 h_c (E_n^{m+\delta l_m, c+\delta l_c} + E_n^{m,c}))$$

$$- (\tilde{E}_n^{m+\delta l_m, c+\delta l_c} - \tilde{E}_n^{m,c}) \cdot (k^2 h_c (\tilde{E}_n^{m+\delta l_m, c+\delta l_c} + \tilde{E}_n^{m,c}))\, dx.$$

From equation (5.1.12) we get

$$G'(m + \delta l_m, c + \delta l_c)(h_m, h_c) - G'(m,c)(h_m, h_c)$$

$$= \delta \sum_{n=1}^{\infty} \int_\Omega L_{m,c}^{-1} (\nabla \times (l_m \nabla \times E^{m+\delta l_m, c+\delta l_c}) + k^2 l_c E^{m+\delta l_m, c+\delta l_c}) \cdot$$

$$\cdot (\nabla \times (h_m \nabla \times (E_n^{m+\delta l_m, c+\delta l_c} + E_n^{m,c})) + k^2 h_c (E_n^{m+\delta l_m, c+\delta l_c} + E_n^{m,c}))$$

$$- \tilde{L}_{m,c}^{-1} (\nabla \times (l_m \nabla \times \tilde{E}^{m+\delta l_m, c+\delta l_c}) + k^2 l_c \tilde{E}^{m+\delta l_m, c+\delta l_c}) \cdot$$

$$\cdot (\nabla \times (h_m \nabla \times (\tilde{E}_n^{m+\delta l_m, c+\delta l_c} + \tilde{E}_n^{m,c})) + k^2 h_c (\tilde{E}_n^{m+\delta l_m, c+\delta l_c} + \tilde{E}_n^{m,c}))\, dx.$$

Taking the limit $\delta \to 0$ yields the desired result. □

This result is slightly discouraging, since compared with the case of interior data, the functional G

is generally not convex. However for a successful minimization procedure the following property is sufficient.

Definition 5.4.

Let X be a Banach space. We call a non-negative functional $F : X \mapsto \mathbb{R}$ essentially convex if it satisfies

$$F'(x)h = 0, \ \forall h \in X \iff F(x) = 0.$$

We investigate the convexity properties of G in the next section.

5.2 Essential convexity

In this section we show that there is a lot of reason to believe that the functional G in the last section is essentially convex. Unfortunately we are not able to show the essential convexity of G itself, but we show the essential convexity of a related functional.

5.2.1 Indication of essential convexity of G

In this subsection we present a result which is useful for numerical purposes. We consider the equation (5.1.3)

$$\nabla \times (\mu_r^{-1} \nabla \times E) + k^2 \epsilon_r E = 0, \quad \text{in } \Omega$$

and the corresponding admittance map (5.1.8)

$$\Lambda_{\mu_r,\epsilon_r}(\boldsymbol{n} \times E|_{\partial\Omega}) = \gamma_T(\nabla \times E),$$

which we consider as a map from $H^{1/2}(\text{div};\partial\Omega) \mapsto H^{1/2}(\text{curl};\partial\Omega)$. Theorem 2.28 states that the operator $\Lambda_{\mu_r,\epsilon_r}$ describes a well-defined isomorphism. $\Lambda_{\mu_r,\epsilon_r}$ is even coercive as can be seen from the result below.

Theorem 5.5.

Let $\mu_r^{-1}(x) \geq \mu_m > 0$ and $\epsilon_r(x) \geq \epsilon_m$. Then $\Lambda_{\mu_r,\epsilon_r} : H^{1/2}(\text{div};\partial\Omega) \mapsto H^{1/2}(\text{curl};\partial\Omega)$ is a positive operator with respect to the L^2-inner product.

Proof.

Let E be the solution of (5.1.3) and boundary condition

$$\boldsymbol{n} \times E|_{\partial\Omega} = f.$$

Then

$$\begin{aligned}
\int_{\partial\Omega} (\Lambda_{\mu_r,\epsilon_r} f) f dS &= \int_{\partial\Omega} \langle \gamma_T(\nabla \times E), \gamma_t(E) \rangle dS \\
&= \int_\Omega \mu_r^{-1} \langle \nabla \times E, \nabla \times E \rangle dx - \int_\Omega \langle \nabla \times (\mu_r^{-1} \nabla \times E), E \rangle dx \\
&= \int_\Omega \mu_r^{-1} \langle \nabla \times E, \nabla \times E \rangle dx + \int_\Omega k^2 \epsilon_r \langle E, E \rangle dx \\
&\geq d \| E \|_{H(curl;\Omega)}
\end{aligned}$$

where $d := \min\{k^2 \epsilon_m, \mu_m\}$. □

One can also show the following monotonicity result.

Lemma 5.6.

Let $\mu_{r,1}^{-1} \geq \mu_{r,2}^{-1}$ *and* $\epsilon_{r,1} \geq \epsilon_{r,2}$. *Then the operator*

$$\Lambda_{\mu_{r,1},\epsilon_{r,1}} - \Lambda_{\mu_{r,2},\epsilon_{r,2}}$$

is non-negative.

Proof.

Let E_i be the solution of (5.1.3) with $\mu_{r,i}$ and $\epsilon_{r,i}$, $i = 1,2$ instead of μ_r and ϵ_r and boundary condition

$$\boldsymbol{n} \times E_i|_{\partial\Omega} = f.$$

Then

$$\begin{aligned}
&\int_{\partial\Omega} (\Lambda_{\mu_{r,1},\epsilon_{r,1}} f - \Lambda_{\mu_{r,2},\epsilon_{r,2}} f) \cdot f dS = \\
&\quad \int_{\Omega} \mu_{r,1}^{-1} \langle \nabla \times E_1, \nabla \times E_1 \rangle - \langle \nabla \times (\mu_{r,1}^{-1} \nabla \times E_1), E_1 \rangle dx \\
&\quad - \int_{\Omega} \mu_{r,2}^{-1} \langle \nabla \times E_2, \nabla \times E_2 \rangle - \langle \nabla \times (\mu_{r,2}^{-1} \nabla \times E_2), E_2 \rangle dx = \\
&\quad \int_{\Omega} \mu_{r,1}^{-1} \langle \nabla \times E_1, \nabla \times E_1 \rangle + k^2 \epsilon_{r,1} \langle E_1, E_1 \rangle dx - \int_{\Omega} \mu_{r,2}^{-1} \langle \nabla \times E_2, \nabla \times E_2 \rangle + k^2 \epsilon_{r,2} \langle E_2, E_2 \rangle dx = \\
&\quad \int_{\Omega} \mu_{r,1}^{-1} \langle \nabla \times E_1, \nabla \times E_1 \rangle + k^2 \epsilon_{r,1} \langle E_1, E_1 \rangle \, dx + \mu_{r,2}^{-1} \langle \nabla \times E_2, \nabla \times E_2 \rangle - k^2 \epsilon_{r,2} \langle E_2, E_2 \rangle dx \\
&\quad - \int_{\Omega} 2\mu_{r,2}^{-1} \langle \nabla \times E_2, \nabla \times E_1 \rangle \, dx \underbrace{-2 \int_{\Omega} \mu_{r,2}^{-1} \langle \nabla \times E_2, \nabla \times E_2 - \nabla \times E_1 \rangle \, dx}_{:=a}.
\end{aligned}$$

By an integration by parts we can easily see that

$$a = -2 \int_{\Omega} \langle \nabla \times (\mu_{r,2}^{-1} \nabla \times E_2), E_2 - E_1 \rangle dx = 2 \int_{\Omega} k^2 \epsilon_{r,2} \langle E_2, E_2 - E_1 \rangle dx$$

and thus

$$\int_{\partial\Omega}(\Lambda_{\mu_{r,1},\epsilon_{r,1}}f - \Lambda_{\mu_{r,2},\epsilon_{r,2}}f)\cdot f dS =$$

$$\int_{\Omega}\mu_{r,1}^{-1}\langle\nabla\times E_1,\nabla\times E_1\rangle + k^2\epsilon_{r,1}\langle E_1,E_1\rangle - 2\mu_{r,2}^{-1}\langle\nabla\times E_2,\nabla\times E_1\rangle$$

$$- 2k^2\epsilon_{r,2}2\langle E_2,E_1\rangle + \mu_{r,2}^{-1}\langle\nabla\times E_2,\nabla\times E_2\rangle + k^2\epsilon_{r,2}\langle E_2,E_2\rangle\, dx.$$

Since

$$\epsilon_{r,1}\langle E_1,E_1\rangle - 2\epsilon_{r,2}\langle E_2,E_1\rangle + \epsilon_{r,2}\langle E_2,E_2\rangle \geq$$

$$\epsilon_{r,2}\langle E_1,E_1\rangle - 2\epsilon_{r,2}\langle E_2,E_1\rangle + \epsilon_{r,2}\langle E_2,E_2\rangle \geq 0$$

and

$$\int_{\Omega}\mu_{r,1}^{-1}\langle\nabla\times E_1,\nabla\times E_1\rangle - 2\mu_{r,2}^{-1}\langle\nabla\times E_2,\nabla\times E_1\rangle + \mu_{r,2}^{-1}\langle\nabla\times E_2,\nabla\times E_2\rangle\, dx =$$

$$\int_{\Omega}\|\mu_{r,2}^{-1/2}\nabla\times E_2\|^2 - 2\langle(\mu_{r,2}^{-1/2}\nabla\times E_1,\mu_{r,2}^{-1/2}\nabla\times E_2\rangle + \mu_{r,1}^{-1}\langle\nabla\times E_1,\nabla\times E_1\rangle\, dx =$$

$$\int_{\Omega}\|\mu_{r,2}^{-1/2}\nabla\times E_2 - \mu_{r,2}{-1/2}\nabla\times E_1\|^2 + \mu_{r,1}^{-1}\langle\nabla\times E_1,\nabla\times E_1\rangle - \mu_{r,2}^{-1}\langle\nabla\times E_1,\nabla\times E_1\rangle\, dx$$

$$\geq \int_{\Omega}(\mu_{r,1}^{-1} - \mu_{r,2}^{-1})\langle\nabla\times E_1,\nabla\times E_1\rangle\, dx \geq 0$$

the result follows. □

Let G be defined as in (5.1.10). We know from Theorem 5.2 that the L^2-gradient of G is given by

$$(\nabla G(m,c))_1 = \sum_{n=1}^{\infty}\theta_n(\|\nabla\times E_n^{m,c}\|^2 - \|\nabla\times \tilde{E}_n^{m,c}\|^2), \tag{5.2.1}$$

$$(\nabla G(m,c))_2 = \sum_{n=1}^{\infty} k^2 \theta_n (\|E_n^{m,c}\|^2 - \|\tilde{E}_n^{m,c}\|^2). \tag{5.2.2}$$

Now we can show the following result.

Lemma 5.7.

If $\nabla G(m,c) = 0$ and $m \leq \mu_r^{-1}$ as well as $c \leq \epsilon_r$ (or $m \geq \mu_r^{-1}$ as well as $c \geq \epsilon_r$) then we also have $G(m,c) = 0$ and thus $m = \mu_r^{-1}$ as well as $c = \epsilon_r$.

Proof.

If $\nabla G(m,c) = 0$ then we also have $m(\nabla G(m,c))_1 = c(\nabla G(m,c))_2 = 0$. Thus we also have

$$0 = \sum_{n=1}^{\infty} \theta_n \int_\Omega m(\|\nabla \times E_n^{m,c}\|^2 - \|\nabla \times \tilde{E}_n^{m,c}\|^2) + k^2 c(\|E_n^{m,c}\|^2 - \|\tilde{E}_n^{m,c}\|^2) dx.$$

After an integration by parts we get

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \theta_n \int_{\partial\Omega} \langle \boldsymbol{n} \times E_n^{m,c}, \gamma_T(\nabla \times E_n^{m,c}) \rangle - \langle \boldsymbol{n} \times \tilde{E}_n^{m,c}, \gamma_T(\nabla \times \tilde{E}_n^{m,c}) \rangle \, dS \\ &= \sum_{n=1}^{\infty} \theta_n \int_{\partial\Omega} (\Lambda_{m^{-1},c} - \Lambda_{\mu_r,\epsilon_r} \Lambda_{m^{-1},c}^{-1} \Lambda_{\mu_r,\epsilon_r}) \phi_n \cdot \phi_n \, dS, \end{aligned}$$

where we used that $\Lambda_{m^{-1},c}^{-1} \Lambda_{\mu_r,\epsilon_r} \phi_n = \boldsymbol{n} \times \tilde{E}_n^{m,c}|_{\partial\Omega}$. One easily sees that

$$\Lambda_{m^{-1},c} - \Lambda_{\mu_r,\epsilon_r} \Lambda_{m^{-1},c}^{-1} \Lambda_{\mu_r,\epsilon_r} = (\Lambda_{m^{-1},c} - \Lambda_{\mu_r,\epsilon_r})(\Lambda_{m^{-1},c}^{-1} + \Lambda_{\mu_r,\epsilon_r}^{-1}) \Lambda_{\mu_r,\epsilon_r}.$$

We know that $\Lambda_{m^{-1},c}^{-1}$ and $\Lambda_{\mu_r,\epsilon_r}^{-1}$ are strictly positive as well as $\Lambda_{\mu_r,\epsilon_r}$. Now if $m \geq \mu$, then $\Lambda_{m^{-1},c} - \Lambda_{\mu_r,\epsilon_r}$ is non-negative and thus

$$(\Lambda_{m^{-1},c} - \Lambda_{\mu_r,\epsilon_r})(\Lambda_{m^{-1},c}^{-1} + \Lambda_{\mu_r,\epsilon_r}^{-1}) \Lambda_{\mu_r,\epsilon_r}$$

is non-negative and because of $\theta_n > 0$, for all $n \in \mathbb{N}$ we have

$$\langle (\Lambda_{m^{-1},c} - \Lambda_{\mu_r,\epsilon_r})(\Lambda_{m^{-1},c}^{-1} + \Lambda_{\mu_r,\epsilon_r}^{-1})\Lambda_{\mu_r,\epsilon_r}\phi_n, \phi_n\rangle = 0 \quad \text{for } n = 1, 2, \ldots .$$

Since the ϕ_n are an orthonormal basis and $H^{1/2}(\text{div}; \partial\Omega)$ is dense in $L_t^2(\partial\Omega)$, we get

$$(\Lambda_{m^{-1},c} - \Lambda_{\mu_r,\epsilon_r})(\Lambda_{m^{-1},c}^{-1} + \Lambda_{\mu_r,\epsilon_r}^{-1})\Lambda_{\mu_r,\epsilon_r} = 0.$$

However because $\Lambda_{m^{-1},c}^{-1}$, $\Lambda_{\mu_r,\epsilon_r}^{-1}$ and $\Lambda_{\mu_r,\epsilon_r}$ are strictly positive we get

$$\Lambda_{m^{-1},c} = \Lambda_{\mu_r,\epsilon_r}$$

which means

$$m = \mu_r^{-1}, \quad c = \epsilon_r.$$

□

Although this result is encouraging it is is certainly not enough to guarantee a successful numerical recovery of the coefficients μ_r and ϵ_r since we cannot expect to approximate the coefficients μ_r^{-1} and ϵ_r strictly from below or above. To show that there is a lot of reason to believe that G is essentially convex we show that if we replace the sum in the definition of G by a supremum, the functional is essentially convex.

5.2.2 Essential convexity for a supremum functional

Now we show that a related functional of G is essentially convex. For this we will use the impedance map, i.e. the inverse of the admittance map. Given the map

$$\Lambda^{-1}_{\mu_r,\epsilon_r} : H^{1/2}(\text{curl};\partial\Omega) \mapsto H^{1/2}(\text{div};\partial\Omega)$$

we define a new functional on $D_G \times H^{1/2}(\text{curl};\partial\Omega)$ (see (5.1.9) for a definition of D_G). Let $f \in H^{1/2}(\text{curl};\partial\Omega)$. We set

$$G(m,c,f) = \int_\Omega m\|\nabla \times (E_f^{m,c} - \tilde{E}_f^{m,c})\|^2 + k^2 c\|(E_f^{m,c} - \tilde{E}_f^{m,c})\|^2\, dx,$$

where $E_f^{m,c}$ solves (5.1.3) with $\mu_r^{-1} = m$ and $\epsilon_r = c$ and $\boldsymbol{n} \times E_f^{m,c}|_{\partial\Omega} = \Lambda^{-1}_{\mu_r,\epsilon_r} f$. $\tilde{E}_f^{m,c}$ is the solution of (5.1.3), with $\mu_r^{-1} = m$ and $\epsilon_r = c$ and $\gamma_T(\nabla \times \tilde{E}^{m,c}|_{\partial\Omega}) = f$. We define the functional H by

$$H(m,c) = \sup_{\|f\|\le 1} G(m,c,f). \tag{5.2.3}$$

As before we assume that

$$0 < \mu_m \le \mu_r^{-1}(x) \le \mu_M, \quad 0 < \epsilon_m \le \epsilon_r(x) \le \epsilon_M, \quad x \in \Omega. \tag{5.2.4}$$

Before we prove any results about the functional H, we need some auxiliary results.

Lemma 5.8.

The map $\Lambda^{-1}_{\mu_r,\epsilon_r}$ is symmetric, with respect to the L^2_t-inner product.

Proof.

We consider solutions u and v of (5.1.3) with

$$\gamma_T(\nabla \times u) = f, \quad \gamma_T(\nabla \times v) = g.$$

We get

$$\begin{aligned}
&\int_{\partial\Omega} \langle \Lambda^{-1}_{\mu_r,\epsilon_r} f, g\rangle - \langle f, \Lambda^{-1}_{\mu_r,\epsilon_r} g\rangle dS = \\
&\int_{\partial\Omega} \langle \gamma_t(u), \gamma_T(\nabla \times v)\rangle - \langle \gamma_T(\nabla \times u), \gamma_t(v)\rangle dS = \\
&\int_\Omega \mu_r^{-1}\langle \nabla \times u, \nabla \times v\rangle - \langle u, \nabla \times (\mu_r^{-1}\nabla \times v)\rangle dx \\
&- \int_\Omega \mu_r^{-1}\langle \nabla \times u, \nabla \times v\rangle + \langle \nabla \times (\mu_r^{-1}\nabla \times u), v\rangle dx = \\
&\int_\Omega k^2\epsilon_r\langle u, v\rangle - k^2\epsilon_r\langle v, u\rangle \, dx = 0.
\end{aligned}$$

□

In the following we put a condition on μ_r and ϵ_r which will be crucial in the further analysis.

Condition 5.9.

