A Normalised Distance Function Considered over the Partition of the Unit Interval Generated by the Points of the Farey Tree

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DECLARATION

This work has not previously been accepted in substance for any degree and is not concurrently submitted in candidature for any degree.

Signed .... (candidate) Date 7th September 2010

STATEMENT 1

This thesis is being submitted in partial fulfillment of the requirements for the degree of **PhD**

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STATEMENT 2

This thesis is the result of my own independent work/investigation, except where otherwise stated. Other sources are acknowledged by explicit references.

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Finally, I'd like to pay special tribute and dedicate my work to the memory of my late grandfather, Doug Smith, without whom none of this would have been possible. You will be forever missed, Grampy.

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CHAPTER 1

Introduction

1. The Farey Tree

Consider two rational numbers, expressed in their lowest terms as $\frac{p}{q}$ and $\frac{p'}{q'}$. Their *mediant* is given by

$$\frac{p''}{q''} = \frac{p+p'}{q+q'}.$$

The Farey Tree of order n is defined as the set of ordered rational points such that

$$\mathrm{FT}_1 := \left\{ \frac{0}{1}, \frac{1}{1} \right\},$$

and for $n > 1$ they are constructed recursively according to:

$$\mathrm{FT}_n := \mathrm{FT}_{n-1} \cup \mathcal{Q}_n,$$

where

$$\mathcal{Q}_n := \left\{ \frac{p''}{q''} = \frac{p+p'}{q+q'}, \quad for\,all\ \ neighbouring\ \ pairs \left\{ \frac{p}{q}, \ \frac{p'}{q'} \right\} \ \ in\,\mathrm{FT}_{n-1} \right\}.$$

These elements may be arranged into a full binary tree also known as the *Stern-Brocot Tree*, named after the mathematician Moritz Stern (see [**39**]) and clockmaker Achille Brocot (see [**5**]). Elements at each level of this tree are the *mediants* of one of their parent vertices and of the nearest element in one of the levels above, see figure 1.1. An elegant property of this construction is that each element of $\mathbb{Q} \cap [0, 1]$ appears in the Tree once only, in reduced form. Indeed extending the idea so that a third vertex $\frac{1}{0}$ denoting infinity lies at top level of the tree, allows for a construction which is in 1 to 1 correspondence with the whole set $\mathbb{Q}$.

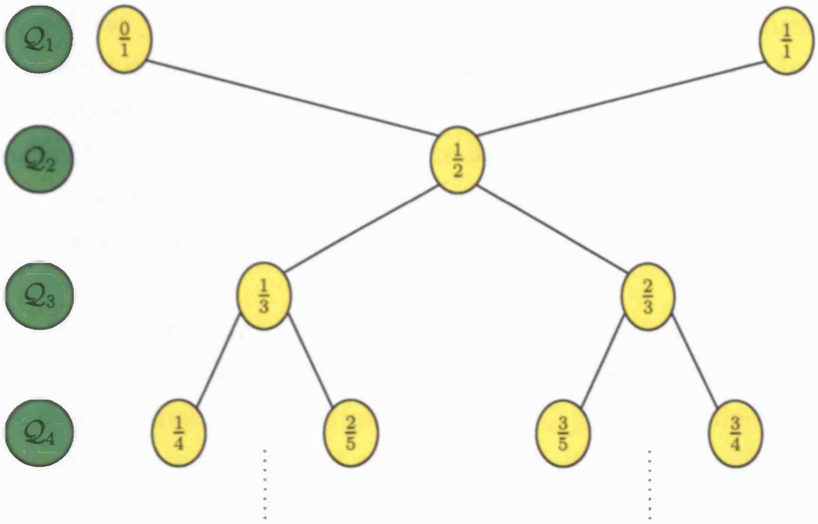


FIGURE 1.1. The Farey Tree

2. The Farey Series

The Farey Tree should not be confused with the Farey *Series*, in fact a sequence of sets of rationals constructed in a similar way to those of the Farey Tree. The chief difference in this construct is that there exists the restriction $q \leq n$ on the element-denominators. Define the Farey Series as:

$$\mathcal{F}_n := \left\{ \frac{p}{q} \ : \ \gcd(p,q) = 1, \text{ and } 0 \leq p \leq q \leq n, \ \ q \neq 0 \right\}.$$

Discussion on this topic supposedly began in 1816 when geologist John Farey wrote in the *Philosophical Magazine*, [**15**], a letter entitled "*On a Curious Property of Vulgar Fractions*":

> *If all the possible vulgar fractions of different values, whose greatest denominator (when in their lowest terms) does not exceed any given number, be arranged in the order of their values, or quotients; then if both the numerator and the denominator of any fraction therein, be added to the numerator and the denominator, respectively, of the fraction next but one to it (on either side), the sums will give the fraction next to it; although, perhaps, not in its lowest terms.*

Formal proof was provided by Cauchy in 1816 after seeing these comments in a French translation of the publication and attributed their discovery to Farey. This was somewhat incorrect as the initial work on these somewhat simple properties of fractions can be attributed to work by Haros some 14 years previously. It is an historical accident therefore, that this now-famous sequence has become synonymous with Farey's name.

Although the nomenclature suggests otherwise, the discoveries of the Farey Series and Tree are unrelated. It should be noted however, that since their constructions are similar it does follow that $\mathcal{F}_n \subseteq \mathrm{FT}_n$. By construction of the Farey Tree, since the number of 'child' fractions doubles with each new iteration of the Farey mediant algorithm one has the cardinality

$$|\mathrm{FT}_n| = 1 + 2^{n-1}.$$

This is in contrast to that of the Farey series for which it is easily seen that

$$|\mathcal{F}_n| = 1 + \sum_{k=1}^{n} \phi(k) = \frac{3n^2}{\pi^2} + \mathcal{O}\left(n \log(n)\right)$$

(see for example [**1**] or [**21**]).

3. Continued Fractions and Continuants

The Farey Tree sequence is related to the *Simple* Continued Fraction (SCF) expansion in a suprisingly simple way, which is discussed briefly here. The SCF expansions are generated via the mapping - known as the Gauss Map - defined

$$T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = \left\{ \frac{1}{x} \right\}, \quad T0 := 0,$$

for $x \in \mathbb{R}$ and where $\lfloor \theta \rfloor$ denotes the least integer part of θ. Figure 1.2 illustrates the behaviour of T.

Set $x - a_0$ such that it lies in $[0, 1)$, and define the following sequence of mappings:

$$T_0 := x - a_0, \ \ T_1 = T(x - a_0), \ \ T_2 = T(T_1), \ldots$$

Thus, setting

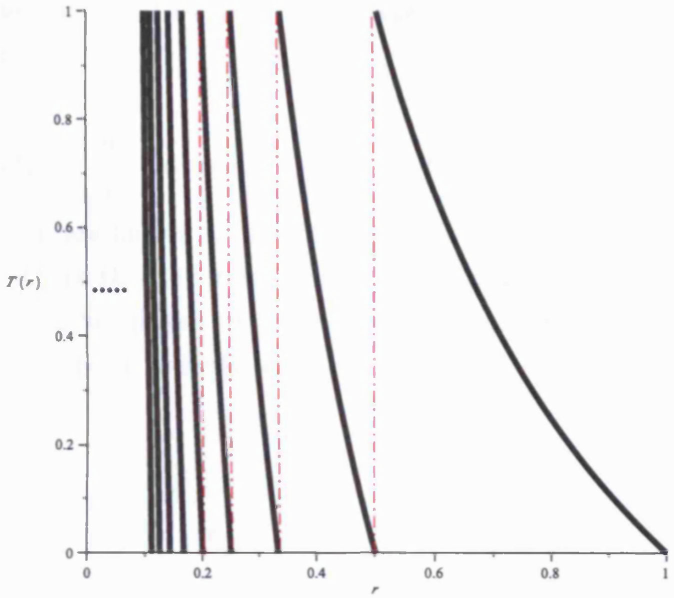


FIGURE 1.2. The Gauss Map

$$a_n = a_n(x) := \left\lfloor \frac{1}{T^{n-1}} \right\rfloor, \quad n \geq 1,$$

yields the expansion

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, a_3 \dots],$$

and the n-th convergent to this (finite or inifinite) expansion is defined $[a_0; a_1, \dots, a_n] = \frac{p_n}{q_n}$, with $a_m \in \mathbb{N}, \quad m = 1, \dots n$, and $a_0 \in \mathbb{Z}$. Furthermore, for irrational x in particular one has

$$x = \lim_{n \to \infty} [a_0; a_1, \dots, a_n].$$

Note that a finite expansion is not necessarily unique since:

$$[a_0; a_1, a_2, \dots, a_n] = [a_0; a_1, a_2, \dots, a_n - 1, 1].$$

A simple combinatorial argument based on the construction of the tree, such as that eluded to in the book [**37**] leads to the conclusion that the set $\mathcal{Q}_n$

contains those continued fraction whose constituent elements a_n have sum n only, formally

$$(1)\qquad \mathcal{Q}_n = \left\{ \frac{p}{q} = [a_1, a_2, \dots, a_r] : a_r \geq 2 \text{ and } \sum_{i=1}^{r} a_i = n \right\}.$$

The elements a_i are known as 'partial quotients' and have $a_i \geq 1$ for all $i = 1, \dots, r-1$, where the constraint $a_r > 1$ is imposed to maintain uniqueness of elements. Further linked to these continued fractions are the *continuant polynomials*, defined recursively as

$$(2)\qquad \langle x_1, \dots, x_n \rangle := \begin{cases} 1, & \text{if } n = 0 \\ x_1, & \text{if } n = 1 \\ x_n \langle x_1, \dots, x_{n-1} \rangle + \langle x_1, \dots, x_{n-2} \rangle & \text{if } n > 1 \end{cases}.$$

These constructs were introduced by Euler in **[14]**; further properties are discussed extensively in **[21]**. Furthermore the continued fraction $[a_1, \dots, a_r]$ can now be expressed as follows:

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots \cfrac{1}{a_r}}} := [a_1, a_2, \dots, a_r] = \frac{\langle 0, a_1, a_2, \dots, a_r \rangle}{\langle a_1, a_2, \dots, a_r \rangle}.$$

One should note also the correspondence of the partial quotients in the Farey continued fraction $[a_1, a_2, \dots, a_r]$ with the product with the matrix product $LR^{a_1} L^{a_2} \dots R^{a_r - 1}$, where

$$(3)\qquad L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The product of these matrices denote a path along edges of the Farey Tree moving to the *L*eft and *R*ight respectively. This is well explored in **[21]** and **[37]**. As an example, consider the Farey point calculated by following a path of moves to the left, then right, then left and so on. The point yielded is that of the ratio of consecutive Fibonacci numbers $\frac{\mathbf{F}(n)}{\mathbf{F}(n+1)}$, which has the form

$$\frac{p}{q} = [1, 1, 1, 1, \dots, 2].$$

Moreover, note that $\max\limits_{\frac{p}{q} \in \mathbf{FT}_n} q = \mathbf{F}(n+1)$.

The Farey Tree points may be generated also by the Farey Map, itself an extension to the Gauss map discussed above. This is defined for $U : [0,1] \to [0,1]$ as follows:

$$U(x) := \begin{cases} \dfrac{x}{1-x}, & x \in [0, \frac{1}{2}] \\ \dfrac{1-x}{x}, & x \in [\frac{1}{2}, 1], \end{cases}$$

and has inverse branches:

$$\Phi_1 \ : \ [0,1] \to \left[0, \frac{1}{2}\right] \ ; \ \Phi_1(x) := \frac{1}{2} - \frac{1}{2}\left(\frac{1-x}{1+x}\right),$$
$$\Phi_2 \ : \ [0,1] \to \left[\frac{1}{2}, 1\right] \ ; \ \Phi_1(x) := \frac{1}{2} + \frac{1}{2}\left(\frac{1-x}{1+x}\right).$$

Figure 1.3 illustrates the behaviour of U.

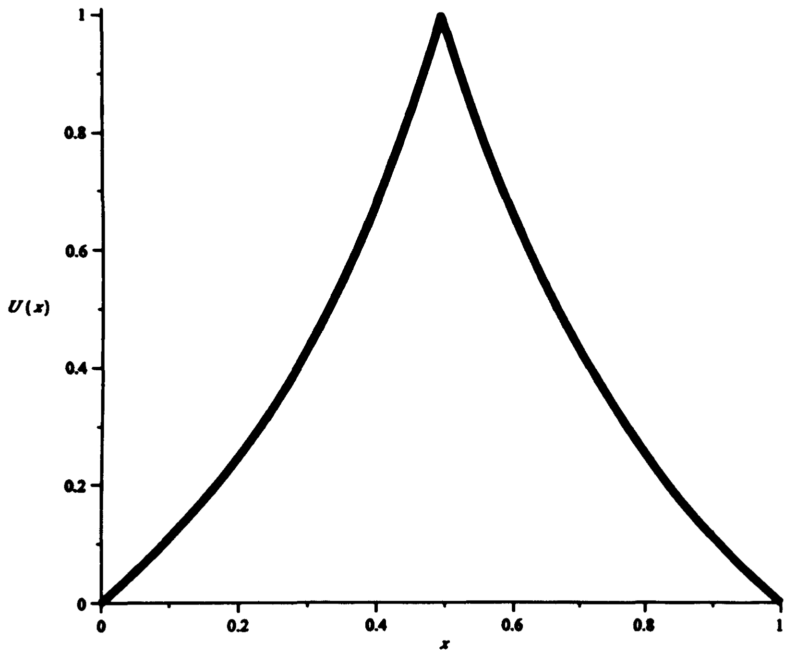


FIGURE 1.3. The Farey Map

The relation between the map and the Farey Tree fractions at level n is simple:

$$\mathcal{Q}_n = U^{-n+1}(1) = \{x \in [0,1] : U^{n-1}(x) = 1\} \quad \text{for all } n \geq 2,$$

implying $\mathrm{FT}_n = U^{-n}(0)$ for all $n \geq 1$. There exists a simple relationship between the Gauss map and the Farey map. Consider first the map

$$T_1(x) = \begin{cases} \frac{1}{x}, & \text{if } x \in (0,1], \\ x-1, & \text{if } x \in (1,\infty). \end{cases}$$

Next, initialise a value $\alpha \in (0,1)$. One iteration of the map T_1 yields a value in the second branch; in fact further iterations remain in the second branch until the value $\{1/\alpha\}$ is produced. Moreover, the number of such iterations spent in the second branch is equal to the final partial quotient in the continued fraction expansion of α. Obviously, prior to the trajectory $T_1^n(x)$ returning to $(0,1]$ (where n is the minimum number of iterations upon which the return is made - this map is then known as the *first return* map), the value of $T_1^{n-1}(x)$ lies in $(1,2]$. Consider a second mapping for when x, defined as a join of two iterations for when x lies in this interval:

$$T_2(x) = \begin{cases} T_1(T_1(x)) = \frac{1}{x-1}, & \text{if } x \in (1,2], \\ T_1(x) = x-1, & \text{if } x \in (2,\infty). \end{cases}$$

Then, the simple change of variable $y = \frac{1}{x}$ in T_2 gives the Farey map. This, and associated results are discussed further in, for example, [**35**].

4. Outline of this Thesis

The central aim of this thesis will be to prove a theorem concerning the moments of the distribution of a normalised metric on the unit interval partitioned by the Farey points. The structure of its presentation is outlined in the following subsections, which describe briefly each of the key chapters and sections.

4.1. Chapter Two. Chapter Two sets the scene for main theorem, including defining the metrics ρ_n and $\rho'_n(x)$ and describing a number of their fundamental properties. The properties of these metrics for $x \in [0,1]$ and $n \in \mathbb{N}$ (the level of the Farey Tree whose points partition the interval) are also described. Particularly relevant are the plots contained in figures 2.1 and 2.2 on pages 35 and 36 which illustrate the behaviour of ρ and ρ' respectively.

The concept of the *Farey Cell*, that is the sub-partition of the unit interval with endpoints a pair of consecutive Farey points, is also introduced. Further illustration of the width and height of Farey cells, yielded by the rho-prime metric is included in Appendix B in the form of histograms. These are figures B.1, B.2 and B.3.

4.2. Chapter Three. Chapter Three is a large chapter which proves the main theorem. This is a lengthy proof consisting of several individual lemmas each linked to a specific property of the denominators of the Farey tree points. For ease of reading, each lemma is labelled with a colour with which a link is made to specific property of interest. The key to these colours may be followed from the guide of figure 3.3 on page 55: section and lemma headings relevant to each property are coloured according to this code.

Theorem 3.3 - see page 50 - first concerns the sum of separate interest of $\frac{1}{q^{\alpha+2}}$, where the denominators q are the denominators of the Farey points at level N and $\alpha > 1$. The theorem is also an auxiliary result to the main proof, and its result is utilised in many of the following lemmas. Here it is proved - indeed in a similar manner to the main proof discussed below - that:

$$\sum_{a \in \mathcal{A}_N} \frac{1}{q^{\alpha+2}} = \mathcal{O}\left(\frac{1}{N^{\alpha+2}}\right).$$

The set $\mathcal{A}_N$ contains vectors a with natural-valued elements whose final entry is greater than one and sum of elements (partial quotients) is N. An improvement is suggested in Appendix A using a similar method to that proposed by Dushistova in [**12**]. The main bulk of this proof begins on page 67.

The structure of the main proof is dependent on each of the lemmas. Each result considers an element of the partition of the sum yielded by the integral of the distance function and the aforementioned scheme of figure 3.3:

$$\int_0^1 \rho_n'(x)^\alpha dx = C_\alpha \sum_{(q,q')} \frac{1}{qq'(q+q')^\alpha}. \tag{4}$$

Here C_α is a constant and (q, q') are the respective denominator-neighbours for Farey Tree points at level N. The proof of (4) is stated on page 47. Each of the denominators q, q' and $q + q'$ will be associated with a continuant - themselves displaying properties yielded by the section of the partition of consideration.

Each Lemma is described briefly below - the opening text of each item in this breakdown respects the colour-coding described earlier.

- Lemma 3.14 on page 76, considers the part of the sum in (4) which contains the largest denominators. The key finding here is that this section of the partition contributes very little to the final result. Indeed the sum of consideration in this Lemma is found to have asymptotic value $\mathcal{O}\left(\frac{1}{n^{2\alpha+1}}\right)$ in the final result.

- Lemma 3.15 investigates the sum with denominators whose continuants have associated largest partial quotient bounded above by $N-w$, where N is the most 'recent' level of the sequence and $w \leq N$ is a parameter. It is shown in this section that there are at least two partial quotients bounded in such a way that causes the sum to again be of small contribution to the final result. In the final result, this Lemma yields a sum with asymptotic value $\mathcal{O}\left(\frac{\log^{2\alpha+3}(N)}{N^{2\alpha+1}}\right)$.

- Lemma 3.16 and Lemma 3.17 (see pages 84 and 92) consider the sums over continuants whose 'final' partial quotient is the largest. The lemmas are distinguished by the split of the sum (4) into 'plus' and 'minus' parts, which are characterised by the origin of denominator continuant from a particular generation of the tree - further details are described on page 48. In the final result, Lemma 3.17 yields the sum of most significant size.

The technique behind the proof of these Lemmas involves the expansion of the sums in a Taylor series, from which the asymptotically-largest parts may be separated. The term associated with the 'minus' denominator is most significant as this denominator originates from a level of the Farey Tree earlier than $N-1$, and hence does not contain the 'large' final partial quotient possessed by denominators whose level of origin is N.

- Lemma 3.18 and Lemma 3.19 (on pages 3.18 and 3.19) are of similar construction: in these results the *largest* partial quotient is the next-to-last. These sums again yield terms of value $\mathcal{O}\left(\frac{1}{N^{\alpha+2}}\right)$ and $\mathcal{O}\left(\frac{1}{N^{\alpha+1}}\right)$. The former of these is most easily derived, and indeed follows in a very similar manner to the results of Lemmas 3.16 and 3.17. The 'largest' term occurs in the case where 'parent' denominator q'' defined on page 48 has value of final partial quotient equal to 2. Each case -i.e. where final partial quotient is equal to 2, and where it is greater - need to be given separate consideration, which serves to lengthen the proofs in this section. The technique involves tracking backward to a previous level of the Farey Tree to investigate the properties of the continuants which make up the denominator at the future level.

- Theorem 3.4 - stated on page 51 and proved on page 118 - has a preliminary result which calculates the asymptotic value of the final unconsidered sum in the partition. This allows for a preliminary proof of the main theorem to be obtained, i.e. that the sum in (4) has main term of order $\mathcal{O}\left(\frac{1}{N^{\alpha+2}}\right)$ and an error with asymptotic value $\mathcal{O}\left(\frac{1}{N^{\alpha+3}}\right)$.

- Lemma 3.21 on page 119 considers the sum in the partition over continuants whose largest partial quotient is neither its last or next-to-last. It is this property that ensures that these sums have asymptotic value $\mathcal{O}\left(\frac{1}{N^{\alpha+2}}\right)$, using the methods of Lemmas 3.18 and 3.19. Essentially, this is due to the fact that the partial quotient of largest magnitude will always be present in both the associated 'plus' and 'minus' denominators.

Upon the completion of the proof of Lemma 3.21, which requires Theorem 3.4, one is able to assemble the main proof of Theorem 3.5. This process is completed on page 131. Indeed the Lemmas themselves follow a hierarchical structure: Lemma 3.14 is used in each of the other proofs, while Lemma 3.18 and Lemma 3.19 both use the results of Lemma 3.17 and Lemma 3.16 respectively. Lemma 3.21 then follows in a similar fashion. Finally, each section is followed by numerical calculations and their related graphical output is colour-coded in a similar fashion.

4.3. Chapter Four. Chapter Four seeks an improvement to the result derived for the ρ-metric first proved in [**31**] and then [**12**]. This utilises the 'new' algorithm which proves the main theorem of Chapter Three, improving the known result from [**12**]. The algorithm proposed allows for additional constant terms to be included in the main term of the asymptotic result; although the error term is unchanged.

4.4. Appendix A. Appendix A offers an improvement to Theorem 3.3 proved first in Chapter Three. The improvement arises as a result of further expansion of the Lemma which yields the term of largest magnitude in that Theorem. This is performed in a similar manner to that investigated in [**12**]. Although this improvement is not essential in the general context of the main proof, it is included here for completeness.

4.5. Appendix B. Appendix B contains a number of additional figures which support the methods of Chapter Three and hence the extension employed in Chapter Four. The first section provides additional illustration of the behaviour of the Farey Cells under the ρ'-metric using histograms. The second section contains a number of scatterplots which plot the relative behaviour of a number of the individual sets considered in Chapter Three. The third section contains two sets of pie graphs whose aim is to illustrate the contribution from each set considered by partition 3.3 to the sum (4). The first set is relevant to the partition with $\alpha = 2$, the second set for $\alpha = 5$.

The fourth section briefly describes the history of the development of the main Lemmas in Chapter Three, and the fifth considers the illustrative impact of varying the parameter s introduced in hierarchy 3.3. The appendix is completed with the inclusion of a number of $\alpha = 5$ analogues of figures included in the main text.

5. Miscellaneous

To conclude the introductory chapter, a number of miscellaneous definitions and results shall be presented. These are to be referred to extensively throughout this thesis. The first are the two variants of the Pochhammer symbol for which the notation of [**29**] will be used. Let $z \in \mathbf{C}$ and $n \in \mathbf{N}$: the symbol $z^{\overline{n}}$ will denote the 'climbing factorial':

$$z^{\overline{n}} = z(z+1)\dots(z+n-1), \tag{5}$$

and similarly $z^{\underline{n}}$ will denote the 'falling factorial':

$$z^{\underline{n}} = z(z-1)\dots(z-n+1). \tag{6}$$

More generally if $\alpha \in \mathbb{C}$, then one has

$$z^{\overline{\alpha}} = \frac{\Gamma(z+\alpha)}{\Gamma(z)} \quad \text{and} \quad z^{\underline{\alpha}} = \frac{z!}{(z-\alpha)!},$$

where $\Gamma(\cdot)$ is the Gamma function.

DEFINITION 1.1. *The hypergeometric series (seen for example in* [**21**]*, and in greater detail in* [**2**]*) F is defined as*

$$F\left(\begin{matrix} a_1, & a_2, & \dots & a_m \\ b_1, & b_2, & \dots & b_n \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{a_1^{\overline{k}}.a_2^{\overline{k}}\dots a_m^{\overline{k}}}{b_1^{\overline{k}}.b_2^{\overline{k}}\dots b_n^{\overline{k}}} \frac{z^k}{k!},$$

with $F(\cdot)_l$ used to denote the summation with upper limit l instead of infinity (also known as a partial hypergeometric series).

The following minor results follow by applying the definitions above.

LEMMA 1.2. *For $|z| < 1$,*

$$F\left(\begin{matrix} a, 1 \\ 1 \end{matrix} \middle| z \right) = \frac{1}{(1-z)^a}$$

PROOF. The proof is easily seen by Taylor's theorem, and is discussed in [**21**], page 206. □

LEMMA 1.3. *The function ψ, defined:*

$$\psi(Y) = \frac{1}{(1-Y)^{\alpha+1}},$$

has, for $|Y| < 1$, and fixed $\alpha > 1$ the following Taylor series expansions (about 0)

$$\psi(Y) = 1 + \sum_{k=1}^{\infty} \frac{\prod_{i=1}^{k}(\alpha+i)}{k!} Y^k$$

PROOF. The task of checking is fairly easy; the imposition of the said condition on Y is such that the remainder term given by the application of the Theorem tends to zero (hence the infinite Taylor series is valid). □

LEMMA 1.4. *For $|X| < 1$ and $\alpha > 1$, one sees that*

$$\sum_{k=1}^{\infty} X^k \sum_{1 \le l \le k} \frac{\prod_{i=1}^{l}(\alpha+i)}{l!} = \frac{1}{(1-X)^{\alpha+2}} - 1 + \frac{X}{X-1}.$$

PROOF. The result is obtained by via a number of binomial expansions. In particular, the internal sum

$$\sum_{1 \le l \le k} \frac{\prod_{i=1}^{l}(\alpha+i)}{l!} = \sum_{0 \le l < k+1} \frac{(\alpha+1)^{\overline{l}}}{l!} - 1,$$

is itself a partial hypergeometric series. We have

$$\sum_{1 \le l \le k} \frac{\prod_{i=1}^{l}(\alpha+i)}{l!} = \frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+2)k!} - 1,$$

and, since both of the following series converge:

$$\sum_{k=1}^{\infty} X^k \sum_{1 \le l \le k} \frac{\prod_{i=1}^{l}(\alpha+i)}{l!} = \sum_{k=1}^{\infty} \left(\frac{X^k}{k!}\right) \frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+2)} - \sum_{k=1}^{\infty} X^k. \tag{7}$$

Now, using the identity:

$$\frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+2)} = \frac{1}{(\alpha+1)}(\alpha+1)^{\overline{k+1}} = (\alpha+2)^{\overline{k}},$$

one sees that the first summation on the right hand side of (7) is $F\left(\begin{matrix}\alpha+2,1\\1\end{matrix}\middle| X\right)$. This implies the end result. □

The following inequality is utilised at length in this thesis.

LEMMA 1.5. *For the continuant polynomial $\langle x_1, x_2, \dots, x_r\rangle$ of arbitrary length r we have the following inequalities:*

$$\begin{aligned}
\langle x_1, x_2, \dots, x_r\rangle &\ge x_l\langle x_1, \dots, x_{l-1}\rangle\langle x_{l+1}, \dots, x_r\rangle \\
\langle x_1, x_2, \dots, x_r\rangle &\ge x_l x_k\langle x_1, \dots, x_{l-1}\rangle\langle x_{l+1}, \dots, x_{k-1}\rangle\langle x_{k+1}, \dots, x_r\rangle,
\end{aligned}$$

and so forth.

These inequalities are seen already in [**31**] and in further detail in [**12**]. Proofs may easily be obtained by the results of [**21**], originating from those of Euler (see [**14**]) and using Cassini's identity (also discussed in, for example [**40**]).

6. Literature Review

A primary source of material, including a detailed discussion of the Farey series is the book of Hardy and Wright [**22**], which has provided much of the background reading for this project. A recent survey of results and the history associated with the Farey sequences is presented in [**8**], which upholds the naming convention *Farey-Haros* sequence.

6.1. Texts Related Directly to the Main Proof. One will note that the main proof of this thesis (Theorem 3.5) is a development of the technique first used by Zhigljavsky and Moshchevitin in [**31**]. Here one has that, for $\beta > 1$

$$\sum_{i=1}^{N(n)} p_{i,n}^{\beta} = \frac{2}{n^{\beta}} \frac{\zeta(2\beta-1)}{\zeta(2\beta)} + \mathcal{O}\left(\frac{\log(n)}{n^{\frac{(\beta+1)(2\beta-1)}{2\beta}}}\right)$$

where $p_{i,n}$ is the distance between consecutive Farey Tree fractions, and $N(n) = |\mathrm{FT}_n| - 1$. These quantities are introduced more formally in Chapter 2. The note by Dushistova - reference [**12**] - offers an improvement to this original result, in particular by obtaining the following error term

$$\mathcal{O}\left(\frac{\log^{3\beta}(n)}{n^{3\beta-2}}\right).$$

This is an improvement since, for $\beta \in (1, 1.5]$ the main terms are equal and

$$3\beta - 2 > \frac{(\beta+1)(2\beta-1)}{2\beta}$$

in the exponent of the error term's denominator.

The result of several texts are used throughout the main proof. Of particular use have been algebraic results from Graham, Knuth and Patashnik in [**21**] and elementary results from analytic number theory contained in [**1**] and [**22**]. Further detail on hypergeometric series may be seen in the text of Bailey in [**2**]. Consideration of the asymptotic distribution of the conventional distance to the Farey Series points is made in [**25**]. Prior metrical results are also constructed in [**26**]; their relation to the current work is discussed in detail in chapter 2. These also depend on the Mellin transform, whose properties and asymptotics are further discussed in [**32**].

A simple formulation of the ρ'-distance, discussed as the *ultra-distance* for $\alpha \in (0,1)$, $q\left|\frac{p}{q}-\alpha\right|$ is made in the article [**36**]. It is seen here that this distance acts as a measure of effectiveness of fraction $\frac{p}{q}$ as a rational approximation to α. Moreover an *ultra-close* distance is also seen to imply the 'best' approximation.

