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Quantum field theory via vertex algebras

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Abstract

We investigate an alternative formulation of quantum field theory that elevates the Wilson-Zimmermann operator product expansion (OPE) to an axiom of the theory. We observe that the information contained in the OPE coefficients may be straightforwardly repackaged into "vertex operators". This way of formulating quantum field theory has quite obvious similarities to the theory of vertex algebras.

As examples of this framework, we discuss the free massless boson in D dimensions and the massless Thirring model.

We set up perturbation theory for vertex algebras. We discuss a general theory of perturbations of vertex algebras, which is similar to the Hochschild cohomology describing the deformation theory of ordinary algebras. We pass on to a more explicit discussion by looking at perturbations of the free massless boson in D dimensions. The perturbations we consider correspond to some interaction Lagrangian $\lambda P(\varphi) = \lambda \sum_p c_p \varphi^p$. We construct the perturbations by exploiting the associativity of the vertex operators and the field equation in perturbative form. We develop a set of graphical rules that display the vertex operators as certain multiple series reminiscent of the hypergeometric series.

Contents

1 **Introduction** **3**

2 **Vertex algebras in QFT** **14**

2.1 Axiomatic framework 14

2.2 Comparison to other notions of vertex algebras 17

2.2.1 Chiral algebras 17

2.2.2 Full field algebras 25

3 **Examples: The free boson and the massless Thirring model** **30**

3.1 The free boson 30

3.1.1 Proof of associativity for the free boson 37

3.2 The massless Thirring model 42

3.2.1 Classical model 45

3.2.2 Klaiber's solution 47

3.3 Construction of the vertex algebra for the massless Thirring model 53

3.3.1 Proof of associativity for the massless Thirring model 59

3.3.2 Current, field equation, primary fields 70

4 **Perturbation theory** **73**

4.1 Perturbation theory via Hochschild cohomology 74

4.1.1 The cohomology of vertex algebra perturbations 75

4.1.2 Gauge theories 78

4.2 Graphical rules for computing vertex operators 84

4.2.1 Right inverse for the Laplacian 88

4.2.2 The case $D = 2$, $P = \varphi^3$: 90

4.2.3 The general case 106

4.2.4 Alternative expressions for tree-like terms 113

4.3 Renormalization 122

4.3.1 Examples 144

5 Conclusions and outlook **148**

A Spherical harmonics and Gegenbauer functions **151**

B Identities for Gegenbauer functions **156**

C The free field vertex operators **168**

Chapter 1

Introduction

Quantum field theory (QFT) is the theoretical formalism that describes collisions of elementary particles and critical phenomena.

There are many ways to define a QFT all of which are equivalent to some extent, at least at a formal level. Among the most common are the Lagrangian approach via path integrals [22, 31, 46, 72], the definition of a set of n-point functions satisfying some set of axioms [63, 68], the so-called bootstrap program [3, 49], the operator algebraic approach [33], to name only a few.

Common to all of these is that one deals with a set of "fields" $\mathcal{O}_a$, $a \in \mathcal{I}$, where $\mathcal{I}$ is some index set. In the axiomatic approach, a QFT is defined by the set of all correlation functions

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_{a_i}(x_i) \right\rangle_{\text{corr.}}, \quad n \in \mathbb{N},\, a_1, \ldots, a_n \in \mathcal{I}$$

which are distributions in the variables $x_i \in \mathbb{R}^D$. The path integral can be understood a tool to define these correlation functions. This does not always work; one instance where it works particularly well is perturbation theory. Here, one tries to give rigorous meaning to limits

$$\Big\langle \prod_i \mathscr{O}_{a_i}(x_i) \Big\rangle_{\text{corr.}} := \lim_{\Lambda \to \infty} \int \prod_i \mathscr{O}_{a_i}(x_i) \exp\left[-S_\Lambda(\varphi)\right] \mathrm{d}\mu_\Lambda(\varphi)\,, \qquad (1.0.1)$$

where $\mathrm{d}\mu_\Lambda(\varphi)$ is a Gaussian measure (associated to the free part of a classical action) and S_Λ is the full action including all counterterms depending on a cutoff scale Λ. This may be done by imposing some boundary conditions (renormalization conditions) on the correlation functions for finite Λ and deriving the limits (1.0.1) by the so-called Polchinski flow equations [50, 51, 65]. Another option is to make a diagrammatic expansion of the right hand side of eq. (1.0.1). The terms in this expansion can be identified with "Feynman diagrams". Each of them has a finite value for $\Lambda < \infty$, but typically diverges for $\Lambda \to \infty$. In fact, the Λ-dependent terms in S_Λ have to be chosen precisely so that any divergence in a Feynman diagram is canceled by another divergence with the opposite sign in some other Feynman diagram (see e.g. [72]).

Formula (1.0.1) is supposed to define the correlation functions of a Euclidean QFT, i.e. a QFT defined on $\mathbb{R}^D$ equipped with the Euclidean metric. The correlation functions of the corresponding relativistic QFT can be found by a so-called "Wick rotation": D-dimensional Euclidean space can be viewed as the D-dimensional Riemannian subspace of a complex D-dimensional space that as a further subspace also possesses D-dimensional Minkowski space. The correlation functions on the Euclidean space can be continued analytically to the whole complex D-dimensional space and the restrictions of these continuations to Minkowski space define the relativistic QFT.

However no such construction exists between Riemannian and Lorentzian QFT when one considers curved spaces, as a generic Lorentzian spacetime cannot be expressed as a section of a complex manifold that also possesses a Riemannian section.

So formula (1.0.1) is not appropriate under these more general conditions, and one has to look at alternative ways to define QFT. Such an alternative framework is provided by Algebraic QFT (AQFT) [24,33]. Here, the main focus is on the algebraic relations between quantum fields. Technically speaking, the basic idea is to associate a $*$-algebra $\mathcal{A}(\mathcal{O})$ to any region $\mathcal{O}$ of the considered Lorentzian spacetime. This may be thought of as the algebra of observables (smeared quantum fields) of the subsystem associated to the region $\mathcal{O}$.
For the case of perturbative quantum field theory in curved spacetime, the construction of these local algebras has been achieved in [12,13,40,41]. The success of AQFT in this context suggests that at the heart of QFT are the algebraic relations between quantum fields.

One way to express these algebraic relations on the level of correlation functions (which also works for curved spaces) is the Kadanoff-Wilson-Zimmermann operator product expansion (OPE) [76,78]. It determines the short distance behavior of the fields of the theory, and is most conveniently expressed as an identity for insertions of field operators into correlation functions,

$$\left\langle \mathcal{O}_a(x)\mathcal{O}_b(0)\prod_i \mathcal{O}_{d_i}(y_i)\right\rangle_{\text{corr.}} \sim \sum_c C_{ab}^c(x)\left\langle \mathcal{O}_c(0)\prod_i \mathcal{O}_{d_i}(y_i)\right\rangle_{\text{corr.}} . \tag{1.0.2}$$

Here, $a, b, c, d_i \in \mathcal{I}$ are again indices labeling the composite fields of the theory. The OPE-coefficients C_{ab}^c are distributions. The OPE has been shown to exist in perturbation theory by Zimmermann [80,81] as an asymptotic expansion, see also the earlier work of Brandt [10,11]. Zimmermann showed that there exist distributions C_{ab}^c such that the difference between the left hand side and the right hand side of eq. (1.0.2), where only indices c such that the composite field $\mathcal{O}_c$ has scaling dimension less or equal to M are included in the sum on the right hand side, is $o(|x|^{M-\dim a-\dim b})$. A simpler proof of this fact basing on the Polchinski flow equations can be found in [50, 51].

The OPE is directly physically relevant for *deep inelastic scattering*, which is the collision process between a very light and a very heavy elementary particle, e.g. a lepton and a hadron [16, 73]. The cross section for the scattering of a lepton by an initial hadron H, with an arbitrary final hadron state, is a linear combination (with known coefficients) of the amplitudes

$$\int \mathrm{d}^4 x\, e^{\mathrm{i}kx} \langle H | J^\mu(x) J^\nu(0) | H \rangle \,, \quad \mu, \nu = 1, \ldots, D \,, \tag{1.0.3}$$

where k is the momentum transferred from the lepton to the hadrons, $J^\mu(x)$ is the electromagnetic current, and $|H\rangle$ is the initial hadron state. For the situation of deep inelastic scattering, one is interested in large momenta, and so the main contributions to eq. (1.0.3) will arise from the singularities of the operator product $J^\mu(x)J^\nu(0)$. Thus the OPE allows one to approximate the cross section for the process.

The present thesis is an investigation of the theoretical aspects of the OPE. More precisely, we investigate a novel formulation of QFT [37, 42] that *elevates the OPE to an axiom of the theory*. This latter idea is motivated by the view that the algebraic relations between quantum fields are the fundament of QFT, as suggested by AQFT, and by conformally invariant QFT in 2 dimensions.

If one combines the OPE with associativity of the composition of field operators, one gets a consistency condition on the OPE coefficients,

$$\sum_d C_{ab}^d(x-y)C_{dc}^e(y) = \sum_d C_{bc}^d(y)C_{ad}^e(x)\,. \qquad (1.0.4)$$

The viewpoint advocated in [37, 42] is to (partly) *define* a quantum field theory by a set of distributions C_{ab}^c satisfying this consistency condition and some other properties of OPE coefficients. We will list the full set of required properties for the Euclidean case in definition 2.1.1. From then on, we are going to stick with the Euclidean setting (with the exceptions of sections 3.2.1 and 3.2.2).

Here we want to discuss the differences between the Minkowskian and the Euclidean case. Let us start from a Wightman field theory possessing an OPE. The spectral condition can be viewed as a condition on the singularity structure of products of (unsmeared) quantum fields. More precisely, the spectral condition is equivalent to a certain condition on the *wave-front sets* of these products, called the *microlocal spectrum condition* [12, 13, 66]. In fact, the replacement of the spectral condition by its microlocal version becomes a necessity when one goes to curved spaces. Being a condition on products of quantum fields, the microlocal spectrum condition also applies to the singularity structure (wavefront set) of the OPE [42]. As the microlocal spectrum condition is independent of the other Wightman axioms, it directly translates into an axiom for the OPE coefficients of a relativistic QFT if one wishes to define the theory by the latter.

The situation is slightly different in the Euclidean case: There is no independent spectral condition in the Osterwalder-Schrader axioms [63]. In the passage from Minkowskian to Euclidean QFT, the spectral condition entails analyticity of the Euclidean OPE coefficients, cf. the discussion in section 2.2.1. However, there is no equivalence between the two. When passing from Euclidean to Minkowskian QFT, the spectral condition arises from an interplay of temperedness, Euclidean covariance and time-reflection positivity of the Schwinger functions. Clearly there must be some implication from these three properties that is equivalent to the spectral condition for the relativistic OPE coefficients, but the technicalities of the proof of the spectral condition in [63] make it difficult to see what it should be. In definition 2.1.1, we will only require analyticity for the Euclidean OPE coefficients.

In the case of Lorentzian QFT, the definition of a QFT by its OPE coefficients can be viewed as equivalent to the usual approaches in the following way [42]: We assume we are given the index set $\mathcal{I}$ labeling the composite fields and the OPE coefficients of the theory, defined on some Lorentzian manifold M. Now we take the free $*$-algebra that is generated by the symbols $\mathcal{O}_a(f), a \in \mathcal{I}$, where f can be any test function on M. The algebra of observables $\mathcal{A}(M)$ is obtained from the free algebra by factoring out a number of relations, most importantly the relations that arise from the OPE. Then one defines the space of states $\mathcal{S}(M)$ to consist of all linear functionals $\mathcal{A}(M) \to \mathbb{C}$ in which the OPE holds[1] as an asymptotic relation as explained above. Apart from the OPE, the states have to fulfill some other obvious conditions such as positivity. The pair $(\mathcal{A}(M), \mathcal{S}(M))$ defines the theory in the sense of AQFT. The explicit construction of states might be very difficult, and we will not discuss it any further here. Of course, the construction of the set of states is necessary to construct the whole theory. Hence the definition of a set of consistent OPE coefficients only partly defines a QFT. However in this thesis we will limit ourselves to an analysis of the OPE coefficients themselves.

[1]The double use of the OPE in the definitions of $\mathcal{A}(M)$ and $\mathcal{S}(M)$ is potentially redundant, see the discussion in [42].

If one considers the abstract vector space V spanned by the field labels from $\mathcal{I}$ (denoted by a, b, c etc.), and equips it with some inner product, then the OPE coefficients $C_{ab}^{c}(x)$ define operators $Y(a, x)$ taking vectors from V to some suitable closure $\bar{V}$ of V, by requiring

$$\langle c, Y(a,x)b \rangle = C_{ab}^{c}(x) \ . \tag{1.0.5}$$

The consistency condition in this notation reads

$$Y(a,x)Y(b,y) = Y(Y(a,x-y)b,y) \, , \tag{1.0.6}$$

which is a well known identity in the theory of vertex algebras [19,25,26,30,48]. Vertex algebras first appeared in the physics context as chiral algebras in two-dimensional conformal field theory. In this thesis, we will reserve the term "vertex algebra" for the more general object defined by the abstract vector space V and the "vertex operators" $Y(a, x)$, see definition 2.1.1. We will call the vertex algebras describing chiral halves of two-dimensional conformal field theory "chiral algebras". We investigate the relation between the two in section 2.2.1.

In this thesis, we will pursue a twofold aim: First, we want to consider well known examples of quantum field theories such as the free boson in $D \geq 2$ dimensions and the massless Thirring model and extract the corresponding vertex algebra. For simplicity and definiteness, we are going to do this in the Euclidean setting. The second (more ambitious) aim is to set up perturbation theory for vertex algebras. We will use a new constructive tool from [37,62] to do so, which is not available in conventional perturbative quantum field theory. It consists in a combination of the associativity condition eq. (1.0.6) and a field equation. The idea is as follows: Assume the vertex operators $Y_0(a,x)$ describe the Euclidean field theory of the free massless boson in D dimensions and are known. We want to construct formal power series in some coupling parameter λ,

$$Y(a,x) = \sum_{i=0}^{\infty} \lambda^i Y_i(a,x)\,,$$

such that they satisfy

a) the axioms for a vertex algebra (definition 2.1.1) – in particular, associativity – in the sense of formal power series

b) a field equation such as
$$\Delta Y(\varphi,x) = \lambda Y(\varphi^3,x)\,.$$

Given the vertex operators of 0-th order, we may obtain candidates for the first order perturbation $Y_1(\varphi,x)$ by solving the differential equation[2]

$$\Delta Y_1(\varphi,x) = Y_0(\varphi^3,x)\,. \tag{1.0.7}$$

The first order perturbations $Y_1(\varphi^p,x)$, $p>1$, can not be obtained in the same way, because the field equation does not relate these perturbations to any of those that are already known. But as we want associativity to hold to first order in λ, we expect, for example,

$$Y_1(\varphi,(1+\epsilon)x)Y_0(\varphi,x) + Y_0(\varphi,(1+\epsilon)x)Y_1(\varphi,x) = \sum_c \langle c, Y_0(\varphi,\epsilon x)\varphi\rangle Y_1(c,x)$$

[2]In conventional perturbative QFT, the field equation is contained in the so-called Schwinger-Dyson equations (see e.g. [46]). These can be used as a constructive tool in exactly the same manner as in eq. (1.0.7). However the Schwinger-Dyson equations do not tell us how to pass from the first order perturbation of the field φ to the first order perturbation of the fields φ^k, $k>1$, and thus we cannot establish an iteration procedure for higher order perturbations.

$$+ \langle c, Y_1(\varphi, \epsilon x)\varphi\rangle Y_0(c, x) \qquad (1.0.8)$$

for $\epsilon > 0$ sufficiently small. Dimensional analysis tells us[3] that for $\dim c > 2\dim\varphi$, $\langle c, Y_i(\varphi, \epsilon x)\varphi\rangle \to 0$ for $\epsilon \to 0$, assuming $\dim\lambda \geq 0$. But this means that in the limit $\epsilon \to 0$, all terms in eq. (1.0.8) are known except for

$$\langle\varphi^2, Y_0(\varphi, \epsilon x)\varphi\rangle Y_1(\varphi^2, x) = Y_1(\varphi^2, x) \qquad (1.0.9)$$

on the right hand side, where we have used $\langle\varphi^2, Y_0(\varphi, \epsilon x)\varphi\rangle = 1$, see section 3.1. Thus the associativity condition eq. (1.0.8) can be used to *define* the first order perturbation $Y_1(\varphi^2, x)$. In a similar way, we can use associativity to define $Y_1(\varphi^p, x)$ for $p > 2$, and all other $Y_1(a, x)$, $a \in V$. Then we solve the differential equation

$$\Delta Y_2(\varphi, x) = Y_1(\varphi^3, x),$$

to go to the second order in λ, and so on. In this manner, we may construct all perturbations of vertex operators $Y_i(a, x)$ for $i \in \mathbb{N}, a \in V$.

To clarify the relation between the above scheme and the traditional way of doing perturbative calculations via Feynman graphs, we would like to make three remarks:

- Feynman graphs are a tool to compute n-point functions, the above procedure calculates OPE coefficients directly. It is a different way of setting up perturbation theory, and it is not clear *a priori* that these two are equivalent.

[3]More precisely, we are making some natural assumptions on the scaling degree of the perturbations of vertex operators, see section 4.

- The vertex operators do not require any renormalization as such. Associativity tells us in principle right from the start how to obtain finite expressions. Nevertheless, it might be that limits such as the limit $\epsilon \to 0$ above eq. (1.0.8) that are necessary for the construction of vertex operators do not exist. This might mirror problems in conventional perturbative QFT where perturbations of massless theories possess infrared divergences. This is of potential concern for us as well because we perturb around the free massless boson in D dimensions. We will not be able to generally prove the existence of the limits in question; however on an intuitive level we do not expect these problems to occur because we deal with the OPE which is a purely "local" concept – we may restrict ourselves to arbitrarily short distances, and infrared problems should not matter.

- It is not clear if this new way of calculating OPE coefficients is more efficient than the traditional way via Feynman graphs. A comparison of these two options in a low-order example can be found in [36], where identical results are found for both options. We will see in section 4.2 that to compute a coefficient in the manner described above, one has to carry out as many infinite sums as one has to compute loop integrals in the traditional way. These infinite sums are quite complicated, and we will only be able to carry them out in the most simple cases. Then again, we only know how to explicitly calculate loop integrals up to a certain order, and this knowledge is due to decades of research by many people. Possibly further study of the infinite sums we encounter in our approach might show that this tool is just as or even more efficient than calculations via Feynman graphs.

We are going to develop a set of graphical rules that result in an explicit formula for the vertex operator $Y_i(\varphi, x)$. The idea behind this is to emulate Feynman diagrams from conventional perturbative quantum field theory. There is no direct analogue of renormalization, as all vertex operators constructed from the above iteration procedure should be finite. Nevertheless, we will see that the $Y_i(\varphi, x)$ can be expressed as a sum over graphical objects, and the contribution of any single one of these objects is divergent in a certain sense. These divergences have to be canceled by other contributions. By analogy to the renormalization of Feynman graphs, it will turn out as natural to identify some of these objects as "unrenormalized" contributions, and others as "counterterms" that cure the divergences.

This thesis is organized as follows: In section 2, we put the considerations above into an axiomatic formulation and give our definition of a vertex algebra. We go on to compare this notion with chiral algebras from 2-dimensional conformal field theory and the notion of "full field algebra" [45] which describes full conformal field theory (without boundaries). In section 3, we extract the vertex algebra structure of well-known quantum field theories: The free massless boson in D dimensions and the massless Thirring model. For the latter, we will have to make a list of fields and prove that the OPE closes among them. In section 4, we start the pursuit of our second aim, the setup of perturbation theory. First we make some general remarks on perturbations of vertex algebras following [37], which allow to classify the possible deformations of a given vertex algebra in cohomological terms. We then go on to develop the graphical rules for perturbations of vertex operators. In the process, we will have to prove a number of identities for functions of hypergeometric kind that we could not find in the literature. Some well known facts about these functions can be found in appendix A, and the proofs for the new identities in appendix B.

Chapter 2

Vertex algebras in QFT

2.1 Axiomatic framework

We repeat and give more details on the ideas of the introduction in the form of a definition. We start with a vector space V with multiple grading, which is to be thought of as the linear space of composite quantum fields (at one point). The gradings stem from the bosonic/fermionic nature of the composite fields, their dimensions and their transformation properties under rotations of the underlying (Euclidean) space $\mathbb{R}^D$. There might be yet more gradings on V from symmetry charges, but we will not be interested in this possibility here.

Thus we have

$$V = \bigoplus_{i\in\{0,1\}} \bigoplus_{\Delta\in\mathbb{R}^+} \bigoplus_{S\in\mathrm{irrep}} V^{i,\Delta,S} \tag{2.1.1}$$

where $i = 0$ ($i = 1$) stands for the bosonic (fermionic) subspace, Δ denotes the scaling dimension, and $\bigoplus_{S\in\mathrm{irrep}}$ runs over all finite-dimensional irreducible unitary representations

S of Spin(D), the double cover of $SO(D)$. For each i, Δ, S, $V^{i,\Delta,S}$ consists of finitely many copies of the representation space of S. The sum $\bigoplus_{\Delta \in \mathbb{R}^+}$ is assumed to be infinite but countable. Also, we assume that for any Δ, only finitely many $V^{i,\Delta,S}$ are non-zero.

For a suitable definition of convergence in V, we introduce the dual space V^* by

$$V^* = \bigoplus_{i \in 0,1} \bigoplus_{\Delta \in \mathbb{R}_{\geq 0}} \bigoplus_{S \in \text{irrep}} \left(V^{i,\Delta,S}\right)^*$$

and set

$$\bar{V} = \text{Hom}(V^*, \mathbb{C}). \tag{2.1.2}$$

Let moreover $P_\Delta : \bar{V} \to V$ denote the projection onto the subspace $V^{\bullet,\Delta,\bullet} := \bigoplus_{i,S} V^{i,\Delta,S}$.

We are now ready to give our definition of vertex algebras.

Definition 2.1.1. A *vertex algebra* is a graded vector space V as in eq. (2.1.1) with a real analytic map, linear in $V^{\otimes 2}$,

$$\begin{aligned} Y : V^{\otimes 2} \times \mathbb{R}^D \setminus \{0\} &\to \bar{V} \\ (a \otimes b, x) &\mapsto Y(a, x)b\,, \end{aligned} \tag{2.1.3}$$

and maps

$$\nabla^\mu : V \to V,\ \mu = 1, \ldots, D, \tag{2.1.4}$$

satisfying the following axioms:

- **Vacuum:**

  There exists $\mathbf{1} \in V^{0,0,e}$ with

$$Y(\mathbf{1}, x) = \mathrm{Id}_V,\ \nabla^\mu \mathbf{1} = 0,\ \mu = 1, \ldots, D, \tag{2.1.5}$$

  where e is the trivial representation of $\mathrm{Spin}(D)$.

- **Grading:**

  For $a \in V^{i,\bullet,\bullet}, b \in V^{j,\bullet,\bullet}, i + j \equiv k \bmod 2$,

$$Y(a, z)b \in \bar{V}^{k,\bullet,\bullet} \tag{2.1.6}$$

  where $V^{i,\bullet,\bullet} = \bigoplus_{\Delta,S} V^{i,\Delta,S}$.

- **Rotation covariance:**

  There exists a representation R of $\mathrm{Spin}(D)$ on V satisfying

$$R(\Lambda)\, Y(R(\Lambda)^{-1}a, x)\, R(\Lambda)^{-1} = Y(a, \Lambda x) \tag{2.1.7}$$

  where $\Lambda \in \mathrm{Spin}(D)$ and Λx is understood as the action of the image of Λ under the double cover $\mathrm{Spin}(D) \to SO(D)$ on x in the fundamental representation on $\mathbb{R}^D$.

- **Associativity:**

  For $a, b, c \in V, d \in V^*$ and $|x| > |y| > |x - y|$, the infinite sums

$$\sum_{\Delta \in \mathbb{R}_{\geq 0}} \langle d, Y(a, x)\, (P_\Delta\, Y(b, y)c) \rangle\, ,$$
$$\sum_{\Delta \in \mathbb{R}_{\geq 0}} \langle d, Y\left((P_\Delta\, Y(a, x - y)b)\, , y \right) c \rangle \tag{2.1.8}$$

converge to the same value, and thus

$$Y(a,x)Y(b,y)c = Y(Y(a,x-y)b,y)c \tag{2.1.9}$$

is a well-defined element of $\bar{V}$.

- **Compatibility:**

  For $a \in V$,

$$Y(\nabla^\mu a, x) = \frac{\partial}{\partial x_\mu} Y(a,x) \tag{2.1.10}$$

- **Skew-symmetry:**

  For $a \in V_i, b \in V_j$,

$$\exp(x \cdot \nabla) Y(b,-x)a = (-1)^{ij} Y(a,x)b \in \bar{V} \tag{2.1.11}$$

  where the exponential has to be understood in the sense of an infinite power series, $\exp(x \cdot \nabla) = \sum_{n=0}^{\infty} (n!)^{-1} (x_\mu \nabla^\mu)^n$ (using the Einstein summation convention).

2.2 Comparison to other notions of vertex algebras

2.2.1 Chiral algebras

Vertex algebras first appeared in the physics context as chiral algebras in two-dimensional conformal field theories, see e.g. [8,18,28,48]. In this subsection, we explain how a vertex algebra can be extracted from a given conformally invariant QFT in 2 dimensions. We mainly follow the introduction of [48].

Consider a relativistic quantum field theory on two-dimensional Minkowski space M with metric $dx^2 = dx_0^2 - dx_1^2$, given by a space of states $\mathcal{H}$, a vacuum vector $|0\rangle$, a unitary representation U of the Poincaré group on $\mathcal{H}$, and a set of fields (i.e. operator valued tempered distributions) Ψ satisfying the Wightman axioms (see e.g. [68]). In particular, $|0\rangle$ is in the domain of any monomial $\phi_1(f_1)\dots\phi_n(f_n)$ where $\phi_1,\dots,\phi_n \in \Psi$ and[1], $f_1,\dots,f_n \in \mathcal{S}(\mathbb{R}^2)$, the space $\mathcal{D}$ spanned by polynomials in the $\phi(f)$'s acting on the vacuum is dense in $\mathcal{H}$, and any two fields $\phi, \chi \in \Psi$ are mutually local,

$$[\phi(x), \chi(y)]_\pm = 0 \text{ for } |x-y|^2 < 0 \tag{2.2.12}$$

where $[\cdot,\cdot]_\pm$ is the commutator or anticommutator, depending on the statistics of ϕ, χ. Moreover, one assumes that the theory is *conformal*: U extends to a unitary representation of the conformal group, that apart from the Lorentz boosts and translations contains the special conformal transformations

$$x \mapsto x^y = \frac{x + |x|^2 y}{1 + 2x \cdot y + |x|^2|y|^2},\ y \in M, \tag{2.2.13}$$

generated by self-adjoint operators Q_0, Q_1 that annihilate the vacuum.

One introduces light cone coordinates $t = x_0 + x_1, t^* = x_0 - x_1$, and $Q = -\frac{1}{2}(Q_0 + Q_1), \bar{Q} = -\frac{1}{2}(Q_0 - Q_1)$, $P = \frac{1}{2}(P_0 - P_1), \bar{P} = \frac{1}{2}(P_0 + P_1)$ where P_0, P_1 are the generators of translations. For $\phi \in \Psi$,

$$\phi(t+q, t^* + q^*)|0\rangle = e^{i(qP + q^*\bar{P})}\phi(t, t^*)|0\rangle \tag{2.2.14}$$

[1] $\mathcal{S}(\mathbb{R}^2)$ denotes Schwartz space, the space of rapidly decaying smooth functions.

By the Wightman axioms, the spectrum of $P, \bar{P}$ is a subset of the non-negative reals, and thus eq. (2.2.14) has a $\mathcal{D}$-valued analytic continuation to $\mathrm{Im}\, t$, $\mathrm{Im}\, t^* \geq 0$. In the following it will be important that for $\mathrm{Im}\, t$,$\mathrm{Im}\, t^* > 0$, eq. 2.2.14 is not only a distribution but a $\mathcal{D}$-valued analytic function. For a discussion of these analyticity properties of Wightman fields, see [68]. The conformal transformation eq. (2.2.13) decouples into

$$t^y = \frac{t}{1 + y^+ t}, \quad t^{*y} = \frac{t^*}{1 + y^- t^*}, \tag{2.2.15}$$

where $y^+ = y_0 + y_1$, $y^- = y_0 - y_1$.

The set of *quasiprimary* fields is the set of fields that transform under eq. (2.2.15) according to

$$U(y)\phi(t, t^*)U(y)^{-1} = (1 + y^+ t)^{-2\Delta_\phi}(1 + y^- t^*)^{-2\Delta^*_\phi}\phi(t^y, t^{*y}).$$

with $\Delta_\phi, \Delta^*_\phi \in \mathbb{R}_{\geq 0}$.

Now we use the coordinate transformation

$$z = \frac{1 + it}{1 - it}, \quad z^* = \frac{1 + it^*}{1 - it^*} \tag{2.2.16}$$

which maps the open upper half plane in t and t^* to the open unit circle in z, z^* respectively. In particular, z, z^* are treated as independent variables. For quasiprimary ϕ, we define

$$\tilde{Y}(\phi, z, z^*) = \frac{1}{(1 + z)^{2\Delta_\phi}(1 + z^*)^{2\Delta^*_\phi}}\phi(t, t^*) \tag{2.2.17}$$

where t, t^* are related to z, z^* as in eq. (2.2.16). Note that $\tilde{Y}(\phi, z, z^*)|0\rangle|_{z=z^*=0}$ is a well-defined vector in $\mathcal{D}$.

Next we define

$$L_{-1} = \frac{1}{2}(P + [P, Q] - Q)$$
$$L_0 = \frac{1}{2}(P + Q)$$
$$L_1 = \frac{1}{2}(P - [P, Q] - Q) \tag{2.2.18}$$

They satisfy

$$\left[L_{-1}, \tilde{Y}(\phi, z, z^*)\right]_{\pm} = \partial_z \tilde{Y}(\phi, z, z^*)$$
$$\left[L_0, \tilde{Y}(\phi, z, z^*)\right]_{\pm} = (z\partial_z + \Delta_\phi)\tilde{Y}(\phi, z, z^*)$$
$$\left[L_1, \tilde{Y}(\phi, z, z^*)\right]_{\pm} = (z^2\partial_z + 2\Delta_\phi z)\tilde{Y}(\phi, z, z^*)\,. \tag{2.2.19}$$

Operators $L_i^*, i = -1, 0, 1$ can be defined analogously.

The operators $L_i, L_i^*, i = -1, 0, 1$ are the generators of the global conformal group, which consists of the special conformal transformations eq. (2.2.15) and the Poincaré group.

Now we limit ourselves to the set of holomorphic fields in Ψ, i.e. those that satisfy $\partial_{t^*}\phi(t, t^*) = 0$. For those fields we write $\phi(t, t^*) = \phi(t)$ in the following, and we write $\tilde{Y}(\phi, z, z^*) = \tilde{Y}(\phi, z)$ if ϕ is quasiprimary and holomorphic. The locality axiom eq. (2.2.12) for two holomorphic fields ϕ, χ reads

$$[\phi(t), \chi(t')]_{\pm} = \sum_{j \geq 0} \delta^j(t - t')\psi_j(t')$$

for some $\psi_j \in \Psi$. Thus

$$[\tilde{Y}(\phi, z), \tilde{Y}(\chi, w)]_{\pm} = \sum_{j \geq 0} \delta^{(j)}(z-w)\tilde{Y}(\eta_j, w), \tag{2.2.20}$$

which has to be understood as a definition of the holomorphic fields $\tilde{Y}(\eta_j, w)$ on the right hand side, as the η_j may not be quasiprimary. The sum on the right hand side is finite, so we get

$$(z-w)^N [\tilde{Y}(\phi, z), \tilde{Y}(\chi, w)]_{\pm} = 0 \text{ for } N \gg 0$$

Finally, for holomorphic fields ϕ there exists the Laurent expansion

$$\tilde{Y}(\phi, z) = \sum_{n \in \mathbf{Z}} \phi_{(n)} z^{-n-1} \tag{2.2.21}$$

for $|z| < 1$ with $\phi_{(n)} \in$End$(\mathcal{D})$. Let V be the subspace of $\mathcal{D}$ spanned by polynomials of the $\phi_{(n)}$'s acting on $|0\rangle$. The decomposition (2.2.21) allows us to view Y as a formal power series in z, i.e. as a End$(V)[[z, z^{-1}]]$ valued map. This is beneficial because in this way, one can treat each order in the variable z independently in a purely algebraic manner.

The 4-tuple $(V, |0\rangle, \tilde{Y}, L_{-1})$ constitutes the chiral algebra, which satisfies the following axiomatic definition.

Definition 2.2.1. A *chiral algebra* is given by a vector space V, a vector $|0\rangle \in V$, a map $L_{-1} : V \to V$ and a map

$$\tilde{Y} : V \to \text{End}(V)[[z, z^{-1}]]$$
$$a \mapsto \tilde{Y}(a, z)$$

called the *state-field correspondence* or *vertex operator*, satisfying the following axioms:

- **Translation covariance:** $[L_{-1}, \tilde{Y}(a,z)] = \partial\tilde{Y}(a,z)$
- **Vacuum:** $L_{-1}|0\rangle = 0$, $\tilde{Y}(|0\rangle, z) = \mathrm{Id}_V$, $\tilde{Y}(a,z)|0\rangle_{z=0} = a$
- **Locality:** $(z-w)^N[\tilde{Y}(\phi,z), \tilde{Y}(\chi,w)] = 0$ for $N \gg 0$

In this definition, we have assumed that the bracket $[\cdot,\cdot]_\pm = [\cdot,\cdot]_- = [\cdot,\cdot]$ is identical to the commutator.

The concept of chiral algebras has been known to physicists since the mid-1970's. The first rigorous definition, equivalent to the one above, was given by Borcherds [9], who used it in his analysis of the representation theory of the monster group. The literature on chiral algebras[2] is vast; we cite the seminal papers [19,25,26] without any claim to completeness.

We come to the comparison of the definitions 2.1.1 and 2.2.1. First of all, we clarify the relation between the vertex operator encoding the OPE as in eq. (1.0.5) and the vertex operator $\tilde{Y}$ defined in eq. (2.2.17).

Assuming there exists an OPE, we have for quasiprimary fields ϕ, χ

$$\tilde{Y}(\phi, z, z^*)\tilde{Y}(\chi, 0, 0)|0\rangle = \sum_\zeta C^\zeta_{\phi\chi}(z, z^*)\tilde{Y}(\zeta, 0, 0)|0\rangle \qquad (2.2.22)$$

where the sum on the right hand side will contain non-quasiprimary fields, so the $\tilde{Y}(\zeta, 0, 0)$ are not of the form eq. (2.2.17); instead they are *defined* by eq. (2.2.22).

It is not obvious at all from what we have said up to now that an equation of the kind eq. (2.2.22) must hold for any conformally invariant quantum field theory. In particular,

[2]In the cited papers, chiral algebras as defined here are called vertex algebras or vertex operator algebras.

we do not know if the OPE holds. In [54] it is shown that an equation such as (2.2.22) always exists in two-dimensional CFT. In this reference, the right hand side is obtained as a decomposition of the left hand side into "conformal partial waves". The vectors $\tilde{Y}(\zeta,0,0)|0\rangle$ are defined by this decomposition. However it does not follow from these arguments that $\tilde{Y}(\zeta,w,w^*)$ are related to local quantum fields as in eq. (2.2.17).

If we write $\tilde{Y}(\chi,0,0)|0\rangle = \vec{\chi} \in \mathcal{D}$ and $\tilde{Y}(\zeta,0,0)|0\rangle = \vec{\zeta} \in \mathcal{D}$ then eq. (2.2.22) reads

$$\tilde{Y}(\phi,z,z^*)\vec{\chi} = \sum_{\zeta} \langle \vec{\zeta}, \tilde{Y}(\phi,z,z^*)\vec{\chi}\rangle \vec{\zeta}.$$

First we restrict the domain of $\tilde{Y}(\phi,z,z^*)$ to $z^* = \bar{z}$ (complex conjugation) and identify $\mathbb{C}$ with $\mathbb{R}^2$. Eq. (2.2.22) is only well-defined for $z \neq 0$. Secondly, we view V as a subspace of $\mathcal{D}$ via the inclusion map $\phi \mapsto \tilde{Y}(\phi,0,0)|0\rangle$. With these restrictions understood, the operator $\tilde{Y}$ is the operator Y defined in eq. (1.0.5) for the (Euclidean version of the) 2-dimensional quantum field theory that we started with.

Thus from a quantum field theoretical point of view, a chiral algebra is a special case of a vertex algebra. In the above deduction, not only did we presume a two-dimensional (globally) conformally invariant theory, but also the existence of holomorphic fields[3]. This fact did allow for the Laurent expansion (2.2.21) and the subsequent interpretation of Y as a "formal distribution", i.e. as a $\mathrm{End}(V)[[z,z^{-1}]]$ valued map. Note that the requirement of real analyticity of Y in definition 2.1.1 is something fundamentally different.

If one does not assume that the fields are holomorphic, one can still show from global con-

[3]It may seem that the existence of quasiprimary fields was another assumption, but this follows from the fact that L_0 is bounded below, which in turn follows from positivity of energy, see [8].

formal invariance that the vertex operators must be of the form

$$\tilde{Y}(a, z, z^*) = \sum_{r,\bar{r}\in\mathbb{R}} a_{r,\bar{r}} z^{-r-1} z^{*-\bar{r}-1} \tag{2.2.23}$$

with $a_{r,\bar{r}} \in \operatorname{End}\mathcal{D}$. This follows from a consideration of the action of the differential operators in eq. (2.2.19) on correlation functions [8]:

$$T_i \langle 0|\tilde{Y}(a_1, z_1, z_1^*) \ldots \tilde{Y}(a_n, z_n, z_n^*)|0\rangle = \bar{T}_i \langle 0|\tilde{Y}(a_1, z_1, z_1^*) \ldots \tilde{Y}(a_n, z_n, z_n^*)|0\rangle = 0 \tag{2.2.24}$$

where $i = -1, 0, 1$, and

$$T_{-1} = \sum_{j=1}^{n} \partial_{z_j} \qquad T_{-1}^* = \sum_{j=1}^{n} \partial_{z_j^*}$$
$$T_0 = \sum_{j=1}^{n} z_j \partial_{z_j} + \Delta_{a_j} \qquad T_0^* = \sum_{j=1}^{n} z_j^* \partial_{z_j^*} + \Delta_{a_j}^*$$
$$T_1 = \sum_{j=1}^{n} z_j^2 \partial_{z_j} + 2\Delta_{a_j} z_j \qquad T_1^* = \sum_{j=1}^{n} z_j^{*2} \partial_{z_j^*} + 2\Delta_{a_j}^* z_j^* .$$

Eq. (2.2.24) follows from eq. (2.2.19) and $L_i|0\rangle = L_i^*|0\rangle = 0$, $i = -1, 0, 1$. From eq. (2.2.24) it follows that the functional dependence of the vertex operators $\tilde{Y}(a_i, z_i, z_i^*)$ on z_i, z_i^* is of the form (2.2.23).

Summarizing, a vertex operator $\tilde{Y}(a, z)$ of a holomorphic field a is a Laurent series in z. These are the vertex operators from chiral algebras. The more general vertex operators of the form (2.2.23) do not appear in chiral algebras, but vertex operators as in definition 2.1.1 describing a globally conformally invariant QFT will have this form.

Finally, if one drops both, the requirements of analyticity and global conformal invariance, not much can be said about the functional form of the vertex operators or OPE coefficients in the variables z, z^*. It is this restriction of the functional form of OPE coefficients that makes conformal field theory so much better treatable (and solvable) than non-conformal QFT.

2.2.2 Full field algebras

In the last subsection, we only considered a part (the analytic or chiral part) of the conformally invariant QFT that we started with gave rise to a vertex algebra. Here we review how to describe the whole theory in a language very akin to our definition 2.1.1 for the special case of a conformally invariant theory. That is, we want to mention a construction introduced by Huang and Kong [45], the so-called *full field algebras.* In this reference the authors give their version of how to define and construct genus-zero conformal field theories. It is part of a program to construct conformal field theories in the sense of Kontsevich and Segal, see the references in [45]. In the context of conformally invariant quantum field theory on genus-zero surfaces, a full field algebra is equivalent to a vertex algebra in $D = 2$ with the representation R of $\mathrm{Spin}(D)$ (cf. eq. (2.1.7)) extending to a representation of the global conformal group.

Let $V, V^*, \bar{V}$ be vector spaces similar to the one in eq. (2.1.1), where we slightly change the grading,

$$\begin{aligned} V &= \bigoplus_{i \in 0,1} \bigoplus_{\Delta, \bar{\Delta} \in \mathbb{R}+} V^{i,\Delta,\bar{\Delta}}, \\ V^* &= \bigoplus_{i \in 0,1} \bigoplus_{\Delta, \bar{\Delta} \in \mathbb{R}+} \left(V^{i,\Delta,\bar{\Delta}}\right)^*, \\ \bar{V} &= \mathrm{Hom}(V^*, \mathbb{C}) \end{aligned} \tag{2.2.25}$$

Let $\mathrm{d}, \bar{\mathrm{d}}$ denote the operators defined by $\mathrm{d}(a) = \Delta a$, $\bar{\mathrm{d}}(a) = \bar{\Delta} a$ for $a \in V^{\bullet,\Delta,\bar{\Delta}}$, $P_{r,q}$ the projection operator $V \to V^{\bullet,r,q}$, and $\mathcal{F}^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ for all } i \neq j\}$. A formal sum $\sum_{n\in\mathbb{N}} v_n, v_n \in V$ is said to be absolutely convergent if for any $f \in V^*$, the sum $\sum_{n\in\mathbb{N}} f(v_n)$ is absolutely convergent. If that is the case, then in particular $\sum_{n\in\mathbb{N}} v_n \in \bar{V}$.

Definition 2.2.2. A *full field algebra* is a vector space V as in eq. (2.2.25) together with a vector $\mathbf{1} \in V$ (the vacuum) and maps

$$\begin{array}{rccc} m_n : & V^{\otimes n} \times \mathcal{F}^n & \to & \bar{V} \\ & (a_1 \otimes \cdots \otimes a_n, (z_1, \dots, z_n)) & \mapsto & m_n(a_1, \dots, a_n; z_1, \dots, z_n) \end{array} \tag{2.2.26}$$

linear in $a_1, \dots, a_n$ and smooth in the real and imaginary part of $z_1, \dots, z_n$, and

$$\nabla, \bar{\nabla} : V \to V\,, \tag{2.2.27}$$

satisfying the following set of axioms:

- **Vacuum**:

$$m_1(a, 0) = a$$
$$m_{n+1}(a_1, \dots, a_n, \mathbf{1}; z_1, \dots, z_{n+1}) = m_n(a_1, \dots, a_n; z_1, \dots, z_n)$$

- **Convergence:**

  The infinite series

$$\sum_{r_1,q_1,\dots,r_k,q_k} m_k\Big(P_{r_1,q_1} m_{l_1}(a_1^{(1)}, \dots, a_{l_1}^{(1)}; z_1^{(1)}, \dots, z_{l_1}^{(1)}), \dots,$$
$$P_{r_k,q_k} m_{l_k}(a_1^{(k)}, \dots, a_{l_k}^{(k)}; z_1^{(k)}, \dots, z_{l_k}^{(k)}); z_1^{(0)}, \dots, z_k^{(0)}\Big) \tag{2.2.28}$$

converges absolutely to

$$m_{l_1+\cdots+l_k}\Big(a_1^{(1)},\ldots,a_{l_1}^{(1)},\ldots,a_1^{(k)},\ldots,a_{l_k}^{(k)};$$
$$z_1^{(1)}+z_1^{(0)},\ldots,z_{l_1}^{(1)}+z_1^{(0)};\ldots,z_1^{(k)}+z_k^{(0)},\ldots,z_{l_k}^{(k)}+z_k^{(0)}\Big) \tag{2.2.29}$$

on the domain $|z_i^r|+|z_j^{r'}|<|z_r^0-z_{r'}^0|$, $r,r'\in 1,\ldots,k, r\neq r', i\in 1,\ldots l_r,, j\in 1,\ldots l_{r'}$.

- **Symmetry:**

  Let $a_i\in V^{p_i,\bullet,\bullet}$, $i=1,\ldots,n$, $\sigma\in \mathrm{Sym}(n)$. Then

$$m_n(a_{\sigma(1)},\ldots,a_{\sigma(n)};z_{\sigma(1)},\ldots,z_{\sigma(n)}) = \prod_{\substack{1\leq i<j\leq n\\ \sigma(i)>\sigma(j)}}(-1)^{p_ip_j}m_n(a_1,\ldots,a_n;z_1,\ldots,z_n) \tag{2.2.30}$$

- **Single-valuedness:**

$$\exp(2\pi i(\mathrm{d}-\bar{\mathrm{d}}))=\mathrm{Id}_V$$

- **Scaling and compatibility:**

  Define $Y:V^{\otimes 2}\times\mathbb{C}\setminus\{0\}\to\bar{V}$ by

$$Y(a,z)b=m_2(a,b;z,0). \tag{2.2.31}$$

  Then for $a\in V^{\bullet,\Delta_a,\bar{\Delta}_a}$,

$$[\nabla,Y(a,z)]=\partial_zY(a,z),\qquad \big[\bar{\nabla},Y(a,z)\big]=\partial_{\bar{z}}Y(a,z),$$
$$[\mathrm{d},Y(a,z)]=(z\partial_z+\Delta_a)Y(a,z)\qquad \big[\bar{\mathrm{d}},Y(a,z)\big]=(\bar{z}\partial_{\bar{z}}+\bar{\Delta}_a)Y(a,z)\,. \tag{2.2.32}$$

Remark: The maps m_n can informally be understood as the OPE for products of n fields,

$$\mathcal{O}_{a_1}(x_1)\ldots\mathcal{O}_{a_n}(x_n) \sim \sum_c \langle c, m_n(a_1,\ldots,a_n;x_1,\ldots,x_n)\rangle \mathcal{O}_c(x_n)\,.$$

The similarity to definition 2.1.1 is obvious. Let us discuss the differences.

The most obvious difference is the one between the "associativity" axiom in definition 2.1.1 and the "convergence" axiom in definition 2.2.2. It is quite obvious that associativity follows from convergence if, given a full field algebra, the vertex operator is defined as in eq. (2.2.31). As is shown in theorem 2.11 of [45], for a special kind of full field algebras, the converse is true as well. The main ingredients of the proof are the definition

$$m_n(a_1,\ldots,a_n;z_1,\ldots,z_n) := Y(a_1,z_1-z_n)\ldots Y(a_{n-1},z_{n-1}-z_n)a_n \quad \text{for } |z_1-z_n| > \cdots > |z_{n-1}-z_n| \tag{2.2.33}$$

and a result on analytic continuations of compositions of vertex operators from [44]. It is shown in the latter reference that for a full field algebra satisfying some additional conditions, compositions of vertex operators that are only defined on certain domains as in eq. (2.2.33) fulfill a system of ordinary differential equations of regular singular points. From the theory of ordinary differential equations, it follows that they have an analytic continuation to the larger domain $\mathcal{F}^n$, with absolutely convergent expansions there.

However the additional conditions one has to impose on the full field algebra seem to be rather restrictive: There have to exist chiral algebras V_L, V_R such that V as a $V_L \otimes V_R$-module satisfies the "C_1-cofiniteness condition" [27, 44, 45]. This will not be satisfied, e.g., for the example of the massless Thirring field in chapter 3. Also, there exists no analogue of this situation for the vertex algebras in perturbation theory that we will consider in chapter 4.