Let $\mu_r, \epsilon_r \in C^3(\overline{\Omega})$. Let further $\Omega' \subset\subset \Omega$ be simply-connected. We assume that in $\Omega\backslash\overline{\Omega'}$ the coefficients ϵ_r and μ_r are constants and equal to 1, i.e.

$$\mu_r(x) = 1, \quad \epsilon_r(x) = 1, \quad \forall x \in \Omega\backslash\overline{\Omega'}$$

and that Ω' contains the support of $\mu_r - 1$ and $\epsilon_r - 1$.

With this condition we can prove the following result.

Lemma 5.10.

The map $\Lambda^{-1}_{\mu_r,\epsilon_r} - \Lambda^{-1}_{1,1}$ *is compact.*

Proof.

For this proof we factorize $\Lambda^{-1}_{\mu_r,\epsilon_r} - \Lambda^{-1}_{1,1}$ as

$$G(L - L_0).$$

Here $G : H^{1/2}(\text{curl}; \partial\Omega') \mapsto H^{1/2}(\text{div}; \partial\Omega)$ is defined as

$$G\psi = \boldsymbol{n} \times v|_{\partial\Omega},$$

where v is a solution of

$$\nabla \times \nabla \times v + k^2 v = 0, \quad \text{in } \Omega\backslash\overline{\Omega'}, \tag{5.2.5}$$

$$\gamma_T(\nabla \times v|_{\partial\Omega'}) = \psi, \quad \gamma_T(\nabla \times v|_{\partial\Omega}) = 0. \tag{5.2.6}$$

The Operator L is given by

$$Lf = \gamma_T(\nabla \times u_+|_{\partial\Omega'}),$$

where u solves (5.1.3) and u_+ denotes the trace from $\Omega\backslash\overline{\Omega'}$. The operator L_0 is defined analogously by

$$L_0 f = \gamma_T(\nabla \times (u_0)_+|_{\partial\Omega'}),$$

where u_0 solves (5.1.3) with μ_r and ϵ_r replaced by 1 and 1. It is easy to see that $L - L_0$ is a bounded operator. This follows from the well-posedness of the boundary value problem associated with (5.1.3) and the continuity of the trace operator. To show the compactness of

$\Lambda^{-1}_{\mu_r,\epsilon_r} - \Lambda^{-1}_{1,1}$ we show that G is compact. For this we choose a domain Ω'' with $\overline{\Omega'} \subset \Omega''$, $\overline{\Omega''} \subset \Omega$ and C^∞-boundary $\partial\Omega''$. Then we decompose G as $G = G_2 G_1$. The operators G_1 and G_2 are defined as

$$G_1 : H^{1/2}(\mathrm{curl}; \partial\Omega') \mapsto H^{1/2}(\mathrm{curl}; \partial\Omega'')$$

with

$$G_1\psi = \gamma_T(\nabla \times v|_{\partial\Omega''}),$$

where v is a solution of (5.2.5), (5.2.6) in $\Omega\backslash\overline{\Omega'}$ and

$$G_2 : H^{1/2}(\mathrm{curl}; \partial\Omega'') \mapsto H^{1/2}(\mathrm{div}; \partial\Omega)$$

with

$$G_2\varphi = \boldsymbol{n} \times u|_{\partial\Omega},$$

where u solves the boundary value problem (5.2.5), (5.2.6) in $\Omega\backslash\overline{\Omega''}$ with boundary data φ instead of $\Omega\backslash\overline{\Omega'}$ and ψ respectively. It is easy to see that G_1 and G_2 are bounded and we can even show that G_1 is compact. This follows from the fact that a solution of

$$\nabla \times \nabla \times v + k^2 v = 0, \quad \text{in } \Omega\backslash\overline{\Omega'}$$

is also a solution of

$$-\Delta v + k^2 v = 0, \quad \text{in } \Omega\backslash\overline{\Omega'}.$$

Since the above equation is elliptic we can get local regularity results (see Theorem 4.17 in

[McL00]) to show that $v \in H^s(U)$ for every open set U such that $\overline{U} \subset \Omega \backslash \overline{\Omega'}$ and $\partial\Omega'' \subset U$ and any $s \in \mathbb{R}$. Thus the trace $\gamma_T(\nabla \times v) \in H^s(\text{curl}; \partial\Omega'')$ for any $s \in \mathbb{R}$ and the compactness follows now from the Sobolev embedding theorem (Theorem 2.10). □

Corollary 5.11.

Let $\alpha(x)$ and $\beta(x)$ satisfy the condition 5.9. Then the map

$$\Lambda^{-1}_{\mu_r,\epsilon_r} - \Lambda^{-1}_{\alpha,\beta}$$

is compact.

Proof.

Note that

$$\Lambda^{-1}_{\mu_r,\epsilon_r} - \Lambda^{-1}_{\alpha,\beta} = \Lambda^{-1}_{\mu_r,\epsilon_r} - \Lambda^{-1}_{1,1} - \Lambda^{-1}_{\alpha,\beta} + \Lambda^{-1}_{1,1}.$$

□

We want to show that the functional H in (5.2.3) is essentially convex. After an integration by parts we get

$$G(m,c,f) = \int_{\partial\Omega} \langle \gamma_t(E_f^{m,c} - \tilde{E}_f^{m,c}), \gamma_T(\nabla \times (E_f^{m,c} - \tilde{E}_f^{m,c}))\rangle = \int_{\partial\Omega} f \cdot R_{m,c} f, \qquad (5.2.7)$$

where

$$R_{m,c} = (\Lambda^{-1}_{\mu_r,\epsilon_r} - \Lambda^{-1}_{m,c})\Lambda_{m,c}(\Lambda^{-1}_{\mu_r,\epsilon_r} - \Lambda^{-1}_{m,c}).$$

Note that $R_{m,c}$ is a non-negative, compact and symmetric operator with respect to the L^2-inner product. The non-negativity follows from Theorem 5.5 and the symmetry and compactness follow from Lemma 5.8 and Corollary 5.11. By applying Theorem 2.8 and Corollary 2.9 we see that R_m allows a spectral decomposition in $L^2_t(\partial\Omega)$ and thus

$$H(m,c) = \sup_{\|f\|\leq 1} G(m,c,f) = G(m,c,f_{m,c}) = \lambda_{m,c} = \|R_{m,c}\|, \tag{5.2.8}$$

where $\lambda_{m,c}$ is the largest eigenvalue of $R_{m,c}$ and $f_{m,c}$ is the corresponding normalized eigenvector. The Gâteaux derivative of the functional H is similar to the Gâteaux derivative of the functional G of the previous subsection.

Theorem 5.12.

For h_m, $h_c \in L^\infty(\Omega)$ with $h_m = h_c = 0$ in a neighbourhood of $\partial\Omega$ we have

$$H'(m,c)[h_m,h_c] = \int_\Omega h_m(\|\nabla \times E^{m,c}_{fm}\|^2 - \|\nabla \times \tilde{E}^{m,c}_{fm}\|) + h_c k^2(\|E^{m,c}_{fm}\|^2 - \|\tilde{E}^{m,c}_{fm}\|^2)\,dx. \tag{5.2.9}$$

Proof.

The proof is quite technical. The interested reader can find it in the appendix A.1. □

We do not show that if we have $H'(m,c)[h_m,h_c] = 0$ for all h_m, $h_c \in L^\infty(\Omega)$ with $h_m = h_c = 0$ in a neighbourhood of $\partial\Omega$ we also have $H(m,c) = 0$. Instead we show a slightly weaker version. If

$$(\nabla H(m,c))_1 = \|\nabla \times E^m_{f_{m,c}}\|^2 - \|\nabla \times \tilde{E}^m_{f_{m,c}}\|^2 = 0,$$

$$(\nabla H(m,c))_2 = \|E^m_{f_{m,c}}\|^2 - \|\tilde{E}^m_{f_{m,c}}\|^2 = 0$$

then

$$H(m,c) = 0.$$

Before we show this, we need a few auxiliary results.

Lemma 5.13.

Let $h_m, h_c \in L^\infty(\Omega)$. Let $E_f^{m,c}$ be the solution of (5.1.3) and boundary condition

$$\boldsymbol{n} \times E_f^{m,c}|_{\partial\Omega} = \Lambda_{\mu_r,\epsilon_r}^{-1} f$$

and $\tilde{E}_f^{m,c}$ the solution of equation (5.1.3) with boundary condition

$$m\boldsymbol{n} \times (\nabla \times E_f^{m,c}|_{\partial\Omega})|_{\partial\Omega} = f.$$

Then for any fixed $f \in H^{1/2}(\mathrm{curl}; \partial\Omega)$ we have

(i)

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(E_f^{m+\varepsilon h_m, c} - E_f^{m,c}) = P_{m,h_m,f},$$

where $u = P_{m,h_m,f}$ is the unique solution of the boundary value problem

$$\nabla \times (m\nabla \times u) + cu = -\nabla \times (h_m \nabla \times E_f^{m,c}), \quad \text{in } \Omega,$$

$$\boldsymbol{n} \times u|_{\partial\Omega} = 0.$$

(ii)

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\tilde{E}_f^{m+\varepsilon h_m, c} - \tilde{E}_f^{m,c}) = \tilde{P}_{m,h_m,f},$$

where $u = \tilde{P}_{m,h_m,f}$ is the unique solution of the boundary value problem

$$\nabla \times (m\nabla \times u) + cu = -\nabla \times (h_m \nabla \times \tilde{E}_f^{m,c}), \quad \text{in } \Omega,$$

$$m\boldsymbol{n} \times (\nabla \times u|_{\partial\Omega}) \times \boldsymbol{n} = -h_m \boldsymbol{n} \times (\nabla \times u|_{\partial\Omega}) \times \boldsymbol{n}.$$

(iii)

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(E_f^{m,c+\varepsilon h_c} - E_f^{m,c}) = P_{c,h_c,f},$$

where $u = P_{c,h_c,f}$ *is the unique solution of the boundary value problem*

$$\nabla \times (m \nabla \times u) + cu = -h_c E_f^{m,c}, \quad in\ \Omega,$$

$$\boldsymbol{n} \times u|_{\partial\Omega} = 0.$$

(iv)

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\tilde{E}_f^{m,c+\varepsilon h_c} - \tilde{E}_f^{m,c}) = \tilde{P}_{c,h_c,f},$$

where $u = \tilde{P}_{c,h_c,f}$ *is the unique solution of the boundary value problem*

$$\nabla \times (m \nabla \times u) + cu = -h_c \tilde{E}_f^{m,c}, \quad in\ \Omega,$$

$$m\boldsymbol{n} \times (\nabla \times u|_{\partial\Omega}) \times \boldsymbol{n} = 0.$$

Proof.

The proofs are standard and therefore omitted. A similar proof can be found in [Jai04][Lemma 3.2.10]. □

We point out that in the above Lemma we did not demand $h_m|_{\partial\Omega} = h_c|_{\partial\Omega} = 0$.

Lemma 5.14.

For $f \in H^{1/2}(\text{curl}; \partial\Omega)$ and $m, c \in D_G$ we define $T_{m,c,f} : H^{1/2}(\text{curl}; \partial\Omega) \mapsto L^1(\Omega) \times L^1(\Omega)$ by

$$T_{m,c,f}(g) = \begin{pmatrix} \langle \nabla \times E_f^{m,c}, \nabla \times E_g^{m,c} \rangle - \langle \nabla \times \tilde{E}_f^{m,c}, \nabla \times \tilde{E}_g^{m,c} \rangle \\ \langle E_f^{m,c}, E_g^{m,c} \rangle - \langle \tilde{E}_f^{m,c}, \tilde{E}_g^{m,c} \rangle \end{pmatrix}.$$

Then $T^*_{m,f} : L^\infty(\Omega) \times L^\infty(\Omega) \mapsto H^{1/2}(\text{div}; \partial\Omega)$ is given by

$$T^*_{m,c,f}(h_m, h_c) = \Lambda^{-1}_{\mu_r,\epsilon_r}(h_m \gamma_t(\nabla \times E_f^m) + m\gamma_T(\nabla \times P_{m,h_m,f})) + \nabla \times P_{c,h_c,f} + \gamma_t(\tilde{P}_{m,h_m,f} + \tilde{P}_{m,h_m,f})$$

and

(i) $T_{m,c,f_{m,c}}(f_{m,c}) = \nabla H(m)$;

(ii) for any $g \in H^{1/2}(\text{curl}; \partial\Omega)$ and $m, c \in D_G$ we have $T^*_{m,c,g}(m, c) = B(g)$, where

$$B = \Lambda^{-1}_{\mu_r,\epsilon_r} \Lambda_{m,c} \Lambda^{-1}_{\mu_r,\epsilon_r} - \Lambda^{-1}_{m,c}.$$

(iii) B is symmetric with respect to the L^2-inner product.

(iv)

$$H^{1/2}(\text{curl}; \partial\Omega) = \overline{R(T^*_{m,c,f})} \oplus N(T_{m,c,f}).$$

Proof.

We have

$$\int_\Omega T_{m,c,f}(g)[h_m, h_c]dx =$$

$$\int_\Omega h_m \langle \nabla \times E_f^{m,c}, \nabla \times E_g^{m,c} \rangle - h_m \langle \nabla \times \tilde{E}_f^{m,c}, \nabla \times \tilde{E}_g^{m,c} \rangle dx$$

$$+\int_\Omega h_c\langle E_f^{m,c}, E_g^{m,c}\rangle - h_c\langle \tilde{E}_f^{m,c}, \tilde{E}_g^{m,c}\rangle dx =$$

$$\int_{\partial\Omega}\langle\gamma_t(E_g^{m,c}), h_m\gamma_T(\nabla\times E_f^{m,c})\rangle - \langle\gamma_t(\tilde{E}_g^{m,c}), h_m\gamma_T(\nabla\times \tilde{E}_f^{m,c})\rangle dS$$

$$+\int_\Omega \langle E_g^{m,c}, \nabla\times(h_m\nabla\times E_f^{m,c}) + h_c E_f^{m,c}\rangle - \langle \tilde{E}_g^{m,c}, \nabla\times(h_m\nabla\times \tilde{E}_f^{m,c}) + h_c\tilde{E}_f^{m,c}\rangle dx =$$

$$\int_{\partial\Omega}\langle\gamma_t(E_g^{m,c}), h_m\gamma_T(\nabla\times E_f^{m,c})\rangle - \langle\gamma_t(\tilde{E}_g^{m,c}), h_m\gamma_T(\nabla\times \tilde{E}_f^{m,c})\rangle dS$$

$$+\int_\Omega \langle E_g^{m,c}, -\nabla\times(m\nabla\times P_{m,h_m,f}) - cP_{m,h_m,f} - \nabla\times(m\nabla\times P_{c,h_c,f}) - cP_{c,h_c,f}\rangle dx$$

$$-\int_\Omega \langle \tilde{E}_g^{m,c}, -\nabla\times(m\nabla\times \tilde{P}_{m,h_m,f}) - c\tilde{P}_{m,h_m,f} - \nabla\times(m\nabla\times \tilde{P}_{c,h_c,f}) - c\tilde{P}_{c,h_c,f}\rangle\, dx.$$

An integration by parts gives

$$\int_\Omega T_{m,c,f}(g)[h_m,h_c]dx =$$

$$\int_{\partial\Omega}\langle\gamma_t(E_g^{m,c}), h_m\gamma_T(\nabla\times E_f^{m,c})\rangle - \langle\gamma_t(\tilde{E}_g^{m,c}), h_m\gamma_T(\nabla\times \tilde{E}_f^{m,c})\rangle dS$$

$$-\int_{\partial\Omega}\langle\gamma_T(E_g^{m,c}), m\gamma_t(\nabla\times P_{m,h_m,f})\rangle - \langle\gamma_T(\tilde{E}_g^{m,c}), m\gamma_t(\nabla\times \tilde{P}_{m,h_m,f})\rangle dS$$

$$-\int_{\partial\Omega}\langle\gamma_T(E_g^{m,c}), m\gamma_t(\nabla\times P_{c,h_c,f})\rangle - \langle\gamma_T(\tilde{E}_g^{m,c}), m\gamma_t(\nabla\times \tilde{P}_{c,h_c,f})\rangle dS$$

$$-\int_\Omega \langle E_g^{m,c}, cP_{m,h,f} + cP_{c,h_c,f}\rangle + \langle \tilde{E}_g^{m,c}, c\tilde{P}_{m,h,f} + c\tilde{P}_{c,h_c,f}\rangle dx$$

$$-\int_\Omega m\langle\nabla\times E_g^{m,c}, \nabla\times P_{m,h,f} + \nabla\times P_{c,h_c,f}\rangle - m\langle\nabla\times \tilde{E}_g^{m,c}, \nabla\times \tilde{P}_{m,h,f} + \nabla\times \tilde{P}_{c,h_c,f}\rangle\, dx.$$

Since $\gamma_T(\nabla\times\tilde{P}_{c,h_c,f}) = 0$ implies $\gamma_t\nabla\times\tilde{P}_{c,h_c,f}) = 0$ (see 2.17) and $\gamma_t(P_{m,h_m,f}) = \gamma_t(P_{m,h_c,f}) = 0$

we get after a further integration by parts

$$\int_{\Omega} T_{m,c,f}(g)[h_m, h_c]dx =$$

$$\int_{\partial\Omega} \langle \gamma_t(E_g^{m,c}), h_m\gamma_T(\nabla \times E_f^{m,c})\rangle - \langle \gamma_t(\tilde{E}_g^{m,c}), h_m\gamma_T(\nabla \times \tilde{E}_f^{m,c})\rangle dS$$

$$- \int_{\partial\Omega} \langle \gamma_T(E_g^{m,c}), m\gamma_t(\nabla \times P_{m,h_m,f})\rangle - \langle \gamma_T(\tilde{E}_g^{m,c}), m\gamma_t(\nabla \times \tilde{P}_{m,h_m,f})\rangle dS$$

$$- \int_{\partial\Omega} \langle \gamma_T(E_g^{m,c}), m\gamma_t(\nabla \times P_{c,h_c,f})\rangle dS$$

$$+ \int_{\partial\Omega} \langle (\gamma_t(\tilde{P}_{m,h_m,f} + \tilde{P}_{c,h_c,f}), m\gamma_T(\nabla \times \tilde{E}_g^{m,c})\rangle \, dS.$$

Since $-h_m\gamma_T(\nabla \times \tilde{E}_f^{m,c}) = m\gamma_T(\nabla \times \tilde{P}_{m,h_m,f})$ and

$$\langle \gamma_T(E_g^{m,c}), m\gamma_t(\nabla \times P_{m,h_m,f} + \nabla \times P_{c,h_c,f}\rangle = -\langle \gamma_t(E_g^{m,c}), m\gamma_T(\nabla \times P_{m,h_m,f} + \nabla \times P_{c,h_c,f}\rangle$$

we get

$$\int_{\Omega} T_{m,c,f}(g)[h_m, h_c]dx =$$

$$\int_{\partial\Omega} \langle \gamma_t(E_g^{m,c}), h_m\gamma_T(\nabla \times E_f^{m,c}) + m\gamma_T(\nabla \times P_{m,h_m,f} + \nabla \times P_{c,h_c,f})\rangle dS$$

$$+ \int_{\partial\Omega} \langle m\gamma_T(\nabla \times \tilde{E}_g^{m,c}), \gamma_t(\tilde{P}_{m,h_m,f} + \tilde{P}_{c,h_c,f})\rangle dS =$$

$$\int_{\partial\Omega} \langle \Lambda_{\mu_r,\epsilon_r}^{-1} g, h_m\gamma_T(\nabla \times E_f^{m,c}) + m\gamma_t(\nabla \times P_{m,h_m,f} + \nabla \times P_{c,h_c,f})$$

$$+ \int_{\partial\Omega} \Lambda_{\mu_r,\epsilon_r}(\gamma_t(\tilde{P}_{m,h_m,f} + \tilde{P}_{c,h_c,f}))\rangle dS =$$

$$\int_{\partial\Omega} T^*_{m,c,f}(h_m, h_c)g \, dS.$$

Property (i) is obvious, property (ii) follows from the fact that $P_{m,m,f} = P_{c,c,f} = 0$ and $\tilde{P}_{m,m,f} + \tilde{P}_{c,c,f} = -\tilde{E}_f^{m,c}$ by definition. The symmetry of B is obvious. To prove (iv) we consider a

$g \in R(T^*_{m,c,f})^\perp$. Then for all $h_m, h_c \in L^\infty$ we get

$$\int_\Omega T_{m,c,f}(g)[h_m, h_c]dx = \int_{\partial\Omega} \langle g, T^*_{m,c,f}(h_m, h_c)\rangle dS = 0.$$

If we choose $h_m = sgn \circ (T_{m,c,f}(g))_1$, $h_c = sgn \circ (T_{m,c,f}(g))_2$ we get $\|T_{m,c,f}(g)\|^2 = 0$. Thus $R(T^*_{m,c,f})^\perp \subset N(T_{m,c,f})$. The reverse inclusion is trivial and thus we get (iv). □

Theorem 5.15 (Essential convexity of H).