6.2. Reformulation in terms of dynamical systems. A discussion of the Gauss map and the continued fraction algorithm is made in [**9**]. This note offers a particularly elegant representation of the Gauss map on a torus; such an illustration is seen here in figure 1.4. Under normal conditions, jump continuities on the Gauss map occur at the points $\frac{1}{n}$ for $n \in \mathbb{N}$; these may be removed by mapping onto the circle $\exp(i\pi x)$. From this, the Gauss map $\exp(i\pi x) \longrightarrow \exp\left(i\pi\frac{1}{x}\right)$ forms a coil on the torus, with the singularity at zero mapped to zero for convenience.
An overview of the role of dynamical systems in number theory is made by Lagarias in [**30**]. For example, consider the Gauss map, which has invariant density:

$$\mu_A(x) = \frac{1}{\log(2)}\int_A \frac{dx}{1+x}.$$

Consider also the Farey map, for which it is known that this transformation has associated invariant density

$$p_B(x) = \int_B \frac{dx}{x}.$$

This density has infinite mass, but is absolutely continuous and is ergodic with respect to the Farey map. This is discussed in further detail below. Similar results and properties of dynamical systems and in ergodicity are explored in the book [**10**].

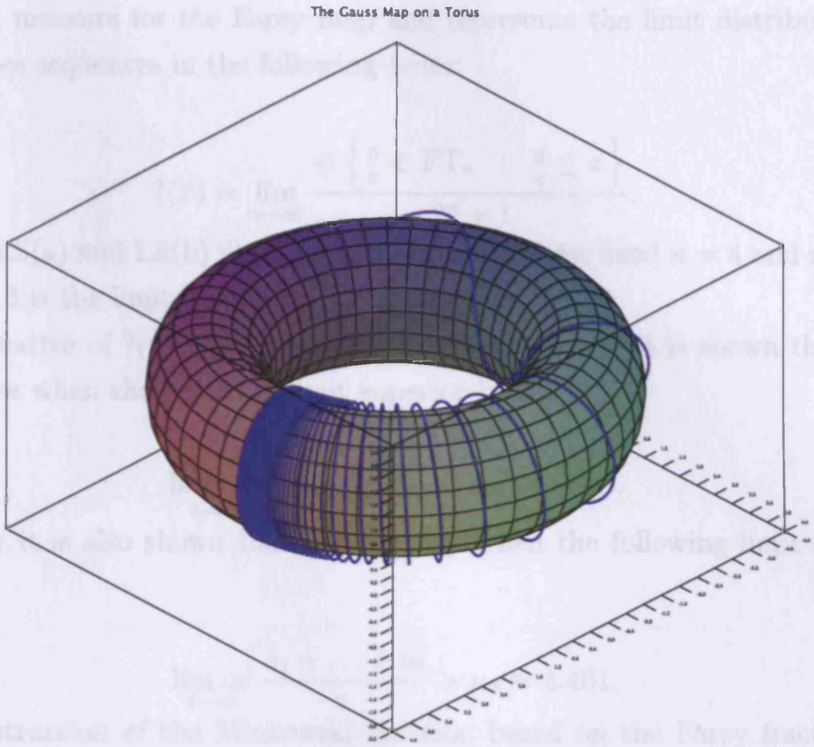


FIGURE 1.4. The Gauss map on a torus

Of particular interest in [**30**] is the Minkowski question-mark function, $?(x)$ defined formally as having values $?(0) = 0$, $?(1)$ and for consecutive Farey neighbours $\frac{p}{q}$, $\frac{p'}{q'}$,

$$?\left(\frac{p+p'}{q+q'}\right) = \frac{?\left(\frac{p}{q}\right) + ?\left(\frac{p'}{q'}\right)}{2}$$

Moreover for irrational $\theta = [a_1, a_2, \ldots]$, one sees that

$$?(\theta) = \sum_{k=1}^{\infty} (-1)^{k+1} 2^{-(a_1+\ldots+a_k-1)}.$$

This function is both strictly increasing and continuous, though its derivative is zero for almost all x on its domain. It is also not absolutely continuous. For rational x, the function takes takes value $\frac{k}{2^s}$ with integers k and s. Moreover if x is a quadratic irrational number - i.e. is of the form $\frac{a \pm \sqrt{b}}{c}$ for $a, b, c \in \mathbb{Z}$ - then $?(x)$ is itself a rational number. It is observed that $d?(x)$ is another invariant measure for the Farey map and represents the limit distribution of Farey Tree sequences in the following sense:

$$?(x) = \lim_{n\to\infty} \frac{\#\left\{\frac{p}{q} \in \mathrm{FT}_n \ : \ \frac{p}{q} \le x\right\}}{2^n + 1}. \tag{8}$$

Figures 1.5(a) and 1.5(b) illustrate the value of (8) for fixed $n = 4$ and $n = 13$. Figure 1.6 is the limit function.

The derivative of $?(x)$ may also take value $+\infty$. In [**13**] it is shown that this is the case when the following limit superior holds:

$$\limsup_{t\to\infty} \frac{a_1 + \ldots + a_t}{t} < \kappa_1 \approx 1.388.$$

Moreover it is also shown that $?'(x)$ is zero when the following limit inferior holds:

$$\liminf_{t\to\infty} \frac{a_1 + \ldots + a_t}{t} > \kappa_2 \approx 4.401.$$

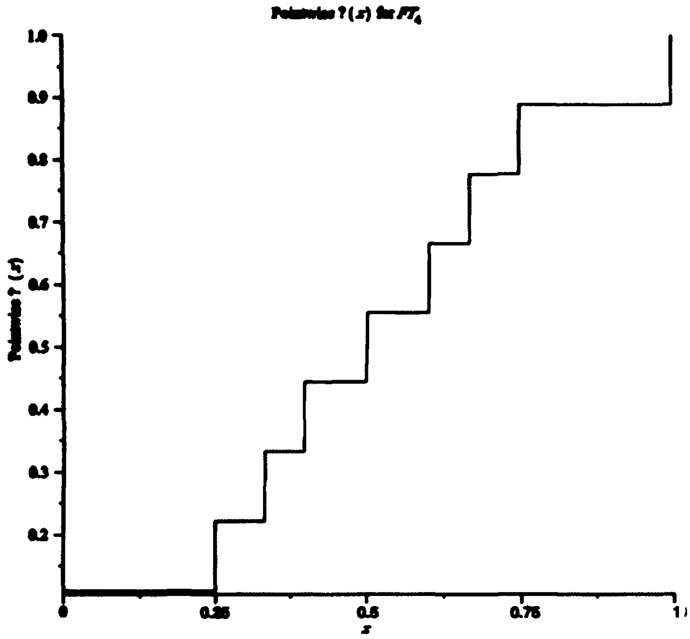
The construction of the Minkowski function based on the Farey fractions is discussed in further detail in [**20**].

Consider again $x \in [0, 1]$ such that $x = [a_1, a_2, \ldots]$. Discussed in [**24**] are the principal convergents generated by the Farey map:

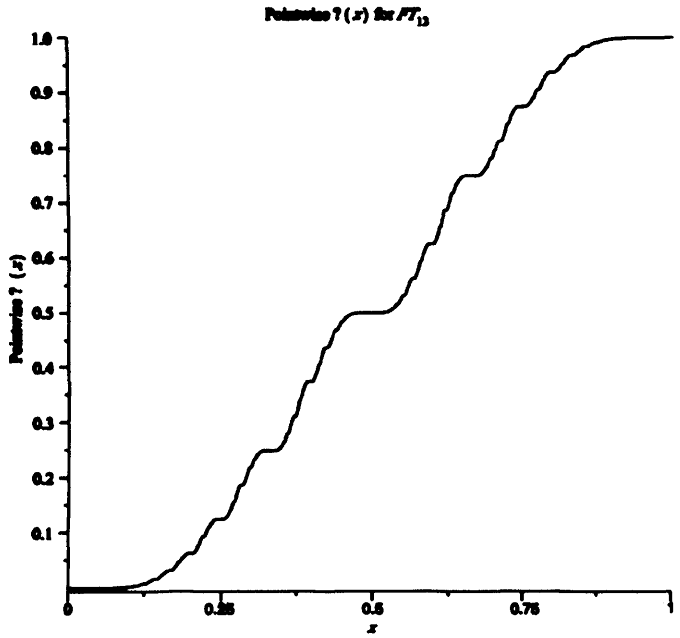
$$\left\{\frac{kp_n + p_{n-1}}{kq_n + q_{n-1}}, \ k = 1, \ldots, a_{n+1} - 1, \ n \in \mathbb{N}\right\}. \tag{9}$$

Since the density p is infinite but σ-finite the (Birkhoff) Ergodic theorem cannot be used to determine the limit distribution of the generated mediant convergents. Ito considers a possible route around this problem by considering the transform U_1 modified from the original Farey map, in order to produce metrical results. It is defined as:

$$U_1(x) := \begin{cases} \dfrac{1-x}{x}, & \text{if } x \in \left[\frac{1}{2}, 1\right), \\[2ex] \dfrac{x}{1-x}, & \text{if } x \in \left[\frac{1}{3}, \frac{1}{2}\right), \\[2ex] \dfrac{x}{1-(k-2)x}, & \text{if } x \in \left[\frac{1}{k+1}, \frac{1}{k}\right), \quad (k \ge 3). \end{cases}$$



(a)



(b)

FIGURE 1.5. Pointwise Minkowski ?-Functions for $n = 4$ and $n = 13$

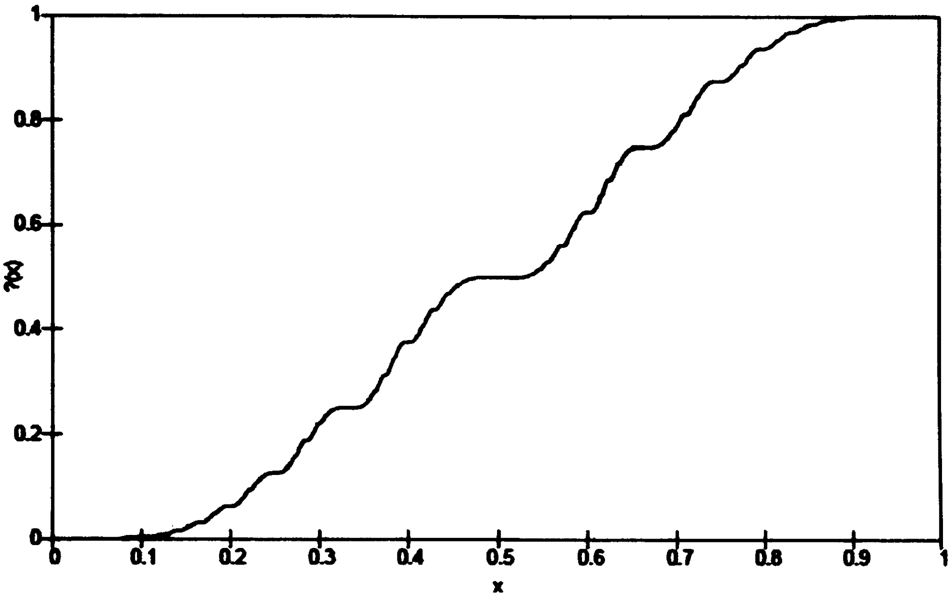


FIGURE 1.6. Minkowski ?-Function

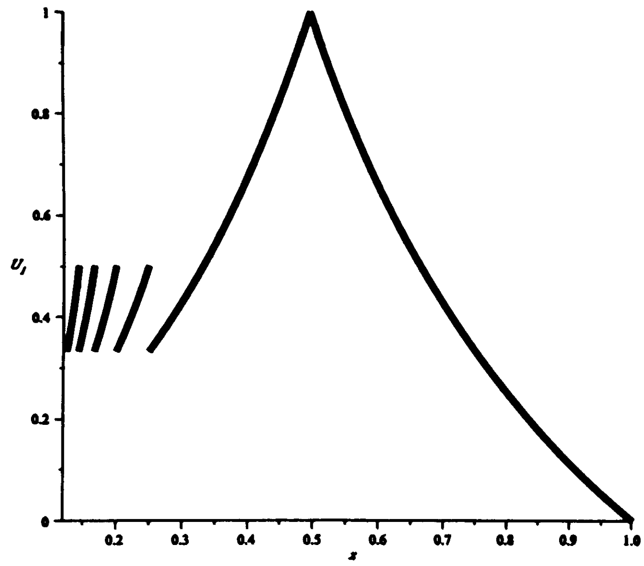


FIGURE 1.7. The modified Farey map

This map generates the so-called *principal* convergents of the Farey map, but also the *first* and *last* mediant convergents, $\frac{p_n+p_{n-1}}{q_n+q_{n-1}}$ and $\frac{p_n-p_{n-1}}{q_n-q_{n-1}}$ respectively: call these $\frac{w_n}{v_n}$. The modified Farey Map of Ito is illustrated in figure 1.7.

As opposed to the conventional Farey map, this transformation is shown to have finite invariant measure μ_1, which has density:

$$d\mu_1 = \begin{cases} \frac{1}{2\log(2)}\frac{dx}{1+x}, & \text{for } x \in \left[0, \frac{1}{3}\right), \\ \frac{1}{2\log(2)}\frac{dx}{x}, & \text{for } x \in \left[\frac{1}{3}, 1\right). \end{cases}$$

By constructing a natural extension of $U_1(x)$, namely

$$(10)\ \overline{U}_1(x,y) = \begin{cases} \left(\frac{1-x}{x}, \frac{1}{1+y}\right), & \text{for } (x,y) \in \left[\frac{1}{2}, 1\right) \times [0,1], \\ \left(\frac{x}{1-x}, \frac{y}{1+y}\right), & \text{for } (x,y) \in \left[\frac{1}{3}, \frac{1}{2}\right) \times [0,1], \\ \left(\frac{x}{1-(k-2)x}, \frac{y}{1+(k-2)y}\right), & \text{for } (x,y) \in \left[\frac{1}{k+1}, \frac{1}{k}\right) \times [0,1], \end{cases}$$

Ito was able to construct metrical results such as the following limit:

$$\lim_{n\to\infty} \frac{1}{n}\log(v_n^{(1)}) = \frac{\pi^2}{24\log(2)},$$

for almost all $x \in [0,1)$. This in particular is analogous to a result of Khinchin in [**41**], whereby for the denominators of the continued fraction convergents $\frac{p_n}{q_n}$ and almost all $x \in [0,1)$,

$$\lim_{n\to\infty} \frac{1}{n}\log(q_n) = \frac{\pi^2}{12\log(2)}.$$

The 'conventional' extensions of the Gauss and Farey maps are considered by Yin and Brown in [**6**]. These are defined:

$$\overline{T}(x,y) = \left(T(x), \frac{1}{\left[\frac{1}{x}\right]+y}\right), \quad (x,y) \in [0,1) \times [0,1)$$

$$\overline{U}(x,y) = \begin{cases} \left(\frac{x}{1-x}, \frac{y}{1+y}\right), & \text{if } , x \in [0,1/2], y \in [0,1], \\ \left(\frac{1-x}{x}, \frac{1}{1+y}\right), & \text{if } .x \in (1/2,1], y \in [0,1]. \end{cases}$$

These maps have associated absolutely continuous and invariant measures $\overline{\mu}$ and $\overline{p}$ respectively. In particular, $\overline{p}$ possesses the infinite, σ-finite characteristic of the Farey map measure p. This means that the Birkhoff ergodic theorem (see, for example [**10**]) is not applicable for $\overline{U}$ either. Moreover it is discussed that $\overline{T}$ can be induced from $\overline{U}$, and it is this fact that is used to produce metrical results for the continued fractions. Consider again, for irrational

$x = [a_1, a_2, \ldots]$ its so-called k-th mediants from set (9) and also those of the form:

$$\frac{(a_{n+1} - k)p_n + p_{n-1}}{(a_{n+1} - k)q_n + q_{n-1}}, \quad a_{n+1} \geq 2k.$$

Denote the sequence of the combined sets of such convergents by $\frac{P_n^{(k)}}{Q_n^{(k)}}$. The main results of [6] apply the Chacon-Ornstein ergodic theorem of [7] to produce the said metrical results, for example the following limit which holds for almost all x and $k = 0, 1, 2, \ldots$:

$$\lim_{n\to\infty} \frac{1}{n} \log(Q_n^{(k)}) = \frac{\pi^2}{12\log(2k+2)}.$$

Other metrical results on the continued fractions are discussed in, for example [28], [3] and [4].

The Farey map belongs to the class of *almost expanding* maps, which are discussed by Prellberg and Slawny in [34]. Consider again the unit interval $[0, 1]$ and a function f which represents a piecewise-monotone transformation of $[0, 1]$; that is on a finite partition of subintervals $I_0, I_1, \ldots, I_k$. Let, for each I_i, the function f extend to a functions f_i on the closures $\overline{I_i}$ that have Hölder-continuous derivatives f_i'. f is defined to be *almost-expanding* if $|f'|$ is greater that or equal to 1 in the interior of each of these subintervals I_i, and equality may be achieved at the endpoints of the intervals.

The *topological pressure* is defined $P_\beta = \log(\lambda_\beta)$, where λ_β is the greatest eigenvalue of the *transfer operator* $\mathcal{L}_\beta$ in the space of continuous functions on $[0, 1]$. The transfer operator $\mathcal{L}_\beta : C[0, 1] \to C[0, 1]$ defined for $\Psi \in C[0, 1]$ is defined:

$$\mathcal{L}_\beta \Psi(x) = \sum_{y\,:\,f(y)=x} \frac{\Psi(y)}{|f'(y)|^\beta}.$$

With $f = U$, the Farey map, one sees that the pressure is zero when $\beta \geq 1$. Furthermore, for $\beta < 1$: $P_\beta = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{(q,q')} \frac{1}{(qq')^\beta}\right)$, where (q, q') are the denominators of successive Farey tree fractions $\frac{p}{q}$ and $\frac{p'}{q'}$. It is thus clear that in [31], Theorem 1, a version of the pressure is considered for $\beta > 1$. Moreover it is proved that non-trivial limits exist for them sum when the normalisation n^β is used rather than $\frac{1}{n}\log(\cdot)$. For figures illustrating approximate behaviour of $P(\beta)$ for the Farey map with $\beta \in (-1.5, 1.5)$, see [19], page 4.

A similar problem to that considered in [31] is explored by Fiala and Kleban in [17]. First, as a matter of continuity with this and associated works, denote the n-th ordered Farey Tree fraction from level k of the tree (i.e. a fraction whose origin is the set $\mathcal{Q}_k$) as $r_k^{(n)} = \frac{n_k^{(n)}}{d_k^{(n)}}$. The problem of interest is that of the sum of lengths of alternate intervals generated by Farey points of of set $\mathcal{Q}_k$;

$$I_k^{(e)} = \sum_{i=1}^{2^{k-2}} (r_k^{(4i)} - r_k^{(4i-2)}).$$

The superscript (e) denotes that this is the sum of so-called *even* intervals. It is shown that $\liminf_{k\to\infty} I_k^{(e)} = 0$ and conjectured that one may indeed replace limit infimum with full limit in this case. Moreover, this is then shown to be equivalent to the limit $\lim_{n\to\infty} S_k = 0$, where

$$S_k = \sum_{n=1}^{2^{k-1}} \frac{1}{(d_k^{(2n)})^2}.$$

This is the sum of the inverse-square of the 'new' denominators at level k. A similar asymptotic result for the sum $\sum_{i=1}^{2^{k-1}} \frac{1}{(d_k^{(2i)})^{\alpha+2}}$, for $\alpha > 1$ is itself considered in this thesis; this is Theorem 3.3.

Motivation for [17] grew from the study of the the so-called *Farey fraction spin-chains*, introduced by Kleban and Özlük in [27]. These spin chains are a set of statistical mechanical models derived from the Farey fractions developed in order to investigate a connection between the discipline and number theory. Consider again the matrices of (3), and the associated products, say

$$M^m(j) = \prod_{k=0}^{m-1} L^{1-a_k} R^{a_k},$$

which are well known to generate Farey neighbours $\frac{c}{d}$ and $\frac{a}{b}$ (the variable j is used to denote a particular configuration of the product which yields the j-th pair of Farey neighbours). In this context, the matrices L and R are known as *spin-states* and level m the *length* of the spin chain. Define next, for $\beta \in \mathbb{R}$,

$$Z_k^F(\beta) = \sum_{j=1}^{2^k} \frac{1}{(q_k^{(j)} + p_k^{(j+1)})^\beta} = \sum_{j=1}^{2^k} \frac{1}{Trace(M^m(j))^\beta},$$

the Farey spin-chain (FSC) partition function, and

$$Z_k^K(\beta) = \sum_{j=1}^{2^k} \frac{1}{(q_k^{(j)})^\beta}$$

is the Knauf spin-chain (KSC) partition function (see, for example, [**18**], equation (2)). Similarly to [**31**], one may express Z_k as the sum of odd and even parts

$$\begin{aligned} Z_k^K(\beta) &= \sum_{j=1}^{2^{k-1}} \frac{1}{(q_k^{(2j)})^\beta} + \sum_{j=1}^{2^{k-1}} \frac{1}{(q_k^{(2j+1)})^\beta} \\ &= Z_{k,e}^K(\beta) + Z_{k,o}^K(\beta). \end{aligned}$$

These functions are of particular interest to the current work due to their connection with the asymptotics of sums over Farey fractions, and their connection with the transfer operator (see below). For example it is shown that, in particular

$$Z_k^K(2\beta) \longrightarrow \frac{\zeta(2\beta - 1)}{\zeta(2\beta)} \qquad (k \to \infty, \ \beta > 1),$$

and $Z_k^{K,e}(\beta) \longrightarrow 0$, $Z_k^F(\beta) \longrightarrow 0$ as $k \to \infty$ in [**18**].

Define finally, the energy of a given state as $E_m = \log(Trace(M^m(j)))$, and the (limiting-)*free energy*:

$$\beta F_{(\cdot)}(\beta) = \lim_{m \to \infty} \frac{-\log(Z_m^{(\cdot)}(\beta))}{m},$$

(the marker $(\cdot)$ is used to denote either the FSC or KSC version). Figure 1 of [**27**] illustrates typical behaviour of $F_{m,F}(\beta)$, the FSC free energy for fixed small values of m. An approach using the transfer operator is considered in **[16]** and **[18]**. In the former it is shown that the Knauf free energy is given by

$$F_K(\beta) = \frac{-1}{\beta}\log(\lambda_\beta),$$

where λ_β is again the largest eigenvalue of $\mathcal{L}_\beta(U)$ on $C[0,1]$. It is thus equal to the log-topological pressure of the Farey map. In this note Fiala and Kleban were also able to calculate values of expectation for certain configurations of spin chains. In **[18]** an approach which considers the Farey map directly is considered. The limiting free energy is considered in further detail in **[33]**. Of particular interest for the current work is the use of a dynamical systems interpretation of the Farey fraction spin chain to obtain a so-called *cluster* approximation. This approximation is obtained by replacing the smooth dynamical system yielded by the Farey map with a piecewise linear map which preserves the essential features of the original dynamical system. Recall the definition of the inverse Farey map branches Φ_1 and Φ_2, described on page 12, then this version of the Farey map is constructed by linearising between its inverse images $\Phi_1^k\left(\frac{1}{2}\right)$. Since this sequence tends to zero, the fixed point of the Farey map which lies at zero is preserved. For an illustration of this new map, see Figure 2, page 461 of **[33]**.

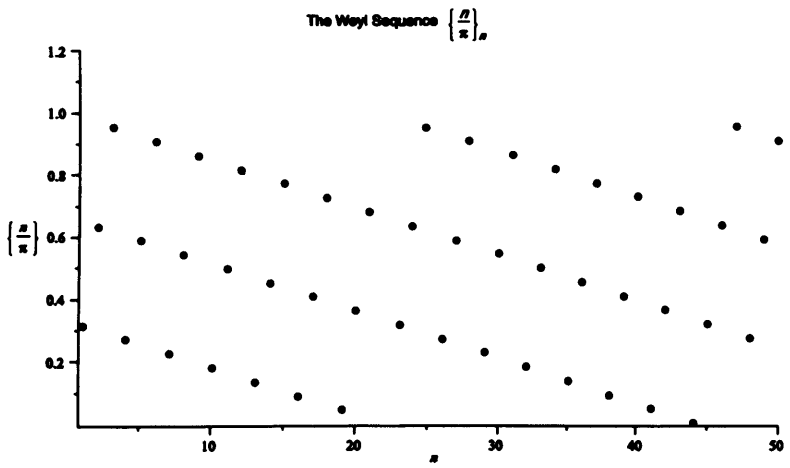


FIGURE 1.8. Example behaviour of a Weyl sequence

6.3. The Weyl Sequences. Define the oft-named *Weyl* sequences in the following way. Let $\theta \in [0,1)$ be an irrational number, and $n = 1, \ldots, m$. Construct the sequence of sets $W_m(\theta) = \{x_1, \ldots, x_m\}$ by taking $x_n = \{n\theta\}$, where $\{\cdot\}$ denotes the fractional part. This is also often seen as $x_n = n\theta \mod 1$. Figure 1.8 illustrates an example for $n = 1, \ldots, 50$ and $\theta = \frac{1}{\pi}$.

In [**42**] an interesting analogy between these sequences, the Farey Series, and the continued fraction algorithm yielded by the Gauss map is made. Assume $y_{0,m} = 0$, $y_{m+1,m} = 1$ and, for $k = 1, \ldots, m$, that $y_{k,m}$ are the points of $W_m(\theta)$ in ascending order. Use the set $\{y_{k,m}\}_{k=0}^{m+1}$ to partition the unit interval $[0,1)$, into subintervals $I_{k,m} = [y_{k,m}, y_{k+1,m}]$ and furthermore denote the following:

$$\begin{aligned}
\delta_m(\theta) &= \min_{n=1,\ldots,m} x_n = y_{1,m} \\
\Delta_m(\theta) &= 1 - \max_{n=1,\ldots,m} x_n = 1 - y_{m,m} \\
\alpha_m(\theta) &= \min_{k=1,\ldots,m} |I_{k,m}| = \min_{k=1,\ldots,m} \{\delta_m(\theta), \Delta_m(\theta)\}, \\
\beta_m(\theta) &= \max_{k=1,\ldots,m} \{\delta_m(\theta), \Delta_m(\theta)\}, \\
\xi_m(\theta) &= \frac{\alpha_m(\theta)}{\beta_m(\theta)}.
\end{aligned}$$

In [**38**] it is shown that the interval lengths $|I_{k,m}|$ may take only one of two or three values: $\delta_m(\theta)$, $\Delta_m(\theta)$ and perhaps $\delta_m(\theta) + \Delta_m(\theta)$. Moreover, let $\{q\theta\}$ and $\{q'\theta\}$ be the smallest and largest elements of $W_m(\theta)$, and let $p = \lfloor q\theta \rfloor$ and $p' = 1 - \lfloor q'\theta \rfloor$. Again, a particular result of the work of [**38**] shows that:

$$\frac{p}{q} < \theta < \frac{p'}{q'}$$

where $\frac{p}{q}$ and $\frac{p'}{q'}$ are consecutive fraction in the Farey Series of order n, $\mathcal{F}_n$.

Furthermore:

$$\begin{aligned}
\delta_m(\theta) &= q\theta - p \\
\Delta_m(\theta) &= p' - q'\theta \\
\Rightarrow \alpha_m(\theta) &= \min_{\frac{p}{q} \in \mathcal{F}_n} |q\theta - p|.
\end{aligned}$$

An elegant analogy with the continued fraction expansion is also discussed in [**42**]. Let the irrational number θ be expressed as $[a_1, a_2, \ldots]$, then using the Gauss map with $T_0 = \theta$ and $T_m = T_m(\theta) = \left\{\frac{1}{T_{m-1}}\right\}$, one sees that the role of $T_m(\theta)$ is played by

$$\xi_m(\theta) = \frac{\min\{|q\theta - p|, |q'\theta - p'|\}}{\max\{|q\theta - p|, |q'\theta - p'|\}}.$$

Similarly, uniform partitions are discussed in fuller detail Drobot: [**11**]. Consider, for the Weyl sequence, $W_r(\theta) = \{x_1, \ldots, x_r\}$ placed in monotone increasing order and the normalised sum of interval lengths

$$A^{(p)}(r) = \frac{1}{(r+1)^{1-p}} \sum_{j=0}^{r} u_{j,r}^p,$$

where $u_{j,r} = x_{j+1,r} - x_{j,r}$. It is shown in [**11**] that, for $p > 1$,

$$\liminf_{r\to\infty} A^{(p)}(r) < \infty \quad \text{and} \quad \limsup_{r\to\infty} A^{(p)}(r) < \infty,$$

where the latter holds if and only if the partial quotients of $\theta = [a_1, a_2, \ldots]$ are bounded.

CHAPTER 2

The Functions ρ and ρ'

1. The Metrics

Consider again the set of Farey Tree points FT_n. These form an ordered set of rational points such that $x_{0,n} = \frac{0}{1}$, $x_{N(n),n} = \frac{1}{1}$ are the parent vertices at level 1 of the Tree, and

$$\mathrm{FT}_n := \left\{x_{0,n}, x_{1,n} \ldots, x_{N(n),n}\right\}, \tag{11}$$

where $N(n) = 2^{n-1}$. For this set we have a number of related definitions. Consider a subinterval of the unit interval given by $I_i := [x_{i,n}, x_{i+1,n})$. Each I_i is known as a 'Farey Cell' which has length

$$p_{i,n} := x_{i+1,n} - x_{i,n} = \frac{1}{qq'}, \tag{12}$$

since $p'q - pq' = 1$, often referred to as *self-modularity* (see for example, page 335 of [**37**]). Moreover, the *Farey Partition* $\mathcal{P}_n$ on $[0,1)$ is defined

$$\mathcal{P}_n : [0,1) \to \bigcup_{i=1}^{N(n)} I_i.$$

Consider the distance functions ρ and ρ' defined:

$$\rho\left(x, \mathrm{FT}_n\right) := \rho_n(x) = \min_{\frac{p}{q} \in \mathrm{FT}_n} \left|x - \frac{p}{q}\right|;$$

$$\rho'\left(x, \mathrm{FT}_n\right) := \rho'_n(x) = \min_{\frac{p}{q} \in \mathrm{FT}_n} q\left|x - \frac{p}{q}\right|;$$

Define also the $\rho^{(\epsilon)}$-distance, for $\varepsilon > 0$ in the following fashion:

$$\rho^{(\epsilon)}\left(x, \mathrm{FT}_n\right) := \rho_n^{(\epsilon)}(x) = q^{1+\epsilon}\left|x - \frac{p}{q}\right|.$$

For Farey points $x_{1,n} = 0$ and $x_{2,n} = \frac{1}{n}$ it is easy to see that

$$\rho_n^{(\epsilon)}\left(0,\frac{1}{n}\right) = n^{1+\epsilon}\left|0-\frac{1}{n}\right| = n^{\epsilon}.$$

In particular, since this value is monotone increasing as n increases, one sees that $\rho_n^{(\epsilon)}$ is not well-behaved as a distance function.

