Some problems associated with sum and integral inequalities

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Mum and Dad
no you, no thesis

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In [2], the following extension of the higher order Rellich inequality

$$\int_{\mathbb{R}^n} |\Delta^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \gamma(n,\alpha,j) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4j}} \tag{1}$$

was proven by W. Allegretto for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. The constant γ is calculated explicitly by the author for all $n \geq 2$, $\alpha \geq 0$ and $j \in \mathbb{N}$, giving the value of the constant in the previously unknown case $n \leq \alpha + 4j$. Hence proving that γ is equal to zero if and only if $n \leq \alpha + 4j$ and $n - \alpha \equiv 0 \pmod 2$. In this problematic case, the author finds that the higher order Rellich inequality (1) can be forced to be non-trivial if further restrictions are placed on the function in $\mathbb{S}^{n-1}$.

An alternative method to restricting the functional class is to look at the Rellich type inequality

$$\int_{\mathbb{R}^n} |\Delta_{\mathbf{A}} f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \Phi(n,\alpha,\tilde{\Psi}) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}} \tag{2}$$

found by W.D. Evans and R.T. Lewis in [15] for $n = 2, 3, 4$. The magnetic Laplacian is of the form $\Delta_{\mathbf{A}} = (\nabla - i\mathbf{A})^2$ where in spherical coordinates

$$\mathbf{A} := \begin{cases} \frac{1}{r}\Psi(\theta_1)\mathbf{e}_1 & \text{if } n = 2, \\ \frac{1}{r\prod_{k=1}^{n-2}\sin\theta_k}\Psi(\theta_{n-1})\mathbf{e}_{n-1} & \text{if } n \geq 3, \end{cases} \tag{3}$$

with $\Psi \in L^\infty(0, 2\pi)$ and $\Psi(0) = \Psi(2\pi)$. The potential $\mathbf{A}$ is of Aharonov-Bohm type and the constant Φ is dependant upon the distance of the magnetic flux $\tilde{\Psi}$ to the integers $\mathbb{Z}$. By finding the discrete spectrum of the Friedrichs extension of $-\Delta_{\mathbf{A}}$ in $L^2(\mathbb{S}^{n-1})$, the author is able to extend the Rellich type inequality (2) to all $n \geq 2$ and $\alpha \geq 0$. Consequently, the higher order Rellich type inequality

$$\int_{\mathbb{R}^n} |\Delta_{\mathbf{A}}^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \Omega(n,\alpha,\tilde{\Psi},j) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4j}} \tag{4}$$

can be constructed. The inequality (4) is shown to be non-trivial for all $n \leq \alpha + 4j$ and $n - \alpha \equiv 0 \pmod 2$, the previously problematic case.

The Rellich type inequality (4) enables an analysis of the spectral properties of perturbations of the magnetic operator $\Delta_{\mathbf{A}}^4$ to be undertaken in $L^2(\mathbb{R}^n)$, $n \geq 2$. Furthermore, a CLR type bound for the number of negative eigenvalues of the operator $\Delta_{\mathbf{A}}^4$ can be found in $L^2(\mathbb{R}^8)$, a space in which there is no CLR bound for the operator Δ^4.

Contents

Chapter 1

Introduction

Please note that some of the results in the proceeding historical review have been reproduced here in an alternative form to the originals to allow for direct comparison of the results. A list of relevant notation appears in section 1.4.

1.1 The Rellich inequality

In lectures at New York University in 1953, published posthumously in [24], Rellich first proved the following inequality

$$\int_{\mathbb{R}^n} |\Delta f(\mathbf{x})|^2 d\mathbf{x} \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^4} \tag{1.1}$$

for $n \geq 2$ and $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ with the further restriction in two dimensions that the functions satisfy

$$\int_0^{2\pi} f(r,\theta) \cos\theta d\theta = \int_0^{2\pi} f(r,\theta) \sin\theta d\theta = 0. \tag{1.2}$$

As it can be seen, the Rellich inequality is trivial in four dimensions. In order to extend several non-oscillation theorems for elliptic equations of order 2 and 4, the Rellich inequality was extended by Allegretto [2], resulting in

$$\int_{\mathbb{R}^n} |\Delta f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq k(n,\alpha) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}} \tag{1.3}$$

where

$$k(n,\alpha) = \frac{(n+\alpha)^2(n-4-\alpha)^2}{16} + \tau(n,\alpha), \tag{1.4}$$

$$\begin{aligned} \tau(n,\alpha) = \inf_{m \in \mathbb{N}_0} \Big\{ m(m+n-2)\Big(m^2 + (n-2)m \\ + \frac{n^2 - 4n - 4\alpha - \alpha^2}{2}\Big)\Big\}. \end{aligned} \tag{1.5}$$

The following was observed; $k(n, \alpha) = 0$ iff for some triplet α, n and m,

$$m(m + n - 2) = -\frac{1}{4}(n^2 - 4n - 4\alpha - \alpha^2). \tag{1.6}$$

Furthermore, Allegretto found the following higher order Rellich inequality by induction,

$$\int_{\mathbb{R}^n} |\Delta^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \prod_{i=0}^{j-1} k(n, \alpha + 4i) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4j}}. \tag{1.7}$$

In Schmincke's investigation [26] of the class of potentials q for which the Schrödinger operator $-\Delta + q$ in $L^2(\mathbb{R}^n)$ is essentially self-adjoint, the following generalisation of the Rellich inequality was proven. For $n \geq 2$, $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and $s \in [\frac{-n(n-4)}{2}, \infty)$, then

$$\begin{aligned}\int_{\mathbb{R}^n} |\Delta f(\mathbf{x})|^2 d\mathbf{x} \geq -s \int_{\mathbb{R}^n} |\nabla f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^2} \\ + \frac{(n^2 + 4s)(n-4)^2}{16} \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^4}\end{aligned} \tag{1.8}$$

where the original Rellich inequality can be recovered for $n \geq 4$ by taking $s = 0$. In the same spirit of the evolution of (1.1) to (1.3), Schmincke's result was extended by Bennet [4]. Suppose $n \geq 2$, $\alpha \in [(-\infty, -n) \cup (-2, \infty)]$ and $\alpha \neq n - 4$. If $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and $s \in [-\frac{(n+\alpha)(n-4-\alpha)}{2}, \infty)$, then

$$\begin{aligned}\int_{\mathbb{R}^n} |\Delta f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq - s \int_{\mathbb{R}^n} |\nabla f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+2}} \\ + \frac{[(n+\alpha)^2 + 4s](n-4-\alpha)^2}{16} \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}}.\end{aligned} \tag{1.9}$$

Schminke's proof inspired Davies and Hinz's impressive paper [9] in which, among other things, sharp constants for the Rellich inequality in $L^p(\mathbb{R}^n)$ were first obtained. For all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and $2 - 2p < \alpha < n - 2p$ then

$$\int_{\mathbb{R}^n} |\Delta f(\mathbf{x})|^p \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq c(n,\alpha,p) \int_{\mathbb{R}^n} |f(\mathbf{x})|^p \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+2p}} \tag{1.10}$$

where the constant c is sharp and defined by

$$c(n,\alpha,p) = \frac{(n-\alpha-2p)((p-1)n+\alpha)}{p^2}. \tag{1.11}$$

By an inductive step, Davies and Hinz found, for $j \in \mathbb{N}$ and $2 - 2p < \alpha < n - 2jp$, the higher order Rellich inequality

$$\int_{\mathbb{R}^n} |\Delta^j f(\mathbf{x})|^p \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq C(n,\alpha,p,j) \int_{\mathbb{R}^n} |f(\mathbf{x})|^p \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+2jp}} \tag{1.12}$$

for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ with a sharp constant

$$C(n,\alpha,p,j) = \prod_{k=0}^{j-1} c(n,\alpha+4k,p). \tag{1.13}$$

In Section 2.1, the exact values of Allegretto's constant $k(n,\alpha)$ are calculated. This enables us in Section 2.2 to explicitly state the higher order $L^2(\mathbb{R}^n)$ Rellich inequality

$$\int_{\mathbb{R}^n} |\Delta^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \gamma(n,\alpha,j) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4j}} \tag{1.14}$$

for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, $j \in \mathbb{N}$ and $\alpha \geq 0$. When $n > \alpha + 4j$, then

$$\gamma(n,\alpha,j) = \prod_{i=0}^{j-1} \frac{(n+\alpha+4i)^2(n-\alpha-4(i+1))^2}{16} \tag{1.15}$$

which agrees with $C(n,\alpha,2,j)$, see (1.13), the constant found by Davies and Hinz. When $n \leq \alpha + 4j$, there was very little information available in the literature. When $j = 1$, Allegretto found that $k(n,\alpha) = 0$ iff the condition

(1.6) is satisfied and that in the case $\alpha = 0$ then (1.6) implies that n is even.

When $n \leq \alpha + 4j$, the constant $\gamma(n, \alpha, j)$ is dependant on the fractional part of $\frac{\alpha-n}{2}$, ε say. In particular, when $\alpha - n$ is an even integer ($\varepsilon = 0$) then the constant $\gamma(n, \alpha, j)$ is equal to zero and (1.14) becomes a trivial inequality. This dependence on the fractional part of $\frac{\alpha-n}{2}$ as opposed to simply n and α means that the constant $\gamma(n, \alpha, j)$ has a very different behaviour depending on whether $n \leq \alpha + 4j$ or $n > \alpha + 4j$. For example, if $j = 1$ and $\alpha = 2$;

n	$\gamma(n,2,1)$	n	$\gamma(n,2,1)$
2	0	6	0
3	$\frac{1}{16}$	7	$\frac{81}{16}$
4	0	8	25
5	$\frac{1}{16}$	9	$\frac{1089}{16}$

Section 2.3 concentrates on the problematic cases when the higher order Rellich inequality (1.14) becomes trivial i.e. when $n \leq \alpha + 4j$ and $\frac{n-\alpha}{2} \in \mathbb{Z}$. It becomes apparent that in these cases a non-trivial higher order Rellich inequality in $L^2(\mathbb{R}^n)$ can be found if some restrictions are placed on the function in $\mathbb{S}^{n-1}$. The Rellich inequality (1.1) in $L^2(\mathbb{R}^2)$ with the restriction (1.2) on the function is a particular case of this (see Remark 2.11). One application of this addresses the problem of there being no non-trivial Rellich inequality in four dimensions. In doing so, for all $f \in C_0^\infty(\mathbb{R}^4)$ and

$$\int_{\mathbb{S}^3} f(r,\omega)\overline{Y_{0,4}(\omega)}d\omega = 0, \tag{1.16}$$

then

$$\int_{\mathbb{R}^4} |\Delta f(x)|^2 dx \geq 9 \int_{\mathbb{R}^4} |f(x)|^2 \frac{dx}{|x|^4} \tag{1.17}$$

where $Y_{0,4}(\omega)$ is the four dimensional spherical harmonic of degree 0.

1.2 Magnetic Potentials

1.2.1 The Aharonov-Bohm effect

The Aharonov-Bohm effect, predicted by Aharonov and Bohm in [1], is a quantum mechanical phenomenon where a charged particle is affected by electromagnetic fields from which the particle is excluded. There has been a suggestion that the Aharonov-Bohm effect demonstrates that the electromagnetic potentials (rather than the electric and magnetic fields) are the fundamental quantities in quantum mechanics.

The specific case of interest in this text is the magnetic Aharonov-Bohm effect, where the wave function of a charged particle passing around a long solenoid experiences a phase shift as a result of the enclosed magnetic field, despite the magnetic field being zero in the region through which the particle passes. The magnetic Aharonov-Bohm effect was experimentally confirmed by Tonomura [29] in 1986. In this case, the magnetic potential $\mathbf{A}$ is taken to be

$$\mathbf{A} := \begin{cases} \frac{1}{r}\Psi(\theta_1)\mathbf{e}_1 & \text{if } n = 2, \\ \frac{1}{r\prod_{k=1}^{n-2}\sin\theta_k}\Psi(\theta_{n-1})\mathbf{e}_{n-1} & \text{if } n \geq 3, \end{cases} \tag{1.18}$$

in spherical coordinates with $\Psi \in L^\infty(0, 2\pi)$ and $\Psi(0) = \Psi(2\pi)$. The associated magnetic field $\mathbf{B} = curl\mathbf{A}$ is equal to zero in the domain $\mathbb{R}^n \setminus \mathcal{L}_n$ where

$$\mathcal{L}_n := \begin{cases} \{0\} & \text{if } n = 2, \\ \left\{\mathbf{x} = (r, \theta_1, ..., \theta_{n-1}) : r\prod_{k=1}^{n-2}\sin\theta_k = 0\right\} & \text{if } n \geq 3. \end{cases} \tag{1.19}$$

1.2.2 Repairing the Hardy inequality in two dimensions

In 1925, Hardy formulated and proved his famous inequality: let $p > 1$ and $f \in L^2(\mathbb{R}_+)$, $f \geq 0$. Then $F(x) = \int_0^x f(t)dt < \infty$ for every $x > 0$ and

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)dx, \tag{1.20}$$

where the constant $\left(\frac{p}{p-1}\right)^p$ is sharp. However it has been noted, see [17], that Landau, Pólya, Riesz and Schur made very important contributions to the development of (1.20). Many aspects of the inequality have been generalised over the years and one form which is of particular interest is the $p = 2$ multi-dimension version of the Hardy inequality, namely

$$\int_{\mathbb{R}^n} |\nabla f(\mathbf{x})|^2 d\mathbf{x} \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^2} \tag{1.21}$$

for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and $n > 2$. The constant $\frac{(n-2)^2}{4}$ is best possible and there is equality if and only if $f(\mathbf{x}) = 0$. When $n = 2$, the Hardy inequality is only trivially true but a non-trivial Hardy type inequality, where the $|\mathbf{x}|^2$ term is replaced with $|\mathbf{x}|^2(1+\log^2|\mathbf{x}|)$, can be found if some additional assumptions are placed on f. For example,

$$\int_{\mathbb{R}^2} |\nabla f(\mathbf{x})|^2 d\mathbf{x} \geq c \int_{\mathbb{R}^2} \frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^2(1+\log^2|\mathbf{x}|)} d\mathbf{x} \quad \text{if} \quad \int_{\{|\mathbf{x}|=1\}} f(\mathbf{x}) d\mathbf{x} = 0. \tag{1.22}$$

This logarithmic factor is not ideal. Solomyak [28] found that the $\log|\mathbf{x}|$ term is only required for functions f depending on $|\mathbf{x}|$ and can be removed if the function satisfies $\int_{\{|x|=r\}} f(\mathbf{x})d\mathbf{x} = 0$ for any $r > 0$. Laptev and Weidl's imaginative approach in [20] to the problem was to introduce a non-trivial magnetic field $\mathbf{A} \in C(\mathbb{R}^2 \setminus \{0\})$ with the condition that $curl\mathbf{A} \in L^1_{loc}(\mathbb{R}^2 \setminus \{0\})$.

Then for $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, the Hardy type inequality

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} f(\mathbf{x})|^2 d\mathbf{x} \geq a \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^2} \tag{1.23}$$

holds where

$$\nabla_{\mathbf{A}} = \nabla - i\mathbf{A}. \tag{1.24}$$

Not only have Laptev and Weidl removed the unpleasant logarithmic factor but they have done so without placing any additional assumptions on f. A magnetic potential $\mathbf{A}$ of Aharonov-Bohm type is an important example of (1.23). More precisely, if in polar coordinates (r, θ) in $\mathbb{R}^2 \setminus \{0\}$, the potential $\mathbf{A}$ is of the form

$$\mathbf{A}(r,\theta) = \frac{\Psi(\theta)}{r}(-\sin\theta, \cos\theta), \quad \Psi \in L^\infty(0, 2\pi), \quad \Psi(0) = \Psi(2\pi), \tag{1.25}$$

then the magnetic field $\mathbf{B} = curl\mathbf{A}$ generated is equal to zero in $\mathbb{R}^2 \setminus \{0\}$. As long as the magnetic flux

$$\tilde{\Psi} := \frac{1}{2\pi} \int_0^{2\pi} \Psi(\theta) d\theta \notin \mathbb{Z} \tag{1.26}$$

then (1.23) is non-trivial with the sharp constant

$$a = dist(\tilde{\Psi}, \mathbb{Z})^2.$$

When the magnetic flux $\tilde{\Psi} \in \mathbb{Z}$, $\nabla_{\mathbf{A}}$ is equivalent by gauge invariance to ∇ and so (1.23) transforms back to (1.21) which is trivial in $\mathbb{R}^2$.

1.2.3 Repairing the Rellich inequality

As discussed in Section 1.1, in order for the Rellich inequality (1.1) to be non-trivial in $n = 2, 4$, additional assumptions had to be placed on the function.

Inspired by ideas from [20] and [9], Evans and Lewis [15] sought to tackle this problem by replacing the Laplacian Δ with a magnetic Laplacian $\Delta_{\mathbf{A}} = \nabla_{\mathbf{A}}^2$ where $\mathbf{A}$ is (1.18), an Aharonov-Bohm potential. If $n = \{2, 3, 4\}$ and $\tilde{\Psi} \in (0, 1)$ then Evans and Lewis found the following Rellich type inequality,

$$\int_{\mathbb{R}^n} |\Delta_{\mathbf{A}} f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \Phi(n, \alpha, \tilde{\Psi}) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}} \tag{1.27}$$

for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. The constants given by

$$\Phi(2, \alpha, \tilde{\Psi}) = \inf_{m \in \mathbb{Z}} \left\{ (m + \tilde{\Psi})^2 - \frac{(\alpha + 2)^2}{4} \right\}^2, \tag{1.28}$$

$$\Phi(3, \alpha, \tilde{\Psi}) = \inf_{m \in \mathbb{Z}} \left\{ (m - \tilde{\Psi})(m - \tilde{\Psi} + 1) - \frac{(\alpha + 1)(\alpha + 3)}{4} \right\}^2, \tag{1.29}$$

$$\Phi(4, \alpha, \tilde{\Psi}) = \inf_{m \in \mathbb{Z}'} \left\{ (m + \tilde{\Psi})^2 - 1 - \frac{\alpha(\alpha + 4)}{4} \right\}^2 \tag{1.30}$$

where $\mathbb{Z}' = \{m : (m + \tilde{\Psi})^2 \geq 1\}$. Similarly to the Laptev-Weidl inequality, Evans and Lewis have found in (1.27), a Rellich type inequality in the critical cases $n = 2, 4$ without making any additional assumptions on f.

As mentioned above and observed in Section 2.2, the Rellich inequality (1.3) is trivial when $n \leq \alpha + 4$ and $\frac{n-\alpha}{2} \in \mathbb{Z}$, so as the value of α increases, the number of critical cases, by which we mean the number of dimensions for which (1.3) is trivial, will increase. With this in mind and using ideas heavily influenced by [15], the purpose of Chapter 3 was to find (1.27) for all $n \geq 2$ and $\alpha \geq 0$. In order to do this, the positive eigenvalues of our magnetic Laplace-Beltrami operator (see Theorem 3.2) were calculated in all dimensions $n \geq 2$. Then Theorem 1 of [15] could be applied thereby resulting in Theorem 3.8 which gives (1.27) for all $n \geq 2$, $\alpha \geq 0$, $\tilde{\Psi} \in (0, 1)$ and $f \in C_0^\infty(\mathbb{R}^n \setminus \mathcal{L}_n)$ where

$$\Phi(n, \alpha, \tilde{\Psi}) = \inf_{m \in \tilde{\mathbb{Z}}^{(n)}} \left\{ (m + \tilde{\Psi})(m + \tilde{\Psi} + n - 2) + \frac{(n + \alpha)(n - \alpha - 4)}{4} \right\}^2 \tag{1.31}$$

and $\hat{\mathbb{Z}}^{(n)} = \{m \in \mathbb{Z} : m \leq 2-n-\tilde{\Psi} \text{ or } m \geq -\tilde{\Psi}\}$. The cut in the domain $\mathcal{L}_n$ is determined by the choice of the magnetic potential $\mathbf{A}$ (see (1.18) and (1.19)). Importantly the Rellich type inequality found has a non-zero constant

$$\Phi(n,\alpha,\tilde{\Psi}) = \begin{cases} \tilde{\Psi}^2(\tilde{\Psi}-\alpha-2)^2 & \text{if} \quad \tilde{\Psi} \in (0,\frac{1}{2}], \\ (\tilde{\Psi}+\alpha+1)^2(\tilde{\Psi}-1)^2 & \text{if} \quad \tilde{\Psi} \in (\frac{1}{2},1) \end{cases} \tag{1.32}$$

when $n \leq \alpha + 4$ and $n - \alpha \equiv 0 \pmod 2$, the case where the ordinary Rellich inequality become trivial. In contrast to Section 1.1, no additional restrictions were placed on the function f. A consequence of Theorem 3.8 is a higher order Rellich type inequality which is non-trivial in the cases in which the higher order Rellich inequality (1.7) is only trivially true.

1.3 CLR type bounds

Consider the higher order Schrödinger operators

$$H_V = (-\Delta)^j - V, \qquad j \in \mathbb{N} \tag{1.33}$$

in $L^2(\mathbb{R}^n)$ where $V \geq 0$ and $V \in L^{\frac{n}{2j}}(\mathbb{R}^n)$. This operator is defined as that associated with a closed semi-bounded quadratic form. It has essential spectrum $[0,\infty)$ and for $n > 2j$,

$$N(H_V) \leq c(n,j) \int_{\mathbb{R}^n} V(\mathbf{x})^{\frac{n}{2j}} d\mathbf{x} \tag{1.34}$$

where $N(H_V)$ denotes the number of negative eigenvalues of the operator H_V. When $j = 1$, (1.34) is known as the Cwikel-Lieb-Rozenblum (CLR) bound, after the authors of the three earliest proofs, see [8], [21] and [25]. More generally, Egorov [12] proved that if the Laplacian is replaced by a positive elliptic differential operator of order $2j$ with sufficiently smooth coefficients, then for $n > 2j$, (1.34) still remains valid. A natural question is, what happens to (1.34) when $n \leq 2j$? The condition $V \in L^{\frac{n}{2j}}(\mathbb{R}^n)$ does not imply the operator H_V is semi-bounded from below, let alone that (1.34) remains true. When $n < 2j$, [5] and [7] found some different estimates for $N(H_V)$ when n is even and n is odd respectively.