Let $\nabla H(m,c) = 0$ (as on Page 126) in Ω and let $m = c = 1$ in a neighbourhood of $\partial\Omega$, then we have $H(m,c) = 0$.

Proof.

Let Q denote the orthogonal projection from $H^{1/2}(\text{curl}; \partial\Omega)$ onto $N(T_{m,c,f_{m,c}})$ and $I - Q$ the projection onto $\overline{R(T^*_{m,f_{m,c}})}$. Since $m = \mu_r^{-1}$ in a neighbourhood of $\partial\Omega$ the operator B is compact since we can write B as

$$B = (\Lambda_{m,c} - \Lambda_{\mu_r,\epsilon_r})(\Lambda^{-1}_{m,c} + \Lambda^{-1}_{\mu_r,\epsilon_r})\Lambda^{-1}_{\mu_r,\epsilon_r}.$$

Again we apply Corollary 2.9 to get an eigensystem for B in $L^2_t(\partial\Omega)$. Thus by the Hilbert-Schmidt theorem it has an eigensystem where the eigenvector f_i correspond to the eigenvalues ρ_i and span the range of Q and the set g_i corresponds to the the eigenvalues λ_i that span the range of $I - Q$. Consequently if we set $f_{m,c} = \sum \alpha_i f_i + \sum \beta_i g_i$ then

$$BQf_{m,c} = B\left(\sum \alpha_i f_i\right) = \sum \rho_i \alpha_i f_i = Q(Bf_{m,c}).$$

So for all $g \in H^{1/2}(\text{curl};\partial\Omega)$ we get

$$0 = \int_\Omega T_{m,c,f_{m,c}} Q T^*_{m,c,g}(m,c) \cdot (m,c) dx = \int_{\partial\Omega} Q T^*_{m,c,g}(m,c) T^*_{m,c,f_{m,c}}(m,c)$$
$$= \int_{\partial\Omega} QB(g) \cdot B(f_{m,c}) dS = \int_{\partial\Omega} g \cdot BQBf_{m,c}.$$

Thus we have $BQBf_{m,c} = 0$ and therefore $(QB)^2 f_{m,c} = 0$ and hence

$$\int_{\partial\Omega} QBf_{m,c} \cdot BQf_{m,c} dS = \int_{\partial\Omega} QBf_{m,c} \cdot QBf_{m,c} dS = 0$$

and thus

$$QBf_{m,c} = 0. \tag{5.2.10}$$

Next for all $g \in H^{1/2}(\text{curl};\partial\Omega)$ we have $(I-Q)T^*_{m,c,g}(m,c) = T^*_{m,f_{m,c}}(h_m, h_c)$ for some $h_m, h_c \in L^\infty(\Omega)$. Since $T_{m,c,f_{m,c}}(f_{m,c}) = \nabla H(m,c) = 0$ we get

$$0 = \int_\Omega (h_m, h_c) \cdot T_{m,f_{m,c}}(f_{m,c}) dx = \int_{\partial\Omega} T^*_{m,f_{m,c}}(h_m, h_c) \cdot f_{m,c} dS = \int_{\partial\Omega} (I-Q) T^*_{m,c,g}(m,c) \cdot f_{m,c} dS$$

and because of

$$\int_{\partial\Omega} (I-Q) T^*_{m,c,g}(m,c) \cdot f_{m,c} dS = \int_{\partial\Omega} (I-Q)B(g) \cdot f_{m,c} dS = \int_{\partial\Omega} g \cdot B(I-Q) f_{m,c} dS$$

for all $g \in H^{1/2}(\text{curl};\partial\Omega)$, we get

$$Bf_{m,c} - BQf_{m,c} = 0.$$

From (5.2.10) we conclude

$$Bf_{m,c} = 0. \tag{5.2.11}$$

Now we set $A := \Lambda_{m,c}\Lambda^{-1}_{\mu_r,\epsilon_r}$ and get

$$(A^2 - I)f_{m,c} = 0. \tag{5.2.12}$$

Using

$$R_{m,c}f_{m,c} = Bf_{m,c} + 2(\Lambda^{-1}_{m,c} - \Lambda^{-1}_{\mu_r,\epsilon_r})f_{m,c} = \lambda_{m,c}f_{m,c}$$

we get

$$(\Lambda^{-1}_{m,c} - \Lambda^{-1}_{\mu_r,\epsilon_r})f_{m,c} = \frac{\lambda_{m,c}}{2}f_{m,c}.$$

Consequently

$$H(m,c) = \lambda_{m,c} = \int_{\partial\Omega}(I-A)f_{m,c}\cdot(\Lambda^{-1}_{m,c} - \Lambda^{-1}_{\mu_r,\epsilon_r})f_{m,c}dS = \int_{\partial\Omega}(I-A)f_{m,c}\cdot\frac{\lambda_{m,c}}{2}f_{m,c}dS.$$

If $\lambda_{m,c} > 0$ then from $f_{m,c}\cdot f_{m,c} = 1$ and (5.2.12) it follows that

$$\int_{\partial\Omega}(A+I)f_{m,c}\cdot f_{m,c} = 0.$$

A further application of (5.2.12) gives

$$\int_{\partial\Omega}(A^2 - I + 2(A+I))f_{m,c}\cdot f_{m,c}dS = \int_{\partial\Omega}(A+I)^2 f_{m,c}\cdot f_{m,c}dS = 0.$$

This leads to the important relation

$$Af_{m,c} = -f_{m,c}, \quad \Lambda^{-1}_{m,c}f_{m,c} = -\Lambda^{-1}_{\mu_r,\epsilon_r}f_{m,c}. \tag{5.2.13}$$

Therefore we get

$$\begin{aligned}\lambda_{m,c} = H(m,c) &= \int_{\partial\Omega}(I-A)f_{m,c}\cdot(\Lambda^{-1}_{m,c} - \Lambda^{-1}_{\mu_r,\epsilon_r})f_{m,c}dS \\ &= 2\int_{\partial\Omega}(\Lambda^{-1}_{m,c}f_{m,c} - \Lambda^{-1}_{\mu_r,\epsilon_r}f_{m,c})\cdot f_{m,c}dS\end{aligned}$$

$$= -4 \int_{\partial\Omega} \Lambda_{\mu_r,\epsilon_r}^{-1} f_{m,c} \cdot f_{m,c} dS \leq 0.$$

which is a contradiction. □

So although we cannot show at the moment that the functional G defined in (5.1.10) is essentially convex, we have seen that if we replace the sum in the definition of G with a supremum, the functional becomes 'almost' essentially convex. Thus we deem it likely that G itself is essentially convex. This assumption is also supported by numerical experiments.

We present a numerical recovery procedure for the coefficients μ_r and ϵ_r in Section 5.4. Before we do this we show another important result for the functional G.

5.3 Further results for the functional G

Finally we show that if G (as defined in (5.1.10) tends to zero, the corresponding admittance maps tend to the given data. We first define the appropriate norm for which we want to show convergence.

Definition 5.16.

We define the norm $||\cdot||_\theta$ *as*

$$||A||_\theta = \sup_{\Phi:\Phi=\sum \beta_n \Phi_n, |\beta|\leq\theta} ||A\Phi||_{H^{-1/2}(\text{curl};\partial\Omega)}.$$

Now we can show the following.

Theorem 5.17.

Suppose we have a sequence $(m_t, c_t)_{t\in\mathbb{N}}$ in D_G. If then $G(m_t, c_t) \to 0$ as $t \to \infty$, then also

$$\|\Lambda_{m_t^{-1},c_t} - \Lambda_{\mu_r,\epsilon_r}\|_\theta \to 0.$$

Proof.

$G(m_t, c_t) \to 0$ as $t \to \infty$ implies that

$$\sum_{n=1}^{\infty} \theta_n \|E_n^{m_t,c_t} - \tilde{E}_n^{m_t,c_t}\|^2_{H(\text{curl};\Omega)} \to 0 \quad \text{as} \quad t \to \infty. \tag{5.3.1}$$

From the trace theorem (Theorem 2.18), we get

$$\|\gamma_t(E_n^{m_t,c_t} - \tilde{E}_n^{m_t,c_t})\|_{H^{-1/2}(\text{div};\partial\Omega)} \leq C\|E_n^{m_t,c_t} - \tilde{E}_n^{m_t,c_t}\|^2_{H(\text{curl};\Omega)} \tag{5.3.2}$$

for some constant C. If follows from the definition of $E_n^{m_t,c_t}$ and $\tilde{E}_n^{m_t,c_t}$ that

$$\gamma_t(E_n^{m_t,c_t} - \tilde{E}_n^{m_t,c_t}) = (I - \Lambda^{-1}_{m_t,c_t}\Lambda_{\mu_r,\epsilon_r})\phi_n. \tag{5.3.3}$$

Therefore we can conclude, that

$$\sum_{n=1}^{\infty} \theta_n \|(I - \Lambda^{-1}_{m_t,c_t}\Lambda_{\mu_r,\epsilon_r})\phi_n\|^2_{H^{-1/2}(\text{div};\partial\Omega)} \to 0 \quad \text{as} \quad t \to \infty.$$

Since the sequence (m_t, c_t) converges, it is bounded. Consequently the $\|\Lambda_{m_t,c_t}\|_\gamma$ are uniformly bounded and from the identity,

$$\Lambda_{m_t,c_t} - \Lambda_{\mu_r,\epsilon_r} = \Lambda_{m_t,c_t}(I - \Lambda^{-1}_{m_t,c_t}\Lambda_{\mu_r,\epsilon_r})$$

we get

$$\sum_{n=1}^{\infty} \theta_n \|(\Lambda_{m_t,c_t} - \Lambda_{\mu_r,\epsilon_r})\phi_n\|^2_{H^{-1/2}(\text{div};\partial\Omega)} \to 0.$$

The result now follows from the definition of the norm $\|\cdot\|_\theta$. □

5.4 Numerical Implementation

In this section we present several results to show that our approach is also effective in the case of given boundary data. First we present a few results with global boundary data, where we either recover one or two coefficients. Then we also show results with local boundary data. In all our calculations we use synthetic data. However we never use any clean data, but always data with at least 3% white noise.

5.4.1 The implementation

As with given interior data, we minimize the functional $G(m, c)$ by a conjugate gradient algorithm. As before we need a Neuberger gradient instead of the L^2-gradient, since we require m and c not to change on the boundary. Since we have shown that there is reason to believe that $G(m, c)$ is essentially convex, we do not have to change the general structure of the algorithm. Our given data consists of electric - magnetic boundary value pairs $(\boldsymbol{n} \times E_n|_\Gamma, \gamma_T(\nabla \times E)_n|_\Gamma)$, $1 \leq n \leq N$, where N is a finite number and Γ is either equal to $\partial\Omega$ or an open subset thereof. In all our computations we chose $\Omega = [-0.5, 0.5]^3$. If we use more than one Laplace transformation we get M sets of electric-magnetic boundary value pairs for each frequency λ_j, $1 \leq j \leq M$. We either produce these pairs by choosing time-dependent boundary conditions, solving the corresponding Maxwell systems in Comsol Multiphysics and applying M finite Laplace transformations, or we choose time-independent boundary conditions and then solve the corresponding time-independent Maxwell systems with frequencies λ_j, $1 \leq j \leq M$. As in the case of interior data we use frequencies between 240 MHz and 900 MHz. In all our calculations we specify magnetic boundary conditions on Γ and compute the corresponding electric boundary conditions. We use either polynomials up to degree four or trigonometric functions to specify the magnetic data on Γ. Apart from different given data and a different gradient we need only a different stopping rule to minimize $G(m, c)$. Since we have shown in Theorem 5.16 that the data converges uniformly as $G(m, c)$ tends to zero, we suggest the following stopping rule.

- Check if the norm of the L^2-gradient $\nabla G(m, c)$ is below a certain value.
- If (i) is true, check whether the difference between the given $(\boldsymbol{n} \times E_i|_\Gamma, \gamma_T(\nabla \times E_i)|_\Gamma)$ boundary data and the current data $(\boldsymbol{n} \times E_n^{m,c}|_\Gamma, \gamma_T(\nabla \times E_n^{m,c})|_\Gamma)$ is sufficiently small.

We do not finish all iterations, since we are mainly interested to check whether our approach is effective and often this can often already be seen after a certain amount of iterations. The reason for this was, that a complete recovery of the coefficients could last up to 6 days on a single PC and up to 3 on a parallel system. One reason for this is that the algorithm itself is expensive, especially since we did the computations in $3D$ and had to use quadratic finite elements. We first tried to implement the algorithm using linear elements. However this proved to be insufficient, since especially the computations of derivatives on the boundary of Ω was unsatisfactory in this case. Other reasons for the long computation times are certainly that although Comsol Multiphysics is a very good software, it is certainly not as fast as a solver specially designed for our coercive Maxwell system. The main reason however is that as in the case of interior data we used a very simple parallel implementation with remote logins. This is by no means the most efficient way to implement a parallel version of our algorithm and we believe that with a more sophisticated method, the computation time could be reduced significantly. Our implementations proved to be sufficient to show that our method works very well and that even in the case of data prescribed only on one face of the cube $[-0.5, 0.5]^3$, we can recover the coefficients μ_r and ϵ_r. Thus although the algorithm is too expensive for any applications at the moment this might change with better hardware in the future.

5.4.2 Numerical results using global boundary data

As in the case of interior data we use synthetic boundary data for our implementations. As before we have a certain amount of noise in the data between 3% and 10%. However as can be seen from the following results, the noise does not ruin the recovery of the coefficients. We mainly try to recover discontinuous functions here, since in most applications this is the case. The first implementation is not done in parallel but on a single pc. We recover only ϵ_r here and set $\mu_r = 0.5$. The true ϵ_r is given by

$$\epsilon_{r,1}(x) = \begin{cases} 2.0, & \text{if } |x_1| < 0.3,\ |x_2| < 0.3 \text{ and } |x_3| < 0.3, \\ 0.5, & \text{otherwise.} \end{cases}$$

We use 7 electric-magnetic boundary value pairs and one frequency λ such that $k^2 = 1$. We apply a noise of 3%. We see that after a 100 iterations that the shape of the function is well approximated and also the overall quality of the recovery is sufficient as can be seen from the

L^2-error.

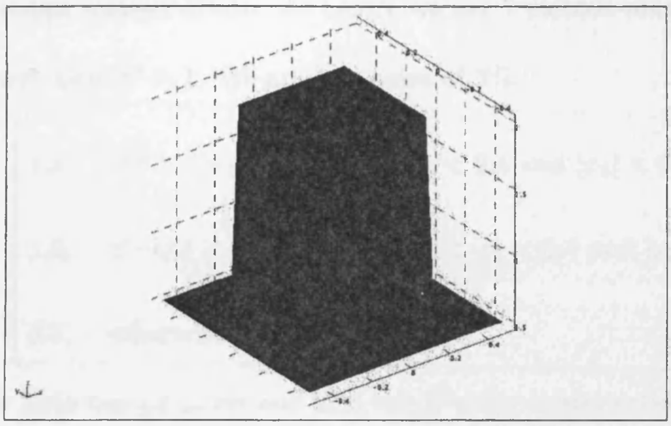

Figure 5.1: True $\epsilon_{r,1}$ at $x_3 = 0$.

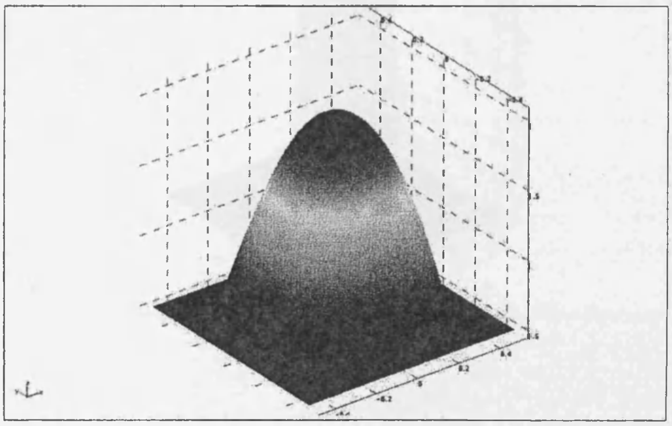

Figure 5.2: Computed $\epsilon_{r,1}$ at $x_3 = 0$, 100 iterations, L^2-error $= 0.114$

Our second implementation was done in parallel. Again we set $\mu_r = 0.5$, but this time we recover a function with two bumps instead of one. As before we use 7 electric-magnetic pairs and only a single frequency λ such that $k^2 = 1$. We apply a noise of 3%.

$$\epsilon_{r,2}(x) = \begin{cases} 1.5, & \text{if } 0.1 < x_1 < 0.4,\ 0.1 < x_2 < 0.4 \text{ and } |x_3| < 0.4, \\ 1.0, & \text{if } -0.4 < x_1 < -0.1,\ -0.1 < x_2 < 0.4 \text{ and } |x_3| < 0.4, \\ 0.5, & \text{otherwise.} \end{cases}$$

We see that we recover both bumps nicely and that the L^2-error is also satisfactory. This is quite encouraging, since it shows that we can also recover two objects at a time.