1.1. The Properties of ρ and ρ'. The behaviour of the functions ρ and ρ' on partition $\mathcal{P}_n$ is illustrated in figures 2.1 and 2.2. such that both have an associated *mediant*; these are the points $m, m' \in \left[\frac{p}{q},\frac{p'}{q'}\right]$ - the Farey Cell - such that:

$$\rho_n^{(\cdot)}\left(m^{(\cdot)},\frac{p}{q}\right) = \rho_n^{(\cdot)}\left(m^{(\cdot)},\frac{p'}{q'}\right).$$

This quantity is illustrated in figure 2.3, and for the Euclidean metric ρ_n this value is

$$m_{i,n} := \frac{1}{2}\left(\frac{p}{q}+\frac{p'}{q'}\right).$$

For the quantity ρ_n' we have, by elementary calculations

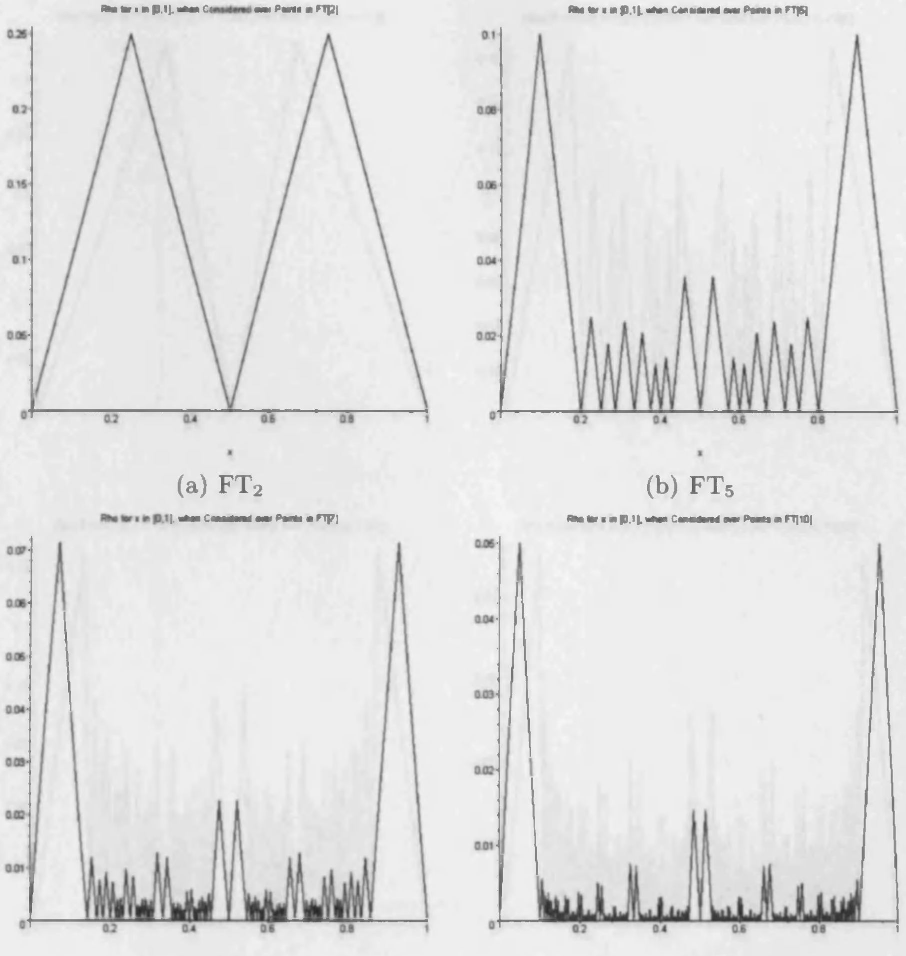
$$m_{i,n}' := \frac{p+p'}{q+q'}.$$

Thus given the set of Farey points FT_n whose origins are at levels at most n, one may form the the set of 'child' fractions $\mathcal{Q}_{n+1}$ by calculating the ρ'-mediants of successive element neighbours of FT_n. Moreover, the metric takes its maximum value, or *height* at the mediant point of that Farey Cell. We define this quantity to be $h^{(\cdot)}$, where

$$\begin{aligned} h_{i,n} &:= \rho(m_{i,n},x_{i,n}) = \rho\left(m,\frac{p}{q}\right) = \frac{1}{2qq'} = \frac{p_{i,n}}{2}, \\ h_{i,n}' &:= \rho'(m_{i,n}',x_{i,n}) = \rho'\left(m',\frac{p}{q}\right) = \frac{1}{q+q'}. \end{aligned} \tag{13}$$

2. Constructing Measures for $\rho_n(x)$ and $\rho_n'(x)$

2.1. The Farey Cells. Consider a Farey Cell in partition $\mathcal{P}_n$ such as that illustrated in Figure 2.3, with endpoints $x_{i,n}$ and $x_{i+1,n}$. Let the variable τ be such that



(a) FT_2 (b) FT_5 (c) FT_7 (d) FT_{10}

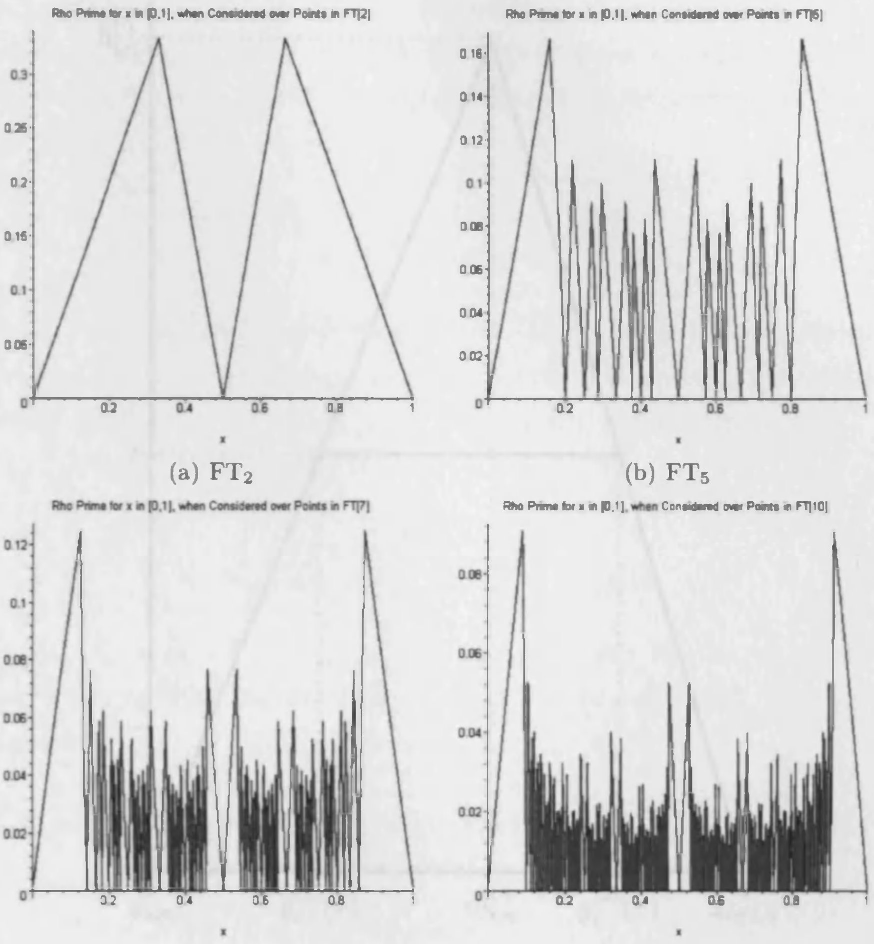
FIGURE 2.1. The Behaviour of ρ on $\mathcal{P}_n$, $n = 2, 5, 7, 10$

$$\rho_n(y) = \tau, \qquad \text{where} \quad y \in \left[\frac{p}{q}, \frac{p'}{q'}\right].$$

For $\tau < h_{i,n}$ this yields two possible distinct values of y, which will be labelled $y_i^{(1)}(\tau)$ and $y_i^{(2)}(\tau)$, and are given by:

$$(14) \qquad \tau = y_i^{(1)}(\tau) - \frac{p}{q} \quad \text{and} \quad \tau = \frac{p'}{q'} - y_i^{(2)}(\tau).$$

Define, for the cell I_i:



(a) FT_2 (b) FT_5

(c) FT_7 (d) FT_{10}

FIGURE 2.2. The Behaviour of ρ' on $\mathcal{P}_n$, $n = 2, 5, 7, 10$

$$\begin{aligned}
\operatorname{meas}\{x \in I_i : \rho_n(x) > \tau\} &:= \left|y_i^{(2)}(\tau) - y_i^{(1)}(\tau)\right| = \left(\frac{p'}{q'} - \tau\right) - \left(\tau + \frac{p}{q}\right) \\
&= \frac{1}{qq'} - 2\tau \\
&= 2\left(\frac{p_{i,n}}{2} - \tau\right).
\end{aligned}$$

Note that if $\tau > h_{i,n}$ the measure in cell I_i is 0. This describes an analogous construction to Lemma 2.1 of [**25**], for measure μ_n defined in this case for the points $p_{i,n}$ given by the Farey Tree, order n. Furthermore, define the measure $\Psi_n(\tau)$ for the metric ρ'_n in an analogous sense:

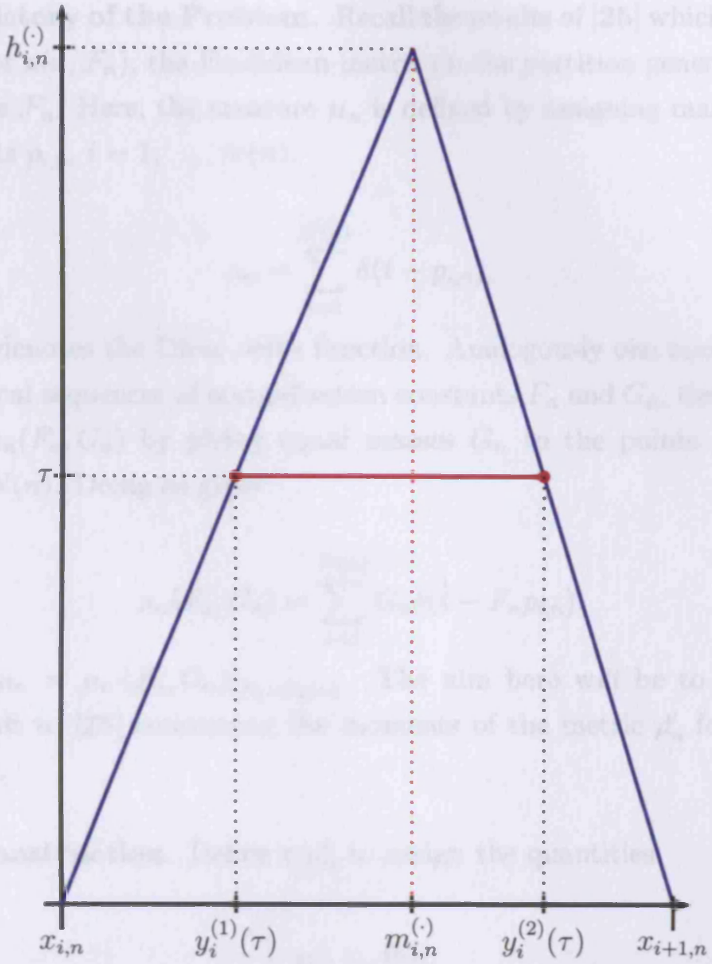


FIGURE 2.3. Constructing the measure

$$\Psi_n(t) := \text{meas}\{x \in [0,1]; \rho'_n(x) \leq t\}.$$

The inverses on this cell where $\rho'_n(x) = \tau$, similarly to (14), are

$$y_i^{(1)}(\tau) = \frac{\tau}{q} + \frac{p}{q} \text{ and } y_i^{(2)}(\tau) = \frac{p'}{q'} - \frac{\tau}{q'},$$

and thus:

$$\text{meas}\{x \in I_i : \rho'_n(x) > \tau\} = \left|y_i^{(2)}(\tau) - y_i^{(1)}(\tau)\right| = \frac{1}{qq'} - t\left(\frac{1}{q} + \frac{1}{q'}\right) = \frac{p_{i,n}}{h'_{i,n}}(h'_{i,n} - t).$$

This is nonzero only for $h'_{i,n} > \tau$.

2.2. History of the Problem. Recall the results of [25] which constructs measures for $\rho(x, \mathcal{F}_n)$, the Euclidean metric on the partition generated by the Farey Series $\mathcal{F}_n$. Here, the measure μ_n is defined by assigning mass 1 to each of the points $p_{i,n}$, $i = 1, \ldots, N(n)$:

$$\mu_n = \sum_{i=1}^{N(n)} \delta(t - p_{i,n}),$$

where $\delta(\cdot)$ denotes the Dirac delta function. Analogously one may assign, for two numerical sequences of normalisation constants F_n and G_n, the normalised measures $\mu_n(F_n, G_n)$ by giving equal masses G_n to the points $F_n p_{i,n}$, over $i = 1, \ldots, N(n)$. Doing so gives

$$\mu_n(F_n, G_n) = \sum_{i=1}^{N(n)} G_n \delta(t - F_n p_{i,n}).$$

Note that $\mu_n = \mu_n (F_n, G_n)|_{F_n = G_n = 1}$. The aim here will be to construct a similar result to [25] concerning the moments of the metric ρ'_n for the Farey Tree points.

2.3. Construction. Define c μ'_n to assign the quantities

$$P_{i,n} = \frac{p_{i,n}}{h'_{i,n}} \tag{15}$$

to each of the points $h'_{i,n}$, which may be written in the following form:

$$\mu'_n = \sum_{i=1}^{N(n)} P_{i,n} \delta(t - h'_{i,n}). \tag{16}$$

For two sequences of positive elements $\{F_n\}$ and $\{G_n\}$ the normalised measures which assign masses $G_n P_{i,n}$ to the points $F_n h'_{i,n}$ is written

$$\mu'_n(F_n, G_n) = \sum_{i=1}^{N(n)} G_n P_{i,n} \delta(t - F_n h'_{i,n}). \tag{17}$$

One should note that the prime notation adopted here does not denote the derivative.

LEMMA 2.1. *The measure $d\Psi_n$ is absolutely continuous with respect to the Lebesgue measure. Moreover one may also define the density function $p'_n(\tau) = \frac{d}{d\tau}\Psi_n(\tau)$ such that*

$$p'_n(\tau) = \begin{cases} \sum_{i: h'_{i,n} > \tau} P_{i,n} & \tau \in [0, \frac{1}{2}] \\ 0 & \text{otherwise.} \end{cases}$$

The quantity $h'_{i,n}$ is defined in (13) *and $P_{i,n}$ in* (15).

PROOF. For any $n \in \mathbf{N}$ and $\tau \geq 0$:

$$\begin{aligned} 1 - \Psi_n(t) &= \text{meas}\{x \in [0,1] : \rho'_n(x) > \tau\} \\ &= \sum_{i: h'_{i,n} > \tau} \frac{p_{i,n}}{h'_{i,n}}(h'_{i,n} - t) \\ &= \sum_{i: h'_{i,n} > \tau} \int_\tau^{h'_{i,n}} \frac{p_{i,n}}{h'_{i,n}} dt \\ &= \int_\tau^\infty \sum_{i: h'_{i,n} > t} P_{i,n} dt, \\ &= \int_\tau^\infty \mu'_n([t, \infty)) dt. \end{aligned}$$

This implies that $d\Psi_n(t)$ is absolutely continuous with respect to the Lebesgue measure. □

COROLLARY 2.2. *Analogously to Corollary* 2.1 *of* **[25]**, *one has, for two sequences of positive elements $\{F_n\}$ and $\{G_n\}$*

$$p'_n\left(\frac{\tau}{F_n}\right) G_n = \sum_{i: h'_{i,n} F_n > \tau} P_{i,n} G_n,$$

for any $\tau > 0$.

LEMMA 2.3. *For the partitions $\mathcal{P}_n$, the sequences $\{F_n\}$ and $\{G_n\}$ chosen so that the measures μ'_n defined in* (16) *$*$-weak converge to some Borel measure μ' for A, a given point of continuity on that measure. Then the measures $p'_n\left(\frac{\tau}{F_n}\right) G_n d\tau$ $*$-weak converge to an absolutely continuous measure $p'(\tau)d\tau$ (with respect to the Lebesgue measure on $(0,\infty)$), where*

$$M(\mu)(s) := \int_0^\infty t^s \, d\mu(t).$$

$$p'(\tau) = \mu'([\tau, \infty)),$$

for any $\tau > 0$ a point of continuity of the measure μ.

PROOF. The original result is discussed as Lemma 2.2 of [**25**], and in fact works analogously for the quantities of the present lemma. This is obvious since the density $p'_n(\tau)$ is monotone decreasing - the quantities $P_{i,n}$ are independent of the selection of τ (for $h'_{i,n} > \tau$). Note also that

$$d\Psi_n(\tau) = \sum_{i:h'_{i,n} > \tau} P_{i,n} d\tau = p'_n(\tau) d\tau.$$

□

Recall the definition of the Mellin transform of function f:

$$M(f)(s) := \int_0^\infty x^{s-1} f(x) \ dx, \ dx,$$

and analogously, for every Borel measure μ:

$$M(\mu)(s) := \int_0^\infty t^s \, d\mu(t).$$

LEMMA 2.4. *Let $p'(\tau) = \mu'([\tau, \infty))$ as defined above (for $\tau > 0$), and*

$$\int_0^\infty t^{\alpha+1} d\mu'(t) < \infty.$$

Then the moments of $p'(\tau)$ may be expressed as a Mellin transform with respect to the measure $d\mu(t)$ and

$$\int_0^\infty t^\alpha p'(t) \, dt = \begin{cases} \infty, & \textit{if } \ \alpha \leq -1, \\ \dfrac{1}{\alpha+1} \displaystyle\int_0^\infty t^{\alpha+1} d\mu'(t) & \textit{if } \ \alpha > -1. \end{cases} \tag{18}$$

PROOF. The proof follows similarly to Lemma 2.3 of [**25**]:

$$\begin{aligned} \int_0^\infty \tau^\alpha p'(\tau) d\tau &= \int_0^\infty \tau^\alpha \mu'([\tau, \infty)) d\tau = \int_0^\infty \left(\tau^\alpha \int_\tau^\infty d\mu'(t) \right) d\tau \\ &= \int_0^\infty \int_0^t \tau^\alpha d\tau \, d\mu'(t) \quad \text{by Fubini's Theorem} \\ &= \frac{1}{\alpha+1} \int_0^\infty t^{\alpha+1} d\mu'(t), \quad \text{for } \alpha > -1. \qquad (19) \end{aligned}$$

Note that for $\alpha \leq -1$ the integral $\int_0^t \tau^\alpha d\tau$ is divergent and hence the above imposition on the value of this constant. □

Define the normalised ρ'_n as follows:

(20)
$$\tilde{\rho}'_n(\theta) = \begin{cases} n\rho'_n\left(\frac{\theta}{n}\right), & 0 \leq \theta \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

For any natural n, Lemma 2.1 implies that the distribution of $\tilde{\rho}'_n$ is $p'_n\left(\frac{\tau}{n}\right) d\tau$. Therefore, as is the case in Theorem 3.1 of **[25]**, the $*$-weak convergence of the measure sequence $\left\{p'_n\left(\frac{\tau}{n}\right) d\tau\right\}_n$ is equivalent to the asymptotic distribution yielded by the function $\tilde{\rho}'_n$. These measures are related to the sequence of $\tilde{\mu}'_n = \mu'_n(n,1)$ which assign the normalised mass $P_{i,n}$ to the points $nh'_{i,n}$ for $i = 1, \ldots, N(n)$, written

$$\tilde{\mu}'_n = \sum_{i=1}^{N(n)} P_{i,n}\delta(t - nh'_{i,n}).$$

We can therefore express the moments of the distance function ρ'_n in terms of the Mellin transform with respect to the measure $d\tilde{\mu}'_n(t)$ as follows. Set the value of τ to be greater than zero, $\alpha > 1$ (noting that the value of the metric is zero outside of the interval $(0,1)$), one has

$$\begin{aligned} \int_0^1 \rho'_n(x)^\alpha \, dx &= \frac{1}{n}\int_0^n \rho'_n\left(\frac{\theta}{n}\right)^\alpha d\theta \quad \text{letting} \quad x = \frac{\theta}{n} \\ &= \frac{1}{n^{\alpha+1}}\int_0^\infty \tau^\alpha p'_n\left(\frac{\tau}{n}\right) d\tau \quad \text{by Lemma 2.1} \\ &= \frac{1}{(\alpha+1)}\frac{1}{n^{\alpha+1}}\int_0^\infty t^{\alpha+1} \, d\tilde{\mu}'_n(t) \\ &= \frac{(\alpha+1)^{-1}}{n^{\alpha+1}}\mathrm{M}(\tilde{\mu}'_n)(\alpha+1), \end{aligned}$$

which implies

$$\mathrm{M}(\tilde{\mu}'_n)(\alpha+1) = n^{\alpha+1}\sum_{i=1}^{N(n)} p_{i,n}h'^{\alpha}_{i,n}.$$

This is calculated from formula (24), on page 47.

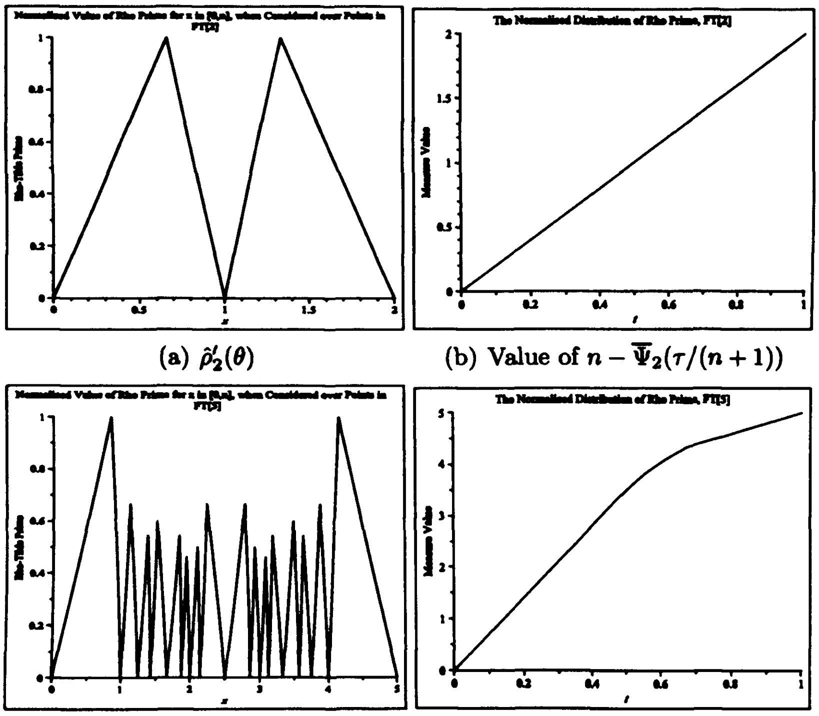
2.4. Figures. The normalised metric $\hat{\rho}'_n$, defined

$$\hat{\rho}'_n(\theta) = \begin{cases} a_n \rho'_n\left(\frac{\theta}{b_n}\right), & 0 \leq \theta \leq b_n, \\ 0, & \text{otherwise,} \end{cases}$$

is associated with the analogous measure $d\overline{\Psi_n}(\tau)$ such that

$$\begin{aligned} 1 - \overline{\Psi_n}(\tau) &:= \text{meas}\left\{x \in [0, b_n] : \hat{\rho}'_n(x) > \frac{\tau}{a_n}\right\} \\ &= \int_\tau^\infty \sum_{i : a_n h'_{i,n} > t} \frac{b_n}{a_n} P_{i,n}. \end{aligned}$$

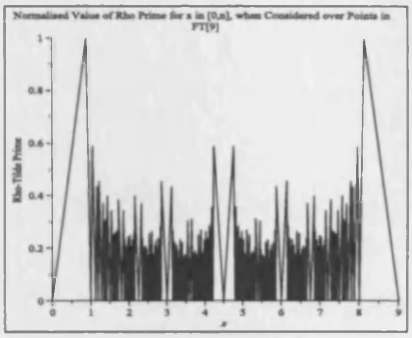
This follows from the previous construction of Lemma 2.1. For illustrative purposes, figures 2.4 and 2.5 are included to highlight the behaviour of the normalised relationship $b_n - \overline{\Psi_n}\left(\frac{\tau}{a_n}\right)$ with $a_n = n + 1$ and $b_n = n$. These fix the interval of τ to be $[0, 1]$ and give the value of the measure in the interval $[0, n]$.



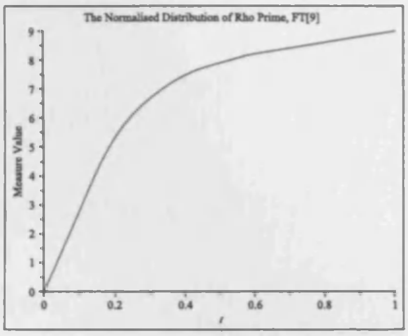
(a) $\hat{\rho}'_2(\theta)$ (b) Value of $n - \overline{\Psi}_2(\tau/(n+1))$

(c) $\hat{\rho}'_5(\theta)$ (d) Value of $n - \overline{\Psi}_5(\tau/(n+1))$

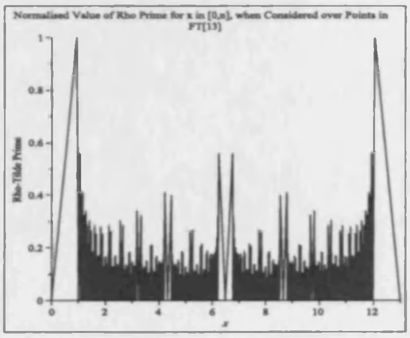
FIGURE 2.4. Normalised distributions, with $\hat{\rho}'$, for $n = 2, 5$



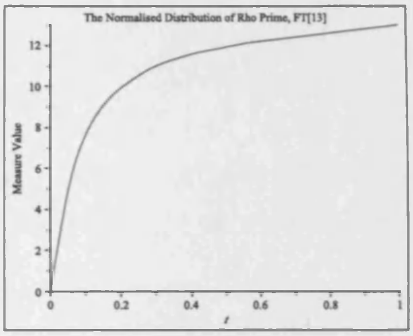
(a) $\hat{\rho}'_9(\theta)$



(b) Value of $n - \overline{\Psi}_9(\tau/(n+1))$



(c) $\hat{\rho}'_{13}(\theta)$



(d) Value of $n - \overline{\Psi}_{13}(\tau/(n+1))$

FIGURE 2.5. Normalised distributions, with $\hat{\rho}'$, for $n = 9, 13$

CHAPTER 3

The Moments of $\rho_n(x)$ and $\rho'_n(x)$

1. The Moments of $\rho_n(x)$

The aim of this this chapter will be a proof that for the ρ' metric, one has

$$\int_0^1 \rho'_n(x)^\alpha dx = \mathcal{O}\left(\frac{1}{n^{\alpha+1}}\right).$$

This will use many of the techniques made familiar in both [**31**] and [**12**], and as an introductory note, a brief overview of this previous work on the ρ metric follows.

1.1. Formulation. Consider first the metric $\rho_n(x)$ and specifically its integral. Let the the pair $\left\{\frac{p}{q}, \frac{p'}{q'}\right\}$ be consecutive elements from FT_n, meaning that the interval $\left[\frac{p}{q}, \frac{p'}{q'}\right]$ is a Farey Cell and that $\sum_{\left\{\frac{p}{q}, \frac{p'}{q'}\right\}}$ is the sum over each Farey cell. Moreover, under $\rho_n(x)$, this Farey Cell has midpoint $m = \frac{1}{2}\left(\frac{p}{q} + \frac{p'}{q'}\right)$. Therefore with constant $\delta > 0$, one has:

$$\begin{aligned}
\int_0^1 \rho_n(x)^\delta \, dx &= \sum_{\left\{\frac{p}{q}, \frac{p'}{q'}\right\}} \left(\int_{\frac{p}{q}}^{m} \left(x - \frac{p}{q}\right)^\delta dx + \int_m^{\frac{p'}{q'}} \left(\frac{p'}{q'} - x\right)^\delta dx \right) \\
&= \frac{1}{\delta+1} \sum_{\left\{\frac{p}{q}, \frac{p'}{q'}\right\}} \left(\left[\left(x - \frac{p}{q}\right)^{\delta+1}\right]_{\frac{p}{q}}^{m} - \left[\left(\frac{p'}{q'} - x\right)^{\delta+1}\right]_m^{\frac{p'}{q'}} \right) \\
&= \frac{1}{\delta+1} \sum_{\left\{\frac{p}{q}, \frac{p'}{q'}\right\}} \left[\left(\frac{1}{2}\left(\frac{p'}{q'} - \frac{p}{q}\right)\right)^{\alpha+1} - \left(\frac{1}{2}\left(\frac{p}{q} - \frac{p'}{q'}\right)\right)^{\delta+1} \right] \\
&= \frac{2^{-\delta}}{\delta+1} \sum_{(q,q')} \frac{1}{(qq')^{\delta+1}}. \qquad (21)
\end{aligned}$$

The sum from (21) is evidently equal to the characteristic of interest in [**31**] and [**12**], and thus for brevity, let $\beta > 1$ and the function $\sigma_\beta(\mathrm{FT}_n)$ be defined as in the previous notes:

$$\sigma_\beta(\mathrm{FT}_n) = \sum_{(q,q')} \frac{1}{(qq')^\beta}. \tag{22}$$

The methods used to determine asymptotic formulae for (22) shall form a basis for this chapter.