The limiting case $n = 2j$ has proved to be more obtuse, see [7] [27], [5], [6] and [18]. Found by Laptev and Netrusov in [19], a CLR type inequality can be obtained for the operator $H_{V,b}$ which is H_V with an additional Hardy term $|x|^{-2j}$,

$$H_{V,b} := (-\Delta)^j + b|\mathbf{x}|^{-2j} - V \tag{1.35}$$

in $\mathbb{R}^{2j}$. If $V \geq 0$ and $V \in L^1(\mathbb{R}_+, L^p(\mathbb{S}^{n-1}))$ for $1 < p \leq \infty$, then by Theorem 1.2 in [19], $H_{V,b}$ is a self-adjoint operator whose essential spectrum coincides with $[0, \infty)$ and

$$N(H_{V,b}) \leq c(b, n, p)\|V\|_{L^1(\mathbb{R}_+, L^p(\mathbb{S}^{n-1}))}. \tag{1.36}$$

Returning to the $j = 1$ case, the CLR inequality is valid for $n \geq 3$ and (1.36) gives a CLR type bound in two dimensions. Using (1.36) and the Laptev-Weidl inequality, Balinsky, Evans and Lewis [3] were able to fulfill the prediction made in [20] in their study of the negative spectrum of magnetic Schrödinger operators. More precisely, let $T_{\mathbf{A}}$ be the self-adjoint realisation in $L^2(\mathbb{R}^2)$ of $-\Delta_{\mathbf{A}} - V$ defined by the associated form where $V \in L^1_{loc}(\mathbb{R}\backslash\{0\})$, $V \geq 0$ and

$$V \in L^1(\mathbb{R}^+, L^\infty(\mathbb{S}^{n-1}, rdr)). \tag{1.37}$$

The magnetic potential $\mathbf{A}$ is defined as (1.25), of Aharonov-Bohm type. Then if the magnetic flux $\tilde{\Psi}$ is not an integer, we have

$$N(T_{\mathbf{A}}) \leq a(\tilde{\Psi})\|V\|_{L^1(\mathbb{R}^+; L^\infty(\mathbb{S}^1); rdr)}. \tag{1.38}$$

Futhermore it is also proven in [3] that the $L^1(\mathbb{R}^+; L^\infty(\mathbb{S}^1); rdr)$-norm cannot be replaced by the $L^1(\mathbb{R}^2)$-norm i.e. an inequality of the form (1.34) cannot be found for the magnetic Schrödinger operator in two dimensions. Evans and Lewis [14] continued the analysis by using the inequality (1.27) found by them to construct a CLR type inequality in the limiting case $n = 4$ and $j = 2$. This is of the form

$$N(L_{\mathbf{A}}) \leq b(\tilde{\Psi})\|V\|_{L^1(\mathbb{R}^+;L^\infty(\mathbb{S}^3);rdr)} \tag{1.39}$$

where $L_{\mathbf{A}}$ is the self-adjoint realisation of $\Delta_{\mathbf{A}}^2 + B_+ - V$ defined by the quadratic form. B_+ is an operator of multiplication by the function b_+, where

$$0 \leq b_+ \in L^1(\mathbb{R}_+; L^\infty(\mathbb{S}^{n-1}); r^3dr) \equiv L^1(\mathbb{R}_+; r^3dr) \otimes L^\infty(\mathbb{S}^{n-1}) \tag{1.40}$$

and

$$0 \leq V \leq L^1((0,\infty); r^3dr). \tag{1.41}$$

Furthermore, Evans and Lewis [14] found that the operator $L_{\mathbf{A}}$ has the essential spectrum $[0,\infty)$. The space $L^2(\mathbb{R}^8)$ having no CLR bound for the operator Δ^4, is another example of a critical case. Consequently in Chapter 4, the spectral perturbations of the magnetic operator $\Delta_{\mathbf{A}}^4$ are investigated. Take K_+ to be the operator of multiplication by the function k_+, where

$$0 \leq k_+ \in L^1(\mathbb{R}_+; L^\infty(\mathbb{S}^{n-1}); r^7dr) \equiv L^1(\mathbb{R}_+; r^7dr) \otimes L^\infty(\mathbb{S}^{n-1}). \tag{1.42}$$

Then the essential spectrum of the self-adjoint realisation of $\Delta_{\mathbf{A}}^4 + K_+$ defined by the quadratic form coincides with $[0,\infty)$. Furthermore the following CLR-type inequality

$$N(J_{\mathbf{A}}) \leq c(\tilde{\Psi})\|V\|_{L^1(\mathbb{R}^+;L^\infty(\mathbb{S}^7);rdr)} \tag{1.43}$$

holds in $L^2(\mathbb{R}^8)$, where $J_{\mathbf{A}}$ is the self-adjoint realisation of $\Delta_{\mathbf{A}}^4 + K_+ - V$

defined by the quadratic form and

$$0 \leq V \leq L^1((0,\infty); r^7 dr). \tag{1.44}$$

1.4 Notation

The following notation will be used throughout the thesis;

$\mathbb{C}$ The set of complex numbers.

$C_0^\infty(\Omega)$ The set of infinitely differential functions of compact support in Ω.

$L^p(\Omega) = \{f : \int_\Omega |f|^p d\mathbf{x} < \infty\}$.

$\mathbb{N}$ The set of natural numbers.

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

$\mathbb{R}$ The set of real numbers.

$\mathbb{R}_+$ The set of nonnegative real numbers.

$\mathbb{R}^n$ n-dimensional Euclidean space.

$Re[z]$ The real part of $z \in \mathbb{C}$.

$\mathbb{S}^{n-1}$ Unit hypersphere in $\mathbb{R}^n$.

$\mathbb{Z}$ The set of integers.

∇	Gradient.
Δ	Laplacian.
$\overline{z}$	The complex conjugate of $z \in \mathbb{C}$.
$a \lesssim b$	a is bounded above by a constant multiple of b, the multiple being independent of any variables in a and b.
$a := b$	a is defined by b.
$T \restriction_\Omega$	Restriction of the operator T to the set Ω.

Further notation will be introduced in the text.

Chapters are divided into sections, with for example Section 3.2 denoting section 2 of chapter 3. Theorems, Corollaries, Lemmas and Remarks are numbered in sequence within each Chapter: Theorem 2.1, Remark 2.2, Corollary 2.3 etc. Equations and formulae are number consecutively within each chapter, so (4.3) denotes the third equation in Chapter 4. The symbol $\blacksquare$ denotes the end of a proof.

A number of references are cited throughout the text, denoted by $[\cdot]$. A full list of which are given in the bibliography.

Chapter 2

The higher order Rellich inequality

2.1 The Rellich inequality and Allegretto's constant

The Rellich inequality

$$\int_{\mathbb{R}^n} |\Delta f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq k(n,\alpha) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}} \tag{2.1}$$

was proven by Allegretto in [2], where

$$k(n,\alpha) = \frac{(n+\alpha)^2(n-4-\alpha)^2}{16} + \tau(n,\alpha), \tag{2.2}$$

$$\tau(n,\alpha) = \inf_{m\in\mathbb{N}_0} \Big\{ m(m+n-2)\Big(m^2+(n-2)m + \frac{n^2-4n-4\alpha-\alpha^2}{2}\Big)\Big\}. \tag{2.3}$$

Define

$$frac[x] \quad := \quad x - \lfloor x \rfloor \tag{2.4}$$

where $\lfloor x \rfloor$ is the floor function (the largest integer less than or equal to x). Under this definition $0 \leq frac[x] < 1$, $x = \lfloor x \rfloor + frac[x]$ and Allegretto's constant can be calculated.

Theorem 2.1. *Suppose* $\varepsilon := frac[\frac{\alpha-n}{2}]$ *and* $\alpha \geq 0$. *Then*

$$k(n,\alpha) \quad = \quad \begin{cases} \frac{(n+\alpha)^2(n-\alpha-4)^2}{16} & \text{if} \quad n > \alpha+4, \\ l(n,\alpha) & \text{if} \quad n \leq \alpha+4 \end{cases} \tag{2.5}$$

where

$$l(n,\alpha)=\begin{cases}0 & \textit{if}\ \ \varepsilon=0,\\ \varepsilon^2(\varepsilon-\alpha-2)^2 & \textit{if}\ \ 0<\varepsilon\le\frac{(\alpha+3)}{2}-\frac{\sqrt{(\alpha+1)(\alpha+3)}}{2},\\ (1-\varepsilon)^2(\varepsilon-\alpha-3)^2 & \textit{if}\ \ \frac{(\alpha+3)}{2}-\frac{\sqrt{(\alpha+1)(\alpha+3)}}{2}<\varepsilon<1.\end{cases} \tag{2.6}$$

Remark 2.2. *The statement, $k(n,\alpha)=0$ iff $n-\alpha\equiv 0 \pmod 2$ and $n\le\alpha+4$, agrees with the condition of Allegretto (Corollary 2, [2]).*

Remark 2.3. *It can be seen from the proof of Theorem 2.1 that*

$$\frac{(\alpha+3)}{2}-\frac{\sqrt{(\alpha+1)(\alpha+3)}}{2}\ge\frac{1}{2}, \tag{2.7}$$

see (2.27), which implies that

$$l(n,\alpha)=\frac{(2\alpha-3)^2}{16} \tag{2.8}$$

when $n-\alpha\equiv 1 \pmod 2$ and $n\le\alpha+4$.

Proof. Rewrite Allegretto's constant

$$k(n,\alpha)=\inf_{m\in\mathbb{N}_0}\left\{m(m+n-2)+\Big(\frac{n+\alpha}{2}\Big)\Big(\frac{n-\alpha-4}{2}\Big)\right\}^2. \tag{2.9}$$

Define for $x\in\mathbb{R}_+$

$$\begin{aligned}\Theta(x,n,\alpha):=&x^2+(n-2)x+\frac{(n+\alpha)(n-\alpha-4)}{4}\\ =&\Big(x+\frac{n-\alpha-4}{2}\Big)\Big(x+\frac{n+\alpha}{2}\Big)\end{aligned} \tag{2.10}$$

and set

$$x_- \ := \ -\frac{n+\alpha}{2}, \tag{2.11}$$

$$x_+ \quad := \quad \frac{\alpha + 4 - n}{2}. \tag{2.12}$$

Now

$$\frac{\partial}{\partial x}\Theta(x,n,\alpha)\Big|_{x=x_\star} = 2x_\star + (n-2) = 0 \qquad \Rightarrow \qquad x_\star = 1 - \frac{n}{2} \tag{2.13}$$

and since the function $\Theta(x,n,\alpha)^2$ is symmetric about its local maximum $x_\star$ and $x_\star$ is zero or negative, we need only look at $x \geq x_\star$. Therefore, finding the $\inf_{m\in\mathbb{N}_0}\Theta(m,n,\alpha)^2$ becomes two different problems depending on whether or not the largest zero x_+ belongs to $\mathbb{R}_+ \cup \{0\}$. The two different cases are given by

$$x_+ \begin{cases} < 0 & \text{if} \quad n > \alpha + 4, \\ \geq 0 & \text{if} \quad n \leq \alpha + 4. \end{cases} \tag{2.14}$$

Case 1: $n > \alpha + 4$

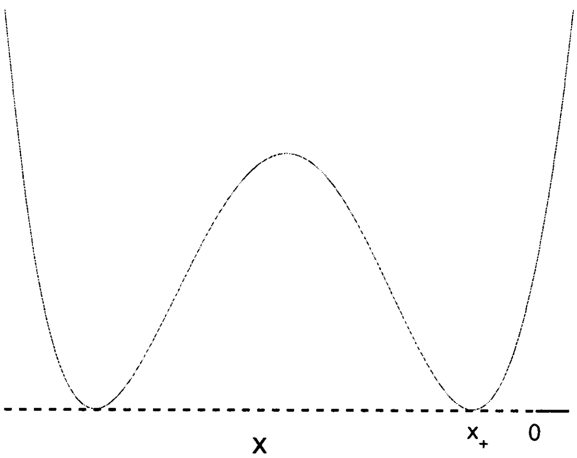

Figure 2.1: $\Theta(x,n,\alpha)^2$ when $n > \alpha + 4$.

The $\inf_{m\in\mathbb{N}_0} \Theta(m,n,\alpha)^2$ occurs when $m = 0$, therefore

$$k(n,\alpha) = \inf_{m\in\mathbb{N}_0} \Theta(m,n,\alpha)^2 = \frac{(n+\alpha)^2(n-\alpha-4)^2}{16}. \tag{2.15}$$

Case 2: $n \leq \alpha + 4$

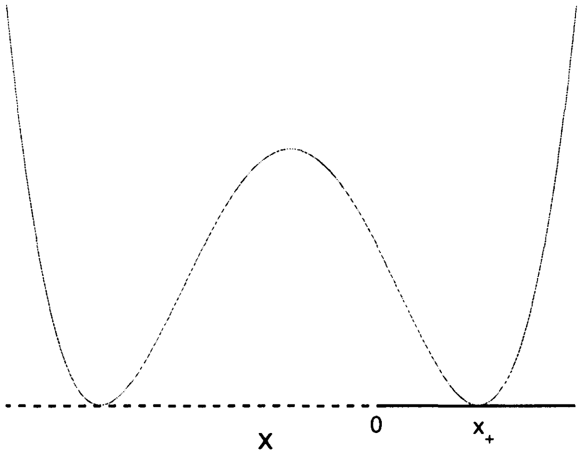

Figure 2.2: $\Theta(x,n,\alpha)^2$ when $n \leq \alpha + 4$.

Evidently from Figure 2.2,

$$\begin{aligned} k(n,\alpha) &= \inf_{m\in\mathbb{N}_0} \Theta(m,n,\alpha)^2 \\ &= \min\{\Theta(m_-,n,\alpha)^2, \Theta(m_+,n,\alpha)^2\} =: l(n,\alpha) \end{aligned} \tag{2.16}$$

where m_- and m_+ are the two integers neighbouring x_+ i.e.

$$m_- := 2 + \delta, \tag{2.17}$$

$$m_+ := 3 + \delta \tag{2.18}$$

where

$$\begin{aligned}\frac{\alpha-n}{2} =& \delta+\varepsilon, && (2.19)\\ \delta :=& \left\lfloor\frac{\alpha-n}{2}\right\rfloor, && (2.20)\\ \varepsilon :=& frac\left[\frac{\alpha-n}{2}\right]. && (2.21)\end{aligned}$$

By calculation

$$\begin{aligned}\Theta(\delta+2,n,\alpha) =& \varepsilon(\varepsilon-\alpha-2), && (2.22)\\ \Theta(\delta+3,n,\alpha) =& (1-\varepsilon)(3+\alpha-\varepsilon). && (2.23)\end{aligned}$$

Noting that

$$\Theta(\delta+3,n,\alpha)+\Theta(\delta+2,n,\alpha) = 2\varepsilon^2-2(\alpha+3)\varepsilon+\alpha+3 =: g(\varepsilon) \qquad (2.24)$$

and

$$\Theta(\delta+3,n,\alpha)-\Theta(\delta+2,n,\alpha) = 3+\alpha-2\varepsilon > 0 \qquad (2.25)$$

since $0 \leq \varepsilon < 1$. The equation $g(\varepsilon_\star) = 0$ has the solution

$$\varepsilon_\star = \frac{(\alpha+3)}{2} - \frac{\sqrt{(\alpha+1)(\alpha+3)}}{2} \qquad (2.26)$$

because the second root $\frac{(\alpha+3)}{2} + \frac{\sqrt{(\alpha+1)(\alpha+3)}}{2} > 1$ is discarded. Observing that

$$\begin{aligned}\frac{1}{2} > \frac{(\alpha+3)}{2} - \frac{\sqrt{(\alpha+2)^2-1}}{2} &= \varepsilon_\star\\ &= \frac{(\alpha+3)}{2} - \frac{\sqrt{(\alpha+1)^2+2(\alpha+1)}}{2} < 1\end{aligned} \qquad (2.27)$$

and

$$g(0) = \alpha + 3 > 0, \tag{2.28}$$
$$g(1) = -(\alpha + 1) < 0, \tag{2.29}$$

we have

$$l(n,\alpha) = \begin{cases} \varepsilon^2(\varepsilon - \alpha - 2)^2 & \text{if} \quad 0 \le \varepsilon \le \frac{(\alpha+3)}{2} - \frac{\sqrt{(\alpha+1)(\alpha+3)}}{2}, \\ (1-\varepsilon)^2(\varepsilon - \alpha - 3)^2 & \text{if} \quad \frac{(\alpha+3)}{2} - \frac{\sqrt{(\alpha+1)(\alpha+3)}}{2} < \varepsilon < 1 \end{cases} \tag{2.30}$$

by the difference of two squares. ■

2.2 The higher order Rellich inequality

Corollary 4 (i) in [2] gives the following extension of the higher order Rellich inequality, where the constant is given in terms of a product of Allegretto's constants $k(n,\alpha)$. This constant can now be calculated explicitly as a consequence of Theorem 2.1.

Theorem 2.4. *If $j \in \mathbb{N}$ and $\alpha \ge 0$ then*

$$\int_{\mathbb{R}^n} |\Delta^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \ge \gamma(n,\alpha,j) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4j}} \tag{2.31}$$

for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ where

- *if $n > \alpha + 4j$,*

$$\gamma(n,\alpha,j) = \prod_{i=0}^{j-1} \frac{(n+\alpha+4i)^2(n-\alpha-4(i+1))^2}{16}. \tag{2.32}$$

- *if* $\alpha + 4 < n \leq \alpha + 4j$ *and* $\varepsilon := frac[\frac{\alpha-n}{2}] \neq 0$,

$$\begin{aligned}\gamma(n,\alpha,j) = &\prod_{i=0}^{\lfloor\frac{n-\alpha}{4}\rfloor-1} \frac{(n+\alpha+4i)^2(n-\alpha-4(i+1))^2}{16} \\ &\cdot \prod_{i=\lfloor\frac{n-\alpha}{4}\rfloor}^{j-1} l(n,\alpha+4i).\end{aligned} \tag{2.33}$$

- *if* $n \leq \alpha + 4$ *and* $\varepsilon \neq 0$,

$$\gamma(n,\alpha,j) = \prod_{i=0}^{j-1} l(n,\alpha+4i). \tag{2.34}$$

- *if* $n \leq \alpha + 4j$ *and* $\varepsilon = 0$ *(i.e. when* $n - \alpha \equiv 0 \pmod 2$*)*

$$\gamma(n,\alpha,j) = 0. \tag{2.35}$$

Remark 2.5. *When* $n > \alpha + 4j$*, Theorem 2.4 agrees with the inequality found by Davies and Hinz (*$p = 2$*) in [9] and as proved by Davies and Hinz,* $\gamma(n,\alpha,j)$ *is optimal when* $n > \alpha + 4j$.

Proof. By Corollary 4 in [2],

$$\int_{\mathbb{R}^n} |\Delta^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \gamma(n,\alpha,j) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4j}} \tag{2.36}$$

for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ where

$$\gamma(n,\alpha,j) = \prod_{i=0}^{j-1} k(n,\alpha+4i). \tag{2.37}$$

Applying Theorem 2.1 gives

$$k(n,\alpha+4i) \;=\; \begin{cases} \frac{(n+\alpha+4i)^2(n-\alpha-4(i+1))^2}{16} & \text{if} \quad n>\alpha+4(i+1), \\ l(n,\alpha+4i) & \text{if} \quad n\le\alpha+4(i+1), \end{cases} \tag{2.38}$$

for $i \in \{0,1,...,j-1\}$. If $n > \alpha+4j$ then $n > \alpha+4(i+1)$ for all $i \in \{0,1,...,j-1\}$ and so

$$\gamma(n,\alpha,j) = \prod_{i=0}^{j-1} \frac{(n+\alpha+4i)^2(n-\alpha-4(i+1))^2}{16} \tag{2.39}$$

as required. If $n \le \alpha+4$ then $n \le \alpha+4(i+1)$ for all $i \in \{0,1,...,j-1\}$ and so

$$\gamma(n,\alpha,j) \;=\; \prod_{i=0}^{j-1} l(n,\alpha+4i). \tag{2.40}$$

When $\alpha+4 < n \le \alpha+4j$ then there exists $b \in \{1,2,...,j-1\}$ such that

$$\alpha+4b < n \le \alpha+4(b+1), \tag{2.41}$$

where

$$b = \begin{cases} \frac{n-\alpha}{4}-1 & \text{if } n-\alpha \equiv 0 \pmod 4, \\ \lfloor\frac{n-\alpha}{4}\rfloor & \text{if } n-\alpha \not\equiv 0 \pmod 4. \end{cases} \tag{2.42}$$

When $i \in \{0,1,...,b-1\}$ then $n > \alpha+4(i+1)$ but when $i \in \{b,b+1,...,j-1\}$ then $n \le \alpha+4(i+1)$ and so

$$\begin{aligned} \gamma(n,\alpha,j) &= \prod_{i=0}^{b-1} k(n,\alpha+4i).\prod_{i=b}^{j-1} k(n,\alpha+4i) \\ &= \prod_{i=0}^{b-1} \frac{(n+\alpha+4i)^2(n-\alpha-4(i+1))^2}{16} \prod_{i=b}^{j-1} l(n,\alpha+4i). \end{aligned} \tag{2.43}$$

Suppose $n - \alpha \equiv 0 \pmod 2$, then

$$\frac{\alpha - n}{2} \in \mathbb{Z} \quad \Rightarrow \varepsilon = 0 \Rightarrow l(n, \alpha + 4i) = 0 \tag{2.44}$$

which implies $\gamma(n, \alpha, j) = 0$ for all $n \leq \alpha + 4j$. Therefore if it is assumed that $n - \alpha \not\equiv 0 \pmod 2$, then $b = \lfloor \frac{n-\alpha}{4} \rfloor$ and the result follows. ■

2.3 Restriction of the class of functions

The Rellich inequality becomes trivial when $n \leq \alpha + 4$ and $n - \alpha \equiv 0 \pmod 2$ according to Theorem 2.1 (see (2.6)). When $\alpha = 0$ and $n = 2$, a non-trivial Rellich inequality can be found if extra restrictions are placed on the function class. Can this idea be extended to all $n \leq \alpha + 4$ and $n - \alpha \equiv 0 \pmod 2$? Firstly we need to investigate why the constant vanishes. In [15], Evans and Lewis developed the following abstract inequality (referred to as Theorem 1 in [15]):

Theorem 2.6 (Evans and Lewis). *Let Λ_ω be a non-negative self-adjoint operator in the space $L^2(\mathbb{S}^{n-1}; d\omega)$ whose spectrum is discrete and consists of isolated eigenvalues $\lambda_m, m \in \mathcal{I}$, where $\mathcal{I}$ is a countable index set. Let*

$$L_r := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} \tag{2.45}$$

and define the operator $D := L_r + \frac{1}{r^2}\Lambda_\omega$ on the domain

$$\begin{aligned}\mathcal{D}_0 := \{f : f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}),& \\ f(r, \cdot) \in \mathcal{D}(\Lambda_\omega) \text{ for } 0 < r < \infty, Df \in L^2(\mathbb{R}^n)\}&\end{aligned} \tag{2.46}$$

in $L^2(\mathbb{R}^n)$, where $\mathcal{D}(\Lambda_\omega)$ denotes the domain of Λ_ω. Then for all $f \in \mathcal{D}_0$ such *that* $|\cdot|^{-\frac{\alpha}{2}} Df \in L^2(\mathbb{R}^n)$, *we have that*

$$\int_{\mathbb{R}^n} |Df(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq C(n,\alpha) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}} \tag{2.47}$$

where

$$C(n,\alpha) = \inf_{m \in \mathcal{I}} \left\{ \lambda_m + \left(\frac{n+\alpha}{2}\right)\left(\frac{n-\alpha-4}{2}\right) \right\}^2. \tag{2.48}$$

For example, if $D = -\Delta$, then Λ_ω is the Laplace-Beltrami operator which has eigenvalues ρ_m, given by

$$\rho_m = m(m+n-2) \tag{2.49}$$

for $m \in \mathbb{N}_0$. In this case, Theorem 2.6 returns to equation (2.1) and illustrates that the constant $k(n,\alpha)$ is of the form

$$k(n,\alpha) = \inf_{m \in \mathbb{N}_0} \left\{ \rho_m + \left(\frac{n+\alpha}{2}\right)\left(\frac{n-\alpha-4}{2}\right) \right\}^2. \tag{2.50}$$

A technique is needed which will "eliminate" the eigenvalues in $k(n,\alpha)$ which cause the constant to become equal to zero. In order to do this, the following adapted version of Theorem 2.6 will be proven.