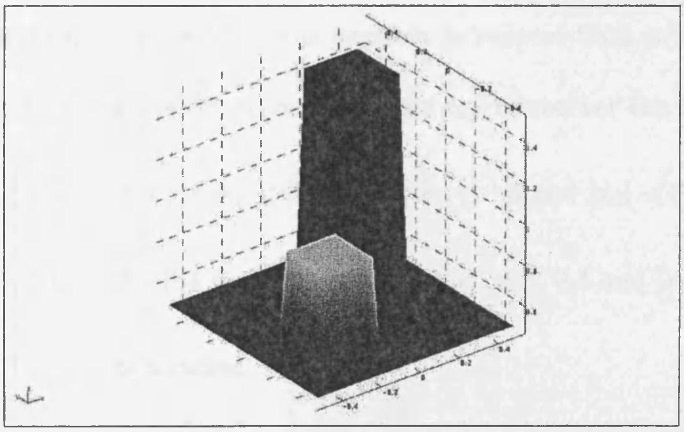

Figure 5.3: True $\epsilon_{r,2}$ at $x_3 = 0$.

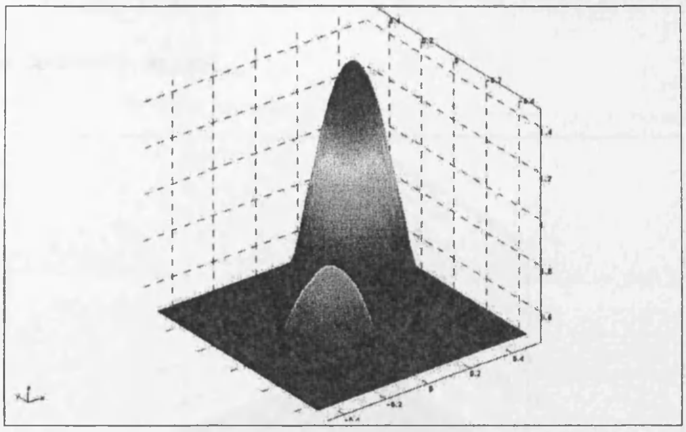

Figure 5.4: Computed $\epsilon_{r,2}$ at $x_3 = 0$, 500 iterations, L^2-error $= 0.106$

In our next implementation we show that it is possible to recover both coefficients μ_r and ϵ_r at the same time. We apply a noise of 5% to our data and try to recover the following coefficients.

$$\epsilon_{r,3}(x) = \begin{cases} 1.5, & \text{if } 0.1 < x_1 < 0.4,\ 0.1 < x_2 < 0.4 \text{ and } |x_3| < 0.4, \\ 1.0, & \text{if } -0.4 < x_1 < -0.1,\ -0.1 < x_2 < 0.4 \text{ and } |x_3| < 0.4, \\ 0.5, & \text{otherwise.} \end{cases}$$

$$\mu_{r,3}(x) = \begin{cases} 2.0, & \text{if } |x_1| < 0.3,\ |x_2| < 0.3 \text{ and } |x_3| < 0.3, \\ 0.5, & \text{otherwise.} \end{cases}$$

As one can see from the Figures 5.6 and 5.8, the recovery of μ_r is by no means as good as the recovery of ϵ_r. However this is due to the fact, that the components of $\nabla G(m, c)$ needed for the update of m in every iteration, involve the computation of $\nabla \times E^{m,c}$ and $\nabla \times \tilde{E}^{m,c}$. Since we only use central differences to compute derivatives and noisy data, this is no surprise. Still the pictures show that the recovery works.

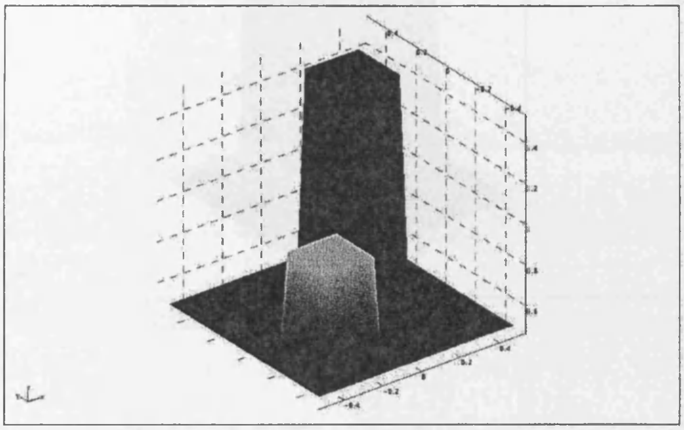

Figure 5.5: True $\epsilon_{r,3}$ at $x_3 = 0$.

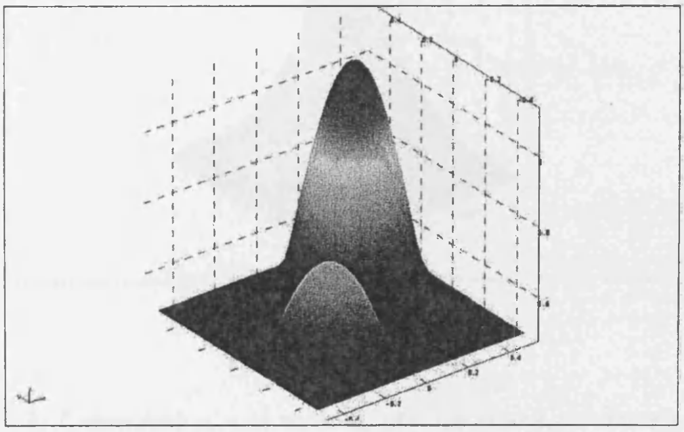

Figure 5.6: Computed $\epsilon_{r,3}$ at $x_3 = 0$, 460 iterations, L^2-error $= 0.102$

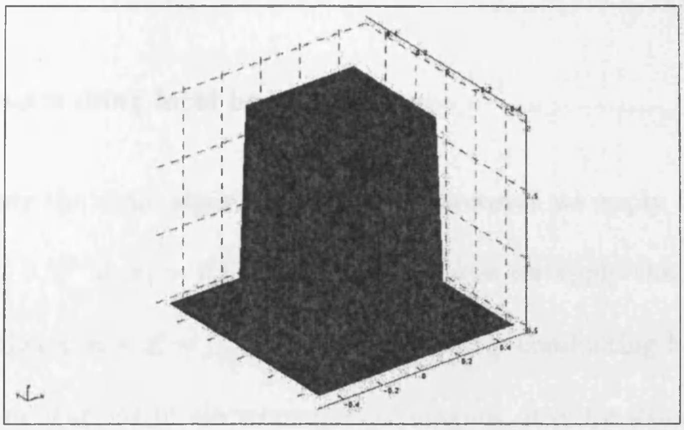

Figure 5.7: True $\mu_{r,3}$ at $x_3 = 0$.

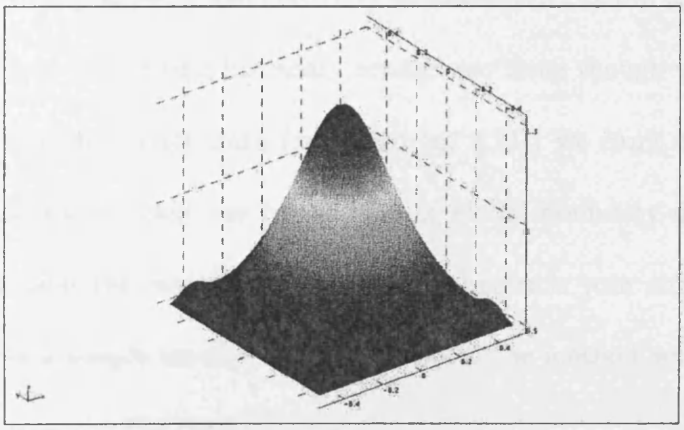

Figure 5.8: Computed $\mu_{r,3}$ at $x_3 = 0$, 460 iterations, L^2-error = 0.297

We see that even with a great amount of noise and without any sophisticated numerical differentiation algorithms our approach produces satisfactory results, even if both coefficients are unknown. Now we want to investigate the more realistic case of given local boundary data.

5.4.3 Numerical results using local boundary data

In this section we apply the same algorithm as before, however we apply the data only on one face of the cube $[-0.5, 0.5]^3$ at $x_3 = 0.5$. On the other faces we apply the perfectly conducting electric boundary condition $\boldsymbol{n} \times E = 0$. Although a perfectly conducting boundary condition is suitable for a lot of applications in electromagnetic imaging, it is for example not suitable for the detection of buried objects like landmines. For these problems it is more realistic to apply symmetric, periodic or some kind of absorbing boundary condition on the other faces. However it is not straightforward to implement these boundary conditions in Comsol Multiphysics and thus we choose simply perfectly conducting boundary conditions. Even though we have a uniqueness result in the case of local boundary data (see Theorem 3.11), we can't expect the recovered coefficients to be of the same quality as in the case of global boundary data. The main rule for inverse problems is that the better your data is, the better is your solution. Thus we first consider the recovery of a simple smooth function to see if the method works in principle. We set $\mu_r = 0.5$ and try to recover the coefficient

$$\epsilon_{r,4} = (x^2 - 1)(y^2 - 1) + 0.5.$$

We apply a noise of 6%. Since we apply the data on the top of the cube $[-0.5, 0.5]^3$ we expect the recovery to be better closer to the face $x_3 = 0.5$. Thus we consider projections at three different levels. We see in Figure 5.10 that we get a good approximation to the true coefficient, however

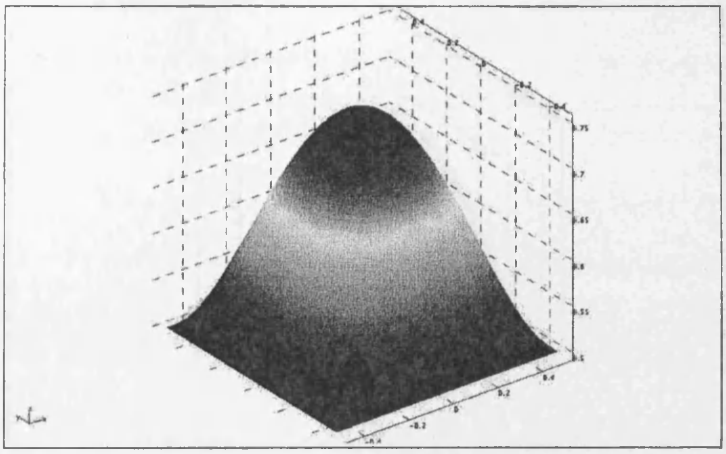

Figure 5.9: True $\epsilon_{r,4}$.

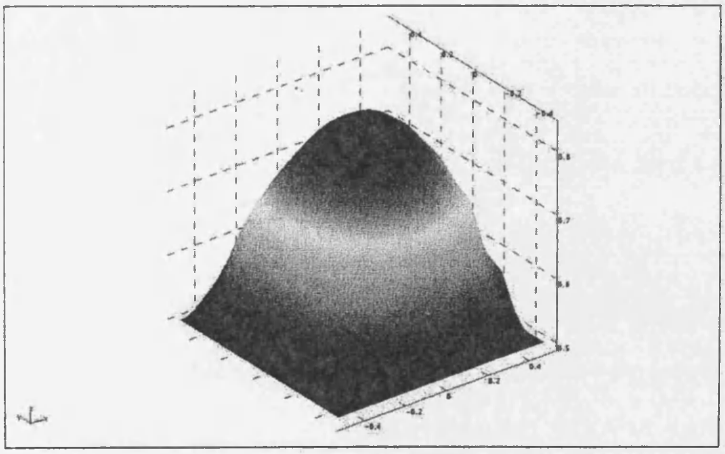

Figure 5.10: Computed $\epsilon_{r,4}$ at $x_3 = 0.35$, 50 iterations, L^2-error $= 0.02$

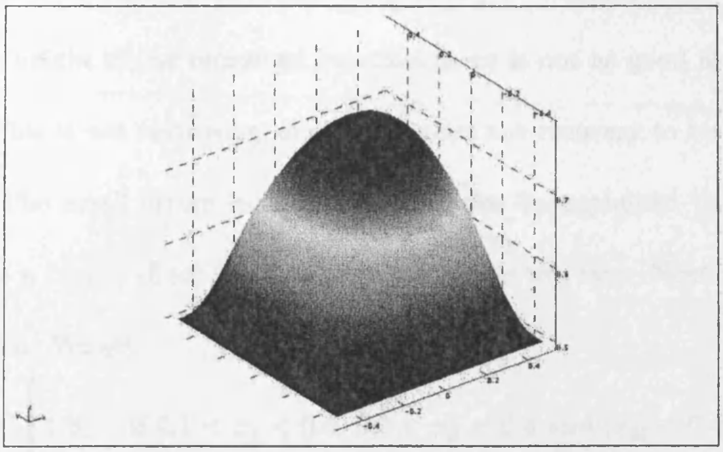

Figure 5.11: Computed $\epsilon_{r,4}$ at $x_3 = 0.23$, 50 iterations

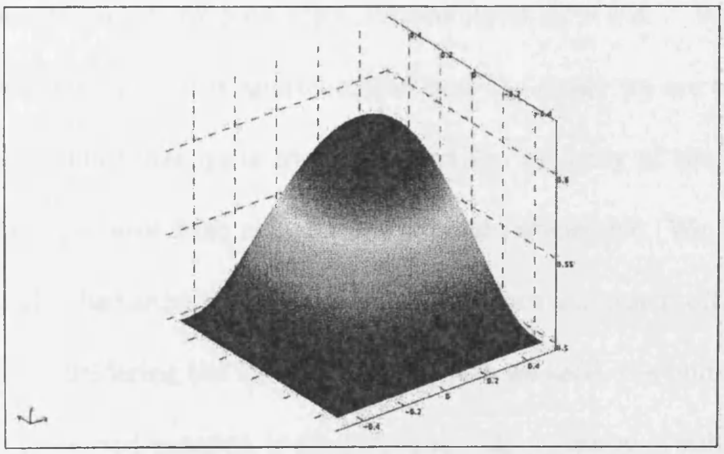

Figure 5.12: Computed $\epsilon_{r,4}$ at $x_3 = 0$, 50 iterations

with a few small errors. At the lower levels the recovery is smoother, however we can see in Figure 5.12 that the height of the recovered function there is not as good as at the projections closer to the data. This is not surprising, since we expect the recovery to be much better closer to the given data. The small errors close to the data can be explained by the fact, that the noise in the data has a bigger effect the closer we are to the top face. Next we try to recover a discontinuous function. We set

$$\epsilon_{r,5}(x) = \begin{cases} 1.5, & \text{if } 0.1 < x_1 < 0.4,\ 0.1 < x_2 < 0.4 \text{ and } |x_3| < 0.4, \\ 1.0, & \text{if } -0.4 < x_1 < -0.1,\ -0.1 < x_2 < 0.4 \text{ and } |x_3| < 0.4, \\ 0.5, & \text{otherwise.} \end{cases}$$

and apply a noise of 6%. Again we look at the recovered function at three different levels. Now this is even more interesting since we have big discontinuity at $x_3 = 0.4$. We see in Figure 5.14 that the height of the true function is approximated best the closer we are to the top face. We also see that the discontinuity has quite some effect on the recovery of the function. However one has to look at these pictures from a three-dimensional perspective. We can see then that a bit further away from the discontinuity the recovery looks much smoother, although the height is not as good. All in all, considering the little amount of data we used, the quite high level of noise and the fact that the recovered function is discontinuous, the recovery is still quite convincing.