1.2. Asymptotic Formula for the $\rho_n(x)$ Moments. An asymptotic formula for $\sigma_\beta(\mathrm{FT}_n)$ is derived in [**31**], and later improved upon by Anna Dushistova in [**12**]. These are Theorems 3.1 and 3.2 respectively. We have

THEOREM 3.1. Moshchevitin and Zhigljavsky, [**31**]: *For any $\beta > 1$, and as $n \to \infty$,*

$$\sigma_\beta(\mathrm{FT}_n) = \sum_{(q,q')} \frac{1}{(qq')^\beta} = \frac{2}{n^\beta} \frac{\zeta(2\beta - 1)}{\zeta(2\beta)} + \mathcal{O}\left(\frac{\log(n)}{n^{\frac{(\beta+1)(2\beta-1)}{2\beta}}}\right),$$

and

THEOREM 3.2. Anna Dusistova, [**12**]: *For any $\beta > 1$, and as $n \to \infty$*

$$\sigma_\beta(\mathrm{FT}_n) = \frac{2}{n^\beta} \frac{\zeta(2\beta - 1)}{\zeta(2\beta)} + \sum_{1 \le k < 2\beta - 2} \frac{C_k}{n^{\beta+k}} + \sum_{0 \le k < \beta - 2} \frac{C_k^*}{n^{2\beta+k}} + \mathcal{O}\left(\frac{\log^{3\beta}(n)}{n^{3\beta-2}}\right),$$

where C_k and C_k^* are positive constants depending on $\beta > 1$ also.

In the notation of [**25**] and the previous chapter, the function σ_β may be written:

$$\sigma_\beta(\mathrm{FT}_n) = \sum_{i=1}^{N(n)} p_{i,n}^\beta,$$

where the quantity $p_{i,n}$ is defined in (12) and $N(n) = 2^{n-1}$ for the Farey Tree. Note that the order of decrease of σ_β is different to that described in Theorems 3.1 and 3.2 for when $\beta \le 1$; in particular it is very easy to see that

$$\sigma_0(\mathrm{FT}_n) = 2^{n-1} = N(n) \text{ and } \sigma_1(\mathrm{FT}_n) = 1, \tag{23}$$

where the latter is true as the sum of all the possible cell lengths partitioning the unit interval is 1.

2. The Moments of $\rho_n'(x)$

2.1. Formulation. Let us first set the scene of the construction. Consider the integral of $\rho_n'(x)$ in a similar fashion, whence:

$$\int_{\forall x \in \mathbf{R}} \rho_n'(x)^\alpha dx = \int_0^1 \rho_n'(x)^\alpha dx = \sum_{\left\{\frac{p}{q},\frac{p'}{q'}\right\}} \int_{\frac{p}{q}}^{\frac{p'}{q'}} \rho_n'(x)^\alpha dx.$$

This implies, where $m' = \frac{p+p'}{q+q'}$ is the ρ' mediant on cell $\left[\frac{p}{q}, \frac{p'}{q'}\right]$, that the integral takes the form

$$\begin{aligned}
\int_0^1 \rho_n'(x)^\alpha dx &= \sum_{\left\{\frac{p}{q},\frac{p'}{q'}\right\}} \left[\int_{\frac{p}{q}}^{m'} q^\alpha \left(x - \frac{p}{q}\right)^\alpha dx + \int_{m'}^{\frac{p}{q}} q'^\alpha \left(\frac{p'}{q'} - x\right)^\alpha dx\right] \\
&= \frac{1}{\alpha+1} \sum_{\left\{\frac{p}{q},\frac{p'}{q'}\right\}} \left(\left[q^\alpha \left(x - \frac{p}{q}\right)^{\alpha+1}\right]_{\frac{p}{q}}^{m'} - \left[q'^\alpha \left(\frac{p'}{q'} - x\right)^{\alpha+1}\right]_{m'}^{\frac{p'}{q'}}\right) \\
&= \frac{1}{\alpha+1} \sum_{\left\{\frac{p}{q},\frac{p'}{q'}\right\}} \left(q^\alpha \left(\frac{1}{q(q+q')}\right)^{\alpha+1} + q'^\alpha \left(\frac{1}{q'(q+q')}\right)^{\alpha+1}\right) \\
&= \frac{1}{\alpha+1} \sum_{(q,q')} \frac{1}{qq'(q+q')^\alpha}. \qquad (24)
\end{aligned}$$

The fact that $p'q - pq' = 1$ ensures that numerators p and p' do not appear in the final formulation. Therefore the change of sum condition to pairs of denominators from consecutive Farey elements (denoted (q, q')) is used without loss of any information.

Calculation of this quantity has been performed using the C programming language, which is indeed the case for all following relevant figures. They have been plotted using the Maple computer algebra system. **For illustrative purposes, we apply a $-\log$ scaling**; indeed as an initial example, raw behaviour of $-\log(\gamma_\alpha(\mathrm{FT}_n))$ for $\alpha = 2$ is shown in figure 3.1.

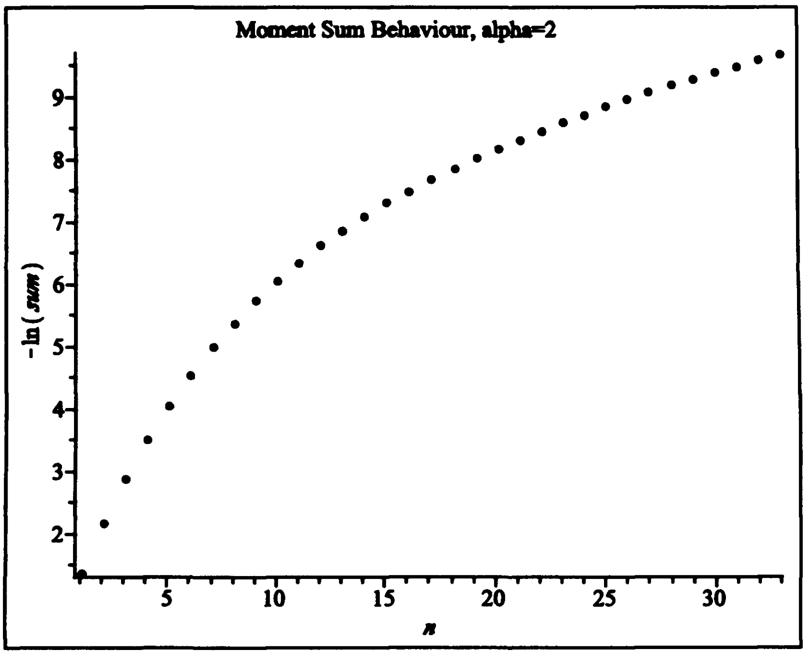


FIGURE 3.1. The ρ' Moments for $\alpha = 2$

3. The Main Theorems

3.1. Statement. There will be two main theorems constructed in this chapter. The first is a result on the sum involving the single denominator q'', while the second is the main result on the ρ' metric. Let

$$\gamma_\alpha(\mathrm{FT}_n) = \sum_{(q,q'',q')} \frac{1}{qq'(q'')^\alpha}, \tag{25}$$

where $q + q' = q''$, is itself a denominator characterised by the previous behaviour of q and q' in the fraction $\frac{p''}{q''} = \frac{p+p'}{q+q'} \in \mathcal{Q}_{n+1}$, defined in (1) on page 11. Now, due to the fact that

$$\mathrm{FT}_{n+1} = \bigcup_{i=1}^{n+1} \mathcal{Q}_i,$$

an ordered triple (q, q'', q') corresponds to consecutive elements $\frac{p}{q}, \frac{p''}{q''}, \frac{p'}{q'}$ where $\frac{p''}{q''} \in \mathcal{Q}_{n+1}$. This fraction has denominator continuant $q'' = q''(a) = \langle a_1, a_2, \dots, a_r \rangle$, where

$$a \in \mathcal{A}_N := \left\{ (a_1, a_2, \dots, a_r) \, : \, a_r \geq 2, \sum_{i=1}^{r} a_i = n+1 = N \right\}.$$

For brevity in notation adopt the convention $n + 1 = N$; though naturally as $n \to \infty$, $N = \mathcal{O}(n)$. Now, a simple rearrangement gives

$$\gamma_\alpha(\mathrm{FT}_n) = \sum_{(q,q'',q')} \left(\frac{1}{q(q'')^{\alpha+1}} + \frac{1}{q'(q'')^{\alpha+1}} \right).$$

We take, without loss of any generality, the denominator q to have origin at level n; and therefore fix the denominator $q'(a)$ to be the smallest of this triple, originating from $\frac{p'}{q'} \in \mathcal{Q}_{n+1-j}$, $j > 1$. For continuity of notation with [**31**] and [**12**], let $q = q_+$, and $q' = q_-$.

In summary, this implies that the continuant $\langle a_1, \dots a_r \rangle$ representing the denominator $q''(a)$ has associated 'parent' denominators $q_-(a)$ and $q_+(a)$ from its neighbouring Farey numbers such that (see for example Lemma 1 of [**31**])

$$q_- = q_-(a) = \langle a_1, \dots, a_{r-1} \rangle \qquad q_+ = q_+(a) = \langle a_1, \dots, a_r - 1 \rangle.$$

Moreover, Lemma 2 of [**31**] states that the following is true:

$$q'' \leq Nq_- \iff q_+ + q_- \leq Nq_- \iff q_+ \leq nq_-.$$

We may assume, without any loss of generality that the vectors $(a_1, \dots, a_{r-1})$ and $(a_1, \dots, a_r - 1)$ upon which these continuants have basis, are elements from $\mathcal{A}_{N-a_r}$ and $\mathcal{A}_{N-1}$ respectively. In the cases where it is known that $a_r = 2$, then one will calculate that the larger 'parent' has

$$q_+ = \langle a_1, a_2, \dots, a_{r-1}, 1 \rangle,$$

to which we may apply the well-known identity

$$[a_1, a_2, \dots, a_{r-1}, 1] = [a_1, a_2, \dots, a_{r-1} + 1].$$

This implies that the resultant 'final' partial quotient is at least 2 also.

Define $\mathbb{N}^r_N$ to be the set of all $r-$dimensional integer vectors whose sum of partial quotients is $N > 0$. This set carries similar definition to $\mathcal{A}_N$ minus the restriction on the value of a_r, and has cardinality 2^{N-1} (this is again easily proven via a combinatorial argument). Moreover $\mathcal{A}_N \subseteq \mathbb{N}^r_N$. Now, consider a vector $a = (a_1, \ldots, a_x) \in \mathcal{A}_N$; the remaining items in $\mathbb{N}^r_N$ are thus of the form $(a_1, \ldots, a_x - 1, 1)$ for all possible a. Therefore

$$|\mathbb{N}^r_N| = 2|\mathcal{A}_N|, \tag{26}$$

with consequence $|\mathcal{A}_N| = 2^{N-2}$, an observation required in the proofs of Lemmas 3.16, 3.17 and onward. In these calculations, when confronted with a summation whose range of action is over integer vectors $(a_1, \ldots, a_x) \in \mathbb{N}^r_N$, then when restricting the value of the final partial quotient (where necessary), one must multiply the sum by factor 2. This compensates for the change in cardinality of the set upon which the sum has action.

Using these arguments it is also easily seen that

$$|\mathcal{A}_N| = 2|\mathcal{A}_n|. \tag{27}$$

The main theorems presented in this chapter are

THEOREM 3.3. *As $n \to \infty$,*

$$\sum_{a \in \mathcal{A}_N} \frac{1}{q(a)^{\alpha+2}} = \frac{C'_0}{N^{\alpha+2}} + \mathcal{O}\left(\frac{\log^{\frac{2\alpha+4}{2\alpha+5}}(N)}{N^{\alpha+3-\frac{\alpha+5}{2\alpha+5}}}\right)$$

where $C'_0 = C'_0(\alpha)$ is defined

$$C'_0 = \frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + 2\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}\right)^2.$$

As a point of note, Theorem 3.3 is a reformulation of that which leads to Lemma 9 in [**12**] and indeed this result may also be seen to be of self-standing interest. The main result is Theorem 3.5, which includes main terms of orders $n^{-(\alpha+1)}$ to $n^{-2\alpha}$ depending on the size of $\alpha > 1$ chosen. However, as a precursor, it is also required to show that $\gamma_\alpha(\mathrm{FT}_n)$ is of order $n^{-(\alpha+1)}$ without the additional terms in order to obtain the improved result. This is Theorem 3.4.

THEOREM 3.4. *For $\alpha > 1$, and as $n \to \infty$*

$$\gamma_\alpha(\mathrm{FT}_n) = \frac{2}{n^{\alpha+1}}\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right).$$

THEOREM 3.5. *For $\alpha > 1$ and as $n \to \infty$,*

$$\begin{aligned}\gamma_\alpha(\mathrm{FT}_n) &= \frac{2}{n^{\alpha+1}}\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + \sum_{1\le k<\alpha}\frac{\mathcal{G}_{k,\alpha}}{n^k}\right) + \frac{1}{n^{\alpha+2}}\left(D_\alpha + \sum_{1\le k<\alpha-1}\frac{\tilde{\mathcal{G}}_{k,\alpha}}{n^k}\right) + \\ &+ \quad \mathcal{O}\left(\frac{\log^{2\alpha+3}(n)}{n^{2\alpha+1}}\right),\end{aligned}$$

where D_α, $\mathcal{G}_{k,\alpha}$, $\tilde{\mathcal{G}}_{k,\alpha}$ are constant for fixed k, α. These are defined by (102), (103) *and* (104) *respectively on page 130.*

Remark: Definitions (102), (103) and (104) arise from the result of several Taylor expansions and associated convergent remainders. As a result they are notationally very large and are referred to here for compactness.

3.2. A Lower Bound for $\gamma_\alpha(\mathrm{FT}_n)$. We may deduce a lower bound for the quantity $\gamma_\alpha(\mathrm{FT}_n)$; this is Lemma 3.6 below. This can be viewed as a precursor to the main result since it is shown that this lower bound is of order $\mathcal{O}(n^{-(\alpha+1)})$. The proof of Lemma 3.6 requires the following formula of Dirichlet concerning the product and convolution of series: suppose $h = f * g$, let:

$$H(x) = \sum_{n\le x} h(n), \quad F(x) = \sum_{n\le x} f(n), \quad G(x) = \sum_{n\le x} g(n).$$

Then we have:

$$H(x) = \sum_{n\le x} f(n)G\left(\frac{x}{n}\right) = \sum_{n\le x} g(n)F\left(\frac{x}{n}\right). \tag{28}$$

LEMMA 3.6. *The moment sum $\gamma_\alpha(\mathrm{FT}_n)$ has the following lower bound:*

$$\gamma_\alpha(\mathrm{FT}_n) \ge \frac{2}{n(n+1)^\alpha}\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + \mathcal{O}\left(\frac{1}{n^{2\alpha+1}}\right).$$

PROOF. The proof requires the following formula, proved as an exercise in [1]. For $x \geq 2$, $c > 1$ ($c \neq 2$), we have

$$\sum_{n \leq x} \frac{\phi(n)}{n^c} = \frac{x^{2-c}}{2-c}\frac{1}{\zeta(2)} + \frac{\zeta(c-1)}{\zeta(c)} + \mathcal{O}\left(\frac{1}{x^{c-1}}\right). \tag{29}$$

Note the following familiar identity:

$$\phi(n) = \sum_{d|n} \frac{\mu(d)n}{d},$$

where $\mu(d)$ is the Möbius function of d. Therefore

$$\begin{aligned}
\sum_{n \leq x} \frac{\phi(n)}{n^c} &= \sum_{n \leq x} \frac{1}{n^{c-1}} \sum_{d|n} \frac{\mu(d)}{d} \\
&= \sum_{n \leq x} \frac{1}{n^{c-1}} \sum_{d} \sum_{\substack{e \\ de=n}} \frac{\mu(d)}{d} \\
&= \sum_{d \leq x} \frac{\mu(d)}{d^c} \sum_{e \leq \frac{x}{d}} \frac{1}{e^{c-1}}, \qquad (30)
\end{aligned}$$

using the Dirichlet product/convolution formula, (28).

We employ Euler's summation formula to show:

$$\sum_{e \leq \frac{x}{d}} \frac{1}{e^{c-1}} = \frac{1}{2-c}\left(\frac{x}{d}\right)^{2-c} + \zeta(c-1) + \mathcal{O}\left(\frac{1}{x^{c-1}}\right), \tag{31}$$

yielding the evaluation below:

$$\sum_{n \leq x} \frac{\phi(n)}{n^c} = \frac{x^{2-c}}{2-c} \sum_{d \leq x} \frac{\mu(d)}{d^2} + \zeta(c-1) \sum_{d \leq x} \frac{\mu(d)}{d^c} + \mathcal{O}\left(\frac{1}{x^{c-1}} \sum_{d \leq x} \frac{1}{d^c}\right).$$

Note also that

$$\sum_{n=1}^{\infty} \frac{\mu(d)}{d^c} = \frac{1}{\zeta(c)},$$

meaning that:

$$\sum_{d \leq x} \frac{\mu(d)}{d^c} = \frac{1}{\zeta(c)} - \sum_{d > x} \frac{\mu(d)}{d^c} = \frac{1}{\zeta(c)} + \mathcal{O}\left(\sum_{d > x} \frac{1}{d^c}\right) = \frac{1}{\zeta(c)} + \mathcal{O}\left(\frac{1}{x^{c-1}}\right),$$

for the given restrictions on c. Moreover, using this and Euler's identity, (31) gives

$$\begin{aligned}\sum_{n\le x}\frac{\phi(n)}{n^c} &= \frac{x^{2-c}}{2-c}\frac{1}{\zeta(2)}+\mathcal{O}\left(\frac{1}{x^{c-1}}\right)+\frac{\zeta(c-1)}{\zeta(c)}-\zeta(c-1)\mathcal{O}\left(\frac{1}{x^{c-1}}\right)\\ &\quad+\mathcal{O}\left(\frac{1}{x^{c-1}}\left[\frac{x^{1-c}}{1-c}+\zeta(c)+\mathcal{O}\left(\frac{1}{x^c}\right)\right]\right)\\ &= \frac{x^{2-c}}{2-c}\frac{1}{\zeta(2)}+\frac{\zeta(c-1)}{\zeta(c)}+\mathcal{O}\left(\frac{1}{x^{c-1}}\right)+\mathcal{O}\left(\frac{1}{x^{2(c-1)}}\right).\end{aligned}$$

Our assumptions dictate that $q_+(a)\ge q_-(a)$, and thus since there are $2\phi(l)$ Farey cells with $q_-(a)=l<n$ as an endpoint (see, for example [**31**]) one sees that:

$$\begin{aligned}\gamma_\alpha(\mathrm{FT}_n) &\ge \sum_{a\in A_N}\left(\frac{1}{(n+1)^{\alpha+1}q_-^{\alpha+1}nq_-}+\frac{1}{(n+1)^{\alpha+1}q_-^{\alpha+2}}\right)\\ &= \left(\frac{1}{n(n+1)^{\alpha+1}}+\frac{1}{(n+1)^{\alpha+1}}\right)\sum_{a\in A_N}\frac{1}{q_-^{\alpha+2}}\\ &\ge \frac{n+1}{n(n+1)^{\alpha+1}}\sum_{a\in A_N:q_-<N}\frac{1}{q_-^{\alpha+2}}\\ &= \frac{2}{n(n+1)^{\alpha}}\sum_{q_-=1}^{n}\frac{\phi(q_-)}{q_-^{\alpha+2}}\end{aligned}$$

Therefore, formula (31) implies that:

$$\begin{aligned}\gamma_\alpha(\mathrm{FT}_n) &\ge \frac{2}{n(n+1)^{\alpha}}\left(\frac{-1}{\alpha}.\frac{n^{-\alpha}}{\zeta(2)}+\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}+\mathcal{O}\left(\frac{1}{n^{\alpha+1}}\right)\right)\\ &= \frac{2}{n(n+1)^{\alpha}}\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}+\mathcal{O}\left(\frac{1}{n^{\alpha+1}}\frac{1}{n^{\alpha}}\right)+\mathcal{O}\left(\frac{1}{n^{\alpha+1}}\frac{1}{n^{\alpha+1}}\right)\\ &= \frac{2}{n(n+1)^{\alpha}}\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}+\mathcal{O}\left(\frac{1}{n^{2\alpha+1}}\right). \qquad (32)\end{aligned}$$

□

The proof of Theorem 3.5 will show that the limit of $\gamma_\alpha(\mathrm{FT}_n)$ is asymptotically equal to its lower bound, i.e. that

$$\liminf_{n\to\infty}\gamma_\alpha(\mathrm{FT}_n)=\lim_{n\to\infty}\gamma_\alpha(\mathrm{FT}_n)=\mathcal{O}\left(\frac{1}{n^{\alpha+1}}\right).$$

Figure 3.2 highlights this supposition by plotting the quantities $n^{\alpha+1}\gamma_\alpha(\mathrm{FT}_n)$.

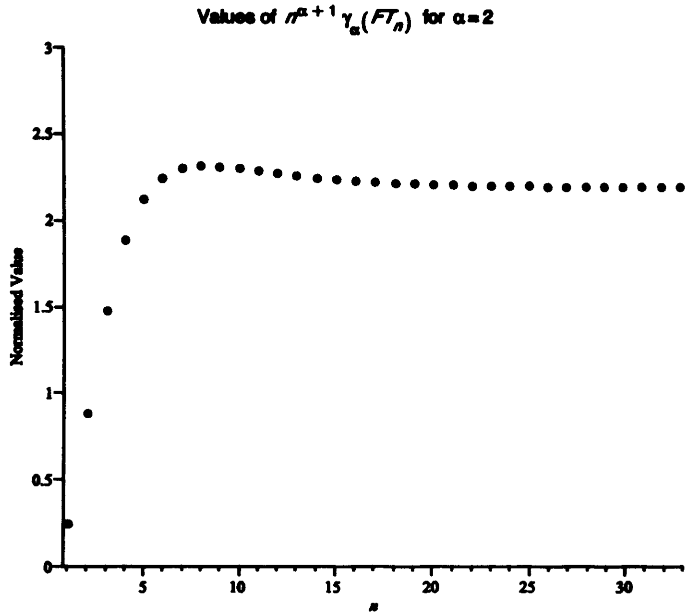


FIGURE 3.2. The behaviour of $n^3\gamma_2$

3.3. Layout of this Proof. The proof of Theorem 3.5 is split into sections and lemmas, one of which comprises the proof of Theorem 3.3. The lemmas making up the main proof will be based upon the sums resulting from the following partitioning of the set $\mathcal{A}_N$:

$$\begin{aligned}
\mathcal{A}^{(1)}_{(N,1)} &:= \{a \in \mathcal{A}_N : \langle a_1, \dots, a_r\rangle < N^s\} \\
\mathcal{A}^{(1)}_{(N,2)} &:= \{a \in \mathcal{A}_N : \langle a_1, \dots, a_r\rangle \geq N^s\} \\
\mathcal{A}^{(2)}_{(N,1)} &:= \{a \in \mathcal{A}^{(1)}_{(N,1)} : \max_{j=1,\dots,r} a_j > N - w\} \\
\mathcal{A}^{(2)}_{(N,2)} &:= \{a \in \mathcal{A}^{(1)}_{(N,1)} : \max_{j=1,\dots,r} a_j \leq N - w\} \\
\mathcal{A}^{(3)}_{(N,j\geq r-1)} &:= \{a \in \mathcal{A}^{(2)}_{(N,1)} : \max_{i=1,\dots,r} a_i = a_j \in \{a_{r-1}, a_r\},\ a_j > N - w\} \\
\mathcal{A}^{(3)}_{(N,j<r-1)} &:= \{a \in \mathcal{A}^{(2)}_{(N,1)} : \max_{i=1,\dots,r} a_i = a_j \in \{a_1, \dots, a_{r-2}\},\ a_j > N - w\} \\
\mathcal{A}^{(4)}_{(N,j=r-1)} &:= \{a \in \mathcal{A}^{(3)}_{(N,j\geq r-1)} : a_j = a_{r-1} > N - w\} \\
\mathcal{A}^{(4)}_{(N,j=r)} &:= \{a \in \mathcal{A}^{(3)}_{(N,j\geq r-1)} : a_j = a_r > N - w\}.
\end{aligned} \tag{33}$$

These are all mutually exclusive sets and are constructed such that

$$\mathcal{A}_N = \mathcal{A}^{(1)}_{(N,2)} \sqcup \mathcal{A}^{(2)}_{(N,2)} \sqcup \mathcal{A}^{(3)}_{(N,j<r-1)} \sqcup \mathcal{A}^{(4)}_{(N,j=r-1)} \sqcup \mathcal{A}^{(4)}_{(N,j=r)},$$

where $\sqcup$ denotes the disjoint set union. For basis of comparison with previous work, the new notation presented in this thesis is such that, for the sets $\mathcal{A}^{(3)}_{(N,1)}$ and $\mathcal{A}^{(3)}_{(N,2)}$ of **[31]** and **[12]**

$$\begin{aligned}\mathcal{A}^{(3)}_{(N,1)} &=: \mathcal{A}^{(4)}_{(N,j=r)} \\ \mathcal{A}^{(3)}_{(N,2)} &=: \mathcal{A}^{(3)}_{(N,j<r-1)} \sqcup \mathcal{A}^{(4)}_{(N,j=r-1)}\end{aligned}$$

This 'new' breakdown of $\mathcal{A}_N$ is illustrated in Figure 3.3.

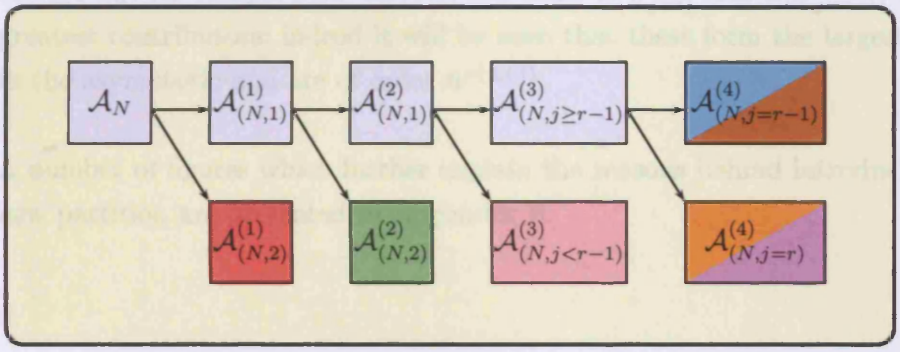


FIGURE 3.3. The hierarchy of denominator sets

The quantities w and s are parameters whose properties and possible selections are considered in at each stage of the proof. Their final values will determine the optimal expression for the final error term upon the reconstruction of the main theorems. As an initial observation, w can take values in the interval $[1, N)$, which will be fairly obvious to the reader since at, for example level L of the Tree, the partial quotients a_i sum to L. The effect of these parameters is discussed in subsection 3.4.1.

For brevity, let:

$$\Sigma^{(j)}_{(N,i)} := \sum_{a \in \mathcal{A}^{(j)}_{(N,i)}} \left(\frac{1}{q_+(q'')^{\alpha+1}} + \frac{1}{q_-(q'')^{\alpha+1}} \right),$$

where $j = 1, 2, 3$ or 4, and i represents one of the items '1', '2', '$j < r-1$', '$j = r-1$' or $j = r$. Hence, using (27), one will see that

$$\gamma_\alpha(\mathrm{FT}_n) = \frac{1}{2}\left(\Sigma^{(1)}_{(N,2)} + \Sigma^{(2)}_{(N,2)} + \Sigma^{(3)}_{(N,j<r-1)} + \Sigma^{(4)}_{(N,j=r-1)} + \Sigma^{(4)}_{(N,j=r)}\right).$$

In addition, the sums $\Sigma^{(4)}_{(N,j=r-1)}$ and $\Sigma^{(4)}_{(N,j=r)}$ are split according to whether each contains the '+' or '−' denominator, for example

$$\Sigma^{(4)}_{(N,j=r)} = \Sigma^{(4)+}_{(N,j=r)} + \Sigma^{(4)-}_{(N,j=r)}.$$

Figures 3.4 and 3.5 (where $s = \frac{\alpha+2}{\alpha}$ and $s = \frac{2\alpha+1}{\alpha}$ respectively; the latter represents the final choice made for this parameter in the main proof) illustrate how each of these sums behaves within the partition and include the points of $\gamma_\alpha(\mathrm{FT}_n)$ for comparison. Clearly, the sums $\Sigma^{(4)-}_{(N,j=r)}$ and $\Sigma^{(4)+}_{(N,j=r-1)}$ are the greatest contributors; indeed it will be seen that these form the largest terms in the asymptotic and are of order $n^{-(\alpha+1)}$.

A number of figures which further explain the reasons behind introducing the new partition are presented in Appendix B.