Another difference between the two definitions is that we demanded real analyticity for the vertex operator (2.1.3) as opposed to the smoothness property of the maps m_n. If the full field algebra describes a quantum field theory, then analyticity is a natural requirement as explained in section 2.2.1.

The single-valuedness axiom above follows from the rotation covariance axiom in definition 2.1.1, the representation R of $\mathrm{Spin}(2) \simeq S^1$ on V given by $R(e^{i\gamma}) = \exp i\gamma(\mathrm{d} - \bar{\mathrm{d}})$. Here and in the following, the double cover $\mathrm{Spin}(2) \to SO(2)$ is given by $e^{i\gamma} \mapsto e^{2i\gamma}$. The action of Spin(2) on $\mathbb{R}^2 \simeq \mathbb{C}$ will be understood to be the action of the image of the double cover on $\mathbb{C}$ by scalar multiplication.

The scaling axiom does not have an analogue in the more general definition 2.1.1, as we do not expect to have scaling invariance in the latter case. If one wants to axiomatically describe conformal field theories, one should include invariance under special conformal transformations as in eq. (2.2.13) into the axioms as well.

Despite these differences, the definition of a full field algebra is just a special case of the definition of a quantum field theory via its operator product expansion as in [37]. In the latter reference, the stronger "convergence" condition (analogous to eq. (2.2.28)) is used in the axiomatic characterization of a quantum field theory instead of associativity (eq. (2.1.8)).

Chapter 3

Examples: The free boson and the massless Thirring model

3.1 The free boson

In this section, we illustrate our abstract framework for the OPE in a simple example. Our example is the free quantum field theory obeying the linear field equation[1]

$$\Delta\varphi := \sum_{\mu=1}^{D}(\nabla_\mu)^2\varphi = 0\,. \tag{3.1.1}$$

The free field vertex algebra in D dimensions can already be found in [61]. In this reference the machinery of "formal distributions" is used. We want to avoid the introduction of these formal objects as they will not be suited for perturbation theory.

[1]We are going to use the symbol Δ for the map $\sum_{\mu=1}^{D}(\nabla_\mu)^2 : V \to V$ and the differential operator $\sum_{\mu=1}^{D}(\partial_\mu)^2$ simultaneously. No confusion will arise from this double use.

The space V of fields in this theory may be taken to be the unital, free, commutative ring generated by φ and its derivatives. In other words, the elements of V are in one-to-one correspondence with linear combinations in the monomials in $\nabla_{\mu_1} \dots \nabla_{\mu_k}\varphi$, and $\nabla_\mu, \mu = 1, ..., D$ are the derivations that act as if they were ordinary derivatives. To implement the field equation, we simply set to zero any expressions containing a factor of the form $\delta^{\mu_i\mu_j}\nabla_{\mu_1} \dots \nabla_{\mu_k}\varphi$, i.e. monomials that would vanish if φ was an actual field satisfying the field equation. Because monomials containing a trace of $\nabla_{\mu_1} \dots \nabla_{\mu_k}\varphi$ are set to zero, V is spanned by all trace-free monomials. Thus, if we denote by curly brackets $t_{\{\mu_1 \dots \mu_k\}}$ the trace-free part of a symmetric tensor, then a basis of V is given by $\mathbf{1}$, together with the set of monomials of the form $\prod \nabla_{\{\mu_1} \dots \nabla_{\mu_k\}}\varphi$.

It is convenient for latter purposes to choose a particular basis. For this, we consider the space of harmonic polynomials in D real variables which are homogeneous of degree l, i.e. the set of all polynomials $h(x)$ in D variables ($x \in \mathbb{R}^D$) with complex coefficients satisfying $h(tx) = t^l h(x)$, and $\Delta h(x) = 0$. Some relevant facts about such polynomials are collected in appendix A. Let the number of linearly independent degree l harmonic polynomials be $N(l, D)$ (see appendix A for an explicit formula). We denote by $h_{(l,m)}, m = 1, \dots, N(l, D)$ a basis of degree l harmonic polynomials. We normalize this basis so that [2]

$$\int_{S^{D-1}} \overline{h}_{(l,m)}(\hat{x}) h_{(l',m')}(\hat{x})\, \mathrm{d}\Omega(\hat{x}) = \delta_{l,l'}\delta_{m,m'} \tag{3.1.2}$$

where $\mathrm{d}\Omega$ is the standard integration element on the sphere. To shorten the notation, we introduce

$$\mathbb{L} = \{(l, m) \,|\, l \in \mathbb{N},\, m = 1, \dots N(l, D)\}. \tag{3.1.3}$$

[2] With this normalization, the harmonic polynomials restricted to S^{D-1} are the $D-1$-dimensional spherical harmonics.

For $\ell = (l, m)$, we set $|\ell| = l$.

A basis of V is then given by $\mathbf{1}$, together with the elements

$$a = \prod_{\ell \in \mathbb{L}} \frac{1}{\sqrt{a_\ell !}} \left(c_\ell^{-1} \bar{h}_\ell(\nabla)\varphi\right)^{a_\ell} \quad , \tag{3.1.4}$$

where the vector $a \in V$ is determined by and identified with the sequence $a = \{a_\ell \in \mathbb{N} \mid \ell \in \mathbb{L}\}$ of non-negative integers, only finitely many of which are non-zero, and c_ℓ is a normalization constant given in appendix C.

As an alternative generating set for V, we take $\mathbb{L}$-valued functions on finite index sets, $\mathfrak{a} : S_\mathfrak{a} \to \mathbb{L}$, where $S_\mathfrak{a}$ is some finite index set. Let $\tilde{V}$ be the set of these functions. When we deal with $\mathfrak{a}^1, \dots, \mathfrak{a}^n \in \tilde{V}$, we will assume that the respective index sets $S_{\mathfrak{a}^1}, \dots, S_{\mathfrak{a}^n}$ are disjoint unless otherwise stated. We identify $\mathfrak{a} \in \tilde{V}$ with

$$\mathfrak{a} := \prod_{i \in S_\mathfrak{a}} \bar{h}_{\mathfrak{a}_i}(\nabla)\varphi \in V \tag{3.1.5}$$

The difference to the vectors defined in eq. (3.1.4) is slight; the latter definition is helpful when one wants to distinguish identical factors $\bar{h}_\ell(\nabla)\varphi$ – in the second definition they are associated to different elements of the index set $S_\mathfrak{a}$.

We will always use standard typeface to denote vectors of the former kind and Fraktur typeface for the latter. The basis eq. (3.1.4) is more convenient for the calculations in perturbation theory, whereas we have introduced vectors of the form eq. (3.1.5) to facilitate the proof of associativity.

The Schwinger functions of the model are well-known. The most convenient way is to state them via sums over "graphs".

Given n finite index sets $S_{\mathfrak{a}^1}, \ldots, S_{\mathfrak{a}^n}$, we define $\bar{\mathcal{G}}(S_{\mathfrak{a}^1}, \ldots, S_{\mathfrak{a}^n})$ to be the set of graphs on $\bigcup_{i=1}^n S_{\mathfrak{a}^i}$ with edges connecting all the elements of these sets,

$$\begin{aligned}\bar{\mathcal{G}}(S_{\mathfrak{a}^1}, \ldots, S_{\mathfrak{a}^n}) = \Big\{ & G \subset \{(i,j) : i,j \in S_{\mathfrak{a}^k}, j \in S_{\mathfrak{a}^l} \text{ for some } 1 \le k < l \le n, \} : \\ & i \in S_{\mathfrak{a}^k} \Rightarrow \exists! e = (j,l) \in G \text{ such that } i = j \text{ or } i = l \Big\} . \end{aligned} \tag{3.1.6}$$

For fields $\mathfrak{a}^1, \ldots, \mathfrak{a}^n$ as in eq. (3.1.5), we have the Schwinger function

$$\langle \mathfrak{a}^1(x_1) \ldots \mathfrak{a}^n(x_n) \rangle = \sum_{G \in \bar{\mathcal{G}}(S_{\mathfrak{a}^1}, \ldots, S_{\mathfrak{a}^n})} P_G(\mathfrak{a}^1, \ldots, \mathfrak{a}^n; x_1, \ldots, x_n) , \tag{3.1.7}$$

where

$$P_G(\mathfrak{a}^1, \ldots, \mathfrak{a}^n; x_1, \ldots, x_n) = \prod_{1 \le k < l \le n} \left(\prod_{\substack{(i,j) \in G \\ i \in S_{\mathfrak{a}^k}, j \in S_{\mathfrak{a}^l}}} \bar{h}_{\mathfrak{a}^k_i} \left(\frac{\partial}{\partial x_k} \right) \bar{h}_{\mathfrak{a}^l_j} \left(\frac{\partial}{\partial x_l} \right) \frac{1}{|x_k - x_l|^{D-2}} \right) \tag{3.1.8}$$

It is not hard to develop the OPE and the vertex operator $Y_0(\mathfrak{a}, x)$ in the notation of eq. (3.1.5), the derivation can be found in appendix C. The result is

$$Y_0(\mathfrak{a}, x)\mathfrak{b} = \sum_{G \in \mathcal{G}(S_\mathfrak{a}, S_\mathfrak{b})} P_G(\mathfrak{a}, \mathfrak{b}, x) \left(\exp(x \cdot \nabla) \mathfrak{a}^G \right) \mathfrak{b}^G \tag{3.1.9}$$

where

$$\mathcal{G}(S_\mathfrak{a}, S_\mathfrak{b}) = \{ G \subset S_\mathfrak{a} \times S_\mathfrak{b} : (i,j), (k,l) \in G \Rightarrow i \neq k, j \neq l \}$$

$$
\begin{aligned}
P_G(\mathfrak{a},\mathfrak{b};x) = & \prod_{(i,j)\in G} \bar{h}_{\mathfrak{a}_i}(\partial)\bar{h}_{\mathfrak{b}_j}(-\partial)|x|^{2-D} \\
\mathfrak{a}^G = & \prod_{i\in S_\mathfrak{a}\setminus G_\mathfrak{a}} \bar{h}_{\mathfrak{a}_i}(\nabla)\varphi \\
\mathfrak{b}^G = & \prod_{i\in S_\mathfrak{b}\setminus G_\mathfrak{b}} \bar{h}_{\mathfrak{b}_i}(\nabla)\varphi \\
G_\mathfrak{a} = & \{i \in S_\mathfrak{a} : \exists j \in S_\mathfrak{b} \text{ so that } (i,j) \in G\} \\
G_\mathfrak{b} = & \{j \in S_\mathfrak{b} : \exists i \in S_\mathfrak{a} \text{ so that } (i,j) \in G\}\,.
\end{aligned} \tag{3.1.10}
$$

The subscript "0" reminds us that we are dealing with a free field in this section. To present the vertex operators in the notation of eq. (3.1.4), it is convenient to view V as a "Fock-space," with a_ℓ (see eq. (3.1.4)) interpreted as the "occupation number" of the "mode" labeled by ℓ, and with $\mathbf{1}$ playing the role of "Fock-vacuum" denoted $|0\rangle$ (vanishing occupation number). On this Fock-space, one can then define creation and annihilation operators $\mathbf{a}_\ell, \mathbf{a}_\ell^+ : V \to V$, see appendix C. They satisfy the standard commutation relations

$$
[\mathbf{a}_\ell, \mathbf{a}_{\ell'}^+] = \delta_{\ell,\ell'}\, \mathrm{Id}_V\,, \quad [\mathbf{a}_\ell^+, \mathbf{a}_{\ell'}^+] = [\mathbf{a}_\ell, \mathbf{a}_{\ell'}] = 0\,. \tag{3.1.11}
$$

In this language, the basis elements of V are written as

$$
a = \prod_{\ell\in\mathbf{L}} \frac{(\mathbf{a}_\ell^+)^{a_\ell}}{\sqrt{a_\ell!}}\,|0\rangle\,. \tag{3.1.12}
$$

We now give the formula for $Y_0(\varphi, x)$ corresponding to the basic field. For $D > 2$, this is given by

$$
Y_0(\varphi, x) = K_D\, r^{-(D-2)/2} \sum_{l\in\mathbf{L}} \frac{1}{\sqrt{\omega(\ell, D)}}
$$

$$\times \left[r^{|\ell|+(D-2)/2} h_\ell(\hat{x})\, \mathbf{a}_\ell^+ + r^{-|\ell|-(D-2)/2} \overline{h_\ell(\hat{x})}\, \mathbf{a}_\ell \right], \tag{3.1.13}$$

where $\hat{x} = x/|x|$, $K_D = \sqrt{D-2}$, and the "frequency" $\omega(\ell, D)$ is given by $2|\ell| + D - 2$, see appendix C. For a general $a \in V$ of the form eq. (3.1.4), the vertex operators $Y_0(a, x) : V \to V$ are

$$Y_0(a,x) = : \prod_{\ell \in \mathrm{L}} \frac{1}{(a_\ell!)^{1/2}} \left\{ c_\ell^{-1} \bar{h}_\ell(\partial) Y_0(\varphi, x) \right\}^{a_\ell} : \ . \tag{3.1.14}$$

Here, double dots $: \cdots :$ mean "normal ordering", i.e., all creation operators are to the right of all annihilation operators.

There is a representation $\tilde{R}$ of Spin(D) on the space of harmonic polynomials given by its action on the basis elements h_ℓ,

$$(\tilde{R}(g) h_\ell)(x) = h_\ell(g^{-1}x) \tag{3.1.15}$$

where we have identified $g \in \mathrm{Spin}(D)$ with its image under the universal cover Spin(D) $\to$ $SO(D)$ in the fundamental representation. Eq. (3.1.15) is automatically unitary due to our choice of harmonic polynomials in eq. (3.1.2). The representation R of Spin(D) on V is given by

$$R(g) \left(\bar{h}_{\ell_1}(\nabla)\varphi \dots \bar{h}_{\ell_n}(\nabla)\varphi \right) = \left((\tilde{R}(g)\bar{h}_{\ell_1})(\nabla)\varphi \dots (\tilde{R}(g)\bar{h}_{\ell_n})(\nabla)\varphi \right) \tag{3.1.16}$$

i.e. composite fields transform in the tensor representation $\tilde{R} \otimes \cdots \otimes \tilde{R}$. This transformation is reducible and can be decomposed into finitely many finite dimensional unitary irreducible representations S of Spin(D) [34]. The product $\bar{h}_{\ell_1}(\nabla)\varphi \dots \bar{h}_{\ell_n}(\nabla)\varphi$ decomposes into vectors with a definite grading S accordingly. In 3 dimensions, this is the well known decomposition of tensor representations of $SU(2)$ into irreducible representations via Clebsch-Gordon coefficients.

The other two gradings can be given more explicitly. The theory is purely bosonic,

$$V = V^{0,\bullet,\bullet}.$$

The grading by the scaling dimension is given by

$$\bar{h}_\ell(\nabla)\varphi \in V^{0,(D-2)/2+|\ell|,\bullet},$$
$$a \in V^{0,r,\bullet}, b \in V^{0,t,\bullet} \Rightarrow (ab) \in V^{0,r+t,\bullet}.$$

The rotation covariance axiom can be checked by writing

$$h_\ell(\Lambda^{-1}x) = (\tilde{R}(\Lambda)h_\ell)(x) = \sum_{\ell'} \mathrm{D}(\Lambda)_\ell^{\ell'} h_{\ell'}(x) \tag{3.1.17}$$

where $\mathrm{D}(g)_\ell^{\ell'}$ are the entries of a unitary matrix. Then we have

$$\begin{aligned}
&R(\Lambda)Y\left(R(\Lambda)^{-1}\bar{h}_\ell(\nabla)\varphi, \Lambda^{-1}x\right)R(\Lambda)^{-1}\\
&\quad = R(\Lambda)\Bigg(\sum_{\ell_1} \mathrm{D}(\Lambda^{-1})_\ell^{\ell_1} h_{\ell_1}\left(\frac{\partial}{\partial(\Lambda^{-1}x)}\right)\\
&\quad\quad \times K_D \sum_{\ell_2} \omega(\ell_2, D)^{-1/2}\left(h_{\ell_2}(x)\mathbf{a}_{\ell_2}^{+} + r^{-2|\ell_2|-D+2}\bar{h}_{\ell_2}(x)\mathbf{a}_{\ell_2}\right)R(\Lambda^{-1})\Bigg)\\
&\quad = \sum_{\ell_1,\ell_3} \mathrm{D}(\Lambda^{-1})_\ell^{\ell_1}\mathrm{D}(\Lambda)_{\ell_1}^{\ell_3} h_{\ell_3}(\partial)\\
&\quad\quad \times K_D \sum_{\ell_2,\ell_4,\ell_5} \omega(\ell_2, D)^{-1/2}\mathrm{D}(\Lambda)_{\ell_2}^{\ell_4}\mathrm{D}(\Lambda)_{\ell_2}^{\ell_5}\left(h_{\ell_4}(x)\mathbf{a}_{\ell_5}^{+} + r^{-2|\ell_4|-D+2}\bar{h}_{\ell_4}(x)\mathbf{a}_{\ell_5}\right)\\
&\quad = Y(\bar{h}_\ell(\nabla)\varphi, x)
\end{aligned} \tag{3.1.18}$$

where in the last equation, we have used the unitarity of D. Thus we have proved rotation covariance for φ-linear vectors $v \in V$. For composite operators, rotation covariance follows from their definition eq. (3.1.14) and eq. (3.1.18).

The compatibility axiom eq. (2.1.10) follows immediately from definition eq. (3.1.14). It remains only to show associativity (eq. (2.1.8)) and skew-symmetry (eq. (2.1.11)), which we give in the following subsection.

It is not difficult to adapt the model of the last subsection to the case $D = 2$. The vertex operator in this case reads

$$Y_0(\varphi, x) = \mathbf{a}_0 \ln r + \mathbf{a}_0^+ + \sum_{\ell=(l,m)\in \mathrm{L}}^{\infty} \frac{1}{\sqrt{2l}} \left[r^l \mathrm{e}^{imla} \, \mathbf{a}_{(l,m)}^+ + r^{-l} \mathrm{e}^{-imla} \, \mathbf{a}_{(l,m)} \right]. \tag{3.1.19}$$

where we have identified $\mathbb{R}^2 \simeq \mathbb{C}$ and set $x = re^{i\alpha}$, $r \in \mathbb{R}_{>0}$, $\alpha \in \mathbb{R}$.

3.1.1 Proof of associativity for the free boson

We show that associativity holds for vectors of the form eq. (3.1.5). This is in fact much easier than showing it using the notation from eq. (3.1.4), which would make the proof very cumbersome.

Let us repeat the formula for the free field vertex operator eq. (3.1.9),

$$Y_0(\mathfrak{a}, x)\mathfrak{b} = \sum_{G \in \mathcal{G}(S_\mathfrak{a}, S_\mathfrak{b})} P_G(\mathfrak{a}, \mathfrak{b}, x) \left(\exp(x \cdot \nabla)\mathfrak{a}^G\right) \mathfrak{b}^G \tag{3.1.20}$$

where this time we include the case $D = 2$ by setting

$$P_G(\mathfrak{a}, \mathfrak{b}; x) = \prod_{(i,j)\in G} \bar{h}_{\mathfrak{a}_i}(\partial) \bar{h}_{\mathfrak{b}_j}(-\partial) g(|x|)$$

with

$$g(r) = \begin{cases} r^{2-D} & \text{for } D > 2 \\ \ln r & \text{for } D = 2\,. \end{cases} \tag{3.1.21}$$

For a harmonic function f on $\mathbb{R}^D$, we have (see eq. (A.0.8))

$$f(x) = \exp(x \cdot \partial) f(0) = \sum_{\ell \in \mathbf{L}} \frac{\Gamma(D/2) h_\ell(x)}{2^{|\ell|}\Gamma(|\ell| + D/2)} \bar{h}_\ell(\partial) f(0)\,. \tag{3.1.22}$$

As $\Delta \bar{h}_{\ell'}(\nabla)\varphi \equiv 0$ by the definition of V, we may use this decomposition to write

$$\exp(x \cdot \nabla)\bar{h}_{\ell'}(\nabla)\varphi = \sum_{\ell \in \mathbf{L}} \frac{\Gamma(D/2) h_\ell(x)}{2^{|\ell|}\Gamma(|\ell| + D/2)} \bar{h}_\ell(\nabla)\bar{h}_{\ell'}(\nabla)\varphi\,. \tag{3.1.23}$$

Thus

$$\begin{aligned} \exp(x \cdot \nabla)\mathfrak{a} &= \prod_{i \in S_\mathfrak{a}} \exp(x \cdot \nabla)\bar{h}_{\mathfrak{a}_i}(\nabla)\varphi \\ &= \prod_{i \in S_\mathfrak{a}} \Big(\sum_{\ell_i \in \mathbf{L}} \frac{\Gamma(D/2) h_{\ell_i}(x)}{2^{|\ell_i|}\Gamma(|\ell_i| + D/2)} \bar{h}_{\ell_i}(\nabla)\bar{h}_{\mathfrak{a}_i}(\nabla)\varphi \Big)\,. \end{aligned} \tag{3.1.24}$$

The very simple idea of the proof of associativity is that the left and right hand side of the associativity condition are both just expansions of a finite sum of terms of the form

$$(\exp(x \cdot \nabla)\mathfrak{d}_1)\,(\exp(y \cdot \nabla)\mathfrak{d}_2)\,\mathfrak{d}_3 P(x - y, x, y) \tag{3.1.25}$$

with $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3 \in V$ and $P \in \mathbb{C}[\log|x-y|, \log|x|, \log|y|, |x-y|^{-1}, |x|^{-1}, |y|^{-1}]$. Our task is only to put both sides back into this form.

For $|x| > |y| > |x-y|$ we have

$$Y(\mathfrak{a},x)Y(\mathfrak{b},y)\mathfrak{c} = \sum_{G\in\mathcal{G}(S_\mathfrak{b},S_\mathfrak{c})} P_G(\mathfrak{b},\mathfrak{c},y) \sum_{\substack{\ell_i\in\mathbf{L}\\ i\in S_\mathfrak{b}\setminus G_\mathfrak{b}}} \prod_{i\in S_\mathfrak{b}\setminus G_\mathfrak{b}} \frac{\Gamma(D/2)h_{\ell_i}(y)}{2^{|\ell_i|}\Gamma(|\ell_i|+D/2)}$$
$$\times \sum_F P_F(\mathfrak{a},\mathfrak{b}^{G,\vec{\ell}}\mathfrak{c}^G,x) \sum_{\substack{\ell_i\in\mathbf{L}\\ i\in S_\mathfrak{a}\setminus F_\mathfrak{a}}} \prod_{i\in S_\mathfrak{a}\setminus F_\mathfrak{a}} \frac{\Gamma(D/2)h_{\ell_i}(x)}{2^{|\ell_i|}\Gamma(|\ell_i|+D/2)}\bar{h}_{\ell_i}(\nabla)\bar{h}_{\mathfrak{a}_i}(\nabla)\varphi$$
$$\times \prod_{i\in S_\mathfrak{b}\setminus(F_\mathfrak{b}\cup G_\mathfrak{b})} \bar{h}_{l_i}(\nabla)\bar{h}_{\mathfrak{b}_i}(\nabla)\varphi \prod_{i\in S_\mathfrak{c}\setminus(F_\mathfrak{c}\cup G_\mathfrak{c})} \bar{h}_{\mathfrak{c}_i}(\nabla)\varphi \qquad (3.1.26)$$

where the sum $\sum_F$ runs over all $F \in \mathcal{G}(S_\mathfrak{a}, S_\mathfrak{b}\cup S_\mathfrak{c}\setminus(G_\mathfrak{b}\cup G_\mathfrak{c}))$ and

$$V \ni \mathfrak{b}^{G,\vec{\ell}}\mathfrak{c}^G = \prod_{i\in S_\mathfrak{b}\setminus G_\mathfrak{b}} \bar{h}_{\ell_i}(\nabla)\bar{h}_{\mathfrak{b}_i}(\nabla)\varphi \prod_{i\in S_\mathfrak{c}\setminus G_\mathfrak{c}} \bar{h}_{\mathfrak{c}_i}(\nabla)\varphi\,. \qquad (3.1.27)$$

In eq. (3.1.26), we can carry out the sum over all ℓ_i for $i\in F_\mathfrak{b}$:

$$\sum_{\substack{l_i\in\mathbf{L}:\\ i\in F_\mathfrak{b}}} \prod_{i\in F_\mathfrak{b}} \frac{\Gamma(D/2)h_{\ell_i}(y)}{2^{|\ell_i|}\Gamma(|\ell_i|+D/2)} P_F(\mathfrak{b}^{G,\vec{\ell}}\mathfrak{c}^G)$$
$$=\sum_{\substack{l_i\in\mathbf{L}\\ i\in F_\mathfrak{b}}} \frac{\Gamma(D/2)h_{\ell_i}(y)}{2^{|\ell_i|}\Gamma(|\ell_i|+D/2)} \prod_{(i,j)\in F:j\in S_\mathfrak{b}} \bar{h}_{\mathfrak{a}_i}(\partial)\bar{h}_{\mathfrak{b}_j}(-\partial)\bar{h}_{\ell_j}(-\partial)g(|x|)$$
$$\times \prod_{(i,j)\in F:j\in S_\mathfrak{c}} \bar{h}_{\mathfrak{a}_i}(\partial)\bar{h}_{\mathfrak{c}_j}(-\partial)g(|x|)$$
$$= \prod_{(i,j)\in F:j\in S_\mathfrak{b}} \bar{h}_{\mathfrak{a}_i}(\partial)\bar{h}_{\mathfrak{b}_j}(-\partial)g(|x-y|) \prod_{(i,j)\in F:j\in S_\mathfrak{c}} \bar{h}_{\mathfrak{a}_i}(\partial)\bar{h}_{\mathfrak{c}_j}(-\partial)g(|x|)\,. \qquad (3.1.28)$$

For the right-hand side of the associativity condition, we have

$$Y(Y(\mathfrak{a},x-y)\mathfrak{b},y)\mathfrak{c} = \sum_{G\in\mathcal{G}(S_\mathfrak{a},S_\mathfrak{b})} P_G(\mathfrak{a},\mathfrak{b},x-y) \sum_{\substack{\ell_i\in\mathbf{L}\\ i\in S_\mathfrak{a}\setminus G_\mathfrak{a}}} \prod_{i\in S_\mathfrak{a}\setminus G_\mathfrak{a}} \frac{\Gamma(D/2)h_{\ell_i}(y)}{2^{|\ell_i|}\Gamma(|\ell_i|+D/2)}$$

$$\times \sum_F P_F(\mathfrak{a}^{G,\vec{\ell}}\mathfrak{b}^G, x) \sum_{\jmath_i \in \mathbf{L}} \prod_i \frac{\Gamma(D/2) h_{\jmath_i}(y)}{2^{|\jmath_i|}\Gamma(|\jmath_i| + D/2)} \bar{h}_{\jmath_i}(\nabla)\bar{h}_{\ell_i}(\nabla)\bar{h}_{\mathfrak{a}_i}(\nabla)\varphi$$

$$\times \prod_{i \in S_\mathfrak{b} \setminus (F_\mathfrak{b} \cup G_\mathfrak{b})} \bar{h}_{\ell_i}(\nabla)\bar{h}_{\mathfrak{b}_i}(\nabla)\varphi \prod_{i \in S_\mathfrak{c} \setminus (F_\mathfrak{c} \cup G_\mathfrak{c})} \bar{h}_{\mathfrak{c}_i}(\nabla)\varphi \tag{3.1.29}$$

where in the second line, the sum $\sum_F$ runs over all $F \in \mathcal{G}(S_\mathfrak{a} \cup S_\mathfrak{b} \setminus (G_\mathfrak{a} \cup G_\mathfrak{b}), S_\mathfrak{c})$, the sum $\sum_{\jmath_i \in \mathbf{L}}$ has to be understood as a multiple sum, one for each $i \in S_\mathfrak{a} \cup S_\mathfrak{b} \setminus (G_\mathfrak{a} \cup G_\mathfrak{b} \cup F_\mathfrak{a} \cup F_\mathfrak{b})$, and the product $\prod_i$ runs over all $i \in S_\mathfrak{a} \cup S_\mathfrak{b} \setminus (G_\mathfrak{a} \cup G_\mathfrak{b} \cup F_\mathfrak{a} \cup F_\mathfrak{b})$. Moreover

$$\mathfrak{a}^{G,\vec{\ell}}\mathfrak{b}^G = \prod_{i \in S_\mathfrak{a} \setminus G_\mathfrak{a}} \bar{h}_{\ell_i}(\nabla)\bar{h}_{\mathfrak{a}_i}(\nabla)\varphi \prod_{i \in S_\mathfrak{b} \setminus G_\mathfrak{b}} \bar{h}_{\mathfrak{b}_i}(\nabla)\varphi. \tag{3.1.30}$$

This time we can carry out the sum over all ℓ_i for $i \in F_\mathfrak{a}$:

$$\begin{aligned}
&\sum_{\ell_i \in \mathbf{L}} \prod_{i \in F_\mathfrak{a}} \frac{\Gamma(D/2) h_{\ell_i}(x-y)}{2^{|\ell_i|}\Gamma(|\ell_i| + D/2)} P_F(\mathfrak{a}^{G,\vec{\ell}}\mathfrak{b}^G, c, y) \\
&= \sum_{\ell_i \in \mathbf{L}} \prod_{i \in F_\mathfrak{a}} \frac{\Gamma(D/2) h_{\ell_i}(x-y)}{2^{|\ell_i|}\Gamma(|\ell_i| + D/2)} \prod_{(i,j) \in F : i \in S_\mathfrak{a}} \bar{h}_{\mathfrak{a}_i + \ell_i}(\partial)\bar{h}_{\mathfrak{c}_j}(-\partial) g(|y|) \\
&\quad \times \prod_{(i,j) \in F : i \in S_\mathfrak{b}} \bar{h}_{\mathfrak{b}_i}(\partial)\bar{h}_{\mathfrak{c}_j}(-\partial) g(|y|) \\
&= \prod_{(i,j) \in F : i \in S_\mathfrak{a}} \bar{h}_{\mathfrak{a}_i + \ell_i}(\partial)\bar{h}_{\mathfrak{c}_j}(-\partial) g(|x|) \prod_{(i,j) \in F : i \in S_\mathfrak{b}} \bar{h}_{\mathfrak{b}_i}(\partial)\bar{h}_{\mathfrak{c}_j}(-\partial) g(|y|)
\end{aligned} \tag{3.1.31}$$

In eq. (3.1.29) we see that for $i \in S_\mathfrak{a} \setminus (G_\mathfrak{a} \cup F_\mathfrak{a})$, we can write the sum over ℓ_i and $\jmath_i$ as a sum over one index only,

$$\begin{aligned}
&\sum_{\jmath_i, \ell_i \in \mathbf{L}} \frac{\Gamma(D/2) h_{\ell_i}(x-y)}{2^{|\ell_i|}\Gamma(|\ell_i| + D/2)} \frac{\Gamma(D/2) h_{\jmath_i}(y)}{2^{|\jmath_i|}\Gamma(|\jmath_i| + D/2)} \bar{h}_{\jmath_i}(\nabla)\bar{h}_{\ell_i}(\nabla)\bar{h}(\nabla)_{\mathfrak{a}_i}\varphi \\
&= \exp\left((x-y) \cdot \nabla\right) \exp(y \cdot \nabla) \bar{h}_{\mathfrak{a}_i}(\nabla)\varphi \\
&= \exp\left(x \cdot \nabla\right) \bar{h}_{\mathfrak{a}_i}(\nabla)\varphi
\end{aligned}$$

$$= \sum_{\ell_i \in \mathrm{L}} \frac{\Gamma(D/2) h_{\ell_i}(x)}{2^{|\ell_i|}\Gamma(|\ell_i| + D/2)} \bar{h}_{\ell_i}(\nabla) h_{\mathfrak{a}_i}(\nabla)\varphi \tag{3.1.32}$$

Inserting eqs. (3.1.28) into eq. (3.1.26), and eqs. (3.1.31) and (3.1.32) into (3.1.29) we get in both cases

$$\sum_{G \in \mathcal{G}(\mathfrak{a},\mathfrak{b},\mathfrak{c})} Q_G(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, x, y, x-y) \prod_{i \in S_\mathfrak{a} \setminus G_\mathfrak{a}} (\exp(x \cdot \nabla) h_{\mathfrak{a}_i}(\nabla)\varphi)$$
$$\times \prod_{i \in S_\mathfrak{b} \setminus G_\mathfrak{b}} (\exp(y \cdot \nabla) h_{\mathfrak{b}_i}(\nabla)\varphi) \prod_{i \in S_\mathfrak{c} \setminus G_\mathfrak{c}} \bar{h}_{\mathfrak{c}_i}(\nabla)\varphi \tag{3.1.33}$$

where

$$\begin{aligned}
\mathcal{G}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) =& \Big\{ G \subset S_\mathfrak{a} \times S_\mathfrak{b} \cup S_\mathfrak{a} \times S_\mathfrak{c} \cup S_\mathfrak{b} \times S_\mathfrak{c} : \\
& (i,j), (k,l) \in G \Rightarrow (i,j) = (k,l) \text{ or } i, j \notin \{k, l\} \Big\} \\
Q_G(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, x, y, x-y) =& \prod_{(i,j) \in G_{\mathfrak{ab}}} \bar{h}_{\mathfrak{a}_i}(\partial_x) \bar{h}_{\mathfrak{b}_j}(\partial_y) |x-y|^{2-D} \prod_{(i,j) \in G_{\mathfrak{ac}}} \bar{h}_{\mathfrak{a}_i}(\partial_x) \bar{h}_{\mathfrak{c}_j}(-\partial_x) |x|^{2-D} \\
& \times \prod_{(i,j) \in G_{\mathfrak{bc}}} \bar{h}_{\mathfrak{a}_i}(\partial_y) \bar{h}_{\mathfrak{c}_j}(-\partial_y) |y|^{2-D} \\
G_\mathfrak{a} =& \{ i \in S_\mathfrak{a} : \exists j \in S_\mathfrak{b} \cup S_\mathfrak{c} \text{ such that } (i,j) \in G \} \\
G_{\mathfrak{ab}} =& G \cap S_\mathfrak{a} \times S_\mathfrak{b}
\end{aligned}$$

and $G_\mathfrak{b}, G_\mathfrak{c}, G_{\mathfrak{ac}}, G_{\mathfrak{bc}}$ are defined similarly.

This proves associativity for the free boson.

Skew-symmetry is easily checked: The right hand side of eq. (2.1.11) reads

$$\exp(x\cdot\nabla)\left(\sum_{G\in\mathcal{G}(S_{\mathfrak{b}},S_{\mathfrak{a}})} P_G(\mathfrak{b},\mathfrak{a},-x)(\exp(-x\cdot\nabla)\mathfrak{b}^G)\mathfrak{a}^G\right). \tag{3.1.34}$$

Let $G\in\mathcal{G}(S_{\mathfrak{b}},S_{\mathfrak{a}})$ and $G':=\{(i,j)\in S_{\mathfrak{a}}\times S_{\mathfrak{b}}:(j,i)\in G\}$. Then $P_G(\mathfrak{b},\mathfrak{a},-x)=P_{G'}(\mathfrak{a},\mathfrak{b},x)$. Furthermore

$$\exp(x\cdot\nabla)\left(\exp(-x\cdot\nabla)\mathfrak{b}^G)\mathfrak{a}^G\right)=\mathfrak{b}^G\exp(x\cdot\nabla)\mathfrak{a}^G$$

and thus eq. (3.1.34) is identical to eq. (3.1.20) which proves eq. (2.1.11) for the free boson.

3.2 The massless Thirring model

The massless Thirring model is a fermionic field theory in two dimensions with a current-current interaction. The classical theory is defined by the Lagrangian

$$\mathcal{L}=\phi^\dagger \mathrm{i}\gamma^\mu\partial_\mu\phi+gj^\mu j_\mu\,, \tag{3.2.35}$$

where ϕ is a two component field and (classically) $j_\mu=\phi^\dagger\gamma^0\gamma_\mu\phi$. Here we use the gamma matrices

$$\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

The massless Thirring model is an example of an exactly solvable model and has been treated by a large number of authors. It can be seen from the Lagrangian eq. (3.2.35) that the model is conformally invariant. A set of n-point functions satisfying the Wightman axioms and the field equation

$$\mathrm{i}\gamma^\mu\partial_\mu\phi=g\gamma^\mu j_\mu\phi \tag{3.2.36}$$

with a suitable definition of the non-linear terms on the right hand side have been given by Klaiber [52] building on an idea by Johnson [47]. Formally, Klaiber gives a solution in terms of Thirring field operators on a positive definite Hilbert space[3]. Klaiber's solution leads to the correct n-point functions, in the sense that they fulfill the Wightman axioms. All Wightman axioms except positivity can be checked directly from the explicit expressions in [52]. The proof of positivity has been given in [15], using ideas from [17].

The definition of composite fields in this framework is problematic. In the earlier papers [47, 52] the issue of composite fields is not treated (apart from the current and the right-hand side of the field equation). This was first addressed by Lowenstein [53] with a definition of "normal products" in the Thirring model. Composite fields can then be defined as limits of coinciding points of normal products. However the set of monomials in the Thirring fields $\phi, \bar{\phi}$ and its derivatives does not close under the OPE. The existence of an OPE for arbitrary composite fields is crucial in our approach, where the OPE is understood to *define* the theory. Hence Lowenstein's framework is not a suitable starting point to construct a vertex algebra of the Thirring model.

Examples of operator product expansions in the massless Thirring model have been given in [17, 53, 77]. A systematic treatment of the operator product expansion in conformally invariant quantum field theory and for the massless Thirring model in particular is due to Lüscher [54]. He showed that the product of two quantum fields applied to the vacuum can be expanded into conformal partial waves. This decomposition is identical to the Wilson operator product expansion of the two quantum fields applied to the vacuum if the OPE exists. For the massless Thirring model, it is also proved in this reference that there exists

[3]Strictly speaking, the construction of the Thirring fields as operators on a Hilbert space in [52] is not rigorous, see the remarks in [75].

a set of local quantum fields that is closed under the conformal wave *viz.* operator product expansion. We are not going to use Lüscher's general results here; instead we are going to take a more direct and explicit approach.

Our treatment of the massless Thirring model will be based on the ideas of Nakanishi [58–60]. He gave an expression for the field operator only in terms of the free massless boson φ and its "dual field" $\tilde{\varphi}$. More precisely, the Thirring field operator is given by normal ordered exponentials of φ and $\tilde{\varphi}$. A rigorous treatment of this setting can be found in [56, 57], where φ, its dual and the exponential are defined as operator valued distributions in an indefinite metric (Krein) space. Here a definition of composite operators consistent with the operator product expansion can be quite easily achieved via bose normal ordering. However we do not want to state the rigorous version of Nakanishi's construction of the massless Thirring model as the definition of the Krein space would require a lot of work. This is not necessary as we can apply Nakanishi's ideas directly in the vertex algebra framework.

This will allow us to give explicit formulas for the vertex operators of the massless Thirring model (section 3.3) and show directly that the axioms of definition 2.1.1 are fulfilled. The main work is the proof of associativity which we give in section 3.3.1.

As the massless Thirring model is a conformally invariant model in two dimensions, one expects that it should also fulfill the conditions for a full field algebra (definition 2.2.2). As we mentioned in our comparison between full field algebras and vertex algebras in $D = 2$ with conformal invariance in chapter 2.2.1, the only notable difference between the two is that the "convergence" condition of the former is slightly stronger than the "associativity" condition of the latter. However from our proof of associativity in section 3.3.1 it will be obvious that the stronger convergence property holds as well.

3.2.1 Classical model

The solution of the massless Thirring model in [47, 52, 59] is based on the following solution of the *classical* model. In this and the next subsection we are going to use a Minkowski signature on $\mathbb{R}^2$ to alleviate the comparison with the references.

Let M be two-dimensional Minkowski space with the metric given by $\mathrm{d}x^2 = \mathrm{d}x_0^2 - \mathrm{d}x_1^2$. We choose not to use the standard notation for the Thirring model which includes vector-valued fields reminiscent of Weyl fermions, gamma matrices, etc. but rather write down all formulas for the 2 components of the Thirring field separately, which we denote by $\phi, \bar{\phi}$. The equations of motion for the fields[4] are given by

$$\begin{aligned} \frac{\mathrm{i}}{2}\left(\partial_0 - \partial_1\right)\phi(x) &= -g\,\bar{\phi}^*\,\bar{\phi}\,\phi \\ \frac{\mathrm{i}}{2}\left(\partial_0 + \partial_1\right)\bar{\phi}(x) &= -g\,\phi^*\,\phi\,\bar{\phi}\,. \end{aligned} \tag{3.2.37}$$

Here $\phi^*, \bar{\phi}^*$ are the complex conjugates of the mutually independent fields $\phi, \bar{\phi}$. The model has a conserved current,

$$\begin{aligned} j^0 = \phi^*\phi + \bar{\phi}^*\bar{\phi}, &\quad j^1 = \bar{\phi}^*\bar{\phi} - \phi^*\phi \\ \partial_\mu j^\mu = \partial_\mu \epsilon^{\mu\nu} j_\nu &= 0 \end{aligned} \tag{3.2.38}$$

[4] $\phi, \bar{\phi}$ are simply taken to be scalar functions $\phi, \phi : M \to \mathbb{C}$ in this subsection.

where $-\epsilon^{01} = \epsilon^{10} = 1$, $\epsilon^{00} = \epsilon^{11} = 0$. Eq. (3.2.38) means that the curl of j^μ vanishes and hence we can write the current as a gradient,

$$j_\mu = \partial_\mu \mathbf{J}\,. \tag{3.2.39}$$

The expressions

$$\begin{aligned} \psi(x) &= \exp(-ig\mathbf{J}(x))\phi(x) \\ \bar{\psi}(x) &= \exp(-ig\mathbf{J}(x))\bar{\phi}(x) \end{aligned} \tag{3.2.40}$$

and the resulting current $k^0 = \psi^*\psi + \bar{\psi}^*\bar{\psi}$, $k^1 = \bar{\psi}^*\bar{\psi} - \psi^*\psi$ satisfy the equations

$$\begin{aligned} \frac{\mathrm{i}}{2}\left(\partial_0 - \partial_1\right)\psi(x) &= 0 \\ \frac{\mathrm{i}}{2}\left(\partial_0 + \partial_1\right)\bar{\psi}(x) &= 0 \\ k^\mu &= j^\mu\,, \end{aligned} \tag{3.2.41}$$

i.e. $(\psi, \bar{\psi})$ satisfies the equations of a free (fermion) field and its current is identical to the current of $(\phi, \bar{\phi})$. Given a particular solution $(\phi_0, \bar{\phi}_0)$ to the first two eqs. (3.2.41), there is a one-parameter family of solutions

$$\begin{aligned} \phi^{(c)}(x) &= \exp\left(ic\left(\mathbf{K}(x) - \tilde{\mathbf{K}}(x)\right)\right)\phi_0(x) \\ \bar{\phi}^{(c)}(x) &= \exp\left(ic\left(\mathbf{K}(x) + \tilde{\mathbf{K}}(x)\right)\right)\bar{\phi}_0(x), \end{aligned} \tag{3.2.42}$$

where $\mathbf{K}, \tilde{\mathbf{K}}$ are the integrated current and pseudocurrent respectively, defined by $\partial^\mu \mathbf{K} = k^\mu$ and $\epsilon^{\mu\nu}\mathbf{K}_\nu = \partial^\mu \tilde{\mathbf{K}}$, and $c \in \mathbb{R}$. In this way, we get a one-parameter family of solutions to eq. (3.2.37),

$$\phi(x) = \exp\left(i\left((g+c)\mathbf{K}(x) - c\tilde{\mathbf{K}}(x)\right)\right)\psi(x)$$
$$\bar{\phi}(x) = \exp\left(i\left((g+c)\mathbf{K}(x) + c\tilde{\mathbf{K}}(x)\right)\right)\bar{\psi}(x), \tag{3.2.43}$$

This simple solution of the classical model is also at the heart of the solutions in [47, 52, 59] for the quantum model. The additional parameter c will be used to fix the statistics of $\phi, \bar{\phi}$ in the quantum theory.