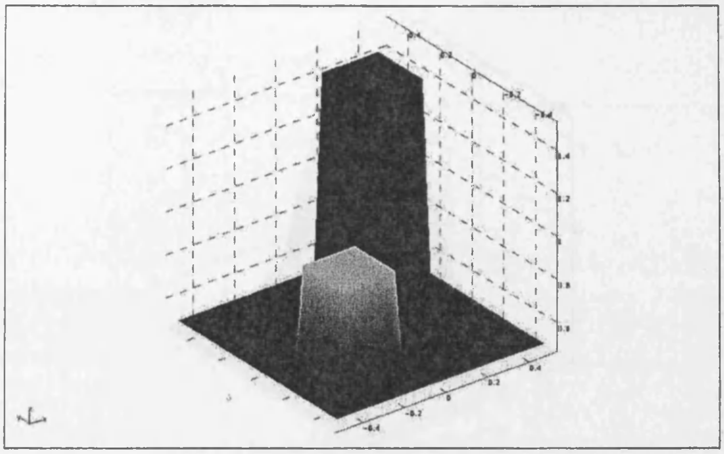

Figure 5.13: True $\epsilon_{r,5}$ for $0.4 > x_3 > -0.4$

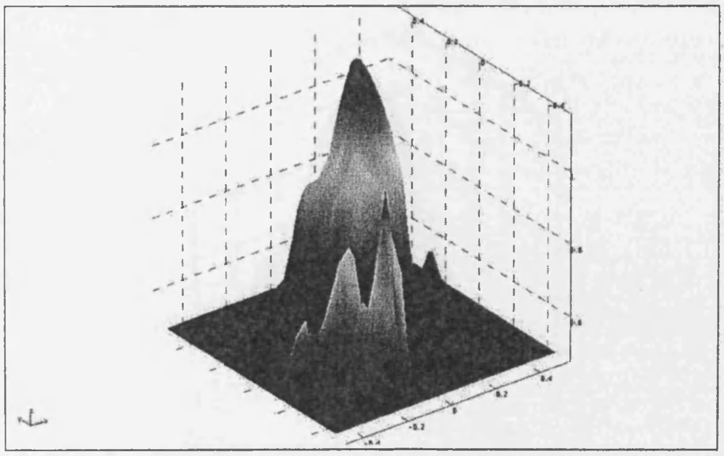

Figure 5.14: Computed $\epsilon_{r,5}$ at $x_3 = 0.35$, 120 iterations, L^2-error $= 0.172$

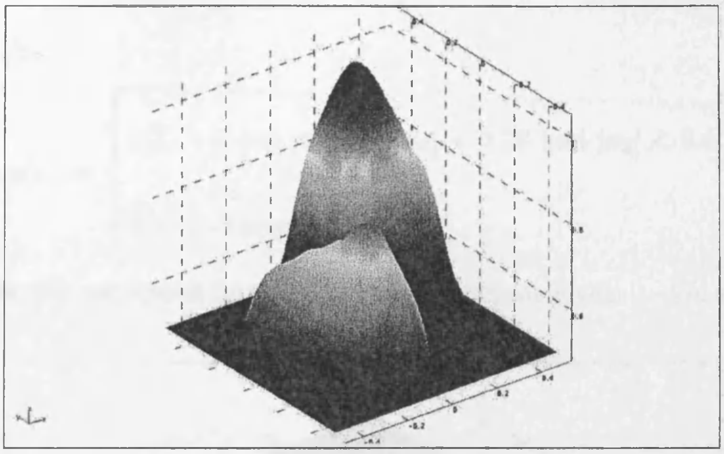

Figure 5.15: Computed $\epsilon_{r,5}$ at $x_3 = 0.23$, 120 iterations

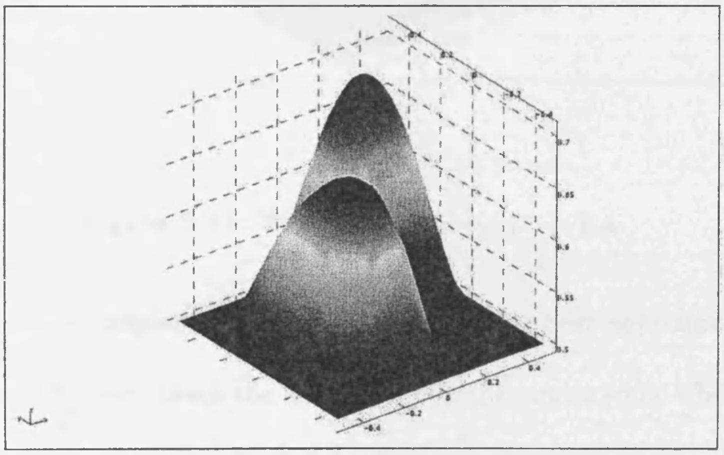

Figure 5.16: Computed $\epsilon_{r,5}$ at $x_3 = 0$, 120 iterations

We confirm these results with another implementation for a discontinuous function. This time we apply a noise of 8%.

$$\epsilon_{r,6}(x) = \begin{cases} 1.2, & \text{if } |x_1| < 0.28,\ |x_2| < 0.28 \text{ and } |x_3| < 0.4, \\ 0.5, & \text{otherwise.} \end{cases}$$

As before we consider the recovered function at three different levels. We see the same effects

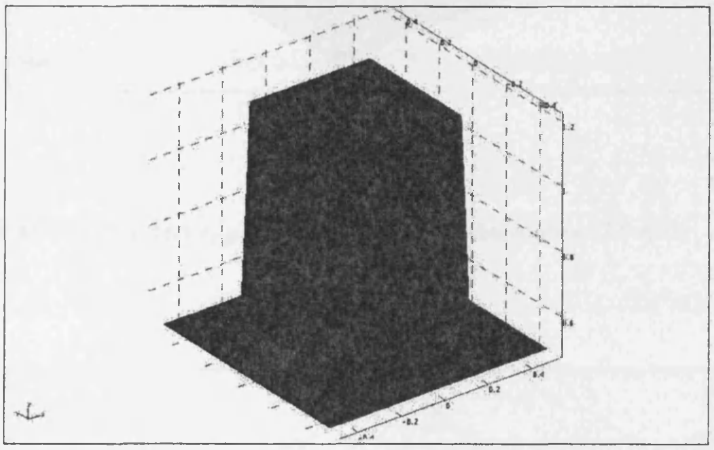

Figure 5.17: True $\epsilon_{r,6}$ for $0.4 > x_3 > -0.4$

as we had with the earlier implementation. The function is best approximated the closer we are to the top surface. However there the discontinuouty has quite some effect on the recovered function and it does not look as smooth as the projections further away from the top surface.

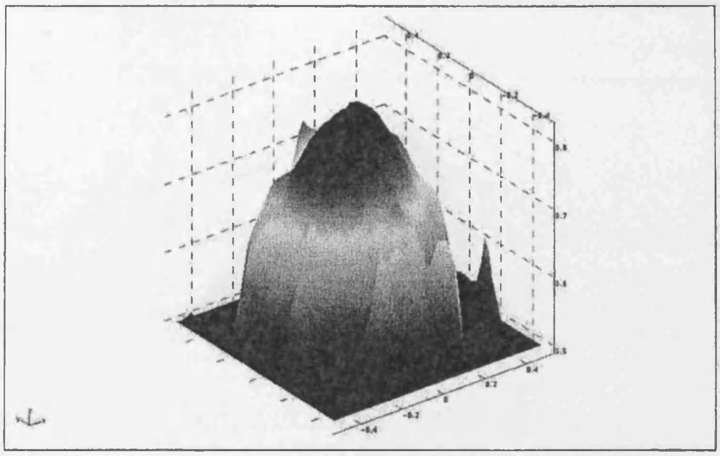

Figure 5.18: Computed $\epsilon_{r,6}$ at $x_3 = 0.35$, 100 iterations, L^2-error $= 0.109$

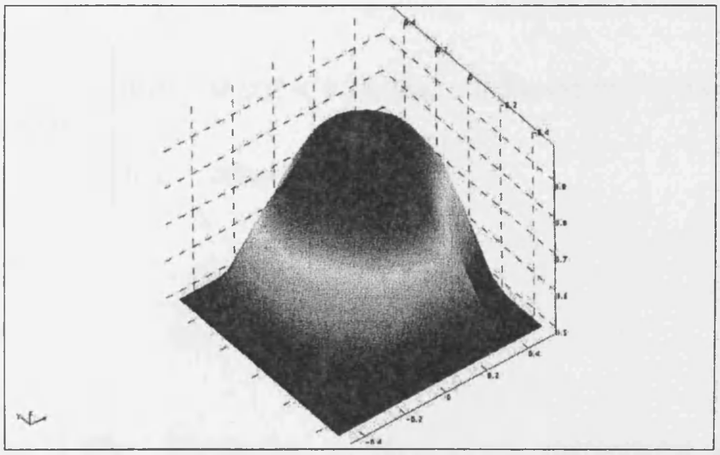

Figure 5.19: Computed $\epsilon_{r,6}$ at $x_3 = 0.23$, 100 iterations

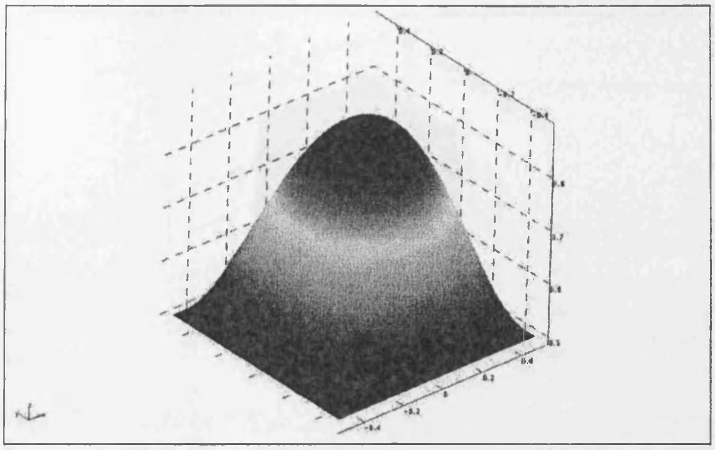

Figure 5.20: Computed $\epsilon_{r,6}$ at $x_3 = 0$, 100 iterations

Finally we present an implementation for the case of two unknown coefficients. We try to recover the coefficients

$$\epsilon_{r,7}(x) = \begin{cases} 1.0, & \text{if } |x_1| < 0.28,\ |x_2| < 0.28 \text{ and } |x_3| < 0.4, \\ 0.5, & \text{otherwise.} \end{cases}$$

and

$$\mu_{r,7} = (x^2 - 1)(y^2 - 1) + 0.5.$$

Again we apply a noise of 8%. We see that as before we can approximate the function ϵ_r quite well even after only 150 iterations. The convergence is again the faster the closer the projection is to the given data. As we have seen in Figure 5.7 and Figure 5.8, the recovery of μ_r works, but is slightly unsatisfactory, due to the error we get in the computation of the functions $\nabla \times E_n^{m,c}$

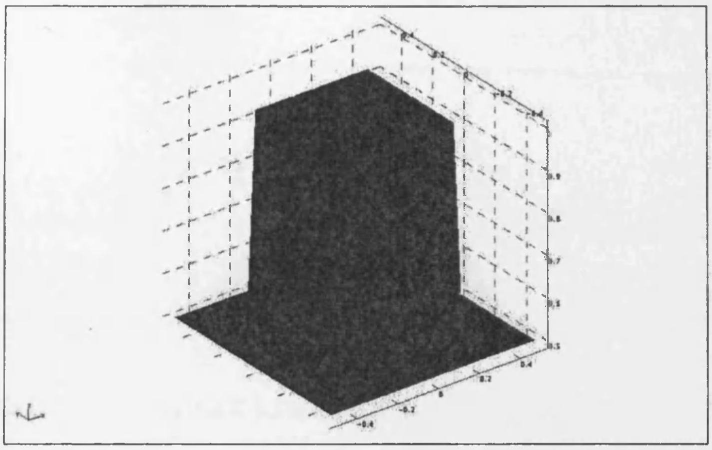

Figure 5.21: True $\epsilon_{r,7}$ for $0.4 > x_3 > -0.4$

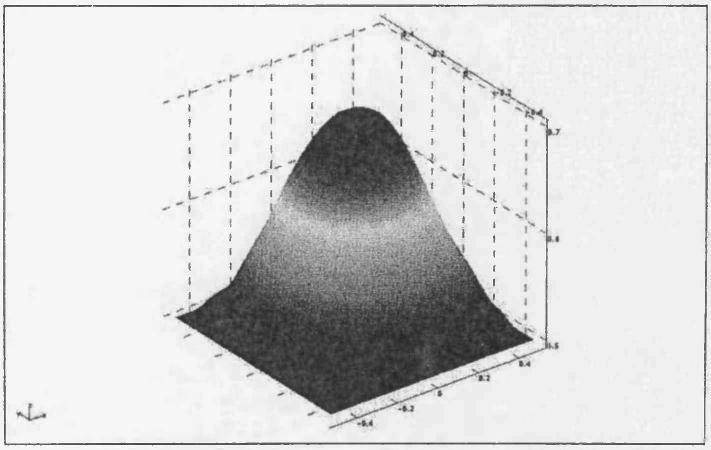

Figure 5.22: Computed $\epsilon_{r,7}$ at $x_3 = 0.35$, 150 iterations, L^2-error $= 0.124$

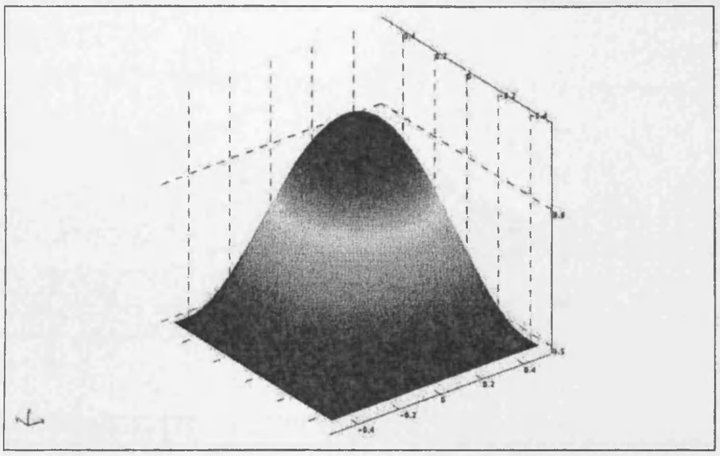

Figure 5.23: Computed $\epsilon_{r,7}$ at $x_3 = 0.23$, 150 iterations

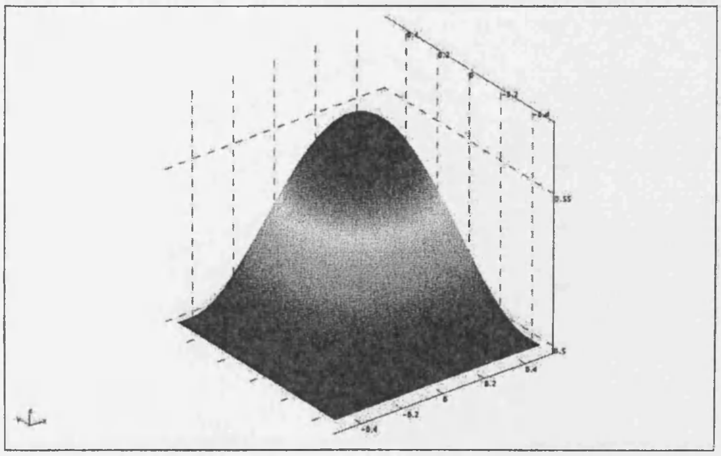

Figure 5.24: Computed $\epsilon_{r,7}$ at $x_3 = 0$, 150 iterations

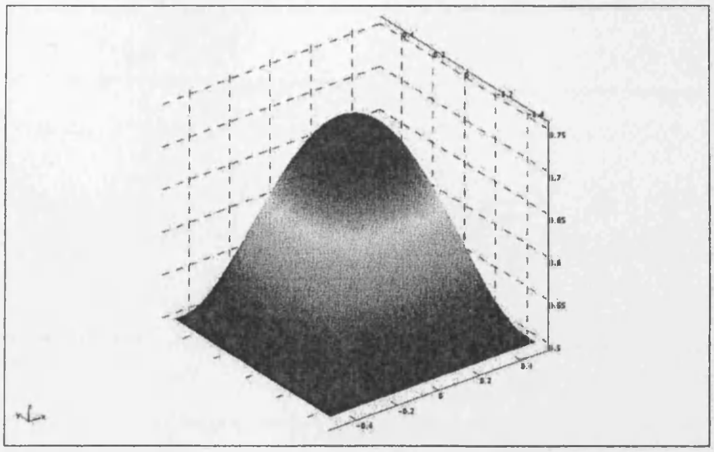

Figure 5.25: True $\mu_{r,7}$ for $0.4 > x_3 > -0.4$

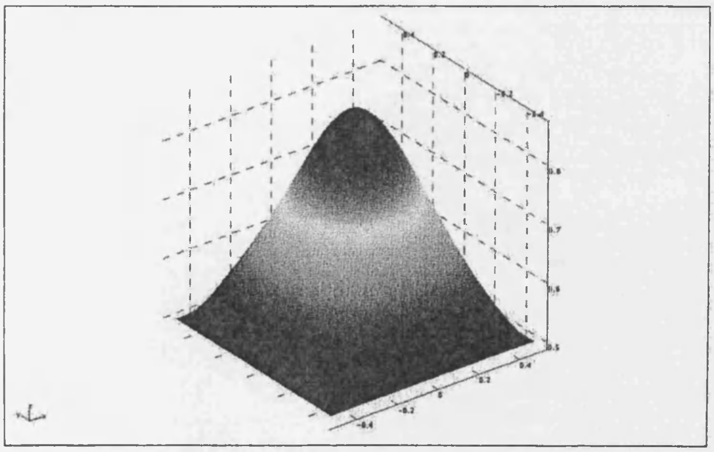

Figure 5.26: Computed $\mu_{r,7}$ at $x_3 = 0.35$, 150 iterations, L^2-error $= 0.048$

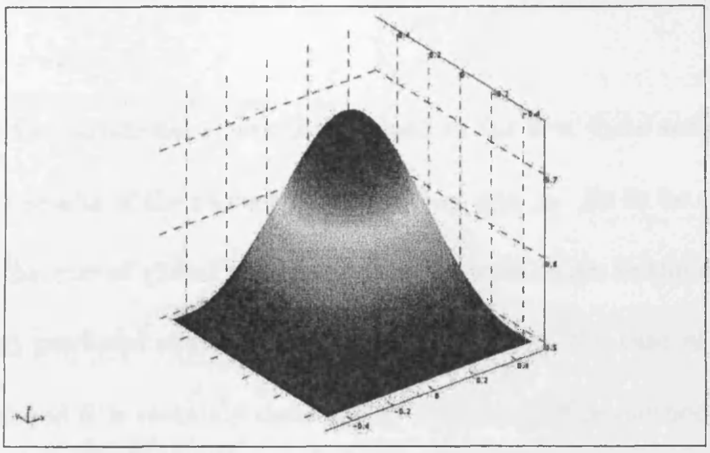

Figure 5.27: Computed $\mu_{r,7}$ at $x_3 = 0.23$, 150 iterations

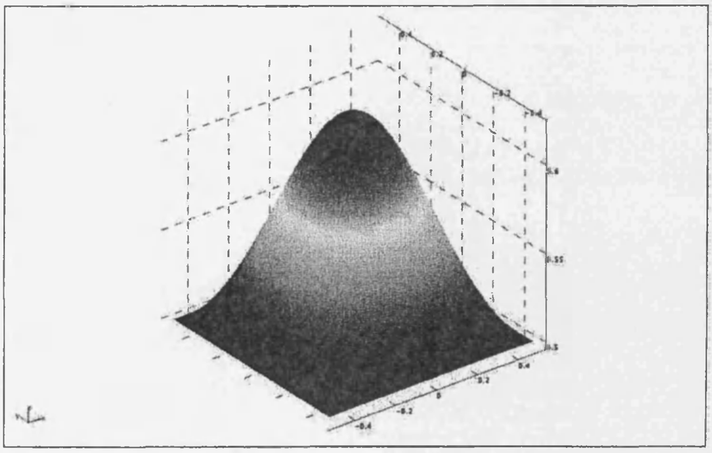

Figure 5.28: Computed $\mu_{r,7}$ at $x_3 = 0$, 150 iterations

and $\nabla \times \tilde{E}_n^{m,c}$. Nevertheless we see that the recovery works well, even in the case of two unknown coefficients.