Theorem 2.7. *Let the assumptions of Theorem 2.6 hold. Let* $\mathcal{G} \subset \mathcal{I}$ *be a countable index subset such that if* $m \in \mathcal{G}$ *then*

$$\int_{\mathbb{S}^{n-1}} f(r,\omega)\overline{u_m(\omega)}d\omega = 0 \tag{2.51}$$

where λ_m *is an eigenvalue (repeated according to multiplicity) of the operator* Λ_ω *with corresponding normalised eigenvector* $u_m(\omega)$. *Then for all* $f \in \mathcal{D}_0$ *such that* $|\cdot|^{-\frac{\alpha}{2}} Df \in L^2(\mathbb{R}^n)$, *we have that*

$$\int_{\mathbb{R}^n} |Df(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \tilde{C}(n,\alpha) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}} \tag{2.52}$$

where

$$\tilde{C}(n,\alpha) = \inf_{m\in\mathcal{I}\setminus\mathcal{G}} \left\{\lambda_m + \left(\frac{n+\alpha}{2}\right)\left(\frac{n-\alpha-4}{2}\right)\right\}^2. \tag{2.53}$$

Proof. Since the spectrum of Λ_ω is assumed to be discrete, its normalised eigenvectors $u_m, m \in \mathcal{I}$ with the eigenvalues $\{\lambda_m\}$ repeated according to multiplicity, form an orthonormal basis of $L^2(\mathbb{S}^{n-1}; d\omega)$. For $f \in \mathcal{D}_0$, set

$$\mathcal{Z}_m[f](r) := \int_{\mathbb{S}^{n-1}} f(r,\omega)\overline{u_m(\omega)}d\omega. \tag{2.54}$$

Then $\mathcal{Z}_m[f] \in C_0^\infty(\mathbb{R}_+)$ and on using Parseval's identity and the fact that $\mathcal{Z}_m[f] = 0$ when $m \in \mathcal{G}$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}} &= \sum_{m\in\mathcal{I}} \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr \\ &= \sum_{m\in\mathcal{I}\setminus\mathcal{G}} \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr \\ &\quad + \sum_{m\in\mathcal{G}} \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr \\ &= \sum_{m\in\mathcal{I}\setminus\mathcal{G}} \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr. \end{aligned} \tag{2.55}$$

Also

$$\begin{aligned}&\sum_{m\in\mathcal{I}} \lambda_m \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr \\ &\qquad = \sum_{m\in\mathcal{I}\setminus\mathcal{G}} \lambda_m \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr,\end{aligned} \tag{2.56}$$

$$\sum_{m\in\mathcal{I}} \lambda_m^2 \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr = \sum_{m\in\mathcal{I}\setminus\mathcal{G}} \lambda_m^2 \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr. \tag{2.57}$$

Now from the proof of Theorem 1 in [15],

$$\begin{aligned}
\int_{\mathbb{R}^n} |Df(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} &\geq \sum_{m\in\mathcal{I}} \lambda_m^2 \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr \\
&+ \frac{1}{2}(n-\alpha-4)(n+\alpha) \sum_{m\in\mathcal{I}} \lambda_m \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr \\
&+ \left\{(n-1)(\alpha+1) + \left(\frac{n-\alpha-2}{2}\right)^2\right\}\left(\frac{n-\alpha-4}{2}\right)^2 \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}} \\
&= \sum_{m\in\mathcal{I}\setminus\mathcal{G}} \lambda_m^2 \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr \\
&+ \sum_{m\in\mathcal{I}\setminus\mathcal{G}} \frac{1}{2}(n-\alpha-4)(n+\alpha)\lambda_m \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr \\
&+ \sum_{m\in\mathcal{I}\setminus\mathcal{G}} \left(\frac{n+\alpha}{2}\right)^2\left(\frac{n-\alpha-4}{2}\right)^2 \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr \\
&= \sum_{m\in\mathcal{I}\setminus\mathcal{G}} \left\{\lambda_m + \left(\frac{n+\alpha}{2}\right)\left(\frac{n-\alpha-4}{2}\right)\right\}^2 \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr \\
&\geq \inf_{m\in\mathcal{I}\setminus\mathcal{G}} \left\{\lambda_m + \left(\frac{n+\alpha}{2}\right)\left(\frac{n-\alpha-4}{2}\right)\right\}^2 \sum_{m\in\mathcal{I}\setminus\mathcal{G}} \int_0^\infty |\mathcal{Z}_m[f](r)|^2 r^{n-\alpha-5} dr \\
&= \inf_{m\in\mathcal{I}\setminus\mathcal{G}} \left\{\lambda_m + \left(\frac{n+\alpha}{2}\right)\left(\frac{n-\alpha-4}{2}\right)\right\}^2 \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}}
\end{aligned} \tag{2.58}$$

and the result follows. ■

Remark 2.8. *It can be seen from the proof of Theorem 2.1 that the constant $k(n,\alpha)$ in the inequality (2.1) vanishes when $\theta(\beta,n,\alpha) = 0$ and $\beta \in \mathbb{N}_0$.*

These conditions are satisfied when $n \leq \alpha + 4$, $n - \alpha \equiv 0 \pmod 2$ *and*

$$\beta = 2 + \frac{\alpha - n}{2}. \tag{2.59}$$

Thus, if the eigenvalue ρ_β *is removed from the constant* $k(n, \alpha)$*, it will no longer vanish when* $n \leq \alpha + 4$ *and* $n - \alpha \equiv 0 \pmod 2$*.*

Let $\tilde{l}(n, \alpha)$ denote the new constant given by

$$\tilde{l}(n, \alpha) = \inf_{m \in \mathbb{N}_0 \backslash \{\beta\}} \left\{ \rho_m + \left(\frac{n+\alpha}{2}\right)\left(\frac{n-\alpha-4}{2}\right) \right\}^2 \tag{2.60}$$

which can be calculated in the same manner as $l(n, \alpha)$, see Theorem 2.1.

Lemma 2.9. *Suppose* $n \leq \alpha + 4$, $n - \alpha \equiv 0 \pmod 2$ *and* $\alpha \geq 0$*. Then*

$$\tilde{l}(n, \alpha) = \begin{cases} (\alpha + 1)^2 & \textit{when} \quad n < \alpha + 4, \\ (\alpha + 3)^2 & \textit{when} \quad n = \alpha + 4. \end{cases} \tag{2.61}$$

Proof. By definition

$$\tilde{l}(n, \alpha) = \inf_{m \in \mathbb{N}_0 \backslash \{\beta\}} \left\{ m(m + n - 2) + \left(\frac{n+\alpha}{2}\right)\left(\frac{n-\alpha-4}{2}\right) \right\}^2. \tag{2.62}$$

From the Proof of Theorem 2.1 it follows that $\beta \in \mathbb{N}_0$ is the only local minimum of $\Theta(x, n, \alpha)^2$ (see (2.10)) in $\mathbb{R}_+$. If $\beta \neq 0$, i.e. when $n < \alpha + 4$, then the new minimum will be at either of the integers neighbouring β, hence

$$\tilde{l}(n, \alpha) = \min\{\Theta(\beta - 1, n, \alpha), \Theta(\beta + 1, n, \alpha)\}. \tag{2.63}$$

By calculation

$$\begin{aligned} \Theta(\beta - 1, n, \alpha) &= -(\alpha + 1), & (2.64) \\ \Theta(\beta + 1, n, \alpha) &= (\alpha + 3). & (2.65) \end{aligned}$$

Therefore

$$\tilde{l}(n,\alpha) = (\alpha+1)^2 \tag{2.66}$$

when $n < \alpha + 4$. If $n = \alpha + 4$ then $\beta = 0$ and so $\beta - 1 \notin \mathbb{N}_0$, hence

$$\tilde{l}(\alpha+4,\alpha) = (\alpha+3)^2 \tag{2.67}$$

and the result follows. ■

Theorem 2.7, together with Lemma 2.9 can be used to find a non-trivial Rellich inequality in the problematic case $n \leq \alpha + 4$ and $n - \alpha \equiv 0 \pmod 2$, see Remark 2.8.

Theorem 2.10. *Suppose $n \leq \alpha+4$, $n-\alpha \equiv 0 \pmod 2$ and $\alpha \geq 0$. Let $\mathcal{Y}_{m,n}$ denote the space of n-dimensional spherical harmonics of degree m, where the dimension of $\mathcal{Y}_{m,n}$ is given by*

$$k_{m,n} := \frac{(2m+n-2)(m+n-3)!}{m!(n-2)!}. \tag{2.68}$$

Then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ which satisfies

$$\int_{\mathbb{S}^{n-1}} f(r,\omega)\overline{Y(\omega)}d\omega = 0 \tag{2.69}$$

for all $Y \in \mathcal{Y}_{2+\frac{\alpha-n}{2},n}$, the following inequality holds

$$\int_{\mathbb{R}^n} |\Delta f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \tilde{l}(n,\alpha) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}} \tag{2.70}$$

where $\tilde{l}(n,\alpha)$ is given by Lemma 2.9.

Remark 2.11. *Suppose $\alpha = 0$ and $n = 2$, then $\Lambda_\theta = -\frac{\partial^2}{\partial\theta^2}$, $\rho_m = m^2$, $\beta = 1$*

and $k_{1,2} = 2$. *So*

$$\begin{aligned} \Lambda_\theta Y(\theta) &= Y(\theta) \\ -\frac{\partial^2}{\partial\theta^2} Y(\theta) &= Y(\theta) \\ &\Downarrow \\ Y(\theta) &= e^{i\theta} = \cos\theta + i\sin\theta. \end{aligned} \tag{2.71}$$

Therefore by Theorem 2.10, the Rellich inequality

$$\int_{\mathbb{R}^2} |\Delta f(\mathbf{x})|^2 d\mathbf{x} \geq \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^4} \tag{2.72}$$

is valid for all $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ *which satisfy*

$$\int_{\mathbb{S}^1} f(r,\theta)\cos\theta d\theta = \int_{\mathbb{S}^1} f(r,\theta)\sin\theta d\theta = 0. \tag{2.73}$$

A result already known in the literature.

Proof. Suppose $D = -\Delta$, in which case Λ_ω is the Laplace-Beltrami operator. It is known that ρ_m (see (2.49)) are the eigenvalues of Λ_ω with corresponding multiplicity $k_{m,n}$ and eigenvectors $Y \in \mathcal{Y}_{m,n}$, the n-dimensional spherical harmonics of degree m, see [13]. On account of

$$\int_{\mathbb{S}^{n-1}} f(r,\omega)\overline{Y(\omega)}d\omega = 0 \tag{2.74}$$

for all $Y \in \mathcal{Y}_{2+\frac{\alpha-n}{2},n}$, Theorem 2.7 gives the Rellich inequality

$$\int_{\mathbb{R}^n} |\Delta f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq c \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}} \tag{2.75}$$

where

$$c = \inf_{m\in\mathbb{N}_0\setminus\{2+\frac{\alpha-n}{2}\}} \left\{m(m+n-2) + \left(\frac{n+\alpha}{2}\right)\left(\frac{n-\alpha-4}{2}\right)\right\}^2$$
$$= \tilde{l}(n,\alpha). \tag{2.76}$$

■

By taking $\alpha = 0$ and $n = 4$ in Theorem 2.10, a non-trivial Rellich inequality in four dimensions is found by placing only one extra condition on the function.

Corollary 2.12. *For all* $f \in C_0^\infty(\mathbb{R}^4 \setminus \{0\})$ *and*

$$\int_{\mathbb{S}^3} f(r,\omega)\overline{Y(\omega)}d\omega = 0, \tag{2.77}$$

where $Y \in \mathcal{Y}_{0,4}$*, then*

$$\int_{\mathbb{R}^4} |\Delta f(\mathbf{x})|^2 d\mathbf{x} \geq 9 \int_{\mathbb{R}^4} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^4}. \tag{2.78}$$

When looking at the higher order Rellich inequality of order j, the constant $\gamma(n,\alpha,j)$ was equal to zero when $n \leq \alpha + 4j$ and $n - \alpha \equiv 0 \pmod 2$ (see Theorem 2.4). Using the same argument as we did in the case of the Rellich inequality, a non-trivial higher order Rellich inequality can be found in the problematic cases by restricting the functional class. Looking at the $n \leq \alpha + 4$ case first;

Theorem 2.13. *Suppose* $n \leq \alpha + 4$, $n - \alpha \equiv 0 \pmod 2$ *and* $\alpha \geq 0$. *Then for all* $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ *which satisfies*

$$\int_{\mathbb{S}^{n-1}} f(r,\omega)\overline{Y(\omega)}d\omega = 0 \tag{2.79}$$

for all $Y \in \bigcup_{i=0}^{j-1} \mathcal{Y}_{2+\frac{\alpha+4i-n}{2},n}$, *we have*

$$\int_{\mathbb{R}^n} |\Delta^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \tilde{\gamma}(n,\alpha,j) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4j}} \tag{2.80}$$

where

$$\tilde{\gamma}(n,\alpha,j) = \prod_{i=0}^{j-1} \tilde{l}(n,\alpha+4i). \tag{2.81}$$

The space $\mathcal{Y}_{m,n}$ *is defined as in Theorem 2.10.*

Proof. For $j = 1$, Theorem 2.13 is precisely Theorem 2.10. Assume (2.80) is true for $j-1$, then

$$\begin{aligned}\int_{\mathbb{R}^n} |\Delta^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} &= \int_{\mathbb{R}^n} |\Delta^{j-1}(\Delta f(\mathbf{x}))|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \\ &\geq \prod_{i=0}^{j-2} \tilde{l}(n,\alpha+4i) \int_{\mathbb{R}^n} |\Delta f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4(j-1)}}.\end{aligned} \tag{2.82}$$

Owing to the assumption that

$$\int_{\mathbb{S}^{n-1}} f(r,\omega)\overline{Y(\omega)}d\omega = 0 \tag{2.83}$$

for all $Y \in \mathcal{Y}_{2+\frac{\alpha+4(j-1)-n}{2},n}$, Theorem 2.10 can be applied and it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} |\Delta^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq& \tilde{l}(n,\alpha+4(j-1)) \\ &\cdot \prod_{i=0}^{j-2} \tilde{l}(n,\alpha+4i) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4j}}\end{aligned} \tag{2.84}$$

and the result then follows by induction. ∎

The case $\alpha+4 < n \leq \alpha+4j$ is constructed using a combination of Theorems 2.4 and 2.13.

Theorem 2.14. *Let $\alpha+4<n\leq\alpha+4j$, $n-\alpha\equiv 0(mod2)$, $\alpha\geq 0$ and*

$$b=\begin{cases}\frac{n-\alpha}{4}-1 & \textit{if} \quad n-\alpha\equiv 0 \pmod 4,\\ \frac{n-\alpha}{4}-\frac{1}{2} & \textit{if} \quad n-\alpha\equiv 2 \pmod 4.\end{cases} \tag{2.85}$$

Then for all $f\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$ which satisfies

$$\int_{\mathbb{S}^{n-1}} f(r,\omega)\overline{Y(\omega)}d\omega=0 \tag{2.86}$$

for all $Y\in\bigcup_{i=b}^{j-1}\mathcal{Y}_{2+\frac{\alpha+4i-n}{2},n}$, we have

$$\int_{\mathbb{R}^n}|\Delta^j f(\mathbf{x})|^2\frac{d\mathbf{x}}{|\mathbf{x}|^\alpha}\geq\tilde{\gamma}(n,\alpha,j)\int_{\mathbb{R}^n}|f(\mathbf{x})|^2\frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4j}} \tag{2.87}$$

where

$$\tilde{\gamma}(n,\alpha,j)=\prod_{i=0}^{b-1}\frac{(n+\alpha+4i)^2(n-\alpha-4(i+1))^2}{16}\cdot\prod_{i=b}^{j-1}\tilde{l}(n,\alpha+4i). \tag{2.88}$$

The space $\mathcal{Y}_{m,n}$ is defined as in Theorem 2.10.

Proof. Identically to (2.41)-(2.42), there exists $b\in\{1,2,...,j-1\}$ such that

$$\alpha+4b<n\leq\alpha+4(b+1), \tag{2.89}$$

where

$$b=\begin{cases}\frac{n-\alpha}{4}-1 & \text{if } n-\alpha\equiv 0 \pmod 4,\\ \lfloor\frac{n-\alpha}{4}\rfloor=\frac{n-\alpha}{4}-\frac{1}{2} & \text{if } n-\alpha\equiv 2 \pmod 4.\end{cases} \tag{2.90}$$

When $i\in\{0,1,...,b-1\}$ then $n>\alpha+4(i+1)$ and so Theorem 2.4 can be applied,

$$\int_{\mathbb{R}^n} |\Delta^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \prod_{i=0}^{b-1} \frac{(n+\alpha+4i)^2(n-\alpha-4(i+1))^2}{16} \int_{\mathbb{R}^n} |\Delta^{j-b} f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4b}}. \tag{2.91}$$

When $i \in \{b, b+1, ..., j\}$ then $n \leq \alpha + 4(i+1)$. Since

$$\int_{\mathbb{S}^{n-1}} f(r,\omega)\overline{Y(\omega)}d\omega = 0 \tag{2.92}$$

for all $Y \in \bigcup_{i=b}^{j-1} \mathcal{Y}_{2+\frac{\alpha+4i-n}{2},n}$, the assumptions of Theorem 2.13 are satisfied and applying it to (2.91) gives

$$\int_{\mathbb{R}^n} |\Delta^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \tilde{\gamma}(n,\alpha,j) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4j}} \tag{2.93}$$

where

$$\begin{aligned} \tilde{\gamma}(n,\alpha,j) &= \prod_{i=0}^{b-1} \frac{(n+\alpha+4i)^2(n-\alpha-4(i+1))^2}{16} \\ &\quad \cdot \prod_{i=0}^{j-b-1} \tilde{l}(n, \alpha+4b+4i) \\ &= \prod_{i=0}^{b-1} \frac{(n+\alpha+4i)^2(n-\alpha-4(i+1))^2}{16} \cdot \prod_{i=b}^{j-1} \tilde{l}(n,\alpha+4i). \end{aligned} \tag{2.94}$$

■

Chapter 3

A Rellich type inequality with magnetic potentials

According to Theorem 2.4, the higher order Rellich inequality

$$\int_{\mathbb{R}^n} |\Delta^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \gamma(n,\alpha,j) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4j}} \tag{3.1}$$

becomes trivial ($\gamma(n,\alpha,j) = 0$) for all $n - \alpha \equiv 0 \pmod 2$ and $n \leq \alpha + 4j$. Each time the Rellich inequality is applied to $\Delta^i f$ (taking it to $\Delta^{i-1} f$) the power of the modulus of $\mathbf{x}$ in the denominator is increased by 4. Let us take for the moment the ordinary Rellich inequality ($j = 1$); the trivial cases occur when $n \leq \alpha + 4$ and $n - \alpha \equiv 0 \pmod 2$ and it can be seen that as α (the power of $|\mathbf{x}|$) increases, the number of dimensions in which the Rellich inequality is trivial grows. This illustrates that the bound $n \leq \alpha + 4j$ is due to the iterative nature of the Rellich inequality. Hence the key to finding a higher order Rellich type inequality for all $n \leq \alpha + 4j$ lies in finding a Rellich type inequality for all $n \leq \alpha + 4$.

In Theorem 2.10, a Rellich inequality in $L^2(\mathbb{R}^n)$ was found in the problem case, $n \leq \alpha + 4$ and $n - \alpha \equiv 0 \pmod 2$ by placing extra assumptions on the function f on $\mathbb{S}^{n-1}$. This in turn produced a higher order Rellich inequality for all $n \leq \alpha + 4j$ and $n - \alpha \equiv 0 \pmod 2$. An alternative solution is to replace the Laplacian Δ with a so-called magnetic Laplacian $\Delta_{\mathbf{A}}$. The idea was first utilised by Laptev and Weidl in [20] who, by replacing the gradient ∇ with a magnetic gradient $\nabla_{\mathbf{A}} = \nabla - i\mathbf{A}$, found a Hardy type inequality in $L^2(\mathbb{R}^2)$, a case in which the Hardy inequality is trivial. The magnetic potential $\mathbf{A}$ is any non-trivial $\mathbf{A} \in C(\mathbb{R}^2 \setminus \{0\})$ with the condition that $curl\mathbf{A} \in L^1_{loc}(\mathbb{R}^2 \setminus \{0\})$ e.g.

$$\mathbf{A}(r,\theta) = \frac{\Psi(\theta)}{r}(-\sin\theta, \cos\theta), \qquad \Psi \in L^\infty(0, 2\pi), \qquad \Psi(0) = \Psi(2\pi), \tag{3.2}$$

a magnetic potential of Aharonov-Bohm type. The Laptev-Weidl inequality is valid for all $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, so no additional assumptions on the function are required.

Analogously to the Laptev-Weidl inequality, Evans and Lewis in [15] found a Rellich type inequality with a magnetic potential in $L^2(\mathbb{R}^n)$ for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ when $n \in \{2,3,4\}$. Here the Laplacian was replaced by a magnetic Laplacian $\Delta_{\mathbf{A}} = \nabla_{\mathbf{A}}^2$, where the potential $\mathbf{A}$ is again of Aharonov-Bohm type. As explained above, to find a non-trivial higher order Rellich type inequality for all $n \leq \alpha + 4j$ and $n - \alpha \equiv 0 \pmod 2$, a Rellich type inequality is needed for all $n \leq \alpha + 4$ and $n - \alpha \equiv 0 \pmod 2$. With this in mind, the inequality of Evans and Lewis is extended to all dimensions n in the proceeding chapter. Firstly a magnetic Laplacian is constructed.

3.1 The magnetic Laplacian

For arbitrary $n \geq 3$, the following spherical coordinates are introduced

$$\begin{aligned} x_1 &= r\cos\theta_1, && (3.3)\\ x_j &= r\cos\theta_j \prod_{k=1}^{j-1} \sin\theta_k, \qquad j \in \{2,3,...,n-1\}, && (3.4)\\ x_n &= r\prod_{k=1}^{n-1} \sin\theta_k. && (3.5)\end{aligned}$$

Define the orthonormal vectors

$$\mathbf{e_0} := \Big(\cos\theta_1, \cos\theta_2\sin\theta_1, ..., \cos\theta_{n-1}\prod_{k=1}^{n-2}\sin\theta_k, \prod_{k=1}^{n-1}\sin\theta_k\Big), \tag{3.6}$$

$$\begin{aligned}\mathbf{e_j} := \Big(\underbrace{0,...,0}_{j-1}, &-\sin\theta_j, \cos\theta_{j+1}\cos\theta_j, \cos\theta_{j+2}\cos\theta_j\sin\theta_{j+1}, \ldots\\ &\ldots, \cos\theta_{n-1}\cos\theta_j \prod_{k=1,k\neq j}^{n-2}\sin\theta_k, \cos\theta_j \prod_{k=1,k\neq j}^{n-1}\sin\theta_k\Big)\end{aligned} \tag{3.7}$$

for all $j \in \{1, ..., n-2\}$ and

$$\mathbf{e_{n-1}} := \Big(\underbrace{0, ..., 0}_{n-2}, -\sin\theta_{n-1}, \cos\theta_{n-1}\Big). \tag{3.8}$$

Then it can be shown that the gradient in spherical coordinates is

$$\nabla = \mathbf{e_0}\frac{\partial}{\partial r} + \frac{1}{r}\mathbf{e_1}\frac{\partial}{\partial\theta_1} + \sum_{j=2}^{n-1}\frac{1}{r\prod_{k=1}^{j-1}\sin\theta_k}\mathbf{e_j}\frac{\partial}{\partial\theta_j}. \tag{3.9}$$

Suppose $\Psi \in C^\infty(\mathbb{S})$ (or equivalently, $\Psi \in C^\infty(0, 2\pi)$ and $\Psi^{(i)}(0) = \Psi^{(i)}(2\pi)$ for all $i \in \mathbb{N}_0$), then the magnetic potential $\mathbf{A}$ is taken to be the 1-form

$$\mathbf{A} := \begin{cases} \frac{1}{r}\Psi(\theta_1)\mathbf{e}_1 & \text{if } n = 2, \\ \frac{1}{r\prod_{k=1}^{n-2}\sin\theta_k}\Psi(\theta_{n-1})\mathbf{e}_{n-1} & \text{if } n \geq 3, \end{cases} \tag{3.10}$$

defined on $\mathbb{R} \setminus \mathcal{L}_n$ where

$$\mathcal{L}_n := \begin{cases} \{0\} & \text{if } n = 2, \\ \big\{\mathbf{x} = (r, \theta_1, ..., \theta_{n-1}\} : r\prod_{k=1}^{n-2}\sin\theta_k = 0\big\} & \text{if } n \geq 3. \end{cases} \tag{3.11}$$

Define the magnetic gradient as

$$\nabla_{\mathbf{A}} \;:=\; \nabla - i\mathbf{A} \tag{3.12}$$

and the magnetic Laplacian to be $\Delta_{\mathbf{A}} = \nabla_{\mathbf{A}}^2$, then (3.12) yields

$$\begin{aligned} \Delta_{\mathbf{A}} f &= (\nabla - i\mathbf{A})(\nabla f - i\mathbf{A}f) \\ &= \nabla(\nabla f) - i\nabla(\mathbf{A}f) - i\mathbf{A}.\nabla f - \mathbf{A}.\mathbf{A}f \\ &= \Delta f - i(\nabla\mathbf{A})f - 2i\mathbf{A}.\nabla f - \mathbf{A}.\mathbf{A}f. \end{aligned} \tag{3.13}$$

Now

$$
\begin{aligned}
\nabla \mathbf{A} =& \Big(\mathbf{e_0}\frac{\partial}{\partial r} + \frac{1}{r}\mathbf{e_1}\frac{\partial}{\partial \theta_1} + \sum_{j=2}^{n-1} \frac{1}{r\prod_{k=1}^{j-1}\sin\theta_k}\mathbf{e_j}\frac{\partial}{\partial \theta_j}\Big) \\
&.\frac{1}{r\prod_{k=1}^{n-2}\sin\theta_k}\Psi(\theta_{n-1})\mathbf{e}_{n-1} \\
=&\frac{1}{r^2\prod_{k=1}^{n-2}\sin^2\theta_k}\mathbf{e_{n-1}}\frac{d}{d\theta_{n-1}}\Big(\Psi(\theta_{n-1})\mathbf{e}_{n-1}\Big) \\
=&\frac{1}{r^2\prod_{k=1}^{n-2}\sin^2\theta_k}\Psi'(\theta_{n-1}) \qquad (3.14)
\end{aligned}
$$

since $\frac{d\mathbf{e_{n-1}}}{d\theta_{n-1}}.\mathbf{e_{n-1}} = 0$. Also

$$
\begin{aligned}
\mathbf{A}.\nabla f =& \frac{1}{r\prod_{k=1}^{n-2}\sin\theta_k}\Psi(\theta_{n-1})\mathbf{e}_{n-1}.\Big(\mathbf{e_0}\frac{\partial}{\partial r} + \frac{1}{r}\mathbf{e_1}\frac{\partial}{\partial \theta_1} \\
&+ \sum_{j=2}^{n-1} \frac{1}{r\prod_{k=1}^{j-1}\sin\theta_k}\mathbf{e_j}\frac{\partial f}{\partial \theta_j}\Big) \\
=&\frac{\Psi(\theta_{n-1})}{r^2\prod_{k=1}^{n-2}\sin^2\theta_k}\frac{\partial f}{\partial \theta_{n-1}} \qquad (3.15)
\end{aligned}
$$

and

$$
\mathbf{A}.\mathbf{A} = \frac{\Psi(\theta_{n-1})^2}{r^2\prod_{k=1}^{n-2}\sin^2\theta_k}. \qquad (3.16)
$$