3.2.2 Klaiber's solution

There is a quite simple analogue to the classical fields $\psi, \bar{\psi}$ in eq. (3.2.40) in quantum field theory, the free massless fermion in two dimensions (for which we use the same symbol). When trying to find a solution via eq. (3.2.43) in this framework, the difficulty lies with the definition of $\mathbf{J}$ as a quantum field and the product on the right hand side. Klaiber [52] resolves these problems within the Fock space representation of the free massless fermion, and we shortly review his work here. Set

$$\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}p)$$

and define $\mathfrak{F}_1, \mathfrak{F}_2$ each as a copy of

$$\bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}_{\text{antisym.}}$$

where the subindex means taking antisymmetrized tensor products. Creation and annihilation operators $\mathbf{b}^*(f), \mathbf{b}(f)$ on $\mathfrak{F}_1$ are defined by

$$(\mathbf{b}^*(f)\Psi)_n(q_1, \ldots, q_n) = \big(f(q_1)\Psi_{n-1}(q_2, \ldots, q_n)\big)_{\text{antisym.}}$$

$$(\mathbf{b}(f)\Psi)_n(q_1,\dots,q_n) = \left(\int_{\mathbb{R}} \mathrm{d}p\, f(p)\Psi_{n+1}(p,q_1,\dots,q_n)\right)_{\text{antisym.}}$$

for $f \in L^2(\mathbb{R}, \mathrm{d}p)$. Analogously we define the action of $\mathbf{c}^*(f), \mathbf{c}(f)$ on $\mathfrak{F}_2$. We identify these operators in an obvious manner with

$$\mathbf{b}^*(f)\otimes\mathbf{1}, \quad \mathbf{b}(f)\otimes\mathbf{1}, \quad \mathbf{1}\otimes\mathbf{c}^*(f), \quad \mathbf{1}\otimes\mathbf{c}^*(f) \tag{3.2.44}$$

acting on $\mathfrak{F} := \mathfrak{F}_1 \otimes \mathfrak{F}_2$. Viewed as distributions, they satisfy the canonical commutation relations

$$\begin{aligned} \{\mathbf{b}(p), \mathbf{b}^*(q)\} &= \{\mathbf{c}(p), \mathbf{c}^*(q)\} = \delta(p-q) \\ \{\mathbf{b}(p), \mathbf{b}(q)\} &= \{\mathbf{b}(p), \mathbf{c}^*(q)\} = 0 \\ \{\mathbf{c}(p), \mathbf{c}(q)\} &= \{\mathbf{c}(p), \mathbf{b}^*(q)\} = 0\,. \end{aligned} \tag{3.2.45}$$

Now we define the free fermion $(\psi, \bar{\psi})$ as operator-valued distributions on $\mathfrak{F}$ by

$$\begin{aligned} \psi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \mathrm{d}p^1 \left[\mathbf{b}^*(p^1)e^{ip\,x} + \mathbf{c}(p^1)e^{-ip\,x}\right], \\ \bar{\psi}(x) &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \mathrm{d}p^1 \left[\mathbf{b}^*(p^1)e^{ip\,x} + \mathbf{c}(p^1)e^{-ip\,x}\right], \end{aligned} \tag{3.2.46}$$

where in the integrands above, $p\,x = p^0 x_0 - p^1 x_1$ and $p^0 = |p^1|$. We now define some field bilinears using Fermi normal ordering[5] $: \cdots :$ of products,

$$j(x) = {:}\psi^\dagger\psi{:}, \quad \bar{j}(x) = {:}\bar{\psi}^\dagger\bar{\psi}{:}$$

[5]The notation $: \cdots :$ is used in sections 3.1 and 4 with a slightly different meaning. Only in this subsection will we use it to denote Fermi normal ordering for operators on a Hilbert space.

$$
\begin{aligned}
J^-(x) &= \frac{1}{2\pi}\int_{-\infty}^{0}\frac{\mathrm{d}p^1}{\sqrt{p^0}}\mathbf{d}(p^1)\left(e^{-\mathrm{i}px}-\theta(\kappa-p_0)\right) \\
\bar{J}^-(x) &= \frac{1}{2\pi}\int_{0}^{\infty}\frac{\mathrm{d}p^1}{\sqrt{p^0}}\mathbf{d}(p^1)\left(e^{-\mathrm{i}px}-\theta(\kappa-p_0)\right) \\
J^+(x) &= \left(J^-(x)\right)^\dagger, \quad \bar{J}^+(x) = \left(\bar{J}^-(x)\right)^\dagger \\
\mathbf{d}(p^1) &= \frac{i}{\sqrt{p^0}}\int \mathrm{d}q^1\left\{\theta(p^1q^1)\left[\mathbf{c}^*(q_1)\mathbf{c}(q_1+p_1)-\mathbf{b}^*(q_1)\mathbf{b}(q_1+p_1)\right]\right. \\
&\quad \left.+\,\theta(q^1(p^1-q^1))\mathbf{b}(p^1-q^1)\mathbf{c}(q^1)\right\},
\end{aligned}
\tag{3.2.47}
$$

where κ is some momentum cutoff. The conserved current is given by $j_0 = j+\bar{j}, j_1 = j-\bar{j}$. $J = J^+ + J^-, \bar{J} = \bar{J}^+ + \bar{J}^-$ have been chosen such that $\partial J = j, \bar{\partial}\bar{J} = \bar{j}$. Note that it was necessary to introduce the cutoff dependent terms to make the integrals well defined. From eq. (3.2.47) follow the commutation relations

$$
\begin{aligned}
\left[J^-(x), J^+(y)\right] &= \frac{1}{\mathrm{i}}D^-_{\kappa\kappa}(x_R, y_R) \\
\left[\bar{J}^-(x), \bar{J}^+(y)\right] &= \frac{1}{\mathrm{i}}D^-_{\kappa\kappa}(x_L, y_L) \\
\left[J^-(x), \bar{J}^+(y)\right] &= 0
\end{aligned}
\tag{3.2.48}
$$

where $x_L = x^0 - x^1$, $x_R = x^0 + x^1$, and

$$
\begin{aligned}
D^\pm_{\kappa\kappa}(r,s) &= D^\pm(r-s) - \Delta^\pm(r) + \Delta^\mp(s) \\
D^-(r) &= \frac{1}{4\pi\mathrm{i}}\log(r-\mathrm{i}0) + \frac{1}{8} \\
D^+(r) &= -D^-(-r) \\
\Delta^\pm(r) &= \pm\frac{1}{2\pi\mathrm{i}}\int_0^\kappa \frac{\mathrm{d}s}{2s}\left(e^{\pm\mathrm{i}sr}-1\right).
\end{aligned}
\tag{3.2.49}
$$

All commutators between J's with identical upper indices $(+,-)$ vanish. The commutation relations of the integrated current with the fermionic fields are

$$
\begin{aligned}
\left[J^{\pm}(x),\psi(y)\right] &= -\,2\sqrt{\pi}D^{\pm}_{\kappa}(x_L,y_L)\psi(y)\\
\left[\bar{J}^{\pm}(x),\bar{\psi}(y)\right] &= -\,2\sqrt{\pi}D^{\pm}_{\kappa}(x_R,y_R)\bar{\psi}(y)\\
\left[J^{\pm}(x),\bar{\psi}(y)\right] &= \left[\bar{J}^{\pm}(x),\psi(y)\right] = 0
\end{aligned}
\tag{3.2.50}
$$

where

$$
D^{\pm}_{\kappa}(r,s) \;= D^{\pm}(r-s)+\Delta^{\mp}(s)\,.
$$

The fermion fields together with the integrated currents are the building blocks of Klaiber's solution. One has to find a way to make the translation variant terms $\Delta^{\pm}(x)$ disappear in physically relevant quantities. In the Klaiber-Thirring model, this is achieved by introducing free charge operators into the formulas for the Thirring fields. Klaiber's ansatz for the Thirring fields $\phi,\bar{\phi}$ (cf. eq. (3.2.43)) is

$$
\begin{aligned}
\phi(x) =& \exp\mathrm{i}\left(g\bar{J}^{+}(x)+(2c+g)J^{+}(x)+\omega^{+}(x)\right)\psi(x)\\
&\times\exp\mathrm{i}\left(g\bar{J}^{-}(x)+(2c+g)J^{-}(x)+\omega^{-}(x)\right)\\
\bar{\phi}(x) =& \exp\mathrm{i}\left(gJ^{+}(x)+(2c+g)\bar{J}^{+}(x)+\bar{\omega}^{+}(x)\right)\bar{\psi}(x)\\
&\times\exp\mathrm{i}\left(gJ^{-}(x)+(2c+g)\bar{J}^{-}(x)+\bar{\omega}^{-}(x)\right)
\end{aligned}
\tag{3.2.51}
$$

where $\omega^{\pm},\bar{\omega}^{\pm}$ are some operators constructed from the free charges

$$
Q=\int \mathrm{d}x^1 j(x),\quad \bar{Q}=\int \mathrm{d}x^1 \bar{j}(x)
$$

and the translation variant functions $\Delta^{\pm}(x_{L/R})$. We refer to [52] for the explicit formulas for $\omega^{\pm}$.

It is not hard to check that products of "fields" as in eq. (3.2.51) define bilinear-form-valued distributions[6] on $\mathfrak{F}$. They are the n-point functions of the model. Writing down the commutators for the fields eq.(3.2.51), the remaining free parameters are fixed by the requirement of locality for the Thirring fields. In [52], the commutators of the Thirring fields are determined by using the formal identities

$$\begin{aligned} e^A e^B &= e^{[A,B]} e^B e^A && \text{for } [A,B] \in \mathbb{C} \\ [A, e^B] &= e^{\lambda} e^B && \text{for } [A,B] = \lambda A,\ \lambda \in \mathbb{C}\,. \end{aligned} \tag{3.2.52}$$

The remaining free parameters are fixed by requiring canonical anticommutation relations. The n-point functions for the Thirring fields are obtained by repeated use of eqs. (3.2.48), (3.2.50) and (3.2.52). Thus Klaiber finds explicit formulas for the n-point functions of $\phi, \bar{\phi}$. Writing $\Phi = (\phi, \bar{\phi})$, $\Phi^* = (\phi^*, \bar{\phi}^*)$, they are given by

$$\begin{aligned} &\Big\langle \Phi_{\mu_1}(x_1) \dots \Phi_{\mu_m}(x_m) \Phi^*_{\nu_1}(y_1) \dots \Phi^*_{\nu_m}(y_m) \Big\rangle \\ &= \exp\Big(\mathrm{i}F(x_1, \dots, x_m, y_1, \dots, y_m)\Big) \langle \Psi_{\mu_1}(x_1) \dots \Psi_{\mu_m}(x_m) \Psi^*_{\nu_1}(y_1) \dots \Psi^*_{\nu_m}(y_m) \rangle \end{aligned} \tag{3.2.53}$$

where $\Psi = (\psi, \bar{\psi})$, $\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_m \in \{1, 2\}$, and

$$\begin{aligned} &F(x_1, \dots, x_m, y_1, \dots, y_m) \\ &\quad = \sum_{1 \le j < k \le m} \rho_1(\mu_j, \mu_k) D^-(y_j^0 - x_k^1 - x_k^0 + x_k^1) + \sigma_1(\mu_j, \mu_k) D^-(x_j^0 + x_k^1 - x_k^0 - x_k^1) \end{aligned}$$

[6] On the other hand it is not obvious whether eq. (3.2.51) really defines operators on $\mathfrak{F}$.

$$+\sum_{1\le j<k\le m} \rho_1(\nu_j,\nu_k)D^-(y_j^0-y_k^1-y_k^0+y_k^1)+\sigma_1(\nu_j,\nu_k)D^-(y_j^0+y_k^1-y_k^0-y_k^1)$$

$$+\sum_{1\le j,k\le m} \rho_2(\mu_j,\nu_k)D^-(x_j^0-x_k^1-y_k^0+y_k^1)+\sigma_2(\mu_j,\nu_k)D^-(x_j^0+x_k^1-y_k^0-y_k^1)$$

with

$$\begin{aligned}
\rho_1(\mu_j,\mu_k) =& \frac{(c+g)^2}{\pi} - 2(c+g) + (-1)^{\mu_j+\mu_k}\left(\frac{c^2}{\pi}-2c\right)\\
&+((-1)^{\mu_j}+(-1)^{\mu_k})\left(\frac{c(c+g)}{\pi}-(2c+g)\right)\\
\sigma_1(\mu_j,\mu_k) =& \frac{(c+g)^2}{\pi} - 2(c+g) + (-1)^{\mu_j+\mu_k}\left(\frac{c^2}{\pi}-2c\right)\\
&-((-1)^{\mu_j}+(-1)^{\mu_k})\left(\frac{c(c+g)}{\pi}-(2c+g)\right)\\
\rho_2(\mu_j,\nu_k) =& -\frac{(c+g)^2}{\pi} - 2(c+g) - (-1)^{\mu_j+\nu_k}\left(\frac{c^2}{\pi}-2c\right)\\
&-((-1)^{\mu_j}+(-1)^{\nu_k})\left(\frac{c(c+g)}{\pi}-(2c+g)\right)\\
\sigma_2(\mu_j,\nu_k) =& -\frac{(c+g)^2}{\pi} - 2(c+g) - (-1)^{\mu_j+\nu_k}\left(\frac{c^2}{\pi}-2c\right)\\
&+((-1)^{\mu_j}+(-1)^{\nu_k})\left(\frac{c(c+g)}{\pi}-(2c+g)\right).
\end{aligned}$$

Strictly speaking the identities (3.2.52) only hold for bounded operators A, B (on some Banach space), so it is unclear if their use is justified here. In [52] these questions of mathematical rigor are left aside. A rigorous proof that the n-point functions found by Klaiber satisfy all Wightman axioms and that eq. (3.2.51) defines operator-valued distributions on $\mathfrak{F}$ is given in [15].

3.3 Construction of the vertex algebra for the massless Thirring model

We will construct the fields of the Thirring model via bosonization. In relativistic field theory, bosonization consists in a representation of the massless free fermion in two dimensions on the Fock space of the massless free boson in two dimensions [67]. The usefulness of this duality between bosons and fermions has, amongst others, been used by Mandelstam [55] in his analysis of the (considerably more complicated) *massive* Thirring model. A solution of the massless Thirring model using bosonization has first been given by Nakanishi [58–60]. A rigorous treatment can be found in [56, 57]. In these papers the Thirring fields are constructed as operators on a Fock-Krein space. We do not want to go into the details of these constructions and directly apply the idea of bosonization to the vertex algebra framework. Below we define the space of composite fields V and the vertex operator Y of the massless Thirring model by formally "exponentiating" two dimensional bosons.
For the special case of chiral algebras, bosonization is a well known concept [18, 48].

Let $\hat{V}$ be the unital differential ring generated by

$$\nabla\chi, \quad \bar{\nabla}\bar{\chi}, \quad \exp(\pm i\kappa), \quad \exp(\pm i\bar{\kappa}) \tag{3.3.54}$$

with derivations[7] $\nabla, \bar{\nabla}$. When taking derivatives, it is informally understood that $\kappa = \alpha\chi + g\bar{\chi}, \bar{\kappa} = g\chi + \alpha\bar{\chi}$ cf. eq. (3.3.57). Here $g \in \mathbb{R}_{\geq 0}$ is the real coupling constant and $\alpha = \sqrt{k+g^2}$, $k \in \mathbb{N}$. The parameter k defines the statistics of the Thirring field, see the discussion of skew-symmetry below.

[7] Our notation here differs slightly from the preceding sections, where we used $\nabla = (\nabla^1, \ldots, \nabla^D)$. Here, $\nabla, \bar{\nabla}$ can informally be understood as the 2 components of the gradient.

Formally, we define V as the quotient of $\hat{V}$ by the following set of relations:

- Anticommutators:

$$\begin{aligned} \exp(\pm i\kappa)\exp(\pm i\bar{\kappa}) &= -\exp(\pm i\bar{\kappa})\exp(\pm i\kappa) \\ \exp(\mp i\kappa)\exp(\pm i\bar{\kappa}) &= -\exp(\pm i\bar{\kappa})\exp(\mp i\kappa) \end{aligned} \tag{3.3.55}$$

- Commutators:

$$[\nabla\chi, \exp(\pm i\kappa)] = [\bar{\nabla}\bar{\chi}, \exp(\pm i\kappa)] = [\nabla\chi, \exp(\pm i\bar{\kappa})] = [\bar{\nabla}\bar{\chi}, \exp(\pm i\bar{\kappa})] = [\nabla\chi, \bar{\nabla}\bar{\chi}] = 0 \tag{3.3.56}$$

- Derivatives, etc.:

$$\begin{aligned} \bar{\nabla}(\nabla\chi) &= \nabla(\bar{\nabla}\bar{\chi}) = 0, & \exp(i\bar{\kappa})\exp(-i\bar{\kappa}) &= \mathbf{1} = \exp(i\kappa)\exp(-i\kappa) \\ \nabla\exp(\pm i\kappa) &= (\pm ig\nabla\chi)\exp(\pm i\kappa) & \bar{\nabla}\exp(\pm i\kappa) &= (\pm i\alpha\bar{\nabla}\bar{\chi})\exp(\pm i\kappa) \\ \nabla\exp(\pm i\bar{\kappa}) &= (\pm i\alpha\nabla\chi)\exp(\pm i\bar{\kappa}) & \bar{\nabla}\exp(\pm i\bar{\kappa}) &= (\pm ig\bar{\nabla}\bar{\chi})\exp(\pm i\bar{\kappa}) \end{aligned} \tag{3.3.57}$$

We define the gradings on V by requiring

$$\nabla^{l_1}\chi\ldots\nabla^{l_m}\chi\bar{\nabla}^{j_1}\bar{\chi}\ldots\bar{\nabla}^{j_n}\bar{\chi}\exp(i(m\kappa+\bar{m}\kappa)) \in V^{m+\bar{m},\rho,\sigma} \tag{3.3.58}$$

where

$$\rho = \sum_{i=1}^{m} l_i + \sum_{i=1}^{n} j_i + \frac{(m^2+\bar{m}^2)(1+2g^2)}{2}$$

$$\sigma = \sum_{i=1}^{m} l_i - \sum_{i=1}^{n} j_i + \frac{m^2 - \bar{m}^2}{2}\,. \tag{3.3.59}$$

The last grading in eq. (3.3.58) is by the irreducible representations of Spin(2). The irreducible representations can be labeled by an integer j and are given by

$$e^{\mathrm{i}\gamma} \mapsto e^{\mathrm{i}j\gamma},$$

where the right hand side is understood as an element of End($\mathbb{C}$). The representation R of Spin(2) on V is given by

$$R(e^{\mathrm{i}\gamma})a = \exp(\mathrm{i}k\gamma)a \quad \text{for } a \in V^{\bullet,\bullet,k}\,. \tag{3.3.60}$$

When inserting this into the covariance axiom eq. (2.1.7), note that the action of $e^{\mathrm{i}\gamma}$ $\in$Spin(2) on $z \in \mathbb{C}$ is given by $z \mapsto e^{2\mathrm{i}\gamma} z$.

As in section 3.1, we use functions on finite index sets to denote composite fields. Let

$$\mathbb{L}' = \{(l,m)\,|\, l \in \mathbb{N},\, m \in \{+1,-1\}\}\,.$$
$$\mathbb{L}'_{>0} = \{(l,m)\,|\, l \in \mathbb{Z}_{>0},\, m \in \{+1,-1\}\}\,.$$

Consider functions $\mathfrak{a}$ that map a finite index set $S_{\mathfrak{a}}$ to $\mathbb{L}'_{>0}$. We identify these functions with

$$\mathfrak{a} = \prod_{i \in S_{\mathfrak{a}}} \nabla^{\mathfrak{a}_i} \varphi \tag{3.3.61}$$

where

$$\nabla^{(l,m)}\varphi = \begin{cases} \nabla^l \chi & \text{if } m = 1 \\ \bar{\nabla}^l \bar{\chi} & \text{if } m = -1\,. \end{cases} \tag{3.3.62}$$

Let $\tilde{V}$ be the set of these functions. When we deal with $\mathfrak{a}^1, \ldots, \mathfrak{a}^n \in \tilde{V}$, we will assume that the respective index sets $S_{\mathfrak{a}^1}, \ldots, S_{\mathfrak{a}^n}$ are disjoint unless otherwise stated. We will say that $i \in S_{\mathfrak{a}^1}, j \in S_{\mathfrak{a}^2}$ are *of the same kind* if $\mathfrak{a}^1_i = (l, m), \mathfrak{a}^2_j = (l', m)$ for some $l, l' \in \mathbb{Z}_{>0}, m \in \{\pm\}$. A generating set of V is given by

$$\left\{ \mathfrak{a} \exp i(m\kappa + \bar{m}\bar{\kappa}) : \mathfrak{a} \in \tilde{V}, m, \bar{m} \in \mathbb{Z} \right\} \tag{3.3.63}$$

Let $\mathfrak{A} = \mathfrak{a} \exp i(m\kappa + \bar{m}\bar{\kappa})$, $\mathfrak{B} = \mathfrak{b} \exp i(n\kappa + \bar{n}\bar{\kappa})$. We define the vertex operator for the Thirring model by setting

$$\begin{aligned} Y(\mathfrak{A}, z)\mathfrak{B} =& f(m, \bar{m}, n, \bar{n}, z) \sum_{G \in \tilde{\mathcal{G}}(S^*_{\mathfrak{a}}, S^*_{\mathfrak{b}})} P_G(\mathfrak{a}, \mathfrak{b}, m, \bar{m}, n, \bar{n}, z) \\ & \times \left(\exp(z\nabla + \bar{z}\bar{\nabla}) \mathfrak{a}^G \right) \mathfrak{b}^G \\ & \times \left(\exp(z\nabla + \bar{z}\bar{\nabla}) \exp(\mathrm{i}(m\kappa + \bar{m}\bar{\kappa})) \right) \exp(\mathrm{i}(n\kappa + \bar{n}\bar{\kappa})) \end{aligned} \tag{3.3.64}$$

where we have used the following notation:

- f is given by

$$\begin{aligned} f(m, \bar{m}, m, \bar{m}, z) =& \exp\left(-(mg + \bar{n}\alpha)(ng + \bar{n}\alpha) \log \bar{z} \right) \\ & \times \exp\left(-(\bar{m}g + n\alpha)(\bar{n}g + n\alpha) \log z \right) \\ =& |z|^{-2\alpha g(m\bar{n} + \bar{m}n) - 2g^2(mn + \bar{m}\bar{n})} \end{aligned}$$

$$\times \, z^{-kmn}\bar{z}^{-k\bar{m}\bar{n}}, \tag{3.3.65}$$

- $S_{\mathfrak{a}}^*, S_{\mathfrak{b}}^*$ are the index sets $S_{\mathfrak{a}}, S_{\mathfrak{b}}$ with one added index,

$$S_{\mathfrak{a}}^* = S_{\mathfrak{a}} \cup \{i_{0,\mathfrak{a}}\}, \quad S_{\mathfrak{b}}^* = S_{\mathfrak{b}} \cup \{i_{0,\mathfrak{b}}\}$$

- $\tilde{\mathcal{G}}(S_{\mathfrak{a}}^*, S_{\mathfrak{b}}^*)$ is the set of graphs connecting vertices from $S_{\mathfrak{a}}^*$ with vertices of the same kind from $S_{\mathfrak{b}}^*$, where the vertices $i_{0,\mathfrak{a}}, i_{0,\mathfrak{b}}$ may be connected to several vertices (of both kinds) from $S_{\mathfrak{b}}, S_{\mathfrak{a}}$ respectively,

$$\begin{aligned}\tilde{\mathcal{G}}(S_{\mathfrak{a}}^*, S_{\mathfrak{b}}^*) = \Big\{ & G \subset \{(i,j) : i \in S_{\mathfrak{a}}^*, j \in S_{\mathfrak{b}}^*\} : \\ & \Big((i,j) \in G, i = (l,m) \in S_{\mathfrak{a}}, j = (l',m') \in S_{\mathfrak{b}} \Rightarrow m = m'\Big), \\ & ((i,j),(k,l) \in G \Rightarrow i \neq k \text{ or } i = k = i_{0,\mathfrak{a}}, j \neq l \text{ or } j = l = i_{0,\mathfrak{b}}), \\ & (i_{0,\mathfrak{a}}, i_{0,\mathfrak{b}}) \notin G \Big\}. \end{aligned} \tag{3.3.66}$$

- P_G is given by

$$\begin{aligned} P_G(\mathfrak{a},\mathfrak{b},m,\bar{m},n,\bar{n},z) = & \prod_{(i_{0,\mathfrak{a}},j)\in G} (-\partial)^{\mathfrak{b}_j} \left((mg+\bar{m}\alpha)\log\bar{z} + (\bar{m}g+m\alpha)\log z\right) \\ & \times \prod_{(i,i_{0,\mathfrak{b}})\in G} \partial^{\mathfrak{a}_i} \left((ng+\bar{n}\alpha)\log\bar{z} + (\bar{n}g+n\alpha)\log z\right) \\ & \times \prod_{\substack{(i,j)\in G \\ i\in S_{\mathfrak{a}}, j\in S_{\mathfrak{b}}}} \partial^{\mathfrak{a}_i}(-\partial)^{\mathfrak{b}_j} \log|z|^2 \end{aligned} \tag{3.3.67}$$

where

$$\partial^{(l,m)} = \begin{cases} \partial^l & \text{if } m = 1 \\ \bar{\partial}^l & \text{if } m = -1 \,. \end{cases} \tag{3.3.68}$$

- $\mathfrak{a}^G = \prod_{i \in S_\mathfrak{a} \setminus G_\mathfrak{a}} \nabla^{\mathfrak{a}_i} \varphi$, where $G_\mathfrak{a} = \{i \in S_\mathfrak{a} : \nexists j \in S_\mathfrak{b}^* \text{ so that } (i,j) \in G\}$.

This seemingly complex definition of the vertex operator eq. (3.3.64) can be motivated as follows: In the fock space representation of the free boson, it is possible to define wick ordered exponentials and thus operators with the same functional form as shown in eq. (3.3.63), see [57]. The OPE, equivalent to eq. (3.3.64), can formally be obtained by applying rules like in eq. (3.2.52) to the product of the operators.

We now come to the verification of the axioms of definition 2.1.1.

First of all we note that the right hand side of eq. (3.3.64) is real-analytic in $z \in \mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\}$.

The vacuum, grading and compatibility axiom follow directly from the definitions of V and Y. Covariance of the vertex operators follows directly from the fact that the representation of Spin(2) on V eq. (3.3.60) is just a multiplication by a phase, and the formula for the vertex operator eq. (3.3.64).

We verify skew-symmetry of the vertex operators: With $\mathfrak{A}, \mathfrak{B}$ as before, the right hand side of eq. (2.1.11) reads in the present model

$$\exp(x\nabla + \bar{x}\bar{\nabla})Y(\mathfrak{B}, -x)\mathfrak{A} = f(n, \bar{n}, m, \bar{m}, -x) \sum_{G \in \bar{\mathcal{G}}(S_\mathfrak{b}^*, S_\mathfrak{a}^*)} P_G(\mathfrak{b}, \mathfrak{a}, n, \bar{n}, m, \bar{m}, -x)$$

$$\times \exp(x\nabla + \bar{x}\bar{\nabla}) \left((\exp(-x\nabla - \bar{x}\bar{\nabla})(\mathfrak{b}^G e^{\mathrm{i}(n\kappa+\bar{n}\bar{\kappa})})\right) \mathfrak{a}^G e^{\mathrm{i}(m\kappa+\bar{m}\bar{\kappa})} \Big) . \tag{3.3.69}$$

We note

$$\begin{aligned}
&\mathfrak{A} \in V^{m+\bar{m},\bullet,\bullet}, \mathfrak{B} \in V^{n+\bar{n},\bullet,\bullet}, \\
f(n,\bar{n},m,\bar{m},-x) &= (-1)^{mn+\bar{m}\bar{n}} f(m,\bar{m},n,\bar{n},x) \\
e^{\mathrm{i}(m\kappa+\bar{m}\bar{\kappa})} e^{\mathrm{i}(n\kappa+\bar{n}\bar{\kappa})} &= (-1)^{mn+\bar{m}\bar{n}} e^{\mathrm{i}(n\kappa+\bar{n}\bar{\kappa})} e^{\mathrm{i}(m\kappa+\bar{m}\bar{\kappa})}
\end{aligned} \tag{3.3.70}$$

Furthermore $G \in \tilde{\mathcal{G}}(S^*_{\mathfrak{b}}, S^*_{\mathfrak{a}})$, let $G' := \{(i,j) : (j,i) \in G\}$. Then $G' \in \tilde{\mathcal{G}}(S^*_{\mathfrak{a}}, S^*_{\mathfrak{b}})$ and $P_G(\mathfrak{B}, \mathfrak{A}, -x) = P_G(\mathfrak{A}, \mathfrak{B}, x)$. Using the third line of eq. (3.3.70), we get

$$\begin{aligned}
&\exp(x\nabla + \bar{x}\bar{\nabla})\left(\left(\exp(-x\nabla - \bar{x}\bar{\nabla})(\mathfrak{b}^G e^{\mathrm{i}\bar{\kappa}})\right)\mathfrak{a}^G e^{\mathrm{i}\bar{\kappa}}\right) \\
&\qquad = (-1)^{k(mn+\bar{m}\bar{n})} \left(\exp(x\nabla + \bar{x}\bar{\nabla})(\mathfrak{a}^G\, e^{\mathrm{i}(m\kappa+\bar{m}\bar{\kappa})})\right) \mathfrak{b}^G\, e^{\mathrm{i}(n\kappa+\bar{n}\bar{\kappa})}
\end{aligned} \tag{3.3.71}$$

Using the second line of eq. (3.3.70) and eq. (3.3.71), we see that the right hand side of eq. (3.3.69) is just $(-1)^{k(m+\bar{m})(n+\bar{n})} Y(\mathfrak{A}, x)\mathfrak{B}$. By the first line of eq. (3.3.70), this is just the skew-symmetry property.

It only remains to show associativity, which we do in the next subsection.

3.3.1 Proof of associativity for the massless Thirring model

From the defining relation for V eq. (3.3.57) it follows that

$$\exp(z \cdot \nabla) \exp(\mathrm{i}\kappa) = \exp(\mathrm{i}(\exp(z \cdot \nabla) - \mathrm{Id}_V)\kappa) e^{\mathrm{i}\kappa}$$

$$=\Big(\sum_{q=0}^{\infty}\frac{\mathrm{i}^q}{q!}\sum_{\substack{j_1,\dots,j_q\in\mathbb{N}:\\ |j_1|,\dots,|j_q|\geq 1}}\frac{z^{j_1+\dots+j_q}}{\vec{j}!}(\nabla^{j_1}\bar{\kappa})\dots(\nabla^{j_q}\bar{\kappa})\Big)e^{\mathrm{i}\kappa}$$

where $\vec{j}! = j_1!\dots j_q!$ and it is understood that $\nabla\kappa = \alpha\nabla\chi$. Similarly, we can write

$$\begin{aligned}
&\exp(z\cdot\nabla+\bar{z}\bar{\nabla})\exp(\mathrm{i}(m\kappa+\bar{m}\bar{\kappa}))\\
&\quad=\exp\left(\mathrm{i}\left(\exp(z\nabla+\bar{z}\bar{\nabla})-\mathrm{Id}_V\right)(m\kappa+\bar{m}\bar{\kappa})\right)e^{\mathrm{i}(m\kappa+\bar{m}\bar{\kappa})}\\
&\quad=\Big(\sum_{q=0}^{\infty}\frac{\mathrm{i}^q}{q!}\sum_{\jmath_1,\dots,\jmath_q\in\mathbb{L}'_{>0}}\frac{z^{\jmath_1+\dots+\jmath_q}}{\vec{\jmath}!}(\nabla^{\jmath_1}(m\kappa+\bar{m}\bar{\kappa}))\dots(\nabla^{\jmath_q}(m\kappa+\bar{m}\bar{\kappa}))\Big)e^{\mathrm{i}(m\kappa+\bar{m}\bar{\kappa})}
\end{aligned} \tag{3.3.72}$$

where for $\mathbb{L}'\ni\jmath=(l,m)$,

$$z^{\jmath}=\begin{cases} z^l & \text{if } m=1\\ \bar{z}^l & \text{if } m=-1\,,\end{cases} \tag{3.3.73}$$

and $z^{\jmath_1+\dots+\jmath_q}=z^{\jmath_1}\dots z^{\jmath_q}$. Similarly,

$$\nabla^{\jmath}=\begin{cases} \nabla^l & \text{if } m=1\\ \bar{\nabla}^l & \text{if } m=-1\,.\end{cases} \tag{3.3.74}$$

Also we have used the notation $\vec{\jmath}! = \prod_{i=1}^{m}|\jmath_i|!$, $\bar{\nabla}\kappa = g\bar{\nabla}\chi$, $\nabla\bar{\kappa} = g\nabla\chi$, $\bar{\nabla}\bar{\kappa} = \alpha\bar{\nabla}\bar{\chi}$ in eq. (3.3.72).

Left hand side of eq. (2.1.8)

Let

$$\mathfrak{A} = \mathfrak{a}\ \exp(\mathrm{i}(m\kappa + \bar{m}\bar{\kappa}))$$
$$\mathfrak{B} = \mathfrak{b}\ \exp(\mathrm{i}(n\kappa + \bar{n}\bar{\kappa}))$$
$$\mathfrak{C} = \mathfrak{c}\ \exp(\mathrm{i}(p\kappa + \bar{p}\bar{\kappa})).$$

The left hand side of the associativity axiom eq. (2.1.8) for these vectors reads

$$\begin{aligned} Y(\mathfrak{A},x)Y(\mathfrak{B},y)\mathfrak{C} =& f(m,\bar{m},n+p,\bar{n}+\bar{p},x)f(n,\bar{n},p,\bar{p},y) \\ &\times \sum_{G\in\bar{\mathcal{G}}(S^*_\mathfrak{b},S^*_\mathfrak{c})} P_G(\mathfrak{b},\mathfrak{c},n,\bar{n},p,\bar{p},y) \\ &\times \sum_{\substack{\boldsymbol{\ell}_i\in\mathbf{L}': \\ i\in S_\mathfrak{b}\setminus G_\mathfrak{b}}} \left(\prod_{i\in S_\mathfrak{b}\setminus G_\mathfrak{b}} \frac{y^{\boldsymbol{\ell}_i}}{|\boldsymbol{\ell}_i|!}\right) \sum_{q=0}^{\infty} \frac{\mathrm{i}^q}{q!} \sum_{\jmath_1,\dots,\jmath_q\in\mathbf{L}'_{>0}} \frac{y^{\jmath_1+\dots+\jmath_q}}{\vec{\jmath}!} \\ &\times \sum_F P_F(\mathfrak{a},(\mathfrak{b}^{G,\vec{\ell}}\mathfrak{c}^G(n\kappa+\bar{n}\bar{\kappa})^{\jmath_1,\dots,\jmath_q}),m,\bar{m},n+p,\bar{n}+\bar{p},x) \\ &\times \left(\exp(x\nabla+\bar{x}\bar{\nabla})\mathfrak{a}^F\right) \prod_{i\in S_\mathfrak{b}\setminus(G_\mathfrak{b}\cup F_\mathfrak{b})} \nabla^{\mathfrak{b}_i+\boldsymbol{\ell}_i}\varphi \prod_{i\in S_\mathfrak{c}\setminus(G_\mathfrak{c}\cup F_\mathfrak{c})} \nabla^{\mathfrak{c}_i}\varphi \\ &\times \prod_{i\in M\setminus F^M} \nabla^{\jmath_i}(n\kappa+\bar{n}\bar{\kappa}) \\ &\times \left(\exp(x\nabla+\bar{x}\bar{\nabla})e^{\mathrm{i}(m\kappa+\bar{m}\bar{\kappa})}\right)e^{i((n+p)\kappa+(\bar{n}+\bar{p})\bar{\kappa})} \end{aligned} \tag{3.3.75}$$

where we have used the following notation:

- $M = \{1,\dots,q\}$
- $(n\kappa+\bar{n}\bar{\kappa})^{\jmath_1,\dots,\jmath_q} = \prod_{i=1}^q \nabla^{\jmath_i}(n\kappa+\bar{n}\bar{\kappa})$.

- The sum $\sum_F$ runs over all $F \in \tilde{\mathcal{G}}\left((S_{\mathfrak{a}}^*, (S_{\mathfrak{b}} \cup S_{\mathfrak{c}} \cup \dot{M} \setminus (G_{\mathfrak{b}} \cup G_{\mathfrak{c}}))^*\right)$

- $\bar{M} = \{j \in M : \nexists i \in S_{\mathfrak{a}}^* \text{ such that } (i,j) \in F\}$

The polynomial P_F in eq. (3.3.75) reads explicitly

$$
\begin{aligned}
&P_F(\mathfrak{a}, (\mathfrak{b}^{G,\vec{\ell}}\mathfrak{c}^{G}\bar{\kappa}^{j_1,\dots,j_q}), m, \bar{m}, n+p, \bar{n}+\bar{p}, x) \\
&\quad = \prod_{\substack{(i_{0,\mathfrak{a}},j)\in F:\\ j\in M}} (-\partial)^{q_j}\Big((\bar{m}g + m\alpha)(\bar{n}g+n\alpha)\log x + (mg+\bar{m}\alpha)(ng+\bar{n}\alpha)\log\bar{x}\Big) \\
&\quad \times \prod_{\substack{(i,j)\in F:\\ i\in S_{\mathfrak{a}}, j\in M}} \partial^{\mathfrak{a}_i}(-\partial)^{q_j}\left((\bar{n}g+n\alpha)\log x + (ng+\bar{n}\alpha)\log\bar{x}\right) \\
&\quad \times \prod_{\substack{(i_{0,\mathfrak{a}},j)\in F:\\ j\in S_{\mathfrak{b}}}} (-\partial)^{\mathfrak{b}_j+\ell_j}\left((\bar{m}g+m\alpha)\log x + (mg+\bar{m}\alpha)\log\bar{x}\right) \\
&\quad \times \prod_{\substack{(i,j)\in F:\\ i\in S_{\mathfrak{a}}, j\in S_{\mathfrak{b}}}} \partial^{\mathfrak{a}_i}(-\partial)^{\mathfrak{b}_j+\ell_j}\log|x|^2 \\
&\quad \times \prod_{\substack{(i_{0,\mathfrak{a}},j)\in F:\\ j\in S_{\mathfrak{c}}}} (-\partial)^{\mathfrak{c}_j+\ell_j}\left((\bar{m}g+m\alpha)\log x + (mg+\bar{m}\alpha)\log\bar{x}\right) \\
&\quad \times \prod_{\substack{(i,j)\in F:\\ i\in S_{\mathfrak{a}}, j\in S_{\mathfrak{c}}}} (-\partial)^{\mathfrak{c}_j+\ell_j}\log|x|^2 \\
&\quad \times \prod_{(i,i_{0,\mathfrak{d}})\in F} \partial^{\mathfrak{a}_i}\Big(((\bar{n}+\bar{p})g + (n+p)\alpha)\log x \\
&\quad + ((n+p)g + (\bar{n}+\bar{p})\alpha)\log\bar{x}\Big)
\end{aligned}
$$

where $\mathfrak{d} = \mathfrak{b}^{G,\vec{\ell}}\mathfrak{c}^{G}(n\kappa + \bar{n}\bar{\kappa})^{j_1,\dots,j_q}$.

We are going to simplify eq. (3.3.75) in two steps. In the first step, we recognize that for some of the edges of F there are associated indices from $\mathbb{L}'$ that can explicitly be summed over. The second step is a partial sum over the graphs F, keeping most of the other objects in the sum eq. (3.3.75) fixed.

Throughout the proof, we will assume that we can freely exchange the order in which the infinite sums are evaluated. At the very end we will see why this is indeed justified.

First step: In eq. (3.3.75), we set

$$\begin{aligned} M_F^* =& \{j \in M : (i_{0,\mathfrak{a}}, j) \in F\} \\ M_F =& \{j \in M : \exists i \in S_\mathfrak{a} \text{ such that } (i,j) \in F\} \,. \end{aligned}$$

To each vertex $j \in M_F^*$ there is an associated index $q_j \in \mathbb{L}'$ that is to be summed over. In fact, this sum is nothing else but a Taylor expansion. Assuming that the line segment connecting x to $x-y$ in the complex plane does not intersect the ray $\mathbb{R}_{\leq 0}$, we have

$$\begin{aligned} \sum_{q_j \in \mathbb{L}'_{>0}} & \frac{y^{q_j}}{q_j!}(-\partial)^{q_j} \left((\bar{m}g + m\alpha)(\bar{n}g + n\alpha)\log x + (mg + \bar{m}\alpha)(ng + \bar{n}\alpha)\log \bar{x}\right) \\ =& (\bar{m}g + m\alpha)(\bar{n}g + n\alpha)\log(x-y) + (mg + \bar{m}\alpha)(ng + \bar{n}\alpha)\log(\bar{x}-\bar{y}) \\ & - \Big((\bar{m}g + m\alpha)(\bar{n}g + n\alpha)\log x + (mg + \bar{m}\alpha)(ng + \bar{n}\alpha)\log \bar{x}\Big) \,. \end{aligned} \tag{3.3.76}$$

In the case that the line segment connecting x and $x-y$ does intersect $\mathbb{R}_{\leq 0}$, we have additional integer multiples of $2\pi\mathrm{i}$ on the right hand side. The fact that these extra terms are indeed *integer* multiples of $2\pi\mathrm{i}$ is due to our particular choice of α below eq. (3.3.54).

For $j \in M_F$, we can carry out the sum

$$\sum_{q_j \in \mathbb{L}'_{>0}} \frac{y^{q_j}}{q_j!}\partial^{\mathfrak{a}_i}(-\partial)^{q_j} \left((\bar{n}g + n\alpha)\log x + (ng + \bar{n}\alpha)\log \bar{x}\right)$$

$$=\partial^{\mathfrak{a}_i}(\bar{n}g+n\alpha)\log(x-y)+(ng+\bar{n}\alpha)\log(\bar{x}-\bar{y})$$
$$-\partial^{\mathfrak{a}_i}(\bar{n}g+n\alpha)\log x+(ng+\bar{n}\alpha)\log x \qquad (3.3.77)$$

For $i=i_{0,\mathfrak{a}}, j\in S_{\mathfrak{b}}$, we have

$$\sum_{l_j\in \mathbf{L}'}\frac{y^{l_j}}{l_j!}(-\partial)^{\mathfrak{b}_j+l_j}\left((\bar{m}g+m\alpha)\log x+(mg+\bar{m}\alpha)\log\bar{x}\right)$$
$$=(-\partial)^{\mathfrak{b}_j}(\bar{n}g+n\alpha)\log(x-y)+(ng+\bar{n}\alpha)\log(\bar{x}-\bar{y})$$

For $i\in S_{\mathfrak{a}}, j\in S_{\mathfrak{b}}$, we have

$$\sum_{l_j\in \mathbf{L}'}\frac{y^{l_j}}{q_j!}(-\partial)^{\mathfrak{b}_j+l_j}\log|x|^2=(-\partial)^{\mathfrak{b}_j}\log|x|^2$$

Second step: After performing the sum eq. (3.3.76), eq. (3.3.75) contains one factor

$$R:=\Big((\bar{m}g+m\alpha)(\bar{p}g+p\alpha)\log(x-y)+(mg+\bar{m}\alpha)(pg+\bar{p}\alpha)\log(\bar{x}-\bar{y})\Big)$$
$$-\Big((\bar{m}g+m\alpha)(\bar{n}g+n\alpha)\log x+(mg+\bar{m}\alpha)(ng+\bar{n}\alpha)\log\bar{x}\Big)+k(2\pi\mathrm{i})$$

for each $i\in M_F^*$. $k\in\mathbb{Z}$ is the parameter that will determine the statistics of the model that we introduced below eq. (3.3.54). In eq. (3.3.75), we fix a graph G, and all indices $\jmath_i,\ell_i$ that have not been summed over in the first step. Now we sum over all q and F such that $F_{\mathfrak{a}},F_{\mathfrak{b}},F_{\mathfrak{c}},M_F$ and $\bar{M}$ remain fixed. The only set of vertices that is not kept constant in this sum is M_F^*. Let $q_1:=\#M_F^*$. For each q, there are $\binom{q}{q_1}$ different graphs F that fulfill the above. Thus eq. (3.3.75) reads

$$\cdots \sum_{q=0}^{\infty} \frac{\mathrm{i}^q}{q!} \binom{q}{q_1} R^{q_1} \cdots = \cdots \exp(\mathrm{i}R) \sum_{q'=0}^{\infty} \frac{\mathrm{i}^{q'}}{q'!} \cdots$$
$$= \cdots \frac{f(m,\bar{m},n,\bar{n},x-y)}{f(m,\bar{m},n,\bar{n},x)} \sum_{q'=0}^{\infty} \frac{\mathrm{i}^{q'}}{q'!} \cdots \tag{3.3.78}$$

where we have transformed the sum over q into a sum over q_1 and $q' = q - q_1$. Note that in eq. (3.3.78), we will have the simplification

$$f(m,\bar{m},n+p,\bar{n}+\bar{p},x) \frac{f(m,\bar{m},n,\bar{n},x-y)}{f(m,\bar{m},n,\bar{n},x)} = f(m,\bar{m},p,\bar{p},x) f(m,\bar{m},n,\bar{n},x-y) \,. \tag{3.3.79}$$

Right hand side of eq. (2.1.8)

The right hand side of eq. (2.1.8) reads

$$\begin{aligned} Y(Y(\mathfrak{A},x-y)\mathfrak{B},y)\mathfrak{C} =& f(m,\bar{m},n,\bar{n},x-y) f(m+n,\bar{m}+\bar{n},p,\bar{p},y) \\ &\times \sum_{G \in \tilde{\mathcal{G}}(S_\mathfrak{a}^*, S_\mathfrak{b}^*)} P_G(\mathfrak{a},\mathfrak{b},m,\bar{m},n,\bar{n},x-y) \\ &\times \sum_{\substack{\boldsymbol{\ell}_i \in \mathbf{L}: \\ i \in S_\mathfrak{a} \setminus G_\mathfrak{a}}} \left(\prod_{i \in S_\mathfrak{a} \setminus G_\mathfrak{a}} \frac{(x-y)^{\boldsymbol{\ell}_i}}{|\boldsymbol{\ell}_i|!} \right) \sum_{q=0}^{\infty} \frac{\mathrm{i}^q}{q!} \sum_{\jmath_1,\ldots,\jmath_q \in \mathbf{L}'_{>0}} \frac{(x-y)^{\jmath_1 + \cdots + \jmath_q}}{\vec{\jmath}!} \\ &\times \sum_{F \in \tilde{\mathcal{G}}(T^*, S_\mathfrak{c})} P_F((\mathfrak{a}^{G,\vec{\ell}} \mathfrak{b}^G (m\kappa + \bar{m}\bar{\kappa})^{\jmath_1,\ldots,\jmath_q}), \mathfrak{c}, m+n, \bar{m}+\bar{n}, p, \bar{p}, y) \\ &\times \sum_{\substack{\wp_i \in \mathbf{L}': \\ i \in T}} \left(\prod_{i \in T_\mathfrak{b} \setminus F_\mathfrak{b}} \frac{y^{\wp_i}}{|\wp_i|!} \right) \prod_{i \in S_\mathfrak{a} \setminus (G_\mathfrak{a} \cup F_\mathfrak{a})} \nabla^{\boldsymbol{\ell}_i + \wp_i + \mathfrak{a}_i} \varphi \\ &\times \prod_{i \in S_\mathfrak{b} \setminus (G_\mathfrak{b} \cup F_\mathfrak{b})} \nabla^{\wp_i + \mathfrak{a}_i} \varphi \prod_{i \in S_\mathfrak{c} \setminus F_\mathfrak{c}} \nabla^{\mathfrak{c}_i} \varphi \prod_{i \in \bar{M}} \nabla^{\jmath_i + \wp_i} (m\kappa + \bar{m}\bar{\kappa}) \\ &\times \left(\exp(y\nabla + \bar{y}\bar{\nabla}) \exp i((m+n)\kappa + (\bar{m}+\bar{n})\bar{\kappa}) \right) e^{i(p\kappa + \bar{p}\bar{\kappa})} \end{aligned} \tag{3.3.80}$$

where $T = S_{\mathfrak{a}} \cup S_{\mathfrak{b}} \cup M \setminus (G_{\mathfrak{a}} \cup G_{\mathfrak{b}})$, $\bar{M} = \{i \in M : \nexists j \in S_{\mathfrak{c}}^* \text{ such that } (i,j) \in F\}$ and $\mathfrak{d} = \mathfrak{a}^{G,\vec{\ell}}\mathfrak{b}^G(m\kappa + \bar{m}\bar{\kappa})^{\jmath_1,\dots,\jmath_q}$. The polynomial P_F is explicitly given by

$$
\begin{aligned}
&P_F((\mathfrak{a}^{G,\vec{\ell}}\mathfrak{b}^G(m\kappa+\bar{m}\bar{\kappa})^{\jmath_1,\dots,\jmath_q}), \mathfrak{c}, m+n, \bar{m}+\bar{n}, p, \bar{p}, y) \\
&\quad = \prod_{\substack{(i,i_{0,\mathfrak{c}})\in F: \\ i\in M}} \partial^{\jmath_i}\Big((\bar{m}g+m\alpha)(\bar{p}g+p\alpha)\log y + (mg+\bar{m}\alpha)(pg+\bar{p}\alpha)\log\bar{y}\Big) \\
&\quad\quad \times \prod_{\substack{(i,j)\in F: \\ i\in M, j\in S_{\mathfrak{c}}}} (-\partial)^{\mathfrak{c}_i}\partial^{\jmath_j}\left((\bar{m}g+m\alpha)\log y + (mg+\bar{m}\alpha)\log\bar{y}\right) \\
&\quad\quad \times \prod_{\substack{(i,i_{0,\mathfrak{c}})\in F: \\ i\in S_{\mathfrak{a}}}} \partial^{\mathfrak{a}_i+\ell_i}\left((\bar{p}g+p\alpha)\log y + (pg+\bar{p}\alpha)\log\bar{y}\right) \\
&\quad\quad \times \prod_{\substack{(i,j)\in F: \\ i\in S_{\mathfrak{a}}, j\in S_{\mathfrak{c}}}} \partial^{\mathfrak{a}_i+\ell_i}(-\partial)^{\mathfrak{c}_j}\log|y|^2 \\
&\quad\quad \times \prod_{\substack{(i,i_{0,\mathfrak{c}})\in F: \\ i\in S_{\mathfrak{b}}}} \partial^{\mathfrak{b}_i+\wp_i}\left((\bar{p}g+p\alpha)\log y + (pg+\bar{p}\alpha)\log\bar{y}\right) \\
&\quad\quad \times \prod_{\substack{(i,j)\in F: \\ i\in S_{\mathfrak{b}}, j\in S_{\mathfrak{c}}}} \partial^{\mathfrak{b}_i}(-\partial)^{\mathfrak{c}_j}\log|y|^2 \\
&\quad\quad \times \prod_{(i_{0,\mathfrak{d}},j)\in F} (-\partial)^{\mathfrak{c}_j}\Big(((\bar{m}+\bar{n})g+(m+n)\alpha)\log y \\
&\quad\quad + ((m+n)g+(\bar{m}+\bar{n})\alpha)\log\bar{y}\Big)
\end{aligned}
$$

Again, we simplify in two steps.