We have shown that the variational algorithm outlined in the first three sections of this chapter produces satisfactory results of the recovered functions ϵ_r and μ_r. As to be expected the results were a lot better in the case of global boundary data. However even in the case of local boundary data, the method produced convincing results, especially in the case of only one unknown coefficient. As mentioned it is certainly desirable to implement this method in a more efficient parallel environment to improve the quality of the recovery as well as to reduce the computation time.

In the next chapter we discuss an alternative approach to the inverse problem, where we develop a direct method to compute the support of the functions $\mu_r^{-1} - 1$ and $\epsilon_r - 1$.

6 A factorization method

We have seen in the last chapter, that a successful recovery of the coefficients μ_r and ϵ_r can be achieved even under the presence of noise. However we have also seen that this is a very time-intensive process. Although this might be overcome in the future by the use of better hardware, it is still unsatisfactory at the moment. To detect a buried object it is often enough just to determine the support of the coefficients μ_r and ϵ_r. In the last few years two of the most promising methods to emerge to solve these problems are the Linear Sampling Method [CCM04], [CHP03b] and the so called Factorization method [Kir03], [Kir04], [GHK+05]. We want to point out especially the works [Kir04] and [GHK+05] since they deal with inverse problems for Maxwell's equations, however in a different form and using different data than in the present work.

6.1 Introduction

We want to develop a factorization method to determine the support of $\epsilon_r - 1$ and $\mu_r^{-1} - 1$ in the equation

$$\nabla \times (\mu_r^{-1}(x)\nabla \times E) + k^2\epsilon_r(x)E = 0, \quad \text{in } B, \tag{6.1.1}$$

where B is a domain in $\mathbb{R}^3$. As before we make the following assumptions.

- B is a bounded domain in $\mathbb{R}^3$ with a $C^{1,1}$-boundary ∂B.
- $0 < \mu_m \leq \mu_r^{-1}(x) \leq \mu_M, \quad 0 < \epsilon_m \leq \epsilon_r(x) \leq \epsilon_M, \quad x \in B.$

Furthermore we assume $\mu_r(x) = \epsilon_r(x) = 1$, for all $x \in \partial B$ and outside of B. We already know from Theorem 3.5 that the impedance map $Z : H^{-1/2}(\text{curl}; \partial B) \mapsto H^{-1/2}(\text{div}; \partial B)$

$$Z(\gamma_T(\nabla \times E)) = \boldsymbol{n} \times E, \quad \text{on } \partial B$$

uniquely identifies the coefficients μ_r and ϵ_r. To ensure a clear notation, we write Z instead of Z_{μ_r,ϵ_r} in this chapter. We can expect $\mu_r^{-1} \neq 1$ or $\epsilon_r \neq 1$ only in regions where a disturbance of the background medium is located. The basic idea behind the factorization method is to make use of some sort of symmetric or self-adjoint factorization

$$Y = GSG^*, \tag{6.1.2}$$

where Y is the measured data, S is a symmetric or self-adjoint operator and the operator G stores the information of the support of the wanted coefficients. The operators G and G^* are either adjoint or at least dual to each other. In this chapter we consider only bounded operators and therefore symmetry is equivalent to self-adjointness. This leads to the following setting.

6.2 The Factorization

We consider the electric boundary value problem

$$\nabla \times (\mu_r^{-1} \nabla \times E) + k^2 \epsilon_r E = 0, \quad \text{in } B, \tag{6.2.1}$$

$$\gamma_T(\nabla \times E)|_{\partial B} = f \in H^{-1/2}(\text{curl}; \partial B) \tag{6.2.2}$$

and the corresponding impedance map $Z : H^{-1/2}(\text{curl}; \partial B) \mapsto H^{-1/2}(\text{div}; \partial B)$

$$Zf = \boldsymbol{n} \times E, \quad \text{on } \partial B. \tag{6.2.3}$$

We assume that μ_r^{-1} and ϵ_r are of the form

$$\mu_r^{-1}(x) = \begin{cases} 1, & x \in B\backslash\overline{\Omega} \\ 1 + \mu_1(x), & x \in \Omega \end{cases} \tag{6.2.4}$$

$$\epsilon_r(x) = \begin{cases} 1, & x \in B\backslash\overline{\Omega} \\ 1 + \epsilon_1(x), & x \in \Omega. \end{cases} \tag{6.2.5}$$

Here Ω denotes a domain with a $C^{1,1}$-boundary such that $\overline{\Omega} \subset B$ and $B\backslash\overline{\Omega}$ is connected. For the factorization method we define the electric field E_0 as the solution of the boundary value problem

$$\nabla \times \nabla \times E_0 + k^2 E_0 = 0, \quad \text{in } B, \tag{6.2.6}$$

$$\gamma_T(\nabla \times E_0)|_{\partial B} = f \in H^{-1/2}(\text{curl}; \partial B) \tag{6.2.7}$$

and the corresponding impedance map $Z_0 : H^{-1/2}(\text{curl}; \partial B) \mapsto H^{-1/2}(\text{div}; \partial B)$ by

$$Z_0 f = \boldsymbol{n} \times E_0, \quad \text{on } \partial B. \tag{6.2.8}$$

We know from Chapter 2 that both boundary value problems (6.2.1), (6.2.2) and (6.2.6), (6.2.7) have unique solutions in $H(\text{curl}; B)$ and we know from Lemma 2.28 that Z and Z_0 are well-defined isomorphisms. Further we know from Theorem 2.17 that $H^{-1/2}(\text{curl};\partial B)$ and $H^{-1/2}(\text{div};\partial B)$ are dual to each other and from Theorems 5.8 and 5.10 that the map $Z - Z_0$ is self-adjoint and compact. We want to make use of a factorization of $Z - Z_0$ to determine the domain Ω. In order to do this we define the operators

$$G : H^{-1/2}(\text{curl};\partial\Omega) \mapsto H^{-1/2}(\text{div};\partial B), \quad T : H^{-1/2}(\text{div};\partial\Omega) \mapsto H^{-1/2}(\text{curl};\partial\Omega).$$

We set $G\psi = \boldsymbol{n} \times A|_{\partial B}$, where $A \in H(\text{curl}; B\backslash\overline{\Omega})$, with $\boldsymbol{n} \times A|_{\partial B} \in H^{-1/2}(\text{div};\partial B)$ solves

$$\nabla \times \nabla \times A + k^2 A = 0, \quad \text{in } B,\backslash\overline{\Omega} \tag{6.2.9}$$

$$\gamma_T(\nabla \times A) = \psi, \text{ on } \partial\Omega, \quad \boldsymbol{n} \times A = 0 \text{ on } \partial B. \tag{6.2.10}$$

We set $Th = \gamma_T(\nabla \times C_+)$, on $\partial\Omega$ where $C \in H(\text{curl}; B\backslash\overline{\Omega}) \cap H(\text{curl};\Omega)$ and C solves the transmission problem

$$\nabla \times (\mu_r^{-1}\nabla \times C) + k^2\epsilon_r C = 0 \quad \text{in } B\backslash\partial\Omega, \quad \gamma_T(\nabla \times C) = 0 \text{ on } \partial B, \tag{6.2.11}$$

$$\gamma_T(\nabla \times C_+) - \gamma_T(\nabla \times C_-) = 0 \text{ on } \partial\Omega, \quad \boldsymbol{n} \times (C_+) - \boldsymbol{n} \times (C_-) = h \text{ on } \partial\Omega. \tag{6.2.12}$$

Here C_+ denotes the trace from the exterior and C_- the trace from the interior of Ω. Both the problems (6.2.9), (6.2.10) and (6.2.11), (6.2.12) have unique solutions. For (6.2.9), (6.2.10) this follows from Theorem 2.22. Uniqueness for (6.2.11), (6.2.12) can easily be seen as follows. If $Th = 0$, then by (6.2.12) and Corollary 2.31 we have $C \in H(\text{curl}; B)$. Therefore C is the unique solution of (6.2.1), (6.2.2) in $H(\text{curl}; B)$ with $f = 0$, which implies $C = 0$ and thus $h = 0$. To show existence we use the variational formulation of the transmission problem, which is to find

$C \in H(\text{curl}; B\backslash\overline{\Omega}) \cap H(\text{curl}; \Omega)$ with $\boldsymbol{n} \times C|_{\partial B} \in H^{-1/2}(\text{div}; \partial B)$ and $\boldsymbol{n} \times (C_+) - \boldsymbol{n} \times (C_-) = h$ such that

$$\int_\Omega \mu_r^{-1}\langle \nabla \times C, \nabla \times \varphi\rangle + k^2\epsilon_r\langle C, \varphi\rangle dx = 0, \quad \forall \varphi \in H(\text{curl}; B).$$

Existence now follows from the Lax-Milgram Theorem 2.21. We also need the operator

$$T_0 : H^{-1/2}(\text{div}; \partial\Omega) \mapsto H^{-1/2}(\text{curl}; \partial\Omega),$$

which is defined in the same way as T, but with μ_r^{-1} and ϵ_r replaced by 1. Before we present a factorization of the map $Z - Z_0$, we need a few auxiliary results.

Theorem 6.1.

G is compact and one-to-one with dense range. Furthermore the adjoint of G is given by G^* : $H^{-1/2}(\text{curl}; \partial B) \mapsto H^{-1/2}(\text{div}; \partial\Omega)$ with $G^*\varphi = \boldsymbol{n} \times D$, on $\partial\Omega$, where $D \in H(\text{curl}; B\backslash\overline{\Omega})$ satisfies

$$\nabla \times \nabla \times D + k^2 D = 0. \quad \text{in } B\backslash\overline{\Omega},$$

$$\gamma_T(\nabla \times D) = 0 \text{ on } \partial\Omega, \quad \gamma_T(\nabla \times D) = \varphi \text{ on } \partial B.$$

Proof.

We first show the form of G^*. Using the duality of $H^{-1/2}(\text{curl}; \partial\Omega)$ and $H^{-1/2}(\text{div}; \partial\Omega)$ and $H^{-1/2}(\text{curl}; \partial B)$ and $H^{-1/2}(\text{div}; \partial B)$ resp. we get

$$\langle G\psi, \varphi\rangle_{L^2} - \langle \psi, G^*\varphi\rangle_{L^2} =$$

$$\int_{\partial B} \langle \gamma_t(A), \gamma_T(\nabla \times D)\rangle dS + \int_{\partial\Omega} \langle \gamma_T(\nabla \times A), \gamma_t(D)\rangle dS.$$

An integration by parts gives

$$\langle G\psi, \varphi\rangle_{L^2} - \langle \psi, G^*\varphi\rangle_{L^2} =$$

$$\underbrace{\int_{B\setminus\overline{\Omega}} \langle \nabla\times A, \nabla\times D\rangle dx}_{a} - \int_{B\setminus\overline{\Omega}} \langle A, \nabla\times\nabla\times D dx + \underbrace{\int_{\partial\Omega} \langle \gamma_t(A), \gamma_T(\nabla\times D)\rangle dS}_{=0}$$

$$-\underbrace{\int_{B\setminus\overline{\Omega}} \langle \nabla\times D, \nabla\times A\rangle dx}_{b} + \int_{B\setminus\overline{\Omega}} \langle \nabla\times\nabla\times A, D\rangle dx + \underbrace{\int_{\partial B} \langle \gamma_T(\nabla\times A), \gamma_t(D)\rangle dS}_{=0}.$$

We have $a = b$ and thus

$$\langle G\psi, \phi\rangle_{L^2} - \langle \psi, G^*\phi\rangle_{L^2} = \int_{B\setminus\overline{\Omega}} \langle A, -k^2 D\rangle - \langle -k^2 A, D\rangle = 0.$$

The compactness of G was already shown in the proof of Theorem 5.10. Injectivity of G is equivalent to the uniqueness of the Cauchy problem for the equation

$$\nabla\times\nabla\times A + k^2 A = 0, \quad \text{in } B\setminus\overline{\Omega},$$

$$\gamma_T(\nabla\times A) = 0, \text{ on } \partial B, \quad \boldsymbol{n}\times\cdot A = 0 \ \partial B$$

which follows from Theorem 2.33. The density of G is equivalent to the injectivity of G^* which again follows from the uniqueness of the Cauchy problem. □

The next theorem is important for the factorization of $Z - Z_0$.

Theorem 6.2.

Let $\mu_1(x) \geq \alpha > 0$ and $\epsilon_1(x) \geq \alpha > 0$ or $\mu_1(x) \leq \beta < 0$ and $\epsilon_1(x) \leq \beta < 0$ for some $\alpha, \beta \in \mathbb{R}$.

Then there exists a $c \in \mathbb{R}_+$ *such that*

$$\langle (T - T_0)h, h\rangle_{L^2} \geq c\|h\|_{H^{-1/2}(\text{div};\partial\Omega)}, \quad \forall h \in H^{-1/2}(\text{div};\partial\Omega), \tag{6.2.13}$$

or

$$\langle (T_0 - T)h, h\rangle_{L^2} \geq c\|h\|_{H^{-1/2}(\text{div};\partial\Omega)}, \quad \forall h \in H^{-1/2}(\text{div};\partial\Omega) \tag{6.2.14}$$

resp.

Proof. Let $\mu_1(x) \geq \alpha > 0$ and $\epsilon_1(x) \geq \alpha > 0$. Then

$$\langle Th, h\rangle_{L^2} - \langle T_0 h, h\rangle_{L^2} =$$

$$\int_{\partial\Omega} \langle \gamma_T(\nabla \times C_+), \gamma_t(C_+ - C_-)\rangle - \langle \gamma_T(\nabla \times C_{0,+}), \gamma_t(C_{0,+} - C_{0,-})\rangle dS.$$

Integration by parts yields

$$\begin{aligned}
&\langle Th, h\rangle_{L^2} - \langle T_0 h, h\rangle_{L^2} = \\
&- \int_\Omega \langle \nabla \times (\mu_r^{-1}\nabla \times C), C\rangle - \mu_r^{-1}\langle \nabla \times C, \nabla \times C\rangle \, dx \\
&+ \int_\Omega \langle \nabla \times \nabla \times C_0, C_0\rangle - \langle \nabla \times C_0, \nabla \times C_0\rangle \, dx \\
&- \int_{B\setminus\overline{\Omega}} \langle \nabla \times (\mu_r^{-1}\nabla \times C), C\rangle - \mu_r^{-1}\langle \nabla \times C, \nabla \times C\rangle \, dx \\
&+ \int_{B\setminus\overline{\Omega}} \langle \nabla \times \nabla \times C_0, C_0\rangle - \langle \nabla \times C_0, \nabla \times C_0\rangle \, dx = \\
&- \int_B \langle \nabla \times (\mu_r^{-1}\nabla \times C), C\rangle - \mu_r^{-1}\langle \nabla \times C, \nabla \times C\rangle \, dx \\
&+ \int_B \langle \nabla \times \nabla \times C_0, C_0\rangle - \langle \nabla \times C_0, \nabla \times C_0\rangle \, dx =
\end{aligned}$$

$$\int_B k^2\epsilon_r\langle C,C\rangle+\mu_r^{-1}\langle\nabla\times C,\nabla\times C\rangle dx-\int_B k^2\langle C_0,C_0\rangle+\langle\nabla\times C_0,\nabla\times C_0\rangle\,dx=$$

$$\int_B k^2\epsilon_r\langle C,C\rangle-k^2\langle C_0,C_0\rangle dx+\int_B \mu_r^{-1}\langle\nabla\times C,\nabla\times C\rangle-\langle\nabla\times C_0,\nabla\times C_0\rangle\,dx.$$

A few simple calculations give

$$\langle Th,h\rangle_{L^2}-\langle T_0h,h\rangle_{L^2}=$$

$$\int_B k^2\epsilon_r\langle C,C\rangle-k^2\langle C_0,C_0\rangle dx-\int_B 2\langle\nabla\times C_0,\nabla\times(C_0-C)\rangle\,dx$$

$$+\int_B \mu_r^{-1}\langle\nabla\times C,\nabla\times C\rangle-2\langle\nabla\times C,\nabla\times C_0\rangle+\langle\nabla\times C_0,\nabla\times C_0\rangle\,dx=$$

$$\int_B k^2\epsilon_r\langle C,C\rangle-k^2\langle C_0,C_0\rangle dx+\int_B 2k^2\langle C_0,C_0-C\rangle\,dx$$

$$+\int_B \mu_r^{-1}\langle\nabla\times C,\nabla\times C\rangle-2\langle\nabla\times C,\nabla\times C_0\rangle+\langle\nabla\times C_0,\nabla\times C_0\rangle\,dx=$$

$$\int_B k^2\epsilon_r\langle C,C\rangle+k^2\langle C_0,C_0\rangle dx-2k^2\langle C_0,C\rangle\,dx$$

$$+\int_B \mu_r^{-1}\langle\nabla\times C,\nabla\times C\rangle-2\langle\nabla\times C,\nabla\times C_0\rangle+\langle\nabla\times C_0,\nabla\times C_0\rangle\,dx.$$

Using $\epsilon_1\geq\alpha>0$, the first integral can be estimated by

$$\int_B k^2(1+\epsilon_1)\langle C,C\rangle+k^2\langle C_0,C_0\rangle dx-2k^2\langle C_0,C\rangle\,dx=$$

$$\int_B \epsilon_1\langle C,C\rangle\,dx+\int_B k^2\|C-C_0\|^2\,dx\geq\alpha\|C\|_{L^2(\Omega)^3}.$$

With $\mu_1\geq\alpha>0$ we get for the second integral

$$\int_B(1+\mu_1)\langle\nabla\times C,\nabla\times C\rangle-2\langle\nabla\times C,\nabla\times C_0\rangle+\langle\nabla\times C_0,\nabla\times C_0\rangle dx=$$

$$\int_B \mu_1|\nabla\times C|^2+|\nabla\times C-\nabla\times C_0\|^2dx\geq\alpha\|\nabla\times C\|_{L^2(\Omega)^3}.$$

So altogether we get

$$\langle Th, h\rangle - \langle T_0 h, h\rangle \geq \alpha \|C\|_{H(\mathrm{curl};\Omega)}.$$

To show the coercivity of $T - T_0$ in $H^{-1/2}(\mathrm{div}; \partial\Omega)$ we argue as follows. If $\langle (T - T_0)h, h\rangle = 0$, then by the above inequality we have $C = 0$ in Ω. Thus $\boldsymbol{n} \times (\nabla \times C_+) = \boldsymbol{n} \times (\nabla \times C_-) = 0$ on $\partial\Omega$, giving $C = 0$ in $B\backslash\overline{\Omega}$ as well. Thus we get $h = \boldsymbol{n} \times C_+ - \boldsymbol{n} \times C_- = 0$. This shows that

$$\langle (T - T_0)h, h\rangle = 0 \geq 0, \quad \forall h \neq 0.$$

Now assume we have a bounded sequence h_j in $H^{-1/2}(\mathrm{div}; \partial\Omega)$ such that

$$\|h_j\|_{H^{-1/2}(\mathrm{div};\partial\Omega)} = 1, \quad \text{and} \quad \langle (T - T_0)h, h\rangle \to 0, \quad j \to \infty.$$

Then $C_j \to 0$ in Ω and thus $\boldsymbol{n} \times (\nabla \times C_{j,+}) = \boldsymbol{n} \times (\nabla \times C_{j,-}) \to 0$, yielding as above that $\boldsymbol{n} \times C_{j,+} - \boldsymbol{n} \times C_{j,-} =\!\!\to 0$ and thus $h_j \to 0$, which is a contradiction. □

We can also show that T and T_0 are self-adjoint.