Now by Egorov and Shubin [13], the Lapacian in spherical coordinates is

$$
\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \sum_{j=1}^{n-1}\frac{1}{r^2 q_j}\Big[(n-j-1)\cot\theta_j\frac{\partial}{\partial \theta_j} + \frac{\partial^2}{\partial \theta_j^2}\Big] \qquad (3.17)
$$

where

$$
q_j = \begin{cases} 1 & \text{if } \ j = 1, \\ \prod_{k=1}^{j-1}\sin\theta_k & \text{if } \ j \geq 2. \end{cases} \qquad (3.18)
$$

Hence

$$
\begin{aligned}
-\Delta_{\mathbf{A}} &= -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} - \sum_{j=1}^{n-2}\frac{1}{r^2 q_j}\Big[(n-j-1)\cot\theta_j\frac{\partial}{\partial\theta_j} + \frac{\partial^2}{\partial\theta_j^2}\Big] \\
&+ \frac{1}{r^2 q_{n-1}}\Big(-\frac{\partial^2}{\partial\theta_{n-1}^2} + i\Psi'(\theta_{n-1}) + 2i\Psi(\theta_{n-1})\frac{\partial}{\partial\theta_j} + \Psi(\theta_{n-1})^2\Big) \\
&= -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} - \sum_{j=1}^{n-2}\frac{1}{r^2 q_j}\Big[(n-j-1)\cot\theta_j\frac{\partial}{\partial\theta_j} + \frac{\partial^2}{\partial\theta_j^2}\Big] \\
&+ \frac{1}{r^2 q_{n-1}}\Big(i\frac{\partial}{\partial\theta_{n-1}} + \Psi(\theta_{n-1})\Big)^2. \qquad (3.19)
\end{aligned}
$$

If n=2, simply use the negative magnetic Laplacian defined by Evans and Lewis in [15],

$$
-\Delta_{\mathbf{A}} = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Big(i\frac{\partial}{\partial\theta_1} + \Psi(\theta_1)\Big)^2. \qquad (3.20)
$$

Denote by $\tilde{\Psi}$ the mean value of the function Ψ over $\mathbb{S}$ (the magnetic flux) i.e.

$$
\tilde{\Psi} := \frac{1}{2\pi}\int_0^{2\pi}\Psi(\theta)d\theta. \qquad (3.21)
$$

Remark 3.1. *Consider the magnetic potentials,*

$$
\mathbf{A_0} := \begin{cases} \frac{1}{r}\Psi(\theta_1)\mathbf{e}_1 & \textit{if } n=2, \\ \frac{1}{r\prod_{k=1}^{n-2}\sin\theta_k}\Psi(\theta_{n-1})\mathbf{e}_{n-1} & \textit{if } n\geq 3, \end{cases} \qquad (3.22)
$$

$$
\mathbf{A_1} := \begin{cases} \frac{1}{r}\tilde{\Psi}\mathbf{e}_1 & \textit{if } n=2, \\ \frac{1}{r\prod_{k=1}^{n-2}\sin\theta_k}\tilde{\Psi}\mathbf{e}_{n-1} & \textit{if } n\geq 3, \end{cases} \qquad (3.23)
$$

$$
\mathbf{A_2} := \begin{cases} \frac{1}{r}(\tilde{\Psi}+m)\mathbf{e}_1 & \textit{if } n=2, \\ \frac{1}{r\prod_{k=1}^{n-2}\sin\theta_k}(\tilde{\Psi}+m)\mathbf{e}_{n-1} & \textit{if } n\geq 3, \end{cases} \qquad (3.24)
$$

where $m \in \mathbb{Z}$. It's an easy exercise to show that $\mathbf{A_i}$ is of the form ∇P_i for *some potential function P_i and $i \in \{0,1,2\}$. For instance, in two dimensions,*

$$\nabla P_0 = \frac{\partial P_0}{\partial r}\mathbf{e}_0 + \frac{1}{r}\frac{\partial P_0}{\partial \theta_1}\mathbf{e}_1 = \mathbf{A_0} = \frac{1}{r}\Psi(\theta_1)\mathbf{e}_1$$

$$\Downarrow$$

$$\frac{\partial P_0}{\partial r} = 0$$

$$\frac{\partial P_0}{\partial \theta_1} = \Psi(\theta_1)$$

$$\Downarrow$$

$$P_0(\theta_1) = \int_0^{\theta_1} \Psi(t)dt + constant \tag{3.25}$$

and so $P_0(2\pi) - P_0(0) = 2\pi\tilde{\Psi}$. It follows that for $i,j \in \{0,1,2\}$ and $i \neq j$ that

$$\mathbf{A_i} = \mathbf{A_j} + \nabla P_g \tag{3.26}$$

where $P_g = P_i - P_j \in C^\infty(\mathbb{S})$ is called the gauge function. Then $U : \psi(\mathbf{x}) \longmapsto e^{-iP_g(\theta_{n-1})}\psi(\mathbf{x})$ is a unitary transformation on $L^2(\mathbb{R}^n)$ and

$$U(-\Delta_{\mathbf{A_i}})U^{-1}\psi = -\Delta_{\mathbf{A_j}}\psi. \tag{3.27}$$

Therefore $-\Delta_{\mathbf{A_0}}$, $-\Delta_{\mathbf{A_1}}$ and $-\Delta_{\mathbf{A_2}}$ are unitarily (gauge) equivalent. This leads to the consequence that we can assume w.l.o.g that $\Psi = \tilde{\Psi} \in [0,1)$ is a constant. Furthermore, it follows that if $\tilde{\Psi} = 0$ (or equivalently $\tilde{\Psi} \in \mathbb{Z}$), then $-\Delta_{\mathbf{A}}$ is gauge equivalent to $-\Delta$.

In order to apply Theorem 2.6 with $D = -\Delta_{\mathbf{A}}$, the discrete eigenvalues of the angular part of the negative magnetic Laplacian need to be found.

Theorem 3.2. *Suppose for $j \in \{2,\ldots,n\}$, the following iterative operators* *are defined;*

$$\Lambda_{j,\omega}(\theta_{n+1-j},\ldots,\theta_{n-1}) := \begin{cases} \left(i\frac{\partial}{\partial\theta_{n-1}} + \Psi(\theta_{n-1})\right)^2 & \textit{if } j=2, \\ -\frac{\partial^2}{\partial\theta_{n+1-j}^2} - (j-2)\cot\theta_{n+1-j}\frac{\partial}{\partial\theta_{n+1-j}} & \\ \quad +\frac{1}{\sin\theta_{n+1-j}}\tilde{\Lambda}_{j-1,\omega}(\theta_{n+2-j},\ldots,\theta_{n-1}) & \textit{if } j\geq 3 \end{cases} \tag{3.28}$$

where $\tilde{\Lambda}_{n-1,\omega}$ is the Friedrichs extension of the non-negative symmetric operator $\Lambda_{n-1,\omega}$ in $L^2(\mathbb{S}^{n-2})$. Then the Friedrichs extension of $\Lambda_{n,\omega}$ in $L^2(\mathbb{S}^{n-1})$ has a discrete spectrum consisting of eigenvalues

$$\rho_m^\Psi = (m+\tilde{\Psi})(m+\tilde{\Psi}+n-2) \tag{3.29}$$

where

$$m \in \hat{\mathbb{Z}}^{(n)} := \{m \in \mathbb{Z} : m \leq 2-n-\tilde{\Psi} \textit{ or } m \geq -\tilde{\Psi}\}. \tag{3.30}$$

Remark 3.3. *To show the motivation behind Theorem 3.2, we recall that*

$$-\Delta_{\mathbf{A}} = L_r - \frac{1}{r}\Lambda_{n,\omega}. \tag{3.31}$$

Remark 3.4. *The operator $\Lambda_{n,\omega}$ can be seen to be a "magnetic Laplace-Beltrami operator" because if $\tilde{\Psi} \in \mathbb{Z}$, then the operator $\Lambda_{n,\omega}$ is gauge invariant to the n-dimensional Laplace-Beltrami operator and as expected the above Theorem gives the exact expression to find the positive eigenvalues (denoted by ρ_m - see (2.49)) of this operator.*

Remark 3.5. *The eigenvalues ρ_m^Ψ have already been calculated by Evans and Lewis in [15] for $n=2,3,4$. The two dimensional case*

$$\rho_m^\Psi = (\Psi+m)^2 \tag{3.32}$$

is especially crucial as it will be shown in the proof of Theorem 3.2 that all *the higher dimensions reduce to the two dimensional case.*

Before the Theorem is proven, the following preliminaries are needed.

Lemma 3.6. *Suppose $n \in \mathbb{N} \setminus \{1,2\}$. The associated Legendre equation*

$$\frac{d^2u}{d\theta_1^2} + (n-2)\cot\theta_1 \frac{du}{d\theta_1} + \Big(\lambda - \frac{\mu}{\sin^2\theta_1}\Big)u = 0 \tag{3.33}$$

can be reduced to

$$\frac{d^2w}{d\theta_1^2} + \cot\theta_1 \frac{dw}{d\theta_1} + \Big(\lambda + \frac{(n-3)(n-1)}{4} - \frac{\mu + \frac{(n-3)^2}{4}}{\sin^2\theta_1}\Big)w = 0 \tag{3.34}$$

by using the transformation, $u = \sin^{-\frac{n-3}{2}}\theta_1 w$.

Proof. Let $u = \sin^{-\frac{n-3}{2}}\theta_1 w$, then

$$\frac{du}{d\theta_1} = \sin^{-\frac{n-3}{2}}\theta_1 \frac{dw}{d\theta_1} - \frac{n-3}{2}\sin^{-\frac{n-3}{2}}\theta_1 \cot\theta_1 w, \tag{3.35}$$

$$\begin{aligned}\frac{d^2u}{d\theta_1^2} =& \sin^{-\frac{n-3}{2}}\theta_1 \frac{d^2w}{d\theta_1^2} - (n-3)\sin^{-\frac{n-3}{2}}\theta_1\cot\theta_1\frac{dw}{d\theta_1} \\ &+ \frac{(n-3)(n-1)}{4}\sin^{-\frac{n-3}{2}}\theta_1\cot^2\theta_1 w + \frac{n-3}{2}\sin^{-\frac{n-3}{2}}\theta_1 w.\end{aligned} \tag{3.36}$$

Then (3.33) becomes

$$\begin{aligned}\sin^{-\frac{n-3}{2}}\theta_1\Big[\frac{d^2w}{d\theta_1^2} + \cot\theta_1\frac{dw}{d\theta_1} - \frac{(n-3)^2}{4}\cot^2\theta_1 w \\ + \Big(\lambda + \frac{n-3}{2} - \frac{\mu}{\sin^2\theta_1}\Big)w\Big] = 0.\end{aligned} \tag{3.37}$$

Multiplying by $\sin^{\frac{n-3}{2}}\theta_1$ and noting that $\cot^2\theta_1 = \frac{1}{\sin^2\theta_1} - 1$, it follows that

$$\frac{d^2w}{d\theta_1^2} + \cot\theta_1\frac{dw}{d\theta_1} + \left(\lambda + \frac{(n-3)^2}{4} + \frac{n-3}{2} - \frac{\mu + \frac{(n-3)^2}{4}}{\sin^2\theta_1}\right)w = 0. \tag{3.38}$$

■

Lemma 3.7. *Let $n \in \mathbb{N}\setminus\{1,2\}$; given that the following iterative relation is true for all $i \in \{3, ..., n\}$*

$$\xi_i := \left(p_i - \left\{\xi_{i-1} + \frac{(i-3)^2}{4}\right\}^{\frac{1}{2}} + \frac{1}{2}\right)^2 - \frac{(i-2)^2}{4} \tag{3.39}$$

where $p_i \in \mathbb{Z}$, then

$$\xi_n = \begin{cases} (m_n - \xi_2^{\frac{1}{2}})^2 - \frac{(n-2)^2}{4} & \text{if } n \text{ is even,} \\ (m_n - \xi_2^{\frac{1}{2}} + \frac{1}{2})^2 - \frac{(n-2)^2}{4} & \text{if } n \text{ is odd.} \end{cases} \tag{3.40}$$

where $m_n \in \mathbb{Z}$.

Proof. Clearly (3.40) is true when $n = 3$ by putting $i = 3$ into (3.39)

$$\xi_3 = \left(p_3 - \xi_2^{\frac{1}{2}} + \frac{1}{2}\right)^2 - \frac{1}{4}. \tag{3.41}$$

Again by (3.39),

$$\xi_4 = \left(p_4 - \left\{\xi_3 + \frac{1}{4}\right\}^{\frac{1}{2}} + \frac{1}{2}\right)^2 - 1. \tag{3.42}$$

Substituting (3.41) into (3.42) gives

$$\begin{aligned} \xi_4 &= \left(p_4 - \left\{\left(p_3 - \xi_2^{\frac{1}{2}} + \frac{1}{2}\right)^2\right\}^{\frac{1}{2}} + \frac{1}{2}\right)^2 - 1 \\ &= \left(p_4 - \left|p_3 - \xi_2^{\frac{1}{2}} + \frac{1}{2}\right| + \frac{1}{2}\right)^2 - 1 \\ &= \begin{cases} (p_4 - p_3 + \xi_2^{\frac{1}{2}})^2 - 1 & \text{if } p_3 - \xi_2^{\frac{1}{2}} + \frac{1}{2} \geq 0, \\ (p_4 + p_3 + 1 - \xi_2^{\frac{1}{2}})^2 - 1 & \text{if } p_3 - \xi_2^{\frac{1}{2}} + \frac{1}{2} < 0. \end{cases} \end{aligned} \tag{3.43}$$

Since $p_3, p_4 \in \mathbb{Z}$, both cases can be written as

$$\xi_4 = (m_4 - \xi_2^{\frac{1}{2}})^2 - 1 \tag{3.44}$$

with $m_4 \in \mathbb{Z}$ and so (3.40) is true for $n = 4$. Again by (3.39),

$$\xi_l = \left(p_l - \left\{\xi_{l-1} + \frac{(l-3)^2}{4}\right\}^{\frac{1}{2}} + \frac{1}{2}\right)^2 - \frac{(l-2)^2}{4}. \tag{3.45}$$

Firstly assume (3.40) is true for $n = l - 1$ where l is even, then

$$\xi_{l-1} = \left(m_{l-1} - \xi_2^{\frac{1}{2}} + \frac{1}{2}\right)^2 - \frac{(l-3)^2}{4}. \tag{3.46}$$

Substituting this into (3.45) gives

$$\begin{aligned}
\xi_l =& \left(p_l - \left\{\left(m_{l-1} - \xi_2^{\frac{1}{2}} + \frac{1}{2}\right)^2\right\}^{\frac{1}{2}} + \frac{1}{2}\right)^2 - \frac{(l-2)^2}{4} \\
=& \left(p_l - \left|m_{l-1} - \xi_2^{\frac{1}{2}} + \frac{1}{2}\right| + \frac{1}{2}\right)^2 - \frac{(l-2)^2}{4} \\
=& \begin{cases} (p_l - m_{l-1} + \xi_2^{\frac{1}{2}})^2 - \frac{(l-2)^2}{4} & \text{if } m_{l-1} - \xi_2^{\frac{1}{2}} + \frac{1}{2} \geq 0, \\ (p_l + m_{l-1} + 1 - \xi_2^{\frac{1}{2}})^2 - \frac{(l-2)^2}{4} & \text{if } m_{l-1} - \xi_2^{\frac{1}{2}} + \frac{1}{2} < 0. \end{cases}
\end{aligned} \tag{3.47}$$

Since $p_l, m_{l-1} \in \mathbb{Z}$, both cases can be generalised as

$$\xi_l = (m_l - \xi_2^{\frac{1}{2}})^2 - \frac{(l-2)^2}{4} \tag{3.48}$$

with $m_l \in \mathbb{Z}$. Secondly, assuming again that (3.40) is true for $n = l - 1$ but in this case l is odd, then

$$\xi_{l-1} = (m_{l-1} - \xi_2^{\frac{1}{2}})^2 - \frac{(l-3)^2}{4}. \tag{3.49}$$

Substituting this into (3.45) gives

$$\begin{aligned}
\xi_l =& \Big(p_l - \Big\{\Big(m_{l-1} - \xi_2^{\frac{1}{2}}\Big)^2\Big\}^{\frac{1}{2}} + \frac{1}{2}\Big)^2 - \frac{(l-2)^2}{4} \\
=& \Big(p_l - |m_{l-1} - \xi_2^{\frac{1}{2}}| + \frac{1}{2}\Big)^2 - \frac{(l-2)^2}{4} \\
=& \begin{cases} (p_l - m_{l-1} + \xi_2^{\frac{1}{2}} + \frac{1}{2})^2 - \frac{(l-2)^2}{4} & \text{if } m_{l-1} - \xi_2^{\frac{1}{2}} + \frac{1}{2} \geq 0, \\ (p_l + m_{l-1} - \xi_2^{\frac{1}{2}} + \frac{1}{2})^2 - \frac{(l-2)^2}{4} & \text{if } m_{l-1} - \xi_2^{\frac{1}{2}} + \frac{1}{2} < 0. \end{cases}
\end{aligned} \tag{3.50}$$

Since $p_l, m_{l-1} \in \mathbb{Z}$ both cases can be generalised as

$$\xi_l = \Big(m_l - \xi_2^{\frac{1}{2}} + \frac{1}{2}\Big)^2 - \frac{(l-2)^2}{4} \tag{3.51}$$

with $m_l \in \mathbb{Z}$. Therefore for all $l \geq 4$, if (3.40) is true for $n = l-1$ then it is true for $n = l$ and the result then follows by induction. ■

3.2 Proof of Theorem 3.2

Take $n \geq 3$, $q \in \{3, ..., n\}$ and denote by $v_{q,j_q}(\theta_{n+1-q}, ..., \theta_{n-1})$ and ξ_{q,j_q}, the eigenvectors and corresponding eigenvalues, respectively, of the operator $\tilde{\Lambda}_{q,\omega}(\theta_{n+1-q}, ..., \theta_{n-1})$. Now since $\tilde{\Lambda}_{q-1,\omega}(\theta_{n+2-q}, ..., \theta_{n-1})$ is a self-adjoint operator, $v_{q-1,j_{q-1}}$ form an orthonormal basis of $L^2(\mathbb{S}^{q-2})$. Therefore, on identifying $L^2(\mathbb{S}^{q-1})$ with $\bigoplus_{j_{q-1}}\{L^2(0,\pi); \sin^{q-2}\theta_1\} \bigotimes \{v_{q-1,j_{q-1}}\}$, we have

$$\begin{aligned}
\Lambda_{q,\omega}(\theta_{n+1-q}, ..., \theta_{n-1}) =& -\frac{\partial^2}{\partial\theta_{n+1-q}^2} - (q-2)\cot\theta_{n+1-q}\frac{\partial}{\partial\theta_{n+1-q}} \\
&+ \frac{1}{\sin\theta_{n+1-q}}\tilde{\Lambda}_{q-1,\omega}(\theta_{n+2-q}, \ldots, \theta_{n-1}) \\
=& \bigoplus_{j_{q-1}}\Big\{\Upsilon_{q,j_{q-1}}(\theta_1) \bigotimes I_{q-1,j_{q-1}}\Big\}
\end{aligned} \tag{3.52}$$

where $I_{q-1,j_{q-1}}$ is the identity on $\{v_{q-1,j_{q-1}}\}$ and $\Upsilon_{q,j_{q-1}}$ is an operator defined on $C_0^\infty(0,\pi)$ by

$$\Upsilon_{q,j_{q-1}}(\theta_1) = -\frac{\partial^2}{\partial\theta_1^2} - (q-2)\cot\theta_1\frac{\partial}{\partial\theta_1} + \frac{\xi_{q-1,j_{q-1}}}{\sin^2\theta_1}. \tag{3.53}$$

Setting $\Upsilon_{q,j_{q-1}}(\theta_1)u = \lambda u$ gives

$$\frac{\partial^2 u}{\partial\theta_1^2} + (q-2)\cot\theta_1\frac{\partial u}{\partial\theta_1} + \Big(\lambda - \frac{\xi_{q-1,j_{q-1}}}{\sin^2\theta_1}\Big)u = 0. \tag{3.54}$$

By Lemma 3.6, this can be reduced to

$$\frac{\partial^2 w}{\partial\theta_1^2} + \cot\theta_1\frac{\partial w}{\partial\theta_1} + \Big(\lambda + \frac{(q-2)^2-1}{4} - \frac{\xi_{q-1,j_{q-1}} + \frac{(q-3)^2}{4}}{\sin^2\theta_1}\Big)w = 0. \tag{3.55}$$

Now in [15], Evans and Lewis found that the Friedrichs extension of the operator

$$-\frac{\partial^2}{\partial\theta_1^2} - \cot\theta_1\frac{\partial}{\partial\theta_1} + \frac{\mu^2}{\sin^2\theta_1} \tag{3.56}$$

has a discrete spectrum consisting of eigenvalues

$$\nu_j(\mu) = (j-\mu)(j+1-\mu), \qquad j \in \{k \in \mathbb{Z} : \nu_k(\mu) \geq 0\}. \tag{3.57}$$

So replacing μ^2 with $\xi_{q-1,j_{q-1}} + \frac{(q-3)^2}{4}$ and subtracting $\frac{(q-2)^2-1}{4}$ gives the eigenvalues of the Friedrichs extension of $\Upsilon_{q,j_{q-1}}(\theta_1)$ to be

$$\begin{aligned}\xi_{q,j_q} =& \Big(j_q - \Big[\xi_{q-1,j_{q-1}} + \frac{(q-3)^2}{4}\Big]^{\frac{1}{2}}\Big)\Big(j_q - \Big[\xi_{q-1,j_{q-1}} + \frac{(q-3)^2}{4}\Big]^{\frac{1}{2}} + 1\Big) \\ &- \frac{(q-2)^2-1}{4} \\ =& \Big(j_q - \Big\{\xi_{q-1,j_{q-1}} + \frac{(q-3)^2}{4}\Big\}^{\frac{1}{2}} + \frac{1}{2}\Big)^2 - \frac{(q-2)^2}{4}\end{aligned} \tag{3.58}$$

where $j_q \in \{k \in \mathbb{Z} : \xi_{q,k} \geq 0\}$. Equation (3.58) is true for all $q \in \{3, ..., n\}$, so by Lemma 3.7

$$\xi_{n,j_n} = \begin{cases} (j_n - \xi_{2,j_2}^{\frac{1}{2}})^2 - \frac{(n-2)^2}{4} & \text{if } n \text{ is even,} \\ (j_n - \xi_{2,j_2}^{\frac{1}{2}} + \frac{1}{2})^2 - \frac{(n-2)^2}{4} & \text{if } n \text{ is odd,} \end{cases} \tag{3.59}$$

with $j_n \in \{k \in \mathbb{Z} : \xi_{n,k} \geq 0\}$. Note that ξ_{2,j_2} are the eigenvalues of the operator $\Lambda_{2,\omega}(\theta_1)$, which from [15] are known to be $\xi_{2,j_2} = (j_2 + \tilde{\Psi})^2$. Hence if n is even

$$\begin{aligned} \xi_{n,j_n} =& (j_n + |j_2 + \tilde{\Psi}|)^2 - \frac{(n-2)^2}{4} \\ =& \begin{cases} (j_n + j_2 + \tilde{\Psi})^2 - \frac{(n-2)^2}{4} & \text{if } j_2 + \tilde{\Psi} \geq 0, \\ (-j_n + j_2 + \tilde{\Psi})^2 - \frac{(n-2)^2}{4} & \text{otherwise.} \end{cases} \end{aligned} \tag{3.60}$$