First step: We set

$$
\begin{aligned}
M_F^* &= \{i \in M : (i, i_{0,\mathfrak{c}}) \in F\} \\
M_F &= \{i \in M : \exists j \in S_{\mathfrak{c}} \text{ such that } (i,j) \in F\}.
\end{aligned}
$$

In eq. (3.3.80), we can carry out some partial sums, which amount to sum up Taylor expansions: For each $i \in M_F^*$,

$$\sum_{\jmath_i \in \mathbf{L}'_{>0}} \frac{(x-y)^{\jmath_i}}{|\jmath_i|!} \partial^{\jmath_i} \Big((\bar{m}g + m\alpha)(\bar{p}g + p\alpha) \log y + (mg + \bar{m}\alpha)(pg + \bar{p}\alpha) \log \bar{y} \Big)$$

$$= \Big((\bar{m}g + m\alpha)(\bar{p}g + p\alpha) \log x + (mg + \bar{m}\alpha)(pg + \bar{p}\alpha) \log \bar{x} \Big)$$

$$- \Big((\bar{m}g + m\alpha)(\bar{p}g + p\alpha) \log y + (mg + \bar{m}\alpha)(pg + \bar{p}\alpha) \log \bar{y} \Big) + k(2\pi \mathrm{i}) \qquad (3.3.81)$$

where $k \in \mathbb{Z}$. For each $i \in M_F$,

$$\sum_{\jmath_i \in \mathbf{L}'_{>0}} \frac{(x-y)^{\jmath_i}}{|\jmath_i|!} (-\partial)^{\mathfrak{c}_i} \partial^{\mathfrak{q}_j} \left((\bar{m}g + m\alpha) \log y + (mg + \bar{m}\alpha) \log \bar{y} \right)$$

$$= (-\partial)^{\mathfrak{c}_i} \left((\bar{m}g + m\alpha) \log x + (mg + \bar{m}\alpha) \log \bar{x} \right)$$

$$- (-\partial)^{\mathfrak{c}_i} \left((\bar{m}g + m\alpha) \log y + (mg + \bar{m}\alpha) \log \bar{y} \right)$$

For each $i \in S_{\mathfrak{a}}$ such that $(i, i_{0,\mathfrak{c}}) \in F$:

$$\sum_{\boldsymbol{\ell}_i \in \mathbf{L}'} \frac{(x-y)^{\boldsymbol{\ell}_i}}{|\boldsymbol{\ell}_i|!} \partial^{\mathfrak{a}_i + \boldsymbol{\ell}_i} \left((\bar{p}g + p\alpha) \log y + (pg + \bar{p}\alpha) \log \bar{y} \right)$$

$$= \partial^{\mathfrak{a}_i} \left((\bar{p}g + p\alpha) \log x + (pg + \bar{p}\alpha) \log \bar{x} \right) \qquad (3.3.82)$$

For each $i \in S_{\mathfrak{a}}$ such that $\exists j \in S_{\mathfrak{c}} : (i, j) \in F$:

$$\sum_{\boldsymbol{\ell}_i \in \mathbf{L}'} \frac{(x-y)^{\boldsymbol{\ell}_i}}{|\boldsymbol{\ell}_i|!} \partial^{\mathfrak{a}_i + \boldsymbol{\ell}_i} (-\partial)^{\mathfrak{c}_j} \log |y|^2 = \partial^{\mathfrak{a}_i} (-\partial)^{\mathfrak{c}_j} \log |x|^2$$

Second step: After performing the sum eq. (3.3.81), eq. (3.3.80) contains one factor

$$
\begin{aligned}
R :=& \Big((\bar{m}g + m\alpha)(\bar{p}g + p\alpha) \log x + (mg + \bar{m}\alpha)(pg + \bar{p}\alpha) \log \bar{x} \Big) \\
&- \Big((\bar{m}g + m\alpha)(\bar{p}g + p\alpha) \log y + (mg + \bar{m}\alpha)(pg + \bar{p}\alpha) \log \bar{y} \Big) + k(2\pi \mathrm{i})
\end{aligned}
$$

for each $i \in M^*_F$. In eq. (3.3.80), we fix a graph G, and all indices $\jmath_i, \ell_i, \wp_i$ that have not been summed over in the first step. Now we sum over all q and F such that $F_{\mathfrak{a}}, F_{\mathfrak{b}}, F_{\mathfrak{c}}, M_F$ and $\bar{M}$ remain fixed. The only set of vertices that is not kept constant in this sum is M^*_F. Let $q_1 := \# M^*_F$. For each m, there are $\binom{q}{q_1}$ different graphs F that fulfill all of the above. Thus eq. (3.3.80) reads

$$
\begin{aligned}
\cdots \sum_{q=0}^{\infty} \frac{\mathrm{i}^q}{q!} \binom{q}{q_1} R^{q_1} \ldots &= \ldots \exp(\mathrm{i}R) \sum_{q'=0}^{\infty} \frac{\mathrm{i}^{q'}}{q'!} \cdots \\
&= \ldots \frac{f(m, \bar{m}, p, \bar{p}, x)}{f(m, \bar{m}, p, \bar{p}, y)} \sum_{q'=0}^{\infty} \frac{\mathrm{i}^{q'}}{q'!} \cdots
\end{aligned} \tag{3.3.83}
$$

where we have transformed the sum over q into a sum over q_1 and $q' = q - q_1$. Note that in eq. (3.3.83), we will have the simplification

$$
f(m + n, \bar{m} + \bar{n}, p, \bar{p}, y) \frac{f(m, \bar{m}, p, \bar{p}, x)}{f(m, \bar{m}, p, \bar{p}, y)} = f(m, \bar{m}, p, \bar{p}, x) f(n, \bar{n}, p, \bar{p}, y) \,. \tag{3.3.84}
$$

Comparison of both sides

After these simplifications, we see that both eq. (3.3.75) and eq. (3.3.80) are identical to

$$
f(m, \bar{m}, n, \bar{n}, x - y) f(m, \bar{m}, p, \bar{p}, x) f(n, \bar{n}, p, \bar{p}, y) \sum_{G \in \mathcal{G}(S^*_{\mathfrak{a}}, S^*_{\mathfrak{b}}, S^*_{\mathfrak{c}})} P_G(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, x, y, x - y)
$$

$$\times \left(\exp(x\cdot\nabla)e^{\mathrm{i}(m\kappa+\bar{m}\bar{\kappa})}\mathfrak{a}^G\right)\left(\exp(y\cdot\nabla)e^{\mathrm{i}(n\kappa+\bar{n}\bar{\kappa})}\mathfrak{b}^G\right)\mathfrak{c}^G e^{\mathrm{i}(p\kappa+\bar{p}\bar{\kappa})} \tag{3.3.85}$$

where $\mathcal{G}(S^*_\mathfrak{a}, S^*_\mathfrak{b}, S^*_\mathfrak{c})$ is the set of graphs connecting vertices of the same kind from $S^*_\mathfrak{a}$, $S^*_\mathfrak{b}$ and $S^*_\mathfrak{c}$, where the vertices $i_{0,\mathfrak{a}}, i_{0,\mathfrak{b}}$ and $i_{0,\mathfrak{c}}$ may be connected to several vertices (of both kinds),

$$\begin{aligned}\tilde{\mathcal{G}}(S^*_\mathfrak{a}, S^*_\mathfrak{b}, S^*_\mathfrak{c}) = \Big\{ G \subset \{(i,j) \in S^*_\mathfrak{a}\times S^*_\mathfrak{b} \cup S^*_\mathfrak{a}\times S^*_\mathfrak{c} \cup S^*_\mathfrak{b}\times S^*_\mathfrak{c}\} : \\ \Big((i,j)\in G, i=(l,m), j=(l',m') \Rightarrow m=m'\Big), \\ \Big(((i,j),(k,l)\in G \Rightarrow i\neq k \text{ or } i=k\in\{i_{0,\mathfrak{a}}, i_{0,\mathfrak{b}}\}, \\ j\neq l \text{ or } j=l\in\{i_{0,\mathfrak{b}}, i_{0,\mathfrak{c}}\}\Big), \\ (i_{0,\mathfrak{a}}, i_{0,\mathfrak{b}}), (i_{0,\mathfrak{a}}, i_{0,\mathfrak{c}}), (i_{0,\mathfrak{b}}, i_{0,\mathfrak{c}}) \notin G\Big\}.\end{aligned}$$

P_G is given by

$$\begin{aligned}&P_G(\mathfrak{A},\mathfrak{B},\mathfrak{C},x,y,x-y)\\ &=P_{G_{\mathfrak{a},\mathfrak{b}}}(\mathfrak{a},\mathfrak{b},m,\bar{m},n,\bar{n},x-y)P_{G_{\mathfrak{a},\mathfrak{c}}}(\mathfrak{a},\mathfrak{c},m,\bar{m},p,\bar{p},x)P_{G_{\mathfrak{b},\mathfrak{c}}}(\mathfrak{b},\mathfrak{c},n,\bar{n},p,\bar{p},y)\end{aligned}$$

where $G_{\mathfrak{a},\mathfrak{b}} = G\cap S^*_\mathfrak{a}\times S^*_\mathfrak{b}$, $G_{\mathfrak{a},\mathfrak{c}} = G\cap S^*_\mathfrak{a}\times S^*_\mathfrak{c}$, $G_{\mathfrak{b},\mathfrak{c}} = G\cap S^*_\mathfrak{b}\times S^*_\mathfrak{c}$.

Eq. (3.3.85) is obviously an element of $\bar{V}$, as the sum over graphs $G\in\tilde{\mathcal{G}}(S^*_\mathfrak{a}, S^*_\mathfrak{b}, S^*_\mathfrak{c})$ is a finite sum. Consider

$$\langle v^*, \text{eq. (3.3.85)}\rangle \tag{3.3.86}$$

for any $v^*\in V^*$. This is a polynomial in x^{-1}, y^{-1} and $(x-y)^{-1}$ times $f(m,\bar{m},n,\bar{n},x-y)f(m,\bar{m},p,\bar{p},x)f(n,\bar{n},p,\bar{p},y)$. If we reexpand eq. (3.3.86) according to either eq. (3.3.75) or eq. (3.3.80), this amounts to replacing finitely many analytic functions by their Taylor expansions. More precisely, to get the left hand side of the associativity condition, we replace for each graph G, $P_{G_{a,b}}$ and $f(m, \bar{m}, n, \bar{n}, x-y)$ by their respective Taylor expansions in y around $y = 0$. To get the right hand side, we replace $P_{G_{a,c}}$ and $(m, \bar{m}, p, \bar{p}, x)$ by their Taylor expansion in $x - y$ around $x - y = 0$. As we are considering the domain $|x| > |y| > |x - y|$, these expansions are absolutely convergent. Thus both sides are sums of products of absolutely convergent series, thus absolutely convergent. This justifies the exchange of limits throughout the proof and completes the proof of associativity for the massless Thirring model.

3.3.2 Current, field equation, primary fields

We briefly discuss some objects of physical interest.

The Thirring fields $\phi, \bar{\phi}$ and their conjugates are

$$\begin{aligned} \phi &= \exp(\mathrm{i}\kappa) & \phi^* &= \exp(-\mathrm{i}\kappa) \\ \bar{\phi} &= \exp(\mathrm{i}\bar{\kappa}) & \bar{\phi}^* &= \exp(-\mathrm{i}\bar{\kappa}) \end{aligned} \tag{3.3.87}$$

We have $\phi, \bar{\phi}^* \in V^{1,(k+g^2)/2,1/2}$, $\bar{\phi}, \phi^* \in V^{1,(k+g^2)/2,-1/2}$. In the following we set $k = 1$ for definiteness, which means that the Thirring fields and their conjugates anticommute (in the sense that eq. (2.1.11) holds with the factor $(-1)^{ij}$ equal to -1). The $\phi\phi^*$-OPE is

$$Y(\phi^*, z)\phi = |z|^{2g^2} z^{-1} \left(\mathbf{1} + \mathrm{i}(\alpha z \nabla\chi + g\bar{z}\bar{\nabla}\bar{\chi}) + O(|z|^2)\right) \tag{3.3.88}$$

We identify the "left"current j with $\nabla\chi$ and the "right" current $\bar{j}$ with $\bar{\nabla}\bar{\chi}$. The equations $\nabla\bar{j} = \bar{\nabla}j = 0$ which are part of the defining relations of V are equivalent to the conservation equations of the current j_μ (defined by $j_0 = \bar{j}, j_1 = j$) and the pseudocurrent $\epsilon_{\mu\nu} j^\nu$,

$$\nabla^\mu j_\mu = \bar{\nabla} j + \nabla \bar{j} = 0$$

$$\nabla^\mu \epsilon_{\mu\nu} j^\nu = \bar{\nabla} j - \nabla \bar{j} = 0\,.$$

The equations of motion of the massless Thirring model follow from the defining relations of V as well,

$$\bar{\nabla}\phi = \mathrm{i}\, g\, \bar{j}\, \phi$$

$$\nabla\bar{\phi} = \mathrm{i}\, g\, j\, \bar{\phi}\,.$$

The holomorphic and anti-holomorphic components of the stress-energy tensor are given by [23]

$$T = -\frac{1}{2}(\nabla\chi)^2$$

$$\bar{T} = -\frac{1}{2}(\bar{\nabla}\bar{\chi})^2$$

This is identical to the stress energy tensor of the free boson. Thus $\nabla\chi$, $\bar{\nabla}\bar{\chi}$ are primary fields of conformal weights $(1,0)$ and $(0,1)$ respectively. The vertex algebra of the holomorphic part of the free boson with its representation of the Virasoro algebra is a sub-vertex algebra of the massless Thirring model. By this we mean that there is a subspace V_χ of V (the differential unital ring generated by $\nabla\chi$) thus that the vertex operator $Y(\cdot, z)$ restricts to a map $V_\chi \to \bar{V}_\chi \otimes \mathbb{C}[z]$. An analogous statement holds for the anti-holomorphic part. This is very much the same situation as in the well known boson-fermion correspondence [67]. Thus the massless Thirring model carries two representations of the Virasoro algebra with central charge 1.

The OPE of the stress energy tensor with the Thirring field $\phi = \exp(\mathrm{i}\kappa)$ is given by

$$\begin{aligned}Y(T,z)\phi =& -\frac{1}{2}Y\left((\nabla\chi)^2, z\right)\exp(\mathrm{i}\kappa)\\ =&\frac{\alpha^2\exp(\mathrm{i}\kappa)}{2z^2}+\frac{\mathrm{i}\alpha}{z}(\nabla\chi)\exp(\mathrm{i}\kappa)+\dots\\ =&\frac{\alpha^2\phi}{2z^2}+\frac{\nabla\phi}{z}+\dots,\end{aligned}$$

where the dots stand for non-singular terms. A similar OPE holds for the anti-holomorphic part of the stress energy tensor and thus ϕ is a primary field of conformal weight $(\alpha^2/2, g^2/2)$. Similarly, $\bar{\phi}$ is primary with conformal weight $(g^2/2, \alpha^2/2)$. The other primary fields in the theory are given by $\exp(\pm\mathrm{i}m\kappa), \exp(\pm\mathrm{i}n\bar{\kappa})$, $m, n \in \mathbb{Z}_{>0}$, with conformal weights $(m^2\alpha^2/2, m^2g^2/2)$ and $(n^2g^2/2, n^2\alpha^2/2)$ respectively.

Chapter 4

Perturbation theory

One of the most important constructions of quantum field theory is perturbation theory. In Euclidean field theory, this means the following: One starts with the known Schwinger functions of some theory, usually a free field theory. To these Schwinger functions there is an associated measure on the space of field configurations. Then one tries to construct Schwinger functions as formal power series in one or more parameters which may be $\hbar$, masses, coupling constants or the rank N of the "color gauge group" $U(N)$ [70] of the theory. The zeroth order is typically given by some free theory. The set of all such Schwinger functions is supposed to satisfy the axioms of Euclidean field theory in the sense of formal power series (i.e. order by order in the parameters). One way to do so is to give meaning to a path integral as discussed in the introduction.

Here we want to use the same idea for vertex algebras. This means we start from the vertex algebra of the free boson from section 3.1, say, given by the vector space V and the vertex operator map Y_0. Then we are going to define vertex operators in the sense of formal power series,

$$\begin{array}{rcl} Y: \; V^{\otimes 2} \times \mathbb{R}^D \setminus \{0\} & \rightarrow & \bar{V}[[\lambda]] \\ (a \otimes b, x) & \mapsto & \sum_{i=0}^{\infty} \lambda^i Y_i(a,x)b \end{array} \tag{4.0.1}$$

that satisfy the axioms 2.1.1 order by order in the parameter λ. Here, $V[[\lambda]]$ denotes the ring of V-valued formal power series in λ.

4.1 Perturbation theory via Hochschild cohomology

In [37], an interesting formulation of allowed perturbations of vertex algebras in terms of certain cohomology rings has been given. We want to review it here briefly, translating the statements made there into the vertex algebra language.

Consider linear invertible maps $Z : \bar{V} \rightarrow \bar{V}$ that preserve all gradings of V. They induce a map transforming the vertex operator,

$$Y \mapsto \tilde{Y} = Z \circ Y \circ (Z^{-1})^{\otimes 2} \tag{4.1.2}$$

It is quite obvious that the pair $(V, \tilde{Y})$ fulfills all axioms of definition 2.1.1. We want to consider the vertex algebras (V, Y) and $(V, \tilde{Y})$ as equivalent. In the renormalization of perturbation theory, one naturally has to deal with automorphisms of the space of fields (field redefinitions) that in our framework amount to letting the map Z be an element of $\mathrm{End}(\bar{V})[[\lambda]]$. Such a map induces a perturbation of the free field vertex operator map Y_0 as in eq. (4.0.1) via eq. (4.1.2). The resulting vertex algebra $(V, \tilde{Y})$ is equivalent to the free field (V, Y_0), and we call such a $\tilde{Y}$ a *trivial* deformation of Y. In the following, we are going to be interested in deformations of vertex algebras modulo trivial deformations. It turns out that this set is a certain cohomology ring.

4.1.1 The cohomology of vertex algebra perturbations

The only condition of definition 2.1.1 that is not (at most) linear in Y is associativity. Thus all other conditions, when formulated for the deformed vertex operator eq. (4.0.1), will not impose any relations between the perturbations Y_i of different order. The only condition that restricts the possible perturbations of first order Y_1 that depends on the given 0-th order Y_0 is associativity in first order of λ,

$$Y_0(a, x_1 - x_3)Y_1(a_2, x_2 - x_3)a_3 + Y_1(a_1, x_1 - x_3)Y_0(a_2, x_2 - x_3)a_3 = \\ Y_0(Y_1(a_1, x_1 - x_2)a_2, x_2 - x_3)a_3 + Y_1(Y_0(a_1, x_1 - x_2)a_2, x_2 - x_3)a_3 \,. \quad (4.1.3)$$

In this subsection only, we will understand vertex operators as maps

$$\begin{aligned} Y: \quad V^{\otimes 2} \times \mathcal{F}^2 &\rightarrow \bar{V} \\ (a_1, a_2, x_1, x_2) &\mapsto Y(a_1, x_1 - x_2)a_2 \,, \end{aligned} \quad (4.1.4)$$

where $\mathcal{F}^2 = \{(x, y) \in \mathbb{R}^{2D} : x \neq y\}$. Both sides of eq. (4.1.3) can be understood as maps $V^{\otimes 3} \times \mathcal{F}^3 \rightarrow \bar{V}$ applied to $(a_1 \otimes a_2 \otimes a_3, x_1, x_2, x_3)$, where $\mathcal{F}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^{3D} : |x_1 - x_3| > |x_2 - x_3| > |x_1 - x_2| > 0\}$. We write $\Omega^i(V)$ for the space of maps $V^{\otimes i} \times \mathcal{F}^i \rightarrow \bar{V}$ that is linear in $V^{\otimes i}$ and analytic on $\mathcal{F}^i$. Then we can rewrite eq. (4.1.3) as

$$\mathrm{b}Y_1 = 0 \quad (4.1.5)$$

where $\mathrm{b} : \Omega^2(V) \rightarrow \Omega^3(V)$ and $(\mathrm{b}Y_1)(a_1 \otimes a_2 \otimes a_3, x_1, x_2, x_3)$ is given by the left hand side minus right hand side of eq. (4.1.3).

Now let us consider a trivial deformation that is induced by some field redefinition $Z \in \mathrm{End}(\bar{V})[[\lambda]]$. We may assume $Z_0 = \mathrm{Id}_V$. Then

$$Y_1 = Z_1 Y_0 - Y_0 \circ (\mathrm{Id}_V \otimes Z_1 + Z_1 \otimes \mathrm{Id}_V) \tag{4.1.6}$$

If we think of Z_1 as an element of $\Omega^1(V)$ (where $\mathcal{F}^1 = \mathbb{R}^D$ is understood), then eq. (4.1.6) is the prescription for a linear map $\Omega^1(V) \to \Omega^2(V)$ that maps Z_1 to Y_1. In fact, we can give a redefinition of b as a linear map $\Omega_n(V) \to \Omega_{n+1}(V)$ such that both eq. (4.1.3) and eq. (4.1.6) are special cases,

$$\begin{aligned}
(\mathrm{b}X)(a_1 \otimes \cdots \otimes a_{n+1}, x_1, \ldots, x_{n+1}) = \\
& Y_0(a_1, x_1 - x_{n+1}) X(a_2 \otimes \cdots \otimes a_{n+1}, x_2, \ldots, x_{n+1}) \\
& + \sum_{i=1}^{n} (-1)^i X\Big(a_1 \otimes \cdots \otimes a_{i-1} \otimes Y(a_i, x_i - x_{i+1}) a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{n+1}, \\
& x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}\Big) \\
& + (-1)^{n+1} Y_0(X(a_1 \otimes \cdots \otimes a_n, x_1, \ldots, x_n), x_n - x_{n+1}) a_{n+1} \,.
\end{aligned} \tag{4.1.7}$$

It just so happens that the b thus defined is a *differential*, i.e. $\mathrm{b}^2 = 0$. This whole construction is very similar to the Hochschild complex, a concept that arises in the analysis of deformations of finite dimensional associative algebras. The proof for $\mathrm{b}^2 = 0$ is a straightforward computation, see [37]. Before, one has to assure that b is defined on the spaces $\Omega^i(V), i > 2$ by a suitable choice of $\mathcal{F}^i$.

We arrive at the following conclusion: The space of perturbations fulfilling associativity to first order modulo the space of trivial perturbations is given by the cohomology ring

$$H^2(V) = \frac{\mathrm{Ker}\,\mathrm{b} : \Omega^2(V) \to \Omega^3(V)}{\mathrm{Im}\,\mathrm{b} : \Omega^1(V) \to \Omega^2(V)}\,. \tag{4.1.8}$$

The next question is: Given $Y_j, j = 0, \ldots, i$ fulfill associativity up to order i,

$$\begin{aligned}&\sum_{j=0}^{k} Y_j(a_1, x_1 - x_3) Y_{k-j}(a_2, x_2 - x_3) a_3 \\ &\quad = \sum_{j=0}^{k} Y_j(Y_{k-j}(a_1, x_1 - x_2) a_2, x_2 - x_3) a_3 \text{ for } k \leq i\,,\end{aligned} \tag{4.1.9}$$

does there exist Y_{i+1} so that associativity holds in order $i+1$ in λ? This can be reformulated as follows: Let

$$\begin{aligned} w(a_1 \otimes a_2 \otimes a_3, x_1, x_2, x_3) = &\sum_{j=1}^{i} Y_j(a_1, x_1 - x_3) Y_{k-j}(a_2, x_2 - x_3) a_3 \\ &- Y_j(Y_{k-j}(a_1, x_1 - x_2) a_2, x_2 - x_3) a_3 \end{aligned} \tag{4.1.10}$$

Then eq. (4.1.9) reads

$$\mathrm{b} Y_{i+1} = w\,. \tag{4.1.11}$$

This has to be solved for Y_{i+1}. A necessary condition for the existence of a solution is that w as defined in eq. (4.1.10) is b-closed, i.e. $\mathrm{b}w = 0$. It can be shown by another computation (again see [37]) that this is indeed so. Eq. (4.1.11) has a solution if and only if w is a b-exact element of $\Omega^3(V)$. A sufficient condition for the existence of a solution would be that there are no b-closed elements in $\Omega^3(V)$ apart from the b-exact ones, i.e. $H^3(V) = 0$. We conclude that the possible obstructions to construct higher order perturbations are elements of the cohomology ring

$$H^3(V) = \frac{\mathrm{Ker}\,\mathrm{b} : \Omega^3(V) \to \Omega^4(V)}{\mathrm{Im}\,\mathrm{b} : \Omega^2(V) \to \Omega^3(V)} \,. \tag{4.1.12}$$

4.1.2 Gauge theories

In the case of local gauge theories, the above analysis is slightly more complicated. The classical solutions of these theories are invariant under local gauge transformations. As an example, we write down the infinitesimal gauge transformation for a Yang-Mills gauge potential $A = A^I_\mu \mathrm{d}x^\mu T_I$ where T_I, $I = 1, \ldots, N$ are the generators of a semi-simple gauge group G:

$$\delta A^I_\mu = D_\mu \varepsilon^I = \partial_\mu \varepsilon^I + e f^I_{JK} \varepsilon^J A^K_\mu \,,$$

where f^I_{JK} are the structure constants of the Lie algebra of G, $[T_J, T_K] = f^I_{JK} T_I$, e is the gauge coupling constant, and ε is the gauge parameter.

Because of the invariance under gauge transformations, the classical solutions are not determined by the field equation and boundary conditions alone. In other words, the field equation fails to be globally hyperbolic (or elliptic, in the Euclidean case). This property causes some complications in the quantization process, which normally starts from a well defined classical initial value problem [4]. One way to solve this problem is to introduce additional fields and equations to the considered classical system. These additional "ghost fields" and/or "antifields" are chosen such that firstly the enlarged system of equations constitutes a well defined initial value problem, and secondly, it possesses an additional symmetry, the so-called BRST symmetry [6,7]. On the space of fields V, it is implemented by a *differential* $s : V \to V$, $s^2 = 0$. The gauge-invariant observables of the original theory are then identified with elements in a certain cohomology ring of s. Reviews of the BRST formalism with many references can be found in [5, 35]. The major advantage of this formalism is that BRST automatically makes "gauge fixing" compatible with the quantization and renormalization process. See [39] for a discussion of this point. We do not want to review the details of the BRST formalism in relativistic or Euclidean quantum field theory here, and refer the reader to the literature cited above and the references cited therein. See also [64].

Instead, we start directly with the formulation of a gauge theory in the vertex algebra framework. Let V be the space of fields as before. It has to be thought of as containing all ghost- and antifields. Let γ denote the map $V \to V$ defined by $\gamma = \mathrm{Id}$ on $V^{0,\bullet,\bullet}$ and $\gamma = -\mathrm{Id}$ on $V^{1,\bullet,\bullet}$. Let $s : V \to V$ be a *graded derivation* on V, i.e.

$$s(ab) = (sa)b + (-1)^j a\, s(b) \text{ for } a \in V^{j,\bullet,\bullet}\,. \tag{4.1.13}$$

Moreover we want s to be a differential, $s^2 = 0$, and to satisfy

$$s\gamma + \gamma s = 0\,, \tag{4.1.14}$$

which can be paraphrased by saying that s is the generator of a "fermionic" symmetry. Also let there be another grading map $g : V \to V$ with spectrum in $\mathbb{Z}$. The eigenvalue under g is called the "ghost number" of an element of V. We demand that s raises the ghost number by one,

$$gs - sg = s\,. \tag{4.1.15}$$

Finally, we demand that the action of s, g on vertex operators is given by

$$\begin{aligned} s(Y(a,x)b) &= Y(sa,x)b + (-1)^j Y(a,x)(sb) && \text{for } a \in V^{j,\bullet,\bullet} \\ g(Y(a,x)b) &= (g_a + g_b)Y(a,x)b && \text{for } g(a) = g_a a,\ g(b) = g_b g\,. \end{aligned} \tag{4.1.16}$$

This reflects the fact that $Y(a,x)b$ can be understood as an operator product expansion, i.e. a product of two fields at points x and 0 respectively. For a more thorough discussion of these assumptions on s, g, Y see [37].

Presume we have a vertex algebra (V, Y) and maps s, g satisfying all of the above. Then we can define the space of *physical fields*,

$$\hat{V} = \frac{\mathrm{Ker}\, s|_{V_0}}{\mathrm{Im}\, s|_{V_{-1}}} \tag{4.1.17}$$

where $V_j = \{a \in V : g(a) = j\, a\}$, ensuring that physical fields have ghost number 0. Then it is possible to show that Y induces a map

$$\hat{Y} : \hat{V}^{\otimes 2} \times \mathbb{R}^D \setminus \{0\} \to \bar{\hat{V}}, \tag{4.1.18}$$

where $\bar{\hat{V}}$ is defined in the same way as $\bar{V}$ was defined in eq. (2.1.2). See [37] for a proof. Thus $(\hat{V}, \hat{Y})$ is the vertex algebra of gauge invariant objects associated to (V, Y).

Now assume we are given a vertex algebra (V, Y_0) with maps s_0, g as above. As before, we want to classify the possible perturbations of first order and then the obstructions to construct higher order perturbations to the vertex operators. This analysis becomes slightly more involved because simultaneously to Y_0 we have to deform s_0. More precisely, we have to consider formal power series

$$Y = \sum_{i=0}^{\infty} \lambda^i Y_i, \qquad s = \sum_{i=0}^{\infty} \lambda^i s_i \tag{4.1.19}$$

such that $s^2 = 0$ and eq. (4.1.16) hold in each order $i \in \mathbb{N}$ of λ,

$$\sum_{j=0}^{i} s_j s_{i-j} = 0$$

$$\sum_{j=0}^{i} s_j Y_{i-j}(a,x) - Y_{i-j}(s_j a, x) - Y_{i-j}(a,x) s_j = 0\,, \tag{4.1.20}$$

in addition to the perturbative form of associativity,

$$\sum_{j=0}^{i} Y_j(a,x) Y_{i-j}(b,y) - Y_j(Y_{i-j}(a,x-y)b,y) = 0\,. \tag{4.1.21}$$

We want to introduce an operator B that extends the action of s to $\Omega^n(V)$. To do so, we first define

$$f^\gamma = \gamma \circ f \circ \gamma^{\otimes n}$$

for $f \in \Omega^n(V)$. Then we set

$$Bf = s_0 f - \sum_{i=1}^{n} f^\gamma \circ \left(\gamma^{\otimes i-1} \otimes s_0 \otimes \mathrm{Id}_V^{\otimes n-i}\right) \tag{4.1.22}$$

This particular form of B is chosen to ensure that firstly, the right hand sides of eq. (4.1.20) for $i = 1$ are identical to Bs_1 and BY_1, and secondly,

$$B^2 = \mathrm{b}B + B\mathrm{b} = 0\,. \tag{4.1.23}$$

For a proof of this latter statement, see [37]. Eq. (4.1.23) is crucial for what follows because it means that $\delta := \mathrm{b} + B$ is a differential on $\bigoplus_{n=1}^{\infty} \Omega^n(V)$, and we can form the associated *double complex*. To do so, we have to introduce an additional grading of $\Omega^n(V)$ by the ghost number. Let

$$\Omega^{n,m}(V) = \{f \in \Omega^n(V) : g(f) = m\,f\}$$

for $m \in \mathbb{Z}$. By $g(f) = m\,f$, we mean that $g \circ f - f \circ G^n f = m\,f$ where

$$\begin{aligned} G^n =& g \otimes \mathrm{Id}_V \otimes \ldots \mathrm{Id}_V + \mathrm{Id}_V \otimes g \otimes \mathrm{Id}_V \otimes \ldots \mathrm{Id}_V \\ &+ \cdots + \mathrm{Id}_V \otimes \ldots \mathrm{Id}_V \otimes g \\ \in& \mathrm{End}(V^{\otimes n}) \end{aligned} \tag{4.1.24}$$

Thus we have $Y_i \in \Omega^{2,0} \forall i \in \mathbb{N}$ and $s_i \in \Omega^{0,1} \forall i \in \mathbb{N}$. Also note that $B : \Omega^{i,j}(V) \rightarrow \Omega^{i,j+1}(V)$. The double complex is given by the space $\bigoplus_{n\in\mathbb{N},m\in\mathbb{Z}} \Omega^{n,m}(V)$ and the two differentials b, B. We have the diagram

$$\begin{array}{ccccccccc} \xrightarrow{B} & \Omega^{0,0}(V) & \xrightarrow{B} & \Omega^{0,1}(V) & \xrightarrow{B} & \Omega^{0,2}(V) & \xrightarrow{B} \\ & \Big\downarrow \mathrm{b} & & \Big\downarrow \mathrm{b} & & \Big\downarrow \mathrm{b} & \\ \xrightarrow{B} & \Omega^{1,0}(V) & \xrightarrow{B} & \Omega^{1,1}(V) & \xrightarrow{B} & \Omega^{1,2}(V) & \xrightarrow{B} \\ & \vdots & & \vdots & & \vdots & \end{array}$$

in which the rows and columns are exact sequences. We introduce the quotient spaces

$$H^n(V, \delta) = \frac{\bigoplus_{i+j=n} \Omega^{i,j}(V) \cap \mathrm{Ker}\,\delta}{\bigoplus_{i+j=n} \Omega^{i,j}(V) \cap \mathrm{Im}\,\delta}\,. \tag{4.1.25}$$

We write down eqs. (4.1.20) and (4.1.21) in more compact form: Set

$$\alpha_i = (s_i, Y_i, 0, \ldots) \in \bigoplus_{n,m}^{\infty} \Omega^{n,m}(V)$$

$$
\begin{aligned}
u_i &= \sum_{j=1}^{i-1} s_j s_{i-j} \\
v_i(a_1, a_2, x_1, x_2) &= \sum_{j=1}^{i-1} s_j\, Y_{i-j}(a_1, x_1 - x_2)b - Y_{i-j}(s_j a_1, x_1 - x_2)a_2 \\
&\quad - Y_{i-j}(a_1, x_1 - x_2) s_j a_2 \\
w_i(a_1, b_1, c_1, x_1, x_2, x_3) &= \sum_{j=1}^{i-1} Y_j(a_1, x_1 - x_3) Y_{i-j}(a_2, x_2 - x_3) a_3 \\
&\quad - Y_j(Y_{i-j}(a_1, x_1 - x_2)a_2, x_2 - x_3)a_3
\end{aligned}
\tag{4.1.26}
$$

for all $(x_1, x_2) \in \mathcal{F}^2$, $(x_1, x_2, x_3) \in \mathcal{F}^3$ in the third and fourth line of eq. (4.1.26) respectively, so that $u_i \in \Omega^1(V)$, $v_i \in \Omega^2(V)$ and $w_i \in \Omega^3(V)$, provided all u_j, v_j, w_j exist for $j < i$. Then we set

$$
\beta_i = (u_i, v_i, w_i, 0, \dots) \in \bigoplus_{n,m}^{\infty} \Omega^{n,m}(V) \tag{4.1.27}
$$

and we can write eqs. (4.1.20) and (4.1.21) as

$$
\delta\, \alpha_i = \beta_i\,. \tag{4.1.28}
$$

This is the desired cohomological formulation of the problem to construct the i-th order perturbations of s, Y given all perturbations of order $j < i$.

The possible first order perturbations $\alpha_1 = (s_1, Y_1, \dots)$ have to be solutions to the equation $\delta\alpha_1 = \beta_1$. As $\beta_1 = 0$, this means α_1 is δ-closed. A trivial deformation α_1 induced by $Z \in \operatorname{End} V$ reads in first order of λ

$$
s_1 = Z_0 s_1 + Z_1 s_0, \qquad Y_1 = Z_1 Y_0 - Y_0 \circ (Z_1 \otimes \mathrm{Id}_V + \mathrm{Id}_V \otimes Z_1) \tag{4.1.29}
$$

or in more compact form

$$\alpha_1 = \delta\zeta_1 \tag{4.1.30}$$

with $\zeta_1 = (Z_1, 0, \dots)$. Thus we obtain that the space of first order perturbations that fulfill associativity and BRST invariance is given by $H^2(V, \delta)$. Given all perturbations of order $j < i$, the question if there exist s_i, Y_i such that associativity and BRST invariance hold up to order i is the same as asking if there exists a solution α_i to eq. (4.1.28). A necessary criterion for the existence of a solution is $\delta\beta_i = 0$. That this is indeed the case can again be looked up in [37]. It is not sufficient however; if there are δ-closed forms $(f_1, f_2, f_3, 0, 0, \dots)$ that are not δ-exact, then β_i might be of that kind. In this case there is no solution to eq. (4.1.28) and the construction of the deformed vertex algebra fails. Thus the obstructions to the perturbative construction of vertex algebras lie in the space $H^3(V, \delta)$. The elements of this space can be interpreted as potential "gauge anomalies".

4.2 Graphical rules for computing vertex operators

Usually, perturbations of a free quantum field theory are characterized by an interaction Lagrangian, which, together with appropriate counterterms, is inserted into the path integral eq. (1.0.1) in order to obtain the perturbation series of the Schwinger functions. To make this work, one first considers a regulated path integral, with a regulated interaction, S_Λ, and one then removes the regulator Λ. This procedure is explained in many textbooks, see e.g. [82], and it leads also to the definition of the OPE coefficients, C_{ab}^c, see e.g. [50, 51] for a derivation using the Polchinski RG flow equations.

As we have explained, we want to pursue an approach wherein the OPE is elevated to the status of a fundamental relation, and we should therefore also have a method to calculate the perturbations of the OPE coefficients *viz.* vertex operators directly, without recourse to the Schwinger functions. In principle, we have outlined how this works in the preceding subsections. But for this we would need to understand more explicitly the – rather abstractly defined – cohomology rings $H^2(V)$ and $H^3(V)$.

We now turn our attention to a more concrete investigation of perturbations of free field theories. We will investigate the case where the parameter λ in eq. (4.0.1) is identified with a coupling constant. Let us assume that the "bare interaction" (in the usual QFT parlance) is given by a polynomial $\lambda P(\varphi) = \lambda \sum c_p \varphi^p$. In order to get a well-defined perturbative definition of the Schwinger functions, we also assume that the interaction is renormalizable, i.e. $\deg(P)\frac{D-2}{2} \leq D$. It is a well-known fact in standard perturbation theory that the Schwinger functions of the theory may then be defined so that

$$\Big\langle [\Delta\varphi(x) - \lambda P'(\varphi(x))]\mathcal{O}_a(0) \prod_i \mathcal{O}_{d_i}(y_i) \Big\rangle_{\text{corr.}} = 0\,. \tag{4.2.31}$$

Here, $\mathcal{O}_a, \mathcal{O}_{d_i}$ are arbitrary composite fields, and the arguments satisfy $|y_i| > |x| > 0$. The expression in brackets $[\dots]$ is of course the non-linear field equation. Let us now apply the OPE to the expression $[\Delta\varphi(x) - \lambda P'(\varphi(x))]\mathcal{O}_a(0)$ in the above Schwinger function. Then we get a relation between the OPE coefficients involving $\Delta\varphi$ and those involving $P'(\varphi)$. In terms of the vertex operators this relation is

$$\Delta Y(\varphi, x) = \lambda Y(P'(\varphi), x)\,, \tag{4.2.32}$$

where we are now viewing $P'(\varphi)$ as an element in V, the abstract vector space whose elements are in one-to-one correspondence with the composite fields.

Next, we expand the vertex operators in a (formal) perturbation series in λ as in eq. (4.0.1), and this immediately leads to the relation

$$\Delta Y_i(\varphi, x) = Y_{i-1}(P'(\varphi), x). \tag{4.2.33}$$

The evident strategy is now to try to design an iterative scheme from this equation, by calculating the order i vertex operator on the left in terms of the lower order $i-1$ vertex operator on the right, starting inductively with $i = 1$, as all vertex operators of order $i-1 = 0$ are given by the free field operators from section 3.1. However, such a procedure runs into the immediate difficulty that the right side involves the vertex operator associated with the composite field $P'(\varphi)$, whereas the left side only gives the vertex operator of the basic field φ. We must therefore introduce a second induction loop which constructs, iteratively, the vertex operators of an arbitrary $a \in V$ from those of φ. It is the essential strength of the present approach that this is possible, using the associativity in perturbative form,

$$\sum_{j=0}^{i} Y_j(a, x) Y_{i-j}(b, y) = \sum_{j=0}^{i} Y_{i-j}(Y_j(a, x-y)b, y), \qquad 0 < |x-y| < |y| < |x|.$$

Thus, suppose that we are given $Y_i(\varphi, x)$, and all other vertex operators up to order $1, \dots, i-1$. Consider points x, y such that $0 < |x-y| < |y| < |x|$ and the following special case of the i-th order consistency condition:

$$\sum_{j=0}^{i} Y_j(\varphi, x) Y_{i-j}(\varphi, y) = \sum_{j=0}^{i} Y_{i-j}(Y_j(\varphi, x-y)\varphi, y). \tag{4.2.34}$$

Under the hypothesis that this condition indeed holds for the – yet to be constructed – terms in this equation that are not known at this point, we can draw the following conclusion. Let us look at the term with $j = 0$ on the right side of the equation. Using the known form of the Y_0's (from the free theory), we have $Y_0(\varphi, x-y)\varphi = |x-y|^{-(D-2)}\mathbf{1} + \varphi^2 + \dots$, where the dots stand for terms that are smooth in $x - y$ and vanish for $x = y$. Using $Y_i(\mathbf{1}, y) = 0$ for $i > 0$, we hence arrive at the relation

$$Y_i(\varphi^2, y) = \sum_{j=0}^{i} Y_j(\varphi, x) Y_{i-j}(\varphi, y) - \sum_{j=1}^{i} Y_{i-j}(Y_j(\varphi, x - y)\varphi, y) + \dots . \tag{4.2.35}$$

Again, the dots represent terms that vanish in the limit as $|x - y| \to 0$. The singular terms with the minus-signs may be thought of as some sort of "counterterms", which cancel off the divergence of the first term on the right side in the limit. The key thing to note is now that all terms on the right side that do not vanish in the limit $|x-y| \to 0$ are known, by induction. Thus, taking the limit, we obtain the desired vertex operator $Y_i(\varphi^2, x)$, and, by iterating this procedure in an obvious way, all other i-th order vertex operators $Y_i(\varphi^3, x), Y_i(\varphi^4, x), \dots$.

In summary, our iterative scheme is set up in such a way that, at order i, we have to perform one "inversion" of the Laplace operator in eq. (4.2.33) to get $Y_i(\varphi, x)$, and then we subsequently have to construct all other vertex operators at order i via the consistency condition. Unfortunately, it is not evident from what we have said that the vertex operators constructed in this way will satisfy the consistency condition to all orders. Furthermore, there is certainly the freedom to add to the i-th order solution to the inhomogeneous Laplace equation $Y_i(\varphi, x)$ a solution to the homogeneous equation. It would seem that both issues are connected, and that one must impose the validity of the i-th order consistency condition in order to (partly) fix this ambiguity. One would then, furthermore, expect that the ambiguity is equivalent to the usual sort of renormalization ambiguity, which in our framework is given by eq. (4.1.2) (with $Z = \sum \lambda^i z_i$) and a change in the coupling constant λ by a (formal) diffeomorphism.

We will not address these issues here but rather focus on the kind of mathematical expressions that one obtains following the above iterative scheme. We divide our discussion into several parts. First, in section 4.2.1 we discuss how to define a right inverse of the Laplace operator in a way that is suitable in our setting. Then in sections 4.2.2, 4.2.3, we will discuss how to organize the terms that appear in the iteration process by a graphical notation involving trees. In section 4.2.4, we find another representation of the vertex operators in terms of infinite sums of hypergeometric type. In section 4.3, we explain how the "counterterms" may be incorporated into the graphical notation.

4.2.1 Right inverse for the Laplacian

In our inductive scheme, we have to apply the right inverse of the Laplacian at each step of the induction. According to the general setup, the vertex operators are analytic functions on $\mathbb{R}^D \setminus \{0\}$, so we should define a right inverse on this function space. However, in perturbation theory, the space of functions is actually much more restricted. At zeroth order, the vertex operators are in fact infinite sums of products of creation/anhihilation operators, harmonic polynomials $h_\ell(\hat{x})$ and powers r^k, where $\ell \in \mathbb{L}, k \in \mathbb{Z}$, see eqs. (3.1.13) and (3.1.14). Unfortunately, such functions are not stable under the application of the right inverse of the Laplace operator – we also get factors of $\ln r$. For example, it is not possible to choose $k \in \mathbb{Z}, \ell \in \mathbb{L}, A \in \mathbb{R}$ such that $\Delta A\, r^k\, h_\ell(\hat{x}) = r^{-2}$. However $\Delta \frac{\ln r}{D-2} = r^{-2}$ (assuming $D > 2$), i.e. we should choose $G(r^{-2}) = \frac{\ln r}{D-2} + H(x)$, where H is some harmonic function that can be written as $A\, r^k\, h_\ell(\hat{x})$ for some A, k, ℓ. By repeated application, we get factors of $\ln^2 r$, and so on. To incorporate the logarithms, we are thus led to introduce the following spaces of functions for $i \in \mathbb{N}$:

$$\mathcal{E}_i = \operatorname{span}\left\{ r^k \ln^j r\, h_\ell(\hat{x}) : \ \ell \in \mathbb{L},\ k \in \mathbb{Z},\ \mathbb{N} \ni j \leq i \right\}. \tag{4.2.36}$$

Evidently, the union $\cup_j \mathcal{E}_j$ is a filtered ring ($\mathcal{E}_i \mathcal{E}_j \subset \mathcal{E}_{i+j}$). Any right inverse of the Laplace operator, G, maps $\mathcal{E}_i \to \mathcal{E}_{i+1}$ for all i. In order to give an explicit formula for G we introduce a residue integral trick for computing right inverses of the Laplacian which we are going to use extensively. We first define

$$\begin{aligned} G\left(r^k h_\ell(\hat{x})\right) =& G\left(\frac{1}{2\pi \mathrm{i}} \oint_C \frac{\mathrm{d}\delta}{\delta} r^{k+\delta} h_\ell(\hat{x})\right) \\ :=& \frac{1}{2\pi \mathrm{i}} \oint_C \frac{\mathrm{d}\delta}{\delta} \frac{r^{k+2+\delta}}{(k+2+\delta)(k+D+\delta) - |\ell|(|\ell|+D-2)} h_\ell(\hat{x}) . \end{aligned} \tag{4.2.37}$$

This defines G as an operator on $\mathcal{E}_0 \to \mathcal{E}_1$. To extend the trick (4.2.37) to all of $\mathcal{E} = \cup_j \mathcal{E}_j$, assume that we are given $f(x) \in \mathcal{E}_j$ as a j-fold residue integral of the form

$$f(x) = \prod_{i=1}^{j} \left(\frac{1}{2\pi \mathrm{i}} \oint_{C_i} \frac{r^{\delta_i} \mathrm{d}\delta_i}{\delta_i}\right) F(\delta_1, \dots, \delta_j)\, r^k h_\ell(\hat{x}) , \tag{4.2.38}$$

where $F(\delta_1, \dots, \delta_j)$ does not depend on x. It is always possible to represent $f \in \mathcal{E}_j$ as a linear combination of expressions of the form (4.2.38), and in fact, this is the form we will encounter. We define the Laplace inverse of $f(x)$ by

$$\begin{aligned} (Gf)(x) :=& \prod_{i=1}^{j+1} \left(\frac{1}{2\pi \mathrm{i}} \oint_{C_i} \frac{r^{\delta_i} \mathrm{d}\delta_i}{\delta_i}\right) F(\delta_1, \dots, \delta_j) \\ & \times \left[(k+2+\sum_{i=1}^{j+1} \delta_i)(k+D+\sum_{i=1}^{j+1} \delta_i) - |\ell|(|\ell|+D-2)\right]^{-1} r^{k+2} h_\ell(\hat{x}). \end{aligned} \tag{4.2.39}$$

We remark that the order of the residue integrals is not arbitrary here; the integral over δ_{j+1} has to be performed first.