Theorem 6.3.

The operators T and T_0 are self-adjoint.

Proof.

We show it just for T, the calculation for T_0 is the same. Let $Th = \gamma_T(\nabla \times C_+)$ and let F be the solution corresponding to the transmission problem (6.2.11), (6.2.12) for g instead of h. Then

$$\langle Th, g\rangle_{L^2} - \langle h, Tg\rangle_{L^2} =$$

$$\int_{\partial\Omega} \langle \gamma_T(\nabla \times C_+), \gamma_t(F_+ - F_-)\rangle dS - \int_{\partial\Omega} \langle \gamma_t(C_+ - C_-), \gamma_T(\nabla \times F_+)\rangle dS.$$

An integration by parts and the transmission conditions in (6.2.11) and (6.2.12) give

$$\langle Th, g\rangle_{L^2} - \langle h, Tg\rangle_{L^2} =$$

$$\int_{\partial\Omega} \langle \gamma_t(C_-), \gamma_T(\nabla \times F_+)\rangle - \langle \gamma_T(\nabla \times C_-), \gamma_t(F_-)\rangle dS$$

$$+ \int_{B\setminus\overline{\Omega}} -\langle \nabla \times \nabla \times C, F\rangle + \langle \nabla \times C, \nabla \times F\rangle \, dx + \int_{\partial B} \langle \gamma_T(\nabla \times C), \gamma_t(F)\rangle dS$$

$$+ \int_{B\setminus\overline{\Omega}} \langle C, \nabla \times \nabla \times F\rangle - \langle \nabla \times F, \nabla \times C\rangle \, dx - \int_{\partial B} \langle \gamma_T(\nabla \times F), \gamma_t(C)\rangle dS =$$

$$\int_{B\setminus\overline{\Omega}} k^2 \langle C, F\rangle \, dx + \int_{\partial B} \langle \gamma_T(\nabla \times C), \gamma_t(F)\rangle dS$$

$$- \int_{B\setminus\overline{\Omega}} k^2 \langle C, F\rangle \, dx - \int_{\partial B} \langle \gamma_T(\nabla \times F_+), \gamma_t(C_+)\rangle dS =$$

$$\int_{\partial B} \langle \gamma_T(\nabla \times C), \gamma_t(F)\rangle - \langle \gamma_T(\nabla \times F), \gamma_t(C)\rangle dS =$$

$$\int_B \langle -\nabla \times (\mu_r^{-1} \nabla \times C), F\rangle + \langle \mu_r^{-1} \nabla \times C, \nabla \times F\rangle \, dx$$

$$+ \int_B \langle \nabla \times (\mu_r^{-1} \nabla \times F), C\rangle - \langle \mu_r^{-1} \nabla \times F, \nabla \times C\rangle \, dx = 0.$$

□

Equipped with these results we can prove the desired factorization.

Theorem 6.4.

The following factorization holds:

$$Z - Z_0 = G(T - T_0)G^*. \tag{6.2.15}$$

Proof.

We have already seen in the proof of Lemma 5.10, that we can write

$$Z - Z_0 = G(L - L_0),$$

where $L : H^{-1/2}(\text{curl}; \partial B) \mapsto H^{-1/2}(\text{curl}; \partial\Omega)$ is given by

$$Lf = \gamma_T(\nabla \times E_+|_{\partial\Omega}),$$

where E solves (6.2.1), (6.2.2) and L_0 by

$$L_0 f = \gamma_T(\nabla \times (E_0)_+|_{\partial\Omega}),$$

where E_0 solves (6.2.6), (6.2.7). Two integration by parts show that the adjoint of L is given by $L^* : H^{-1/2}(\text{div}; \partial\Omega) \mapsto H^{-1/2}(\text{div}; \partial B)$ with $Lh = \boldsymbol{n} \times C|_{\partial B}$, where C solves (6.2.11), (6.2.12). Thus we get

$$L^* h = GTh, \quad L_0^* h = GT_0,$$

implying

$$L - L_0 = (T - T_0)G^*,$$

and therefore

$$Z - Z_0 = G(L - L_0) = G(T - T_0)G^*.$$

□

This factorization is the main tool for the determination of Ω.

6.3 Determination of Ω

The idea behind the factorization method is to prove that the range of G determines Ω. If one can additionally show that the ranges of G and $Z - Z_0$ or for example $G^{1/2}$ and $(Z - Z_0)^{1/2}$ coincide then one has shown that the given data, determines Ω and one also gets a direct method to determine all of Ω. We show here only that the range of G determines Ω and thus using the factorization (6.2.15), that the given data determines a subset Ω_1 of Ω. We do this because at the moment we have no proof that the range of G and $Z - Z_0$ coincide.

The task is now to construct singular solutions of (6.1.1) to characterize Ω. For this we need a fundamental solution of

$$\nabla \times \nabla \times E + k^2 E = 0. \qquad (6.3.1)$$

We note that the fundamental solution of the equation

$$-\Delta E + k^2 E = 0 \qquad (6.3.2)$$

is given by $g(x,y)I$, where I is the identity matrix in $\mathbb{R}^3$ and

$$g(x,y) = -\frac{e^{-k\|x-y\|}}{4\pi\|x-y\|}. \tag{6.3.3}$$

Using the relation

$$\nabla \times \nabla \times E = -\Delta E + \nabla\nabla \cdot E,$$

we can conclude that the fundamental solution for (6.3.1) is given by

$$\Phi(x,y) = g(x,y)I - \frac{1}{k^2}\nabla_y \nabla_y \cdot (g(x,y)I). \tag{6.3.4}$$

To construct the magnetic Maxwell function $M(x,y)$ we set

$$f_y(x) = \gamma_T(\nabla_x \times \Phi(x,y)).$$

For a fixed y let E_y be the unique solution of (6.1.1) with boundary condition

$$\gamma_T(\nabla \times E_y) = f_y.$$

Now we set

$$M(x,y) = \Phi(x,y) - E_y(x). \tag{6.3.5}$$

$M(x,y)$ inherits the singular behaviour of $\Phi(x,y)$ i.e. we have

$$\lim_{x \to y} \|M(x,y)\| = \infty. \tag{6.3.6}$$

Let b be a unit vector in $\mathbb{R}^3$. We set

$$\varphi_y(x) = M(x,y)b, \quad x \text{ on } \partial B. \tag{6.3.7}$$

Now we show the following.

Theorem 6.5.

Let φ_y be defined by (6.3.7) for $y \in B$. Then φ_y belongs to the range of G if and only if $y \in \Omega$.

Proof.

If $y \in \Omega$, then obviously, $G\psi = \varphi_y$ for $\psi = M(x,y)b$ on $\partial\Omega$. Now we consider the case $y \notin \Omega$. Assume that $G\psi = \varphi_y$ for some $\psi \in H^{-1/2}(\text{curl};\partial\Omega)$. By $A \in H(\text{curl}; B\backslash\overline{\Omega})$ we denote the corresponding solution of (6.2.9), (6.2.10). We set $E = M(x,y)b$ in $B\backslash\overline{\Omega}$. From $\gamma_T(\nabla \times E) = 0 = \gamma_T(\nabla \times A)$ and $\boldsymbol{n} \times A|_{\partial B} = \varphi_y = \boldsymbol{n} \times E|_{\partial B}$ and the uniqueness of the Cauchy problem, we conclude that $E = A$ in $B\backslash\{\overline{\Omega} \cup b_\varepsilon(y)\}$ for every $\varepsilon > 0$, where $b_\varepsilon(y)$ is the ball around y with radius ε. However due to the fact that A solves the equation

$$-\Delta A + k^2 A, \quad \text{in } B\backslash\{\overline{\Omega} \cup b_\varepsilon(y)\},$$

we can conclude from standard regularity estimates for elliptic system (see [McL00][Theorem 4.17]) that A has to be continuous on every subset $U \subset\subset B\backslash\overline{\Omega}$. Therefore we get $\|A\|_{L^\infty} < c$ for some constant c on $b_{\varepsilon_0}(y)$ for a distinct ε_0. But this leads to a contradiction since

$$\|E\|_{L^\infty} \to \infty, \quad \text{as } x \to y$$

and therefore $\|E\|_{L^\infty} > c$ in $b_\delta(y)$ for a $\delta < \varepsilon_0$. Thus E and A cannot coincide on $B\backslash\{\Omega \cup b_\delta(y)\}$. This completes the proof. □

We know from the factorization (6.2.15), that

$$\mathcal{R}(Z - Z_0) \subset \mathcal{R}(G) \subset H^{-1/2}(\text{div}; \partial B) \tag{6.3.8}$$

where $\mathcal{R}$ denotes the range of the operator. We also know that these embeddings are dense. In this respect the next result is quite encouraging.

Corollary 6.6.

Let φ_y be defined by (6.3.7) for $y \in B$. Then if φ_y belongs to the range of $Z - Z_0$ we have $y \in \Omega$.

This shows that we can guarantee to find a subset $\Omega_1 \subset \Omega$ with the given data. The corollary also provides a method to solve the inverse problem. One defines the functions φ_y for every $y \in B$ and then defines

$$W(y) := \left(\sum_{n=1}^{\infty} \frac{|\langle \varphi_y, \psi_n \rangle|^2}{|\lambda_n|} \right)^{-1}, \tag{6.3.9}$$

where (λ_n, ψ_n) is an eigensystem of $Z - Z_0$. The function W is then the characteristic function of Ω_1. We will not present any implementations of this method here, however we would like to refer the interested reader to works where similar methods for inverse scattering problems have been implemented successfully (see for example [KR00], [GK02], [GHK+05]).

Remark: To show that all of Ω is determined by the range of $Z - Z_0$, we would have to show that the range of $Z - Z_0$ and G coincide. That this might be the case is supported by Theorem 6.2, where we have shown that $T - T_0$ is an isomorphism if $\mu_1(x) \geq \alpha > 0$ and $\epsilon_1(x) \geq \alpha > 0$ holds.

Indeed if we could show that $Z - Z_0$ and $S := T - T_0$ have well-defined square roots in $L_t^2(\partial B)$ and $L_t^2(\partial\Omega)$ resp. we could argue as follows. Note that with well-defined square roots we get the decomposition

$$(Z - Z_0)^{1/2}((Z - Z_0)^{1/2})^* = G(S^{1/2})^* S^{1/2} G^* = (G(S^{1/2})^*)(G(S^{1/2})^*)^*. \tag{6.3.10}$$

With this decomposition we can make use of the following theorem.

Theorem 6.7.

Let H_1 and H_2 be Hilbert spaces, $A : H_2 \mapsto H_2$, $G : H_2 \mapsto H_1$ and $B : H_1 \mapsto H_1$ be real, linear and bounded operators with

$$A = G^* B G$$

and let the following conditions hold:

- *G is one-to-one and compact.*
- *B is coercive and thus an isomorphism.*

Then the ranges of the operators $A^{1/2}$ and of G^* coincide.

Proof.

See [Kir04][Theorem 2.4]. □

An application of Theorem 6.7 then yields that the ranges of the operators $(Z - Z_0)^{1/2}$ and $G(S^{1/2})^*$ coincide. The problem with this approach is, that now we would like to argue that $S^{1/2}$ is an isomorphism from $L_t^2(\partial\Omega)$ to $H^{-1/2}(\text{curl}; \partial\Omega)$. However this cannot be done, since $H^{-1/2}(\text{curl}; \partial\Omega)$ and $H^{-1/2}(\text{div}; \partial\Omega)$ are not even proper subspaces of $L_t^2(\partial\Omega)$ and thus we cannot guarantee that $S^{1/2}$ is well-defined. Even if we could do that, it would not be guaranteed that $S^{1/2}$ is an isomorphism.

An alternative idea for this approach is to define the square root of S and $Z - Z_0$ simply via the functional calculus, i.e.

$$S^{1/2} = \frac{1}{2\pi} \int_\delta (S - \zeta I)^{-1} \zeta^{1/2} d\zeta,$$

where δ is a simple, closed, smooth curve surrounding the spectrum $\sigma(S)$ in a clockwise direction. Since S and $Z - Z_0$ are positive operators, this is certainly well-defined. However the problem in this case again would be to determine the range of $S^{1/2}$, which implies that as before we cannot guarantee that $S^{1/2}$ is an isomorphism.

This ends our discussion of a factorization for Maxwell's equations given near field boundary data. In the last chapter of this work we look at possible extensions and open questions of this work.

7 Open Questions

7.1 Theoretical problems and extensions

Since many applications rely on partial boundary data, an improvement of the result in Theorem 3.11 is of great interest.

Problem 7.1.

Show that partial boundary on a subset $\Gamma \subset \partial\Omega$ uniquely identifies both coefficients μ_r and ϵ_r in

$$\nabla \times E + k\mu_r H = 0,$$

$$\nabla \times H - k\epsilon_r E = 0.$$

Since the idea behind the proof of Theorem 3.11 cannot be used for this problem, a different approach has to be researched to solve this problem.

Another open question is to show that the range of $Z - Z_0$ and G in Chapter 6 coincide, or that at least the range of G is completely determined by $Z - Z_0$.

Problem 7.2.

Show that the range of the operator G in Chapter 6 is determined by the operator $Z - Z_0$.

Apart from these theoretical questions we also present a few ideas for future numerical extensions.

7.2 Numerical improvements

As we have mentioned in Chapter 4 and Chapter 5, the time needed to recover the functions μ_r and ϵ_r using our variational algorithm is not satisfactory at the moment. Therefore the following things could be done to improve this.

Problem 7.3.

To improve the computational time of the variational algorithm presented in Chapter 5 we suggest the following.

- *The use of a Maxwell solver written especially for the coercive system*

$$\nabla \times E + k\mu_r H = 0,$$

$$\nabla \times H - k\epsilon_r E = 0.$$

- *A parallel implementation on a cluster consisting of M parallel cpus, given M electric magnetic boundary values pairs $(\boldsymbol{n} \times E_n, \boldsymbol{n} \times H_n)$. In particular an advanced parallel algorithm without any remote logins and saving and loading of intermediate results should improve the computation time of the recovery.*

To improve the quality of the recovered functions a better numerical differentiation algorithm to compute the terms $\nabla \times E_n^{m,c}$ and $\nabla \times \tilde{E}_n^{m,c}$ in each iteration is needed to recover the function μ_r. Since this is more expensive than using central differences, this should be done using a good parallel implementation like the one outlined above.

Finally a project for the near future is to implement the factorization method discussed in Chapter 6.

Problem 7.4.

Design a numerical implementation of the factorization method in Chapter 6 to recover the support of the functions $\mu_r^{-1} - 1$ *and* $\epsilon_r - 1$ *in*

$$\nabla \times E + k\mu_r H = 0,$$

$$\nabla \times H - k\epsilon_r E = 0.$$

A Appendix

A.1 Proof of Theorem 5.12

To proof the form of the Fréchet derivative we first need an auxiliary result.

Lemma A.1.

Suppose $(m, c) \in D_G$ *and* $(h_m, h_c) \in L^\infty(\Omega) \times L^\infty(\Omega)$ *with small enough norm. Then we have*

$$||R_{m+h_m,c+h_c} - R_{m,c}|| \leq K||(h_m, h_c)||_{L^\infty}.$$

Proof.