Both of these can be enumerated as the following

$$\hat{\xi}_{n,\hat{j}_n} = (\hat{j}_n + \tilde{\Psi})^2 - \frac{(n-2)^2}{4}, \tag{3.61}$$

with $\hat{j}_n \in \{k \in \mathbb{Z} : \hat{\xi}_{q,k} \geq 0\}$. Since n is an even integer and $\hat{j}_n \in \mathbb{Z}$, setting $\mathbb{Z} \ni m = (\hat{j}_n - \frac{n}{2} + 1)$ gives the eigenvalues of the operator $\tilde{\Lambda}_{n,\omega}$ to be

$$\rho_m^{\Psi} = (m + \tilde{\Psi})(m + \tilde{\Psi} + n - 2), \tag{3.62}$$

with $m \in \hat{\mathbb{Z}}^{(n)} := \{k \in \mathbb{Z} : \rho_m^{\Psi} \geq 0\}$. Similarly if n is odd

$$\begin{aligned} \xi_{n,j_n} =& \left(j_n + |j_2 + \tilde{\Psi}| + \frac{1}{2}\right)^2 - \frac{(n-2)^2}{4} \\ =& \begin{cases} (j_n + j_2 + \tilde{\Psi} + \frac{1}{2})^2 - \frac{(n-2)^2}{4} & \text{if } j_2 + \tilde{\Psi} \geq 0, \\ (-j_n + j_2 + \tilde{\Psi} - \frac{1}{2})^2 - \frac{(n-2)^2}{4} & \text{otherwise.} \end{cases} \end{aligned} \tag{3.63}$$

Both of these can be enumerated as the following

$$\hat{\xi}_{n,\hat{j}_n} = \left(\hat{j}_n + \tilde{\Psi} + \frac{1}{2}\right)^2 - \frac{(n-2)^2}{4}. \tag{3.64}$$

Since n is an odd integer and $\hat{j}_n \in \mathbb{Z}$, setting $\mathbb{Z} \ni m = (\hat{j}_n - \frac{n}{2} + \frac{3}{2})$ again gives (3.62). What remains is to find $\hat{\mathbb{Z}}^{(n)}$. When $\xi_{n,m} \geq 0$ there are two possibilities. Either

$$\begin{array}{ccc} m + \tilde{\Psi} \geq 0 & \text{and} & m + \tilde{\Psi} + n - 2 \geq 0 \\ \Downarrow & & \Downarrow \\ m \geq -\tilde{\Psi} & \text{and} & m \geq 2 - n - \tilde{\Psi} \end{array} \tag{3.65}$$

or

$$\begin{array}{ccc} m + \tilde{\Psi} \leq 0 & \text{and} & m + \tilde{\Psi} + n - 2 \leq 0 \\ \Downarrow & & \Downarrow \\ m \leq -\tilde{\Psi} & \text{and} & m \leq 2 - n - \tilde{\Psi}. \end{array} \tag{3.66}$$

Since $2 - n - \tilde{\Psi} \leq -\tilde{\Psi}$ when $n \geq 2$, then

$$\hat{\mathbb{Z}}^{(n)} = \{m \in \mathbb{Z} : m \leq 2 - n - \tilde{\Psi} \text{ or } m \geq -\tilde{\Psi}\}. \tag{3.67}$$

■

3.3 A Rellich type inequality with a magnetic potential

Theorem 2.6, together with Theorem 3.2, gives a Rellich type inequality with a magnetic potential. We now impose the condition that the flux $\tilde{\Psi} \notin \mathbb{Z}$, or equivalently (by gauge invariance) that $\tilde{\Psi} \in (0, 1)$, see Remark 3.1.

Theorem 3.8. *Suppose $\alpha \geq 0$, $n \geq 2$ and $\tilde{\Psi} = \frac{1}{2\pi}\int_0^{2\pi} \Psi(t)dt \in (0,1)$. Then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \mathcal{L}_n)$ (defined in (3.11)),*

$$\int_{\mathbb{R}^n} |\Delta_{\mathbf{A}} f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \Phi(n,\alpha,\tilde{\Psi}) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}} \tag{3.68}$$

where

$$\Phi(n,\alpha,\tilde{\Psi}) = \inf_{m\in\hat{\mathbb{Z}}^{(n)}} \left\{(m+\tilde{\Psi})(m+\tilde{\Psi}+n-2)+\frac{(n+\alpha)(n-\alpha-4)}{4}\right\}^2 \tag{3.69}$$

and $\hat{\mathbb{Z}}^{(n)}$ is defined in Theorem 3.2. Suppose $\varepsilon := frac[\frac{\alpha-n}{2}]$, then the constant $\Phi(n,\alpha,\tilde{\Psi})$ is equal to zero if and only if

$$n \leq \alpha + 4 \tag{3.70}$$

and

$$\tilde{\Psi} \in \{\varepsilon, 1-\varepsilon\}. \tag{3.71}$$

Remark 3.9. *When $n - \alpha \equiv 0 \pmod 2$, $\varepsilon = 0$ and so Theorem 3.8 gives a non-trivial Rellich type inequality for all $\tilde{\Psi} \in (0,1)$ when $n \leq \alpha + 4$ and $n - \alpha \equiv 0 \pmod 2$. In this case, the ordinary Rellich inequality is only trivially true.*

Remark 3.10. *Evans and Lewis [15] found that for $\alpha = 0$ and $n = 3$, even though the ordinary Rellich inequality was non-trivial, the Rellich type inequality (3.68) is trivial when $\tilde{\Psi} = \frac{1}{2}$. The conditions (3.70) and (3.71) imply that this is actually true for all $n \leq \alpha+4$ and $n-\alpha \equiv 1 \pmod 2$ when $\tilde{\Psi} = \frac{1}{2}$.*

Proof. From (3.19),

$$-\Delta_{\mathbf{A}} = L_r - \frac{1}{r^2}\left\{\frac{\partial^2}{\partial\theta_1^2} + (n-2)\cot\theta_1 \frac{\partial}{\partial\theta_j}\right.$$

$$
\begin{aligned}
&+\sum_{j=1}^{n-2}\frac{1}{\prod_{k=1}^{j-1}\sin\theta_k}\Big[\frac{\partial^2}{\partial\theta_j^2}+(n-j-1)\cot\theta_j\frac{\partial}{\partial\theta_j}\Big]\\
&+\frac{1}{\prod_{k=1}^{n-2}\sin\theta_k}\Lambda_{2,\omega}(\theta_{n-1})\Big\}\\
=&L_r-\frac{1}{r^2}\Big\{\frac{\partial^2}{\partial\theta_1^2}+(n-2)\cot\theta_1\frac{\partial}{\partial\theta_j}\\
&+\sum_{j=1}^{n-3}\frac{1}{\prod_{k=1}^{j-1}\sin\theta_k}\Big[\frac{\partial^2}{\partial\theta_j^2}+(n-j-1)\cot\theta_j\frac{\partial}{\partial\theta_j}\Big]\\
&+\frac{1}{\prod_{k=1}^{n-2}\sin\theta_k}\Lambda_{3,\omega}(\theta_{n-2},\theta_{n-1})\\
=&\ldots=L_r-\frac{1}{r^2}\Lambda_{n,\omega}(\theta_1,\ldots,\theta_{n-1})
\end{aligned}
\tag{3.72}
$$

where in accordance with the definition given in Theorem 3.2,

$$
\Lambda_{j,\omega}(\theta_{n+1-j},\ldots,\theta_{n-1}) := \begin{cases} \left(i\frac{\partial}{\partial\theta_{n-1}}+\Psi(\theta_{n-1})\right)^2 & \text{if } j=2,\\ -\frac{\partial^2}{\partial\theta_{n+1-j}^2}-(j-2)\cot\theta_{n+1-j}\frac{\partial}{\partial\theta_{n+1-j}} & \\ \quad +\frac{1}{\sin\theta_{n+1-j}}\Lambda_{j-1,\omega}(\theta_{n+2-j},\ldots,\theta_{n-1}) & \text{if } j\geq 3. \end{cases}
\tag{3.73}
$$

Take the extended operator

$$
\begin{aligned}
\Lambda_{n,\omega}(\theta_1,...,\theta_{n-1}) = -\frac{\partial^2}{\partial\theta_1^2}-(n-2)\cot\theta_1\frac{\partial}{\partial\theta_1}&\\
+\frac{1}{\sin\theta_1}\tilde{\Lambda}_{n-1,\omega}(\theta_2,...,\theta_{n-1})&
\end{aligned}
\tag{3.74}
$$

where $\tilde{\Lambda}_{n-1,\omega}$ is the Friedrich's extension in $L^2(\mathbb{S}^{n-2})$ of the operator $\Lambda_{n-1,\omega}$. Then applying Theorem 3.2 gives the eigenvalues of the Friedrichs extension in $L^2(\mathbb{S}^{n-1})$ of the operator $\Lambda_{n,\omega}$ as

$$
\rho_m^{\Psi}=(m+\tilde{\Psi})(m+\tilde{\Psi}+n-2)
\tag{3.75}
$$

for $m \in \hat{\mathbb{Z}}^{(n)}$. Applying Theorem 1 from [15] (see Theorem 2.6) gives (3.68) with

$$\begin{aligned}\Phi(n,\alpha,\tilde{\Psi}) &= \inf_{m\in\hat{\mathbb{Z}}^{(n)}} \left\{\rho_m^{\Psi} + \frac{(n+\alpha)(n-\alpha-4)}{4}\right\}^2 \\ &= \inf_{m\in\hat{\mathbb{Z}}^{(n)}} \left\{\left(m+\frac{n+\alpha}{2}+\tilde{\Psi}\right)\left(m+\frac{n-\alpha}{2}-2+\tilde{\Psi}\right)\right\}^2. \qquad (3.76)\end{aligned}$$

Suppose that $\Phi(n,\alpha,\tilde{\Psi})$ is equal to zero, then either

$$m+\frac{n+\alpha}{2}+\tilde{\Psi}=0 \qquad (3.77)$$

or

$$m+\frac{n-\alpha}{2}-2+\tilde{\Psi}=0. \qquad (3.78)$$

By noting the definition of $\hat{\mathbb{Z}}^{(n)}$, the condition (3.77) is satisfied when

$$0=m+\frac{n+\alpha}{2}+\tilde{\Psi}\le 2-\frac{\alpha-n}{2} \quad \Rightarrow \quad n\le\alpha+4. \qquad (3.79)$$

Rearranging (3.77) gives

$$m-n+\frac{\alpha-n}{2}=-\tilde{\Psi}. \qquad (3.80)$$

Since $\tilde{\Psi}\in(0,1)$, $m\in\mathbb{Z}$ and the fractional parts of both sides of (3.80) must be equal, it follows

$$frac[\frac{\alpha-n}{2}]=1-\tilde{\Psi}. \qquad (3.81)$$

The condition (3.78) is similarly true iff $n\le\alpha+4$, rearranging (3.78)

$$m-2+\tilde{\Psi}=\frac{\alpha-n}{2}. \qquad (3.82)$$

Taking the fractional parts of both sides gives

$$frac[\frac{\alpha - n}{2}] = \tilde{\Psi} \tag{3.83}$$

as required. ■

The Rellich type inequality is in itself an interesting object but the need for it arose from the fact that the Rellich inequality is only trivially true when $n \leq \alpha + 4$ and $\frac{n-\alpha}{2} \in \mathbb{Z}$. To this end, the value of $\Phi(n, \alpha, \tilde{\Psi})$ is calculated explicitly in this case.

Corollary 3.11. *Suppose* $2 \leq n \leq \alpha + 4$, $n - \alpha \equiv 0 \pmod 2$ *and* $\tilde{\Psi} \in (0, 1)$. *Then if* $n < \alpha + 4$,

$$\Phi(n, \alpha, \tilde{\Psi}) = \begin{cases} \tilde{\Psi}^2(\tilde{\Psi} - \alpha - 2)^2 & \text{if } \tilde{\Psi} \in (0, \frac{1}{2}], \\ (\tilde{\Psi} + \alpha + 1)^2(\tilde{\Psi} - 1)^2 & \text{if } \tilde{\Psi} \in (\frac{1}{2}, 1) \end{cases} \tag{3.84}$$

and

$$\Phi(\alpha + 4, \alpha, \tilde{\Psi}) = \begin{cases} \tilde{\Psi}^2(\tilde{\Psi} + \alpha + 2)^2 & \text{if } \tilde{\Psi} \in (0, \frac{1}{2}], \\ (\tilde{\Psi} - \alpha - 3)^2(\tilde{\Psi} - 1)^2 & \text{if } \tilde{\Psi} \in (\frac{1}{2}, 1). \end{cases} \tag{3.85}$$

Remark 3.12. *For* $\alpha = 0$ *and* $n = 2, 4$, *the same constants are found by Evans and Lewis in [15].*

Proof. By Theorem 3.8

$$\Phi(n, \alpha, \tilde{\Psi}) = \inf_{m \in \hat{\mathbb{Z}}^{(n)}} \left\{ (m + \tilde{\Psi})(m + \tilde{\Psi} + n - 2) + \frac{(n + \alpha)(n - \alpha - 4)}{4} \right\}^2. \tag{3.86}$$

Define for all $x \in \mathbb{R}$,

$$\begin{aligned} \tau(x, n, \alpha, \tilde{\Psi}) := & x^2 + (n - 2 + 2\tilde{\Psi})x + \tilde{\Psi}(\tilde{\Psi} + n - 2) \\ & + \frac{(n + \alpha)(n - \alpha - 4)}{4} \end{aligned}$$

$$=\left(x+\frac{n+\alpha}{2}+\tilde{\Psi}\right)\left(x+\frac{n-\alpha}{2}-2+\tilde{\Psi}\right) \tag{3.87}$$

and set

$$x_- := \frac{-n-\alpha}{2}-\tilde{\Psi}, \tag{3.88}$$

$$x_+ := \frac{\alpha-n}{2}+2-\tilde{\Psi}. \tag{3.89}$$

It is evident that the points x_- and x_+ are the global minimums of the function $\tau(x,n,\alpha,\tilde{\Psi})^2$ and so the integer minimum of $\tau(x,n,\alpha,\tilde{\Psi})^2$ $(\min_{m\in\mathbb{Z}}\tau(x,n,\alpha,\tilde{\Psi})^2)$ occurs at one of the integers neighbouring x_- and x_+ i.e.

$$m_-^l := \lfloor x_- \rfloor, \tag{3.90}$$

$$m_-^u := \lfloor x_- \rfloor + 1, \tag{3.91}$$

$$m_+^l := \lfloor x_+ \rfloor, \tag{3.92}$$

$$m_+^u := \lfloor x_+ \rfloor + 1. \tag{3.93}$$

Taking into consideration that $\tilde{\Psi}\in(0,1)$ and $\frac{n-\alpha}{2}\in\mathbb{Z}$, it follows that

$$\begin{aligned} m_-^l &= \left\lfloor \frac{-n-\alpha}{2}-\tilde{\Psi}\right\rfloor = \left\lfloor \frac{n-\alpha}{2}-n-\tilde{\Psi}\right\rfloor \\ &= \frac{-n-\alpha}{2}-1 \end{aligned} \tag{3.94}$$

and

$$\begin{aligned} m_+^l &= \left\lfloor \frac{\alpha-n}{2}+2-\tilde{\Psi}\right\rfloor \\ &= \frac{\alpha-n}{2}+2-1 = \frac{\alpha-n}{2}+1. \end{aligned} \tag{3.95}$$

Hence

$$m_-^u = \frac{-n-\alpha}{2}, \tag{3.96}$$
$$m_+^u = \frac{\alpha-n}{2} + 2. \tag{3.97}$$

Take $n < \alpha + 4$ which implies that $n \le \alpha + 2$ because $n - \alpha \equiv 0 \pmod 2$ and so

$$m_-^u \le 1-n, \tag{3.98}$$
$$m_+^l \ge 0. \tag{3.99}$$

Therefore m_-^l, m_-^u, m_+^l, $m_+^u \in \hat{\mathbb{Z}}^{(n)}$. By direct calculation

$$\tau(m_-^l, n, \alpha, \tilde{\Psi}) = (\tilde{\Psi} - \alpha - 3)(\tilde{\Psi} - 1), \tag{3.100}$$
$$\tau(m_-^u, n, \alpha, \tilde{\Psi}) = \tilde{\Psi}(\tilde{\Psi} - \alpha - 2), \tag{3.101}$$
$$\tau(m_+^l, n, \alpha, \tilde{\Psi}) = (\tilde{\Psi} + \alpha + 1)(\tilde{\Psi} - 1), \tag{3.102}$$
$$\tau(m_+^u, n, \alpha, \tilde{\Psi}) = \tilde{\Psi}(\tilde{\Psi} + \alpha + 2). \tag{3.103}$$

Noting that

$$\tau(m_-^u, n, \alpha, \tilde{\Psi}) - \tau(m_+^u, n, \alpha, \tilde{\Psi}) = -2\tilde{\Psi}(\alpha + 2) < 0, \tag{3.104}$$
$$\tau(m_-^u, n, \alpha, \tilde{\Psi}) + \tau(m_+^u, n, \alpha, \tilde{\Psi}) = 2\tilde{\Psi}^2 > 0, \tag{3.105}$$
$$\tau(m_-^l, n, \alpha, \tilde{\Psi}) - \tau(m_+^l, n, \alpha, \tilde{\Psi}) = 2(1 - \tilde{\Psi})(\alpha + 2) > 0, \tag{3.106}$$
$$\tau(m_-^l, n, \alpha, \tilde{\Psi}) + \tau(m_+^l, n, \alpha, \tilde{\Psi}) = 2(\tilde{\Psi} - 1)^2 > 0, \tag{3.107}$$

it follows by the difference of two squares that

$$\min\{\tau(m_-^u, n, \alpha, \tilde{\Psi})^2, \tau(m_+^u, n, \alpha, \tilde{\Psi})^2\} = \tau(m_-^u, n, \alpha, \tilde{\Psi})^2 \tag{3.108}$$

and

$$\min\{\tau(m_-^l, n, \alpha, \tilde{\Psi})^2, \tau(m_+^l, n, \alpha, \tilde{\Psi})^2\} = \tau(m_+^l, n, \alpha, \tilde{\Psi})^2. \tag{3.109}$$

Hence the infimum occurs at either m_+^l or m_-^u. Again by calculation

$$\tau(m_-^u, n, \alpha, \tilde{\Psi}) + \tau(m_+^l, n, \alpha, \tilde{\Psi}) = -\alpha - 1 - 2\tilde{\Psi}(1 - \tilde{\Psi}) < 0 \tag{3.110}$$

$$\tau(m_-^u, n, \alpha, \tilde{\Psi}) - \tau(m_+^l, n, \alpha, \tilde{\Psi}) = (\alpha + 1)(1 - 2\tilde{\Psi}) \begin{cases} \geq 0 & \text{if} \quad \tilde{\Psi} \in (0, \frac{1}{2}], \\ < 0 & \text{if} \quad \tilde{\Psi} \in (\frac{1}{2}, 1). \end{cases} \tag{3.111}$$

Therefore when $n < \alpha + 4$,

$$\begin{aligned} \Phi(n, \alpha, \tilde{\Psi}) &= \inf_{m \in \hat{\mathbb{Z}}^{(n)}} \tau(m, n, \alpha, \tilde{\Psi})^2 \\ &= \begin{cases} \tau(m_-^u, n, \alpha, \tilde{\Psi})^2 & \text{if } \tilde{\Psi} \in (0, \frac{1}{2}], \\ \tau(m_+^l, n, \alpha, \tilde{\Psi})^2 & \text{if } \tilde{\Psi} \in (\frac{1}{2}, 1). \end{cases} \end{aligned} \tag{3.112}$$

Take $n = \alpha + 4$, then

$$m_-^l = 1 - n, \tag{3.113}$$
$$m_+^l = -n, \tag{3.114}$$
$$m_-^u = -1, \tag{3.115}$$
$$m_+^u = 0. \tag{3.116}$$

In this case m_-^l, $m_+^u \in \hat{\mathbb{Z}}^{(n)}$ but m_-^u, $m_+^l \notin \hat{\mathbb{Z}}^{(n)}$, and so the infimum occurs at either m_-^l or m_+^u. By calculation

$$\tau(m_-^l, n, \alpha, \tilde{\Psi}) + \tau(m_+^u, n, \alpha, \tilde{\Psi}) = \alpha + \frac{5}{2} + 2(\tilde{\Psi} - \frac{1}{2}) > 0, \tag{3.117}$$

$$\tau(m_-^l, n, \alpha, \tilde{\Psi}) - \tau(m_+^u, n, \alpha, \tilde{\Psi}) = (\alpha+3)(1-2\tilde{\Psi}) \begin{cases} \geq 0 & \text{if } \tilde{\Psi} \in (0, \frac{1}{2}], \\ < 0 & \text{if } \tilde{\Psi} \in (\frac{1}{2}, 1). \end{cases} \tag{3.118}$$

Therefore

$$\begin{aligned} \Phi(\alpha+4, \alpha, \tilde{\Psi}) &= \inf_{m \in \hat{\mathbb{Z}}^{(n)}} \tau(m, \alpha+4, \alpha, \tilde{\Psi})^2 \\ &= \begin{cases} \tau(m_+^u, \alpha+4, \alpha, \tilde{\Psi})^2 & \text{if } \tilde{\Psi} \in (0, \frac{1}{2}], \\ \tau(m_-^l, \alpha+4, \alpha, \tilde{\Psi})^2 & \text{if } \tilde{\Psi} \in (\frac{1}{2}, 1) \end{cases} \end{aligned} \tag{3.119}$$

as required. ■

3.4 A higher order Rellich type inequality

As a consequence of Theorem 3.8, the following higher order Rellich type inequality can be constructed.

Corollary 3.13. *Suppose $\alpha \geq 0$, $j \in \mathbb{N}$, $n \geq 2$ and $\tilde{\Psi} \in (0,1)$. Then*

$$\int_{\mathbb{R}^n} |\Delta_{\mathbf{A}}^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \geq \Omega(n, \alpha, \tilde{\Psi}, j) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4j}} \tag{3.120}$$

for all $f \in C_0^\infty(\mathbb{R}^n \setminus \mathcal{L}_n)$, where

$$\Omega(n, \alpha, \tilde{\Psi}, j) = \prod_{i=0}^{j-1} \Phi(n, \alpha+4i, \tilde{\Psi}) \tag{3.121}$$

and Φ is given by (3.69) in Theorem 3.8.

Remark 3.14. *When $n \leq \alpha + 4j$ and $\frac{n-\alpha}{2} \in \mathbb{Z}$, the constant $\Omega(n, \alpha, \tilde{\Psi}, j)$ is non-zero for all $\tilde{\Psi} \in (0,1)$ and so Corollary 3.13 gives a non-trivial higher order Rellich type inequality for all $f \in C_0^\infty(\mathbb{R}^n \setminus \mathcal{L}_n)$. In this case the higher order Rellich inequality is only trivially true, see Theorem 2.4.*

Proof. For $j = 1$, (3.120) is precisely Theorem 3.8. Assume (3.120) is true for $j-1$, then

$$\begin{aligned}\int_{\mathbb{R}^n} |\Delta_{\mathbf{A}}^j f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} &= \int_{\mathbb{R}^n} |\Delta_{\mathbf{A}}^{j-1}(\Delta_{\mathbf{A}} f(\mathbf{x}))|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} \\ &\geq \prod_{i=0}^{j-2} \Phi(n, \alpha+4i, \tilde{\Psi}) \int_{\mathbb{R}^n} |\Delta_{\mathbf{A}} f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4(j-1)}} \\ &\geq \Phi(n, \alpha+4(j-1), \tilde{\Psi}) \\ &\quad \cdot \prod_{i=0}^{j-2} \Phi(n, \alpha+4i, \tilde{\Psi}) \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4j}} \qquad (3.122)\end{aligned}$$

and the result then follows by induction. ■

Chapter 4

Counting Eigenvalues

The Rellich type inequality found in Theorem 3.8 enables an analysis of the spectral properties of the magnetic quad-harmonic operator $\Delta_{\mathbf{A}}^4$ to be undertaken. Furthermore in $L^2(\mathbb{R}^8)$, a space in which there is no CLR type bound for the number of negative eigenvalues of the operator Δ^4, a CLR type bound can be found for the operator $\Delta_{\mathbf{A}}^4$. In this Chapter the convention (unless otherwise indicated by a subscript) will be that $\|\cdot\|$ and $(\cdot,\cdot)$ denote the $L^2(\mathbb{R}^n)$ norm and inner-product respectively.