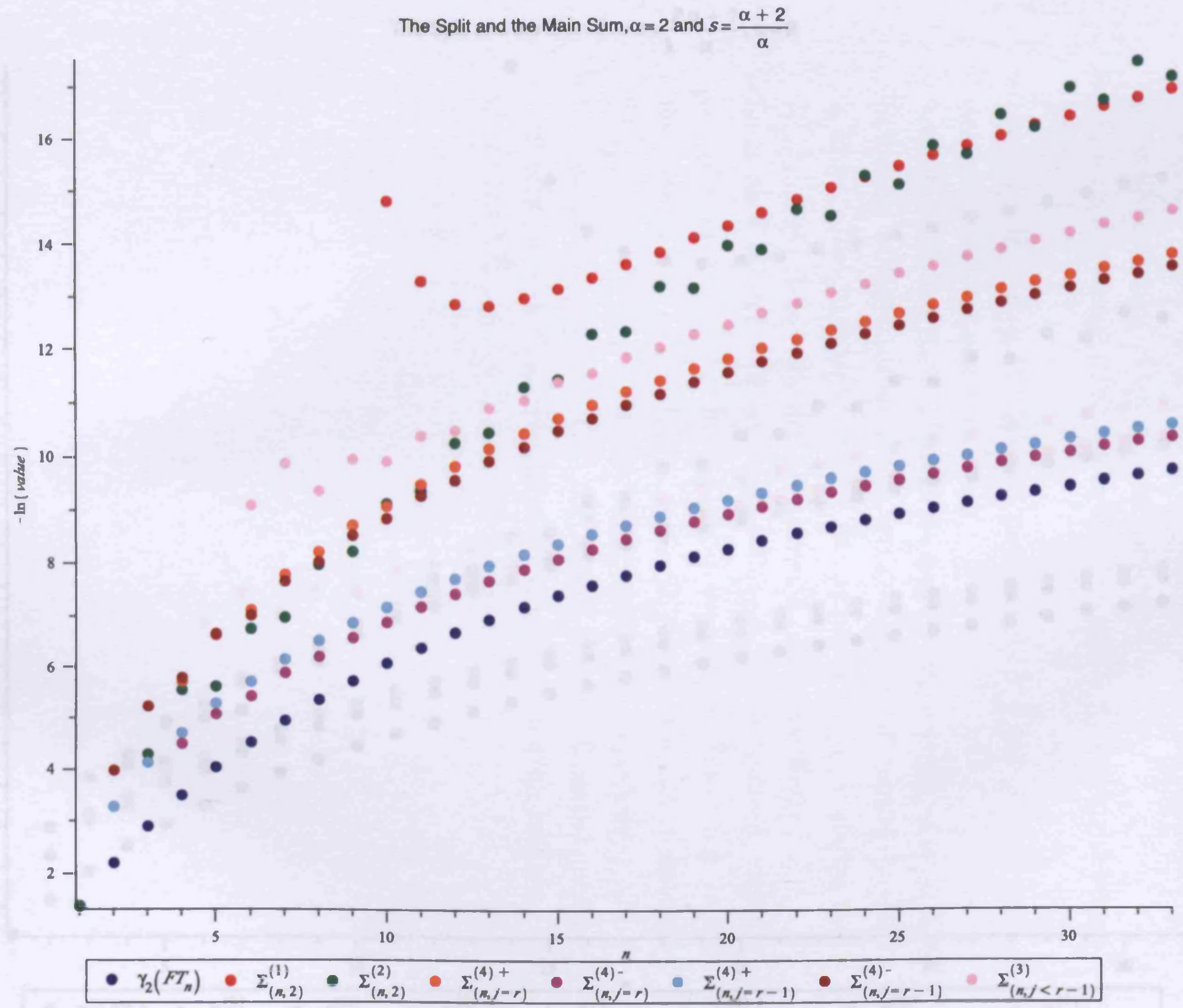


FIGURE 3.4. The partition $-\log(\cdot)$-scale. $\alpha = 2$ and $s = \frac{\alpha+2}{\alpha}$

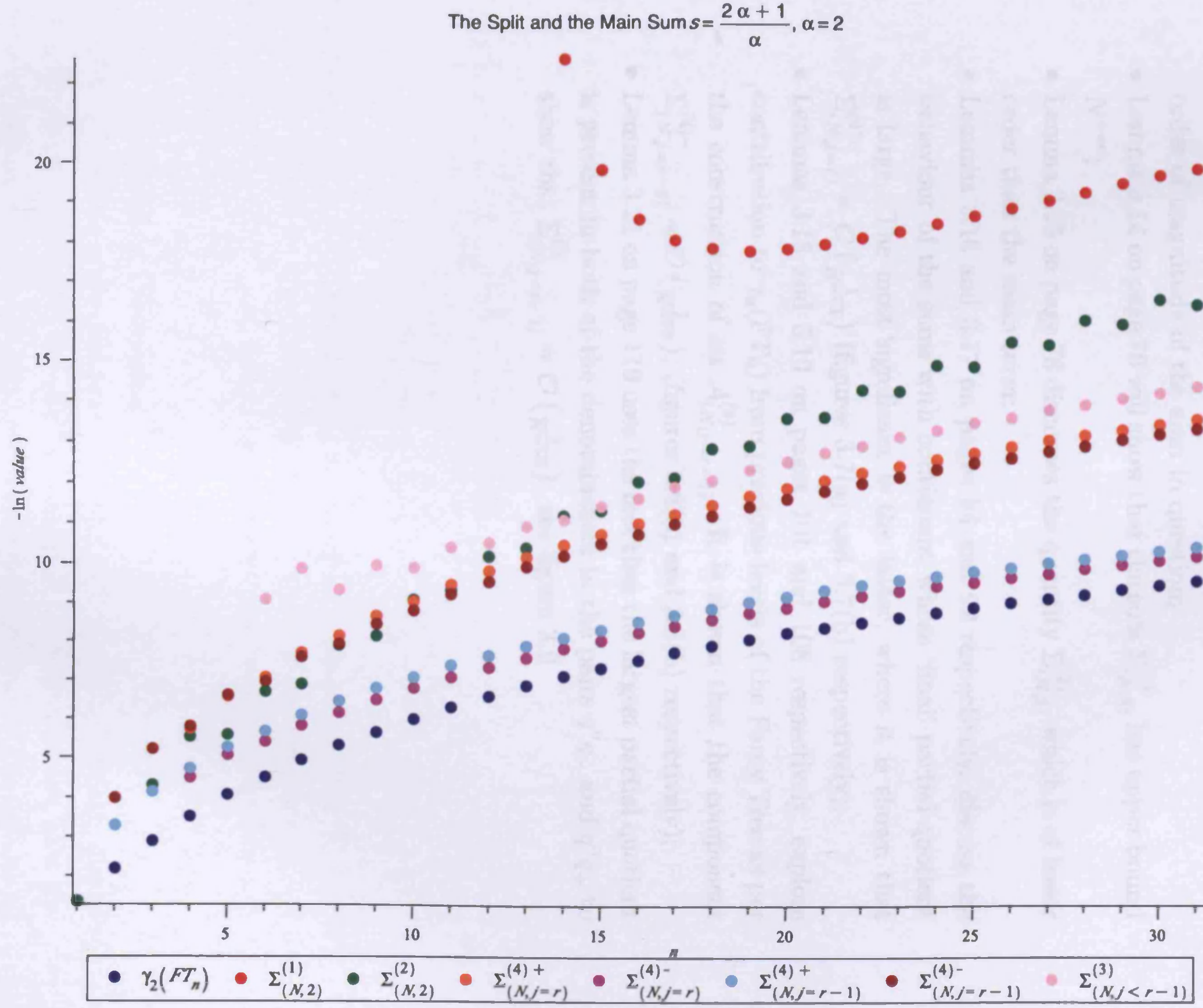


FIGURE 3.5. The partition, $-\log(\cdot)$-scale. $\alpha = 2$ and $s = \frac{2\alpha+1}{\alpha}$

Each of the sums yielded by hierarchy (33) is considered individually and thus the main proof is organised as follows:

- Theorem 3.3, which is indeed an auxiliary result required by Theorem 3.5 is proved in section 4 on page 67, figure 3.6 highlights the order of magnitude of the sum in question;
- Lemma 3.14 on page 76 will show that the sum $\Sigma^{(1)}_{(N,2)}$ has upper bound $N^{-s\alpha}$;
- Lemma 3.15 on page 78 discusses the quantity $\Sigma^{(2)}_{(N,2)}$ which is of lesser order than the main term;
- Lemmas 3.16 and 3.17 on pages 84 and 92 respectively, discuss the behaviour of the sums with continuant whose 'final' partial quotient is large. The most significant is the latter, where it is shown that $\Sigma^{(4)-}_{(N,j=r)} = \mathcal{O}\left(\frac{1}{N^{\alpha+1}}\right)$ (figures 3.7(a) and 3.7(b) respectively);
- Lemmas 3.18 and 3.19 on pages 101 and 108 respectively, explore contribution to $\gamma_\alpha(\mathrm{FT}_n)$ from previous levels of the Farey Tree as per the construction of set $\mathcal{A}^{(4)}_{(N,j=r-1)}$. It is shown that the component $\Sigma^{(4)-}_{(N,j=r-1)} = \mathcal{O}\left(\frac{1}{N^{\alpha+1}}\right)$, (figures 3.8(a) and 3.8(b) respectively);
- Lemma 3.21 on page 119 uses the fact that the largest partial quotient is present in both of the denominators in the pairs $q''q_-$ and $q''q_+$ to show that $\Sigma^{(3)}_{(N,j<r-1)} = \mathcal{O}\left(\frac{1}{N^{\alpha+2}}\right)$, see figure 3.9

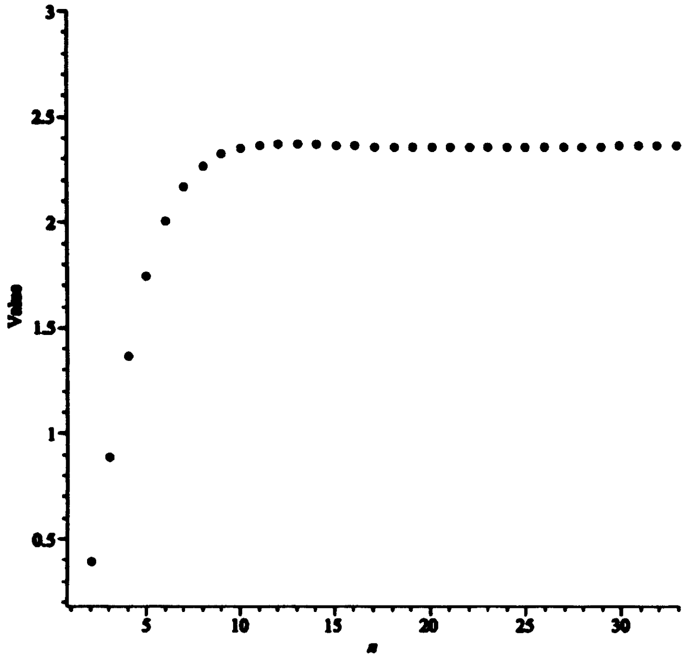


FIGURE 3.6. Values of $n^{\alpha+2}\sum_{a\in\mathcal{A}_n}\frac{1}{q(a)^{\alpha+2}}$, $\alpha=2$

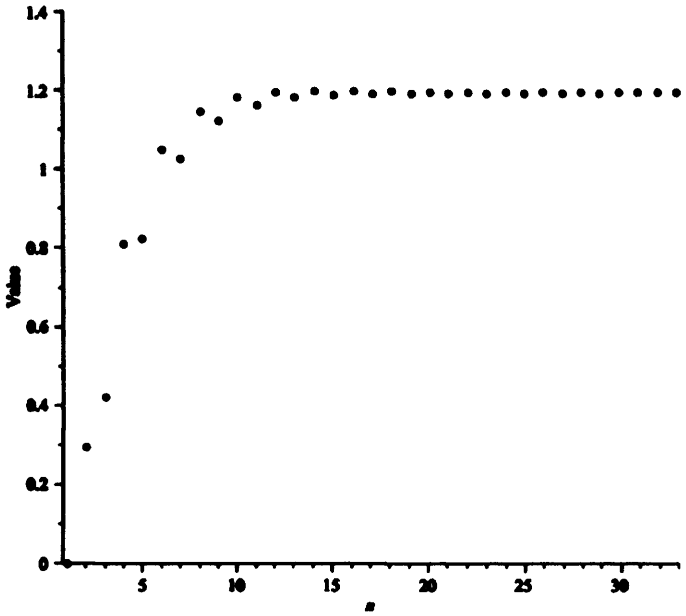
This information will combine to show that

$$\begin{aligned}\gamma_\alpha(\mathrm{FT}_n) &= \frac{K^-_{\alpha,n}+\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}}{n^{\alpha+1}}+\frac{\widehat{K}_{\alpha,n}+\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}}{n^{\alpha+2}}+\\ &\quad+\ \mathcal{O}\left(\frac{1}{n^{s\alpha}}+\frac{\log(n)}{n^{2\alpha+1}}+\frac{n^2\log^{2\alpha+3}(n)}{w^{2\alpha+3}}+\frac{1}{n^{\alpha+3}w^{\alpha-2}}\right),\end{aligned}$$

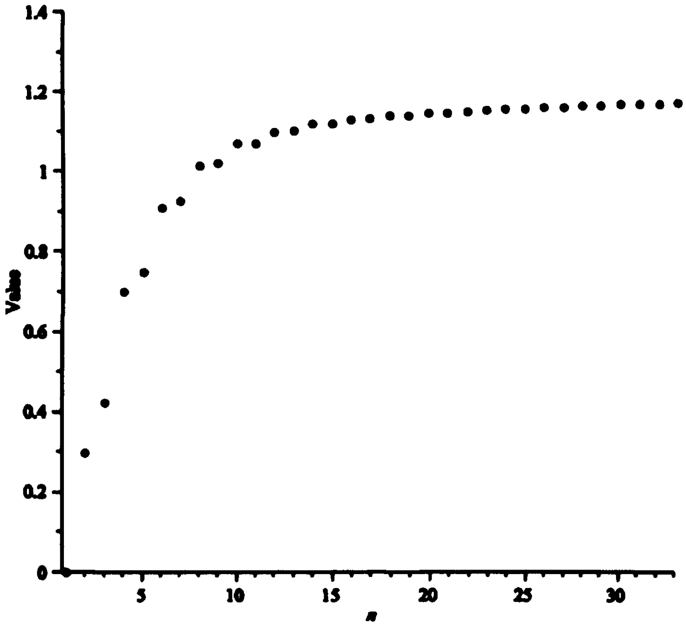
where $\widehat{K}_{\alpha,n}=K^+_{\alpha,n}+K^{*+}_{\alpha,n}+K^{*-}_{\alpha,n}+K^{**}_{\alpha,n}$ as defined in Lemmas 3.16, 3.17, 3.18, 3.19 and 3.21. It should be noted that these are not constant and contain terms of orders $n^{-(\alpha+2)}$ to $n^{-(2\alpha)}$, and this in part dictates the choice of parameter s in the final proof.

3.4. Preliminary Discussions.

3.4.1. *The Effect of the Parameters w and s.* One will notice that, due to the hierarchical structure of the sets making up $\mathcal{A}_N$ described above, that the choice of parameters s and w can have a profound effect on the behaviour of these constituent parts, and therefore on the sums based upon them.

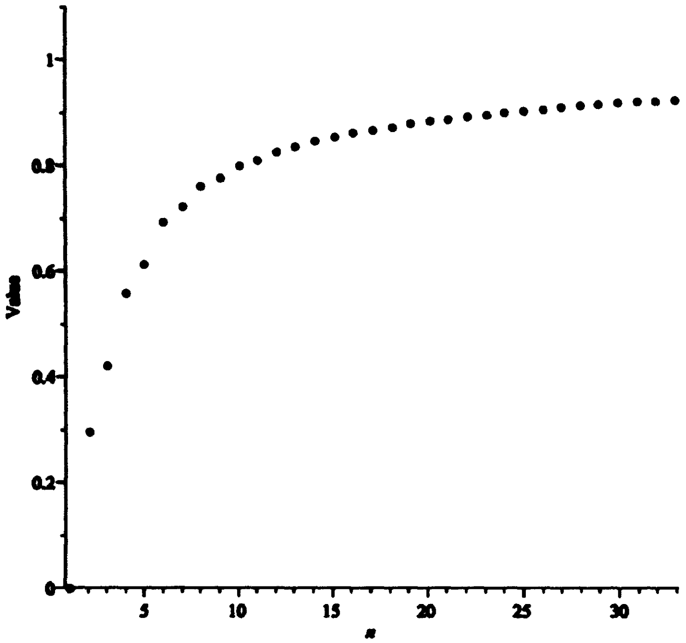


(a) Values of $n^{\alpha+2}\Sigma^{(4)+}_{(n,j=r)}$, $\alpha = 2$

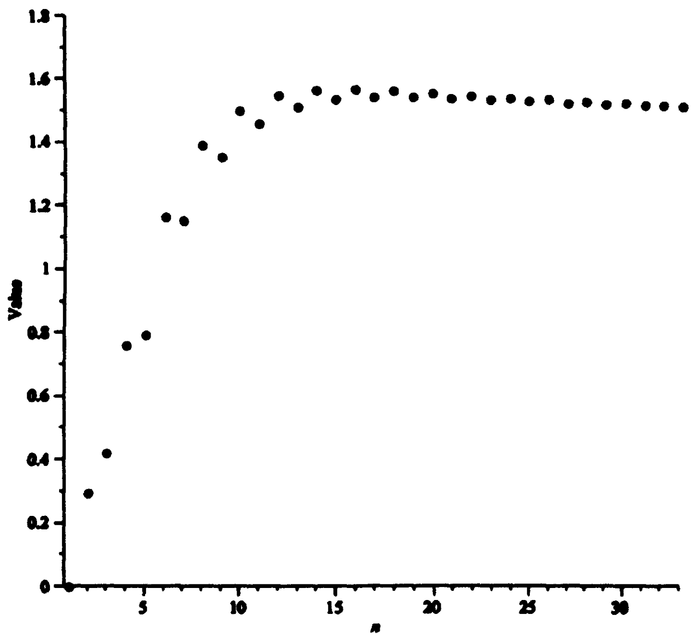


(b) Values of $n^{\alpha+1}\Sigma^{(4)-}_{(n,j=r)}$, $\alpha = 2$

FIGURE 3.7.



(a) Values of $n^{\alpha+1}\Sigma^{(4)+}_{(n,j=r-1)}$, $\alpha = 2$



(b) Values of $n^{\alpha+2}\Sigma^{(4)-}_{(n,j=r-1)}$, $\alpha = 2$

FIGURE 3.8.

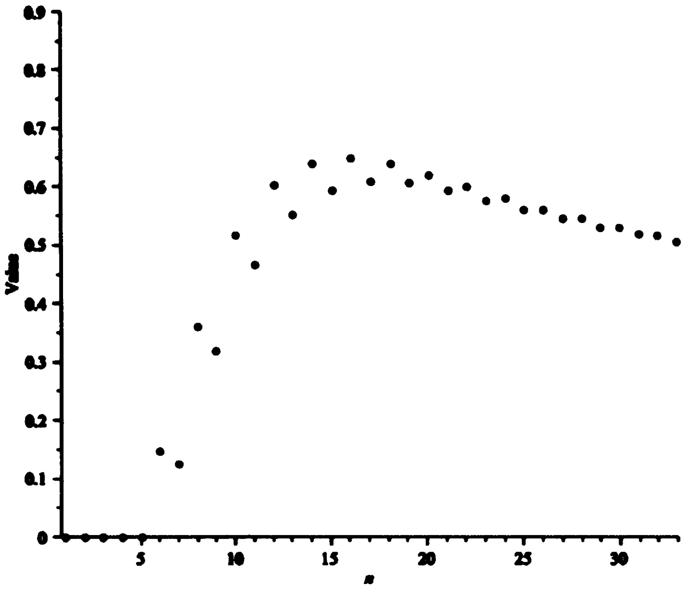


FIGURE 3.9. Values of $n^{\alpha+2}\Sigma^{(3)}_{(n,j<r)}$, $\alpha = 2$

The existence of lower bound $s > 1$ requires an important observation. The choice of $s \leq 1$ renders the set $\mathcal{A}^{(1)}_{(N,1)}$ empty, as it is easily seen that that the sum of two consecutive denominators q and q' at level n has $q + q' > n$ (this is also true of the Farey Series, where $\mathcal{F}_n \subseteq \mathrm{FT}_n$, see for example [**23**]). Hence it is impossible for q'' to be less than N. Moreover, it is easily seen that the largest denominator present in any level L of the Tree is equal to the $(L+1)^{\text{th}}$ Fibonacci number $\mathbf{F}(L+1)$ and thus the choice

$$s \geq \frac{\log(\mathbf{F}(L+1))}{\log(L)}$$

renders $\mathcal{A}^{(1)}_{(N,2)}$ empty. Figure 3.10 highlights the effect of increasing the value of s (in $-\log(\cdot)$ scale) on the sum $\Sigma^{(1)}_{(N,2)}$. The assertion that w must lie in the interval $[1, N)$ is obvious since at, for example level N of the Farey Tree, the partial quotients a_i sum to N. Moreover, note that the choice $w \leq \frac{N}{2}$ in any of the sets (33) where a partial quotient has $a_j > N - w$, guarantees that it is the only such partial quotient.

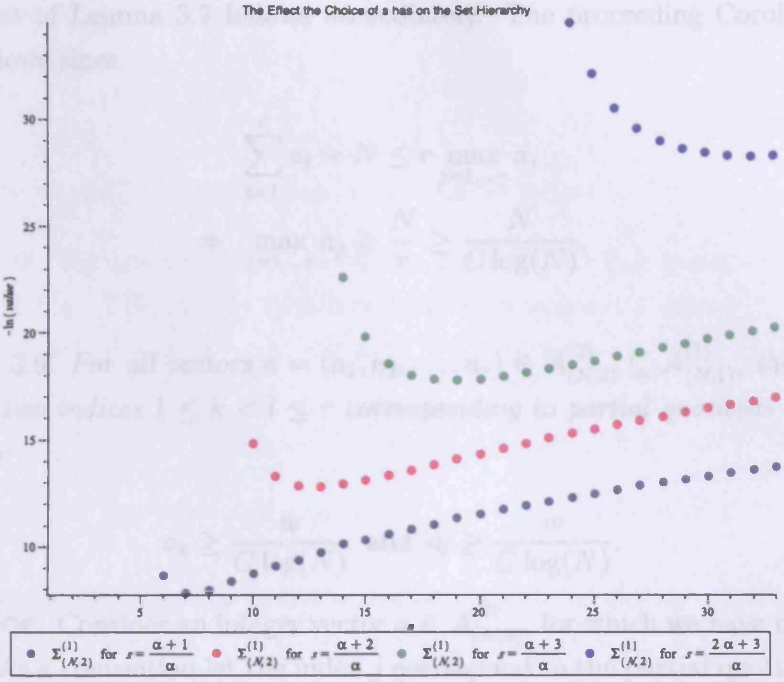


FIGURE 3.10. The effect of s on $\Sigma^{(1)}_{(N,2)}$, $\alpha = 2$

3.4.2. *Lemmas From* [**31**] *and* [**12**]. Some related lemmas are presented in this subsection. The major results in this chapter require their use and are thus included and explained here for completion.

LEMMA 3.7. *For all vectors* $a = (a_1, a_2, \ldots, a_r) \in \mathcal{A}^{(1)}_{(N,1)}$ *with* $N \geq 3$, *the value of the last index is such that*

$$r \leq C\log(N), \text{ where } C = s/\log\left(\frac{\sqrt{5}+1}{2}\right).$$

COROLLARY 3.8. *For the vectors* a *of Lemma 3.7,* $\max_{i=1,\ldots,r} a_i \geq \dfrac{N}{C\log(N)}$

PROOF. The following inequality holds for $a \in \mathcal{A}^{(1)}_{(N,1)}$:

$$\left(\frac{\sqrt{5}+1}{2}\right)^r \leq \langle a_1, a_2, \ldots a_r \rangle < N^s,$$

where the left hand side occurs by construction of the Farey numbers and their relation to the Fibonacci sequence (see for example [**37**]). Therefore, the statement of Lemma 3.7 follows immediately. The proceeding Corollary 3.8 then follows since

$$\sum_{i=1}^{r} a_i = N \leq r \max_{j=1,\ldots r} a_j$$
$$\Rightarrow \max_{j=1,\ldots r} a_j \geq \frac{N}{r} \geq \frac{N}{C\log(N)}.$$

□

LEMMA 3.9. *For all vectors $a = (a_1, a_2, \ldots, a_r) \in \mathcal{A}^{(2)}_{(N,2)} \subseteq \mathcal{A}^{(1)}_{(N,1)}$, there exist at least two indices $1 \leq k < l \leq r$ corresponding to partial quotients with the property:*

$$a_k \geq \frac{w}{C\log(N)} \text{ and } a_l \geq \frac{w}{C\log(N)}.$$

PROOF. Consider an integer vector $a \in \mathcal{A}^{(2)}_{(N,2)}$, for which we have $\max a_i \leq N - w$. As a convention let the index j correspond to the partial quotient with property $a_j = \max\{a_1, \ldots, a_r\}$, where

$$\frac{N}{C\log(N)} \leq a_j \leq N - w.$$

This implies

$$\sum_{i \neq j} a_i = N - a_j \geq w, \tag{34}$$

Denote the next largest partial quotient as $a_k = \max_{i \neq j} a_i$ (which is not necessarily unique). Using (34) we have

$$\sum_{i \neq j} a_i < r \max_{i \neq j} a_i = r a_k \quad \Rightarrow \quad N - a_j < r a_k,$$
$$\text{so } a_k > \frac{N - a_j}{r} \quad \Rightarrow \quad a_k > \frac{N - N + w}{C\log(N)} \quad = \quad \frac{w}{C\log(N)}.$$

Hence there must exist at least two indices $1 \leq k < l \leq r$ corresponding to partial quotients with the prescribed property. □

LEMMA 3.10. *Define the sets $\Omega_N := \{a \in \mathcal{A}_N | \exists j : a_j > N - w\}$, and $P(u, v, X) := \{a \in \mathcal{A} | a = (a_1, \ldots, a_x, X, a'_1, \ldots, a'_{x'})$, with $u = a_1 + \ldots + a_x,\ v = a'_1 + \ldots + a'_{x'}\}$.*

Then, when $w \le \frac{N}{2}$, we have

$$\Omega_N = \bigsqcup_{X=N-w}^{N} \bigsqcup_{u+v=N-X} P(u,v,X),$$

where the symbol $\bigsqcup$ again denotes a disjoint set union.

Proof. Let the parameter w be less than $\frac{N}{2}$ and first consider the case where $a \in \Omega_N$. This implies that there exists some index j with $1 \le j \le x$ to a partial quotient $a_j = \mathbf{Y} > N - w > \frac{N}{2}$. Therefore

$$\Rightarrow a \in P(a_1 + \ldots + a_x, a'_1 + \ldots + a'_{x'}, \mathbf{Y}).$$

Conversely, let the vector a belong to $\bigsqcup_{X=N-w}^{n} \bigsqcup_{u+v=N-X} P(u,v,X)$.

This automatically means that the vector belongs to the set $\mathcal{A}_N$, and as a condition there exists a partial quotient with $a_i > N - w$. What remains to prove is that such an $a \in \Omega_N$ belongs uniquely to $P(u,v,X)$. This is shown by the following contradiction: let $a \in \Omega_N$ such that:

$$a \in P(u,v,X) \text{ and } a \in P(u^*,v^*,X^*).$$

which implies:

$$a = (a_1, \ldots, a_{i-1}, X, a_{i+1}, \ldots, a_x),$$

with $u = a_1 + \ldots + a_{i-1}$ and $v = a_{i+1} + \ldots + a_x \Rightarrow u + v = N - X$;
and

$$a = (a_1^*, \ldots, a_{j-1}^*, X^*, a_{j+1}^*, \ldots, a_x^*),$$

with $u^* = a_1^* + \ldots + a_{j-1}^*$ and $v^* = a_{j+1}^* + \ldots + a_x^* \Rightarrow u^* + v^* = N - X^*$.
Moreover, suppose that $i \ne j$,

$$\Rightarrow \exists \text{ two partial quotients in the vector } a \text{ such that :}$$
$$a_i > N - w \ge \frac{N}{2}, \quad a_j > N - w \ge \frac{N}{2},$$
$$\Rightarrow \sum_{k=1}^{x} a_k > N \quad \Rightarrow a \notin \Omega_N,$$

(since $w \le \frac{N}{2}$). This contradicts the original assumption, and thus indices i and j are indeed equal. Furthermore,

$$X = X^*, \quad (u,v) = (u^*,v^*), \quad \text{and} \quad P(u,v,X) = P(u^*,v^*,X^*).$$

□

4. On the Sum with the Single Denominator

The aim in this section is to prove Theorem 3.3. This result is obtained by similar partitioning of the sum using (33), where

$$\sum_{a\in\mathcal{A}_N} \frac{1}{q(a)^{\alpha+2}} = \tilde{\Sigma}^{(1)}_{(N,2)} + \tilde{\Sigma}^{(2)}_{(N,1)} + \tilde{\Sigma}^{(2)}_{(N,2)}, \tag{35}$$

and

$$\tilde{\Sigma}^{(j)}_{(N,i)} = \sum_{a\in\mathcal{A}^{(j)}_{(N,i)}} \frac{1}{q(a)^{\alpha+2}},$$

The behaviour of the individual sums in (35) is considered in each of Lemmas 3.11, 3.12 and 3.13.