The above formula is not the only possible choice for G. In fact, any other choice G' of the right inverse compatible with $\Delta G' = id_{\mathcal{E}_i}, G' : \mathcal{E}_i \to \mathcal{E}_{i+1}$ can be parameterized by constants $A_{j,\ell}, B_{j,\ell}, j, l \in \mathbb{N}, m \in \{1, ..., N(l, D)\}$:

$$G'\left[r^k \ln^j r\, h_\ell(\hat{x})\right] = G\left[r^k \ln^j r\, h_\ell(\hat{x})\right] + \delta_{|\ell|}^{k+2} A_{j,\ell}\, r^{|\ell|}\, h_\ell(\hat{x}) + \delta_{-|\ell|-D+2}^{k+2} B_{j,\ell}\, r^{-|\ell|-D+2}\, h_\ell(\hat{x}) \,, \tag{4.2.40}$$

where δ_b^a is the Kronecker delta. At each iteration step, we are free to choose one set of constants $A_{j,\ell}, B_{j,\ell}$.

This freedom is partly restricted by associativity. One expects that the remaining freedom corresponds to the renormalization ambiguities in the conventional framework. As already said, we will not prove here that our particular choice for G in eq. (4.2.39) actually leads to a set of vertex operators that fulfill associativity.

We now want to consider in more detail the iterative scheme for calculating vertex operators in perturbation theory described at the beginning of this section, based on the definition of the right inverse G of the Laplacian Δ that we have just given.

4.2.2 The case $D = 2$, $P = \varphi^3$:

We will explain this first for the case $D = 2$, with interaction $\lambda P(\varphi) = \lambda\varphi^3$. What makes the construction in this theory simple is that – as we will see – the "counterterms" in eq. (4.2.35) have no effect. As usual in perturbation theory, it is of no concern that the interaction polynomial is not semi-bounded, even though this would render impossible the non-perturbative construction of the Schwinger functions.

In $D = 2$, the angular part and the $N(\ell, D) = 2$ harmonic polynomials at order $|\ell| > 0$ are

$$\hat{x} = \mathrm{e}^{\mathrm{i}\alpha}, \quad h_{(l,\pm)}(\hat{x}) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\pm il\alpha} \tag{4.2.41}$$

At zeroth order, the vertex operator linear in φ is given by

$$Y_0(\varphi, x) = \mathrm{a}_0 \ln r + \mathrm{a}_0^+ + \sum_{l=1}^{\infty} \sum_{m=\pm 1} \frac{1}{\sqrt{2l}} \Big[r^l \mathrm{e}^{\mathrm{i}ml\alpha}\, \mathrm{a}_\ell^+ + r^{-l} \mathrm{e}^{-\mathrm{i}ml\alpha}\, \mathrm{a}_\ell \Big] . \qquad (4.2.42)$$

When determining the vertex operator $Y_{i+1}(\varphi, x)$ at order $i+1$, we have to calculate $Y_i(\varphi^2, x)$, see eq. (4.2.35), and then apply the inverse G of Δ as in eq. (4.2.33). Eq. (4.2.35) contains what we called "counterterms", which are the terms with the minus sign in front.

We look at the behavior of OPE coefficients/vertex operator matrix elements under rescalings. The "dimension" (denoted by "dim") of the function $r^l h_\ell(\hat{x})(\log r)^k$ is defined[1] to be $-l$. From the definition of the free field vertex operators, we see that

$$\dim\langle c, Y_0(a, x) b\rangle = \dim(a) + \dim(b) - \dim(c) , \qquad (4.2.43)$$

where

$$\dim(a) := \sum_{\ell \in \mathrm{L}} a_\ell (|\ell| + \tfrac{D-2}{2}) \qquad (4.2.44)$$

i.e., each factor of φ counts as having dimension $\frac{D-2}{2}$, and each derivative as 1, which gives $|\ell|$ for the $|\ell|$-th order differential operator $\bar{h}_\ell(\partial)$. No confusion can arise from this double use of "dim". We now assume that all higher order vertex operators matrix elements have definite dimension as well. Each inversion of the Laplacian lowers the dimension by 2, so that we should have

$$\dim\langle c, Y_i(a, x) b\rangle = \dim(a) + \dim(b) - \dim(c) + i[(\deg(P) - 2)\tfrac{D-2}{2} - 2] , \qquad (4.2.45)$$

[1]The "dimension" as defined here coincides with the scaling degree [12,38] for the kind of functions treated here.

We will assume from now on that the vertex operators fulfill eq. (4.2.45). It follows that, in $D = 2$ dimension with a polynomial interaction, the dimension of all terms $\langle c, Y_j(\varphi, x-y)\varphi\rangle$ is strictly negative for $j > 0$, hence they will vanish in the limit $x \to y$. The only counterterm which is allowed for in eq. (4.2.35) by dimensional analysis is

$$\langle \varphi, Y_0(\varphi, \epsilon x)\varphi\rangle Y_{i-1}(\varphi, x) \tag{4.2.46}$$

and this vanishes as $\langle \varphi, Y_0(\varphi, \epsilon x)\varphi\rangle = 0$. For $P(\varphi) = \varphi^4$ we would have had to determine $Y_{i-1}(\varphi^3, x)$ before $Y_i(\varphi, x)$ and thus we would have to deal with counterterms of the form

$$Y_{i-1}(\varphi^3, x) = \cdots - \langle \varphi, Y_0(\varphi, \epsilon x)\varphi^2\rangle Y_{i-1}(\varphi, x) - \ldots\,, \tag{4.2.47}$$

where $\langle \varphi, Y_0(\varphi, \epsilon x)\varphi^2\rangle = \ln|\epsilon x|$ is non-vanishing. For this reason, we will stick with the simpler case $P(\varphi) = \varphi^3$, where we may drop the counterterms. Thus, in this special case, (4.2.35) reads

$$Y_i(\varphi^2, x) = \sum_{j=0}^{i} Y_j(\varphi, x) Y_{j-i}(\varphi, x)\,. \tag{4.2.48}$$

We may thus form $Y_i(\varphi^2, x)$, and hence, using the field equation (4.2.32), we get $Y_{i+1}(\varphi, x)$. For $i = 0$, the formula is instead [compare eq. (3.1.14)]

$$Y_1(\varphi, x) = G\colon Y_0(\varphi, x)^2\colon . \tag{4.2.49}$$

Eqs. (4.2.48) and (4.2.49) have an obvious graphical representation by binary trees. Adopting such a graphical notation immediately helps one to see that if we iterate eq. (4.2.48) starting from $i = 0$ to arbitrary i, then the resulting expression will be organized in terms of binary trees. As we have to heed the order of the creation and annihilation operators in the term

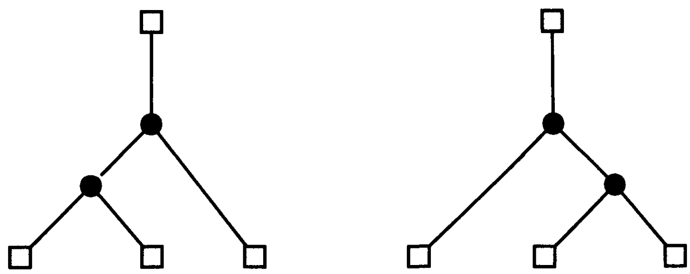


Figure 4.1: Two trees representing the terms $G((G : Y_0(\varphi, x)^2 :) Y_0(\varphi, x))$ and $G(Y_0(\varphi, x)(G : Y_0(\varphi, x)^2 :))$ respectively. Even though the trees are related by a reflection, they have to be counted as different. Both make a contribution to the vertex operator $Y_2(\varphi, x)$.

represented by a tree, we have to count as different trees that are related to each other by a reflection or similar symmetry operation, see fig. 4.1.

These considerations motivate the following definition of trees representing contributions to vertex operators:

Definition 4.2.1. A *tree* T is a 4-tuple (R_T, V_T, L_T, E_T), where

- R_T is the *root*
- V_T is the set of *vertices*
- $L_T = L_T^- \cup L_T^+$ is the set of *leaves*, where L_T^- is called the set of *annihilation leaves* and L_T^- is called the set of *creation leaves*, and $L_T^- \cap L_T^+ = \emptyset$
- $E_T \subset (\{R_T\} \cup V_T) \times (V_T \times L_T)$ is the set of *edges*

such that

- There exists exactly one edge $e = (i, j) \in E(T)$ with $i = R_T$
- For all $v \in V_T \cup L_T$, there exists exactly one sequence $\{e_i\}_{i=1}^n$ of edges $e_i \in E_T$, $i = 1, \dots, n$ connecting the root to v, i.e. fulfilling $(e_1)_1 = R_T$, $(e_i)_2 = (e_{i+1})_1$ for $i =$

$1, \ldots, n-1$ and $(e_n)_2 = v$, where we have used the notation $(e_i)_1 = j, (e_i)_2 = k$ for $e_i = (j,k)$. The sequence $\{e_i\}_{i=1}^n$ is called the *path* from R_T to v, and for the set of vertices on the path we write $P(R_T, v) = \{w \in V_T : w = (e_i)_1 \text{ for some } i = 1, \ldots, n\}$.

Before we further explain the relation between trees and vertex operators, we are going to make some auxiliary definitions concerning vertices and trees. If for two vertices v, w, there exists a path from v to w, we say that w is a *descendant* of v and v is the *antecedent* of w. If $(v,w) \in E_T$, we say that w is a *direct* descendant of v, and v the *direct* antecedent of w.

Because of the necessity to distinguish between trees that are related by a symmetry operation as in figure 4.1, there has to be some order on the vertices and leaves of trees. This is achieved by the next definition.

Definition 4.2.2. An *ordered tree* is a tree T with a total order relation $\prec$ on $V_T \cup L_T \cup \{R_T\}$ such that for $e = (i,j) \in E_T$, $i \prec j$, and for any two direct descendants w_1, w_2 of v with $w_1 \prec w_2$, we have $u \prec w_2$ for any descendant u of w_1. (In a graphical representation, we will draw w_1 to the left of w_2 in this situation.)

We denote the set of ordered trees by $\mathcal{T}$, and the set of trees $T \in \mathcal{T}$ with i vertices, $|V_T| = i$, by $\mathcal{T}_i$.
Furthermore, for a given $T \in \mathcal{T}$, we define the set of *momentum carrying edges* as $E_T^{\mathbf{p}} = E_T \cap ((V_T \cup \{R_T\}) \times V_T)$.

In the remainder of the subsection, we want to describe in somewhat more detail what the mathematical expression is for each such ordered tree. Our choice $P = \varphi^3$ means that we deal with binary trees in this subsection, and we denote the set of binary trees with i vertices by $\mathcal{T}_i^2$.

Let $T \in \mathcal{T}_i^2$ and let $Y_i(T, \varphi, x)$ be the contribution to $Y_i(\varphi, x)$ coming from that tree. Thus,

$$Y_i(\varphi, x) = \sum_{T \in \mathcal{T}_i^2} Y_i(T, \varphi, x) \,. \tag{4.2.50}$$

If we start from the leaves of the tree, then to each leaf j we have to associate a pair $\ell_j = (l_j, m_j) \in \mathbb{Z}_+ \times \{\pm 1\}$ and one of the following factors:

$$\begin{aligned} &\frac{1}{\sqrt{2l_j}} r^{l_j} \mathrm{e}^{\mathrm{i} m_j l_j \alpha}\, \mathbf{a}^+_{l_j, m_j} \text{ if } j \in L_T^+ \\ &\frac{1}{\sqrt{2l_j}} r^{-l_j} \mathrm{e}^{-\mathrm{i} m_j l_j \alpha}\, \mathbf{a}_{l_j, m_j} \text{ if } j \in L_T^- \,. \end{aligned} \tag{4.2.51}$$

Or, if $l_j = 0$, we have to associate one of the factors $\mathbf{a}_0^+$ or $\oint \frac{\mathrm{d}\delta_j}{\delta_j} r^{\delta_j} \mathbf{a}_0$, using the residue trick from section 4.2.1 to generate the logarithm. For each leaf j in a tree a tree $T \in \mathcal{T}_i^2, i > 0$, there is exactly one vertex $v \in V_T$ such that $(v, j) \in E_T$ the direct antecedent of j. The creation/annihilation operators of the leaves that are direct descendants of the same vertex must be normal ordered, by eq. (4.2.49). Let us now consider a vertex v that has no further vertices as descendant; i.e. if we follow a line downwards starting from v, we arrive at a leaf. At v, we have to multiply the factors in eq. (4.2.51) associated to the leaves attached to v, put this product into normal order and then apply the right inverse of the Laplacian G. It is efficient to take care of the phase factors by introducing, for each vertex, an auxiliary integration variable $0 \leq \beta \leq 2\pi$, and to use the trivial identity

$$\begin{aligned} &\prod_{j \in L_T^+} \mathrm{e}^{\mathrm{i} m_j l_j \alpha} \prod_{j \in L_T^-} \mathrm{e}^{-\mathrm{i} m_j l_j \alpha} \\ &= 2 \sum_{l=1}^{\infty} \int_0^{2\pi} \frac{\mathrm{d}\beta}{2\pi} \cos[l(\alpha - \beta)] \prod_{j \in L_T^+} \mathrm{e}^{\mathrm{i} m_j l_j \beta} \prod_{j \in L_T^-} \mathrm{e}^{-\mathrm{i} m_j l_j \beta} \end{aligned} \tag{4.2.52}$$

$$+\int_0^{2\pi}\frac{\mathrm{d}\beta}{2\pi}\prod_{j\in L_T^+}\mathrm{e}^{\mathrm{i}m_jl_j\beta}\prod_{j\in L_T^-}\mathrm{e}^{-\mathrm{i}m_jl_j\beta}\,, \tag{4.2.53}$$

which follows immediately using that the functions $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\pm il\alpha}$ form an orthonormal basis on $[0,2\pi]$. The right inverse of the Laplace operator has to be applied to an expression of the form $r^{\tilde{\nu}}\cos(l(\alpha-\beta))$, where $r^{\tilde{\nu}}$ results from collecting the powers of r of the factors (4.2.51) associated with the leaves below v. The power is thus

$$\tilde{\nu}=\sum_{j\in L_T^+}l_j-\sum_{j\in L_T^-}l_j\,, \tag{4.2.54}$$

where we have assumed that none of the l_j's is zero. Now we introduce $\delta\in\mathbb{C}\setminus\mathbb{Z}$ and apply our residue trick from section 4.2.1 to evaluate $G\left(r^{\tilde{\nu}}\cos(l(\alpha-\beta))\right)$. (For each annihilation leaf line with $l_j=0$, we must replace the corresponding term with another such δ_j.) The result is a contour integral with integrand $2\frac{\cos(l(\alpha-\beta))}{\nu^2-l^2}r^{\nu+2}$, with $\nu=\tilde{\nu}+\delta$. (This holds for $l\neq 0$, for $l=0$ we have $\frac{1}{\nu^2}r^{\nu+2}$.) Now we interchange the order of summation over l and the integration over β. Having introduced the additional integration parameter β now pays off, as we can carry out the sum over l using the following formula[2] for $\nu\in\mathbb{C}\setminus\mathbb{Z}$:

$$\frac{1}{2\nu^2}+\sum_{l=1}^{\infty}\frac{\cos(l(\alpha-\beta))}{\nu^2-l^2}=\frac{\pi}{\sin(\nu\pi)}\frac{\cos(\nu(\alpha-\beta)-\nu\pi)}{2\nu}\,. \tag{4.2.55}$$

The right hand side is bounded in β on $[0,2\pi]$ and in δ on C (which is some circle of fixed radius in the complex plane, cf. eq. (4.2.37)). Thus the interchange of summation and

[2]This may be viewed as a degenerate case of the Dougall identity, see appendix A. To prove the identity, consider the contour integral

$$\oint_C\frac{\cos\left(z(\alpha-\pi)\right)\mathrm{d}z}{z(\nu-z)\sin\pi z}$$

where C is a circle around the origin with radius $\pi(M+1/2)$, $M\in\mathbb{N}$. For $M\to\infty$, the integral vanishes and we obtain eq. (4.2.55) via the residue theorem.

integration is allowed by the dominated convergence theorem. We can repeat this procedure for each of the remaining vertices of the tree, moving upwards in the tree. At each new vertex v we introduce a new integration variable β_v, and a new summation variable l_v – which is summed over using the above cosine identity – as well as a variable ν_v defined similarly as above. More precisely, when we use the residue trick to apply the right inverse of the Laplace operator, we must first introduce for each vertex v in the tree a new variable $\delta_v \in \mathbb{C} \setminus \mathbb{Z}$, and the residue in this variable then has to be taken in the end. If we do all this, we hence arrive at the following *graphical rules* for calculating $Y_i(T, \varphi, x)$, and hence $Y_i(\varphi, x)$:

1. Draw the tree T with i vertices. Label the vertices by an index, v, and the lines by pairs of indices (v, w). The leaves also carry indices.

2. With each leaf j with direct antecedent v, associate a pair $(l_j, m_j) \in \mathbb{Z}_+ \times \{\pm 1\}$ and one of the following factors

$$\begin{aligned} &\frac{1}{\sqrt{2l_j}} r^{l_j} \mathrm{e}^{\mathrm{i} m_j l_j \alpha} \, \mathbf{a}^+_{l_j, m_j} \text{ if } j \in L^+_T \\ &\frac{1}{\sqrt{2l_j}} r^{-l_j} \mathrm{e}^{-\mathrm{i} m_j l_j \alpha} \, \mathbf{a}_{l_j, m_j} \text{ if } j \in L^-_T \, . \end{aligned} \tag{4.2.56}$$

   For the zero modes ($l_j = 0$), we have to associate one of the factors $\mathbf{a}^+_0$ or $\oint \frac{\mathrm{d}\delta_j}{\delta_j} r^{\delta_j} \mathbf{a}_0$, where $\delta_j \in \mathbb{C} \setminus \mathbb{Z}$, again depending on the orientation. The creation/annihilation operators of the leaves connected to the same vertex are to be normal ordered. To the edge $e = (v, j)$ associate $\nu_e := l_j$ if $j \in L^+_T$ and $\nu_e := -l_j$ if $j \in L^-_T$.

3. With each vertex v associate a parameter $\delta_v \in \mathbb{C} \setminus \mathbb{Z}$, and a parameter $\beta_v \in [0, 2\pi]$.

4. To each $e \in E_T^{\mathrm{p}}$, associate $\nu_e \in \mathbb{C} \setminus \mathbb{Z}$, which is determined by

$$\nu_{(u,v)} = 2 + \delta_v + \sum_{\mathrm{d.d.}\, w} \nu_{(v,w)}\,, \tag{4.2.57}$$

to be imposed at each $v \in V_T$, where u is the direct antecedent of v and the sums run over all direct descendants of v. The "2" results from the inversion of the Laplacian, which at each inversion step (i.e., each vertex) raises the power of the radial coordinate by 2. The δ_w arises from the residue trick for the Laplace inversion.

5. With the root, associate the parameter $\alpha \in [0, 2\pi]$, and the factor $r^{\nu_{\mathrm{root}}}$ collecting all the factors of r, where $x = r\mathrm{e}^{\mathrm{i}\alpha}$. The number $\nu_{\mathrm{root}} := \nu_{(R_T,v)} \in \mathbb{C}$, where v is the direct descendant of R_T, is defined by eq. (4.2.57).

6. With each momentum carrying edge $e = (v, w) \in E_T^{\mathrm{p}}$ associate a factor

$$\frac{\pi}{\sin(\pi\nu_e)} \frac{\cos[(\beta_v - \beta_w)\nu_e - \pi\nu_e]}{\nu_e}.$$

This results from the application of the cosine identity (4.2.55).

7. Perform the sum over all $\ell_j \in \mathbb{L}$. Furthermore, perform the integrals

$$\prod_{v \in V_T} \int_0^{2\pi} \mathrm{d}\beta_v \quad \text{and} \quad \prod_{v \in V_T} \frac{1}{2\pi\mathrm{i}} \oint_{C_v} \frac{\mathrm{d}\delta_v}{\delta_v}.$$

The last step requires some further comment. Our first comment concerns the choice of the integration contours C_v in the residue integral. They can be chosen as small circles around the origin with radii $|\delta_v|$ chosen so that for any subset $\mathcal{V}$ of the vertices in the tree, and for any values of the phases of the $\delta_v, v \in \mathcal{V}$, we have $\sum_{v \in \mathcal{V}} \delta_v \in \mathbb{C} \setminus \mathbb{Z}$. We need

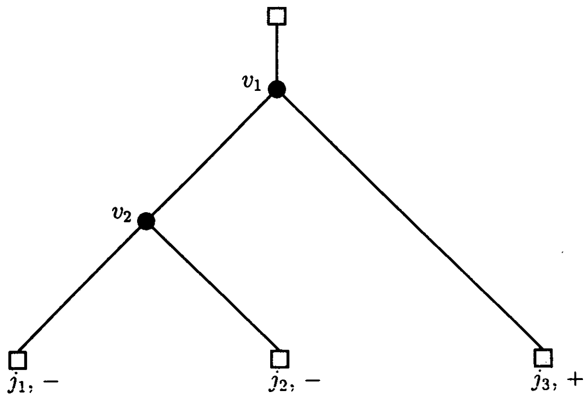


Figure 4.2: Tree contributing to $Y_2(\varphi, x)$.

this in order for our residue trick to work; the exponents ν_v have to be non-integer. The second remark concerns the convergence of the multiple summation over the counters l_j. To illustrate the general point, we consider the example tree given in fig. 4.2.

We apply the above rules and obtain the following expression for fig. 4.2:

$$\begin{aligned}\sum_{\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\boldsymbol{\ell}_3\in\mathbf{L}} \oint_{C_1}\oint_{C_2} \frac{\mathrm{d}\delta_{v_1}}{\delta_{v_1}}\frac{\mathrm{d}\delta_{v_2}}{\delta_{v_2}} \int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\beta_{v_1}\mathrm{d}\beta_{v_2} \\ \times \frac{\pi\cos\left[(\beta_1-\beta_2-\pi)\nu_2\right]}{\nu_2\sin\pi\nu_2}\frac{\pi\cos\left[(\alpha-\beta_1-\pi)\nu_1\right]}{\nu_1\sin\pi\nu_1} \\ \times e^{\mathrm{i}\beta_2(-m_1l_1-m_2l_2)}e^{\mathrm{i}\beta_1 m_3 l_3}r^{\nu_1}\mathbf{a}_{\boldsymbol{\ell}_1}\mathbf{a}_{\boldsymbol{\ell}_2}\mathbf{a}^+_{\boldsymbol{\ell}_3}\end{aligned} \tag{4.2.58}$$

with $\nu_2 = -l_1 - l_2 + 2 + \delta_2$, $\nu_1 = -l_1 - l_2 + l_3 + 4 + \delta_1 + \delta_2$, $\boldsymbol{\ell}_i = (l_i, m_i)$, $i = 1, 2, 3$, which makes a contribution to the vertex operator $Y_2(\varphi, x)$. From eq. (4.2.58), it is not immediately obvious that the left hand side should converge. We will not prove this here, but we expect convergence because of the lack of counterterms in the iteration that led to eq. (4.2.58). In the next section, it will be necessary to introduce regulators to assure that the amplitudes of trees are convergent.

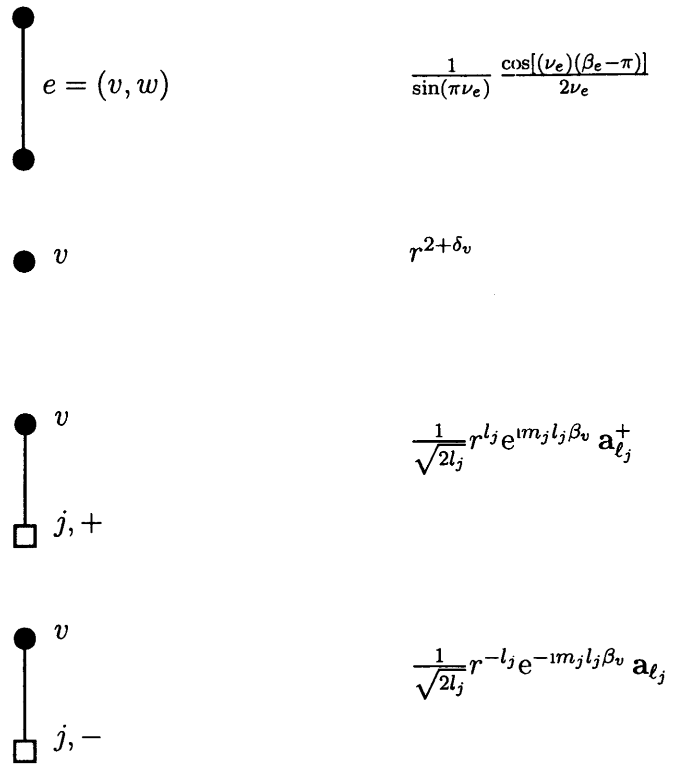


Figure 4.3: The rules for the "amplitude" of a graph. The appropriate summations and integrals have to be understood.

Matrix elements

Now that we have found an expression for the vertex operators, the next step is to calculate matrix elements. The aim is to find rules that amend those from the last subsection so that for given vectors $b, c \in V$, we obtain an expression for the matrix element $\langle c, Y_i(\varphi, x)b\rangle$. We do not discuss the more general matrix element $\langle c, Y_i(a, x)b\rangle$ for $a \in V$ here to avoid the occurrence of counterterms.

First we introduce an additional grading of V by the "field number",

$$V = \oplus_j^\infty V_j$$

$$V_j = \{a \in V : \#\{a_\ell : \ell \in \mathbb{L}, a_\ell > 0\} = j\}. \tag{4.2.59}$$

We write $\#a = j$ for $a \in V_j$. Now let

$$b = \prod_{j=1}^{n} \mathbf{a}^+_{\ell_j} |0\rangle \in V, \quad c = \prod_{k=n+1}^{n+m} \mathbf{a}^+_{\ell_k} |0\rangle\,, \tag{4.2.60}$$

thus $\#b = n, \#c = m$.

Let $\langle 0| \in V^*$ be given by $\langle 0|0\rangle = 1, \langle 0|a = 0$ for $a \in V^{\bullet,k,\bullet}, k > 0$, and let $V^* \ni c^* = \langle 0| \prod_{j=n+1}^{m+n} \mathbf{a}_{\ell_k}$. Let $A \in \mathrm{End}(V)$ be a (not necessarily normal ordered) monomial in creation and annihilation operators, with $\#A$ factors overall. The matrix elements $\langle c,\, A\, b\rangle = c^*(A\, b)$ will only be non-0 if there exists a "pairing" of creation and annihilation operators occurring in c^*, A, b, where each pair consists of one creation operator and one annihilation operator carrying the same index from $\mathbb{L}$. There might be more than one of those pairings for given b, c, A. b, c, A may be represented by $\#b, \#c, \#A$ vertices[3] respectively. A pairing may be represented as a graph on these vertices, where each vertex is connected to precisely one other vertex by an edge. Two vertices connected by an edge must carry the same index $\ell \in \mathbb{L}$.

As we have seen in the last subsection, to any tree T that makes a contribution to $Y_i(\varphi, x)$ according to eq. (4.2.50), there is an associated sum over monomials of creation and annihilation operators with x-dependent functions as coefficients, $Y_i(T, \varphi, x)$. When calculating matrix elements $\langle c,\, Y_i(T, \varphi, x)\, b\rangle$, we have to perform several sums: First the sum over all indices $\ell \in \mathbb{L}$ carried by each of the leaves, and then for each monomial A in creation and annihilation operators specified by a set of indices $\in \mathbb{L}$, over all pairings of the creation and annihilation operators occurring in c^*, A and b. In our graphical representation, the latter

[3] These vertices are not the same as the vertices $v \in V_T$ in definition 4.2.1. We apologize for the double use of the term.

sum is over all graphs which have as vertices the $i+1$ leaves of the tree, and moreover $\#b$ vertices representing creation operators, and $\#c$ vertices representing annihilation operators. The latter two groups of vertices will be called *contravariant leaves* and *covariant leaves* respectively. Now we interchange the order of the sums, i.e. we specify the pairing before we take the sum over the indices carried by the leaves.

The foregoing discussion motivates the following definitions.

Definition 4.2.3. A *contracted tree* is a 7-tuple $(R_T, V_T, L_T, E_T, L_T^b, L_T^c, P_T)$ where

- $T = (R_T, V_T, L_T, E_T)$ is a tree as in definition 4.2.2
- L_T^b is the set of *contravariant leaves*
- L_T^c is the set of *covariant leaves*
- $P_T \subset (L_T^- \cup L_T^c) \times (L_T^+ \cup L_T^b)$ is the *pairing*

satisfying the following conditions,

- For every $i \in L_T^c$ there is exactly one $j \in L_T^+ \cup L_T^b$ with $(i,j) \in P_T$, and for every $j \in L_T^b$ there is exactly one $i \in (L_T^- \cup L_T^c)$ with $(i,j) \in P_T$
- For every $i \in L_T^+$ there is exactly one $j \in (L_T^- \cup L_T^c)$ with $(j,i) \in P_T$ and for every $i \in L_T^-$ there is exactly one $j \in (L_T^+ \cup L_T^b)$ with $(i,j) \in P_T$.

See figure 4.4 for the graphical representation of an example.

In situations where no confusion can arise from it, we will denote a contracted tree $(R_T, V_T, L_T, E_T, L_T^b, L_T^c, P_T)$ simply by T.

Now let $b, c \in V$ as above.

Definition 4.2.4. A *contracted tree on* b, c is a contracted tree T with maps $\ell_b : L_T^b \to \mathbb{L}, \ell_c : L_T^c \to \mathbb{L}$ such that

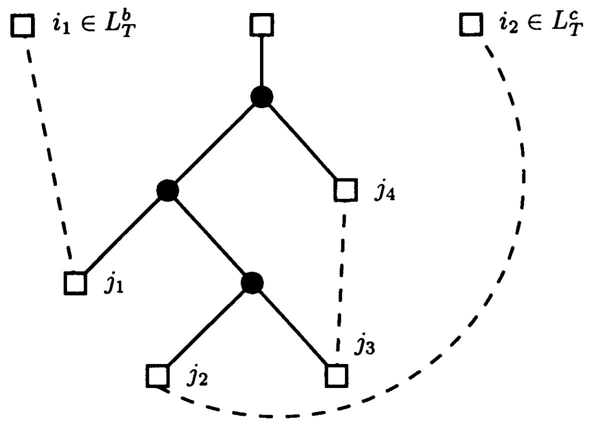


Figure 4.4: Example of a contracted tree. Here, $L_T^b = \{i_1\}$, $L_T^c = \{i_2\}$, $P_T = \{(j_1, i_1), (i_2, j_2), (j_3, j_4)\}$. We have drawn L_T^b to the left of the tree and L_T^c to the right – we will follow this convention in the following figures.

- $b = \prod_{j \in L_T^b} \mathbf{a}^+_{\ell_b(j)}|0\rangle$, $c = \prod_{j \in L_T^c} \mathbf{a}^+_{\ell_c(j)}|0\rangle$
- $\#b + |L_T^+| = \#c + |L_T^-|$
- For each $(i,j) \in P_T \cap (L_T^c \times L_T^b)$, $\ell_c(i) = \ell_b(j)$

Two contracted trees on b, c, $(T_1, {\ell_b}^1, {\ell_c}^1)$, $(T_2, {\ell_b}^2, {\ell_c}^2)$ count as identical if they are isomorphic, in the sense that there exists an isomorphism

$$\iota : V_{T_1} \cup L_{T_1} \cup \{R_{T_1}\} \cup L_{T_1}^b \cup L_{T1}^c \to V_{T_2} \cup L_{T_2} \cup \{R_{T_2}\} \cup L_{T_2}^b \cup L_{T2}^c \tag{4.2.61}$$

with $\iota(V_{T_1}) = V_{T_2}$, $\iota(L_{T_1}) = L_{T_2}$, $\iota(R_{T_1}) = R_{T_2}$, $\iota(L_{T_1}^b) = L_{T_2}^b$, $\iota(L_{T_1}^c) = L_{T_2}^c$, $\{(\iota(v), \iota(w)) : (v,w) \in E_{T_1}\} = E_{T_2}$, $\{(\iota(i), \iota(j)) : (v,w) \in P_{T_1}\} = P_{T_2}$, ${\ell_b}^1(i) = {\ell_b}^2(\iota(i)) \, \forall i \in L_{T_1}^b$, ${\ell_c}^1(i) = {\ell_c}^2(\iota(i)) \, \forall i \in L_{T_1}^c$.

We also denote the contracted tree (T, ℓ_b, ℓ_c) by T^{ℓ_b, ℓ_c}. Let $\mathcal{T}_{i;b,c}$ denote the set of all contracted trees on b, c with $|V_T| = i$. Using the above definitions, we have

$$\langle c, Y_i(\varphi, x) b\rangle = \sum_{T^{\ell_b,\ell_c} \in \mathcal{T}_{i;b,c}} Y_i(T^{\ell_b,\ell_c}, \varphi, x) \tag{4.2.62}$$

where $Y_i(T^{\ell_b,\ell_c}, \varphi, x)$ is the contribution of $T^{\ell_b,\ell_c} \in \mathcal{T}_{i;b,c}$, determined by the following rules:

1. Draw the contracted tree T^{ℓ_b,ℓ_c} with i vertices, $\#b$ covariant leaves, and $\#c$ contravariant leaves.

2. The tree-leaves $i \in L_T$ that are paired with $j \in L_T^b \cup L_T^c$ carry the index $\ell_b(i) = \ell_b(j)$.

3. To each $e = (i, j) \in P_T \cap (L_T \times L_T)$, assign the index ℓ_e, and set $\ell_i = \ell_j := \ell_e$.

4. Follow rules 3-6 from the last subsection.

5. For all pairs $e \in P_T \cap (L_T \times L_T)$ perform the sum over ℓ_e. Furthermore, perform the integrals
$$\prod_{v \in V_T} \int_0^{2\pi} \mathrm{d}\beta_v \quad \text{and} \quad \prod_{v \in V_T} \frac{1}{2\pi \mathrm{i}} \oint_{C_v} \frac{\mathrm{d}\delta_v}{\delta_v}.$$

Each $e \in P_T \cap (L_T \times L_T)$ closes a loop in the contracted tree T. The sum over the associated index ℓ_e will be the source of divergences when we pass to higher order interactions $P = \varphi^p, p > 3$ or a bigger dimension $D > 2$. These divergences will have to be canceled by counterterms as explained in the last subsection.

Summarizing these rules, we get

$$\begin{aligned} Y_i(T^{\ell_b,\ell_c}, \varphi, x) = &\sum_{l_e \in \mathbb{N}: e \in P_T'} \left(\prod_{v \in V_T} \int_{C_v} \frac{\mathrm{d}\delta_v}{\delta_v} \int_0^{2\pi} \mathrm{d}\beta_v \right) \\ &\times \prod_{e \in E_T^{\mathrm{P}}} \frac{\pi}{\sin \pi(l_e + \delta_e)} \frac{\cos[(l_e + \delta_e)(\beta_e - \pi)]}{l_e + \delta_e} \end{aligned}$$

$$\times \prod_{e\in P_T'} \frac{\cos(\beta_e l_e)}{l_e} \exp\left(\ln r \sum_{v\in V_T} (2+\delta_v)\right)$$
$$\times \prod_{i\in L_T^b} \frac{r^{l_i} e^{\mathrm{i} m_i l_i \beta_e}}{\sqrt{2l_e}} \prod_{i\in L_T^c} \frac{r^{-l_i} e^{-\mathrm{i} m_i l_i \beta_e}}{\sqrt{2l_e}}. \qquad (4.2.63)$$

This formula requires several explanations.

The first sum $\sum_{l_e\in\mathbb{N}}$ is a multiple sum. There is one index l_e for each $e \in P_T' = P_T \cap (L_T \times L_T)$ (i.e. one for each "loop" in the contracted tree). Each of these sums runs over all $l_e \in \mathbb{N}$. This results from the sum over all $\ell_e = (l_e, m_e)$ mentioned in rule number 6 above; the sums over m_e have already been performed in eq. (4.2.63).

Each momentum carrying edge $e \in E_T^{\mathbf{p}} = E_T \cap (V_T \cup \{R_T\}) \times V_T$ also carries an $l_e \in \mathbb{Z}$, which is determined by the "conservation rule"

$$l_{(u,v)} = 2 + \sum_{\mathrm{d.d.}\, w} l_{(v,w)} \qquad (4.2.64)$$

which holds at every vertex v, where u is the direct antecedent of v and the sum runs over the direct descendants w of v. Hence, the numbers $l_e, e \in T$ are determined by the L loop momenta $l_e, e \in P_T'$ and the momenta l_i associated to the leaves $i \in L_T$ via the momentum conservation rule at the vertices of T. This momentum conservation rule comes from eq. (4.2.57). For $e = (v, v')$, we have also introduced δ_e as the sum $\sum_w \delta_w$ of all those $\delta_w \in \mathbb{C} \setminus \mathbb{Z}$ that are associated with vertices that are descendants of v, and this accounts for the corresponding term in eq. (4.2.57). Finally, we have set $\beta_e = \beta_v - \beta_w$ if $e = (v, w) \in E_T$, with $\beta_v = \alpha$ if v is the root, and $\beta_e = \beta_v$ if e is not momentum carrying (i.e. $w \in L_T$).

For $i \in L_T^b$ we have used the notation $\ell_b(i) = (l_i, m_i)$ and similarly for $i \in L_T^c$.

As always, x is related to r and α by $x = r e^{\mathrm{i}\alpha}$.

4.2.3 The general case

In the last subsection, we have treated a very special case. No counterterms appeared in the iterative process that starts from the free field vertex operators $Y_0(a, x)$ and successively builds higher order vertex operators $Y_i(\varphi, x)$, $i = 1, 2, \ldots$. This is related to the fact that $D = 2$, $P(\varphi) = \varphi^3$ defines a super-renormalizable theory. In super-renormalizable theories, the coupling λ has positive dimension,

$$\text{dim}\lambda = D - p\frac{D-2}{2} > 0$$

where p defines the interaction polynomial $P(\varphi) = \varphi^p$, and thus there are only finitely many combinations of $i \in \mathbb{N}, a, b, c \in V$ such that $\langle c, Y_i(a, \epsilon x)b\rangle$ will be a counterterm at some point of the iteration process[4]. For renormalizable ($\text{dim}\lambda = 0$) or non-renormalizable theories ($\text{dim}\lambda < 0$) there are infinitely many of these combinations.

The case $D = 2$, $P(\varphi) = \varphi^3$ is special even among the super-renormalizable theories in that the number of possible counterterms is not only finite but 0. Any other choice of D, P will be more complicated – and the particular choice will not matter much in the following. Not even the distinction between renormalizable and super-renormalizable theories will be important for us. The only thing that matters is that compositions of vertex operators like

$$Y_j(\varphi, (1+\epsilon)x)Y_{i-j}(\varphi^k, x), \quad k = 1, \ldots, p-2$$

will occur in the iterative process and in general they will be divergent for $\epsilon \to 0$. The divergences need to be canceled by counterterms. Even though we have entitled this subsection

[4]Here we are talking only about the construction of vertex operators $Y_i(\varphi^k, x), k = 1, \ldots, p-1, i = 1, 2, \ldots$.

"general" we choose $P(\varphi) = \varphi^4$ in the following for definiteness. It will be obvious how to generalize to any other interaction polynomial. Also, we exclude $D = 2$ because in this case most formulas we are going to develop take on a slightly different form.

Our aim is to express the vertex operator $Y_i(\varphi, x)$ as a function of the already known 0-th order vertex operators, and then find a graphical representation for this expression.
We list the equations that we need to decompose $Y_i(\varphi, x)$ into 0-th order operators. Eq. (4.2.33) reads

$$Y_{i+1}(\varphi, y) = G Y_i(\varphi^3, y) . \tag{4.2.65}$$

Moreover, we need the equations (4.2.35) and

$$Y_i(\varphi^3, y) = \sum_{j=0}^{i} Y_j(\varphi, x) Y_{i-j}(\varphi^2, y) - \sum_{j=1}^{i} Y_{i-j}(Y_j(\varphi, x-y)\varphi^2, y) - \frac{1}{|x-y|^{D-2}} Y_i(\varphi, y) + \dots , \tag{4.2.66}$$

where the dots stand again for terms vanishing in the limit $x - y \to 0$.
We do not perform the limit $x - y \to 0$ for the moment, so each time we use either of the eqs. (4.2.35) or (4.2.66), we have to introduce a new variable from $\mathbb{R}^D$. We can choose this new variable (x above) to be $(1+\epsilon)$ times the old variable (y above, i.e. $x = (1+\epsilon)y$), where a new regulator $\epsilon_v > 0$ has to be introduced each time we apply either of eqs. (4.2.35) or (4.2.66).
The result of repeatedly applying eqs. (4.2.65), (4.2.35) and (4.2.66) is a sum of products of nested 0-th order vertex operators whose arguments from $\mathbb{R}^D$ depend on the initial variable from $\mathbb{R}^D$ and the ϵ_v's. We now focus on a special partial sum, the sum of "tree-like" summands. We call a summand tree-like if, when tracing back its path through the iteration process, at each iteration step it does neither belong to the counterterms nor to the smooth

terms represented by dots. Another way to put this is to say that we are only interested in the terms that we would have obtained by dropping the counterterms and dots in eqs. (4.2.35) and (4.2.66). Obviously our focus on tree-like summands is motivated by the fact that we have already developed a formalism for their treatment in section 4.2.2. It will only need some adjustments to account for the occurrence of counterterms and the choice $D > 2$.

We would now like to find a closed form expression for the tree-like terms, applying the same kind of reasoning as in section 4.2.2. The analogue of relation (4.2.52) is

$$\prod_{j\in L_T^+} h_{\ell_j}(\hat{x}) \prod_{j\in L_T^-} \bar{h}_{\ell_j}(\hat{x}) = \sum_{l=0}^{\infty} \frac{2l+D-2}{\sigma_D} \int_{S^{D-1}} \mathrm{d}\Omega(\hat{y})\, \mathrm{P}(D,l,\hat{x}\cdot\hat{y}) \prod_{j\in L_T^+} h_{\ell_j}(\hat{y}) \prod_{j\in L_T^-} \bar{h}_{\ell_j}(\hat{y}) \,, \tag{4.2.67}$$

using this time the orthogonality of the harmonic polynomials, as well as the formula for the Gegenbauer polynomials $\mathrm{P}(z,l,D)$ given in appendix A. We proceed as in section 4.2.2. Let us now consider a vertex v that has no further vertices attached to it downwards in the tree. Let the leaves attached to v be labeled by j. We collect corresponding factors of r, and the harmonic polynomials. The harmonic polynomials are multiplied by the formula just given, while the factors of r work out as r^{l_v}, where l_v is now given by

$$l_v = \sum_{j\in L_T^+} l_j - \sum_{j\in L_T^-} (l_j + D - 2)\,. \tag{4.2.68}$$

Thus, in total we get a factor of $(2l+D-2)r^{l_v}\,\mathrm{P}(\hat{x}\cdot\hat{y},l,D)$. We have to apply the inverse of the Laplacian to this expression using the residue trick, introducing δ_v and $\nu_v = l_v + \delta_v$.

When we do this, we get a contour integral with integrand

$$\frac{(2l+D-2)\mathrm{P}(\hat{x}\cdot\hat{y},l,D)r^{\nu+2}}{\nu_v(\nu_v+D-2)-l(l+D-2)}\,. \tag{4.2.69}$$

The sum over l is now readily performed[5] using the *generalized Dougall's identity*, see appendix B, which states that for any $\nu \in \mathbb{C}\setminus\mathbb{Z}$ and $-1 \le z \le +1$ and $D \ge 3$, we have the identity

$$\sum_{l=0}^{\infty} \frac{(2l+D-2)\,\mathrm{P}(z;l,D)}{\nu(\nu+D-2)-l(l+D-2)} = \frac{\pi}{\sin\pi\nu}\,\mathrm{P}(-z,\nu,D)\,. \tag{4.2.70}$$

From the vertices connected to the leaves, we then work our way upwards, repeating for each new vertex the same procedure. We will then end up with a similar set of graphical rules. The main difference to section 4.2.2 is that we have to use Gegenbauer functions instead of the trigonometric functions, and that we must take into account the regulators ϵ_v. As one can see, this means that we have to introduce extra factors

$$(R_{\epsilon,j})^{|\ell_j|} = \Bigg(\prod_{\text{antecedents}\,v}(1+\epsilon_v)\Bigg)^{|\ell_j|} \quad \text{if } j\in L_T^+$$

$$(R_{\epsilon,j})^{-|\ell_j|-D+2} = \Bigg(\prod_{\text{antecedents}\,v}(1+\epsilon_v)\Bigg)^{-|\ell_j|-D+2} \quad \text{if } j\in L_T^-$$

for each leaf j, where the product runs over all antecedents $v\in V_T$ of j.

To summarize, we have the following *graphical rules* for calculating $Y_i(T^{\ell_b,\ell_c},\varphi,x)$, the contribution of a contracted tree $T\in\mathcal{T}^3_{i;,\ell_b,\ell_c}$ to the vertex operator $Y_i(\varphi,x)$ for the theory with interaction $\lambda P(\varphi) = \lambda\varphi^4$ in $D>2$ (a general interaction polynomial is completely analogous):

1. With each vertex $v\in V_T$ associate a parameter $\delta_v\in\mathbb{C}\setminus\mathbb{Z}$, a parameter $\hat{y}_v\in S^{D-1}$ and

[5]This key observation is due to [36].

a regulator $\epsilon_v > 0$.

2. For each tree-leaf $j \in L^+$ that is contracted with some $i \in L_T^c$ (i.e. $(i,j) \in P_T$) write down

$$R_{\epsilon,j}^{|\ell_c(i)|} \frac{K_D}{\sqrt{\omega(\ell_c(i)), D}} h_{\ell_c(i)}(\hat{y}_v)$$

For each tree-leaf $j \in L^-$ that is contracted with some $i \in L_T^b$ (i.e. $(j,i) \in P_T$) write down

$$R_{\epsilon,j}^{-|\ell_b(i)|-D+2} \frac{K_D}{\sqrt{\omega(\ell_b(i), D)}} \bar{h}_{\ell_b(i)}(\hat{y}_v)$$

To the edge $e = (v,j)$, associate $\nu_e := |\ell_c(i)|$ in the first case and $\nu_e := -|\ell_b(i)| - D + 2$ in the second.

3. To each $e = (i,j) \in P_T' = P_T \cap (L_T \times L_T)$, associate an index $l_e \in \mathbb{N}$ and write down a factor

$$R_{\epsilon,i}^{2-D} \left(\frac{R_{\epsilon,j}}{R_{\epsilon,i}} \right)^{l_e} \mathrm{P}(\hat{y}_i \cdot \hat{y}_j, l_e, D) \tag{4.2.71}$$

If v is the direct antecedent of i and w the direct antecedent of j, then set $\nu_{(v,i)} := -l_e - D + 2$, $\nu_{(w,j)} := l_e$.

4. With each momentum carrying edge e we associate $\nu_e \in \mathbb{C} \setminus \mathbb{Z}$ defined by

$$\nu_{(u,v)} = 2 + \delta_v + \sum_{\text{d.d.}\, w} \nu_{(v,w)}, \tag{4.2.72}$$

to be imposed at each $v \in V_T$, where u is the direct antecedent of v and the sum runs over all direct descendants of v. The "2" results from the inversion of the Laplacian, which at each inversion step (i.e., each vertex) raises the power of the radial coordinate by 2. The δ_v arises from the residue trick for the right inverse of the Laplacian.

5. With the root R, associate the parameter $\hat{x} \in S^{D-1}$, and the factor $r^{\nu(R,v)}$, where $x = r\hat{x}$ and v is the direct descendant of the root.