4.1 An upper bound for $\| \|f\|^2_{L^2(\mathbb{S}^{n-1})} r^{n-8} \|_{L^\infty(\mathbb{R}_+)}$

As defined in (2.54), we take $\mathcal{Z}_m[f](r) : \mathbb{R}_+ \mapsto \mathbb{C}$ to be the L^2 inner-product on the hypersphere $\mathbb{S}^{n-1}$ of the function $f : \mathbb{R}^n \mapsto \mathbb{C}$ with the m-th eigenvector $u_m : \mathbb{S}^{n-1} \mapsto \mathbb{C}$ of the non-negative self-adjoint operator Λ_ω (see Theorem 2.6), i.e.

$$\mathcal{Z}_m[f](r) := \Big(f(r,\cdot), u_m\Big)_{L^2(\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} f(r,\omega)\overline{u_m(\omega)}d\omega, \tag{4.1}$$

$$\mathcal{Z}_m^{(j)}[f](r) := \frac{\partial^j}{\partial r^j} \int_{\mathbb{S}^{n-1}} f(r,\omega)\overline{u_m(\omega)}d\omega, \tag{4.2}$$

for $j \in \mathbb{N}$ and

$$\begin{aligned}\mathcal{Z}_m[\Lambda_\omega f] &= \Big(\Lambda_\omega f(r,\cdot), u_m\Big)_{L^2(\mathbb{S}^{n-1})} = \lambda_m \Big(f(r,\cdot), u_m\Big)_{L^2(\mathbb{S}^{n-1})} \\ &= \lambda_m \mathcal{Z}_m[f].\end{aligned} \tag{4.3}$$

Lemma 4.1. *Suppose $D = L_r + \frac{1}{r^2}\Lambda_\omega$ where*

$$L_r = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r}. \tag{4.4}$$

Then for all $f \in \mathcal{D}_0$ where

$$\begin{aligned}\mathcal{D}_0 = \{f : f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}),\\ f(r,\cdot) \in \mathcal{D}(\Lambda_\omega) \text{ for } 0 < r < \infty, Df \in L^2(\mathbb{R}^n)\},\end{aligned} \tag{4.5}$$

we have

$$\begin{aligned}&\int_0^\infty |\mathcal{Z}_m[Df]|^2 r^{n-1-\alpha} dr\\ &\quad = \int_0^\infty \Big(|\mathcal{Z}_m^{(2)}[f]|^2 + \frac{2\lambda_m + (n-1)(\alpha+1)}{r^2} |\mathcal{Z}_m^{(1)}[f]|^2\\ &\quad + \frac{\lambda_m^2 + (n-4-\alpha)(\alpha+2)\lambda_m}{r^4} |\mathcal{Z}_m[f]|^2 \Big) r^{n-1-\alpha} dr.\end{aligned} \tag{4.6}$$

Proof. The inner-product $(u,v)_{L^2(\mathbb{S}^{n-1})}$ is linear in u, therefore

$$\mathcal{Z}_m[h_1 + h_2] = \mathcal{Z}_m[h_1] + \mathcal{Z}_m[h_2] \tag{4.7}$$

and so

$$\begin{aligned}\int_0^\infty |\mathcal{Z}_m[Df]|^2 r^{n-1-\alpha} dr &= \int_0^\infty \left| \mathcal{Z}_m\left[L_r f + \frac{1}{r^2} \Lambda_\omega f \right] \right|^2 r^{n-1-\alpha} dr\\ &= \int_0^\infty \left| \mathcal{Z}_m[L_r f] + \frac{1}{r^2} \mathcal{Z}_m[\Lambda_\omega f] \right|^2 r^{n-1-\alpha} dr\\ &= \int_0^\infty |\mathcal{Z}_m[L_r f]|^2 r^{n-1-\alpha} dr\\ &\quad + 2Re\Big[\int_0^\infty \mathcal{Z}_m[L_r f] \overline{\mathcal{Z}_m[\Lambda_\omega f]} r^{n-3-\alpha} dr \Big]\\ &\quad + \int_0^\infty |\mathcal{Z}_m[\Lambda_\omega f]|^2 r^{n-5-\alpha} dr.\end{aligned} \tag{4.8}$$

The assumption $f \in \mathcal{D}_0$ implies that $\mathcal{Z}_m[f] \in C_0^\infty(\mathbb{R}_+)$ and so by using the fact that $\mathcal{Z}_m$ commutes with L_r we obtain

$$\int_0^\infty |\mathcal{Z}_m[L_r f]|^2 r^{n-1-\alpha} dr = \int_0^\infty |L_r \mathcal{Z}_m[f]|^2 r^{n-1-\alpha} dr$$

$$
\begin{aligned}
&= \int_0^\infty |\mathcal{Z}_m^{(2)}[f]|^2 r^{n-1-\alpha} dr \\
&\quad + (n-1).2Re\Big[\int_0^\infty \mathcal{Z}_m^{(2)}[f].\overline{\mathcal{Z}_m^{(1)}[f]} r^{n-2-\alpha} dr\Big] \\
&\quad + (n-1)^2 \int_0^\infty |\mathcal{Z}_m^{(1)}[f]|^2 r^{n-3-\alpha} dr \\
&= \int_0^\infty |\mathcal{Z}_m^{(2)}[f]|^2 r^{n-1-\alpha} dr \\
&\quad + (n-1)(\alpha+1) \int_0^\infty |\mathcal{Z}_m^{(1)}[f]|^2 r^{n-3-\alpha} dr \qquad (4.9)
\end{aligned}
$$

by applying integration by parts. Noting (4.3), it follows

$$
\begin{aligned}
2Re\Big[\int_0^\infty & \mathcal{Z}_m[L_r f]\overline{\mathcal{Z}_m[\Lambda_\omega f]} r^{n-3-\alpha} dr\Big] \\
&= 2\lambda_m Re\Big[\int_0^\infty L_r \mathcal{Z}_m[f]\overline{\mathcal{Z}_m[f]} r^{n-3-\alpha} dr\Big] \\
&= -2\lambda_m Re\Big[\int_0^\infty \mathcal{Z}_m^{(2)}[f]\overline{\mathcal{Z}_m[f]} r^{n-3-\alpha} dr\Big] \\
&\quad - 2(n-1)\lambda_m Re\Big[\int_0^\infty \mathcal{Z}_m^{(1)}[f]\overline{\mathcal{Z}_m[f]} r^{n-4-\alpha} dr\Big] \\
&= 2\lambda_m \int_0^\infty |\mathcal{Z}_m^{(1)}[f]|^2 r^{n-3-\alpha} dr \\
&\quad - 2(\alpha+2)\lambda_m Re\Big[\int_0^\infty \mathcal{Z}_m^{(1)}[f]\overline{\mathcal{Z}_m[f]} r^{n-4-\alpha} dr\Big] \\
&= 2\lambda_m \int_0^\infty |\mathcal{Z}_m^{(1)}[f]|^2 r^{n-3-\alpha} dr \\
&\quad + (\alpha+2)(n-4-\alpha)\lambda_m \int_0^\infty |\mathcal{Z}_m[f]|^2 r^{n-5-\alpha} dr \qquad (4.10)
\end{aligned}
$$

and

$$
\int_0^\infty |\mathcal{Z}_m[\Lambda_\omega f]|^2 r^{n-5-\alpha} dr = \lambda_m^2 \int_0^\infty |\mathcal{Z}_m[f]|^2 r^{n-5-\alpha} dr. \qquad (4.11)
$$

Substituting (4.9)-(4.11) into (4.8) gives the appropriate result. ∎

Lemma 4.2. *For any $r, \delta \in (0,\infty)$, $j \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$, we have the inequality*

$$|\mathcal{Z}_m^{(j)}[f](r)|^2 r^{\beta+1} \le \frac{1}{\delta}\int_0^\infty |\mathcal{Z}_m^{(j+1)}[f](s)|^2 s^{\beta+2} ds + (\beta+1+\delta)\int_0^\infty |\mathcal{Z}_m^{(j)}[f](s)|^2 s^{\beta} ds \tag{4.12}$$

for all $f \in \mathcal{D}_0$.

Proof. By integration by parts

$$2Re\Big[\int_0^r \mathcal{Z}_m^{(j+1)}[f](s).\overline{\mathcal{Z}_m^{(j)}[f](s)}s^{\beta+1}ds\Big] = |\mathcal{Z}_m^{(j)}[f](r)|^2 r^{\beta+1} - (\beta+1)\int_0^r |\mathcal{Z}_m^{(j)}[f](s)|^2 s^{\beta} ds. \tag{4.13}$$

Noting that $Re\ u \le |u|$ and

$$2ab \le \frac{a^2}{\delta} + \delta b^2 \tag{4.14}$$

for all $\delta > 0$, it follows that

$$\begin{aligned} &|\mathcal{Z}_m^{(j)}[f](r)|^2 r^{\beta+1} \\ &\le 2\int_0^r |\mathcal{Z}_m^{(j+1)}[f](s)s^{\frac{\beta}{2}+1}|.|\mathcal{Z}_m^{(j)}[f](s)s^{\frac{\beta}{2}}|ds + (\beta+1)\int_0^r |\mathcal{Z}_m^{(j)}[f](s)|^2 s^{\beta} ds \\ &\le \frac{1}{\delta}\int_0^r |\mathcal{Z}_m^{(j+1)}[f](s)|^2 s^{\beta+2} ds + (\beta+1+\delta)\int_0^r |\mathcal{Z}_m^{(j)}[f](s)|^2 s^{\beta} ds \\ &\le \frac{1}{\delta}\int_0^\infty |\mathcal{Z}_m^{(j+1)}[f](s)|^2 s^{\beta+2} ds + (\beta+1+\delta)\int_0^\infty |\mathcal{Z}_m^{(j)}[f](s)|^2 s^{\beta} ds \end{aligned} \tag{4.15}$$

since $|\mathcal{Z}_m^{(j+1)}[f](s)|^2 s^{\beta+2} > 0$ and $|\mathcal{Z}_m^{(j)}[f](s)|^2 s^{\beta} > 0$ for all $s \in [0,\infty]$. ■

Combining Lemma 4.1 and Lemma 4.2 gives the following Corollary.

Corollary 4.3. *Suppose $f \in \mathcal{D}_0$ and*

$$C(n,\alpha) = \inf_{m \in \mathcal{I}} \Big\{ \lambda_m + \Big(\frac{n+\alpha}{2}\Big)\Big(\frac{n-\alpha-4}{2}\Big)\Big\}^2 \neq 0 \tag{4.16}$$

as given in [15]. Then

$$\begin{aligned} \Big\| \frac{Df}{|\mathbf{x}|^{\frac{\alpha}{2}}} \Big\|^2 \geq \mathcal{M}(n,\alpha) \Big\| (\alpha+1) \Big\| \frac{\partial f(r,\cdot)}{\partial r} \Big\|^2_{L^2(\mathbb{S}^{n-1})} r^{n-2-\alpha} \\ + (\alpha+2) \min_{m \in \mathcal{I}} \{\lambda_m\} \|f(r,\cdot)\|^2_{L^2(\mathbb{S}^{n-1})} r^{n-4-\alpha} \Big\|_{L^\infty(0,\infty)} \end{aligned} \tag{4.17}$$

where

$$\mathcal{M}(n,\alpha) := \frac{C(n,\alpha)}{C(n,\alpha) + [\lambda_m(\lambda_m - \frac{(\alpha+2)^2}{2})]_{m,-}} \tag{4.18}$$

and $[a]_{m,-} := \max_{m \in \mathcal{I}}\{0, -a\}$.

Remark 4.4. *It can be seen that (4.17) is a generalisation of Corollary 1 in [14] where the $\alpha = 0$ case was considered.*

Proof. By Lemma 4.1,

$$\begin{aligned} \int_0^\infty |\mathcal{Z}_m[Df](s)|^2 s^{n-1-\alpha} ds &= \int_0^\infty |\mathcal{Z}_m^{(2)}[f](s)|^2 s^{n-1-\alpha} ds \\ &\quad + (n-1)(\alpha+1) \int_0^\infty |\mathcal{Z}_m^{(1)}[f](s)|^2 s^{n-3-\alpha} ds \\ &\quad + \lambda_m \Big(2 \int_0^\infty |\mathcal{Z}_m^{(1)}[f](s)|^2 s^{n-3-\alpha} ds \\ &\quad + (n-4-\alpha)(\alpha+2) \int_0^\infty |\mathcal{Z}_m[f](s)|^2 s^{n-1-\alpha} ds \Big) \\ &\quad + \lambda_m^2 \int_0^\infty |\mathcal{Z}_m[f](s)|^2 s^{n-1-\alpha} ds \\ &=: I_1 + \lambda_m I_2 + \lambda_m^2 \int_0^\infty |\mathcal{Z}_m[f](s)|^2 s^{n-1-\alpha} ds. \end{aligned} \tag{4.19}$$

For arbitrary $\delta_1 > 0$,

$$
\begin{aligned}
I_1 = \delta_1\Big(\frac{1}{\delta_1}\int_0^\infty & |\mathcal{Z}_m^{(2)}[f](s)|^2 s^{n-1-\alpha} ds \\
&+ (n-2-\alpha+\delta_1)\int_0^\infty |\mathcal{Z}_m^{(1)}[f](s)|^2 s^{n-3-\alpha} ds\Big) \\
&+ (\alpha+1-\delta_1)(n+\delta_1-1)\int_0^\infty |\mathcal{Z}_m^{(1)}[f](s)|^2 s^{n-3-\alpha} ds.
\end{aligned}
\tag{4.20}
$$

Taking $\delta_1 = \alpha+1$, it follows by Lemma 4.2 that

$$
I_1 \geq (\alpha+1)|\mathcal{Z}_m^{(1)}[f](r)|^2 r^{n-2-\alpha} \tag{4.21}
$$

for some $r \in (0,\infty)$. Similarly, for $\delta_2 > 0$

$$
\begin{aligned}
I_2 = 2\delta_2\Big(\frac{1}{\delta_2}\int_0^\infty & |\mathcal{Z}_m^{(1)}[f](s)|^2 s^{n-3-\alpha} ds \\
&+ (n-4-\alpha+\delta_2)\int_0^\infty |\mathcal{Z}_m[f](s)|^2 s^{n-5-\alpha} ds\Big) \\
&+ [(\alpha+2-2\delta_2)(n-4-\alpha)-2\delta_2^2]\int_0^\infty |\mathcal{Z}_m[f](s)|^2 s^{n-5-\alpha} ds
\end{aligned}
\tag{4.22}
$$

and

$$
I_2 \geq (\alpha+2)|\mathcal{Z}_m[f](r)|^2 r^{n-4-\alpha} - \frac{(\alpha+2)^2}{2}\int_0^\infty |\mathcal{Z}_m[f](s)|^2 s^{n-5-\alpha} ds \tag{4.23}
$$

by taking $\delta_2 = \frac{\alpha+2}{2}$. Therefore

$$
\begin{aligned}
\int_0^\infty & |\mathcal{Z}_m[Df](s)|^2 s^{n-1-\alpha} ds \\
&\geq (\alpha+1)|\mathcal{Z}_m^{(1)}[f](r)|^2 r^{n-2-\alpha} + (\alpha+2)\lambda_m|\mathcal{Z}_m[f](r)|^2 r^{n-4-\alpha} \\
&\quad + \lambda_m\Big(\lambda_m - \frac{(\alpha+2)^2}{2}\Big)\int_0^\infty |\mathcal{Z}_m[f](s)|^2 s^{n-5-\alpha} ds
\end{aligned}
$$

$$\geq(\alpha+1)|\mathcal{Z}_m^{(1)}[f](r)|^2 r^{n-2-\alpha} + (\alpha+2)\lambda_m |\mathcal{Z}_m[f](r)|^2 r^{n-4-\alpha}$$
$$- \max\left\{\lambda_m\Big(\frac{(\alpha+2)^2}{2} - \lambda_m\Big), 0\right\} \int_0^\infty |\mathcal{Z}_m[f](s)|^2 s^{n-5-\alpha} ds. \quad (4.24)$$

The operator Λ_ω has a discrete spectrum consisting of eigenvalues $\lambda_m, m \in \mathcal{I}$, where $\mathcal{I}$ is a countable index set. Therefore summing m over $\mathcal{I}$ gives

$$\begin{aligned}
&\sum_{m\in\mathcal{I}} \int_0^\infty |\mathcal{Z}_m[Df](s)|^2 s^{n-1-\alpha} ds \\
&\quad \geq (\alpha+1) \sum_{m\in\mathcal{I}} |\mathcal{Z}_m^{(1)}[f](r)|^2 r^{n-2-\alpha} \\
&\quad + (\alpha+2) \min_{m\in\mathcal{I}}\{\lambda_m\} \sum_{m\in\mathcal{I}} |\mathcal{Z}_m[f](r)|^2 r^{n-4-\alpha} \\
&\quad - \max_{m\in\mathcal{I}} \left\{\lambda_m\Big(\frac{(\alpha+2)^2}{2} - \lambda_m\Big), 0\right\} \sum_{m\in\mathcal{I}} \int_0^\infty |\mathcal{Z}_m[f](s)|^2 s^{n-5-\alpha} ds
\end{aligned} \quad (4.25)$$

and by Parseval's identity

$$\sum_{m\in\mathcal{I}} |\mathcal{Z}_m^{(1)}[f](r)|^2 = \int_{\mathbb{S}^{n-1}} \left|\frac{\partial f(r,\omega)}{\partial r}\right|^2 d\omega, \quad (4.26)$$

$$\sum_{m\in\mathcal{I}} |\mathcal{Z}_m[f](r)|^2 = \int_{\mathbb{S}^{n-1}} |f(r,\omega)|^2 d\omega, \quad (4.27)$$

$$\sum_{m\in\mathcal{I}} \int_0^\infty |\mathcal{Z}_m[f](s)|^2 s^{n-5-\alpha} ds = \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}}, \quad (4.28)$$

$$\sum_{m\in\mathcal{I}} \int_0^\infty |\mathcal{Z}_m[Df](s)|^2 s^{n-1-\alpha} ds = \int_{\mathbb{R}^n} |Df(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha}}. \quad (4.29)$$

Finally by Theorem 1 from [15]

$$\int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}} \leq \frac{1}{C(n,\alpha)} \int_{\mathbb{R}^n} |Df(\mathbf{x})|^2 \frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha}}. \quad (4.30)$$

Substituting (4.26)-(4.30) into (4.25) gives the required result. ■

Theorem 4.5. *Suppose $f, Df \in \mathcal{D}_0$ and $C(n,0)C(n,4) > 0$. Then*

$$\|D^2 f\|^2 \geq \kappa_D(n) \Big\| 5 \Big\| \frac{\partial f}{\partial r} \Big\|^2_{L^2(\mathbb{S}^{n-1})} r^{n-6} + 6 \min_{m \in \mathcal{I}} \{\lambda_m\} \|f\|^2_{L^2(\mathbb{S}^{n-1})} r^{n-8} \Big\|_{L^\infty(0,\infty)} \tag{4.31}$$

where

$$\kappa_D(n) := \frac{C(n,0)C(n,4)}{C(n,4) + [\lambda_m(\lambda_m - 18)]_{m,-}} > 0. \tag{4.32}$$

Proof. Applying Theorem 1 from [15] and then Corollary 4.3 gives

$$\begin{aligned} \|D^2 f\|^2 \geq & C(n,0) \Big\| \frac{Df}{|\cdot|^2} \Big\|^2 \\ \geq & C(n,0)\mathcal{M}(n,4) \Big\| 5 \Big\| \frac{\partial f(r,\cdot)}{\partial r} \Big\|^2_{L^2(\mathbb{S}^{n-1})} r^{n-6} \\ & + 6 \min_m \{\lambda_m\} \|f(r,\cdot)\|^2_{L^2(\mathbb{S}^{n-1})} r^{n-8} \Big\|_{L^\infty(0,\infty)}. \end{aligned} \tag{4.33}$$

Finally, the assumption $C(n,0)C(n,4) > 0$ implies that $\kappa_D(n) > 0$. ■

Corollary 4.6. *If $\tilde{\Psi} \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 0)$ when $n \in \{3,5,7\}$ and $\tilde{\Psi} \in (0,1)$ otherwise, then*

$$\Big\| \Big\| \frac{\partial^t f}{\partial r^t} \Big\|^2_{L^2(\mathbb{S}^{n-1})} r^{n-8+2t} \Big\|_{L^\infty(0,\infty)} \lesssim \|\Delta^2_{\mathbf{A}} f\|^2 \tag{4.34}$$

for $f \in C_0^\infty(\mathbb{R}^n \setminus \mathcal{L}_n)$ and $t \in \{0,1\}$.

Proof. Take $D = -\Delta_{\mathbf{A}}$, then Theorem 3.8 implies that the product

$$C(n,0)C(n,4) = \Phi(n,0,\tilde{\Psi})\Phi(n,4,\tilde{\Psi}) > 0 \tag{4.35}$$

except when $n \in \{3,5,7\}$ and $\tilde{\Psi} = \frac{1}{2}$, see Remarks 3.9 and 3.10. Therefore, Theorem 4.5 with $\lambda_m = \rho_m^{\Psi}$ (see Theorem 3.2) and $\mathcal{D}_0 = C_0^\infty(\mathbb{R}^n \setminus \mathcal{L}_n)$ gives

$$\begin{aligned} \|\Delta_{\mathbf{A}}^2 f\|^2 \geq \kappa_{-\Delta_{\mathbf{A}}}(n)\Big\| 5\Big\|\frac{\partial f}{\partial r}\Big\|^2_{L^2(\mathbb{S}^{n-1})} r^{n-6} \\ + 6 \min_{m\in\hat{\mathbb{Z}}^{(n)}}\{\rho_m^{\Psi}\}\|f\|^2_{L^2(\mathbb{S}^{n-1})} r^{n-8}\Big\|_{L^\infty(0,\infty)}. \end{aligned} \tag{4.36}$$

All that needs to be shown is that $\min_{m\in\hat{\mathbb{Z}}^{(n)}}\{\rho_m^{\Psi}\}$ is positive. By definition, $m \in \hat{\mathbb{Z}}^{(n)}$ implies that $\rho_m^{\Psi} \geq 0$. Furthermore by Theorem 3.2

$$\min_{m\in\hat{\mathbb{Z}}^{(n)}}\{\rho_m^{\Psi}\} = \min_{m\in\hat{\mathbb{Z}}^{(n)}}\{(m+\tilde{\Psi})(m+\tilde{\Psi}+n-2)\}, \tag{4.37}$$

which is non-zero since $\tilde{\Psi}$, $\tilde{\Psi}+n-2 \notin \mathbb{Z}$, hence

$$\min_{m\in\hat{\mathbb{Z}}^{(n)}}\{\rho_m^{\Psi}\} > 0. \tag{4.38}$$

■

4.2 Forms and Operators

Let $F_{\mathbf{A}}$ denote the Friedrichs extension of $\Delta_{\mathbf{A}}^4 \restriction_{\mathcal{D}_0}$ where $\mathcal{D}_0 = C_0^\infty(\mathbb{R}^n \setminus \mathcal{L}_n)$ and set $\Gamma_{\mathbf{A}} = F_{\mathbf{A}}^{\frac{1}{2}}$. We denote by $\mathcal{H}(\Gamma_{\mathbf{A}})$, the Hilbert Space determined by $\mathcal{D}(\Gamma_{\mathbf{A}})$ and the graph inner-product

$$\begin{aligned} (\mu,\nu)_{\Gamma_{\mathbf{A}}} =& (\Gamma_{\mathbf{A}}\mu, \Gamma_{\mathbf{A}}\nu) + (\mu,\nu) \\ =& ((\Gamma_{\mathbf{A}}+i)\mu, (\Gamma_{\mathbf{A}}+i)\nu) \end{aligned} \tag{4.39}$$

with norm

$$\|\mu\|_{\Gamma_{\mathbf{A}}} = (\mu,\mu)^{\frac{1}{2}}_{\Gamma_{\mathbf{A}}} = (\|\Gamma_{\mathbf{A}}\mu\|^2 + \|\mu\|^2)^{\frac{1}{2}}. \tag{4.40}$$

Note that $\mathcal{D}(\Gamma_{\mathbf{A}})$ is the form domain of $F_{\mathbf{A}}$, $\mathcal{D}_0$ is dense in $\mathcal{H}(\Gamma_{\mathbf{A}})$ and for $\mu \in \mathcal{D}_0$,

$$\begin{aligned}\|\Delta_{\mathbf{A}}^2\mu\|^2 &= (\Delta_{\mathbf{A}}^4\mu,\mu) = (\Gamma_{\mathbf{A}}^2\mu,\mu)\\ &= \|\Gamma_{\mathbf{A}}\mu\|^2\end{aligned} \tag{4.41}$$

since $\mathcal{D}_0$ lies in $\mathcal{D}(\Gamma_{\mathbf{A}})$.