4.1. Lemmas.

LEMMA 3.11. *For $s > 1$, $\alpha > 1$,*

$$\tilde{\Sigma}^{(1)}_{(N,2)} \le \frac{1}{N^{\frac{\alpha}{2}(2s-1)}}.$$

PROOF. From (23) and [31] one has that

$$\sum_{a\in\mathcal{A}_N} \left[\frac{1}{qq_+} + \frac{1}{qq_-}\right] = 1. \tag{36}$$

It is shown in [31] that

$$\sigma_\alpha(\mathrm{FT}_n) = \sum_{a\in\mathcal{A}_n} \left[\frac{1}{(qq_+)^\alpha} + \frac{1}{(qq_-)^\alpha}\right] \le 1,$$

and thus:

$$\begin{aligned}
\tilde{\Sigma}^{(1)}_{(N,2)} &= \sum_{a\in\mathcal{A}^{(1)}_{(N,2)}} \frac{1}{(q(a)q(a))^{\frac{\alpha+2}{2}}} \\
&\le \sum_{a\in\mathcal{A}^{(1)}_{(N,2)}} \frac{1}{(q(a)q_-(a))^{\frac{\alpha+2}{2}}} \\
&\le \max_{a\in\mathcal{A}^{(1)}_{(N,2)}} \frac{1}{(q(a)q_-(a))^{\frac{\alpha+2}{2}-1}} \sum_{a\in\mathcal{A}^{(1)}_{(N,2)}} \frac{1}{q(a)q_-(a)} \\
&\le \max_{a\in\mathcal{A}^{(1)}_{(N,2)}} \frac{1}{(q(a)q_-(a))^{\frac{\alpha+2}{2}-1}} \sum_{a\in\mathcal{A}^{(1)}_{(N,2)}} \left(\frac{1}{q(a)q_-(a)} + \frac{1}{q(a)q_+(a)}\right) \\
&\le \max_{a\in\mathcal{A}^{(1)}_{(N,2)}} \frac{1}{(q(a)q_-(a))^{\frac{\alpha}{2}}} = \frac{N^{\frac{\alpha}{2}}}{N^{s\alpha}},
\end{aligned}$$

using the fact that $q \le Nq_-$, which completes the proof. □

LEMMA 3.12. *As $N \to \infty$, and for $w \le \frac{N}{2}$*

$$\tilde{\Sigma}^{(2)}_{(N,2)} \ll \frac{N^2 \log^{2(\alpha+2)}(N)}{w^{2(\alpha+2)}}.$$

PROOF. By Lemma 3.9, there exists for each $a \in \mathcal{A}^{(2)}_{(N,2)} \subseteq \mathcal{A}^{(1)}_{(N,1)}$ at least two partial quotients a_k, a_l with $1 \le k < l \le r$ in the continuant $\langle a_1, \ldots, a_r\rangle$, such that

$$a_k, a_l \ge \frac{w}{C\log(N)}.$$

Hence

$$\tilde{\Sigma}^{(2)}_{(N,2)} \le \sum_{\substack{a\in\mathcal{A}_N \\ \langle a_1,\ldots,a_r\rangle < N^s \\ \exists k,l:\, a_k,a_l \ge \frac{w}{C\log(N)}}} \frac{1}{\langle a_1, \ldots, a_r\rangle^{\alpha+2}}.$$

Using the following identity (see for example, [**21**]):

$$\langle a_1, \ldots, a_r\rangle \ge a_k a_l \langle a_1, \ldots, a_{k-1}\rangle \langle a_{k+1}, \ldots, a_{l-1}\rangle \langle a_{l+1}, \ldots, a_r\rangle,$$

for which we set

$$\left.\begin{aligned} u &= a_1 + \ldots + a_{k-1} \\ v &= a_{k+1} + \ldots + a_{l-1} \\ p &= a_{l+1} + \ldots + a_r, \end{aligned}\right. \qquad \Rightarrow u + v + p = N - a_k - a_l.$$

one may deduce that

$$\begin{aligned} \tilde{\Sigma}^{(2)}_{(N,2)} &\le \sum_{\substack{a_k+a_l\le N \\ a_k,a_l\ge \frac{w}{C\log(N)}}} \sum_{\substack{a\in A_N \\ q''(a)<N^s}} \frac{1}{(a_k a_l)^{\alpha+2}\langle a_1,\ldots,a_{k-1}\rangle^{\alpha+2}\langle a_{k+1},\ldots,a_{l-1}\rangle^{\alpha+2}\langle a_{l+1},\ldots,a_r\rangle^{\alpha+2}} \\ &\le \sum_{\substack{a_k+a_l\le N \\ a_k,a_l\ge \frac{w}{C\log(N)}}} \frac{1}{(a_k a_l)^{\alpha+2}} \sum_{\substack{u+v+p \\ =N-a_k-a_l}} \left(\sum_u \frac{1}{\langle a_1,\ldots,a_{k-1}\rangle^{\alpha+2}} \sum_v \frac{1}{\langle a_{k+1},\ldots,a_{l-1}\rangle^{\alpha+2}} \right. \\ &\qquad \left. \times \sum_p \frac{1}{\langle a_{l+1},\ldots,a_r\rangle^{\alpha+2}} \right). \end{aligned}$$

For notational ease, denote

$$S_E := \sum_{\substack{a_k+a_l\le N \\ a_k,a_l\ge \frac{w}{C\log(N)}}} \frac{1}{(a_k a_l)^{\alpha+2}}, \quad \text{and}$$

$$\begin{aligned} S_I &:= \sum_{\substack{u+v+p \\ =N-a_k-a_l}} \sum_{u=a_1+\ldots+a_{k-1}} \frac{1}{\langle a_1,\ldots,a_{k-1}\rangle^{\alpha+2}} \sum_{v=a_{k+1}+\ldots+a_{l-1}} \frac{1}{\langle a_{k+1},\ldots,a_{l-1}\rangle^{\alpha+2}} \times \\ &\qquad \times \sum_{\substack{p=a_{l+1}+\ldots+a_r \\ a_r\ge 2}} \frac{1}{\langle a_{l+1},\ldots,a_r\rangle^{\alpha+2}}. \end{aligned}$$

First consider the *internal sum* S_I, for which we have

$$
\begin{aligned}
S_I &\le 4 \sum_{u+v+p\le N} \sum_{\substack{u=x_1+\ldots+x_f\\ x_f\ge 2}} \frac{1}{\langle x_1,\ldots,x_f\rangle^{\alpha+2}} \sum_{\substack{v=y_1+\ldots+y_g\\ y_g\ge 2}} \frac{1}{\langle y_1,\ldots,y_g\rangle^{\alpha+2}} \times \\
&\qquad \times \sum_{\substack{p=z_1+\ldots+z_h\\ z_h\ge 2}} \frac{1}{\langle z_1,\ldots,z_h\rangle^{\alpha+2}} \qquad (37)\\
&\le 4 \sum_{u+v+p\le \infty} \sum_{\substack{u=x_1+\ldots+x_f\\ x_f\ge 2}} \frac{1}{\langle x_1,\ldots,x_f\rangle^{\alpha+2}} \sum_{\substack{v=y_1+\ldots+y_g\\ y_g\ge 2}} \frac{1}{\langle y_1,\ldots,y_g\rangle^{\alpha+2}} \times \\
&\qquad \times \sum_{\substack{p=z_1+\ldots+z_h\\ z_h\ge 2}} \frac{1}{\langle z_1,\ldots,z_h\rangle^{\alpha+2}} \\
&\le 4 \sum_{\substack{x_1+\ldots+x_f\le\infty\\ x_f\ge 2}} \frac{1}{\langle x_1,\ldots,x_f\rangle^{\alpha+2}} \sum_{\substack{y_1+\ldots+y_g\le\infty\\ y_g\ge 2}} \frac{1}{\langle y_1,\ldots,y_g\rangle^{\alpha+2}} \sum_{\substack{z_1+\ldots+z_h\le\infty\\ z_h\ge 2}} \frac{1}{\langle z_1,\ldots,z_h\rangle^{\alpha+2}} \\
&= 4 \left(\sum_{\substack{x_1+\ldots+x_f\le\infty\\ x_f\ge 2}} \frac{1}{\langle x_1,\ldots,x_f\rangle^{\alpha+2}} \right)^3 \\
&\le 4 \left(\sum_{l=1}^{\infty} \frac{\phi(l)}{l^{\alpha+2}} \right)^3 = 4 \left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} \right)^3 .
\end{aligned}
$$

The factor 4 at line (37) appears as a result of setting the restriction $a_{k-1}, a_{l-1} \ge 2$ (which are then replaced by generic partial quotients x_i and y_i). This process recalls use of the identity

$$\left| \left\{ (a_1,\ldots,a_x) \,:\, a_i \ge 1,\ \forall i = 1,\ldots,x \text{ and } \sum_{i=1}^{x} a_i = v \right\} \right| = 2|\mathcal{A}_v|.$$

The discussion of this combinatorial identity is seen on page 11 and is easy to prove. Furthermore, consider the *external sum* S_E - effectively a double sum over the variables a_k and a_l. When one of these is fixed, the length of the corresponding sum is $N - \frac{2w}{C\log(N)}$, and hence

$$S_E \le \frac{1}{\left(\frac{w^2}{C^2\log^2(N)}\right)^{\alpha+2}} \left(N - \frac{2w}{C\log(N)} \right)^2 .$$

For $N \geq 3$, the value of $N - \dfrac{2w}{C\log(N)}$ never exceeds N. Therefore

$$\tilde{\Sigma}^{(2)}_{(N,2)} \ll \frac{N^2 \log^{2(\alpha+2)}(N)}{w^{2(\alpha+2)}},$$

as required. □

LEMMA 3.13. *For $w \leq \frac{N}{2}$ and as $N \to \infty$,*

$$\tilde{\Sigma}^{(2)}_{(N,1)} = \frac{C_0'}{N^{\alpha+2}} + \mathcal{O}\left(\frac{w}{N^{\alpha+3}} + \frac{1}{w^{\alpha}N^{\alpha+2}} + \frac{1}{N^{\frac{\alpha}{2}(2s-1)}}\right).$$

PROOF. One may recalculate the result of Lemma 7 of [12] for the analogous quantity $\mathfrak{R}'$ as

$$\mathfrak{R}' = \sum_{\substack{a\in A_n \\ \exists j: a_j > N-w}} \frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+2}} = C_0' + \mathcal{O}\left(\frac{1}{w^{\alpha}}\right), \tag{38}$$

where

$$C_0' := \frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + 2\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}\right)^2.$$

Let us continue by rearranging the sum $\tilde{\Sigma}^{(2)}_{(N,1)}$ as follows:

$$\begin{aligned}
\tilde{\Sigma}^{(2)}_{(N,1)} &= \sum_{\substack{a\in A_N \\ q''(a)<N^s \\ \exists i: a_i > N-w}} \frac{1}{\langle a_1,\dots,a_r\rangle^{\alpha+2}} \\
&= \sum_{\substack{a\in A_N \\ \exists i: a_i > N-w}} \frac{1}{\langle a_1,\dots,a_r\rangle^{\alpha+2}} - \sum_{\substack{a\in A_N \\ q''(a)\geq N^s \\ \exists i: a_i > N-w}} \frac{1}{\langle a_1,\dots,a_r\rangle^{\alpha+2}} \\
&= \sum_{\substack{a\in A_N \\ \exists i: a_i > N-w}} \frac{1}{\langle a_1,\dots,a_r\rangle^{\alpha+2}} + \mathcal{O}\left(\frac{1}{N^{\frac{\alpha}{2}(2s-1)}}\right).
\end{aligned}$$

Moreover, let $a_i = N - v$ for $v = 1, \dots \lfloor w \rfloor$. Using the continuant identity

$$q(a) = (a_i + [a_{i-1},\dots,a_1] + [a_{i+1},\dots,a_r])\langle a_1,\dots,a_{i-1}\rangle\langle a_{i+1},\dots,a_r\rangle$$

one may ascertain that

$$q(a) = N\left(1 - \frac{v}{N} + \frac{A}{N}\right)\langle a_1, \dots, a_{i-1}\rangle\langle a_{i+1}, \dots, a_r\rangle,$$

where $A = [a_{i-1}, \dots, a_1] + [a_{i+1}, \dots, a_r] = \mathcal{O}(1)$ for brevity. Expansion into a Taylor series according to $\frac{v}{N} - \frac{A}{N}$ of the function

$$\frac{1}{q(a)^{\alpha+2}} = \frac{1}{N^{\alpha+2}}\frac{1}{\langle a_1, \dots, a_{i-1}\rangle^{\alpha+2}\langle a_{i+1}, \dots, a_r\rangle^{\alpha+2}}\frac{1}{\left(1 - \frac{v}{N} + \frac{A}{N}\right)^{\alpha+2}}.$$

yields:

$$\begin{aligned} \frac{1}{q(a)^{\alpha+2}} &= \frac{1}{N^{\alpha+2}}\frac{1}{\langle a_1, \dots, a_{i-1}\rangle^{\alpha+2}\langle a_{i+1}, \dots, a_r\rangle^{\alpha+2}} \\ &\qquad \times \left(1 + \sum_{k=1}^{\infty}\frac{\prod_{i=1}^{k}(\alpha+i+1)}{k!}\frac{(v-A)^k}{N^k}\right) \\ &= \frac{1}{N^{\alpha+2}}\frac{1}{\langle a_1, \dots, a_{i-1}\rangle^{\alpha+2}\langle a_{i+1}, \dots, a_r\rangle^{\alpha+2}}\left(1 + \mathcal{O}\left(\frac{v}{N}\right)\right) \end{aligned}$$

(where $\left|\frac{v-A}{N}\right| \le 1$ by construction). As a result of Lemma 3.10, and (38) one has

$$\begin{aligned} \tilde{\Sigma}^{(2)}_{(N,1)} &= \sum_{\substack{a\in\mathcal{A}_N \\ \exists i: a_i > N-w}} \frac{1}{N^{\alpha+2}}\frac{1}{\langle a_1, \dots, a_{i-1}\rangle^{\alpha+2}\langle a_{i+1}, \dots, a_r\rangle^{\alpha+2}}\left(1 + \mathcal{O}\left(\frac{w}{N}\right)\right) + \\ &\qquad + \mathcal{O}\left(\frac{1}{N^{\frac{\alpha}{2}(2s-1)}}\right) \\ &= \frac{C_0'}{N^{\alpha+2}} + \mathcal{O}\left(\frac{w}{N^{\alpha+3}}\right) + \mathcal{O}\left(\frac{1}{w^{\alpha}N^{\alpha+2}}\right) + \mathcal{O}\left(\frac{1}{N^{\frac{\alpha}{2}(2s-1)}}\right), \end{aligned}$$

since $v = \mathcal{O}(w)$. □

Remark: One will note that the error term $\frac{w}{N^{\alpha+3}}$, yielded by the term in the Taylor expansion

$$\sum_{k=1}^{\infty}\frac{\prod_{i=1}^{k}(\alpha+i+1)}{k!}\frac{(v-A)^k}{N^k}$$

is essentially of same order of magnitude as the main term for $w \le \frac{N}{2}$. However in the final proof of Theorem 3.3 the selection of this parameter yields a quantity of lesser order of magnitude, which is also true of Lemma 9 of **[12]** (from which this Lemma was derived).

4.2. Proof of Theorem 3.3. The quantity (35) can be reassembled as

$$\sum_{a\in\mathcal{A}_N}\frac{1}{q(a)^{\alpha+2}}=\frac{C_0'}{N^{\alpha+2}}+\mathcal{O}\left(\frac{w}{N^{\alpha+3}}+\frac{1}{w^{\alpha}N^{\alpha+2}}+\frac{1}{N^{\frac{\alpha}{2}(2s-1)}}+\frac{N^2\log^{2(\alpha+2)}(N)}{w^{2(\alpha+2)}}\right).$$

With the necessary conditions that $s>1$ and $w\leq\frac{N}{2}$ it is seen that

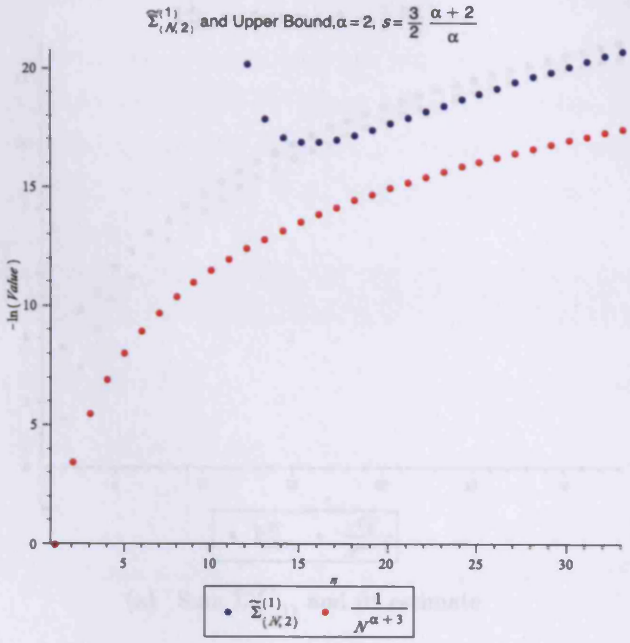
$$s=\frac{3(\alpha+2)}{2\alpha}$$

and

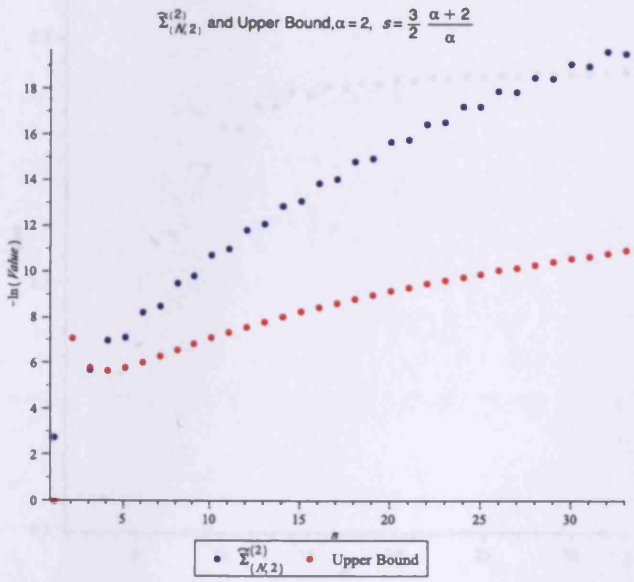
$$w=N^{\frac{\alpha+5}{2\alpha+5}}\log^{\frac{2\alpha+4}{2\alpha+5}}(N)$$

will give the stated error term. This is a consequence of the dominance of the first and fourth items in the error term. Note that this is not the best possible solution, and an improvement based on the original work of Dushistova is proposed in Appendix A. □

4.3. Numerical Evidence. Figures 3.11(a), 3.11(b) and 3.12(a) highlight the behaviour of the sums at each Lemma in the proof of Theorem 3.3. Figures 3.12(b) highlights the order of magnitude of the sum in Lemma 3.13. One should note that, where appropriate, upper bounds appear *below* the characteristic of interest in the following figures. This is due to the use of $-\log(\cdot)$ scaling.

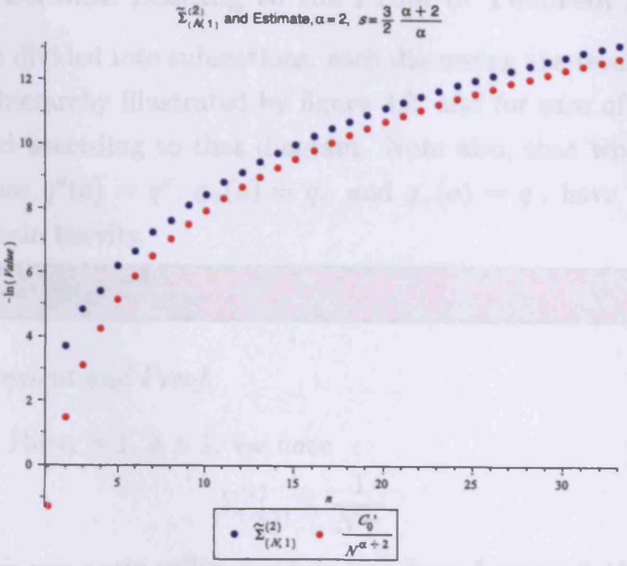


(a) Sum $\tilde{\Sigma}^{(1)}_{(N,2)}$ and its upper bound

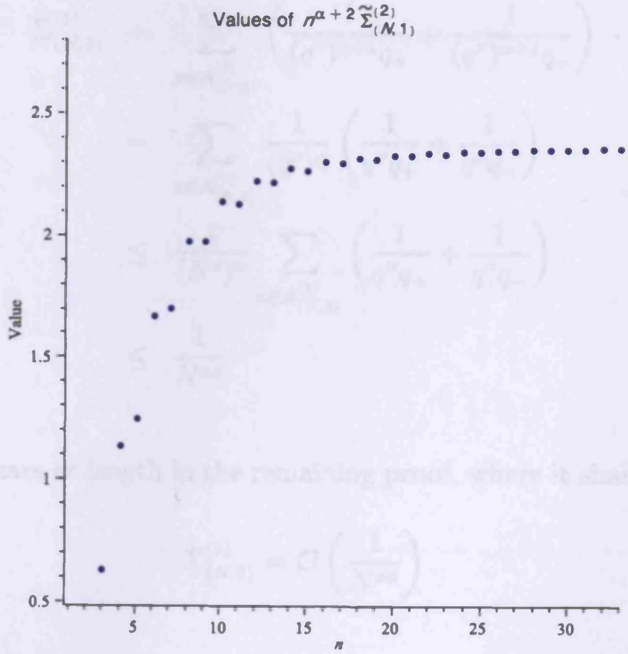


(b) Sum $\tilde{\Sigma}^{(2)}_{(N,2)}$ and its upper bound

FIGURE 3.11.



(a) Sum $\tilde{\Sigma}^{(2)}_{(N,1)}$ and its estimate



(b) Normalised sum $\tilde{\Sigma}^{(2)}_{(N,1)}$, the main term in Theorem 3.3

FIGURE 3.12.

5. Lemmas Leading to the Proof of Theorem 3.5

This section is divided into subsections, each discussing the result of each sum based on the hierarchy illustrated by figure 3.3, and for ease of reading, each is colour coded according to that diagram. Note also, that where applicable the suppressions $q''(a) = q''$, $q_+(a) = q_+$ and $q_-(a) = q_-$ have been made in order to maintain brevity.

5.1. On the Sum $\Sigma^{(1)}_{(N,2)}$.

5.1.1. *Statement and Proof.*

LEMMA 3.14. *For $\alpha > 1$, $s > 1$, we have*

$$\Sigma^{(1)}_{(N,2)} \leq \frac{1}{N^{s\alpha}}.$$

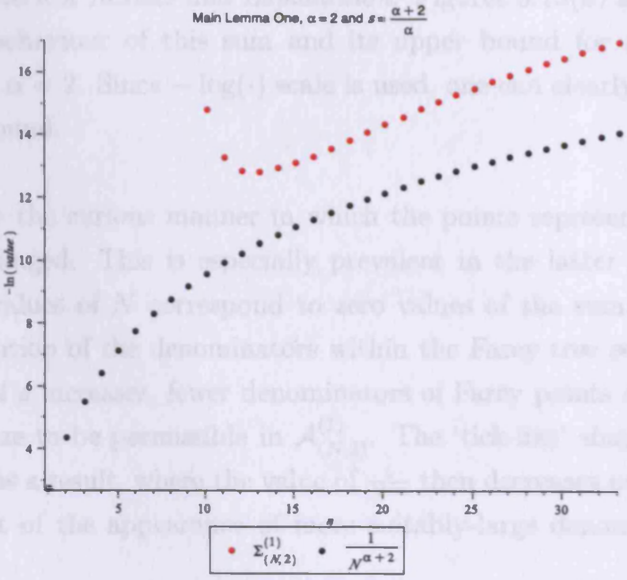
PROOF. We can again utilise formula (36) from Lemma 3.11 to show that

$$\begin{aligned}
\Sigma^{(1)}_{(N,2)} &= \sum_{a\in\mathcal{A}^{(1)}_{(N,2)}} \left(\frac{1}{(q'')^{\alpha+1}q_+} + \frac{1}{(q'')^{\alpha+1}q_-}\right) \\
&= \sum_{a\in\mathcal{A}^{(1)}_{(N,2)}} \frac{1}{(q'')^{\alpha}} \left(\frac{1}{q''q_+} + \frac{1}{q''q_-}\right) \\
&\leq \frac{1}{(N^s)^\alpha} \sum_{a\in\mathcal{A}^{(1)}_{(N,2)}} \left(\frac{1}{q''q_+} + \frac{1}{q''q_-}\right) \\
&\leq \frac{1}{N^{s\alpha}}
\end{aligned}$$

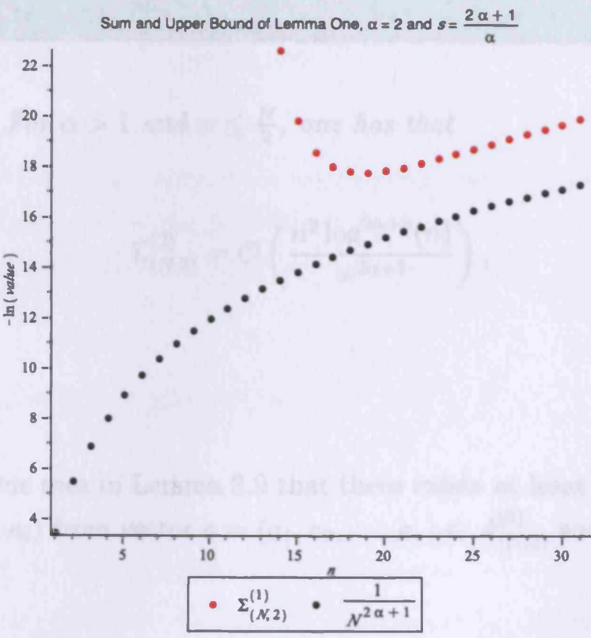
□

This fact appears at length in the remaining proof, where it shall be stated as:

$$\Sigma^{(1)}_{(N,2)} = \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right).$$



(a) $\Sigma^{(1)}_{(N,2)}$ and upper bound, for $s = \frac{\alpha+2}{\alpha}$



(b) $\Sigma^{(1)}_{(N,2)}$ and upper bound, for $s = \frac{2\alpha+1}{\alpha}$

FIGURE 3.13.

5.1.2. *Numerical Results and Explanation.* Figures 3.13(a) and 3.13(b) illustrate the behaviour of this sum and its upper bound for $s = \frac{\alpha+2}{\alpha}$ and $s = \frac{2\alpha+1}{\alpha}$ with $\alpha = 2$. Since $-\log(\cdot)$ scale is used, one can clearly see that $\frac{1}{N^{s\alpha}}$ is an upper bound.

One may note the curious manner in which the points representing the sum $\Sigma^{(1)}_{(N,2)}$ are arranged. This is especially prevalent in the latter case whereby the smallest values of N correspond to zero values of the sum. This is due to the distribution of the denominators within the Farey tree sequence itself; as the value of s increases, fewer denominators of Farey points at level N are of sufficient size to be permissible in $\mathcal{A}^{(1)}_{(N,2)}$. The 'tick-like' shape of this line thus appears as a result, where the value of $\frac{1}{N^{s\alpha}}$ then decreases at a far quicker rate than that of the appearance of more suitably-large denominators as N increases.

5.2. On the Sum $\Sigma^{(2)}_{(N,2)}$.

LEMMA 3.15. *For $\alpha > 1$ and $w \leq \frac{N}{2}$, one has that*

$$\Sigma^{(2)}_{(N,2)} = \mathcal{O}\left(\frac{n^2 \log^{2\alpha+3}(n)}{w^{2\alpha+3}}\right),$$

as $N \to \infty$.

PROOF. One sees in Lemma 3.9 that there exists at least two partial quotients (say a_k, a_l) from vector $a = (a_1, a_2, \ldots, a_r) \in \mathcal{A}^{(2)}_{(N,2)}$ such that

$$a_k \geq \frac{w}{C\log(N)} \quad \text{and} \quad a_l \geq \frac{w}{C\log(N)}.$$

When there are exactly two such partial quotients, the number of possible combinations of them is at its maximum. Therefore

$$
\begin{aligned}
\Sigma^{(2)}_{(N,2)} &\leq \sum_{\substack{a\in\mathcal{A}_N \\ q''(a)<N^s \\ \exists 1\leq k<l\leq r: a_k,a_l\geq \frac{w}{C\log(N)}}} \left[\frac{1}{(q'')^{\alpha+1}q_-}+\frac{1}{(q'')^{\alpha+1}q_+}\right] \\
&= \sum_{\substack{a\in\mathcal{A}_N \\ q''(a)<N^s \\ \exists k\leq r-1: a_k,a_r\geq \frac{w}{C\log(N)}}} \left[\frac{1}{(q'')^{\alpha+1}q_-}+\frac{1}{(q'')^{\alpha+1}q_+}\right] + \\
&\qquad + \sum_{\substack{a\in\mathcal{A}_N \\ q''(a)<N^s \\ \exists k,l\leq r-1: a_k,a_l\geq \frac{w}{C\log(N)}}} \left[\frac{1}{(q'')^{\alpha+1}q_-}+\frac{1}{(q'')^{\alpha+1}q_+}\right] \qquad (39) \\
&\leq 2\sum_{\substack{a\in\mathcal{A}_N \\ q''(a)<N^s \\ \exists k\leq r-1: a_k,a_r\geq \frac{w}{C\log(N)}}} \frac{1}{(q'')^{\alpha+1}q_-} + 2\sum_{\substack{a_1+\ldots+a_r=N \\ q''(a)<N^s \\ \exists k,l\leq r-1: a_k,a_l\geq \frac{w}{C\log(N)}}} \frac{1}{(q'')^{\alpha+1}q_-} \qquad (40) \\
&= 2(\mathbf{S}_1+\mathbf{S}_2), \quad \text{for brevity.}
\end{aligned}
$$

Where line (39) is an equality since we only separate out into sub-sums the possibilities where one of the prescribed partial quotients is the final entry, and where this is not the case. Line (40) follows by use of the assumption that $q_-\leq q_+$.