6. For each momentum carrying edge $e = (v, w)$ write down the factor

$$\frac{\pi}{\sin \pi\nu_e} \mathrm{P}(-\hat{y}_v \cdot \hat{y}_w; \nu_e, D).$$

This results from the application of the Dougall formula.

7. Perform the sum over all l_e for $e \in P'_T$. Furthermore, perform the integrals

$$\prod_{\text{vertices } v} \int_{S^{D-1}} \mathrm{d}\Omega(\hat{y}_v) \quad \text{and} \quad \prod_{\text{vertices } v} \frac{1}{2\pi \mathrm{i}} \oint_{C_v} \frac{\mathrm{d}\delta_v}{\delta_v}.$$

If we proceed as in section 4.2.2 and perform the sum over all contracted trees, similar to eq. (4.2.50), we do not get the matrix element of the complete vertex operator $Y_i(\varphi, x)$. To get the complete vertex operator, we should also incorporate the counterterms (see next section). A sum over graphs analogous to eq. (4.2.50) will depend on the regulators ϵ_v, and will be divergent for $\epsilon_v \to 0$. The expectation is that these divergences are canceled by counterterms.

To assure that the amplitude of each contracted tree is finite, we must choose the regulators not only to be non-zero, but also in such a way that the sums over the counters l_e, $e \in P'_T$, converge. Looking at eq. (4.2.71), we see that this will be so if $R_{\epsilon,i} > R_{\epsilon,j}$. To obtain this for all $e \in P'_T$, it suffices to choose the set of regulators $\{\epsilon_v\}$ such that $\epsilon_v > \epsilon_w$ for $v \prec w$.

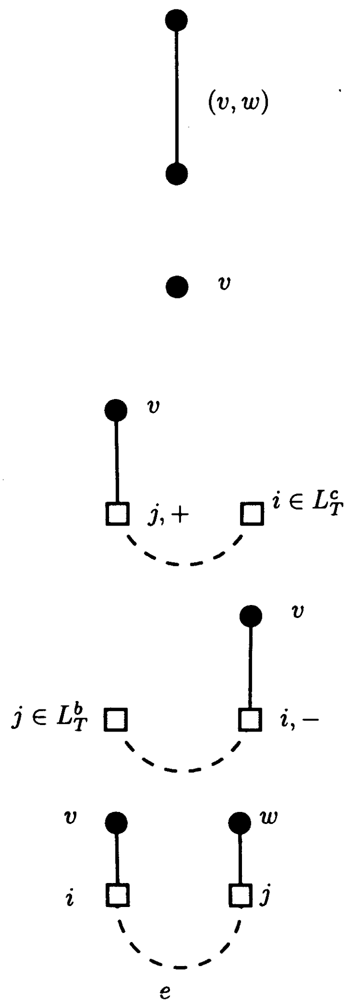
Let us write down explicitly the contributions $Y_i(T^{\ell_b,\ell_c}, \varphi, x)$ from contracted trees T^{ℓ_b,ℓ_c}.

The formula is

$$Y_i(T^{\ell_b,\ell_c},\varphi,x) = \sum_{\substack{l_e\in\mathbb{N}:\\ e\in P'_T}} \left(\prod_{v\in V_T}\int_{C_v}\frac{\mathrm{d}\delta_v}{\delta_v}\int_{S^{D-1}}\mathrm{d}\Omega(\hat{y}_v)\right) \tag{4.2.73}$$
$$\times \prod_{e=(v,w)\in E_T^{\mathrm{P}}} \frac{\pi}{\sin\pi(l_e+\delta_e)}\mathrm{P}(-\hat{y}_v\cdot\hat{y}_w, l_e+\delta_e, D)$$
$$\times \prod_{(i,j)\in P'_T} R_{\epsilon,i}^{2-D}\left(\frac{R_{\epsilon,j}}{R_{\epsilon,i}}\right)^{l_e}\mathrm{P}(\hat{y}_i\cdot\hat{y}_j, l_e, D)$$
$$\times \exp\left(\ln r\left\{\sum_{v\in V_T}(2+\delta_v)+\sum_{i\in L_T^{c}{}'}|\ell_c(i)|-\sum_{i\in L_T^{b}{}'}(|\ell_b(i)|+D-2)-(D-2)|P'_T|\right\}\right)$$
$$\times \prod_{i\in L_T^{c'}}\sqrt{\frac{D-2}{2|\ell_c(i)|+D-2}}h_{\ell_c(i)}(\hat{y}_i)R_{\epsilon,i'}^{|\ell_c(i)|}$$
$$\times \prod_{i\in L_T^{b'}}\sqrt{\frac{D-2}{2|\ell_b(i)|+D-2}}\bar{h}_{\ell_b(i)}(\hat{y}_i)R_{\epsilon,i'}^{-|\ell_b(i)|-D+2}$$

As in eq. (4.2.63), the first sum $\sum_{l_e\in\mathbb{N}}$ is a multiple sum with one index l_e for each $e\in P'_T := P_T\cap(L_T\times L_T)$ (one for each "loop" in the contracted tree). Each of these sums runs over all $l_e\in\mathbb{N}$. In the last two lines, we have set $\hat{y}_i=\hat{y}_v$ for the leaf i with direct antecedent v, and i' is the tree leaf that i is paired with, i.e. the unique tree-leaf such that $(i,i')\in P_T$ or $(i',i)\in P_T$. Also we have used

$$\begin{aligned} L_T^{b'} &= L_T^b\setminus\{j\in L_T : \exists i\in L_T \text{ such that } (i,j)\in P_T\} \\ L_T^{c'} &= L_T^b\setminus\{i\in L_T : \exists j\in L_T \text{ such that } (i,j)\in P_T\} \end{aligned} \tag{4.2.74}$$



$$\frac{\pi}{\sin \pi \nu_{(v,w)}} \mathrm{P}(-\hat{y}_v \cdot \hat{y}_w, \nu_{(v,w)}, D)$$

$$r^{2+\delta_v}$$

$$K_D \omega(|\ell^c(i)|, D)^{-1/2} h_{\ell^c(i)}(\hat{y}_v) r^{|\ell^c(i)|} R_{\epsilon,j}^{|\ell^c(i)|}$$

$$K_D \omega(|\ell^b(j)|, D)^{-1/2} \bar{h}_{\ell^b(j)}(\hat{y}_v) r^{-|\ell^b(j)|-D+2} R_{\epsilon,i}^{-|\ell^b(j)-D+2|}$$

$$R_{\epsilon,i}^{2-D} \left(\frac{R_{\epsilon,j}}{R_{\epsilon,i}}\right)^{l_e} \mathrm{P}(\hat{y}_i \cdot \hat{y}_j, l_e, D)$$

Figure 4.5: The rules for the "amplitude" $Y_i(T^{\ell_c,\ell_b}, \varphi, x, \Theta_i)$ of a contracted tree T^{ℓ_b,ℓ_c} in dimension $D \geq 3$. The appropriate summations and integrals have to be understood.

4.2.4 Alternative expressions for tree-like terms

Before we go on to discuss counterterms, we want to present two alternative forms of formula (4.2.73) to express $Y_i(T^{\ell_b,\ell_c}, \varphi, x)$. First, we perform the sums over the "loop momenta" $\sum_{l_e \in \mathbb{N}}$ in eq. (4.2.73) to arrive at an expression that is a multiple integral without any sums. As a second alternative, we will perform the integrals $\int_{S^{D-1}} \mathrm{d}\Omega(\hat{y}_v)$ to end up with a multiple sum over ratios of Gamma functions, with only the $\oint_{C_v} \mathrm{d}\delta_v$-integrals remaining. These alternative forms are of potential value in explicit calculations.

To derive the first alternative, we observe that in rule 4 above, we associated a complex number ν_e to each edge $e = (v, w)$ linking vertices (v, w) in a contracted tree. ν_e is the sum of an integer l_e and some δ_e that we introduced for the residue trick that defines the right inverse for the Laplacian. l_e is given by the "momentum conservation" rule eq. (4.2.72). This means we can rewrite eq. (4.2.73) by treating each index $l_e \in \mathbb{Z}$ as an independent index for each internal edge e, and take the sum over it, if we write down an additional factor

$$\delta\left(l_{u,v},\, 2 + \sum_{\mathrm{d.d.}\,w} l_{(v,w)}\right) = \int_0^{2\pi} \frac{\mathrm{d}t_v}{2\pi} \exp\left(\mathrm{i}t_v \left\{ l_{(u,v)} - 2 - \sum_{\mathrm{d.d.}\,w} l_{(v,w)} \right\}\right) , \tag{4.2.75}$$

for each vertex v, where δ is the Kronecker delta, u is the direct antecedent of v and the sum runs over all direct descendants w of v. The sum eq. (4.2.73) is absolutely convergent thanks to the regulators $R_{i,\epsilon}$. The insertion of eq.(4.2.75) does not change this fact. Now we introduce a positive number θ_e for each internal edge e, and insert a factor $1 = \lim_{\theta_e \to 0} e^{-\theta_e |l_e|}$. As the sum is absolutely convergent, and the convergence is uniform in θ, we may interchange the summation over the l_e's and the limits $\lim_{\theta_e \to 0}$.

The summations over the l_e's also commute with the multiple integral

$$\prod_{\text{vertices}\, v} \left(\int_0^{2\pi} \mathrm{d}t_v \int_{S^{D-1}} \mathrm{d}\Omega(\hat{y}_v) \oint_{C_v} \mathrm{d}\delta_v \right) ,$$

as the sum of the integrands running over all l_e's converges in $L_1(\Xi)$, where Ξ is the domain of integration in the integral above. Thus we may perform the sums over the indices l_e first and independently of each other, for the momentum carrying edges e and also for the pairings of leaves $e \in P_T \cap (L_T \times L_T)$. More specifically, for a momentum carrying edge

$e \in E_T^{\mathbf{P}}$, we have to perform the sums

$$g_D(\delta_e, \cos\beta_e, t_e, \theta_e) = \sum_{l_e \in \mathbf{Z}} e^{\mathrm{i}(t_e+\pi)l_e - \theta_e|l_e|} \mathrm{P}(-\hat{y}_v \cdot \hat{y}_w, l_e + \delta_e, D) \qquad (4.2.76)$$

and for $e \in P_T' = P_T \cap (L_T \times L_T)$ (i.e. for each closed "loop") we have to sum

$$\sum_{l_e \in \mathbf{N}} R_{\epsilon,j}^{2-D} \left(\frac{R_{\epsilon,i}}{R_{\epsilon,j}}\right)^{l_e} \mathrm{P}(\cos\beta_e, l_e, D)\,. \qquad (4.2.77)$$

The latter equation is easily recognized to be the generating functional for the Gegenbauer polynomials, and equals

$$\left(R_{\epsilon,i}^2 + R_{\epsilon,j}^2 - 2R_{\epsilon,j}R_{\epsilon,i}\cos\beta_e\right)^{-(D-2)/2}\,. \qquad (4.2.78)$$

We assumed $D > 2$ here, for $D = 2$ we get a logarithmic expression. Eq. (4.2.76) can be thought as a two-sided generalization to non-integer indices of the Gegenbauer functions of eq. (4.2.77). The result of eq. (4.2.76) is derived in appendix B. For even D, it is

$$g_D(\delta_e, \cos\beta_e, t_e, \theta_e) = \frac{1}{2^{D/2}\delta_D\Gamma(D/2)} \left(\frac{\partial}{\partial\cos\beta}\right)^{(D-2)/2} \Bigg\{ e^{\mathrm{i}\beta_e\delta_D}\Big({}_2F_1\Big(\delta_D, 1; \delta_D + 1; e^{\mathrm{i}(t_e+\pi+\beta_e)-\theta_e}\Big) \qquad (4.2.79)$$

$$+ {}_2F_1\Big(-\delta_D, 1; 1-\delta_D; e^{-\mathrm{i}(t_e+\pi+\beta_e)-\theta_e}\Big) - 1\Big)$$
$$+ e^{-\mathrm{i}\beta\delta_D}\Big({}_2F_1\Big(\delta_D, 1; 1+\delta_D; e^{\mathrm{i}(t_e+\pi-\beta_e)-\theta_e}\Big)$$
$$+ {}_2F_1\Big(-\delta_D, 1; 1-\delta_D; e^{-\mathrm{i}(t_e+\pi-\beta_e)-\theta_e}\Big) - 1\Big)\Bigg\}$$

where $\delta_D = \delta_e + (D-2)/2$ and ${}_2F_1$ is the Gauss hypergeometric function, see eq. (A.0.15). For odd D, we have

$$
\begin{aligned}
g_D(\delta_e, z, t_e, \theta_e) =& \frac{\sqrt{\pi}}{2^{(D-3)/2}\Gamma(D/2)} \left(\frac{\partial}{\partial z}\right)^{(D-3)/2} \\
& \times \Bigg\{ \frac{-1}{\sqrt{1+e^{2(\mathrm{i}(t_e+\pi)-\theta_e)} + 2e^{\mathrm{i}(t_e+\pi)-\theta_e} z}} \Bigg(F_1\Big(-\delta_D, \delta_D, 1, 1; \frac{1-z}{2}, \frac{1-t_-}{2}\Big) \\
& + \frac{z-t_-}{2} F_1\Big(1-\delta_D, 1+\delta_D, 1, 2; \frac{1-z}{2}, \frac{1-t_-}{2}\Big)\Bigg) \\
& + \frac{-e^{-\mathrm{i}(t_e+\pi)-\theta_e}}{\sqrt{1+e^{2(-\mathrm{i}(t_e+\pi)-\theta_e)} + 2e^{-\mathrm{i}(t_e+\pi)-\theta_e} z}} \Bigg(F_1\Big(\delta_D, -\delta_D, 1, 1; \frac{1-z}{2}, \frac{1-\bar{t}_-}{2}\Big) \\
& + \frac{z-\bar{t}_-}{2} F_1\Big(1+\delta_D, 1-\delta_D, 1, 2; \frac{1-z}{2}, \frac{1-\bar{t}_-}{2}\Big)\Bigg)\Bigg\}.
\end{aligned}
$$

where $t_- = e^{-it_e+\theta_e}\left(1-\sqrt{1+e^{2(it_e+\theta_e)} - 2ze^{it_e+\theta_e}}\right)$, $\delta_D = \delta + (D-3)/2$ and F_1 is the hypergeometric function with two arguments, see appendix B. In this manner, we have gotten rid of all sums over discrete variables at the expense of additional integrals arising from eq. (4.2.75). The resulting formula, equivalent to eq. (4.2.73), is

$$
\begin{aligned}
Y_i(T^{\ell_b,\ell_c}, \varphi, x) =& \left(\prod_{e\in E_T^{\mathrm{P}}} \lim_{\theta_e\to 0+}\right) \left(\prod_{v\in V_T} \int_{C_v} \frac{d\delta_v}{\delta_v} \int_{S^{D-1}} d\Omega(\hat{y}_v) \int_0^{2\pi} dt_v\right) \qquad (4.2.80) \\
& \times \prod_{e=(v,w)\in E_T^{\mathrm{P}}} \frac{\pi}{\sin\pi\delta_e} g_D(\delta_e, \cos\beta_e, t_e, \theta_e) \\
& \times \prod_{(i,j)\in P_T'} \left(R_{\epsilon,i}^2 + R_{\epsilon,j}^2 - 2R_{\epsilon,j}R_{\epsilon,i}(\hat{y}_i\cdot\hat{y}_j)\right)^{-(D-2)/2} \\
& \times \exp\left(\ln r \left\{\sum_{v\in V_T}(2+\delta_v) + \sum_{i\in L_T^{c'}} |\ell_c(i)| - \sum_{i\in L_T^{b'}} (|\ell_b(i)| + D - 2)\right\} - (D-2)|P_T'|\right)
\end{aligned}
$$

$$\times \prod_{i \in L_T^{c\prime}} \sqrt{\frac{D-2}{2|\ell_c(i)|+D-2}} h_{\ell_c(i)}(\hat{y}_i) R_{\epsilon,i}^{|\ell_c(i)|}$$
$$\times \prod_{i \in L_T^{b\prime}} \sqrt{\frac{D-2}{2|\ell_b(i)|+D-2}} h_{\ell_b(i)}(\hat{y}_i) R_{\epsilon,i}^{-|\ell_b(i)|-D+2} .$$

We come to the second alternative which we believe is interesting not only because it might be convenient for calculations, but also because it might hint at an interesting relation between vertex algebras and certain special functions of hypergeometric type.

Again we start from eq. (4.2.73). Our aim is now to take linear combinations of contracted trees as in definition 4.2.4 so that the sum of their "amplitudes" as in eq. (4.2.73) can be expressed without using the spherical harmonics $h_\ell(\hat{x})$ but only the Gegenbauer functions $\mathrm{P}(z, \nu, D)$.

We choose a particular (non-orthogonal) basis of V which is defined as follows. For any $p \in \mathbb{R}^D, l \in \mathbb{N}$, we define $\mathbf{a}_l(p)^+ = \omega(l, D)^{-1/2} \sum h_{l,m}(p)\, \mathbf{a}_{l,m}^+$. For $\vec{p} = (p_1, \ldots, p_n) \in \mathbb{R}^{nD}$ and $\vec{l} = (l_1, \ldots, l_n) \in \mathbb{N}^n$, we then define

$$\begin{aligned} |\vec{p}, \vec{l}\rangle &:= \prod_{i=1}^{n} \mathbf{a}_{l_i}^+(p_i)|0\rangle . \\ &= \prod_{i=1}^{n} \mathrm{P}(p_i \cdot \partial, l_i, D)\varphi . \end{aligned}$$

It is evident from this expression that the vectors $|\vec{p}, \vec{l}\rangle$ form an (overcomplete) basis of V.

Let L be a finite set with $|L| = n$, for each $i \in L$, let $l_i \in \mathbb{N}$ and let $\mathcal{L}_{L,\{l_i\}}$ denote the set of maps $\ell_L : L \to \mathbb{L}$ such that $|\ell(i)| = l_i$. Given a contracted tree (T, L_T^b, L_T^c, P_T), let $L_T^{b\prime} = \{i \in L_T^b : \exists j \in L_T^c : (i,j) \in P_T\}$ and $L_T^{c\prime} = \{j \in L_T^c : \exists i \in L_T^b : (i,j) \in P_T\}$. Also, for each $i \in L_T^b{}' \cup L_T^c{}'$, let $l_i \in \mathbb{N}, p_i \in \mathbb{R}^D$. We want to provide an alternative expression for

$$Y(T, L_T^b, L_T^c, \vec{l}, \vec{p}) = \sum_{\ell_b \in \mathcal{L}_{L_T^b{}', \{l_i : i \in L_T^b{}'\}}} \sum_{\ell_c \in \mathcal{L}_{L_T^c{}', \{l_j : j \in L_T^c{}'\}}} \left(\prod_{i \in L_T^b} h_{\ell_b(i)}(p_i) \prod_{j \in L_T^c} h_{\ell_c(j)}(p_j) \right) Y(T^{\ell_b, \ell_c}, \varphi, x, \{\epsilon_i\}) \,. \tag{4.2.81}$$

By taking partial derivatives of $Y(T, L_T^b, L_T^c, \vec{l}, \vec{p})$ with respect to the p_i, and taking appropriate sums, one can obtain any matrix element of the form $\langle c, Y_i(T, \varphi, x) b \rangle$.

The basic idea is now to insert eq. (4.2.73) into eq. (4.2.81) and carry out the angular integrals $\int \mathrm{d}\Omega(\hat{y}_v)$ first, or rather, to turn these integrals into sums. This is done by first expanding the Gegenbauer functions using the formula (valid for $\nu \notin \mathbb{Z}$)

$$\mathrm{P}(z, \nu, D) = \frac{\sin \pi\nu}{\pi} \frac{2^{-(D+1)/2}}{\Gamma(D/2)} \sum_{n=0}^{\infty} (-2z)^n \frac{\Gamma(-\nu/2 + n/2)\Gamma(\nu/2 + n/2 + D/2 - 1)}{n!} \,. \tag{4.2.82}$$

which we prove in appendix B. For $\nu = l \in \mathbb{Z}$, we have instead the well-known formula

$$\mathrm{P}(z, l, D) = \frac{2^{-(D-3)/2}}{\Gamma(D/2)} \sum_{j=0}^{\lfloor l/2 \rfloor} (-2z)^{l-2j} \frac{\Gamma(l - j + D/2 - 1)}{j!(l - 2j)!} \,. \tag{4.2.83}$$

If we perform these expansions for all the Gegenbauer functions appearing in eq. (4.2.73), then we end up with a multiple sum, whose terms contain powers $(\hat{y}_v \cdot \hat{y}_w)^{n_e}$, where $e = (v, w)$ is an edge between v, w, and where each n_e is the summation counter from the power series expansion of the Gegenbauer functions associated with the edge e. To perform the angular integrals, we now further expand each such power using the multinomial formula,

$$(\hat{y}_v \cdot \hat{y}_w)^{n_e} = \sum_{k_{e,1}+\cdots+k_{e,D}=n_e} \frac{n_e!}{\prod_\mu k_{e,\mu}!} \prod_\mu (\hat{y}_{v,\mu}\hat{y}_{w,\mu})^{k_{e,\mu}} . \tag{4.2.84}$$

Here, and in the following, μ runs from 1 to D. After the combined expansions, each term in eq. (4.2.73) will now consist of a prefactor times $\hat{y}_{v,\mu}$, raised to some power $a_{v,\mu}$. The power is

$$a_{v,\mu} = \sum_{e=(v,w) \text{ or } e=(w,v)} k_{e,\mu} , \tag{4.2.85}$$

where the sum is over all edges e that are of the form (v,w) or (w,v) for some $w \in \{R_T\} \cup V_T \cup L_T$. Thus, the integrals we have to consider are of the type ($a_i \in \mathbb{N}$)

$$\int_{S^{D-1}} \mathrm{d}\Omega(\hat{x})\, \hat{x}_1^{a_1} \cdots \hat{x}_D^{a_D} = \begin{cases} 2\dfrac{\prod_\mu \Gamma\left(\frac{a_\mu+1}{2}\right)}{\Gamma\left(\frac{\sum_\mu a_\mu + D}{2}\right)} & \text{if all } a_i \text{ even} , \\ 0 & \text{otherwise} . \end{cases} \tag{4.2.86}$$

This formula can be viewed as a multi-dimensional generalization of the standard formula for the Euler Beta-function ($D = 2$) and can be proved e.g. by induction in D, expressing $\mathrm{d}\Omega$ in D-dimensional polar coordinates. If we combine all the steps we have described so far, then we end up with the following expression for the vertex operator:

$$Y(T, L_T^b, L_T^c, \vec{l}, \vec{p}) = \sum_{\substack{k_e \in \mathbb{N}^D: \\ e \in E_T^{\mathrm{p}} \cup P_T'}} \sum_{\substack{j_e \in \mathbb{N}: e \in P_T', \\ 2j_e \le |k_e|}} \sum_{\substack{k_i \in \mathbb{N}^D: \\ i \in L_T^{b'} \cup L_T^{c'}, |k_i| = l_i}} \sum_{\substack{j_i \in \mathbb{N}: i \in L_T^{b'} \cup L_T^{c'}, \\ 2j_i \le l_i}} \left(\prod_{v \in T} \frac{1}{2\pi \mathrm{i}} \int_{C_v} \frac{\mathrm{d}\delta_v}{\delta_v} \right)$$
$$\times \prod_{e \in E_T^{\mathrm{p}}} \frac{2^{|k_e|-(D+1)/2} \Gamma(-\nu_e/2 + |k_e|/2) \Gamma(\nu_e/2 + |k_e|/2 + D/2 - 1)}{\Gamma(D/2) k_e!}$$

$$
\begin{aligned}
&\times \prod_{v \in V_T} \frac{\prod_\mu \Gamma((\sum_{e\,\mathrm{on}\,v} k_{e,\mu} + 1)/2)}{\Gamma((\sum_{e\,\mathrm{on}\,v} |k_e| + D)/2)} \\
&\times \prod_{e=(i,j)\in P'_T} \frac{\Gamma(|k_e| + j_e + D/2 - 1)}{j_e!(|k_e|)!} R_{\epsilon,i}^{2-D} \left(\frac{R_{\epsilon,j}}{R_{\epsilon,i}}\right)^{|k_e|-2j_e} \\
&\times \prod_{i\in L^b_T} \frac{\Gamma(|k_i| + j_i + D/2 - 1)}{j_i!|k_i|!} \prod_{i\in L^c_T} \frac{\Gamma(|k_i| + j_i + D/2 - 1)}{j_i!|k_i|!} \\
&\times \exp\left(\ln r \left(+\sum_{v\in V_T}(2+\delta_v) + \sum_{i\in L^{c'}_T} l_i - \sum_{i\in L^{b'}_T}(l_i + D - 2)\right)\right) \\
&\times \hat{x}^{k_0} \prod_{i\in L^{b'}_T} p_i^{k_i} \prod_{i\in L^{c'}_T} p_i^{k_i} \qquad (4.2.87)
\end{aligned}
$$

Eq. (4.2.87) requires several comments. The sums in the first row are to be understood as follows: For each momentum carrying edge $e \in E^{\mathbf{p}}_T$ and for each contraction $e \in P'_T$, there is an index $k_e \in \mathbb{N}^D$, which is to be summed over. In order to be able to apply eq. (4.2.83) to the Gegenbauer polynomials associated with a contraction $e \in P'_T$, we have to introduce another index $j_e \in \mathbb{N}$, that is being summed over with the condition $2j_e \leq |k_e|$, where we are using multiindex notation,

$$|k| = \sum_\mu k_\mu \, .$$

Similarly, for each $i \in L^{b'}_T \cup L^{c'}_T$, we have an indices $k_i \in \mathbb{N}^D, j_i \in \mathbb{N}$ that are being summed over subject to the conditions $|k_i| = l_i, 2j_i \leq l_i$. For $e \in E^{\mathbf{p}}_T$, the value of ν_e is again determined by the momentum conservation rule (4.2.64). We write "e on v" to mean that the sum/product is running over those edges e going out from the vertex v, cf. eq.(4.2.85). The integer vector $k_0 \in \mathbb{N}^D$ is the counter associated with the internal edge that connects the root to the rest of the tree. Some more multiindex notation that we have used in eq. (4.2.87) is given by

$$p^k = \prod_\mu p_\mu^{k_\mu}\,, \quad k! = \prod_\mu k_\mu!\,.$$

Formula (4.2.87) is the desired alternative representation of the contribution to the vertex operator. The residue integrals $\int \mathrm{d}\delta_v/\delta_v$ can be performed straightforwardly using the well-known Laurent expansion of the Gamma-function around integer values, which can be inferred from the standard formula

$$\Gamma(1+\delta) = \frac{1}{1+\delta}\mathrm{e}^{\delta(1-\gamma_\mathrm{E})}\exp\left\{\sum_{n=2}^{\infty}(-\delta)^n(\zeta_n - 1)/n\right\}\,, \tag{4.2.88}$$

where ζ_n are the values of the Riemann Zeta-function. Thus, we see that we get a representation involving only (multiple) sums.

It is fair to ask what is the value of having the alternative representations (4.2.80) and (4.2.87). It is not clear that either representation has much of an advantage computationally, as there is essentially an equivalent number of summations as there are integrations in both formulae (4.2.80) and (4.2.87). However, the alternative representation (4.2.87) brings out a striking feature that was far from obvious when we started the construction of the vertex operators, namely that it can be represented in terms of (multiple) infinite series of a very special form, with each term being a monomial in r and $\hat{x}^\mu$, $\mu = 1, \ldots, D$ times a ratio of Gamma-functions. Because of this feature, the above series can be viewed as a generalization of the Gauss hypergeometric series, associated to a contracted tree.

In the next section, we are going to include the counterterms and write down explicit formulas for the vertex operators that result from our iterative procedure. We will not develop a form analogous to eq. (4.2.87) for the formulas including counterterms, but this would be straightforward. These formulas will have to be taken with a grain of salt: We will not be able to show that they satisfy all axioms of definition 2.1.1. In particular, associativity is an open problem, and we will not resolve the question whether or not our formulas satisfy it here, cf. the remarks at the end of section 4.2.1. However, we expect that there exists a choice of a right inverse G of the Laplacian Δ such that associativity holds for vertex operators constructed from the iterative procedure. These vertex operators will have a representation very similar to eq. (4.2.87); in the derivation of these formulas, our particular choice of G did not play any role, and counterterms can be included as well. In this situation, associativity will mean that there exist many highly non-trivial relations between these "hypergeometric functions associated to contracted trees".

4.3 Renormalization

In conventional perturbative quantum field theory, renormalization is necessary to make the terms in the perturbation series well defined. Two examples have already been mentioned in the introduction: If one defines the path integral eq. (1.0.1) that prescribes the correlation functions of the theory by the Polchinski flow equations, then renormalization consists in choosing appropriate boundary values, and determining the cutoff dependent counterterms in the action S_Λ from the flow. If one defines the path integral via Feynman diagrams, the counterterms in S_Λ have to be chosen such that the divergences in the diagrams cancel each other. As a particularly prominent example of this latter approach, we mention dimensional regularization [71], where $\Lambda = \epsilon^{-1}$ and $4 + \epsilon$ is viewed as the dimension of the "regularized" theory. The action of the regularized theory is ϵ-independent. In the renormalized theory, the ϵ-dependent counterterms in the action merely lead to the subtraction of the pole parts in ϵ of the Feynman graphs of the regularized theory.

Thirdly, we mention the Epstein-Glaser approach to renormalization [12,20], which is suited in particular to curved spacetimes. Here one does not define correlation functions via a path integral, but rather one is interested in the field operators for an interaction that takes place in a bounded region of a globally hyperbolic spacetime M. The local interaction is understood to be characterized by a smooth function f with compact support[6] that is 1 in some neighborhood of a Cauchy surface of M. The field operators $\mathcal{O}_a(x; f)$ take values in the formal power series in $\hbar$ and the coupling constant λ , where each coefficient is an element of the algebra generated by the (smeared) free fields, their Wick and time-ordered products. $\mathcal{O}_a(x; f)$ is known once all time-ordered products of the free field theory are known (via Bogoliubov's formula, see [39]), and the latter can be constructed by making certain natural assumptions on them and then proceeding inductively in the number of arguments of the time-ordered products [40,41]. The crucial technical ingredient in this inductive procedure is the extension of distributions that are only defined on the complement of the small diagonal of the product space M^n to the whole of M^n [12].

In our approach the situation is somewhat different to all of the above, because associativity tells us in principle right from the start how to obtain well defined perturbations of arbitrary order. We think this is a remarkable feature of the present framework. Nevertheless, we have already borrowed some vocabulary from renormalization such as "counterterms", and the reason for this is that we also need to perform various limits in our approach which are quite reminiscent of certain operations in conventional renormalization theory. In fact, the inclusion of the counterterms into the rules may be thought of as the "renormalization" of the tree-like contributions that we have treated in the last subsection and that diverge when

[6]For simplicity, we have assumed that the Cauchy surfaces of M are compact here.

the regulators are sent to 0. The counterterms cure these divergences.

For definiteness, we only treat the case $p = D = 4$ here. This defines a renormalizable theory. The generalization of the constructions below to other renormalizable theories (e.g. $D = 3, p = 6$ or $D = 6, p = 3$) will be quite obvious. At the end, we will comment on the differences to the super-renormalizable and the non-renormalizable case.
So let $p = D = 4$ and let us go back to the recursion procedure for evaluating $Y_{i+1}(\varphi, x)$. We assume that $Y_i(\varphi, x)$ and $Y_j(c, x), j \leq i-1, c \in \{\varphi, \varphi^2, \varphi^3\}$ are known. By eq. (4.2.65),

$$Y_{i+1}(\varphi, x) = G\,Y_i(\varphi^3, x)\,. \tag{4.3.89}$$

We assumed that the right hand side is not known yet. By eq. (4.2.66),

$$\begin{aligned} Y_i(\varphi^3, x) = &\sum_{j=0}^{i} Y_j(\varphi, (1+\epsilon)x) Y_{i-j}(\varphi^2, x) \\ &- \langle \varphi, Y_0(\varphi, \epsilon x)\varphi^2 \rangle Y_i(\varphi, x) \\ &- \sum_{j=1}^{i} \sum_{\dim(c) \leq 3} \langle c, Y_j(\varphi, \epsilon x)\varphi^2 \rangle Y_{i-j}(c, x) + \ldots \end{aligned} \tag{4.3.90}$$

where here and in the following dots stand for terms vanishing for $\epsilon \longrightarrow 0$. We claim that $\langle c, Y_i(\varphi, \epsilon x)\varphi^2 \rangle \neq 0$ only if $\#c \equiv 1 \bmod 2$. We will prove this in a moment. Thus we only need to consider $c \in \{\bar{h}_\ell(\nabla)\varphi : |\ell| \leq 2\} \cup \{\varphi^3\}$ and eq. (4.3.90) reads

$$\begin{aligned} Y_i(\varphi^3, x) = &\sum_{j=0}^{i} Y_j(\varphi, (1+\epsilon)x) Y_{i-j}(\varphi^2, x) \\ &- \sum_{j=0}^{i} \Bigg(\langle \varphi Y_j(\varphi, \epsilon x)\varphi^2 \rangle Y_{i-j}(\varphi, x) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=1}^{N(1,4)} \langle \bar{h}_{(1,m)}(\nabla)\varphi,\, Y_j(\varphi,\epsilon x)\varphi^2\rangle \bar{h}_{(1,m)}(\partial)Y_{i-j}(\varphi,x) \\
&+\sum_{m=1}^{N(2,4)} \langle \bar{h}_{(2,m)}(\nabla)\varphi,\, Y_j(\varphi,\epsilon x)\varphi^2\rangle \bar{h}_{(2,m)}(\partial)Y_{i-j}(\varphi,x) \Bigg) \\
&-\sum_{j=1}^{i} \langle \varphi^3, Y_j(\varphi,\epsilon x)\varphi^2\rangle Y_{i-j}(\varphi^3,x) + \ldots
\end{aligned}
\tag{4.3.91}
$$

Our assumption was that all terms on the right hand side are known except for $Y_i(\varphi^2, x)$. We claim that $\langle c, Y_i(\varphi^2, \epsilon x)\varphi\rangle \neq 0$ only if $\#c \equiv 0 \bmod 2$, and thus

$$
\begin{aligned}
Y_i(\varphi^2, x) = &\sum_{j=0}^{i} Y_j(\varphi, (1+\epsilon)x) Y_{i-j}(\varphi, x) \\
&- \langle 0|Y_i(\varphi,\epsilon x)\varphi\rangle \mathrm{Id}_V - \sum_{j=1}^{i} \langle \varphi^2, Y_j(\varphi,\epsilon x)\varphi\rangle Y_{i-j}(\varphi^2, x) + \ldots .
\end{aligned}
\tag{4.3.92}
$$

The proof for the above claims on vanishing matrix elements is an easy induction argument: For $i = 0$, we have $\langle c, Y_0(\varphi, \epsilon x)\varphi\rangle = 0$ for $\#c \equiv 0 \bmod 2$ and $\langle c, Y_0(\varphi^2, \epsilon x)\varphi\rangle = 0$ for $\#c \equiv 1 \bmod 2$ by the explicit formula for the free field vertex operators eq. (3.1.14). The induction step follows from eqs. (4.3.92), (4.3.91) and (4.3.89).

We now want to give a graphical representation of the iterative procedure described above, that leads to an explicit expression for $Y_i(\varphi, x)$ including counterterms. The aim is to find the equivalent to the rules that we gave in section 4.2.2. In other words, given a perturbation order i and a variable $x \in \mathbb{R}^D$, we want to define

- a set of graphical objects $\mathcal{G}$
- a set of regulators $\Theta = \{\epsilon_j\}$

- a map $Y_i(\cdot, \varphi, x, \Theta)$ from $\mathcal{G}$ to $\mathrm{Hom}(V, \bar{V}) \otimes \mathcal{O}(x, \Theta)$ where $\mathcal{O}(x, \Theta)$ is the set of functions that are analytic in $x \in \mathbb{R}^D \setminus \{0\}$ and analytic in $(\epsilon_1, \dots, \epsilon_{|\Theta|})$ on some open domain of $\mathbb{R}^{|\Theta|}$ to be specified

such that the vertex operator $Y_i(\varphi, x)$ is given by

$$\left(\prod_{\epsilon \in \Theta} \lim_{\epsilon \to 0}\right) \sum_{G \in \mathcal{G}} Y_i(G, \varphi, x, \Theta) . \qquad (4.3.93)$$

The multiple limit will have to be taken in the appropriate order.

The definition of the sets and maps mentioned in the bullet points above are what we call "rules for the construction of $Y_i(\varphi, x)$". The ingredients will be the rules from section 4.2.3 and eqs. (4.3.91), (4.3.92). The multiplication of operators in these formulas will again be performed by the help of eq. (4.2.67) and the right inverse of the Laplacian can again be defined as in section 4.2.1. So in principle, the construction of the rules is straightforward. Unfortunately, the inclusion of the counterterms will make it necessary to introduce a lot of additional notation. This will make the rules quite complicated. In a first reading, the potential reader may therefore skip the following constructions and get acquainted with the concept of *renormalization trees*, which is the set of graphical objects $\mathcal{G}$ above, by studying the example in figure 4.6, the allowed vertices in renormalization trees in table 4.1, the graphical rules for $Y_i(\cdot, \varphi, x, \Theta)$ in figures 4.7 and 4.8 and the examples of section 4.3.1.

The idea is as follows: The only feature of the iterative procedure for the construction of perturbations of vertex operators that has not been included in the graphical rules so far is the occurrence of products of the form $\langle c, Y_j(\varphi, \epsilon x)\varphi^k\rangle Y_{i-j}(c, x)$. We can represent contributions to these terms by drawing two trees next to each other: To the left, a contracted tree T_1 as in definition 4.2.3 contributing to the matrix element $\langle c, Y_j(\varphi, \epsilon x)\varphi^k\rangle$. To the right we draw $\#c$ ordered trees $T_2, \ldots, T_{1+\#c}$ as in definition 4.2.2 contributing to $Y_{i-j}(c,x)$. (The vector c in the matrix element determines the number of trees that represent $Y_{i-j}(c,x)$.) In our graphical representation, we identify the covariant leaves of T_1 representing c with the roots of the $\#c$ trees representing $Y_{i-j}(c,x)$.

As we have seen in eqs. (4.3.91) and (4.3.92), there is only a finite number of covariant vectors c that will appear in the counterterms of the iteration scheme. More precisely, $c \in \{|0\rangle, \varphi, \bar{h}_{(1,m)}(\nabla)\varphi, \bar{h}_{(2,m)}(\nabla)\varphi, \varphi^2, \varphi^3\}$. This means we can make a list of the "allowed" contracted trees that represent contributions to matrix elements by specifying their covariant and contravariant leaves. This list will be the main point of the definition below.

The amplitude of the trees $T_1, \ldots, T_{1+\#c}$ is just the product of the individual amplitudes. Similar to what we have done before, we indicate multiplication by drawing a vertex v above all those trees, and edges between v and each of the roots. The contracted tree T_1 plays a slightly different role to those of the others, in that the spatial variable associated to it is ϵx. Thus we have to distinguish the edge between v and the root of T_1 from the other edges: We call it an *ϵ-edge* and draw a little ϵ next to it.

As before, the vertex v stands not only for multiplication, but also for the application of the right inverse G of the Laplacian. This works precisely as before, applying the residue trick from section 4.2.1.

The foregoing discussion is the motivation for the following definition.

Definition 4.3.1. A *renormalization tree* T consists of

- A root R_T, and a set of *internal roots* $R_{\text{int.}}$

- A set of vertices V_T
- A set of tree-leaves L_T consisting of two disjoint sets L_T^+, L_T^-
- A set $L^b_{\text{int.}}$, called the set of *internal contravariant leaves* and a set $L^c_{\text{int.}} \subset R_{\text{int.}} \cup L^b_{\text{int.}}$, called the set of *internal covariant leaves*
- A set of edges $E_T \subset (V_T \cup L_T \cup \{R_T\} \cup R_{\text{int.}})^2 := (\bar{V}_T)^2$, consisting of two disjoint sets E_T^ϵ, E_T^{plain}, called *ϵ-edges* and *plain edges* respectively
- A pairing $P_T \subset (L_T \cup L^b_{\text{int.}}) \times (L_T \cup L^c_{\text{int.}})$.
- A total order relation $\prec$ on $\bar{V}_T$

satisfying the following conditions:

- $R_{\text{int.}}, V_T, L_T, L^b_{\text{int.}}$ are mutually disjoint
- For all $v \in \bar{V}_T$, there exists exactly one sequence $\{e_i\}_{i=1}^n$ of edges $e_i \in E_T$, $i = 1, \dots, n$ connecting R_T to v, i.e. fulfilling $(e_1)_1 = R_T$, $(e_i)_2 = (e_{i+1})_1$ for $i = 1, \dots, n-1$ and $(e_n)_2 = v$, where we have used the notation $(e_i)_1 = j, (e_i)_2 = k$ for $e_i = (j,k)$. The sequence $\{e_i\}_{i=1}^n$ is called the *path* from R_T to v.
- If for two vertices v, w, there exists a path from v to w, we say that w is a *descendant* of v and v is an *antecedent* of w. In this case, $v \prec w$. If $(v,w) \in E_T$, we say that w is a *direct* descendant of v. For any two direct descendants w_1, w_2 of v with $w_1 \prec w_2$, we have $u \prec w_2$ for any descendant u of w_1.
- For each tree-leaf $i \in L_T$, either of the following cases holds:
  - $i \in L_T^+$ and there exists exactly one $j \in L^c_{\text{int.}}$ such that $(j,i) \in P_T$

- $i \in L_T^-$ and there exists exactly one $j \in L^b_{\text{int.}}$ such that $(i,j) \in P_T$
- $i \in L_T^+$ and there exists exactly one $j \in L_T$ such that $(j,i) \in P_T$
- $i \in L_T^-$ and there exists exactly one $j \in L_T$ such that $(i,j) \in P_T$
- There is no $j \in L_T \cup L^b_{\text{int.}} \cup L^c_{\text{int.}}$ such that $(i,j) \in P_T$ or $(j,i) \in P_T$

In the latter case, we say i is an *external leaf.*

- For the following condition, we must first make an auxiliary definition: Assuming that $R_{\text{int.}}, V_T, E_T^{\text{plain}}, L_T, L^b_{\text{int.}}, L^c_{\text{int.}}, P_T$ are given, we define for $v \in R_{\text{int.}}$, $\tilde{L}^b \subset L^b_{\text{int.}}$, $\tilde{L}^c \subset L^c_{\text{int.}}$ the *plain descendant tree with root v, contravariant leaves $\tilde{L}^b$ and covariant leaves $\tilde{L}^c$* to be the 7-tuple $(v, \tilde{V}, \tilde{L}, \tilde{L}^b, \tilde{L}^c, \tilde{E}, \tilde{P})$ where
  - $\tilde{V}$ is given by the set of descendants $w \in V_T$ of v for which there exists a path $\{e_i\}_{i=1}^m$ of edges $e_i \in E_T^{\text{plain}}$, $i = 1, \dots, m$ connecting v_2 to w
  - $\tilde{L}$ is given by the set of descendants $i \in L_T$ of v for which there exists a path $\{e_i\}_{i=1}^m$ of edges $e_i \in E_T^{\text{plain}}$, $i = 1, \dots, m$ connecting v_2 to w
  - The set of edges $\tilde{E}$ is given by

$$\begin{aligned}\tilde{E} =& \{(v,v')\} \cup (E_T^{\text{plain}} \cap (\tilde{V} \times (\tilde{V} \cup \tilde{L}))) \\ & \cup \Big\{(w,w') \in V_T \times (V_T \cup L_T) : \exists u \in L^c_{\text{int.}} \text{ such that } (w,u),(u,w') \in E_T\Big\}\end{aligned}$$

    where v' is the direct descendant of v
  - $\tilde{P} = P_T \cap ((\tilde{L} \times \tilde{L}) \cup (\tilde{L} \times \tilde{L}^b) \cup (\tilde{L}^c \times \tilde{L}) \cup (\tilde{L}^c \times \tilde{L}^b)$ is the pairing

  Now the set of direct descendants of a vertex $v \in V_T$, D_v, must be of either of the following forms:

1. $D_v = \{v_1, v_2, v_3\} \subset V_T \cup L_T$

2. $D_v = \{v_1, \dots, v_n\}$, $n \geq 2$, $v_1 \in R_{\text{int.}}$, $v_2, \dots, v_n \in R_{\text{int.}} \cap L^c_{\text{int.}}$ with $v_1 \prec \dots \prec v_n$, $(v, v_1), \dots, (v, v_{n-1}) \in E^{\epsilon}_T$, $(v, v_n) \in E^{\text{plain}}_T$, and there exist $w_{1,j}, w_{2,j} \in L^b_{\text{int.}}$, $j = 1, \dots, n-1$ such that for $j = 1, \dots, n-2$, the plain descendant tree with root v_j, contravariant leaves $\{w_{1,j}, w_{2,j}\}$ and covariant leaves $\{w_{1,j+1}, w_{2,j+1}, v_{j+1}\}$ defines a contracted tree in the sense of definition 4.2.3, and the plain descendant tree with root v_{n-1}, contravariant leaves $\{w_{1,n-1}, w_{2,n-1}\}$ and covariant leaves $\{v_n\}$ defines a contracted tree.

3. $D_v = \{v_1, \dots, v_n,\}, n \geq 4$ with $v_1 \in R_{\text{int.}}$, $v_2, \dots, v_n \in R_{\text{int.}} \cap L^c_{\text{int.}}$, $v_1 \prec \dots \prec v_n$, $(v, v_1), \dots, (v, v_{n-3}) \in E^{\epsilon}_T$, $(v, v_{n-2}), \dots, (v, v_n) \in E^{\text{plain}}_T$, and There exist $w_{1,j}, w_{2,j} \in L^b_{\text{int.}}, j = 1, \dots, n-3$ such that for $j = 1, \dots, n-2$ the plain descendant tree with root v_j, contravariant leaves $\{w_{1,j}, w_{2,j}\}$ and covariant leaves $\{w_{1,j+1}, w_{2,j+1}, v_{j+1}\}$ is a contracted tree, where $w_{1,n-2} := v_{n-1}, w_{2,n-2} := v_n$.

4. $D_v = \{v_1, \dots, v_n, u_1, \dots, u_m\}, n \geq 0, m \geq 4$ with $v_1 \in R_{\text{int.}}$, $D_v \backslash \{v_1\} \subset R_{\text{int.}} \cap L^c_{\text{int.}}$, $v_1 \prec \dots \prec v_n \prec u_1 \prec \dots \prec u_m$, $(v, v_1), \dots, (v, v_n), (v, u_2), \dots, (v, u_{m-2}) \in E^{\epsilon}_T$, $(v, u_1), (v, u_{m-1}), (v, u_m) \in E^{\text{plain}}_T$. There exist $w_{1,j}, w_{2,j} \in L^b_{\text{int.}}$, $j = 1, \dots, n$, and there exist $\hat{w}_j \in L^b_{\text{int.}}$, $j = 2, \dots, m-2$, such that for $j = 1, \dots, n$ the plain descendant tree with root v_j, contravariant leaves $\{w_{1,j}, w_{2,j}\}$ and covariant leaves $\{w_{1,j+1}, w_{2,j+1}, v_{j+1}\}$ is a contracted tree, where $w_{1,n+1} := u_1, w_{2,n+1} := \hat{w}_2, v_{n+1} := u_2$, and there exist $\hat{w}_j \in L^b_{\text{int.}}$, $j = 1, \dots, m-2$, such that for $j = 2, \dots, m-2$ the plain descendant tree with root v_j, contravariant leaves $\hat{w}_j$ and covariant leaves $\{\hat{w}_{j+1}, u_{j+1}\}$ is a contracted tree, where $\hat{w}_{m-1} := u_m$.