Lemma 4.7. *Suppose that the hypothesis of Corollary 4.6 is satisfied and let K_+ be the operator of multiplication by the real-valued function k_+, where*

$$0 \leq k_+ \in L^1(\mathbb{R}_+; L^\infty(\mathbb{S}^{n-1}); r^7dr) \equiv L^1(\mathbb{R}_+; r^7dr) \otimes L^\infty(\mathbb{S}^{n-1}). \tag{4.42}$$

Then $K_+^{\frac{1}{2}} : \mathcal{H}(\Gamma_{\mathbf{A}}) \to L^2(\mathbb{R}^n)$

i. is bounded;

ii. is $\Gamma_{\mathbf{A}}$-compact on $L^2(\mathbb{R}^n)$;

iii. has $\Gamma_{\mathbf{A}}$-bound zero.

Proof. i. Define the sesquilinear form

$$\mathfrak{K}[\mu,\nu] := (K_+\mu,\nu), \quad \mu,\nu \in \mathcal{H}(\Gamma_{\mathbf{A}}). \tag{4.43}$$

Then the associated quadratic form is

$$\begin{aligned}\mathfrak{K}[\mu] &= \int_{\mathbb{R}^n} k_+(\mathbf{x})\mu(\mathbf{x}).\overline{\mu(\mathbf{x})}d\mathbf{x}\\ &= \int_{\mathbb{R}^n} k_+(\mathbf{x})^{\frac{1}{2}}\mu(\mathbf{x}).\overline{k_+(\mathbf{x})^{\frac{1}{2}}\mu(\mathbf{x})}d\mathbf{x} = \|K_+^{\frac{1}{2}}\mu\|^2.\end{aligned} \tag{4.44}$$

Corollary 4.6 shows that for $\mu \in \mathcal{D}_0$,

$$
\begin{aligned}
|(K_+\mu,\mu)| &= \int_0^\infty \int_{\mathbb{S}^{n-1}} k_+(r,\omega)|\mu(r,\omega)|^2 d\omega r^{n-1} dr \\
&\leq \int_0^\infty \|k_+(r,\cdot)\|_{L^\infty(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} |\mu(r,\omega)|^2 d\omega r^{n-1} dr \\
&\leq \int_0^\infty \|k_+(r,\cdot)\|_{L^\infty(\mathbb{S}^{n-1})} r^7 \| \|\|\mu\|^2_{L^2(\mathbb{S}^{n-1})} r^{n-8} \|_{L^\infty(\mathbb{R}_+)} dr \\
&\lesssim \|k_+\|_{L^1(\mathbb{R}_+;L^\infty(\mathbb{S}^{n-1};r^7dr))} \|\Gamma_{\mathbf{A}}\mu\|^2.
\end{aligned} \tag{4.45}
$$

Since $\mathcal{D}_0$ is dense in $\mathcal{H}(\Gamma_{\mathbf{A}})$, (4.45) holds for all $\mu \in \mathcal{H}(\Gamma_{\mathbf{A}})$ by continuity, and so

$$
\|K_+^{\frac{1}{2}}\mu\|^2 \lesssim \|k_+\|_{L^1(\mathbb{R}_+;L^\infty(\mathbb{S}^{n-1};r^7dr))} \|\mu\|^2_{\Gamma_{\mathbf{A}}}. \tag{4.46}
$$

Hence K_+ is bounded.

ii. Let $\mu_l \rightharpoonup 0$ in $L^2(\mathbb{R}^n)$ i.e.

$$
(\mu_l, f) \to 0 \tag{4.47}
$$

for all $f \in L^2(\mathbb{R}^n)$. Set $\nu_l = (\Gamma_{\mathbf{A}} + i)^{-1}\mu_l$, then for all $f \in \mathcal{D}(\Gamma_{\mathbf{A}})$

$$
\begin{aligned}
(\nu_l, f)_{\Gamma_{\mathbf{A}}} &= ((\Gamma_{\mathbf{A}} + i)\nu_l, (\Gamma_{\mathbf{A}} + i)f) \\
&= ((\Gamma_{\mathbf{A}} + i)(\Gamma_{\mathbf{A}} + i)^{-1}\mu_l, (\Gamma_{\mathbf{A}} + i)f) \\
&= (\mu_l, (\Gamma_{\mathbf{A}} + i)f) \to 0
\end{aligned} \tag{4.48}
$$

since $(\Gamma_{\mathbf{A}} + i)f \in L^2(\mathbb{R}^n)$ and so $\nu_l \rightharpoonup 0$ in $\mathcal{H}(\Gamma_{\mathbf{A}})$. Given $\varepsilon > 0$, choose $\tilde{k}_+$ such that

$$
\begin{aligned}
&\tilde{k}_+ \in C_0^\infty(\mathbb{R}_+, L^\infty(\mathbb{S}^{n-1})), \text{ supp } \tilde{k}_+ \subset \Omega_\varepsilon = B(0; c_\varepsilon) \setminus B\Big(0; \frac{1}{c_\varepsilon}\Big), \\
&\|\tilde{k}_+\|_{L^\infty(\mathbb{R}^n)} < c_\varepsilon \text{ and } \| \|k_+ - \tilde{k}_+\|_{L^\infty(\mathbb{S}^{n-1})} \|_{L^1(\mathbb{R}_+;r^7dr)} < \varepsilon
\end{aligned} \tag{4.49}
$$

for some $c_\varepsilon > 1$. Now

$$\begin{aligned}
\|K_+^{\frac{1}{2}}(\Gamma_{\mathbf{A}} + i)^{-1}\mu_l\|^2 =& \|K_+^{\frac{1}{2}}\nu_l\|^2 \\
=& \int_{\mathbb{R}^n} k_+(\mathbf{x})|\nu_l(\mathbf{x})|^2 d\mathbf{x} \\
=& \int_{\mathbb{R}^n} \tilde{k}_+(\mathbf{x})|\nu_l(\mathbf{x})|^2 d\mathbf{x} \\
&+ \int_{\mathbb{R}^n} (k_+ - \tilde{k}_+)(\mathbf{x})|\nu_l(\mathbf{x})|^2 d\mathbf{x}. \qquad (4.50)
\end{aligned}$$

where

$$\begin{aligned}
&\left| \int_0^\infty \int_{\mathbb{S}^{n-1}} (k_+(r,\omega) - \tilde{k}_+(r,\omega))|\nu_l(r,\omega)|^2 d\omega r^{n-1} dr \right| \\
&\leq \int_0^\infty \|k_+(r,\cdot) - \tilde{k}_+(r,\cdot)\|_{L^\infty(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} |\nu_l(r,\omega)|^2 d\omega r^{n-1} dr \\
&= \int_0^\infty r^7 \|k_+(r,\cdot) - \tilde{k}_+(r,\cdot)\|_{L^\infty(\mathbb{S}^{n-1})} . r^{n-8} \|\nu_l(r,\cdot)\|^2_{L^2(\mathbb{S}^{n-1})} dr \\
&\leq \| r^{n-8} \|\nu_l\|^2_{L^2(\mathbb{S}^{n-1})} \|_{L^\infty(\mathbb{R}_+)} \\
&\quad . \int_0^\infty r^7 \|k_+(r,\cdot) - \tilde{k}_+(r,\cdot)\|_{L^\infty(\mathbb{S}^{n-1})} dr. \\
&\leq \| \|k_+ - \tilde{k}_+\|_{L^\infty(\mathbb{S}^{n-1})} \|_{L^1(\mathbb{R}_+; r^7 dr)} \| r^{n-8} \|\nu_l\|^2_{L^2(\mathbb{S}^{n-1})} \|_{L^\infty(\mathbb{R}_+)} \\
&\leq \varepsilon \| r^{n-8} \|\nu_l\|^2_{L^2(\mathbb{S}^{n-1})} \|_{L^\infty(\mathbb{R}_+)} \\
&\leq \varepsilon C \|\Gamma_{\mathbf{A}} \nu_l\| : \qquad (4.51)
\end{aligned}$$

this is due to the extension of Corollary 4.6 by continuity to $\mathcal{H}(\Gamma_{\mathbf{A}})$, since $\mathcal{D}_0$ is dense in $\mathcal{H}(\Gamma_{\mathbf{A}})$. Also

$$\begin{aligned}
\int_{\mathbb{R}^n} \tilde{k}_+(\mathbf{x})|\nu_l(\mathbf{x})|^2 d\mathbf{x} =& \int_{\Omega_\varepsilon} \tilde{k}_+(\mathbf{x})|\nu_l(\mathbf{x})|^2 d\mathbf{x} \\
\leq& \|\tilde{k}_+\|_{L^\infty(\Omega_\varepsilon)} \|\nu_l\|^2_{L^2(\Omega_\varepsilon)} \leq c_\varepsilon \|\nu_l\|^2_{L^2(\Omega_\varepsilon)}. \qquad (4.52)
\end{aligned}$$

and so consequently

$$\|K_{+}^{\frac{1}{2}}(\Gamma_{\mathbf{A}}+i)^{-1}\mu_l\|^2 \le c_\varepsilon\|\nu_l\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon C\|\Gamma_{\mathbf{A}}\nu_l\|. \tag{4.53}$$

For $\psi \in \mathcal{D}_0$, we have

$$\begin{aligned}
\|\nabla_{\mathbf{A}}\psi\|^2 =&(\Delta_{\mathbf{A}}\psi,\psi) \le \|\Delta_{\mathbf{A}}\psi\|\|\psi\|\\
\le&\frac{1}{\sqrt{3}}\|\Delta_{\mathbf{A}}\psi\|^2 + \frac{\sqrt{3}}{4}\|\psi\|^2 = \frac{1}{\sqrt{3}}(\Delta^2_{\mathbf{A}}\psi,\psi) + \frac{\sqrt{3}}{4}\|\psi\|^2\\
\le&\frac{1}{\sqrt{3}}\|\Delta^2_{\mathbf{A}}\psi\|\|\psi\| + \frac{\sqrt{3}}{4}\|\psi\|^2\\
\le&\frac{1}{\sqrt{3}}\|\Delta^2_{\mathbf{A}}\psi\|^2 + \Big(\frac{1}{4\sqrt{3}} + \frac{\sqrt{3}}{4}\Big)\|\psi\|^2\\
=&\frac{1}{\sqrt{3}}\Big(\|\Delta^2_{\mathbf{A}}\psi\|^2 + \|\psi\|^2\Big)\\
=&\frac{1}{\sqrt{3}}\Big(\|\Gamma_{\mathbf{A}}\psi\|^2 + \|\psi\|^2\Big) = \frac{1}{\sqrt{3}}\|(\Gamma_{\mathbf{A}}+i)\psi\|^2\\
=&\frac{1}{\sqrt{3}}\|\psi\|^2
\end{aligned} \tag{4.54}$$

since $\|\Gamma_{\mathbf{A}}\psi\|^2 = (\Delta^4_{\mathbf{A}}\psi,\psi) = \|\Delta^2_{\mathbf{A}}\psi\|^2$. Since $\mathcal{D}_0$ is dense in $\mathcal{H}(\Gamma_{\mathbf{A}})$, (4.54) holds for all $\psi \in \mathcal{H}(\Gamma_{\mathbf{A}})$. Following the steps of the Proof of Theorem 7.21 in Lieb and Loss [22], suppose $0 \ne f \in \mathcal{D}_0$, then

$$\begin{aligned}
\frac{\partial}{\partial x_j}|f| =&\frac{1}{2}\Big(\frac{\bar{f}}{|f|}\frac{\partial f}{\partial x_j} + \frac{f}{|f|}\frac{\partial \bar{f}}{\partial x_j}\Big)\\
=&Re\Big[\frac{\bar{f}}{|f|}\frac{\partial f}{\partial x_j}\Big] = Re\Big[\frac{\bar{f}}{|f|}\Big(\frac{\partial}{\partial x_j} - iA_j\Big)f\Big].
\end{aligned} \tag{4.55}$$

Therefore

$$|\nabla|f|| = \Big|Re\Big[\frac{\bar{f}}{|f|}\nabla_{\mathbf{A}}f\Big]\Big| \le |\nabla_{\mathbf{A}}f| \tag{4.56}$$

and applying this diamagnetic inequality to (4.54) gives

$$\|\nabla|\nu_l|\|^2 \le \|\nabla_{\mathbf{A}}\nu_l\|^2 \le \frac{\|\mu_l\|^2}{\sqrt{3}} \tag{4.57}$$

and so $\{|\nu_l|\} \in H^1(\mathbb{R}^n)$. The continuity of the embedding $\mathcal{H}(\Gamma_{\mathbf{A}}) \hookrightarrow H^1(\mathbb{R}^n)$ just established and $\nu_l \rightharpoonup 0$ in $\mathcal{H}(\Gamma_{\mathbf{A}})$ implies that $|\nu_l| \rightharpoonup 0$ (see Dunford and Schwarz [10], Theorem V.3.15). Furthermore, since $H^1(\mathbb{R}^n)$ is compactly embedded in $L^2(\Omega_\varepsilon)$, see [11], it follows by Rellich's Theorem that $\nu_l \to 0$ in $L^2(\Omega_\varepsilon)$. Hence (4.53) becomes

$$\lim_{l\to\infty} \|K_+^{\frac{1}{2}}(\Gamma_{\mathbf{A}} + i)^{-1}\mu_l\|^2 \;\le\; \varepsilon C \lim_{l\to\infty} \|\Gamma_{\mathbf{A}}\nu_l\|. \tag{4.58}$$

But ε can be chosen arbitrarily small, therefore

$$\lim_{l\to\infty} \|K_+^{\frac{1}{2}}(\Gamma_{\mathbf{A}} + i)^{-1}\mu_l\|^2 = 0 \tag{4.59}$$

and $K_+^{\frac{1}{2}}(\Gamma_{\mathbf{A}} + i)^{-1}$ is compact on $L^2(\mathbb{R}^n)$ and by definition in [23], $K_+^{\frac{1}{2}}$ is relatively compact with respect to $\Gamma_{\mathbf{A}}$ ($\Gamma_{\mathbf{A}}$-compact).

iii. Furthermore by Lemma III.7.7 in [11], $\Gamma_{\mathbf{A}}$-compact implies that $K_+^{\frac{1}{2}}$ has $\Gamma_{\mathbf{A}}$-bound zero.

■

Theorem 4.8. *Assume the hypothesis of lemma 4.7, then*

i. the form $\mathfrak{g}_{\mathbf{A}}[\mu, \nu] = (\Gamma_{\mathbf{A}}\mu, \Gamma_{\mathbf{A}}\nu)$ is closed and $\Gamma_{\mathbf{A}}^2$ is the associated self-adjoint operator.

ii. the symmetric form $\mathfrak{h}_{\mathbf{A}}[\mu, \nu] = (\Gamma_{\mathbf{A}}\mu, \Gamma_{\mathbf{A}}\nu) + (K_+\mu, \nu)$ is closed and bounded below. Let $H_{\mathbf{A}}^2 = \Gamma_{\mathbf{A}}^2 + K_+$ denote the operator associated with $\mathfrak{h}_{\mathbf{A}}$. It has form domain

$$\mathcal{D}(\mathfrak{h}_{\mathbf{A}}) = \mathcal{Q}(H_{\mathbf{A}}^2) = \mathcal{Q}(\Gamma_{\mathbf{A}}^2) = \mathcal{D}(\Gamma_{\mathbf{A}}) \tag{4.60}$$

and

$$\sigma_{ess}(H_{\mathbf{A}}^2) = \sigma_{ess}(\Gamma_{\mathbf{A}}^2) = [0, \infty). \tag{4.61}$$

Proof. i. This follows from Examples VI.1.23 and VI.2.13 in Kato [16] but for completeness, a proof is constructed. Define the form

$$\mathfrak{g}_{\mathbf{A}}[\mu, \nu] := (\Gamma_{\mathbf{A}}\mu, \Gamma_{\mathbf{A}}\nu) \,, \quad \mathcal{D}(\mathfrak{g}_{\mathbf{A}}) = \mathcal{D}(\Gamma_{\mathbf{A}}). \tag{4.62}$$

So

$$\mathfrak{g}_{\mathbf{A}}[\mu_n - \mu_m] = \|\Gamma_{\mathbf{A}}\mu_n - \Gamma_{\mathbf{A}}\mu_m\|^2. \tag{4.63}$$

Since $\Gamma_{\mathbf{A}}$ is self-adjoint and hence, in particular, closed, $\mathcal{H}(\Gamma_{\mathbf{A}})$ is complete. To prove that $\mathfrak{g}_{\mathbf{A}}$ is closed, it is necessary to show that

$$\mu_n \to \mu, \quad \mathfrak{g}_{\mathbf{A}}[\mu_n - \mu_m] \to 0 \quad \text{as } n, m \to \infty \tag{4.64}$$

imply that $\mathfrak{g}_{\mathbf{A}}[\mu_n - \mu] \to 0$. From (4.63) and (4.64), it follows that $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{H}(\Gamma_{\mathbf{A}})$ and hence goes to a limit, ν say. But $\nu = \mu$ since

$$\|\mu_n - \nu\| \leq \|\mu_n - \nu\|_{\Gamma_{\mathbf{A}}} \to 0. \tag{4.65}$$

The fact that $\mathfrak{g}_{\mathbf{A}}$ is closed now follows since $\mathfrak{g}_{\mathbf{A}}[\mu_n - \mu] = \|\Gamma_{\mathbf{A}}\mu_n - \Gamma_{\mathbf{A}}\mu\|^2$. The adjoint $\mathfrak{g}_{\mathbf{A}}^*$ of the form $\mathfrak{g}_{\mathbf{A}}$ is

$$\mathfrak{g}_{\mathbf{A}}^*[\nu, \mu] = \overline{\mathfrak{g}_{\mathbf{A}}[\nu, \mu]} = \overline{(\Gamma_{\mathbf{A}}\nu, \Gamma_{\mathbf{A}}\mu)} = (\Gamma_{\mathbf{A}}\mu, \Gamma_{\mathbf{A}}\nu) = \mathfrak{g}_{\mathbf{A}}[\mu, \nu] \tag{4.66}$$

and

$$\mathfrak{g}_{\mathbf{A}}[\mu] = (\Gamma_{\mathbf{A}}\mu, \Gamma_{\mathbf{A}}\mu) = \|\Gamma_{\mathbf{A}}\mu\|^2 \geq 0, \tag{4.67}$$

so $\mathfrak{g}_{\mathbf{A}}$ is a semi-bounded symmetric form. Consequently by the First

Representation Theorem, in [16], there exists a unique self-adjoint operator G such that $\mathcal{D}(G) \subset \mathcal{D}(\mathfrak{g}_{\mathbf{A}})$ and

$$\mathfrak{g}_{\mathbf{A}}[\mu, \nu] = (\Gamma_{\mathbf{A}}\mu, \Gamma_{\mathbf{A}}\nu) = (G\mu, \nu) \tag{4.68}$$

for all $\mu \in \mathcal{D}(G), \nu \in \mathcal{D}(\mathfrak{g}_{\mathbf{A}})$. Since G is unique and

$$(\Gamma_{\mathbf{A}}\mu, \Gamma_{\mathbf{A}}\nu) = (\Gamma_{\mathbf{A}}^2\mu, \nu) \tag{4.69}$$

for $\mu \in \mathcal{D}(\Gamma_{\mathbf{A}}^2), \nu \in \mathcal{D}(\mathfrak{g}_{\mathbf{A}})$ then $G = \Gamma_{\mathbf{A}}^2$ since $\mathcal{D}(\Gamma_{\mathbf{A}}^2)$ is dense in $\mathcal{D}(\mathfrak{g}_{\mathbf{A}})$ - it is a core of $\mathfrak{g}_{\mathbf{A}}$.

ii. By Lemma 4.7, $K_+^{\frac{1}{2}}$ has $\Gamma_{\mathbf{A}}$-bound zero i.e.

$$||K_+^{\frac{1}{2}}\mu||^2 \leq a||\mu||^2 + b||\Gamma_{\mathbf{A}}||^2 \tag{4.70}$$

with the greatest lower bound of b, $b_0 = 0$. Due to (4.44), (4.70) can be rewritten as

$$\mathfrak{K}[\mu] \leq a||\mu||^2 + b(\Gamma_{\mathbf{A}}\mu, \Gamma_{\mathbf{A}}\mu) \tag{4.71}$$

so the form $\mathfrak{K}$ is relatively (form) bounded with respect to $\mathfrak{g}_{\mathbf{A}}$ with $\mathfrak{g}_{\mathbf{A}}$-bound zero. Define the form

$$\mathfrak{h}_{\mathbf{A}} = \mathfrak{g}_{\mathbf{A}} + \mathfrak{K} \tag{4.72}$$

on $\mathcal{D}(\mathfrak{h}_{\mathbf{A}}) = \mathcal{D}(\mathfrak{g}_{\mathbf{A}}) \cap \mathcal{D}(\mathfrak{K}) = \mathcal{D}(\mathfrak{g}_{\mathbf{A}})$ since $\mathcal{D}(\mathfrak{g}_{\mathbf{A}}) \subset \mathcal{D}(\mathfrak{K})$. It is a consequence of $\mathfrak{K}$ being $\mathfrak{g}_{\mathbf{A}}$-bounded with $b < 1$ and Theorem VI.1.33 in [16] that $\mathfrak{h}_{\mathbf{A}}$ is closed since $\mathfrak{g}_{\mathbf{A}}$ is a closed form and that they have the same domain.