From for example [**21**], it follows that

$$
\begin{aligned}
q''(a)=\langle a_1,\ldots,a_r\rangle &\geq a_k a_r\langle a_1,\ldots,a_{k-1}\rangle\langle a_{k+1},\ldots,a_{r-1}\rangle \\
\text{and } \langle a_1,\ldots,a_r\rangle &\geq a_k a_l\langle a_1,\ldots,a_{k-1}\rangle\langle a_{k+1},\ldots,a_{l-1}\rangle\langle a_{l+1},\ldots,a_r\rangle \\
q_-(a)=\langle a_1,\ldots,a_{r-1}\rangle &\geq a_k\langle a_1,\ldots,a_{k-1}\rangle\langle a_{k+1},\ldots,a_{r-1}\rangle \\
\text{and } \langle a_1,\ldots,a_{r-1}\rangle &\geq a_k a_l\langle a_1,\ldots,a_{k-1}\rangle\langle a_{k+1},\ldots,a_{l-1}\rangle\langle a_{l+1},\ldots,a_{r-1}\rangle
\end{aligned}
$$

Thus

$$\begin{aligned}
\mathbf{S}_1 &\le \sum_{\substack{a\in\mathcal{A}_N\\ q''(a)<N^s\\ \exists k\le r-1: a_k,a_r\ge\frac{w}{C\log(N)}}} \frac{1}{a_k^{\alpha+2}a_r^{\alpha+1}\langle a_1,\dots,a_{k-1}\rangle^{\alpha+2}\langle a_{k+1},\dots,a_{r-1}\rangle^{\alpha+2}}\\
&= \sum_{\substack{a_k+a_r\le N\\ a_k,a_r\ge\frac{w}{C\log(N)}}} \sum_{\substack{a\in\mathcal{A}_N\\ q''(a)<N^s}} \frac{1}{a_k^{\alpha+2}a_r^{\alpha+1}\langle a_1,\dots,a_{k-1}\rangle^{\alpha+2}\langle a_{k+1},\dots,a_{r-1}\rangle^{\alpha+2}},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{S}_2 &\le \sum_{\substack{a\in\mathcal{A}_N\\ q''(a)<N^s\\ \exists k\le r-1: a_k,a_r\ge\frac{w}{C\log(N)}}} \Bigg(\frac{1}{(a_ka_l)^{\alpha+2}\langle a_1,\dots,a_{k-1}\rangle^{\alpha+2}\langle a_{k+1},\dots,a_{l-1}\rangle^{\alpha+2}}\\
&\qquad\qquad\times\frac{1}{\langle a_{l+1},\dots,a_{r-1}\rangle\langle a_{l+1},\dots,a_r\rangle^{\alpha+1}}\Bigg)\\
&=\sum_{\substack{a_k+a_l\le N\\ a_k,a_l\ge\frac{w}{C\log(N)}}} \frac{1}{(a_ka_l)^{\alpha+2}} \sum_{\substack{a_1+\dots+a_r=N\\ q''(a)<N^s}} \Bigg(\frac{1}{\langle a_1,\dots,a_{k-1}\rangle^{\alpha+2}\langle a_{k+1},\dots,a_{l-1}\rangle^{\alpha+2}}\\
&\qquad\qquad\times\frac{1}{\langle a_{l+1},\dots,a_{r-1}\rangle\langle a_{l+1},\dots,a_r\rangle^{\alpha+1}}\Bigg).
\end{aligned}$$

First study the inequality on $\mathbf{S}_1$. To this end, let $u = a_1+\dots+a_{k-1}$, $v = a_{k+1}+\dots+a_{r-1}$, such that $u+v = N-a_k-a_r$. Thus

$$\begin{aligned}
\mathbf{S}_1 &\le \sum_{\substack{a_k+a_r\le N\\ a_k,a_r\ge\frac{w}{C\log(N)}}} \frac{1}{a_k^{\alpha+2}a_r^{\alpha+1}} \sum_{u+v=N-a_k-a_r}\Bigg(\sum_{u=a_1+\dots+a_{k-1}}\frac{1}{\langle a_1,\dots,a_{k-1}\rangle^{\alpha+2}}\times\\
&\qquad\qquad\times\sum_{v=a_{k+1}+\dots+a_{r-1}}\frac{1}{\langle a_{k+1},\dots,a_{r-1}\rangle^{\alpha+2}}\Bigg).
\end{aligned}\tag{41}$$

The internal part of this sum is bounded above in the following manner; let

$$\mathbf{S}_{1,I} = \sum_{u+v=N-a_k-a_r}\Bigg(\sum_{u=a_1+\dots+a_{k-1}}\frac{1}{\langle a_1,\dots,a_{k-1}\rangle^{\alpha+2}}\sum_{v=a_{k+1}+\dots+a_{r-1}}\frac{1}{\langle a_{k+1},\dots,a_{r-1}\rangle^{\alpha+2}}\Bigg),$$

for which we have the following:

$$\begin{aligned}
\mathbf{S}_{1,I} &\leq 4 \sum_{u+v\leq N} \sum_{\substack{u=x_1+\ldots+x_f \\ x_f\geq 2}} \frac{1}{\langle x_1,\ldots,x_f\rangle^{\alpha+2}} \sum_{\substack{v=y_1+\ldots+y_g \\ y_g\geq 2}} \frac{1}{\langle y_1,\ldots,y_g\rangle^{\alpha+2}} \\
&\leq 4 \sum_{u+v\leq\infty} \sum_{\substack{u=x_1+\ldots+x_f \\ x_f\geq 2}} \frac{1}{\langle x_1,\ldots,x_f\rangle^{\alpha+2}} \sum_{\substack{v=y_1+\ldots+y_g \\ y_g\geq 2}} \frac{1}{\langle y_1,\ldots,y_g\rangle^{\alpha+2}} \\
&\leq 4 \left(\sum_{\substack{x_1+\ldots+x_f\leq\infty \\ x_f\geq 2}} \frac{1}{\langle x_1,\ldots,x_f\rangle^{\alpha+2}} \right)^2 = 4\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}\right)^2. \qquad (42)
\end{aligned}$$

The vectors $\underline{x} = (x_1, x_2, \ldots, x_f)$ and $\underline{y} = (y_1, y_2, \ldots, y_g)$ have arbitrary element-sums u and v respectively. Moreover, the restriction $x_f, y_g \geq 2$ is made, which necessitates the introduction of the factor 4. The external sum of (41),

$$\mathbf{S}_{1,E} = \sum_{\substack{a_k+a_r\leq N \\ a_k,a_r\geq \frac{w}{C\log(N)}}} \frac{1}{a_k^{\alpha+2} a_r^{\alpha+1}}$$

is a double sum for variables a_k and a_r over elements $\frac{2w}{C\log(N)} \leq a_i \leq N$. Hence

$$\sum_{\substack{a_k+a_r\leq N \\ a_k,a_r\geq \frac{w}{C\log(N)}}} \frac{1}{a_k^{\alpha+2} a_r^{\alpha+1}} \leq \frac{1}{\left(\frac{w}{C\log(N)}\right)^{2\alpha+3}} \left(N - \frac{2w}{C\log(N)}\right)^2. \qquad (43)$$

We perform a similar procedure on the second sum $\mathbf{S}_2$. Let $u' = a_1 + \ldots + a_{k-1}$, $v' = a_{k+1} + \ldots + a_{l-1}$ and $p' = a_{l+1} + \ldots + a_r$, such that $u' + v' + p' = N - a_k - a_l$ (it is assumed without any loss of generality, that $l > k$). Thus

$$\begin{aligned}
\mathbf{S}_2 \leq \sum_{\substack{a_k+a_l\leq N \\ a_k,a_l\geq \frac{w}{C\log(N)}}} \frac{1}{(a_k a_l)^{\alpha+2}} \sum_{\substack{u'+v'+p' \\ =N-a_k-a_l}} &\left(\sum_{a_1+\ldots+a_{k-1}=u'} \frac{1}{\langle a_1,\ldots,a_{k-1}\rangle^{\alpha+2}} \right. \\
&\times \sum_{a_{k+1}+\ldots+a_{l-1}=v'} \frac{1}{\langle a_{k+1},\ldots,a_{l-1}\rangle^{\alpha+2}} \\
&\left. \times \sum_{a_{l+1}+\ldots+a_r=p'} \frac{1}{\langle a_{l+1},\ldots,a_{r-1}\rangle\langle a_{l+1},\ldots,a_r\rangle^{\alpha+1}} \right)
\end{aligned}$$

As with (43), one has

$$\sum_{\substack{a_k+a_l\le N\\ a_k,a_l\ge \frac{w}{C\log(N)}}} \frac{1}{(a_k a_l)^{\alpha+2}} \le \frac{1}{\left(\frac{w}{C\log(N)}\right)^{2(\alpha+2)}} \left(N - \frac{2w}{C\log(N)}\right)^2,$$

for the outer part of the summation. The inner part is considered similarly to the preceding calculation; let

$$\begin{aligned} \mathbf{S}_{2,I} = \sum_{\substack{u'+v'+p'\\ =N-a_k-a_l}} \Bigg(&\sum_{u'=a_1+\ldots+a_{k-1}} \frac{1}{\langle a_1,\ldots,a_{k-1}\rangle^{\alpha+2}} \sum_{v'=a_{k+1}+\ldots+a_{l-1}} \frac{1}{\langle a_{k+1},\ldots,a_{l-1}\rangle^{\alpha+2}} \\ &\times \sum_{\substack{p'=a_{l+1}+\ldots+a_r\\ a_r\ge 2}} \frac{1}{\langle a_{l+1},\ldots,a_{r-1}\rangle \langle a_{l+1},\ldots,a_r\rangle^{\alpha+1}} \Bigg), \end{aligned}$$

for which one has

$$\begin{aligned} \mathbf{S}_{2,I} &\le 4 \sum_{u'+v'+p'\le N} \sum_{\substack{u'=x_1+\ldots+x_f\\ x_f\ge 2}} \frac{1}{\langle x_1,\ldots,x_f\rangle^{\alpha+2}} \sum_{\substack{v'=y_1+\ldots+y_g\\ y_g\ge 2}} \frac{1}{\langle y_1,\ldots,y_g\rangle^{\alpha+2}} \\ &\qquad \times \sum_{\substack{p'=z_1+\ldots+z_h\\ z_h\ge 2}} \frac{1}{\langle z_l,\ldots,z_h\rangle^{\alpha+1}\langle z_1,\ldots,z_{h-1}\rangle} \qquad (44) \\ &\le 4 \sum_{u'+v'+p'\le\infty} \sum_{\substack{u'=x_1+\ldots+x_f\\ x_f\ge 2}} \frac{1}{\langle x_1,\ldots,x_f\rangle^{\alpha+2}} \sum_{\substack{v'=y_1+\ldots+y_g\\ y_g\ge 2}} \frac{1}{\langle y_1,\ldots,y_g\rangle^{\alpha+2}} \\ &\qquad \times \sum_{\substack{p'=z_1+\ldots+z_h\\ z_h\ge 2}} \frac{1}{\langle z_l,\ldots,z_h\rangle^{\alpha+1}\langle z_1,\ldots,z_{h-1}\rangle} \\ &\le 4 \left(\sum_{\substack{x_1+\ldots+x_f\le\infty\\ x_f\ge 2}} \frac{1}{\langle x_1,\ldots,x_f\rangle^{\alpha+2}} \right)^2 \sum_{\substack{z_1+\ldots+z_h\le\infty\\ z_h\ge 2}} \frac{(\langle z_1,\ldots,z_{h-1}\rangle)^{-1}}{\langle z_l,\ldots,z_h\rangle^{\alpha+1}}. \end{aligned}$$

Since $\langle z_1,\ldots z_h\rangle \ge \langle z_1,\ldots z_{h-1}\rangle$, we see that the sums are each bounded above by the constant $\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}$, hence the inner sum has constant upper bound $4\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}\right)^3$ (these calculations are explained in both [**31**], [**12**]). Again the factor 4 is introduced at (44) following the imposition of $x_f, y_g \ge 2$. Combining these results yields

$$\Sigma^{(2)}_{(N,2)} \ll \frac{N^2 \log^{2\alpha+3}(N)}{w^{2\alpha+3}}$$

(as $N \to \infty$). □

Remark: One may replace N with n in the final error terms since the difference between the result of this substitution and the original term is of lesser order than the replacement (essentially $N = \mathcal{O}(n)$).

5.2.1. *Numerical Results and Explanation.* The quantity $\Sigma^{(2)}_{(N,2)}$ and this upper bound with $w = \frac{N}{2}$ is plotted in figure 3.14. It is clear from this figure that Lemma 3.15 does not produce the tightest bound. However the result suffices in the context of overall proof of Thoerem 3.5, since the bound in question is of lesser order of magnitude than the main term.

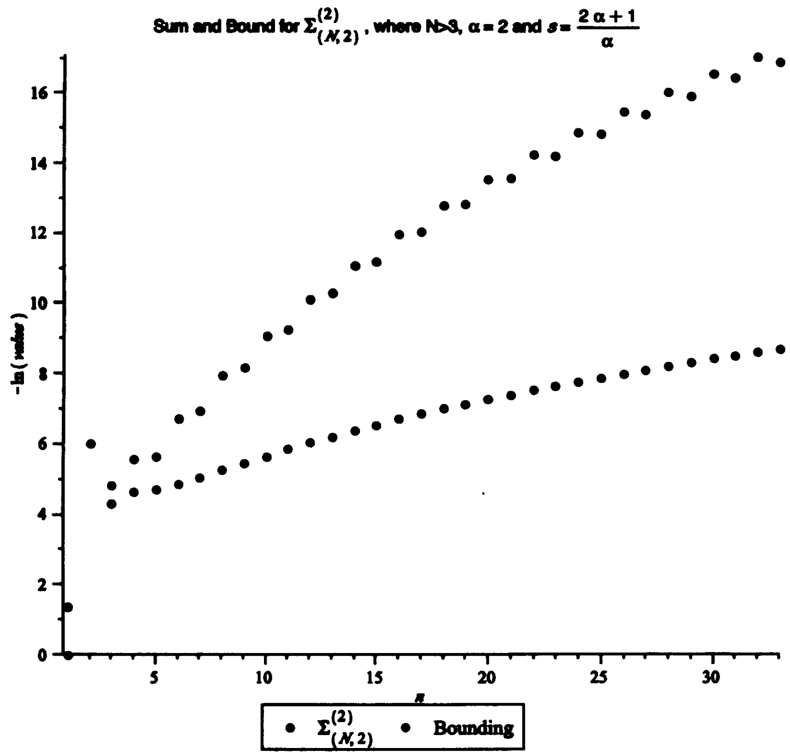


FIGURE 3.14. The sum $\Sigma^{(2)}_{(N,2)}$

The shape of this graph is curious and worthy of some explanation. Consider a continuant $\langle a_1, \ldots, a_r \rangle \in \mathcal{A}^{(4)}_{(N,j=r)}$, i.e. such that $a_r > N - w$. Under Farey Tree arithmetic, this denominator has a 'child' vertex at level $N+1$ defined as

$$\langle a_1, \ldots, a_r - 1, 2\rangle.$$

This clearly does not satisfy the conditions required to be denominator with origin $\mathcal{A}^{(4)}_{(N+1,j=r)}$. Moreover if $a_r = N - w$ then the 'child' continuant described above now becomes a member of the denominator set $\mathcal{A}^{(2)}_{(N+1,2)}$. The resultant affect on the cardinality of this set explains the non-monotonic behaviour of sum $\Sigma^{(2)}_{(N,2)}$.

5.3. On the Sum $\Sigma^{(4)}_{(N,j=r)}$. Recall the definition of the sum of interest:

$$\Sigma^{(4)}_{(N,j=r)} = \sum_{a \in \mathcal{A}^{(4)}_{(N,j=r)}} \left[\frac{1}{q''(a)^{\alpha+1} q_-(a)} + \frac{1}{q''(a)^{\alpha+1} q_+(a)}\right], \tag{45}$$

where $\mathcal{A}^{(4)}_{(N,j=r)} := \{a \in \mathcal{A}^{(3)}_{(N,j\geq r-1)} : a_j = a_r > N - w\}$. This section discusses proof of an asymptotic formula for (45). We consider the sub-sum yielded by each of the terms with denominator q_+ and q_- in turn; these are the calculations of Lemmas 3.16 and 3.17 respectively.

5.3.1. *For* $\Sigma^{(4)+}_{(N,j=r)}$.

For the sub-sum $\Sigma^{(4)+}_{(N,j=r)} = \sum_{a\in\mathcal{A}^{(4)}_{(N,j=r)}} \frac{1}{q''(a)^{\alpha+1}q_+(a)}$, we have the following asymptotic result.

LEMMA 3.16. *When $w \leq \frac{N}{2}$, and for $s > 1$, $\alpha > 1$:*

$$\Sigma^{(4)+}_{(N,j=r)} = \frac{2}{n(n+1)^{\alpha+1}}\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + K^+_{\alpha,n}\right) + \mathcal{O}\left(\frac{1}{n^{s\alpha}} + \frac{\log(n)}{n^{2\alpha+3}} + \frac{1}{n^{\alpha+2}w^{\alpha+1}}\right),$$

where $$K^+_{\alpha,n} = \sum_{1\leq k<\alpha+1}\sum_{v=1}^{\infty}\sum_{\substack{a_1+\ldots+a_{r-1}=v\\ a_{r-1}\geq 2}} \frac{(v-A)^k}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}\sum_{j+l=k}\frac{\theta_j(\alpha)}{n^l(n+1)^j};$$

with $\theta_j(\alpha) = \frac{1}{k!}\prod_{i=1}^{j}(\alpha+i)$ *and* $A = [a_{r-1},\ldots,a_1]$ *which lies in the interval* $(0, \frac{1}{2})$.

PROOF. We follow a scheme of working similar to Lemma 14 of [**12**]. By definition of $\mathcal{A}^{(4)}_{(N,j=r)}$ the imposition of $w \leq \frac{N}{2}$ ensures $a_r > N - w$;

$$
\begin{aligned}
\Sigma^{(4)+}_{(N,j=r)} &= \sum_{\substack{a\in\mathcal{A}_N;\ a_r>N-w\\ q''(a)<N^s}} \frac{1}{q''(a)^{\alpha+1}q_+(a)}\\
&= \left(\sum_{a\in\mathcal{A}_N;\ a_r>N-w} - \sum_{\substack{a\in\mathcal{A}_N;\ a_r>N-w\\ q''(a)\geq N^s}}\right)\frac{1}{q''(a)^{\alpha+1}q_+(a)}\\
&= \sum_{a\in\mathcal{A}_N;\ a_r>N-w} \frac{1}{q''(a)^{\alpha+1}q_+(a)} + \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right), \text{ by Lemma 3.14.}
\end{aligned}
$$

Now, let $a = (a_1, \ldots, a_r) \in \mathcal{A}_N$ and set $a_r = N - v$ for $v = 1, \ldots, [w]$ in the continuants $q'' = a_r q_- + (q_-)_-$ and $q_+ = (a_r - 1)q_- + (q_-)_-$. Since $(q_-)_- = \langle a_1, \ldots, a_{r-2}\rangle$ then

$$(q_-)_- = q_-[a_{r-1}, \ldots, a_1],$$

which yields

$$
\begin{aligned}
q''(a) &= N\, q_-(a)\left(1 - \frac{v}{N} + \frac{[a_{r-1}, a_{r-2}, \ldots a_1]}{N}\right)\\
q_+(a) &= n\, q_-(a)\left(1 - \frac{v}{n} + \frac{[a_{r-1}, a_{r-2}, \ldots a_1]}{n}\right)
\end{aligned}
\tag{46}
$$

(where $N = n + 1$). The continued fraction in these equalities will, under the restriction $a_{r-1} \geq 2$ to be introduced shortly, lie in the interval $(0, \frac{1}{2})$. It will be denoted by A for brevity.

(47)

$$\Sigma^{(4)+}_{(N,j=r)} = \frac{1}{nN^{\alpha+1}} \sum_{v=1}^{[w]} \sum_{\substack{a_1+\ldots+a_{r-1}=v\\ a\in\mathcal{A}_N}} \frac{1}{q_-(a)^{\alpha+2}} \frac{1}{\left(1 - \frac{(v-A)}{N}\right)^{\alpha+1}\left(1 - \frac{(v-A)}{n}\right)} + \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right).$$

Let $X = \frac{v-A}{N}$ and $Y = \frac{v-A}{n}$; for these we have

$$\frac{v-A}{N} < \frac{v}{N} \leq \frac{w}{N} \leq \frac{1}{2},$$

and

$$\frac{v-A}{n} < \frac{v}{n} \leq \frac{w}{N} \leq \frac{(n+1)}{2n} < 1,$$

(using the restriction $n \geq 2$ introduced in Lemma 3.15). This means that both $|X| < 1$ and $|Y| < 1$ by construction. Now, recall the Taylor expansions of Lemma 1.3; one may now expand the main term in 47 using the following:

$$\begin{aligned}\frac{1}{\left(1-\frac{(v-A)}{N}\right)^{\alpha+1}\left(1-\frac{(v-A)}{n}\right)} &= \left(1+\sum_{k=1}^{\infty}\frac{1}{j!}\frac{\prod_{i=1}^{j}(\alpha+i)(v-A)^j}{N^j}\right)\sum_{l=0}^{\infty}\left(\frac{v-A}{n}\right)^l \\ &= 1+\sum_{k=1}^{\infty}\sum_{j+l=k}\frac{\theta_j(\alpha)(v-A)^j(v-A)^l}{n^l N^j},\end{aligned}$$

where $\theta_j(\alpha) = \frac{1}{j!}\prod_{i=1}^{j}(\alpha+i)$. Moreover,

$$\begin{aligned}\Sigma^{(4)+}_{(N,j=r)} &= \frac{2}{n(n+1)^{\alpha+1}}\sum_{v=1}^{[w]}\sum_{\substack{a_1+\ldots+a_{r-1}=v\\ a_{r-1}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}+ \\ &\quad+\frac{2}{n(n+1)^{\alpha+1}}\sum_{k=1}^{\infty}\sum_{v=1}^{[w]}\sum_{\substack{a_1+\ldots+a_{r-1}=v\\ a_{r-1}\geq 2}}\frac{(v-A)^k}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}\sum_{j+l=k}\frac{\theta_j(\alpha)}{n^l N^j} \qquad (48)\\ &\quad+ \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right).\end{aligned}$$

To continue, we apply the main result of Theorem 3.3: that there exists some constant C'_1 (not necessarily related to the value C'_0 which appears in the original main term) such that

$$\sum_{a\in A_X}\frac{1}{\langle a_1,\ldots,a_x\rangle^{\alpha+2}} \leq \frac{C'_1}{X^{\alpha+2}} \qquad (49)$$

(as $X \to \infty$). Further discussion on the constants in the result of Theorem 3.3 is made in Appendix A. Despite being a somewhat weaker variant of the Theorem, inequality (49) is indeed sufficient to calculate the asymptotic behaviour of the sums in (48). The inequality is indeed utilised at length throughout the remaining proofs also.

One will again notice that a factor 2 has been introduced coincidentally with the restriction on the partial quotient $a_{r-1} \geq 2$. Recall that doing so requires the use of the following identity

$$\left|\left\{(a_1,\dots,a_x)\ :\ a_i \geq 1,\ \forall i = 1,\dots,x \text{ and } \sum_{i=1}^{x} a_i = v\right\}\right| = 2|\mathcal{A}_v|, \tag{50}$$

which was introduced earlier in (26) and is also utilised in Lemmas 3.12 and 3.15.

Let us next consider the value of the first of the two sums given as the right hand side of identity (48), which may be expressed as :

$$\begin{aligned}\sum_{v=1}^{[w]} \sum_{\substack{a_1+\dots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{q_-(a)^{\alpha+2}} &= \left(\sum_{v=1}^{\infty} - \sum_{v=[w]}^{\infty}\right) \sum_{\substack{a_1+\dots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{\langle a_1,\dots,a_{r-1}\rangle^{\alpha+2}} \\ &= \frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} - \sum_{v=[w]}^{\infty} \sum_{\substack{a_1+\dots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{\langle a_1,\dots,a_{r-1}\rangle^{\alpha+2}}. \end{aligned} \tag{51}$$

Moreover, Theorem 3.3 states that $\sum_{\substack{a_1+\dots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{\langle a_1,\dots,a_{r-1}\rangle^{\alpha+2}} \ll \frac{1}{v^{\alpha+2}}$, so the second entity of (51) is $\mathcal{O}(w^{-(\alpha+1)})$. We thus have

$$\frac{2}{n(n+1)^{\alpha+1}} \sum_{v=1}^{[w]} \sum_{\substack{a_1+\dots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{q_-(a)^{\alpha+2}} = \frac{2}{n(n+1)^{\alpha+1}} \frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + \mathcal{O}\left(\frac{1}{N^{\alpha+2} w^{\alpha+1}}\right).$$

Let us next investigate the properties of the second series where the 'inner sum' S_k^+ is defined:

$$\begin{aligned} S_k^+ &= \left(\sum_{v=1}^{\infty} - \sum_{v=[w]}^{\infty}\right) \sum_{\substack{a_1+\dots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{(v-A)^k}{\langle a_1,\dots,a_{r-1}\rangle^{\alpha+2}} \sum_{j+l=k} \frac{\theta_j(\alpha)}{n^l N^j} \\ &= S_{k,1}^+ - S_{k,w}^+. \end{aligned}$$

Consider at first, the case where α is not integer valued. Now for fixed α and k, $\sum_{j\leq k} \theta_j(\alpha)$ is constant. Moreover we see that $S_{k,1}^+$ is convergent for $k < \alpha + 1$, since:

$$\begin{aligned}
S_{k,1}^{+} &= \sum_{v=1}^{\infty} \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{(v-A)^k}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}} \sum_{j+l=k} \frac{\theta_j(\alpha)}{n^l N^j} \\
&\ll \sum_{v=1}^{\infty} \frac{v^k}{n^k} \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}} \\
&\ll \frac{1}{n^k} \sum_{v=1}^{\infty} \frac{1}{v^{\alpha+2-k}}, \quad \text{(from (49))}.
\end{aligned}$$

Since C_1' is a constant, this justifies the use of the Vinogradov symbol above. Furthermore, the series $S_{k,w}^{+}$ displays a similar property, whereby

$$\begin{aligned}
S_{k,w}^{+} &= \sum_{v=[w]}^{\infty} \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{(v-A)^k}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}} \sum_{j+l=k} \frac{\theta_j(\alpha)}{n^l N^j} \\
&\ll \sum_{v=[w]}^{\infty} v^k \frac{1}{v^{\alpha+2}} \\
&= \mathcal{O}\left(\frac{1}{w^{\alpha+1-k}}\right).
\end{aligned}$$

What remains is to investigate the behaviour of S_k^{+} for the remaining k, which we will see has same order of magnitude as the main term. We have an upper bound for S_k^{+} given by:

$$\begin{aligned}
S_k^{+} &\leq \frac{1}{n^k} \sum_{v=1}^{[w]} v^k \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}} \sum_{j\leq k} \theta_j(\alpha) \\
&\leq \frac{C_1'}{n^k} \left(\sum_{j\leq k} \theta_j(\alpha)\right) \sum_{v=1}^{[w]} \frac{v^k}{v^{\alpha+2}} \\
&\leq \frac{C_1'}{n^k} \left(\sum_{j\leq k} \theta_j(\alpha)\right) \int_1^{[w]} \frac{dv}{v^{\alpha+2-k}} \qquad (52) \\
&\leq \frac{C_1'}{n^k} \left(\sum_{j\leq k} \theta_j(\alpha)\right) \frac{1}{w^{\alpha+1-k}}. \qquad (53)
\end{aligned}$$

Therefore

(54)
$$\begin{aligned}\frac{2}{n(n+1)^{\alpha+1}}\sum_{k>\alpha+1} S_k^+ &\leq \frac{2C_1'}{n^{\alpha+2}}\sum_{k>\alpha+1}\frac{1}{n^k}\frac{1}{w^{\alpha+1-k}}\sum_{j\leq k}\theta_j(\alpha)\\ &\leq \frac{2C_1'}{n^{\alpha+2}w^{\alpha+1}}\sum_{k=1}^{\infty}\left(\frac{w}{n}\right)^k\sum_{j\leq k}\theta_j(\alpha)\\ &\leq \frac{2C_1'}{n^{\alpha+2}w^{\alpha+1}}\left[\frac{1}{\left(1-\frac{w}{n}\right)^{\alpha+2}}-1+\frac{\frac{w}{n}}{\frac{w}{n}-1}\right].\end{aligned}$$

The sum at (54) is a hypergeometric series, for which we use Lemma 1.4 to obtain the stated upper bound. This is valid since $\frac{w}{n} \leq \frac{(n+1)}{2n} \leq \frac{2}{3}$ for the values $n \geq 2$ required by Lemma 3.15. Moreover, this now implies that

$$\frac{2}{n(n+1)^{\alpha+1}}\sum_{k>\alpha+1} S_k^+ = \mathcal{O}\left(\frac{1}{n^{\alpha+2}w^{\alpha+1}}\right).$$

In the particular case where $\alpha \in \mathbb{N}\setminus\{1\}$, (52) implies that $S_k^+ = \mathcal{O}\left(\frac{\log(w)}{n^{\alpha+1}}\right)$ for $\alpha = k-1$. Therefore, in this case:

$$\frac{2}{n(n+1)^{\alpha+1}}\sum_{k>\alpha+1} S_k^+ = \mathcal{O}\left(\frac{\log(n)}{n^{2\alpha+3}}\right).$$

This yields the final result stated.

Remark: the result shown is that with α chosen to be integer valued. This represents the largest error term possible. □

5.3.2. *Numerical Results.* The values to be plotted depend on the chosen value of parameter s. For example, when $s = \frac{\alpha+3}{\alpha}$, Lemma 3.16 becomes:

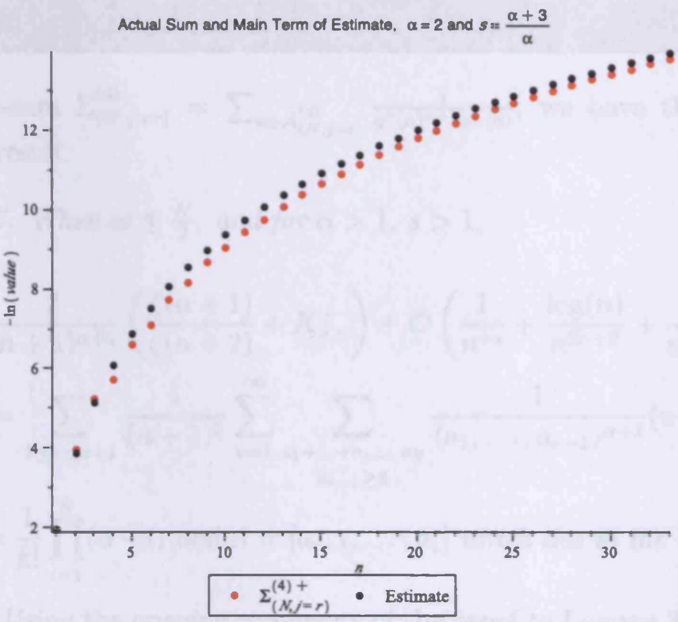
$$\Sigma^{(4)+}_{(N,j=r)} = \frac{2}{n(n+1)^{\alpha+1}}\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + \mathcal{O}\left(\frac{1}{n^{\alpha+3}}\right).$$

The main term of this, along with the actual value of $\Sigma^{(4)+}_{(N,j=r)}$ is plotted in figure 3.15(a). Conversely, with $s = \frac{2\alpha+3}{\alpha}$ one has the full main term seen in the Lemma, and for $\alpha = 2$ this is:

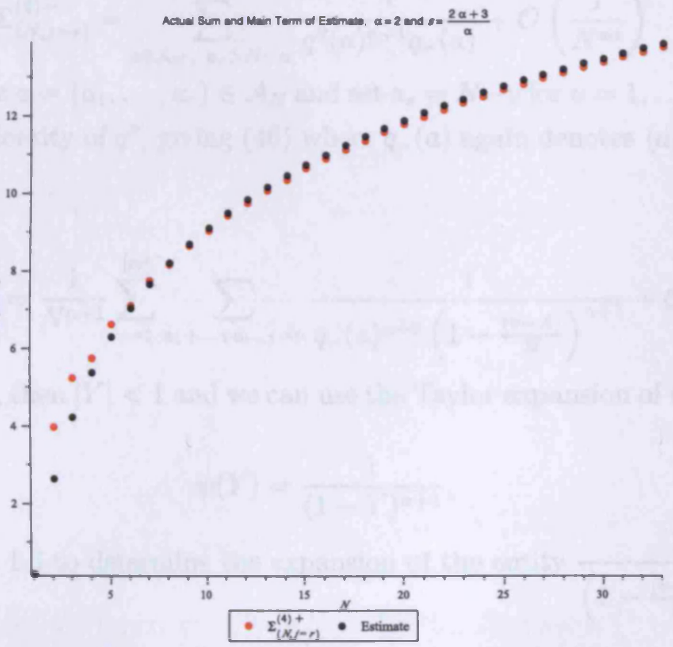
$$\frac{2}{n(n+1)^3}\left(\frac{\zeta(3)}{\zeta(4)} + K^+_{2,n}\right).$$

Figure 3.15(b) plots the quantity, along with the true value of $\Sigma^{(4)+}_{(N,j=r)}$ using the main term from the estimate:

$$\begin{aligned} K^+_{\alpha,n} &\leq \sum_{1\leq k<\alpha+1} \frac{1}{(n+1)^k} \sum_{v=1}^{\infty} v^k \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}} \sum_{j\leq k} \theta_k(\alpha) \\ &\leq \sum_{1\leq k<\alpha+1} \frac{1}{(n+1)^k} \sum_{j\leq k} \theta_k(\alpha) \sum_{v=2}^{\infty} v^k \left(\frac{C_0}{v^{\alpha+2}} + \mathcal{O}\left(\frac{\log^{\frac{2\alpha+4}{2\alpha+5}}(v)}{v^{\alpha+3-\frac{\alpha+5}{2\alpha+5}}} \right) \right). \end{aligned}$$



(a) $\Sigma^{(4)+}_{(N,j=r)}$ with Main Term, $s = \frac{\alpha+3}{\alpha}$



(b) $\Sigma^{(4)+}_{(N,j=r)}$ with Main Term, $s = \frac{2\alpha+3}{\alpha}$

FIGURE 3.15.