5. $D_v = \{v_1, \dots, v_n, u_1, \dots, u_m\}, n \geq 0, m \geq 2$ with $v_1 \in R_{\text{int.}}$, $D_v \backslash \{v_1\} \subset R_{\text{int.}} \cap L^c_{\text{int.}}$, $v_1 \prec \dots \prec v_n \prec u_1 \prec \dots \prec u_m$, $(v, v_1), \dots, (v, v_n), (v, u_2), \dots, (v, u_m) \in E^{\epsilon}_T$,

$(v, u_1) \in E_T^{\text{plain}}$. There exist $w_{1,1}, w_{2,1} \in L^b_{\text{int.}}$, $j = 1, \ldots, n$, and there exist $\hat{w}_j \in L^b_{\text{int.}}$, $j = 2, \ldots, m$ such that for $j = 1, \ldots, n$ the plain descendant tree with root v_j, contravariant leaves $\{w_{1,j}, w_{2,j}\}$ and covariant leaves $\{w_{1,j+1}, w_{2,j+1}, v_{j+1}\}$ is a contracted tree, where $w_{1,n+1} := u_1, w_{2,n+1} := \hat{w}_2, v_{n+1} := u_2$, and for $j = 1, \ldots, m-1$, the plain descendant tree with root v_j, contravariant leaves $\{\hat{w}_j\}$ and covariant leaves $\{\hat{w}_{j+1}, u_{j+1}\}$ is a contracted tree, and the plain descendant tree with root u_m, contravariant leaves $\{\hat{w}_m\}$ and covariant leaves $\emptyset$ is a contracted tree.

We call vertices satisfying one of the conditions 1.-5. "allowed vertices".

The set of all renormalization trees T with $|V_T| = i$ is denoted by $\mathcal{T}_i^{\text{ren.}}$.

This lengthy definition is best digested by looking at table 4.1. There we show pictorially how the above case distinction is related to the different counterterms occurring in eqs. (4.3.91) and (4.3.92). An example of a renormalization tree contributing to a high-order perturbation of the free vertex operator can be found in figure 4.6.

With this definition, the trees of section 4.2.2 are a special case of renormalization trees: They are renormalization trees where every single vertex is of the form 1. above. The allowed vertices 2.-5. represent counterterms.

Before we go on to define the regulators $\{\epsilon_j\}$ that are associated to the set of all renormalization trees $T \in \mathcal{T}_i^{\text{ren.}}$, we make some auxiliary definitions concerning edges in renormalization trees:

$$E_T^{\mathbf{p}} = E_T \cap (V_T \times V_T)$$

$$E_T^- = E_T \cap V_T \times L_T^-$$

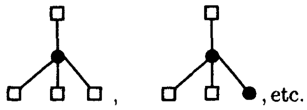
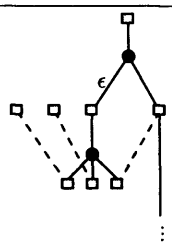
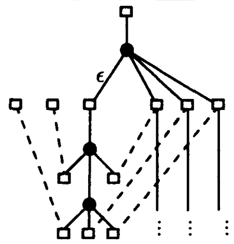
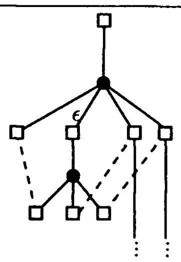
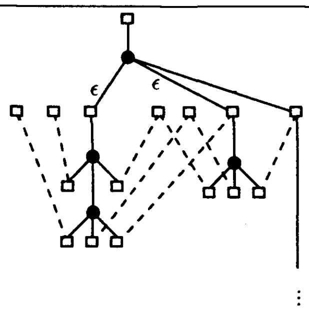
Case	Allowed vertex	Matrix element
1.		$G\left(Y_j(\varphi,(1+\epsilon)x)Y_{i-j}(\varphi^2,x)\right)$
2.		$G\left(\langle\varphi,Y_j(\varphi,\epsilon x)\varphi^2\rangle Y_{i-j}(\varphi,x)\right)$
3.		$G\left(\langle\varphi^3,Y_j(\varphi,\epsilon x)\varphi^2\rangle Y_{i-j}(\varphi^3,x)\right)$
4.		$G\left(\langle\varphi^2,Y_j(\varphi,\epsilon x)\varphi\rangle Y_{i-j}(\varphi^2,x)\right)$
2.		$G\Big(\langle\varphi^3,Y_j(\varphi,\epsilon_1 x)\varphi^2\rangle$ $\times\langle\varphi,Y_k(\varphi,\epsilon_2 x)\varphi^2\rangle Y_{i-j-k}(\varphi,x)\Big)$

Table 4.1: Examples for allowed vertices in a renormalization tree. The first row is case number one from definition 4.3.1. The second row is case number two with $n = 2$. The third row depicts case number 3 with $n = 4$. In the fourth row, we have case number 4 with $n = 0$, $m = 4$. The fifth row depicts case number 2 with $n = 3$. At the bottom of each tree, we have drawn dots to indicate that we have only drawn part of a tree. In the third column, we have written down the composition of vertex operators that the corresponding tree contributes to (without specifying the perturbation orders i, j, k).

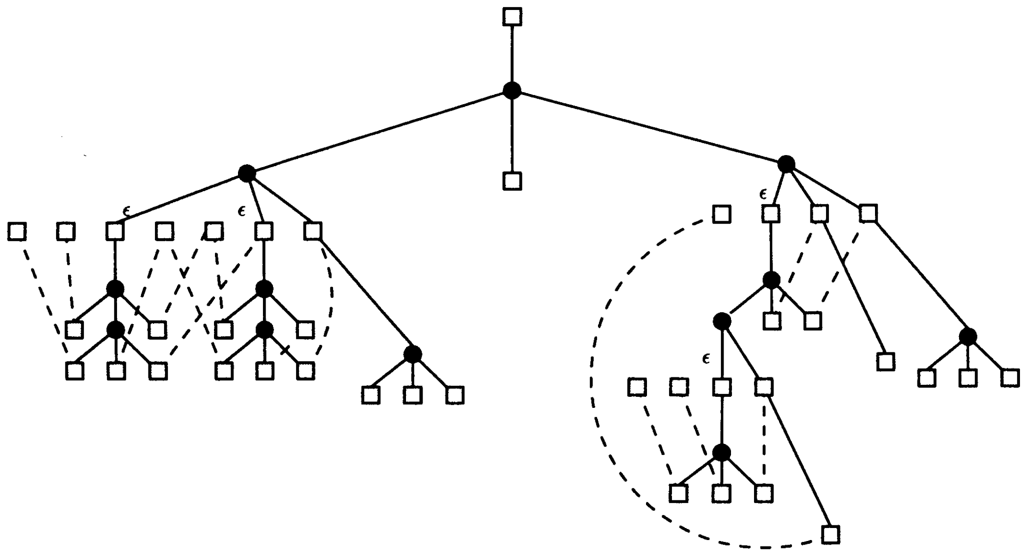


Figure 4.6: An example of a renormalization tree. ϵ-edges are indicated. Below each of the ϵ-edges, we have a contracted tree as in figure 4.4 representing a counterterm. This renormalization tree makes a contribution to the vertex operator $Y_{12}(\varphi, x)$.

$$
\begin{aligned}
E_T^+ &= E_T \cap V_T \times L_T^+ \\
E_T^{\text{plain}*} &= \Big\{ (v,w) \in V_T \times V_T : \exists j \in L^c_{\text{int.}} \text{ such that } (v,j), (j,w) \in E_T^{\text{plain}} \Big\} \\
E_T^{\epsilon *} &= \Big\{ (v,w) \in V_T \times V_T : \exists j \in L^c_{\text{int.}} \text{ such that } (v,j) \in E_T^{\epsilon}, (j,w) \in E_T^{\text{plain}} \Big\} \\
E_T^{-,*} &:= \Big\{ (v,i) \in V_T \times L_T^- : \exists j \in L^c_{\text{int.}} \text{ such that } (v,j), (j,i) \in E_T \Big\} \\
E_T^{+,*} &:= \Big\{ (v,i) \in V_T \times L_T^+ : \exists j \in L^c_{\text{int.}} \text{ such that } (v,j), (j,i) \in E_T \Big\} \\
E_T^* &= E_T^{\text{plain}*} \cup E_T^{\epsilon *} \cup E_T^{-,*} \cup E_T^{+,*} \\
E^T_{\text{red.}} &= E_T^{\mathbf{p}} \cup E_T^{\text{plain}*} \cup E_T^{\epsilon *} \cup E_T^- \cup E_T^+ \cup E_T^{-,*} \cup E_T^{+,*}
\end{aligned}
$$

These definitions will be handy later on to keep the explicit formula for the amplitude

$Y_i(T, \varphi, x, \Theta)$ of a renormalization tree T reasonably short. Note that for a given $T \in \mathcal{T}^{\text{ren.}}$, (R_T, V_T, L_T, E_T^*) defines a tree in the sense of definition 4.2.1. We will call this tree the *reduction* of T and denote it by $T_{\text{red.}}$.

We come to the definition of the set of regulators Θ in eq. (4.3.93). Let us look again at eq. (4.3.91). The terms on the right hand side all depend on the regulator ϵ. When continuing recursively and decomposing the terms on the right hand side into compositions of 0-th order (free field) vertex operators that have a graphical representation by renormalization trees, it has to be made sure that the same regulator ϵ is associated to distinct renormalization trees in the appropriate fashion.

In particular this has to be so when eq. (4.3.91) (or eq. (4.3.92)) appear in the recursion for determining $Y_{i'}(\varphi, y)$, $i' > i+1$. In this situation, some of the renormalization trees $\mathcal{T}_{i'}^{\text{ren.}}$ will have "subtrees" that represent contributions to $Y_i(\varphi, x)$. These subtrees are the renormalization trees below a certain vertex v, which in turn is part of a bigger renormalization tree contributing to $Y_{i'}(\varphi, y)$. To make sure the *same* regulator ϵ is associated to these subtrees of distinct renormalization trees, we have to make some sort of identification of vertices v, w where $v \in V_T, w \in V_{T'}$, $T, T' \in \mathcal{T}_{i'}^{\text{ren.}}$.

This works as follows: We consider the set of all vertices of renormalization trees of order i',

$$\tilde{\Theta} = \coprod_{T \in \mathcal{T}_{i'}^{\text{ren.}}} V_T \,.$$

Let $T, T' \in \mathcal{T}_{i'}^{\text{ren.}}$, $v \in V_T, w \in V_{T'}$ as before. Moreover let $\{e_j\}_{j=1}^{n}$, $\{e'_j\}_{j=1}^{n'}$ be the paths from the roots of T, T' to v, w respectively (cf. definition 4.3.1).

We say $v \sim w$ with $v \in V_T$, $w \in V_{T'}$ if the paths are equivalent in the following sense:

- $n = n'$

- For $k = 1, \dots, n-1$, we have the following situation: Let $u_1 \prec \cdots \prec u_m$ be the direct descendants of $(e_k)_1$ in $T_{\text{red.}}$, and $u'_1 \prec \cdots \prec u'_{m'}$ the direct descendants of $(e'_k)_1$ in $T'_{\text{red.}}$. Also let $(e_k)_2 = u_p$ and $(e'_k)_2 = u'_{p'}$. Then $p = p'$ and for $j = 1, \dots, p$, either $((e_k)_1, u_j) \in E_T^{\text{p}}, ((e'_k)_1, u'_j) \in E_{T'}^{\text{p}}$ or $((e_k)_1, u_j) \in E_T^{\text{plain}*}, ((e'_k)_1, u'_j) \in E_{T'}^{\text{plain}*}$ or $((e_k)_1, u_j) \in E_T^{\epsilon *}, ((e'_k)_1, u'_j) \in E_{T'}^{\epsilon *}$

We denote the equivalence class of v by $[v]$. Now the set of regulators in eq. (4.3.93) is chosen as follows:

$$\Theta = \left\{ \epsilon_{[v]} : \exists T \in \mathcal{T}_i^{\text{ren.}} \text{ such that } v \in V_T \right\} \tag{4.3.94}$$

In the following, we will often omit the brackets and just write ϵ_v for the regulator associated to v. It will be understood that $\epsilon_v = \epsilon_w$ for $v \sim w$.

The regularizing factor associated to $j \in L_T$ as above is given by

$$R_{\epsilon,j} = \prod_{\text{antecedents } v} R_{\epsilon,v} \tag{4.3.95}$$

where the products runs over the antecedents $v \in V_T$ of j, and

$$R_{\epsilon,v} = \begin{cases} \epsilon_v & \text{if } (u,v) \in E_T^{\epsilon *} \\ 1 + \epsilon_v & \text{if } (u,v) \in E_T^{\text{plain}*} \end{cases} \tag{4.3.96}$$

where u is the direct antecedent of v in $T_{\text{red.}}$.

We come to the definition of

$$Y_i(\cdot, \varphi, x, \Theta_i) : \mathcal{T}_i^{\text{ren.}} \to \text{Hom}(V, \bar{V}) \otimes \mathcal{O}(x; \Theta_i) \tag{4.3.97}$$

which will complete the construction of the vertex operator according to eq. (4.3.93).

As in the preceding sections, we first state a set of "rules" that describe how to translate a renormalization tree T into a multiple sum of multiple integrals.

1. First we consider a special set of internal leaves. Let v be a vertex of the form 2. above. If $v_1 \prec \cdots \prec v_n$ are the direct descendants of v, then associate an index $l_{v_n} \in \{0,1,2\}$ to v_n. (v_n is the root of a tree to whose amplitude the differential operator $\sum_{m_i} h_{(l_i,m_i)}(y_u)\bar{h}_{(l_i,m_i)}(\partial)$ will be applied, cf. ll. 2-5 of eq. (4.3.91) and rule number 7 below.)

2. To any other internal leaf $i \in L^b_{\text{int.}}, L^c_{\text{int.}}$ associate the fixed index $l_i = 0$. To each tree-leaf i with (i,j) or $(j,i) \in P_T$ for some $j \in L^b_{\text{int.}}$ or $L^c_{\text{int.}}$ respectively, associate $l_i := l_j$. To each external leaf $i \in L_T$ (cf. definition 4.3.1), associate an index $\ell_i = (l_i, m_i) \in \mathbb{L}$. To the edge $e = (v,i) \in E_T^{\pm,*}$ connecting a tree-leaf i with its antecedent v in $T_{\text{red.}}$, associate $l_e := l_i$ if $i \in L_T^+$ and $l_e := -l_i - 2$ if $i \in L_T^-$.

3. To tree-leaves i,j with $e = (i,j) \in P'_T = P_T \cap (L_T \times L_T)$, associate an index $l_e \in \mathbb{N}$. Write down a factor
$$R_{\epsilon,i}^{l_e} R_{\epsilon,j}^{-l_e-2}\,\mathrm{P}(\hat{y}_v \cdot \hat{y}_w, l_e, 4)\,. \tag{4.3.98}$$
If v is the direct antecedent of i and w the direct antecedent of j, set $l_{(v,i)} = l_e$ and $l_{(w,j)} = -l_e - 2$.

4. With each vertex $v \in V_T$ associate a parameter $\delta_v \in \mathbb{C} \setminus \mathbb{Z}$, a parameter $\hat{y}_v \in S^{D-1}$ and a regulator $\epsilon_v > 0$.

5. To each $e \in E_T^*$ of the form $(v,w) \in V_T \times (V_T \cup L_T) : \exists j \in L^c_{\text{int.}} \cup R_{\text{int.}}$ such that $(v,j),(j,w) \in E_T$, associate the number $q_e := l_j$. q_e is the order of the differential operator mentioned in bullet point 1 that will act on the amplitude of the tree with root j.

6. To each momentum carrying edge $e \in E^{\rm p}_{T,\rm red.}$ associate $\nu_e \in \mathbb{C} \setminus \mathbb{Z}$ such that the conservation rule

$$\nu_{(u,v)} = 2 + \delta_v + \sum_{\text{d.d.}\, w} (\nu_{(v,w)} - q_{(v,w)}) \tag{4.3.99}$$

holds at every vertex v, where u is the direct antecedent of v in $T_{\rm red.}$, the sum runs over the direct descendants w of v in $T_{\rm red.}$, and $q_{(v,w)} := 0$ if not specified otherwise in the last bullet point.

7. For each $e = (v,w) \in E^*_T$, there is precisely one $j \in L^c_{\rm int.}$ such that $(v,j),(j,w) \in E_T$. There is a unique $i \in L^+_T \cup L^b_{\rm int.}$ such that $(i,j) \in P_T$. Let u be the antecedent of i. For each such e, i, write down a factor

$$\begin{aligned} \frac{\pi\, |y_v|^{-\nu_e + l_i}}{\sin \pi \nu_e} \mathrm{P}(\hat{y}_u \cdot \partial_{y_v}, l_i, 4)\, (\mathrm{P}(-\hat{y}_v \cdot \hat{y}_w, \nu_e, 4) |y_v|^{\nu_e}) \text{ if } e \in E^{\text{plain}*}_T\,, \\ |y_v|^{-|\ell_w| + l_i} \mathrm{P}(\hat{y}_u \cdot \partial_{y_v}, l_i, 4)\, \big((R_{\epsilon,w}|y_v|)^{|\ell_w|} h_{\ell_w}(\hat{y}_v)\big) \text{ if } e \in E^{+,*}_T\,, \\ |y_v|^{|\ell_w| + 2 + l_i} \mathrm{P}(\hat{y}_u \cdot \partial_{y_v}, l_i, 4)\, \big((R_{\epsilon,w}|y_v|)^{-|\ell_w| - 2} h_{\ell_w}(\hat{y}_v)\big) \text{ if } e \in E^{-,*}_T\,. \end{aligned} \tag{4.3.100}$$

In the latter two cases, w is an external leaf. Factors as in eq. (4.3.100) arise from the application of the differential operator $\sum_{m_i} h_{(l_i,m_i)}(y_u)\bar{h}_{(l_i,m_i)}(\partial)$ to $Y(T_j, \varphi, r\hat{y}_v, \tilde{\Theta})$, where T_j is the renormalization tree with root j. This represents terms as in ll. 2-5 of eq. (4.3.91).

8. For each tree-leaf i paired with an internal contravariant leaf, write down the factor $R^{-2}_{\epsilon,i}$. For each external tree-leaf i not covered in the preceding bullet point, write down the factor

$$R_{\epsilon,i}^{-|\ell_i| - D + 2} \sqrt{\frac{2}{|\ell_i| + 2}}\, \bar{h}_{\ell_i}(\hat{y}_v) \mathbf{a}_{\ell_i} \text{ if } i \in L^-_T$$

$$R_{\epsilon,i}^{|\ell_i|}\sqrt{\frac{2}{|\ell_i|+2}}\,h_{\ell_i}(\hat{y}_v)\mathbf{a}_{\ell_i}^{+} \text{ if } i \in L_T^{+}$$

where $v \in V_T$ is the direct antecedent of i.

9. For edge $e = (v, w) \in E_T^{\mathbf{P}}$, write down the propagator

$$\frac{\pi}{\sin \pi\nu_e}\,\mathrm{P}(-\hat{y}_v \cdot \hat{y}_w; \nu_e, 4).$$

10. Perform the integrals

$$\prod_{v\in V_T}\int_{S^3} \mathrm{d}\Omega(\hat{y}_v) \quad \text{and} \quad \prod_{v\in V_T}\frac{1}{2\pi \mathrm{i}}\oint_{C_v}\frac{\mathrm{d}\delta_v}{\delta_v}.$$

Take the sum over all $l_e \in \mathbb{N}$ for $e \in P_T \cap (L_T \times L_T)$. Finally, take the sum over all $l_i \in \{0, 1, 2\}$ mentioned in 1.

We summarize these rules in a single formula:

$$\begin{aligned}
Y(T,\varphi,x,\Theta) = &\sum_{l_i\in\{0,1,2\}}\sum_{\substack{l_e\in\mathbb{N}:\\ e\in P_T'}}\left(\prod_{v\in V_T}\int_{C_v}\frac{\mathrm{d}\delta_v}{\delta_v}\int_{S^{D-1}}\mathrm{d}\Omega(\hat{y}_v)\right) \qquad (4.3.101)\\
&\times \prod_{e=(v,w)\in E_T^{\mathbf{P}}}\frac{\pi}{\sin\pi\nu_e}\,\mathrm{P}(-\hat{y}_v\cdot\hat{y}_w,\nu_e,4)\\
&\times \prod_{(i,j)\in P_T'} R_{\epsilon,i}^{-2}\left(\frac{R_{\epsilon,j}}{R_{\epsilon,i}}\right)^{l_e}\mathrm{P}(\hat{y}_i\cdot\hat{y}_j,l_e,4)\\
&\times \prod_{e=(v,w)\in E_T^{\mathrm{plain}*}}\frac{\pi\, r^{-\nu_e+l_i}}{\sin\pi\nu_e}\,\mathrm{P}(\hat{y}_u\cdot\partial_{y_w},l_i,4)\,(\mathrm{P}(\hat{y}_v\cdot\hat{y}_w,\nu_e,4)r^{\nu_e})\\
&\times \prod_{e=(v,w)\in E_T^{+,*}} r^{-|\ell_w|+l_i}\mathrm{P}(\hat{y}_u\cdot\partial_{y_v},l_i,4)\left((R_{\epsilon,w}r)^{|\ell_w|}h_{\ell_w}(\hat{y}_v)\right)
\end{aligned}$$

$$\times \prod_{e=(v,w)\in E_T^{-,*}} r^{|\ell_w|+2+l_i} \mathrm{P}(\hat{y}_u \cdot \partial_{y_v}, l_i, 4)\left((R_{\epsilon,w} r)^{-|\ell_w|-2} h_{\ell_w}(\hat{y}_v)\right)$$

$$\times \exp\left(\ln r \left\{\sum_{v\in V_T}(2+\delta_v) + \sum_{i\in L_T^+}|\ell_i| - \sum_{i\in L_T^-}(|\ell_i|+2) - 2|L^b_{\text{int.}}|\right\}\right)$$

$$\times \prod_{i\in L^+_{T,\text{ext.}}} \sqrt{\frac{2}{2|\ell_i|+2}} h_{\ell_i}(\hat{y}_i) R_{\epsilon,i}^{|\ell_i|}$$

$$\times \prod_{i\in L^-_{T,\text{ext.}}} \sqrt{\frac{2}{2|\ell_i|+2}} h_{\ell_i}(\hat{y}_i) R_{\epsilon,i}^{-|\ell_b(i)|-2},$$

where in the first line, the sum $\sum_{l_i\in\{0,1,2\}}$ is a multiple sum that runs over $l_i \in 0,1,2$ for all $i \in L^c_{\text{int.}}$ that were described in rule number 1 above. In the third line, we have set $\hat{y}_i := \hat{y}_v, \hat{y}_j := \hat{y}_w$ where v is the direct antecedent of i in $T_{\text{red.}}$ and w is the direct antecedent of j in $T_{\text{red.}}$. In the last two lines, we have used the notation $L^+_{T,\text{ext.}}, L^-_{T,\text{ext.}}$ to denote external creation and annihilation leaves respectively, cf. rule number 8.

We want to make one further comment on rule number 7: The expression

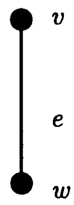
$$\mathrm{P}(\hat{y}_u \cdot \partial_{y_v}, l_i, 4)\left(\mathrm{P}(-\hat{y}_v \cdot \hat{y}_w, \nu_e, 4)|y_v|^{\nu_e}\right) \tag{4.3.102}$$

which results from terms as in ll. 2-5 of eq. (4.3.91), is an analytic function in $\hat{y}_u \cdot \hat{y}_v, \hat{y}_u \cdot \hat{y}_w, \hat{y}_v \cdot \hat{y}_w$ and homogeneous in y_v of degree $\nu_e - l_i$, as $\mathrm{P}(\hat{y}_u \cdot \partial_{y_v}, l_i, 4)$ is a differential operator of degree l_i, and $\mathrm{P}(-\hat{y}_v \cdot \hat{y}_w, \nu_e, 4)|y_v|^{\nu_e}$ is homogeneous in y_v of degree ν_e.

Thus the "momentum conservation" eq. (4.3.99) makes sense as it stands. Similar remarks apply to the other two expressions in eq. (4.3.100).

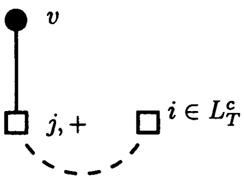
In figures 4.7 and 4.8, we give a pictorial representation of eq. (4.3.101).

The vertex operator $Y_i(\varphi, x)$ is now obtained by combining formulas (4.3.93) and (4.3.101), where $\mathcal{G} = \mathcal{T}_i^{\text{ren.}}$. The order in which the limits $\epsilon_v \to 0$ have to be taken is given by the order

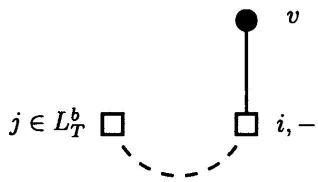


$$\frac{\pi}{\sin \pi\nu_e} \mathrm{P}(-\hat{y}_v \cdot \hat{y}_w, \nu_e, 4)$$

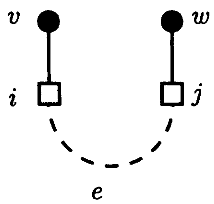
$$r^{2+\delta_v}$$



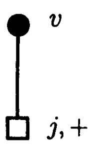
$$\sqrt{\frac{2}{|\ell_c(i)|+2}} h_{\ell_c(i)}(\hat{y}_v) r^{|\ell_c(i)|} R_{\epsilon,j}^{|\ell_c(i)|}$$



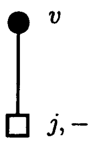
$$\sqrt{\frac{2}{|\ell_b(j)|+2}} \bar{h}_{\ell_b(j)}(\hat{y}_v) r^{-|\ell_b(j)|-2} R_{\epsilon,i}^{-|\ell_b(j)|-2}$$



$$R_{\epsilon,i}^{-2} \left(\frac{R_{\epsilon,j}}{R_{\epsilon,i}}\right)^{l_e} \mathrm{P}(\hat{y}_v \cdot \hat{y}_w, l_e, 4)$$

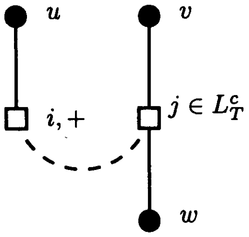


$$\sqrt{\frac{2}{|\ell_j|+2}} h_{\ell_j}(\hat{y}_v) r^{|\ell_j|} \mathbf{a}^+_{\ell_j} R_{\epsilon,j}^{|\ell_j|}$$

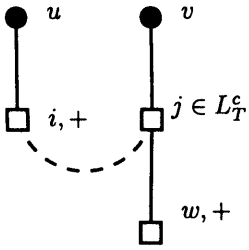


$$\sqrt{\frac{2}{|\ell_j|+2}} \bar{h}_{\ell_j}(\hat{y}_v) r^{-|\ell_j|-2} \mathbf{a}_{\ell_j} R_{\epsilon,j}^{-|\ell_j|-2}$$

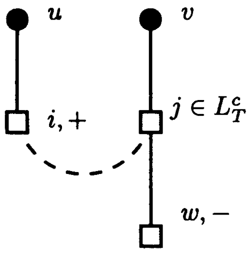
Figure 4.7: First part of the rules for the amplitude of a renormalization tree T.



$$r^{-\nu_e+l_i}\mathrm{P}(\hat{y}_u\cdot\partial_{y_w}, l_i, 4)\,(\mathrm{P}(\hat{y}_v\cdot\hat{y}_w, \nu_e, 4)r^{\nu_e})$$



$$r^{-|\ell_w|+l_i}\mathrm{P}(\hat{y}_u\cdot\partial_{y_v}, l_i, 4)\left((R_{\epsilon,w}r)^{|\ell_w|}h_{\ell_w}(\hat{y}_v)\right)\mathbf{a}^+_{\ell_w}$$



$$r^{|\ell_w|+2+l_i}\mathrm{P}(\hat{y}_u\cdot\partial_{y_v}, l_i, 4)\left((R_{\epsilon,w}r)^{-|\ell_w|-2}h_{\ell_w}(\hat{y}_v)\right)\mathbf{a}_{\ell_w}$$

Figure 4.8: Second part of the rules for the amplitude of a renormalization tree T. The above objects correspond to bullet point number 7 of the rules in the main text.

relation $\prec$ on the vertices:

Consider again eqs. (4.3.91), (4.3.92). On the right hand side of these equations the regulator ϵ is understood to be small but non-zero. Each of the vertex operators on the right hand side has to be thought of as the result of some other limit $\epsilon' \to 0$ that has been performed in the previous iteration step. Obviously this latter limit has to be performed *before* the limit $\epsilon \to 0$. In our graphical representation by renormalization trees, the vertices further down in the tree represent the multiplication of vertex operators and the associated limits that have to be performed first. Thus the order in which limits are taken is determined by the order relation $\prec$ on the vertices: If $v \prec w$, then $\epsilon_{[w]} \to 0$ has to be taken before $\epsilon_{[v]} \to 0$. If the regulators are kept at a non-zero value, one has to choose $\epsilon_{[v]} > \epsilon_{[w]}$ for $v \prec w$ to make sure that all expressions are finite and analytic in Θ.

Composite operators

So far, we only have developed formulas for the perturbations of vertex operators $Y_i(a,x)$ for $a \in \{\varphi, \varphi^2, \varphi^3\}$. It would not be a problem to generalize the iterative procedure for the construction of vertex operators in such a way that one obtains the operator $Y_i(a,x)$ for general $a \in V$. The compatibility axiom eq. (2.1.10) of definition 2.1.1 tells us how to obtain the operators $Y_i(\bar{h}_\ell(\nabla)\varphi, x)$, and the operator $Y_i\left((\bar{h}_\ell(\nabla)\varphi)a, x\right)$ can be obtained from $Y_i(a,x)$ by looking at the associativity condition for $Y\left((\bar{h}_\ell(\nabla)\varphi), (1+\epsilon)x\right)$ and $Y(a,x)$ in i-th order, assuming that all operators $Y_j(b,x)$ are already known for either $j \leq i, \#b < \#a$ or $j < i, \#b \leq \#a$. Again, one can use the trick of section 4.2.1 to obtain a right inverse for the Laplacian, multiply harmonic polynomials as in eq. (4.2.67) and use the generalized Dougall identity eq. (4.2.70). The terms in the resulting formula can again be represented graphically, where now we must allow for more than one root in the resulting trees. More precisely, a contribution to the operator $Y_i(a,x)$ will have $\#a$ roots. Each root will be the root of a renormalization tree. The higher the dimension $\dim a$, the more counterterms will possibly occur, and the list of "allowed vertices" as specified in definition 4.3.1 will become longer. Nevertheless, the construction follows the same principles, and the amplitude of such a "renormalization forest" can be obtained by applying the same graphical rules as depicted in figures 4.7 and 4.8.

Super-renormalizable and non-renormalizable theories

The rules for other renormalizable theories such as $D = 6, p = 3$ or $D = 3, p = 6$ are very similar, where the interaction polynomial is given by $P(\varphi) = \varphi^p$. If one is only interested in the perturbations $Y_i(\varphi^k, x), k = 1, \ldots, p-1$, there is only a finite number of possible covariant vectors appearing in the matrix elements of counterterms, and the rules have to specify a corresponding set of allowed vertices similar to the way it has been done in definition 4.3.1. In the case of super-renormalizable theories, the rules become somewhat more restrictive: There is only a finite number of matrix elements that appear as counterterms (i.e. there exists some M such that $\lim_{\epsilon\to 0}\langle c,\, Y_i(\varphi, \epsilon x)b\rangle = 0$ for $i > M$ and all b, c appearing in the iteration procedure). As a consequence, there are only finitely many contracted trees that may appear as subtrees of renormalization trees. Other than that, the iteration scheme works the same.

In non-renormalizable theories, there appear more and more distinct covariant vectors in matrix elements of counterterms the higher one goes in the perturbation order. The iteration procedure still works out fine, but the list of "allowed" vertices analogous to definition 4.3.1 depends on the considered perturbation order.

This situation is completely analogous to ordinary perturbative quantum field theory (see e.g. [46]): In super-renormalizable theories, there is only a finite number of divergent diagrams. In renormalizable theories their number is infinite, but there is only a finite number of external leg structures that yield divergent diagrams. Finally, there exist divergent diagrams for any external leg structure in non-renormalizable theories.

4.3.1 Examples

As an example, we are going consider in some more detail the explicit expression for the right hand side of eq. (4.3.93) for the vertex operator $Y_2(\varphi, x)$. As in the last subsection, we assume $p = D = 4$. In figure 4.9, we have drawn all renormalization trees $\mathcal{T}_2^{\text{ren.}}$ making a contribution to $Y_2(\varphi, x)$. Strictly speaking, we would have to draw $2^{|L_T|}$ copies of each tree T, in order to account for all possible choices of $L_T^+, L_T^- \subset L_T$, which we have not done here. Thus, for example, each of the trees T_1, T_2, T_3 stands for 32 different renormalization trees as in definition 4.3.1.

The set of regulators is $\{\epsilon_{v_1}, \epsilon_{v_2}\} = \{\epsilon_1, \epsilon_2\}$. For each choice of creation and annihilation leaves, the contributions of each of the first three trees T_1, T_2, T_3 is of the form

$$\sum_{\ell_1,\dots,\ell_5 \in \mathbf{L}} \oint_{C_1} \frac{d\delta_1}{\delta_1} \oint_{C_2} \frac{d\delta_2}{\delta_2} \int_{S^3} d\Omega(\hat{y}_{v_1}) \int_{S^3} d\Omega(\hat{y}_{v_2})$$
$$\times \frac{\pi}{\sin \pi\nu_1} \mathrm{P}(-\hat{y}_{v_1} \cdot \hat{x}, \nu_1, 4) \frac{\pi}{\sin \pi\nu_2} \mathrm{P}(-\hat{y}_{v_1} \cdot \hat{y}_{v_2}, \nu_2, 4)$$
$$\times A(\hat{y}_{v_1}, \hat{y}_{v_2}) r^{\nu_1}. \tag{4.3.103}$$

For example, for T_1, with $L_{T_1} = L_{T_1}^+$, we have $\nu_2 = (\sum_{i=1}^3 |\ell_i|) + 2 + \delta_2$, $\nu_1 = (\sum_{i=1}^5 |\ell_i|) + 4 + \delta_1 + \delta_2$ and

$$A = \prod_{i=1}^{3} h_{\ell_i}(\hat{y}_{v_2}) R_{\epsilon,j_i}^{|\ell_i|} \mathbf{a}_{\ell_i}^+ \prod_{i=4}^{5} h_{\ell_i}(\hat{y}_{v_1}) R_{\epsilon,j_i}^{|\ell_i|} \mathbf{a}_{\ell_i}^+$$

For T_2 and $L_{T_2}^- = \{j_1, \dots, j_4\}$, we have $\nu_2 = -(\sum_{i=2}^4 (|\ell_i| + 2)) + 2 + \delta_2$, $\nu_1 = -(\sum_{i=1}^4 (|\ell_i| + 2)) + |\ell_5| + 4 + \delta_1 + \delta_2$ and

$$A = \bar{h}_{\ell_1}(\hat{y}_{v_1}) R_{\epsilon,j_1}^{-|\ell_1|-D+2} \mathbf{a}_{\ell_1} \left(\prod_{i=2}^{4} \bar{h}_{\ell_i}(\hat{y}_{v_2}) R_{\epsilon,j_i}^{-|\ell_i|-D+2} \mathbf{a}_{\ell_i} \right) h_{\ell_5}(\hat{y}_{v_1}) R_{\epsilon,j_5}^{|\ell_5|} \mathbf{a}_{\ell_5}^+$$

For the latter choice, there is a potential divergence for coinciding $\ell_1 = \ell_5$, but convergence

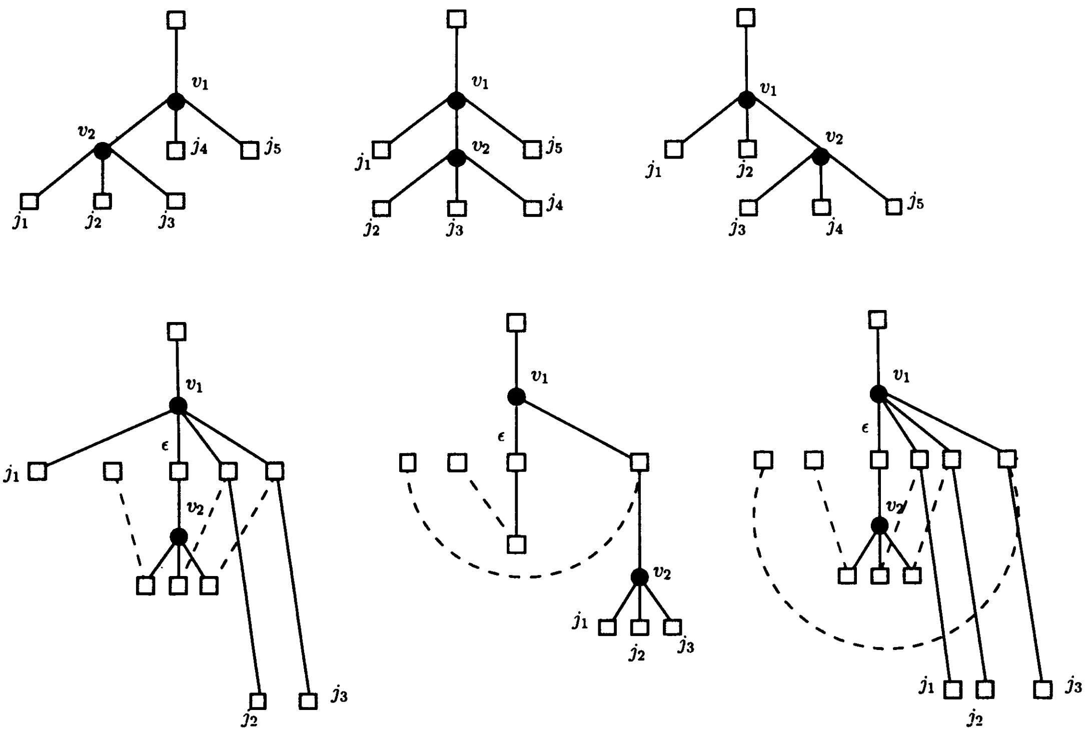


Figure 4.9: The renormalization trees from $\mathcal{T}_2^{\text{ren.}}$ making a contribution to $Y_2(\varphi, x)$. In the main text, we call the trees in the first row $T_1, \ldots, T_3$ and those in the second row $T_3, \ldots, T_6$. Each of them has to be understood as representing all renormalization trees that can be obtained by assigning the leaves to either L_T^+ or L_T^- (creation and annihilation leaves). Strictly speaking, we should not use the same variable names v_1, v_2, etc. in different renormalization trees. The reason is that the set of regulators $\{\epsilon_j\}$ is constructed from the set $\{v : \exists T \in \mathcal{T}_i^{\text{ren.}} \text{ such that } v \in V_T\}$, where we assume that $V_T \cap V_{T'} = \emptyset$ for $T \neq T'$, see eq. (4.3.94). However, this is of no consequence here and we use the same variable names repeatedly in order to alleviate the notation.

of the sum eq. (4.3.103) is assured by $(1 + \epsilon_1) = R_{\epsilon, j_1} > R_{\epsilon, j_5} = 1$. For obvious reasons, we refrain from writing down ν_1, ν_2, A for the other 94 renormalization trees that result from choosing a set of creation and annihilation leaves for any of the first three trees in figure 4.9.

T_4 stands for $2^3 = 8$ renormalization trees. Their amplitudes will again be of the form eq. (4.3.103), but there are only three indices ℓ_1, ℓ_2, ℓ_3 to be summed over. For the choice

$L_{T_4} = L^+_{T_4}$, we have $\nu_1 = (\sum_{i=1}^3 |\ell_i|) + 2 + \delta_1 + \delta_2$, $\nu_2 = -2 + \delta_2$, and

$$A = \prod_{i=1}^{3} h_{\ell_i}(\hat{y}_{v_1}) \mathrm{a}^+_{\ell_i} R^{-2}_{\epsilon, j_4}$$

with $R_{\epsilon,j_4} = \epsilon_1$. The other 7 renormalization trees represented by T_4 have similar amplitudes. T_5 represents 8 renormalization trees. For the choice $L_{T_5} = L^+_{T_5}$, we get the same amplitude as for the last example, except for an extra factor $R^{|\ell_1|}_{\epsilon,j_1} = (1+\epsilon_1)^{|\ell_1|}$ and that $R_{\epsilon,j_4} = \epsilon_2$. T_6 represents 8 renormalization trees. For $L_{T_6} = L^+_{T_6}$, we get again an amplitude of the form eq. (4.3.103), where the sum is only over $\ell_1, \ell_2, \ell_3 \in \mathbb{L}$, with $\nu_1 = (\sum_{i=1}^3 |\ell_i|) + 2 + \delta_1 + \delta_2$, $\nu_2 = (\sum_{i=1}^3 |\ell_i|) + 2 + \delta_2$, and

$$A = \prod_{i=1}^{3} h_{\ell_i}(\hat{y}_{v_2}) \mathrm{a}^+_{\ell_i} R^{-2}_{\epsilon, j_4}$$

where $R_{\epsilon,j_4} = \epsilon_1$.

The sum over the amplitudes eq. (4.3.103) of all 120 renormalization trees that are obtained from assigning creation and annihilation leaves to the objects from figure 4.9 will be an element of $\mathrm{Hom}(V, \bar{V}) \otimes \mathcal{O}(x; \epsilon_1, \epsilon_2)$ for $\epsilon_1 > \epsilon_2$. Taking the limit $\lim_{\epsilon_1 \to 0+} \lim_{\epsilon_2 \to 0+}$ will yield the second order perturbation $Y_2(\varphi, x)$. We see that already at low perturbation orders, there are many terms on the right hand side of eq. (4.3.93). For this reason, we refrain from working out eq. (4.3.93) for higher i and give instead an example for the amplitude of a single renormalization tree contributing to the vertex operator $Y_4(\varphi, x)$, see figure 4.10.

We abbreviate $\epsilon_j := \epsilon_{v_j}, y_j := y_{v_j}$. The amplitude of the renormalization tree of figure 4.10 is

$$Y(T, \varphi, x, \Theta_4) = \sum_{l_{i_1}=0}^{2} \sum_{l_{(i_2,i_3)} \in \mathbb{N}} \sum_{\ell_{i_4} \in \mathbb{L}} \prod_{j=1}^{4} \left(\oint \frac{\mathrm{d}\delta_j}{\delta_j} \int_{S^3} \mathrm{d}\Omega(\hat{y}_j) \right)$$

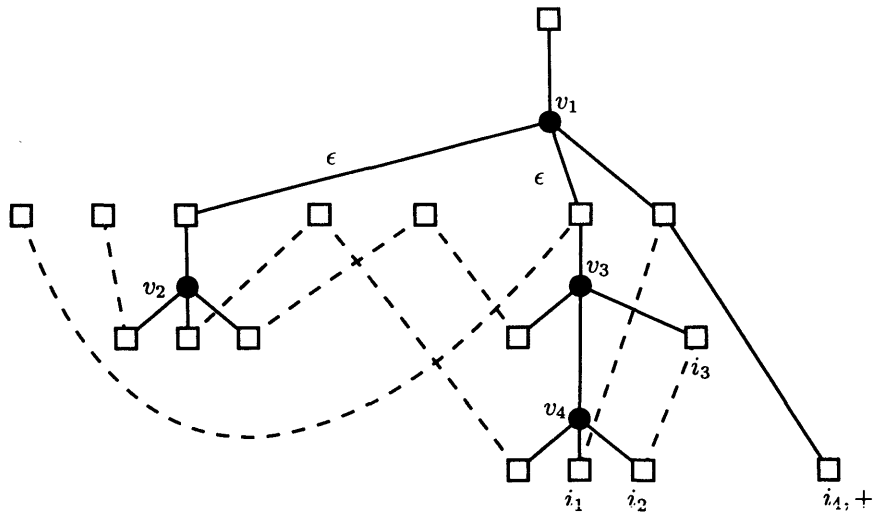


Figure 4.10: A renormalization tree contributing to $Y_4(\varphi, x)$.

$$
\begin{aligned}
&\times \frac{\pi}{\sin \pi\nu_1} \mathrm{P}(-\hat{y}_1 \cdot \hat{x}, \nu_1, 4) \frac{\pi}{\sin \pi\nu_2} \mathrm{P}(-\hat{y}_1 \cdot \hat{y}_2, \nu_2, 4) \\
&\times \frac{\pi}{\sin \pi\nu_3} \mathrm{P}(-\hat{y}_1 \cdot \hat{y}_3, \nu_3, 4) \frac{\pi}{\sin \pi\nu_4} \mathrm{P}(-\hat{y}_3 \cdot \hat{y}_4, \nu_4, 4) \\
&\times r^{l_{i_1} - |\boldsymbol{\ell}_{i_4}|} \left(\mathrm{P}(-\hat{y}_4 \cdot \partial_{y_1}, l_{i_1}, 4) h_{\boldsymbol{\ell}_{i_4}}(y_1) \right) \left(2/(2 + |\boldsymbol{\ell}_{i_4}|) \right)^{1/2} \mathbf{a}^+_{\boldsymbol{\ell}_{i_4}} \\
&\times \epsilon_2^{-2} \epsilon_3^{-6+l_{i_1}} (1 + \epsilon_4)^{-l_{(i_2,i_3)} - 4 + l_i} r^{\nu_1}, \qquad (4.3.104)
\end{aligned}
$$

where

$$
\begin{aligned}
\nu_1 &= |\boldsymbol{\ell}_{i_4}| + \sum_{j=1}^{4} \delta_j \\
\nu_2 &= \delta_2 \\
\nu_3 &= l_{i_1} - 2 + \delta_3 + \delta_4 \\
\nu_4 &= l_{i_1} - l_{(i_2,i_3)} - 2 + \delta_4 .
\end{aligned}
$$

Chapter 5

Conclusions and outlook

As we have explained in several places in this thesis, a quantum field theory can be defined via its operator product expansion *viz.* vertex algebra. In the present work, we investigated several aspects of this approach. First we extracted the vertex algebra of the free massless boson in D dimensions and the massless Thirring model. We verified all axioms for a vertex algebra for these models, in particular associativity. We then considered deformations of vertex algebras and showed that they can be classified as the elements of certain cohomology rings. Similarly, obstructions to the construction of higher order deformations were identified as elements in some cohomology ring. In a more explicit way, we described how to construct these perturbations for deformations of the vertex algebra of the free massless boson in D dimensions φ governed by a field equation $\Delta Y(\varphi, x) = \lambda Y(P'(\varphi), x)$ where P is some polynomial in φ. We developed a set of graphical rules that lead to an explicit formula for the vertex operator $Y_i(\varphi, x)$. First, we did this in terms of trees for the special case $D = 2, P(\varphi) = \varphi^3$, and then in terms of slightly more complicated graphical objects ("renormalization trees") for the general case. We explained how to obtain several alternative forms of the explicit formula for $Y_i(\varphi, x)$, one of which leading to an infinite sum of ratios of Gamma functions, which is why we called the summands of the resulting formula "hypergeometric functions associated to renormalization trees". We conjectured that associativity of the vertex operators (which we did not prove in this thesis) entails highly non-trivial relations for these generalizations of the hypergeometric function. In order to arrive at these alternative representations, we had to prove various identities from the theory of special functions. In particular, we proved a generalization of the so called Dougall formula and found a generalization of the generating functional for Gegenbauer functions with non-integer indices.

There are several interesting possible future research topics linked to this thesis.