We denote the non-negative self-adjoint operator associated with $\mathfrak{h}_{\mathbf{A}}$ by $H_{\mathbf{A}}^2$. Recall that $\Gamma_{\mathbf{A}}^2$ is the self-adjoint operator associated with $\mathfrak{g}_{\mathbf{A}}$

(see (4.67)). In Lemma 4.7, the operator $K_+^{\frac{1}{2}}(\Gamma_{\mathbf{A}} - i)^{-1}$ is shown to be compact in $L^2(\mathbb{R}^n)$. Also $\mathcal{H}(\Gamma_{\mathbf{A}}^2)$ is continuously embedded in $\mathcal{H}(\Gamma_{\mathbf{A}})$ since, if $f \in \mathcal{D}(\Gamma_{\mathbf{A}}^2)$, we have

$$\begin{aligned} \|\mu\|_{\Gamma_{\mathbf{A}}}^2 =&\|\Gamma_{\mathbf{A}}\mu\|^2 + \|\mu\|^2 \\ =&(\Gamma_{\mathbf{A}}^2\mu, \mu) + \|\mu\|^2 \\ \leq&\|\Gamma_{\mathbf{A}}^2\mu\|^2 + \frac{5}{4}\|\mu\|^2. \end{aligned} \tag{4.73}$$

It follows that

$$K_+^{\frac{1}{2}}(\Gamma_{\mathbf{A}}^2 - i)^{-1} \in \mathcal{K}(L^2(\mathbb{R}^n)) \tag{4.74}$$

where $\mathcal{K}(L^2(\mathbb{R}^n)) = \mathcal{K}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ is the set of all compact mappings from $L^2(\mathbb{R}^n)$ into itself. Theorem IV.4.4 shows that (4.74) implies the assumptions of Theorem IV.4.2.(iv) (both of [11]), which gives

$$(\Gamma_{\mathbf{A}}^2 - i)^{-1} - (H_{\mathbf{A}}^2 - i)^{-1} \in \mathcal{K}(L^2(\mathbb{R}^n), \mathcal{Q}(\Gamma_{\mathbf{A}}^2)) \subset \mathcal{K}(L^2(\mathbb{R}^n). \tag{4.75}$$

Since the operators $\Gamma_{\mathbf{A}}^2$ and $H_{\mathbf{A}}^2$ are both self-adjoint then the complex number i belongs to the intersection of their resolvent sets and it follows by (4.75) and Theorem IX.2.4, also from [11], that

$$\sigma_{ess}(\Gamma_{\mathbf{A}}^2) = \sigma_{ess}(H_{\mathbf{A}}^2). \tag{4.76}$$

Alternatively (4.76) follows from Reed and Simon [23], pg 369. To calculate $\sigma_{ess}(\Gamma_{\mathbf{A}}^2)$, consider the following scale transformation

$$S : u \mapsto u_c \ ; \ u_c(\mathbf{x}) = c^{\frac{n}{2}}u(c\mathbf{x}), \ c > 0. \tag{4.77}$$

In view of the denseness of $\mathcal{D}_0$ in $\mathcal{H}(\Gamma_{\mathbf{A}})$, it is sufficient to assume

$u \in \mathcal{D}_0$ in the following argument. Let $\mathbf{y} = c\mathbf{x}$, then

$$\begin{aligned}\|u\|^2 &= \int_{\mathbb{R}^n} |u(\mathbf{y})|^2 d\mathbf{y} = \int_{\mathbb{R}^n} c^{-n} |u_c(\mathbf{x})|^2 c^n d\mathbf{x} \\ &= \int_{\mathbb{R}^n} |u_c(\mathbf{x})|^2 d\mathbf{x} = \|u_c\|^2. \end{aligned} \tag{4.78}$$

Hence S is a unitary transformation, moreover

$$\frac{\partial u_c(\mathbf{x})}{\partial x_j} = c^{\frac{n}{2}} \sum_k \frac{\partial u(\mathbf{y})}{\partial y_k} \frac{\partial y_k}{\partial x_j} = c^{\frac{n}{2}+1} \frac{\partial u(\mathbf{y})}{\partial y_j} \tag{4.79}$$

which implies that

$$\nabla_{\mathbf{x}} u_c(\mathbf{x}) = c^{\frac{n}{2}+1} \nabla_{\mathbf{y}} u(\mathbf{y}), \tag{4.80}$$

$$\Delta_{\mathbf{x}}^4 u_c(\mathbf{x}) = c^{4n+8} \Delta_{\mathbf{y}}^4 u(\mathbf{y}). \tag{4.81}$$

Since

$$\mathbf{A}_{\mathbf{x}} = \frac{\tilde{\Psi}(\theta)}{|\mathbf{x}|} \vec{e}_\theta = c \frac{\tilde{\Psi}(\theta)}{|\mathbf{y}|} \vec{e}_\theta = c\mathbf{A}_{\mathbf{y}}, \tag{4.82}$$

it follows that

$$\begin{aligned}\nabla_{\mathbf{A}} u_c(\mathbf{x}) =& (\nabla_{\mathbf{x}} + i\mathbf{A}_{\mathbf{x}}) u_c(\mathbf{x}) = \nabla_{\mathbf{x}} u_c(\mathbf{x}) + i\mathbf{A}_{\mathbf{x}} u_c(\mathbf{x}) \\ =& c^{\frac{n}{2}+1} \nabla_{\mathbf{y}} u_c(\mathbf{y}) + ic\mathbf{A}_{\mathbf{y}} c^{\frac{n}{2}} u(\mathbf{y}) = c^{\frac{n}{2}+1} \nabla_{\mathbf{A}} u(\mathbf{y}) \end{aligned} \tag{4.83}$$

and so

$$\Delta_{\mathbf{A}}^4 u_c(\mathbf{x}) = c^{4n+8} \Delta_{\mathbf{A}}^4 u(\mathbf{y}). \tag{4.84}$$

Therefore on $\mathcal{D}_0$

$$S^{-1}(\Delta_{\mathbf{A}}^4) S \; = \; c^{\frac{7n}{2}+8} \Delta_{\mathbf{A}}^4. \tag{4.85}$$

This gives for $u, v \in \mathcal{D}_0$,

$$(\Gamma_{\mathbf{A}} S u, \Gamma_{\mathbf{A}} S v) = c^{\frac{7n}{2}+8} (\Gamma_{\mathbf{A}} u, \Gamma_{\mathbf{A}} v). \tag{4.86}$$

Since S maps $\mathcal{D}_0$ to $\mathcal{D}_0$, it follows from the uniqueness property of the Friedrichs extension that

$$S^{-1}(\Gamma_{\mathbf{A}}^2) S \;=\; c^{\frac{7n}{2}+8} \Gamma_{\mathbf{A}}^2 \tag{4.87}$$

and hence, since $\Gamma_{\mathbf{A}}^2 \geq 0$ and c is arbitrary, $\sigma(\Gamma_{\mathbf{A}}^2) = [0, \infty)$.

■

4.3 A CLR type inequality for $\Delta_{\mathbf{A}}^4 + K_+ - V$ in $L^2(\mathbb{R}^8)$

In [14], Evans and Lewis considered the problem of finding a bound for the number of negative eigenvalues of $\Delta_{\mathbf{A}}^2 - V$ in four dimensions where

$$V \in L^1(\mathbb{R}_+; r^3 dr) \otimes L^\infty(\mathbb{S}^3). \tag{4.88}$$

A necessary step of the method employed in [14] was to show that $D^2 = (L_r + \frac{1}{r^2}\Lambda_\omega)^2$ is of the form

$$D^2 = \bigoplus_{m \in \mathcal{I}} \{\chi_n(\lambda_m) \bigotimes I_m\} \tag{4.89}$$

where

$$\begin{aligned}\chi_n(\lambda_m) =& \frac{1}{r^{n-1}} \frac{d^2}{dr^2}\left(r^{n-1} \frac{d^2}{dr^2}\right) - \frac{2\lambda_m + (n-1)}{r^{n-1}} \frac{d}{dr}\left(r^{n-3} \frac{d}{dr}\right) \\ &+ \frac{\lambda_m(\lambda_m + 2(n-4))}{r^4}.\end{aligned} \tag{4.90}$$

Suppose $\{u_m\}_{m\in\mathcal{I}}$ are the normalised eigenvectors of the operator Λ_ω, where $\mathcal{I}$ is a countable index set, then I_m is the identity on the orthonormal basis $\{u_m\}_{m\in\mathcal{I}}$ of $L^2(\mathbb{S}^{n-1})$. (4.89) and (4.90) was used to show in [14] that for $c < \min\{\Psi^4, (1-\Psi)^4\}$, we have

$$\begin{aligned}\Delta_{\mathbf{A}}^2 - V &= \bigoplus_{m\in\mathbb{Z}^{(4)}} \{(\chi_n(\rho_m^\Psi) - V(r)) \bigotimes I_m^\Psi\}\\ &\geq \bigoplus_{|m|>1} \left\{\left(\chi_n(\rho_m) + \frac{c}{r^4}\right) \bigotimes I_m^0\right\} = \Delta^2 + \frac{c}{r^4}\end{aligned} \tag{4.91}$$

where

$$\rho_m = m(m+n-2), \tag{4.92}$$

$$\rho_m^\Psi = (m+\tilde{\Psi})(m+\tilde{\Psi}+n-2), \tag{4.93}$$

see Theorem 3.2. I_m^0 and I_m^Ψ are identities on the orthonormal bases of $L^2(\mathbb{S}^{n-1})$ formed by the normalised eigenvectors of the Laplace-Beltrami operator and the magnetic Laplace-Beltrami operator respectively. The inequality (4.91) implies that

$$N(\Delta_{\mathbf{A}}^2 - V) \leq N\left(\Delta^2 + \frac{c}{r^4}\right) \tag{4.94}$$

where $N(\cdot)$ is the number of negative eigenvalues of the operator. It follows that to look at $\Delta_{\mathbf{A}}^4 - V$, an expression of the form (4.89) for the operator D^4 is needed. It follows from (4.89) that

$$\begin{aligned}\|D^2 f\|^2 &= \int_{\mathbb{R}^n} D^2 f(\mathbf{x}).\overline{D^2 f(\mathbf{x})} d\mathbf{x}\\ &= \sum_m \int_0^\infty \left(\int_{\mathbb{S}^{n-1}} D^2 f(r,\omega)\overline{u_m(\omega)} d\omega\right)\\ &\overline{\cdot\left(\int_{\mathbb{S}^{n-1}} D^2 f(r,\omega)\overline{u_m(\omega)} d\omega\right)}\end{aligned}$$

$$
\begin{aligned}
&= \sum_m \int_0^\infty \left(\int_{\mathbb{S}^{n-1}} \chi_n(\lambda_m) f(r,\omega) \overline{u_m(\omega)} d\omega \right) \\
&\quad \cdot \overline{\left(\int_{\mathbb{S}^{n-1}} \chi_n(\lambda_m) f(r,\omega) \overline{u_m(\omega)} d\omega \right)} \\
&= \sum_m \int_{\mathbb{R}^n} \chi_n(\lambda_m) f(\mathbf{x}) . \overline{\chi_n(\lambda_m) f(\mathbf{x})} d\mathbf{x} \\
&= \sum_m \int_{\mathbb{R}^n} \chi_n(\lambda_m)^2 f(\mathbf{x}) . \overline{f(\mathbf{x})} d\mathbf{x} \qquad (4.95)
\end{aligned}
$$

and so with this in mind, the operator χ_n^2 is investigated.

Lemma 4.9.

$$
\begin{aligned}
\chi_n^2(\lambda) = &\sum_{j=1}^{4} \frac{(-1)^j [P_j(n,\lambda) + c_j(n)]}{r^{n-1}} \frac{d^j}{dr^j} \left(r^{n-9+2j} \frac{d^j}{dr^j} \right) \\
&+ \frac{P_0(\lambda)}{r^8}
\end{aligned} \qquad (4.96)
$$

where

$$
\begin{aligned}
P_4(n,\lambda) &= 0, && (4.97)\\
c_4(n) &= 1, && (4.98)\\
P_3(n,\lambda) &= 4\lambda, && (4.99)\\
c_3(n) &= 6(n-1), && (4.100)\\
P_2(n,\lambda) &= 6\lambda^2 + 6(3n-13)\lambda, && (4.101)\\
c_2(n) &= 6(n-1)(2n-7), && (4.102)\\
P_1(n,\lambda) &= 4\lambda^3 + (30n-158)\lambda + (53n^2 - 479n + 1146)\lambda, && (4.103)\\
c_1(n) &= 15(n-1)(n-3)(n-5), && (4.104)\\
P_0(n,\lambda) &= \lambda(\lambda + 2(n-4))(\lambda + 4(n-6))(\lambda + 6(n-8)). && (4.105)
\end{aligned}
$$

Proof. The proof consists of applying the operator $\chi_n(\lambda)$ to itself and using the operator relation

$$\frac{1}{r^\alpha}\frac{d}{dr} = \frac{d}{dr}\Big(\frac{\cdot}{r^\alpha}\Big) + \frac{\alpha}{r^{\alpha+1}} \tag{4.106}$$

to tidy the original expression into a summation of the form

$$\sum_j \frac{a_j}{r^{n-1}}\frac{d^j}{dr^j}\Big(r^l\frac{d^j}{dr^j}\Big). \tag{4.107}$$

For ease of reading, define

$$Q(j) = \frac{d^j}{dr^j}\Big(r^{n-9+2j}\frac{d^j}{dr^j}\Big). \tag{4.108}$$

Then

$$\chi_n^2(\lambda) = \chi_n(\lambda)\chi_n(\lambda) = \Sigma_{j=1}^8 a_j I_j + \frac{\lambda^2(\lambda+2(n-4))(\lambda+4(n-6))}{r^6} \tag{4.109}$$

where

$$a_1 = 1, \tag{4.110}$$

$$\begin{aligned} I_1 &= \frac{1}{r^{n-1}}\frac{d^2}{dr^2}\Big(r^{n-1}\frac{d^2}{dr^2}\Big(\frac{1}{r^{n-1}}\frac{d^2}{dr^2}r^{n-1}\frac{d^2}{dr^2}\Big)\Big)\\ &= Q(4) - 4(n-1)Q(3) + \frac{(n-1)(n-2)}{r^{n-1}}\frac{d^2}{dr^2}\Big(r^{n-4}\frac{d^3}{dr^3}\Big), \end{aligned} \tag{4.111}$$

$$a_2 = 2\lambda + (n-1), \tag{4.112}$$

$$\begin{aligned} I_2 &= -\frac{1}{r^{n-1}}\frac{d^2}{dr^2}\Big(r^{n-1}\frac{d^2}{dr^2}\Big(\frac{1}{r^{n-1}}\frac{d}{dr}r^{n-3}\frac{d}{dr}\Big)\Big)\\ &= -Q(3) + \frac{4}{r^{n-1}}\frac{d^2}{dr^2}\Big(r^{n-4}\frac{d^3}{dr^3}\Big) + 6(n-4)Q(2)\\ &\quad + \frac{12(n-3)}{r^{n-1}}\frac{d}{dr}\Big(r^{n-6}\frac{d^2}{dr^2}\Big) - 12(n-3)(n-6)Q(1), \end{aligned} \tag{4.113}$$

$$a_3 = \lambda(\lambda + 2(n-4)), \tag{4.114}$$

$$I_3 = \frac{1}{r^{n-1}}\frac{d^2}{dr^2}\Big(r^{n-1}\frac{d^2}{dr^2}\Big(\frac{\cdot}{r^4}\Big)\Big)$$

$$
\begin{aligned}
&=Q(2)-\frac{8}{r^{n-1}}\frac{d}{dr}\left(r^{n-6}\frac{d^2}{dr^2}\right)-[8(n-6)-20]Q(1)\\
&\quad+\frac{20(n-7)}{r^7}\frac{d}{dr}+\frac{20(n-7)(n-8)}{r^8}, && (4.115)\\
a_4 &=2\lambda+(n-1), && (4.116)\\
I_4 &=-\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-3}\frac{d}{dr}\left(\frac{1}{r^{n-1}}\frac{d^2}{dr^2}r^{n-1}\frac{d^2}{dr^2}\right)\right)\\
&=-Q(3)-\frac{4}{r^{n-1}}\frac{d^2}{dr^2}\left(r^{n-4}\frac{d^3}{dr^3}\right)+2(n-4)Q(2)\\
&\quad+\frac{12(n-3)}{r^{n-1}}\frac{d}{dr}\left(r^{n-6}\frac{d^2}{dr^2}\right), && (4.117)\\
a_5 &=(2\lambda+(n-1))^2, && (4.118)\\
I_5 &=\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-3}\frac{d}{dr}\left(\frac{1}{r^{n-1}}\frac{d}{dr}r^{n-3}\frac{d}{dr}\right)\right)\\
&=Q(2)-3(n-3)Q(1), && (4.119)\\
a_6 &=\lambda(2\lambda+(n-1))(\lambda+2(n-4)), && (4.120)\\
I_6 &=\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-3}\frac{d}{dr}\left(\frac{\cdot}{r^4}\right)\right)\\
&=-Q(1)+\frac{4}{r^7}\frac{d}{dr}+\frac{4(n-8)}{r^8}, && (4.121)\\
a_7 &=\lambda(\lambda+2(n-4)), && (4.122)\\
I_7 &=\frac{1}{r^{n+3}}\frac{d^2}{dr^2}r^{n-1}\frac{d^2}{dr^2}\\
&=Q(2)+\frac{8}{r^{n-1}}\frac{d}{dr}\left(r^{n-6}\frac{d^2}{dr^2}\right)+20Q(1)-\frac{20(n-7)}{r^7}\frac{d}{dr} && (4.123)
\end{aligned}
$$

and

$$
\begin{aligned}
a_8 &=\lambda(2\lambda+(n-1))(\lambda+2(n-4)), && (4.124)\\
I_8 &=\frac{1}{r^{n+3}}\frac{d}{dr}r^{n-3}\frac{d}{dr}\\
&=-Q(1)-\frac{4}{r^7}\frac{d}{dr}. && (4.125)
\end{aligned}
$$

Substituting (4.110)-(4.125) into (4.109) gives the required result. ■

Theorem 4.10. *Let* $J_{\mathbf{A}} := \Gamma_{\mathbf{A}}^2 + K_+ - V$, $0 \leq V \leq L^1((0,\infty); r^7 dr)$, $\tilde{\Psi} \in (0,1)$ *and* $n = 8$. *Then there exists a positive constant* $C(\tilde{\Psi})$ *such that the number* $N(J_{\mathbf{A}})$ *of negative eigenvalues of* $J_{\mathbf{A}}$ *satisfies*

$$N(J_{\mathbf{A}}) \leq C(\tilde{\Psi}) \| \|V\|_{L^\infty(\mathbb{S}^7)} \|_{L^1((0,\infty);r^7 dr)} \tag{4.126}$$

where $C(\tilde{\Psi})$ *depends on the distance of* $\tilde{\Psi}$ *from* $\{0,1\}$.

Proof. From (4.95)

$$\Delta_{\mathbf{A}}^4 + K_+ - V \geq \Delta_{\mathbf{A}}^4 - V = \bigoplus_{m \in \mathbb{Z}^{(n)}} \{[\chi_8(\rho_m^\Psi)^2 - V(r)] \bigotimes I_m^\Psi\} \tag{4.127}$$

where $\chi_8(\rho_m^\Psi)^2$ is given by Lemma 4.9 and due to (3.61)

$$\rho_m^\Psi = (m + \tilde{\Psi})^2 - 9, \tag{4.128}$$

$$\hat{\mathbb{Z}}^{(n)} = \{m \in \mathbb{Z} : \rho_m^\Psi \geq 0\} = \{m \in \mathbb{Z} : m \leq -4 \text{ or } m \geq 3\} \tag{4.129}$$

since $\tilde{\Psi} \in (0,1)$. By identical steps,

$$\Delta^4 + \frac{c}{r^8} - V = \bigoplus_{|m| \geq 3} \left\{ \left[\chi_8(\rho_m)^2 + \frac{c}{r^8} - V(r)\right] \bigotimes I_m^0 \right\} \tag{4.130}$$

where

$$\rho_m = m^2 - 9. \tag{4.131}$$

Now

$$
\begin{aligned}
\chi_8(\rho_m^\Psi)^2 - \chi_8(\rho_m)^2 - \frac{c}{r^8} \\
&= \sum_{j=1}^{3} \frac{(-1)^j [P_j(8,\rho_m^\Psi) - P_j(8,\rho_m)]}{r^7} \frac{d^j}{dr^j}\left(r^{2j-1}\frac{d^j}{dr^j}\right) \\
&+ \frac{P_0(8,\rho_m^\Psi) - P_0(8,\rho_m) - c}{r^8}.
\end{aligned} \tag{4.132}
$$

If $m \geq 3$ then

$$
\begin{aligned}
\rho_m^\Psi &= m^2 + 2m\tilde{\Psi} + \tilde{\Psi}^2 - 9 \\
&\geq m^2 - 9 + \tilde{\Psi}^2 = \rho_m + \tilde{\Psi}^2
\end{aligned} \tag{4.133}
$$

which implies that for $j \in \{1,2,3,4\}$

$$
(\rho_m^\Psi)^j \geq (\rho_m)^j + \tilde{\Psi}^{2j}. \tag{4.134}
$$

It follows from (4.134) that

$$
P_3(8,\rho_m^\Psi) - P_3(8,\rho_m) = 4[\rho_m^\Psi - \rho_m] \geq 0, \tag{4.135}
$$

$$
P_2(8,\rho_m^\Psi) - P_2(8,\rho_m) = 6[(\rho_m^\Psi)^2 - (\rho_m)^2] + 66[\rho_m^\Psi - \rho_m] \geq 0, \tag{4.136}
$$

$$
\begin{aligned}
P_1(8,\rho_m^\Psi) - P_1(8,\rho_m) &= 4[(\rho_m^\Psi)^3 - (\rho_m)^3] + 22[(\rho_m^\Psi)^2 - (\rho_m)^2] \\
&+ 706[\rho_m^\Psi - \rho_m] \geq 0.
\end{aligned} \tag{4.137}
$$

Also

$$
\begin{aligned}
P_0(8,\rho_m^\Psi) - P_0(8,\rho_m) &= [(\rho_m^\Psi)^4 - (\rho_m)^4] + 16[(\rho_m^\Psi)^3 - (\rho_m)^3] \\
&+ 48[(\rho_m^\Psi)^2 - (\rho_m)^2] - c \\
&\geq \tilde{\Psi}^8 + 16\tilde{\Psi}^6 + 48\tilde{\Psi}^4 - c
\end{aligned} \tag{4.138}
$$

and therefore if $c < \tilde{\Psi}^4(\tilde{\Psi}^2+4)(\tilde{\Psi}^2+12)$ then

$$\chi_8(\rho_m^{\Psi})^2 - \chi_8(\rho_m)^2 - \frac{c}{r^8} \geq 0. \tag{4.139}$$

If $m \leq -4$ then

$$\begin{aligned}\rho_m^{\Psi} =& m^2 + 2m\tilde{\Psi} + \tilde{\Psi}^2 - 9\\ =&(m+1)^2 - 9 + 2m\tilde{\Psi} + \tilde{\Psi}^2 - 2m - 1\\ =&\rho_{m+1} + (\tilde{\Psi}-1)^2 + 2(\tilde{\Psi}-1)(m+1)\\ \geq&\rho_{m+1} + (\tilde{\Psi}-1)^2 \end{aligned}\tag{4.140}$$

since $(\tilde{\Psi}-1)(m+1) > 0$. This implies that for $j \in \{1,2,3,4\}$

$$(\rho_m^{\Psi})^j \geq (\rho_{m+1})^j + (\tilde{\Psi}-1)^{2j} \tag{4.141}$$

which similarly to above, shows that

$$P_i(8,\rho_m^{\Psi}) - P_i(8,\rho_{m+1}) \geq 0 \tag{4.142}$$

when $i \in \{1,2,3\}$. Again

$$\begin{aligned}P_0(8,\rho_m^{\Psi}) - P_0(8,\rho_{m+1}) - c =& [(\rho_m^{\Psi})^4 - (\rho_{m+1})^4] + 16[(\rho_m^{\Psi})^3 - (\rho_{m+1})^3]\\ &+ 48[(\rho_m^{\Psi})^2 - (\rho_{m+1})^2] - c\\ \geq&(\tilde{\Psi}-1)^8 + 16(\tilde{\Psi}-1)^6 + 48(\tilde{\Psi}-1)^4 - c\end{aligned}\tag{4.143}$$

and

$$\chi(\rho_m^{\Psi})^2 - \chi(\rho_{m+1})^2 - \frac{c}{r^8} \geq 0 \tag{4.144}$$

if $c < (\tilde{\Psi}-1)^4[(\tilde{\Psi}-1)^2+4][(\tilde{\Psi}-1)^2+12]$. Therefore

$$
\begin{aligned}
&N\Big(\bigoplus_{m\geq 3}\{[\chi_8(\rho_m^{\Psi})^2 - V(r)]\bigotimes I_m^{\Psi}\}\Big)\\
&\qquad\leq N\Big(\bigoplus_{m\geq 3}\Big\{\Big[\chi_8(\rho_m)^2+\frac{c}{r^8}-V(r)\Big]\bigotimes I_m^0\Big\}\Big)
\end{aligned}
\tag{4.145}
$$

and

$$
\begin{aligned}
&N\Big(\bigoplus_{m\leq -4}\{[\chi_8(\rho_m^{\Psi})^2 - V(r)]\bigotimes I_m^{\Psi}\}\Big)\\
&\qquad\leq N\Big(\bigoplus_{m\leq -3}\Big\{\Big[\chi_8(\rho_m)^2+\frac{c}{r^8}-V(r)\Big]\bigotimes I_m^0\Big\}\Big)
\end{aligned}
\tag{4.146}
$$

if $c < \min\{\tilde{\Psi}^4[\tilde{\Psi}^2+4][\tilde{\Psi}^2+12], (\tilde{\Psi}-1)^4[(\tilde{\Psi}-1)^2+4][(\tilde{\Psi}-1)^2+12]\}$. Note that from (4.128) and (4.129), since $\tilde{\Psi}\in(0,1)$, only $m\geq 3$ and $m\leq -4$ needs to be considered, consequently

$$
N(J_{\mathbf{A}}) \leq N(\Gamma_{\mathbf{A}}^2 - V) \leq N\Big(\Delta^4+\frac{c}{r^8}-V(r)\Big). \tag{4.147}
$$

The operator $\Delta^4+\frac{c}{r^8}-V(r)$ is of the type for which Laptev and Netrusov in [19] established a CLR type bound, and so by Laptev and Netrusov's Theorem 1.2

$$
N(J_{\mathbf{A}}) \leq N\Big(\Delta^4+\frac{c}{r^8}-V(r)\Big) \leq C\|\|V\|_{L^\infty(\mathbb{S}^7)}\|_{L^1((0,\infty);r^7dr)} \tag{4.148}
$$

where the constant C depends on only c, which in turn depends on $\tilde{\Psi}$ and so $C = C(\tilde{\Psi})$. ∎

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