First note that

$$R_{m+h_m,c+h_c} - R_{m,c} = \Lambda^{-1}_{\mu_r,\epsilon_r}(\Lambda_{m+h_m,c+h_c} - \Lambda_{m,c})\Lambda^{-1}_{\mu_r,\epsilon_r} + (\Lambda^{-1}_{m+h_m,c+h_c} - \Lambda^{-1}_{m,c}).$$

Now for any $f, g \in H^{1/2}(\text{curl}; \partial\Omega)$,

$$\int_{\partial\Omega} \Lambda^{-1}_{\mu_r,\epsilon_r}(\Lambda_{m+h_m,c+h_c} - \Lambda_{m,c})\Lambda^{-1}_{\mu_r,\epsilon_r} f \cdot g \, dS =$$
$$\int_{\partial\Omega} (\Lambda_{m+h_m,c+h_c} - \Lambda_{m,c})\Lambda^{-1}_{\mu_r,\epsilon_r} f \cdot \Lambda^{-1}_{\mu_r,\epsilon_r} g \, dS =$$
$$\int_{\partial\Omega} \gamma_T(\nabla \times (E_f^{m+h_m,c+h_c} - E_f^{m,c})) \cdot \gamma_t(E_g^{m,c}) \, dS. \tag{A.1.1}$$

Also

$$\int_{\partial\Omega}(\Lambda^{-1}_{m+h_m,c+h_c}-\Lambda^{-1}_{m,c})f\cdot g\,dS=\int_{\partial\Omega}\gamma_t(\tilde{E}_f^{m+h_m,c+h_c}-\tilde{E}_f^{m,c})\cdot\gamma_T(\nabla\times(\tilde{E}_g^{m,c}))\,dS. \tag{A.1.2}$$

To estimate the terms in (A.1.1) and (A.1.2) we multiply the identity

$$\begin{aligned}\nabla\times(m\nabla\times(E_f^{m+h_m,c+h_c}-E_f^{m,c}))+c(E_f^{m+h_m,c+h_c}-E_f^{m,c})\\ =-\nabla\times(h_m\nabla\times(E_f^{m+h_m,c+h_c}))-h_cE_f^{m+h_m,c+h_c}\end{aligned} \tag{A.1.3}$$

with $(E_f^{m+h_m,c+h_c}-E_f^{m,c})$ and integrate over Ω to obtain

$$\begin{aligned}&\int_\Omega m|\nabla\times(E_f^{m+h_m,c+h_c}-E_f^{m,c})|^2+c|E_f^{m+h_m,c+h_c}-E_f^{m,c}|^2\,dx=\\ &-\int_\Omega h_m\nabla\times E_f^{m+h_m,c+h_c}\cdot\nabla\times(E_f^{m+h_m,c+h_c}-E_f^{m,c})\,dx\\ &-\int_\Omega h_cE_f^{m+h_m,c+h_c}\cdot(E_f^{m+h_m,c+h_c}-E_f^{m,c})\,dx.\end{aligned} \tag{A.1.4}$$

Using (5.2.4) and the continuous dependence of $E_f^{m,c}$ on m and c we get

$$\begin{aligned}\|E_g^{m+h_m,c+h_c}\|_{H(\mathrm{curl};\Omega)}\le K_1\|E_g^{m,c}\|_{H(\mathrm{curl};\Omega)}\le K_2\|\Lambda^{-1}_{\mu_r,\epsilon_r}g\|_{H^{-1/2}(\mathrm{div};\partial\Omega)}\\ \le K_2\|g\|_{H^{1/2}(\mathrm{curl};\partial\Omega)}\end{aligned} \tag{A.1.5}$$

and

$$\begin{aligned}\|(E_f^{m+h_m,c+h_c}-E_f^{m,c})\|_{H(\mathrm{curl};\Omega)}\le K_3\|(h_m,h_c)\|_{L^\infty}\|E_g^{m,c}\|_{H(\mathrm{curl};\Omega)}\\ \le K_4\|(h_m,h_c)\|_{L^\infty(\Omega)^2}\|f\|_{H^{1/2}(\mathrm{curl};\partial\Omega)},\end{aligned} \tag{A.1.6}$$

where the K_i, $1\le i\le 4$ do not depend on h with similar estimates for the solutions $\tilde{E}_g^{m+h_m,c+h_c}$ and $(\tilde{E}_f^{m+h_m,c+h_c}-\tilde{E}_f^{m,c})$. Using the trace estimates, equation (A.1.4) and the boundedness of

$m + h_m$ and $c + h_c$ we conclude

$$\|\gamma_T(\nabla \times (E_f^{m+h_m,c+h_c} - E_f^{m,c}))\|_{H^{-1/2}(\mathrm{curl};\partial\Omega)} \le C^* \|\nabla \times (E_f^{m+h_m,c+h_c} - E_f^{m,c})\|_{H(\mathrm{curl};\Omega)}$$
$$\le C\|E_f^{m+h_m,c+h_c} - E_f^{m,c}\|_{H(\mathrm{curl};\Omega)}$$

and

$$\|\gamma_t(E_g^{m,c})\|_{H^{-1/2}(\mathrm{div};\partial\Omega)} = \|\Lambda_{\mu_r,\epsilon_r}^{-1} g\|_{H^{-1/2}(\mathrm{div};\partial\Omega)} \le \|\Lambda_{\mu_r,\epsilon_r}^{-1}\|\|g\|_{H^{-1/2}(\mathrm{div};\partial\Omega)}.$$

Now we can estimate the term in (A.1.1) by

$$|\int_{\partial\Omega} \gamma_T(E_f^{m+h_m,c+h_c} - E_f^{m,c})\cdot\gamma_t(E_g^{m,c})\, dS| \le CK_2\|\Lambda_{\mu_r,\epsilon_r}^{-1}\|\|f\|_{L^2(\mathrm{curl};\partial\Omega)}\|g\|_{H^{-1/2}(\mathrm{div};\partial\Omega)}\|(h_m, h_c)\|_{L^\infty(\Omega)^2}.$$

We can estimate (A.1.2) in a similar fashion. □

Now we show the Fréchet differentiability of H.

Proof. Let $\mathcal{E}_{m,c}$ be the eigenspace for the largest eigenvalue $\lambda_{m,c} > 0$ of $R_{m,c}$. From the above lemma and [Kat76][Theorem 2.14, page 203] we get for any $(h_m, h_c) \in L^\infty(\Omega) \times L^\infty(\Omega)$

$$\hat{\delta}(R_{m+h_m,c+h_c}, R_{m,c}) \le \|R_{m+h_m,c+h_c} - R_{m,c}\| \le K\|(h_m, h_c)\|_{L^\infty}, \qquad (A.1.7)$$

where K does not depend on (h_m, h_c) and $\hat{\delta}(R_{m+h_m,c+h_c}, R_{m,c})$ represents the gap of the operators $R_{m+h_m,c+h_c}$ and $R_{m,c}$ defined by

$$\hat{\delta}(R_{m+h_m,c+h_c} - R_{m,c}) = \max\{\delta(R_{m+h_m,c+h_c}, R_{m,c}), \delta(R_{m,c,R_{m+h_m,c+h_c}})\},$$

where $\delta(R_{m+h_m,c+h_c}, R_{m,c}) = \delta(\mathcal{G}_{m+h_m,c+h_c}, \mathcal{G}_{m,c})$. Here $\mathcal{G}_{m+h_m,c+h_c}$ is the graph of the operator

$R_{m+h_m,c+h_c}$ and for closed linear manifolds M and N, the metric $\delta(M,N)$ is given by

$$\delta(M,N) = \sup_{f \in S_M} \operatorname{dist}(f,N),$$

where $S_M = \{f \in M : \ \|f\| = 1\}$ (see [Kat76][page 197]). Since $\lambda_{m,c} = \|R_{m,c}\|$ we have

$$|\lambda_{m+h_m,c+h_c} - \lambda_{m,c}| = |\|R_{m+h_m,c+h_c}\| - \|R_{m,c}\|| \leq \|R_{m+h_m,c+h_c} - R_{m,c}\|. \tag{A.1.8}$$

Now we take $f_{m,c} \in \mathcal{E}_{m,c}$ with $\|f_{m,c}\| = 1$ and project it onto $\mathcal{E}_{m+h_m,c+h_c}$:

$$f_{m,c} = g_{m,c,h} + f^*_{m,c,h}, \tag{A.1.9}$$

where $g_{m,c,h} \in \mathcal{E}^{\perp}_{m,c,h}$ and $f^*_{m,c,h} \in \mathcal{E}_{m+h_m,c+h_c}$. Then

$$(R_{m+h_m,c+h_c} - \lambda_{m+h_m,c+h_c})f_{m,c} = (R_{m+h_m,c+h_c} - R_{m,c})f_{m,c} - (\lambda_{m+h_m,c+h_c} - \lambda_{m,c})f_{m,c}$$

and thus

$$(R_{m+h_m,c+h_c} - \lambda_{m+h_m,c+h_c})g_{m,c,h} = (R_{m+h_m,c+h_c} - R_{m,c})f_{m,c} - (\lambda_{m+h_m,c+h_c} - \lambda_{m,c})f_{m,c} \tag{A.1.10}$$

On $\mathcal{E}^{\perp}_{m,c,h}$ the operator $(R_{m+h_m,c+h_c} - \lambda_{m+h_m,c+h_c})$ is boundedly invertible and if we denote the spectrum of $R_{m+h_m,c+h_c}|_{\mathcal{E}^{\perp}_{m,c,h}}$ by Σ, then by a standard property of the spectrum we get

$$\|(R_{m+h_m,c+h_c} - \lambda_{m+h_m,c+h_c})|^{-1}_{\mathcal{E}^{\perp}_{m,c,h}}\| = 1/(\operatorname{dist}(\lambda_{m+h_m,c+h_c}, E)) = 1/(\lambda_{m+h_m,c+h_c} - \lambda^{(2)}_{m+h_m,c+h_c}),$$

where $\lambda^{(2)}_{m+h_m,c+h_c}$ denotes the second largest eigenvalue of $R_{m+h_m,c+h_c}$. Since $\lambda_{m+h_m,c+h_c} - \lambda_{m,c} \to 0$, for $\|(h_m, h_c)\| \to 0$ by (A.1.8) and Lemma A.1 it follows from (A.1.7) and [Kat76][Theorem 3.1, page 208] that

$$\lambda_{m+h_m,c+h_c} - \lambda^{(2)}_{m+h_m,c+h_c} \geq \lambda_{m,c} - \lambda^{(2)}_{m,c}$$

for all (h_m, h_c) with small enough norm. Thus we infer from (A.1.10) for all (h_m, h_c) with small enough norm that

$$\|g_{m,c,h}\| \leq K\|(h_m, h_c)\|, \tag{A.1.11}$$

where K is independent of (h_m, h_c). Since for small enough (h_m, h_c) we have $f^*_{m,c,h} \neq 0$ we can set $f_{m,c,h} = \frac{f^*_{m,c,h}}{\|f^*_{m,c,h}\|}$ as well as $\|g_{m,c,h}\| = \sin\theta$, where $0 \leq \theta \leq \pi/2$. Then $f_{m,c,h} \in \mathcal{E}_{m+h_m,c+h_c}$, $\|f_{m,c,h}\| = 1$, and noting that $(2/\pi)\theta \leq \sin\theta \leq \theta$ for $0 \leq \theta \leq \pi/2$ we get

$$\|f_{m,c,h} - f^*_{m,c,h}\| = 1 - \|f^*_{m,c,h}\| = 1 - \cos\theta \leq \frac{\pi^2}{8}\sin^2\theta = \frac{\pi^2}{8}\|g_{m,c,h}\|^2. \tag{A.1.12}$$

Let $f_{m,c}$ be an arbitrary, fixed and normalized eigenvector of $R_{m,c}$ corresponding to the eigenvalue $\lambda_{m,c}$. Note that

$$\begin{aligned}(H(m+h_m, c+h_c) - H(m,c))f_{m,c,h} &= (\lambda_{m+h_m,c+h_c} - \lambda_{m,c})f_{m,c,h} \\ &= (R_{m,c} - \lambda_{m,c})(f_{m,c,h} - f_{m,c}) + (R_{m+h_m,c+h_c} - R_{m,c})f_{m,c,h}\end{aligned}$$

Multiplying this identity with $f_{m,c}$, integrating over $\partial\Omega$ and noting that by the symmetry of $(R_{m,c} - \lambda_{m,c})$ the inner product $(R_{m,c} - \lambda_{m,c})(f_{m,c,h} - f_{m,c})f_{m,c}$ equals 0, we get

$$H(m+h_m, c+h_c) - H(m,c) = \tag{A.1.13}$$

$$(\lambda_{m+h_m,c+h_c} - \lambda_{m,c})\int_{\partial\Omega} f^2_{m,c}\, dS =$$

$$\begin{aligned}&\int_{\partial\Omega}(R_{m+h_m,c+h_c} - R_{m,c})f_{m,c}\cdot f_{m,c}\, dS \\ &- (\lambda_{m+h_m,c+h_c} - \lambda_{m,c})\int_{\partial\Omega}(f_{m,c,h} - f_{m,c})\cdot f_{m,c}\, dS \\ &+ \int_{\partial\Omega}(R_{m+h_m,c+h_c} - R_{m,c})(f_{m,c,h} - f_{m,c})\cdot f_{m,c}\, dS\end{aligned} \tag{A.1.14}$$

From (A.1.11), (A.1.12) we can conclude that

$$\|f_{m,c,h} - f_{m,c}\| \le \|f_{m,c,h} - f^*_{m,c,h}\| + \|g_{m,c,h}\| = O(\|(h_m, h_c)\|). \tag{A.1.15}$$

Thus for the Fréchet differentiability of $H(m,c)$ we only need the following estimate. It follows from (A.1.1) and (A.1.2)

$$\begin{aligned}
&\int_{\partial\Omega} (R_{m+h_m,c+h_c} - R_{m,c}) f_{m,c} \cdot f_{m,c}\, dS = \\
&\quad \int_{\partial\Omega} \gamma_T(\nabla \times (E^{m+h_m,c+h_c}_{f_{m,c}} - E^{m,c}_{f_{m,c}})) \cdot \gamma_t(E^{m,c}_{f_{m,c}})\, dS \\
&\quad + \int_{\partial\Omega} \gamma_t(\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}} - \tilde{E}^{m,c}_{f_{m,c}}) \cdot \gamma_T(\nabla \times \tilde{E}^{m,c}_{f_{m,c}})\, dS = \\
&\quad \int_{\Omega} -\nabla \times (m\nabla \times (E^{m+h_m,c+h_c}_{f_{m,c}} - E^{m,c}_{f_{m,c}})) \cdot E^{m,c}_{f_{m,c}} + m\nabla \times (E^{m+h_m,c+h_c}_{f_{m,c}} - E^{m,c}_{f_{m,c}}) \cdot \nabla \times E^{m,c}_{f_{m,c}}\, dx \\
&\quad - \int_{\Omega} \nabla \times (m\nabla \times \tilde{E}^{m,c}_{f_{m,c}}) \cdot (\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}} - \tilde{E}^{m,c}_{f_{m,c}}) + m\nabla \times \tilde{E}^{m,c}_{f_{m,c}} \cdot \nabla \times (\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}} - \tilde{E}^{m,c}_{f_{m,c}})\, dx.
\end{aligned}$$

Now using relation (A.1.3) we get

$$\begin{aligned}
&\int_{\partial\Omega} (R_{m+h_m,c+h_c} - R_{m,c}) f_{m,c} \cdot f_{m,c}\, dS = \\
&\quad \int_{\Omega} (\nabla \times (h_m \nabla \times (E^{m+h_m,c+h_c}_f)) + h_c E^{m+h_m,c+h_c}_f) \cdot E^{m,c}_{f_{m,c}}\, dx \\
&\quad + \int_{\Omega} c(E^{m+h_m,c+h_c}_{f_{m,c}} - E^{m,c}_{f_{m,c}}) \cdot E^{m,c}_{f_{m,c}}\, dx \\
&\quad + \int_{\Omega} m\nabla \times (E^{m+h_m,c+h_c}_{f_{m,c}} - E^{m,c}_{f_{m,c}}) \cdot \nabla \times E^{m,c}_{f_{m,c}}\, dx \\
&\quad + \int_{\Omega} c\tilde{E}^{m,c}_{f_{m,c}} \cdot (\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}} - \tilde{E}^{m,c}_{f_{m,c}}) + m\nabla \times \tilde{E}^{m,c}_{f_{m,c}} \cdot \nabla \times (\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}} - \tilde{E}^{m,c}_{f_{m,c}})\, dx = \\
&\quad \int_{\Omega} h_m \nabla \times (E^{m+h_m,c+h_c}_f) \cdot \nabla \times E^{m,c}_{f_{m,c}} + h_c E^{m+h_m,c+h_c}_f \cdot E^{m,c}_{f_{m,c}}\, dx
\end{aligned}$$

$$+\int_\Omega cE^{m,c}_{f_{m,c}}\cdot(E^{m+h_m,c+h_c}_{f_{m,c}}-E^{m,c}_{f_{m,c}})\,dx$$
$$+\int_\Omega m\nabla\times E^{m,c}_{f_{m,c}}\cdot\nabla\times(E^{m+h_m,c+h_c}_{f_{m,c}}-E^{m,c}_{f_{m,c}})\,dx$$
$$+\int_\Omega c\tilde{E}^{m,c}_{f_{m,c}}\cdot(\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}}-\tilde{E}^{m,c}_{f_{m,c}})+m\nabla\times\tilde{E}^{m,c}_{f_{m,c}}\cdot\nabla\times(\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}}-\tilde{E}^{m,c}_{f_{m,c}})\,dx$$

Since $\gamma_t(E^{m+h_m,c+h_c}_{f_{m,c}}-E^{m,c}_{f_{m,c}})=0$ on $\partial\Omega$ we get after another integration by parts

$$\int_{\partial\Omega}(R_{m+h_m,c+h_c}-R_{m,c})f_{m,c}\cdot f_{m,c}\,dS=$$
$$\int_\Omega h_m\nabla\times(E^{m+h_m,c+h_c}_f)\cdot\nabla\times E^{m,c}_{f_{m,c}}+h_cE^{m+h_m,c+h_c}_f\cdot E^{m,c}_{f_{m,c}}\,dx$$
$$+\int_\Omega c\tilde{E}^{m,c}_{f_{m,c}}\cdot(\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}}-\tilde{E}^{m,c}_{f_{m,c}})+m\nabla\times\tilde{E}^{m,c}_{f_{m,c}}\cdot\nabla\times(\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}}-\tilde{E}^{m,c}_{f_{m,c}})\,dx=$$
$$\int_\Omega h_m\nabla\times(E^{m+h_m,c+h_c}_f)\cdot\nabla\times E^{m,c}_{f_{m,c}}+h_cE^{m+h_m,c+h_c}_f\cdot E^{m,c}_{f_{m,c}}\,dx$$
$$+\int_\Omega(\nabla\times(m\nabla\times(\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}}-\tilde{E}^{m,c}_{f_{m,c}}))+c(\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}}-\tilde{E}^{m,c}_{f_{m,c}}))\cdot\tilde{E}^{m,c}_{f_{m,c}}\,dx,$$

since $\gamma_T(\nabla\times(\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}}-\tilde{E}^{m,c}_{f_{m,c}}))=0$ on $\partial\Omega$. Using relation (A.1.3) once more we conclude

$$\int_{\partial\Omega}(R_{m+h_m,c+h_c}-R_{m,c})f_{m,c}\cdot f_{m,c}\,dS=$$
$$\int_\Omega h_m\nabla\times(E^{m+h_m,c+h_c}_f)\cdot\nabla\times E^{m,c}_{f_{m,c}}+h_cE^{m+h_m,c+h_c}_f\cdot E^{m,c}_{f_{m,c}}\,dx$$
$$+\int_\Omega(-\nabla\times(h_m\nabla\times(\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}}))-h_c\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}})\cdot\tilde{E}^{m,c}_{f_{m,c}}\,dx=$$
$$\int_\Omega h_m\nabla\times(E^{m+h_m,c+h_c}_{f_{m,c}})\cdot\nabla\times E^{m,c}_{f_{m,c}}+h_cE^{m+h_m,c+h_c}_{f_{m,c}}\cdot E^{m,c}_{f_{m,c}}\,dx$$
$$-\int_\Omega h_m\nabla\times(\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}})\nabla\times\tilde{E}^{m,c}_{f_{m,c}}+h_c\tilde{E}^{m+h_m,c+h_c}_{f_{m,c}}\cdot\tilde{E}^{m,c}_{f_{m,c}}\,dx=$$
$$\int_\Omega h_m|\nabla\times(E^{m,c}_{f_{m,c}})|^2+h_c|E^{m,c}_{f_{m,c}}|^2$$

$$- h_m|\nabla \times \tilde{E}^{m,c}_{f_{m,c}}|^2 - h_c|\tilde{E}^{m,c}_{f_{m,c}}|^2 + O(\|(h_m, h_c)\|^2_{L^\infty(\Omega)^2})\, dx,$$

by $(A.1.6)$ and an analogous estimate for $\tilde{E}^{m,c}_{f_{m,c}}$. Together with (A.1.15) the result follows. □

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