- It is desirable to prove that the vertex operators defined in section 4 actually satisfy definition 2.1.1 in the sense of formal power series. Most importantly, one would have to prove associativity. It might be that the key to such a proof is a better understanding of the cancelations of divergences in the regulators ϵ_i in formula eq. (4.3.93). It is a matter of interest for itself to show existence of the limits $\epsilon_i \longrightarrow 0$ and to determine the result. Explicit examples where these limits can be performed explicitly would be of help. Already in perturbation order $i = 2$, they are non-trivial.

- A testing ground for any new concept in QFT – such as the formulation in terms of vertex operators – are exactly solvable systems in 2 or 3 dimensions. The existence of infinitely many conserved charges should lead to restrictions on the possible form of the vertex operators that might make explicit constructions possible. One model of interest would be the massive Thirring or Sine-Gordon model. Here, explicit expressions for n-point functions are not known, but one could try to build on methods from the so-called bootstrap program (see e.g. [79]) that uses the factorization of S-matrices for exactly solvable models. Another option is to try to construct models starting from renormalization group equations.

- The formulation of QFT in terms of OPE is particularly suited to curved spaces, as most other frameworks (such as the Wightman axioms) break down when the symmetries of Minkowski space are absent (see the discussion in [42]). It should be possible to generalize the iterative procedure for calculating the perturbations to vertex operators to curved spaces.

Appendix A

Spherical harmonics and Gegenbauer functions

The following facts about harmonic polynomials can be found in many textbooks, see e.g. [2]. Polynomials $h(x), x \in \mathbb{R}^D$ which are solutions to the Laplace equation $\Delta h(x) = 0$ are called "harmonic polynomials". Since the Laplace operator Δ commutes with dilations $x \mapsto tx$, it follows that any harmonic polynomial can be decomposed into a sum of homogeneous harmonic polynomials. The harmonic polynomials satisfying $h(tx) = t^l h(x), l \in \mathbb{N}$ span a vector subspace of dimension $N(l, D)$ in $\mathbb{C}[x]$, where $N(0, D) = 1$ and

$$N(l, D) = \frac{(2l + D - 2)(l + D - 3)!}{(D - 2)!l!} \quad \text{for } l > 0. \tag{A.0.1}$$

This can be seen for example by noting that the degree l harmonic polynomials $h(x)$ are in one-to-one correspondence with totally symmetric, traceless tensors of rank l on $\mathbb{R}^D$: If $c_{\mu_1 \dots \mu_l}$ are the components of such a tensor, then $h(x) = \sum c_{\mu_1 \dots \mu_l} x_{\mu_1} \cdots x_{\mu_l}$ is a harmonic polynomial of degree l, and vice versa. The spherical harmonics in D dimensions are by definition the restrictions of the harmonic polynomials to S^{D-1}.

In the main text, we consider the set $\{h_\ell(x), \ell \in \mathbb{L}\}$ of homogeneous harmonic polynomials. The members of this set are chosen to satisfy the orthogonality condition eq. (3.1.2). In fact, the h_ℓ form an orthonormal basis of $L^2(S^{D-1}, \mathrm{d}\Omega)$ when restricted to the sphere. It follows immediately from the fact that the $h_\ell(x)$ are harmonic polynomials that their restrictions $h_\ell(\hat{x})$ to the sphere are eigenfunctions of the Laplacian $\hat{\Delta}$ on S^{D-1} with eigenvalue $-|\ell|(|\ell|+D-2)$.

For our calculations in appendix C, we need to know in more detail the relation of the harmonic polynomials h_ℓ to the traceless symmetric tensors of rank $|\ell|$ described above. To state the relevant facts, we use the familiar multi-index notation, $\alpha = (\alpha_1, \ldots, \alpha_D) \in \mathbb{N}^D$, with

$$x^\alpha = \prod_\mu x_\mu^{\alpha_\mu}, \quad \partial_\alpha = \prod_\mu \partial_\mu^{\alpha_\mu}, \quad \alpha! = \prod_\mu \alpha_\mu! \quad \text{etc.,} \tag{A.0.2}$$

and we write

$$h_\ell(x) = \sum_\alpha t_{\ell;\alpha} x^\alpha . \tag{A.0.3}$$

Combining eq. (3.1.2) with theorem 5.14 of [2] we get

$$\sum_\alpha \bar{t}_{\ell;\alpha} t_{\ell';\alpha} \frac{\alpha!}{k_\ell} = \delta_{\ell,\ell'}. \tag{A.0.4}$$

with

$$k_\ell = 2^{|\ell|}\Gamma(|\ell| + D/2)/\Gamma(D/2) . \tag{A.0.5}$$

This can also easily be proved starting from eq. (4.2.86).

The decomposition of a harmonic function f regular at the origin into harmonic polynomials reads

$$f(x) = \sum_{\ell \in \mathbf{L}} \left(\int_{S^{D-1}} d\Omega(\hat{x}) f(\hat{x}) \bar{h}_\ell(\hat{x}) \right) h_\ell(x) \,. \tag{A.0.6}$$

With $\partial^\alpha x^\beta|_{x=0} = \delta_{\alpha\beta}\alpha!$ we have

$$\bar{h}_\ell(\partial) h_{\ell'}(x)|_{x=0} = \sum_\alpha \bar{t}_{\ell;\alpha} t_{\ell';\alpha} \alpha! = \delta_{\ell,\ell'} k_\ell \tag{A.0.7}$$

and thus eq. (A.0.6) reads

$$f(x) = \sum_{\ell \in \mathbf{L}} \frac{\Gamma(D/2) h_\ell(x)}{\Gamma(|\ell| + D/2) 2^{|\ell|}} \bar{h}_\ell(\partial) \left(f(y)\right)|_{y=0}. \tag{A.0.8}$$

We also cite theorem 5.20 of [2], which states that for a harmonic homogeneous polynomial p of degree $|\ell|$,

$$p(\partial)\, g(r) = q_\ell \, r^{2-D} p(x/r^2) \tag{A.0.9}$$

where $r = |x|$,

$$g(r) = \begin{cases} r^{2-D} & \text{for } D > 2 \\ \ln r & \text{for } D = 2 \,, \end{cases} \tag{A.0.10}$$

and

$$q_\ell = \begin{cases} 2^{|\ell|-1}\Gamma(|\ell|) & \text{for } D = 2 \\ 2^{|\ell|}\Gamma(|\ell| + D/2 - 1)/\Gamma(D/2 - 1) & \text{for } D > 2 \,. \end{cases} \tag{A.0.11}$$

The Gegenbauer polynomials in $D \geq 2$ dimensions are defined as the following invariants under $SO(D)$:

$$\sum_{m=1}^{N(l,D)} \bar{h}_{(l,m)}(\hat{x}) h_{(l,m)}(\hat{y}) = \frac{2l + D - 2}{\sigma_D} \mathrm{P}(\hat{x} \cdot \hat{y}, l, D) \,, \tag{A.0.12}$$

where $\sigma_D = \frac{2\pi^{D-2}}{\Gamma(D/2)}$ is the surface area of the $D-1$ dimensional unit sphere. By construction, the Gegenbauer polynomials $\mathrm{P}(z, l, D)$ are polynomials of degree $l \in \mathbb{N}$. The notation $C_l^{(D-2)/2}(z)$ is more common, with notable differences in the normalization convention throughout the literature. A generating function is

$$\frac{1}{D-2}\left(\frac{1}{\sqrt{1-2hz+h^2}}\right)^{D-2} = \sum_{l=0}^{\infty} \mathrm{P}(z,l,D)h^l \,. \tag{A.0.13}$$

This formula holds for $D \geq 3$. For $D = 2$, the left side is to be replaced by $-\ln\sqrt{1-2hz+h^2}$. A generalization of this formula needed in the main text is provided in theorem 2. The Gegenbauer polynomials have the symmetry property $\mathrm{P}(z,l,D) = (-1)^l \mathrm{P}(-z,l,D)$, and satisfy the normalization condition

$$\mathrm{P}(1,l,D) = \frac{(l+D-3)!}{l!(D-2)!} \,. \tag{A.0.14}$$

For complex values of the index $\nu \in \mathbb{C}$ (or D), one can define an analytic continuation by means of the Gauss hypergeometric function

$$\mathrm{P}(z,\nu,D) = \frac{\Gamma(\nu+D-2)}{\Gamma(\nu+1)\,\Gamma(D-1)}\, {}_2F_1\left(-\nu, \nu+D-2, D/2-1/2, \frac{1-z}{2}\right) \tag{A.0.15}$$

The Gauss hypergeometric function is given by the convergent expansion

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!}\, x^n \,, \quad (a)_n = \Gamma(a+n)/\Gamma(a) \,, \tag{A.0.16}$$

for $|x| < 1$. The differential equation satisfied by the Gegenbauer functions is

$$(1-z^2)y'' - (D-1)zy' + \nu(\nu+D-2)y = 0 \,. \tag{A.0.17}$$

Later on, we will use the relation

$$\mathrm{P}(z,\nu,D) = (-1)^{D-3}\mathrm{P}(z,-\nu-D+2,D)\,. \tag{A.0.18}$$

Note that the formula (A.0.16) has a slight anomaly in $D = 2$ dimensions. Here, it gives $\mathrm{P}(\cos\alpha,\nu,2) = \cos(\nu\alpha)/2\nu$ in $D = 2$ dimensions for $\nu \neq 0$, and this evidently does not have a limit as $\nu \to 0$. On the other hand, the generating formula definition gives $\mathrm{P}(\cos\alpha,0,2) = 1$.

Appendix B

Identities for Gegenbauer functions

In the main text, we use certain identities for Gegenbauer functions in D dimensions that we were not able to find in the literature, and which we therefore prove here:

Theorem 1. (*Generalized Dougall's formula*) For any $\nu \in \mathbb{C} \setminus \mathbb{Z}$ and $-1 \leq z \leq +1$ and $D \geq 3$, we have the identity

$$\sum_{l=0}^{\infty} \frac{(2l + D - 2)\, \mathrm{P}(z, l, D)}{\nu(\nu + D - 2) - l(l + D - 2)} = \frac{\pi}{\sin \pi\nu}\, \mathrm{P}(-z, \nu, D)\,. \tag{B.0.1}$$

Proof: For $D = 3$, a proof of the theorem can be given via a contour integral argument, see [21]. We here give a proof for arbitrary $D > 3$ that could easily be adapted to $D = 3$ as well. Let $\hat{\Delta}$ be the Laplacian on the sphere S^{D-1}. This is an elliptic, second order partial differential operator on a compact manifold with analytic coefficients. Using standard results on the functional calculus of such operators, we can form the resolvent operator $R_\nu = [\hat{\Delta} + \nu(\nu + D - 2)]^{-1}$ for any ν such that $\nu(\nu + D - 2)$ is not an eigenvalue, i.e. $\nu \notin \mathbb{Z}$. Let $R_\nu(\hat{x}, \hat{y})$ be the kernel of R_ν, which using general results on the Laplacian on compact Riemannian manifolds is known to be an analytic function on $S^{D-1} \times S^{D-1}$

apart from coincident points. Near coincident points, one has $R_\nu \sim [d(\hat{x},\hat{y})]^{-(D-3)/2}$, where $d(\hat{x},\hat{y}) = \arccos(\hat{x}\cdot\hat{y})$ is the geodesic distance on the sphere. A representation of R_ν in terms of eigenfunctions of the Laplacian is

$$\begin{aligned} R_\nu(\hat{x},\hat{y}) =& \sum_{l=0}^{\infty}\sum_{m=1}^{N(D,l)} \frac{\overline{h}_{l,m}(\hat{x})h_{l,m}(\hat{y})}{\nu(\nu+D-2)-l(l+D-2)} \\ =& \sigma_D^{-1}\sum_{l=0}^{\infty} \frac{(2l+D-2)\,\mathrm{P}(\hat{x}\cdot\hat{y},l,D)}{\nu(\nu+D-2)-l(l+D-2)}. \end{aligned} \tag{B.0.2}$$

In the second line we have used the definition of the Gegenbauer polynomials. Hence we see that the kernel R_ν is, up to a constant, precisely equal to the left side of the Dougall formula.

By definition, the kernel obeys $[\hat{\Delta} + \nu(\nu + D - 2)]R_\nu = \delta$ in the sense of distributions. However, since R_ν is evidently invariant under $SO(D)$-transformations, we may write $R_\nu(\hat{x},\hat{y}) = \tilde{R}(z)$ for some analytic function of $z = \hat{x}\cdot\hat{y}$ when $z \neq 1$. As a consequence of the differential equation satisfied by R_ν, it can easily be seen that $\tilde{R}$ satisfies the differential equation for the Gegenbauer function of dimension D and degree ν, see eq. (A.0.17). Hence we have

$$\tilde{R}(z) = A\,\mathrm{P}(z,\nu,D) + B\,\mathrm{P}(-z,\nu,D) \tag{B.0.3}$$

for some $A, B \in \mathbb{C}$ as $\mathrm{P}(z,\nu,D), \mathrm{P}(-z,\nu,D)$ span the solution space of eq. (A.0.17). Furthermore, $\mathrm{P}(-z,\nu,D)$ is singular at $z = 1$ and regular at $z = -1$ (see e.g. [21]), as is $\tilde{R}(z)$. By contrast, $\mathrm{P}(z,\nu,D)$ is singular at $z = -1$ and regular at $z = 1$. Thus, we must have $A = 0$ in eq. (B.0.3).

In order to determine the constant B, we look at the asymptotic behavior of $\tilde{R}(\hat{x}\cdot\hat{y})$ and $\mathrm{P}(-\hat{x}\cdot\hat{y},\nu,D)$ for $\hat{x}\cdot\hat{y} \to 1$. Let $\beta = d(\hat{x},\hat{y})$. We use an expansion of the hypergeometric function near unit argument (see e.g. [14]) and get

$$\mathrm{P}(-\cos\beta,\nu,D) = \frac{\Gamma((D-1)/2)\,\Gamma((D-3)/2)\,2^{D-3}}{\Gamma(\nu+1)\Gamma(-\nu)\Gamma(D-1)}\,\beta^{3-D} + O(\beta^{4-D})\,. \tag{B.0.4}$$

On the other hand, the asymptotic behavior of the fundamental solution $\tilde{R}$ is explicitly known as well [29],

$$\tilde{R}(\cos\beta) = \frac{\Gamma((D+1)/2)}{(D-1)(D-3)\pi^{(D-1)/2}}\,\beta^{3-D} + O(\beta^{4-D})\,. \tag{B.0.5}$$

Comparing eqs. (B.0.4) and (B.0.5), and using the doubling identity for the Gamma function, we get

$$B = \sigma_D \Gamma(-\nu)\Gamma(\nu+1) = \frac{\sigma_D \pi}{\sin \pi\nu}\,, \tag{B.0.6}$$

which proves the initial claim. □

The next theorem is a generalization of formula (A.0.13).

Theorem 2. (*Shifted generating functional formula*) Let $\delta \in \mathbb{C}\setminus\mathbb{Z}$ and $-1 \leq z = \cos\beta \leq +1, \theta > 0, s \in \mathbb{R}$.

For **even** $D \geq 2$, we have the identity

$$\begin{aligned}\sum_{l\in\mathbf{Z}} e^{\mathrm{i}sl-\theta|l|}\,\mathrm{P}(\cos\beta, l+\delta, D) =&(2\delta_D)^{-1}A_D\Big\{e^{\mathrm{i}\beta\delta_D}\Big({}_2F_1\Big(\delta_D,1;\delta_D+1;e^{\mathrm{i}(s+\beta)-\theta}\Big)\\ &+{}_2F_1\Big(-\delta_D,1;1-\delta_D;e^{-\mathrm{i}(s+\beta)-\theta}\Big)-1\Big)\\ &+e^{-\mathrm{i}\beta\delta_D}\Big({}_2F_1\Big(\delta_D,1;1+\delta_D;e^{\mathrm{i}(s-\beta)-\theta}\Big)\\ &+{}_2F_1\Big(-\delta_D,1;1-\delta_D;e^{-\mathrm{i}(s-\beta)-\theta}\Big)-1\Big)\Big\},\end{aligned} \tag{B.0.7}$$

where

$$A_D = \frac{1}{\Gamma(D/2)} \left(\frac{\partial}{2\,\partial \cos\beta} \right)^{(D-2)/2}$$
$$\delta_D = \delta + (D-2)/2 \,. \tag{B.0.8}$$

For **odd** $D \geq 3$, we have the formula

$$\begin{aligned}\sum_{l\in\mathbf{Z}} e^{\mathrm{i}sl-\theta|l|}\,\mathrm{P}(z, l+\delta, D) = & A_D \Bigg\{ \frac{-1}{\sqrt{1+e^{2(\mathrm{i}s-\theta)}+2e^{\mathrm{i}s-\theta}z}} \Bigg(F_1\Big(-\delta_D, \delta_D, 1, 1; \frac{1-z}{2}, \frac{1-t_-}{2}\Big) \\ & + \frac{z-t_-}{2} F_1\Big(1-\delta_D, 1+\delta_D, 1, 2; \frac{1-z}{2}, \frac{1-t_-}{2}\Big)\Bigg) \\ & + \frac{-e^{-\mathrm{i}s-\theta}}{\sqrt{1+e^{2(-\mathrm{i}s-\theta)}+2e^{-\mathrm{i}s-\theta}z}} \Bigg(F_1\Big(\delta_D, -\delta_D, 1, 1; \frac{1-z}{2}, \frac{1-\bar{t}_-}{2}\Big) \\ & + \frac{z-\bar{t}_-}{2} F_1\Big(1+\delta_D, 1-\delta_D, 1, 2; \frac{1-z}{2}, \frac{1-\bar{t}_-}{2}\Big)\Bigg)\Bigg\}, \end{aligned} \tag{B.0.9}$$

where

$$A_D = \frac{\sqrt{\pi}}{2\Gamma(D/2)} \left(\frac{\partial}{2\,\partial z} \right)^{(D-3)/2}$$
$$\delta_D = \delta + (D-3)/2$$
$$t_- = e^{-\mathrm{i}t+\theta} \left(1 - \sqrt{1+e^{2(\mathrm{i}t-\theta)}+2e^{\mathrm{i}t-\theta}z}\right) \tag{B.0.10}$$

and F_1 is the hypergeometric function with two arguments,

$$F_1(a, b, c; d; v, w) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}m!n!} v^m w^n \,. \tag{B.0.11}$$

Remark: There is an apparent asymmetry in the formulas for even and odd D. One is tempted to believe that both formulas given for the shifted generating function are valid for all D (when appropriately interpreted), but we have not been able to show this.

Proof for even D: For $D = 2$, the proof of the theorem follows immediately from

$$ {}_2F_1(\delta, 1; 1+\delta; h) = \sum_{l\in\mathbb{N}} \frac{\delta}{\delta + l} h^l $$

by the definition of the hypergeometric function. Here we assumed that $|h| < 1$. For D even and $D > 2$, we prove eq. (B.0.7) using the recurrence identity

$$ \frac{\mathrm{d}}{\mathrm{d}z} \mathrm{P}(z, \nu, D) = D\,\mathrm{P}(z, \nu - 1, D + 2)\,. \tag{B.0.12} $$

Proof for odd D: For odd D, we proceed using the same recurrence identity, but in order to be able to do so, we have to evaluate $\sum_{l=0}^{\infty} \mathrm{P}(z, l + \delta, 3)$, and this requires some extra work. The special case of a Gegenbauer function $\mathrm{P}(z, \nu, D)$ with $D = 3$ is called Legendre function in the literature. We start with the Schlaefli integral formula for Legendre functions [74],

$$ P(z, \nu, 3) = \frac{1}{2\pi \mathrm{i}} \oint_{C+} \frac{(t^2 - 1)^{\nu}}{2^{\nu}(t - z)^{\nu+1}} \mathrm{d}t\,. \tag{B.0.13} $$

To make $(t^2 - 1)^{\nu} 2^{-\nu} (t - z)^{-\nu-1}$ single-valued, we have to introduce two cuts in the complex plane, and we follow [69] choosing these cuts as the half-line $\gamma_1 = (-\infty, -1)$ and a curve γ_2 joining the points $t = 1$ and $t = z$, parameterized by

$$ \frac{1 - \eta z}{z - \eta} \quad (-1 \geq \eta > -\infty)\,. \tag{B.0.14} $$

In [69], this particular representation of Legendre functions made it possible to determine derivatives of Legendre functions with respect to their degree ν.

The cuts are obviously related by the transformation

$$\tau(t) = \frac{1 - zt}{z - t} \tag{B.0.15}$$

which is an automorphism of the Riemann sphere. Note that $\tau(z) = \infty$.

Assuming $\left|\frac{h(t^2-1)}{2(t-z)}\right| < 1$ on C^+, we have

$$\begin{aligned}\sum_{l=0}^{\infty} h^l P(z, l+\delta, 3) =& \frac{1}{2^{1+\delta}\pi \mathrm{i}} \oint_{C^+} \mathrm{d}t \left(\frac{t^2-1}{t-z}\right)^{\delta} \frac{1}{t - z - h(t^2-1)/2} \\ =& \frac{1}{2^{1+\delta}\pi \mathrm{i}} \oint_{C^+} \mathrm{d}t\, \chi(t) \frac{1}{t - z - h(t^2-1)/2},\end{aligned} \tag{B.0.16}$$

where in the first equation, we have interchanged the order of summation and integration, and in the second we have set $\chi(t) = (t^2-1)^{\delta}(t-z)^{-\delta}$. In the denominator of the integrand, we have the polynomial

$$\frac{h}{2}(t^2-1) - t + z = \frac{h}{2}(t - t_+(h))(t - t_-(h)) \tag{B.0.17}$$

where

$$t_{\pm}(h) = \frac{1}{h}\left(1 \pm \sqrt{1 + h^2 - 2hz}\right). \tag{B.0.18}$$

Note that

$$\tau(t_+) = t_-, \quad \tau(t_-) = t_+. \tag{B.0.19}$$

Given h with $|h| < 1$, we claim that C^+ can be chosen such that $\left|\frac{h(t^2-1)}{2(t-z)}\right| < 1$ for $t \in C^+$ (so that eq. (B.0.16) holds) and moreover, $t_-(h)$ lies inside and $t_+(h)$ outside of C^+.

For $s > 0$, let $T^- s = \{t_-(h) : |h| = s\}$ and $T^+ s = \{t_+(h) : |h| = s\}$. It is not difficult to

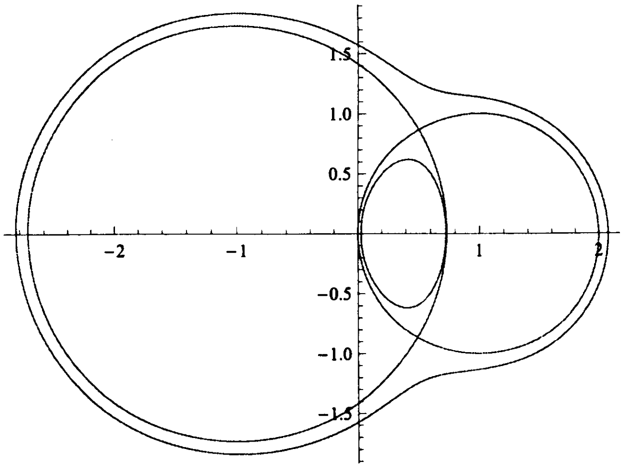


Figure B.1: From the outside to the inside: The contours $T^{+}(s)$, $T^{+}(1)$, $T^{-}(1)$, $T^{-}(s)$ with $s<1$. $T^{+}(1)$, $T^{-}(1)$ are touching each other in the points $z \pm \mathrm{i}\sqrt{1-z^2}$.

show that

$$\begin{aligned} T^{-}(1) =& \{x \in \mathbb{C} : |x-1|^2 = 2(1-z), |x+1|^2 \leq 2(1+z)\} \\ & \cup \{x \in \mathbb{C} : |x-1|^2 \leq 2(1-z), |x+1|^2 = 2(1+z)\}, \\ T^{+}(1) =& \{x \in \mathbb{C} : |x-1|^2 = 2(1-z), |x+1|^2 \geq 2(1+z)\} \\ & \cup \{x \in \mathbb{C} : |x-1|^2 \geq 2(1-z), |x+1|^2 = 2(1+z)\}, \end{aligned} \tag{B.0.20}$$

see figure B.1.

Let $s<1$. By the explicit formula for $t_+(h)$ eq. (B.0.18) one can see that the contour $T^{+}(s)$ lies outside $T^{+}(1)$. As $T^{-}(1), T^{-}(s)$ are the images of $T^{+}(1), T^{+}(s)$ under the transformation (B.0.15) respectively, z lies inside $T^{+}(1), T^{+}(s)$ and $\tau(z)=\infty$, we conclude that $T^{-}(s)$ lies inside $T^{-}(1)$.

Also, for $|h|=s$, $\left|\frac{h(t^2-1)}{2(t-z)}\right| = s$ on $T^{-}(1)$. $T^{-}(1)$ intersects the cut γ_2 in $\sqrt{2(1+z)}-1>z$. On the interval $[\sqrt{2(1+z)}-1, 1]$, we have $\left|\frac{h(t^2-1)}{2(t-z)}\right| \leq s$. Thus we can choose the contour

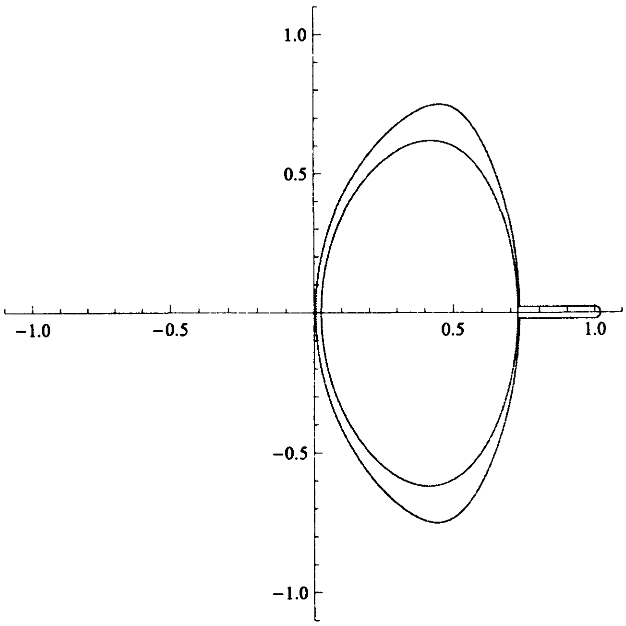


Figure B.2: The inner contour above is $T^-(s)$ with $s < 1$. The outer contour is C^+, encircling $T^-(s)$ and the interval $[z, 1]$. (Of course the two contours are not touching each other.) Also, C^+ has to be sufficiently close to $T^-(s)$ and $[z, 1]$ to ensure $|h(t^2 - 1)/(2(t - z))| < 1$ for $t \in C^+$.

C^+ such that it encircles $T^-(s)$ and γ_2, lies inside $T^+(s)$, and at the same time $\left|\frac{h(t^2-1)}{2(t-z)}\right| < 1$ for $t \in C^+$. We have drawn such a C^+ in figure B.2.

Our next aim will be to express χ as a sum of two functions χ_1, χ_2 that have cuts on γ_1 and γ_2 respectively, and are analytic elsewhere. Then we will be able to carry out the integration in eq. (B.0.16). Assuming $\mathrm{Re}\,\delta < 0$, we apply formula 3.1.11 of [43],

$$\chi(t) = -\pi^{-1} \int \mathrm{d}w \mathrm{d}\bar{w} \left(\bar{\partial}\chi(w, \bar{w})\right) (w - t)^{-1} . \tag{B.0.21}$$

We introduce a function ρ_1 that equals 1 in a small neighborhood of the cut $(-\infty, 1)$ and 0 outside a slightly bigger neighborhood. Also, we introduce ρ_2, smooth, equal to 1 in a small neighborhood of the cut (B.0.14) and 0 in a slightly bigger neighborhood, so that

$\mathrm{supp}\rho_1 \cap \mathrm{supp}\rho_2 = \emptyset$. As χ is analytic away from the cuts (i.e. $\bar{\partial}\chi = 0$ on $\mathbb{C} \setminus (\gamma_1 \cup \gamma_2)$), we can modify eq. (B.0.21) in the following way,

$$\begin{aligned}
\chi(t) =& \chi_1(t) + \chi_2(t)\,, \\
\chi_i(t) =& -\pi^{-1} \int \mathrm{d}w\mathrm{d}\bar{w}\, (\bar{\partial}\chi(w,\bar{w}))\, (w-t)^{-1}\rho_i(w,\bar{w})\,,\ (i=1,2)\,.
\end{aligned} \tag{B.0.22}$$

Now by theorem 3.1.12 of [43]

$$\begin{aligned}
\chi_1(t) =& \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{-1} \mathrm{d}x\, (\chi(x+\mathrm{i}0) - \chi(x-\mathrm{i}0))\, (x-t)^{-1} \\
=& \frac{\sin\delta\pi}{\pi} \int_{-\infty}^{-1} \mathrm{d}x \left(\frac{x^2-1}{z-x}\right)^{\delta} (x-t)^{-1}\,.
\end{aligned} \tag{B.0.23}$$

We substitute $u := 2/(1-x)$ and obtain

$$\chi_1(t) = -\frac{\sin\delta\pi}{\pi} 2^{\delta} \int_0^1 u^{-1-\delta}(1-u)^{\delta} \left(1 - \frac{1-z}{2}u\right)^{-\delta} \left(1 - \frac{1-t}{2}u\right)^{-1} \mathrm{d}u \tag{B.0.24}$$

We next determine χ_2,

$$\begin{aligned}
\chi_2(t) =& -\pi^{-1} \int \mathrm{d}w\mathrm{d}\bar{w} (\partial g)(w-t)^{-1}\rho_2(w,\bar{w}) \\
=& -\pi^{-1} \int \mathrm{d}\tau\mathrm{d}\bar{\tau} \frac{\partial w}{\partial \tau}\frac{\partial \bar{w}}{\partial \bar{\tau}} \left(\frac{\partial \bar{\tau}}{\partial \bar{w}} \partial_{\bar{\tau}}\chi(w(\tau))\right) (w(\tau)-t)^{-1}\rho_2(w(\tau),\bar{w}(\bar{\tau})) \\
=& -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{-1} dx\, (\chi(x+\mathrm{i}0) - \chi(x-\mathrm{i}0)) \frac{1-z^2}{(z-x)^2} \left(\frac{1-zx}{z-x} - t\right)^{-1} \\
=& -\frac{\sin\delta\pi}{\pi} \int_{-\infty}^{-1} \mathrm{d}x \left(\frac{x^2-1}{z-x}\right)^{\delta} \frac{1-z^2}{(z-x)^2} \left(\frac{1-zx}{z-x} - t\right)^{-1}
\end{aligned} \tag{B.0.25}$$

where in the second equation we have performed a change of coordinates according to the Möbius transformation eq. (B.0.15), $w \mapsto \tau(w)$. As mentioned before, τ maps γ_1 on γ_2 and vice versa, and $\tau \circ \tau = \mathrm{Id}$. Again substituting $u = 2/(1-x)$, we get

$$\chi_2(t) = -\frac{\sin\delta\pi}{\pi}\frac{1-z^2}{z-t}2^{-1+\delta}\int_0^1 \mathrm{d}u\, u^{-\delta}(1-u)^{\delta}\left(\frac{1-z}{2}u\right)^{-1-\delta}\left(\frac{1-\tau(t)}{2}u\right)^{-1}$$

In eq. (B.0.16), we replace

$$\frac{1}{t-z-h(t^2-1)/2} = -\frac{2}{h(t-t_+)(t-t_-)}$$

We have chosen C^+ so that t_- lies inside and t^+ outside the contour. We can now calculate the contribution of $\chi_1(t)$ to the integral (B.0.16),

$$\frac{1}{2^{1+\delta}\pi\mathrm{i}}\oint_{C^+}\mathrm{d}t\,\chi_1(t)\frac{1}{t-z-h(t^2-1)/2}\,, \qquad \text{(B.0.26)}$$

which is now a simple residue integral. We obtain

$$\begin{aligned}&2^{1-\delta}\frac{1}{\sqrt{1+h^2+2hz}}\chi_1(t_-)\\ =&\frac{\sin\delta\pi}{\pi}\frac{1}{\sqrt{1+h^2+2hz}}B(1+\delta,-\delta)F_1\left(-\delta,\delta,1,1;(1-z)/2,(1-t_-)/2\right)\,. \end{aligned} \qquad \text{(B.0.27)}$$

where we have used $|1-t_-|/2 < 1$, eq. (B.0.24) and formula 3.211 of [32]. F_1 is the hypergeometric function with two arguments, see eq. (B.0.11), and B is the Beta function $B(x,y) = \Gamma(x+y)/(\Gamma(x)\Gamma(y))$.

In a similar manner, we calculate the contribution of χ_2,

$$\begin{aligned}&\frac{B(1+\delta,1-\delta)\sin\pi\delta(1-z^2)}{4\pi^2\mathrm{i}}\\ &\times\oint_{D^-}\mathrm{d}\tau\left(\frac{1-z^2}{(z-\tau)^2}F_1(1-\delta,1+\delta,1,2;(1-z)/2,(1-\tau)/2)\frac{1}{t(\tau)-z}\right.\end{aligned}$$

$$\times \left(-\frac{2}{h}\right) \frac{z-\tau}{(t_- - z)(\tau - \tau(t_-)} \frac{z-\tau}{(t_+ - z)(\tau - \tau(t_+)} \Bigg), \tag{B.0.28}$$

where we used the coordinate transformation (B.0.15). The contour D^- is the image of C^+ under this transformation. D^- encircles γ_2 and runs clockwise, not crossing the cuts γ_1, γ_2. We can deform D^- into C^- by which we mean the contour C^+ with negative orientation. $\tau(t_+) = t_-$ is on the inside of C^-, $\tau(t_-) = t_+$ on the outside. Thus the residue integral (B.0.28) is

$$\begin{aligned} &\frac{\sin \pi\delta}{\pi} \frac{1}{\sqrt{1+h^2-2hz}} \frac{1-z^2}{2(z-t_+)} \\ &\times B(1+\delta, 1-\delta) F_1(1-\delta, 1+\delta, 1, 2; (1-z)/2, (1-t_)/2)\,. \end{aligned} \tag{B.0.29}$$

Putting together eqs. (B.0.27) and (B.0.29), we obtain

$$\begin{aligned} \sum_{l=0}^{\infty} h^l P(z, l+\delta, 3) = &\frac{\sin \delta\pi}{\pi} \frac{1}{\sqrt{1+h^2+2hz}} \\ &\times \Bigg(B(1+\delta, -\delta) F_1\left(-\delta, \delta, 1, 1; (1-z)/2, (1-t_-)/2\right) \\ &+ \frac{1-z^2}{2(z-t_+)} B(1+\delta, 1-\delta) \\ &\times F_1\left(1-\delta, 1+\delta, 1, 2; (1-z)/2, (1-t_-)/2\right) \Bigg)\,. \end{aligned} \tag{B.0.30}$$

Above, we have used $\mathrm{Re}\,\delta < 0$. However for $|h| < 1$, $|(1-t_)/2| < 1$. Thus both sides of eq. (B.0.30) are analytic in δ throughout the complex plane (with possible exceptions for $\delta \in \mathbb{Z}$), cf. the definition of F_1 in eq. (B.0.11). This means that they are identical and eq. (B.0.30) must hold for $\mathrm{Re}\,\delta \geq 0$ as well. Now eq. (B.0.9) follows from

$$\sum_{l=1}^{\infty} h^l \mathrm{P}(z, -l+\delta, 3) = \sum_{l=1}^{\infty} h^l \mathrm{P}(z, l-1-\delta, 3)$$

$$=h\sum_{l=0}^{\infty} h^l \mathrm{P}(z, l-\delta, 3) \tag{B.0.31}$$

where we used eq. (A.0.18). In the derivation of eq. (B.0.9), we also used formula (B.0.12), standard identities for the gamma function and the relation $(z-t_+)(z-t_-) = z^2 - 1$. □

We finally mention another representation of the Gegenbauer functions used in the main text:

Theorem 3. For $\nu \in \mathbb{C} \setminus \mathbb{Z}$, $|z| < 1$ we have the formula

$$\mathrm{P}(z,\nu,D) = \frac{\sin \pi\nu}{\pi} \frac{2^{-(D+1)/2}}{\Gamma(D/2)} \sum_{n=0}^{\infty} (-2z)^n \frac{\Gamma(-\nu/2+n/2)\Gamma(\nu/2+n/2+D/2-1)}{n!}. \tag{B.0.32}$$

Proof: We prove this first for $D = 2$. Let $z = \cos\alpha$. We have the identities

$$\begin{aligned} \cos\nu\alpha =& {}_2F_1(-\nu/2, \nu/2; 1/2; \sin^2\alpha) \\ =& \frac{\pi}{\Gamma(-\nu/2+1/2)\Gamma(\nu/2+1/2)}\, {}_2F_1(-\nu/2, \nu/2; 1/2; \cos^2\alpha) \\ & - 2\cos\alpha \frac{\pi}{\Gamma(-\nu/2)\Gamma(\nu/2)}\, {}_2F_1(-\nu/2+1/2, \nu/2+1/2; 3/2; \cos^2\alpha)\,, \end{aligned} \tag{B.0.33}$$

where in the second line we have used a standard transformation formula for hypergeometric functions, see [1]. We now use $\mathrm{P}(\cos\alpha, \nu, 2) = \cos(\nu\alpha)/2\nu$, and we expand the hypergeometric series in the second and third line, using the doubling identity of the Gamma function, $\sqrt{\pi}\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+1/2)$, in various ways. Then we obtain the statement of the theorem for $D = 2$. The case $D = 3$ is covered by formula 8.1.4 of [1], together with the use of the doubling identity as above.

For general $D \in \mathbb{N}$, we use the recurrence formula (B.0.12), combined with a standard formula for the derivatives of the hypergeometric function. This then gives the formula for all even D starting from $D = 2$ and all odd D starting from $D = 3$. □

Appendix C

The free field vertex operators

First we derive the expression for the vertex operator $Y_0(\mathfrak{a}, x)$ for $\mathfrak{a} \in \tilde{V}$ as in eq. (3.1.5). Looking at the definition of the Schwinger functions eq. (3.1.7), we can rewrite it using "Wick ordering",

$$\begin{aligned}
&\langle \mathfrak{a}^1(x_1)\mathfrak{a}^2(x_2)\dots\mathfrak{a}^n(x_n)\rangle \\
&\quad = \sum_{G\in\mathcal{G}(S_{\mathfrak{a}^1},S_{\mathfrak{a}^2})} P_G(\mathfrak{a}^1,\mathfrak{a}^2,x)\langle :\mathfrak{a}^{1,G}(x_1)\mathfrak{a}^{2,G}(x_2): \mathfrak{a}^3(x_3)\dots\rangle
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{G}(S_{\mathfrak{a}^1},S_{\mathfrak{a}^2}) &= \{G\subset S_{\mathfrak{a}^1}\times S_{\mathfrak{a}^2} : (i,j),(k,l)\in G \Rightarrow i\neq k, j\neq l\} \\
P_G(\mathfrak{a}^1,\mathfrak{a}^2;x_1,x_2) &= \prod_{(i,j)\in G} \bar{h}_{\mathfrak{a}^1_i}(\partial)\bar{h}_{\mathfrak{a}^2_j}(-\partial)g(|x_1-x_2|) \\
\mathfrak{a}^{k,G} &= \prod_{i\in S_{\mathfrak{a}^k}\setminus G_{\mathfrak{a}^k}} \bar{h}_{\mathfrak{a}_i}(\nabla)\varphi, \quad k=1,2 \\
G_{\mathfrak{a}^1} &= \{i\in S_{\mathfrak{a}^1} : \exists j\in S_{\mathfrak{a}^2} \text{ so that } (i,j)\in G\} \\
G_{\mathfrak{a}^2} &= \{j\in S_{\mathfrak{a}^2} : \exists i\in S_{\mathfrak{a}^1} \text{ so that } (i,j)\in G\}\,, \qquad \text{(C.0.1)}
\end{aligned}$$

where we used the notation from appendix A.

The correlation function $\langle :\mathfrak{a}^1(x_1)\mathfrak{a}^2(x_2): \mathfrak{a}^3(x_3)\dots\mathfrak{a}^n(x_n)\rangle$ is defined by the same formula as in eq. (3.1.7), where in the sum $\sum_G$ all graphs G that possess an edge (i,j) with $i \in S_{\mathfrak{a}^1}, j \in S_{\mathfrak{a}^2}$ are left out. This means $\langle :\mathfrak{a}^{1,G}(x_1)\mathfrak{a}^{2,G}(x_2): \dots\rangle$ is regular in $x_1 - x_2$, and can be Taylor-expanded in $x_1 - x_2$ provided $|x_1 - x_2| < |x_k - x_2|$, $k = 3, \dots, n$. We write this as

$$:\mathfrak{a}^{1,G}(x_1)\mathfrak{a}^{2,G}(x_2): \, = \, :\big(\exp((x_1 - x_2)\cdot\nabla)\mathfrak{a}^{1,G}(x_2)\big)\, \mathfrak{a}^{2,G}(x_2): \tag{C.0.2}$$

leaving aside the "spectator fields" $\mathfrak{a}^3(x_3), \dots, \mathfrak{a}^n(x_n)$. Thus eq. (C.0.2) holds as an equation for insertions into Schwinger functions. Putting together eqs. (C.0.1) and (C.0.2), we get the OPE

$$\begin{aligned} &\mathfrak{a}^1(x_1)\mathfrak{a}^2(x_2) \\ &\quad = \sum_{G\in\mathcal{G}(S_{\mathfrak{a}^1}, S_{\mathfrak{a}^2})} P_G(\mathfrak{a}^1, \mathfrak{a}^2, x_1 - x_2) :\big(\exp((x_1 - x_2)\cdot\nabla)\mathfrak{a}^{1,G}(x_2)\big)\, \mathfrak{a}^{2,G}(x_2): \end{aligned} \tag{C.0.3}$$

Again, this has to be understood as equation for insertions. Slightly changing the notation ($\mathfrak{a}^1 \to \mathfrak{a}$, $\mathfrak{a}^2 \to \mathfrak{b}$, $x_1 \to x$, $x_2 \to 0$) and using relation (1.0.5), this results in the vertex operator

$$Y(\mathfrak{a}, x)\mathfrak{b} = \sum_{G\in\mathcal{G}(S_{\mathfrak{a}^1}, S_{\mathfrak{a}^2})} P_G(\mathfrak{a}, \mathfrak{b}, x)\, \big(\exp(x\cdot\nabla)\mathfrak{a}^G\big)\, \mathfrak{b}^G \tag{C.0.4}$$

Now we want to find a compact expression for the vertex operator associated to $a \in V$ as defined in eq. (3.1.4) using "creation" and "annihilation" operators $\mathbf{a}_\ell^+, \mathbf{a}_\ell$ that give V a Fock space structure.

The main work goes into getting the correct normalization constants.

The non-vanishing partial derivatives of the basic field φ in the theory are

$$\varphi^{\ell} = c_{\ell}^{-1}\bar{t}_{\ell;\alpha}\partial^{\alpha}\varphi \tag{C.0.5}$$

where c_{ℓ} is a numerical constant that will be chosen later. (See appendix A for the definition of $t_{\ell;\alpha}$.) We label composite fields by multiindices

$$\mathcal{O}_a(x) = (a!)^{-1/2}\prod_{\ell\in\mathbf{L}}\left(\varphi^{\ell}\right)^{a_{\ell}}(x)\,. \tag{C.0.6}$$

We use Latin letters for the multiindices denoting composite fields and Greek letters for multiindices when dealing with polynomials in x or ∂. The basic field φ is harmonic by the field equation (3.1.1), so we may use eq. (3.1.22),

$$\varphi(x) = \sum_{\ell\in\mathbf{L}}\frac{c_{\ell}}{k_{\ell}}\,h_{\ell}(x)\,\varphi^{\ell}(0) \tag{C.0.7}$$

This has to be understood as an equation for insertions. Now the OPE of φ with a field $\mathcal{O}_a$ can easily be deduced from eq. (C.0.3) and the definition of φ^{ℓ}, eq. (C.0.5):

$$\begin{aligned}\varphi(x)\mathcal{O}_a(0) = &\sum_{\ell\in\mathbf{L}}\frac{c_{\ell}}{k_{\ell}}\,h_{\ell}(x)\,\left(\varphi^{\ell}\,\mathcal{O}_a\right)(0)\\ &+\sum_{\ell\in\mathbf{L}}c_{\ell}^{-1}\,\bar{h}_{\ell}(\partial)\,g(r)\,\frac{\partial\mathcal{O}_a}{\partial\varphi^{\ell}}(0)\,,\end{aligned} \tag{C.0.8}$$

where $r = |x|$ as always; see eq. (3.1.21) for the definition of the "Euclidean propagator" g. Again, this has to be understood as an equation for insertions.

$$\langle\varphi(x)\mathcal{O}_a(0)\varphi(x_1)\ldots\varphi(x_n)\rangle = \langle(\text{RHS of eq. (C.0.8)})\,\varphi(x_1)\ldots\varphi(x_n)\rangle\,. \tag{C.0.9}$$

The right hand side will be an absolutely convergent series of correlation functions given that the spatial arguments of the spectator fields fulfill $|x_j| > |x|$, $j = 1, \ldots, n$.
Using eq. (A.0.9) in eq. (C.0.8) we get

$$\varphi(x)\mathcal{O}_a(0) = \sum_\ell \sqrt{a_\ell + 1}\,\frac{c_l}{k_l}\,h_\ell(x)\,\mathcal{O}_{a+e_{(\ell)}}(0) + \sum_\ell \sqrt{a_\ell}\,\frac{q_\ell}{c_\ell}\,\bar{h}_\ell(x) r^{-2l-D+2}\,\mathcal{O}_{a-e_{(\ell)}}(0) \qquad \text{(C.0.10)}$$

where by e_ℓ, we mean the multiindex defined by $(e_\ell)_{\ell'} = \delta_{\ell,\ell'}$, and we define $(a - e_\ell) := 0$ for $a_\ell = 0$. To obtain a symmetric form of the OPE, we choose

$$c_\ell = \sqrt{q_\ell k_\ell} = \begin{cases} 2^{|\ell|}\Gamma(|\ell|)\sqrt{|\ell|/2} & \text{for } D = 2 \\ 2^{|\ell|}\Gamma(|\ell| + D/2 - 1)\sqrt{2(|\ell| + D/2 - 1)/(D-2)} & \text{for } D > 2\,. \end{cases} \qquad \text{(C.0.11)}$$

We introduce the abstract vector space V spanned by the field labels a and creation and annihilation operators $\mathbf{a}_\ell^+, \mathbf{a}_\ell$ on V by

$$\begin{aligned} \mathbf{a}_\ell^+ a &= \sqrt{a_\ell + 1}(a + e_\ell) \\ \mathbf{a}_\ell a &= \sqrt{a_\ell}(a - e_\ell)\,. \end{aligned} \qquad \text{(C.0.12)}$$

We also introduce the vertex operator $Y_0(\varphi, x)$ that corresponds to a multiplication of an insertion with the free field $\varphi(x)$. We rewrite the left-hand side of eq. (C.0.10) in this notation by

$$Y_0(\varphi, x)a \qquad \text{(C.0.13)}$$

and we can read off the right hand side of eq. (C.0.10) that

$$Y_0(\varphi, x) = K_D \sum_{\ell} \frac{1}{\sqrt{\omega(D,\ell)}} \left(h_\ell(x)\mathbf{a}_\ell^+ + \bar{h}_\ell(x) r^{-2l-D+2}\mathbf{a}_\ell\right) \tag{C.0.14}$$

with $K_D = 1$ for $D = 2$, $K_D = \sqrt{D-2}$ for $D > 2$ and $\omega(D,\ell) = 2|\ell| + D - 2$.

Having established the appropriate numerical constants for $Y_0(\varphi, x)$, it is easy to see that the vertex operators for composite fields $Y_0(a, x)$ are given by eq. (3.1.14) just by looking at eq. (3.1.7).

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