5.3.3. *For* $\Sigma^{(4)-}_{(N,j=r)}$.

For the sub-sum $\Sigma^{(4)-}_{(N,j=r)} = \sum_{a\in\mathcal{A}^{(4)}_{(N,j=r)}} \frac{1}{q''(a)^{\alpha+1}q_-(a)}$, we have the following asymptotic result.

LEMMA 3.17. *When* $w \leq \frac{N}{2}$, *and for* $\alpha > 1$, $s > 1$,

$$\Sigma^{(4)-}_{(N,j=r)} = \frac{2}{(n+1)^{\alpha+1}}\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + K^-_{\alpha,n}\right) + \mathcal{O}\left(\frac{1}{n^{s\alpha}} + \frac{\log(n)}{n^{2\alpha+2}} + \frac{1}{n^{\alpha+1}w^{\alpha+1}}\right),$$

where $K^-_{\alpha,n} = \sum_{1\leq k<\alpha+1} \frac{1}{(n+1)^k}\sum_{v=1}^{\infty}\sum_{\substack{a_1+\ldots+a_{r-1}=v\\ a_{r-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}(v-A)^k\theta_k(\alpha);$

with $\theta_k(\alpha) = \frac{1}{k!}\prod_{i=1}^{k}(\alpha+i)$ *and* $A = [a_{r-1},\ldots,a_1]$ *which lies in the range* $(0,\frac{1}{2})$.

PROOF. Using the opening comments of the proof to Lemma 3.16, one has

$$\Sigma^{(4)-}_{(N,j=r)} = \sum_{a\in\mathcal{A}_N;\ a_r>N-w} \frac{1}{q''(a)^{\alpha+1}q_-(a)} + \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right).$$

We again take $a = (a_1,\ldots,a_r) \in \mathcal{A}_N$ and set $a_r = N-v$ for $v = 1,\ldots,[w]$ in the continuant identity of q'', giving (46) where $q_-(a)$ again denotes $\langle a_1,\ldots,a_{r-1}\rangle$. This gives

$$\text{(55)}\quad \Sigma^{(4)-}_{(N,j=r)} = \frac{1}{N^{\alpha+1}}\sum_{v=1}^{[w]}\sum_{a_1+\ldots+a_{r-1}=v} \frac{1}{q_-(a)^{\alpha+2}\left(1-\frac{(v-A)}{N}\right)^{\alpha+1}} + \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right).$$

Let $Y = \frac{v-A}{N}$, then $|Y| < 1$ and we can use the Taylor expansion of the function

$$\psi(Y) = \frac{1}{(1-Y)^{\alpha+1}},$$

from Lemma 1.3 to determine the expansion of the entity $\frac{1}{\left(1-\frac{(v-A)}{N}\right)^{\alpha+1}}$.

Thus

$$\frac{1}{\left(1-\frac{(v-A)}{N}\right)^{\alpha+1}} = 1 + \sum_{k=1}^{\infty}\frac{\theta_k(\alpha)(v-A)^k}{N^k},$$

again $\theta_k(\alpha) = \prod_{i=1}^{k}\frac{\alpha+i}{i}$.

Inserting this information into (55) yields

$$\begin{aligned}
\Sigma^{(4)-}_{(N,j=r)} &= \frac{1}{N^{\alpha+1}}\sum_{v=1}^{[w]}\sum_{a_1+\ldots+a_{r-1}=v}\frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}\left(1+\sum_{k=1}^{\infty}\theta_k(\alpha)\frac{(v-A)^k}{N^k}\right)+\\
&+\ \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right)\\
&= \frac{2}{N^{\alpha+1}}\sum_{v=1}^{[w]}\sum_{\substack{a_1+\ldots+a_{r-1}=v\\ a_{r-1}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}+\\
&+\frac{2}{N^{\alpha+1}}\sum_{k=1}^{\infty}\frac{1}{N^k}\sum_{v=1}^{[w]}\sum_{\substack{a_1+\ldots+a_{r-1}=v\\ a_{r-1}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}\theta_k(\alpha)(v-A)^k+ \qquad (56)\\
&+\ \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right).
\end{aligned}$$

The constraint $a_{r-1}\geq 2$ is again introduced at (56), as is seen in Lemma 3.16. Now, let us consider the value of the first of the two sums given as the right hand side of identity (56), which may be expressed as :

$$\begin{aligned}
\sum_{v=1}^{[w]}\sum_{\substack{a_1+\ldots+a_{r-1}=v\\ a_{r-1}\geq 2}}\frac{1}{q_-(a)^{\alpha+2}} &= \left(\sum_{v=1}^{\infty}-\sum_{v=[w]}^{\infty}\right)\sum_{\substack{a_1+\ldots+a_{r-1}=v\\ a_{r-1}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}\\
&= \frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}-\sum_{v=[w]}^{\infty}\sum_{\substack{a_1+\ldots+a_{r-1}=v\\ a_{r-1}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}. \qquad (57)
\end{aligned}$$

By Theorem 3.3

$$\sum_{\substack{a_1+\ldots+a_{r-1}=v\\ a_{r-1}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}\ll\frac{1}{v^{\alpha+2}},$$

and thus the second entity of (57) is $\mathcal{O}(w^{-(\alpha+1)})$. Therefore,

$$\frac{2}{N^{\alpha+1}}\sum_{v=1}^{[w]}\sum_{\substack{a_1+\ldots+a_{r-1}=v\\ a_{r-1}\geq 2}}\frac{1}{q_-(a)^{\alpha+2}}=\frac{2}{N^{\alpha+1}}\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}+\mathcal{O}\left(\frac{1}{N^{\alpha+1}w^{\alpha+1}}\right).$$

Let us next investigate the properties of the second series. For this, we again define the 'inner sum' S_k^- as follows

$$\begin{aligned} S_k^- &= \left(\sum_{v=1}^{\infty} - \sum_{v=[w]}^{\infty}\right) \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}(v-A)^k\theta_k(\alpha) \\ &= S_{k,1}^- - S_{k,w}^-. \end{aligned}$$

Consider at first the case where $\alpha > 1$ is non-integer in value. For these fixed α, k we see that $S_{k,1}^-$ converges for $k < \alpha + 1$ since:

$$\begin{aligned} S_{k,1}^- &= \sum_{v=1}^{\infty} \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}(v-A)^k\theta_k(\alpha) \\ &\ll \sum_{v=1}^{\infty} v^k \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}} \\ &\ll \sum_{v=1}^{\infty} \frac{1}{v^{\alpha+2-k}}, \quad \text{using (49).} \end{aligned}$$

$S_{k,w}^-$ displays similar characteristics, whereby

$$\begin{aligned} S_{k,w}^- &= \sum_{v=[w]}^{\infty} \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}(v-A)^k\theta_k(\alpha) \\ &\ll \sum_{v=[w]}^{\infty} v^k \frac{1}{v^{\alpha+2}} \\ &= \mathcal{O}\left(\frac{1}{w^{\alpha+1-k}}\right). \qquad (58) \end{aligned}$$

What remains is to determine how S_k^- behaves for the remaining k. We again look for an upper bound:

$$
\begin{aligned}
S_k^- &\le \sum_{v=1}^{[w]} v^k \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\ge 2}} \frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}\theta_k(\alpha) \\
&\le C_1'\theta_k(\alpha)\sum_{v=1}^{[w]}\frac{v^k}{v^{\alpha+2}} \qquad \text{using (49)} \\
&\le C_1'\theta_k(\alpha)\int_1^w \frac{dv}{v^{\alpha+2-k}}. \\
&\le C_1'\theta_k(\alpha)\frac{1}{w^{\alpha+1-k}}. \qquad (59)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum_{k>\alpha+1}\frac{S_k^-}{N^{k+\alpha+1}} &\le C_1'\sum_{k>\alpha+1}\frac{\theta_k(\alpha)}{N^{k+\alpha+1}}\frac{1}{w^{\alpha+1-k}} \\
&\le \frac{C_1'}{w^{\alpha+1}N^{\alpha+1}}\sum_{k=1}^{\infty}\theta_k(\alpha)\left(\frac{w}{N}\right)^k. \qquad (60)
\end{aligned}
$$

The series in (60) is again of hypergeometric type - we treat this in a similar manner to that performed in Lemma 3.16. That is, Lemma 1.2 with $z = \frac{w}{N} < 1$ gives

$$
\frac{1}{(1-z)^b} = \sum_{k=0}^{\infty}\frac{(b)^{\overline{k}}}{k!}z^k = F\left(\begin{matrix} b,1 \\ 1\end{matrix}\,\middle|\, z\right).
$$

Hence it follows that

$$
\begin{aligned}
\sum_{k=1}^{\infty}\theta_k(\alpha)\left(\frac{w}{N}\right)^k &= \frac{1}{\left(1-\frac{w}{N}\right)^{\alpha+1}} - 1 \\
&\le 2^{\alpha+1} - 1,
\end{aligned}
$$

since w is at most $\frac{N}{2}$. We see therefore that, for α non-integer valued

$$
\sum_{k>\alpha+1}\frac{S_k^-}{N^{k+\alpha+1}} = \mathcal{O}\left(\frac{1}{w^{\alpha+1}N^{\alpha+1}}\right). \qquad (61)
$$

Should α be chosen such that $\alpha \in \mathbb{N}\setminus\{1\}$, then from (59) one sees that, when $\alpha = k-1$,

$$
S_k^- = \mathcal{O}(\log(w)).
$$

Hence $\sum_{k\geq\alpha+1} \frac{S_k^-}{N^{k+\alpha+1}} = \mathcal{O}\left(\frac{\log(w)}{N^{2\alpha+2}}\right)$ which has uniform upper bound $\frac{\log(N)}{N^{2\alpha+2}}$ since $w \leq \frac{N}{2}$. This gives the stated result, with the entity $K_{\alpha,n}^-$ defined as

$$K_{\alpha,n}^- = \sum_{1\leq k<\alpha+1} \frac{1}{N^k} \sum_{v=1}^{\infty} \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}} (v-A)^k \theta_k(\alpha).$$

Remark: The final error term given in the statement of the lemma is the *worst-possible* case. One has, when $\alpha \notin \mathbb{N}\setminus\{1\}$ and improvement in the error term to

$$\mathcal{O}\left(\frac{1}{n^{s\alpha}} + \frac{1}{w^{\alpha+1}n^{\alpha+1}}\right).$$

The stated result is the error term of greatest possible magnitude. □

5.3.4. *Numerical Results.* Consider again the effect of the chosen value of parameter s. When $s = \frac{\alpha+2}{\alpha}$, Lemma 3.17 becomes:

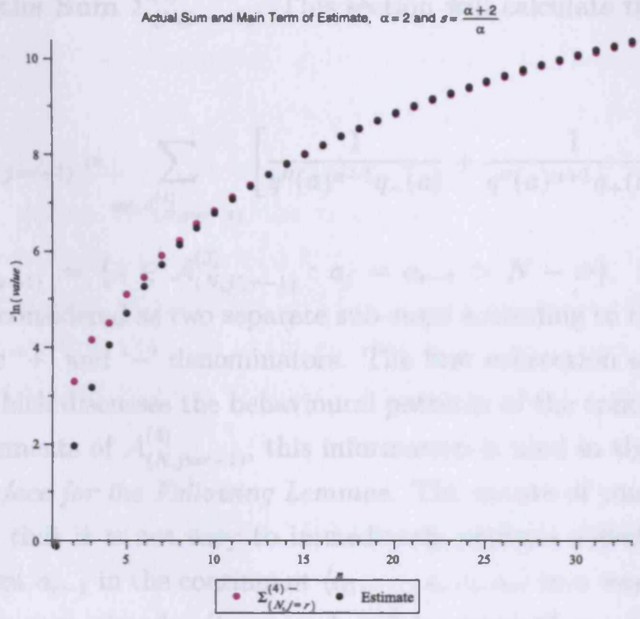
$$\Sigma_{(N,j=r)}^{(4)-} = \frac{2}{(n+1)^{\alpha+1}} \frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right).$$

The main term of this, along with the actual value of $\Sigma_{(N,j=r)}^{(4)+}$ is plotted in figure 3.16(a). Conversely, with $s = \frac{2\alpha+2}{\alpha}$ one has the full main term seen in the Lemma, and for $\alpha = 2$ this is:

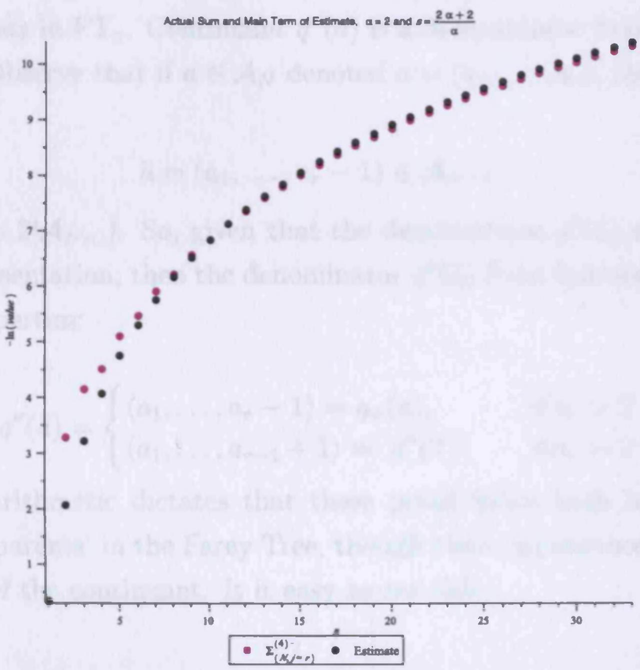
$$\frac{2}{(n+1)^3}\left(\frac{\zeta(3)}{\zeta(4)} + K_{2,n}^-\right).$$

Figure 3.16(b) plots the quantity, along with the true value of $\Sigma_{(N,j=r)}^{(4)+}$ using the main term of the following estimate:

$$\begin{aligned} K_{\alpha,n}^- &\leq \sum_{1\leq k<\alpha+1} \frac{1}{(n+1)^k} \sum_{v=1}^{\infty} v^k \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}} \theta_k(\alpha) \\ &\leq \sum_{1\leq k<\alpha+1} \frac{\theta_k(\alpha)}{(n+1)^k} \sum_{v=2}^{\infty} v^k \left(\frac{C_0'}{v^{\alpha+2}} + \mathcal{O}\left(\frac{\log^{\frac{2\alpha+4}{2\alpha+5}}(v)}{v^{\alpha+3-\frac{\alpha+5}{2\alpha+5}}}\right)\right). \end{aligned}$$



(a) $\Sigma^{(4)-}_{(N,j=r)}$ with Main Term, $s = \frac{\alpha+2}{\alpha}$



(b) $\Sigma^{(4)-}_{(N,j=r)}$ with Main Term, $s = \frac{2\alpha+2}{\alpha}$

FIGURE 3.16.

5.4. On the Sum $\Sigma^{(4)}_{(N,j=r-1)}$. This section will calculate the asymptotic behaviour of

$$\Sigma^{(4)}_{(N,j=r-1)} = \sum_{a\in\mathcal{A}^{(4)}_{(N,j=r-1)}} \left[\frac{1}{q''(a)^{\alpha+1}q_-(a)} + \frac{1}{q''(a)^{\alpha+1}q_+(a)}\right], \tag{62}$$

where $\mathcal{A}^{(4)}_{(N,j=r-1)} := \{a \in \mathcal{A}^{(3)}_{(N,j\geq r-1)} : a_j = a_{r-1} > N - w\}$. Sum (62) will again will be considered as two separate sub-sums according to the presence of the respective '+' and '−' denominators. The first subsection of these proofs is a preface which discusses the behavioural patterns of the continuants whose entries are elements of $\mathcal{A}^{(4)}_{(N,j=r-1)}$; this information is used in the final proof.

5.4.1. *Preface for the Following Lemmas.* The nature of continuant arithmetic is such that it is not easy to immediately perform calculations on the partial quotient a_{r-1} in the continuant $\langle a_1, \dots, a_{r-1}, a_r\rangle$ in a way analogous to what has been seen already. Some work will be required in order to achieve this. Let us refer again to the algorithm which generates the elements of FT_{n+1} using the points in FT_n. Continuant $q''(a)$ is a denominator from a fraction in the former. Observe that if $a \in \mathcal{A}_N$ denoted $a = (a_1, \dots, a_r)$, then

$$\tilde{a} = (a_1, \dots, a_r - 1) \in \mathcal{A}_{N-1},$$

where $|\mathcal{A}_N| = 2|\mathcal{A}_{N-1}|$. So, given that the denominator $q''(a)$ at level N has the said representation, then the denominator $q''(\tilde{a})$ from fraction $\frac{p''(\tilde{a})}{q''(\tilde{a})}$ has the following properties:

$$q''(\tilde{a}) = \begin{cases} \langle a_1, \dots, a_r - 1\rangle = q_+(a), & \text{if } a_r > 2 \\ \langle a_1, \dots, a_{r-1} + 1\rangle =: q''(\tilde{a}'), & \text{if } a_r = 2. \end{cases}$$

Elementary arithmetic dictates that these possibilities both have similarly-constructed 'parents' in the Farey Tree, though their appearances differ in the final entries of the continuant. It it easy to see that

$$\begin{aligned} q''(\tilde{a}) \qquad & \text{has associated } q_+(\tilde{a}) = \langle a_1, \dots, a_r - 2\rangle \\ & \text{and } q_-(\tilde{a}) = \langle a_1, \dots, a_{r-1}\rangle \end{aligned}$$

and

$$q''(\tilde{a}') \quad \text{has associated} \quad q_+(\tilde{a}') = \langle a_1, \dots, a_{r-1}\rangle = q_-(a)$$
$$\text{and} \quad q_-(\tilde{a}) = \langle a_1, \dots, a_{r-2}\rangle = (q_-)_-(a).$$

As an example to highlight this logic, an entity $\langle a_1, \dots, a_{r-1}, 2\rangle$ at level N, has associated q_- parent denoted by the continuant $\langle a_1, \dots, a_{r-1}\rangle$. Clearly this item is the denominator of a fraction from set $\mathcal{Q}_{N-2}$ and forms the 'new' $q_+(\tilde{a})$ in this counting back process (see [**37**] for further illustration). It is due to this phenomenon that the rôles of the equivalent sums denoted with + and − in this section appear to swap with their analogues in Lemmas 3.16 and 3.17 in calculation.

We proceed by setting $a_{r-1} = N-1-v$ for $v = 1, \dots, [w]$ and use the following identity on the continuants:

(63)
$$\langle x_1, \dots, x_f\rangle = \langle x_1, \dots, x_{i-1}\rangle\langle x_{i+1}, \dots, x_f\rangle(x_i + [x_{i-1}, \dots, x_1] + [x_{i+1}, \dots x_f]),$$

seen, for example in [**21**], to show that for $a_r = 2$

$$\begin{aligned} q''(\tilde{a}') &= \langle a_1, \dots, a_{r-2}\rangle(a_{r-1} + 1 + [a_{r-2}, \dots, a_1]) \\ &= N\langle a_1, \dots, a_{r-2}\rangle\left(1 - \frac{v - [a_{r-2}, \dots, a_1]}{N}\right) \\ q_+(\tilde{a}') &= \langle a_1, \dots, a_{r-1}\rangle = n\langle a_1, \dots, a_{r-2}\rangle\left(1 - \frac{v - [a_{r-2}, \dots, a_1]}{n}\right) \\ q_-(\tilde{a}') &= \langle a_1, \dots, a_{r-2}\rangle, \end{aligned}$$

where $N = n+1$ as usual. For the case $a_r > 2$ one has that

$$\begin{aligned} q''(\tilde{a}) &= \langle a_1, \dots, a_{r-2}\rangle(a_{r-1} + [a_{r-2}, \dots, a_1] + [a_r - 1]) = \\ &= n(a_r - 1)\langle a_1, \dots, a_{r-2}\rangle\left(1 - \frac{v - ([a_{r-2}, \dots, a_1] + [a_r - 1])}{n}\right) \\ q_+(\tilde{a}) &= \langle a_1, \dots, a_{r-2}\rangle(a_{r-1} + [a_{r-2}, \dots, a_1] + [a_r - 1]) \\ &= n(a_r - 2)\langle a_1, \dots, a_{r-2}\rangle\left(1 - \frac{v - ([a_{r-2}, \dots, a_1] + [a_r - 2])}{n}\right) \\ q_-(\tilde{a}) &= N\langle a_1, \dots, a_{r-2}\rangle\left(1 - \frac{v - [a_{r-2}, \dots, a_1]}{N}\right). \end{aligned}$$

For brevity, let the following quantities be denoted as:

$$\begin{aligned} A_1 &= [a_{r-2}, \ldots, a_1] \\ A_2 &= [a_{r-2}, \ldots, a_1] + [a_r - 1] \\ A_3 &= [a_{r-2}, \ldots, a_1] + [a_r - 2]. \end{aligned}$$

Under restriction a_{r-2}, then $A_1 \in (0, \frac{1}{2})$, and both $A_2, A_3 \in (0,1)$. For $a_r = 2$ this gives:

(64)
$$\frac{1}{q''(\tilde{a}')^{\alpha+1} q_+(\tilde{a}')} = \frac{1}{nN^{\alpha+1}} \frac{1}{\langle a_1, \ldots, a_{r-2} \rangle^{\alpha+2}} \frac{1}{\left(1 - \frac{(v-A_1)}{N}\right)^{\alpha+1} \left(1 - \frac{(v-A_1)}{n}\right)},$$

$$\frac{1}{q''(\tilde{a}')^{\alpha+1} q_-(\tilde{a}')} = \frac{1}{N^{\alpha+1}} \frac{1}{\langle a_1, \ldots, a_{r-2} \rangle^{\alpha+2}} \frac{1}{\left(1 - \frac{(v-A_1)}{N}\right)^{\alpha+1}}, \tag{65}$$

and for $a_r > 2$;

$$\begin{aligned} \frac{1}{q''(\tilde{a})^{\alpha+1} q_+(\tilde{a})} &= \frac{1}{n^{\alpha+2} \langle a_1, \ldots, a_{r-2} \rangle^{\alpha+2} (a_r - 1)^{\alpha+1} (a_r - 2)} \times \\ &\qquad \times \frac{1}{\left(1 - \frac{v-A_2}{n}\right)^{\alpha+1} \left(1 - \frac{v-A_3}{n}\right)}, \end{aligned} \tag{66}$$

$$\begin{aligned} \frac{1}{q''(\tilde{a})^{\alpha+1} q_-(\tilde{a})} &= \frac{1}{n^{\alpha+2} \langle a_1, \ldots, a_{r-2} \rangle^{\alpha+2} (a_r - 1)^{\alpha+1}} \times \\ &\qquad \times \frac{1}{\left(1 - \frac{v-A_2}{n}\right)^{\alpha+1} \left(1 - \frac{v-A_1}{n}\right)}. \end{aligned} \tag{67}$$

To continue with full calculation of $\Sigma^{(4)}_{(N,j=r-1)}$ split the sum into its constituent '+' and '−' parts:

$$\begin{aligned} \Sigma^{(4)}_{(N,j=r-1)} &= \sum_{\substack{a_1+\ldots+a_r=N \\ a_{r-1}>N-w;\, a_r \geq 2 \\ q''(a)<N^s}} \left[\frac{1}{q''(a)^{\alpha+1} q_+(a)} + \frac{1}{q''(a)^{\alpha+1} q_-(a)} \right] \\ &= \Sigma^{(4)+}_{(N,j=r-1)} + \Sigma^{(4)-}_{(N,j=r-1)}, \end{aligned}$$

for which we have:

$$\Sigma^{(4)+}_{(N,j=r-1)} = \left(\sum_{\substack{a_1+\ldots+a_r=N \\ a_{r-1}>N-w \\ a_r\geq 2}} - \sum_{\substack{a_1+\ldots+a_r=N \\ a_{r-1}>N-w;\, a_r\geq 2 \\ q''(a)\geq N^s}} \right) \frac{1}{q''(a)^{\alpha+1} q_+(a)}$$
$$= \sum_{\substack{a\in\mathcal{A}_N \\ a_{r-1}>N-w,\, a_r>2}} \frac{1}{q''(a)^{\alpha+1} q_+(a)} + \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right),$$

and similarly for the $\Sigma^{(4)-}_{(N,j=r-1)}$ version.

5.4.2. *For* $\Sigma^{(4)+}_{(N,j=r-1)}$.

For the sub-sum $\Sigma^{(4)+}_{(N,j=r-1)} = \sum_{a\in\mathcal{A}^{(4)}_{(N,j=r-1)}} \frac{1}{q''(a)^{\alpha+1} q_+(a)}$, we have the following asymptotic result.

LEMMA 3.18. *When $w \leq \frac{N}{2}$ and for $\alpha > 1$, $s > 1$,*

$$\Sigma^{(4)+}_{(N,j=r-1)} = \frac{2}{(n+2)^{\alpha+1}}\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + K^{-}_{\alpha,n+1}\right) + \frac{K^{*-}_{\alpha,n+1}}{(n+1)^{\alpha+2}} +$$
$$+ \mathcal{O}\left(\frac{\log(n)}{n^{2\alpha+1}} + \frac{1}{n^{\alpha+2}w^{\alpha-1}} + \frac{1}{n^{s\alpha}}\right),$$

where

$$K^{*-}_{\alpha,n} = T_\alpha + \sum_{1\leq k<\alpha-1} \frac{1}{n^k} \sum_{v=1}^{\infty} \sum_{\Omega_v} \frac{1}{(a_r-1)^{\alpha+1}} \frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}} \sum_{j+l=k} \theta_j(\alpha)(v-A_2)^j (v-A_1)^l,$$

T_α is a constant, $\theta_j(\alpha)$ retains its definition from Lemma 3.16 and

$$\Omega_v = \{\overline{a} = (a_1,\ldots,a_{r-2},a_r) \,:\, a_1+\ldots+a_{r-2}+a_r = v,\, a_r \geq 3\}.$$

PROOF. As is described earlier, the denominator $q''(\tilde{a})$ calculated by moving backward a level in the Farey sequence (i.e. this supposedly 'older' denominator belongs to a parent vertex of the originally-considered quantity) itself has parent denominators, one of which in the case $a_r = 2$ becomes the parent at the nearest level $N-2$, despite being the original 'furthest away' denominator. This effectively reverses the rôles of the of plus and minus from a level to its set of predecessors. Hence we require the identities of (65) and (67) which yield

$$\Sigma^{(4)-}_{(N-1,j=r-1)} = \frac{1}{n^{\alpha+2}} \sum_{\substack{a\in\mathcal{A}_{N-1}\\ a_r\geq 3,\, a_{r-1}=N-1-v\\ v=1,\dots,[w]}} \frac{(a_r-1)^{-(\alpha+1)}}{\langle a_1,\dots,a_{r-2}\rangle^{\alpha+2}} \frac{1}{\left(1-\frac{v-A_2}{n}\right)^{\alpha+1}\left(1-\frac{v-A_1}{n}\right)} +$$

$$(68)\qquad +\frac{1}{N^{\alpha+1}} \sum_{\substack{a\in\mathcal{A}_{N-1}\\ a_r=2,\, a_{r-1}=N-1-v\\ v=1,\dots,[w]}} \frac{\langle a_1,\dots,a_{r-2}\rangle^{-(\alpha+2)}}{\left(1-\frac{(v-A_1)}{N}\right)^{\alpha+1}} + \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right).$$

As a note of caution, one should interpret the sum conditions $a\in\mathcal{A}_{N-1}; a_r\geq 3$ or $a\in\mathcal{A}_{N-1}; a_r=2$ as the criteria that one will attain such values of a_r when performing the appropriate algorithm on the the members of $\mathcal{A}_{N-1}$ to derive set $\mathcal{A}_N$.

Let us first concentrate on the series in (68), for which one has

$$\frac{1}{N^{\alpha+1}} \sum_{\substack{a\in\mathcal{A}_{N-1}\\ a_r=2,\, a_{r-1}=N-1-v\\ v=1,\dots,[w]}} \frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{\alpha+2}} \frac{1}{\left(1-\frac{(v-A_1)}{N}\right)^{\alpha+1}} =$$

$$(69)\qquad = \frac{1}{N^{\alpha+1}} \sum_{v=1}^{[w]} \sum_{a_1+\dots+a_{r-2}=v-1} \frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{\alpha+2}} \frac{1}{\left(1-\frac{(v-A_1)}{N}\right)^{\alpha+1}}$$

$$= \frac{2}{N^{\alpha+1}} \sum_{v=1}^{[w]-1} \sum_{\substack{a_1+\dots+a_{r-2}=v\\ a_{r-2}\geq 2}} \frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{\alpha+2}} \frac{1}{\left(1-\frac{(v-A_1)}{N}\right)^{\alpha+1}}.$$

Using Lemma 3.17 one will see that the characteristic at line (69) takes value:

$$\frac{2}{(n+1)^{\alpha+1}}\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + K^-_{\alpha,n}\right) + \mathcal{O}\left(\frac{1}{n^{s\alpha}} + \frac{\log(n)}{n^{2\alpha+2}} + \frac{1}{n^{\alpha+1}w^{\alpha+1}}\right).$$

Now let us turn our attention to the summation with coefficient $\frac{1}{n^{\alpha+2}}$. This is

$$B_1 := \sum_{v=1}^{[w]} \sum_{\Omega_v} \frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}} \frac{1}{\left(1-\frac{v-A_2}{n}\right)^{\alpha+1}\left(1-\frac{v-A_1}{n}\right)}.$$

Using Lemma 1.3 we can expand the latter quantities into Taylor series as:

$$\begin{aligned}
\frac{1}{\left(1-\frac{v-A_2}{n}\right)^{\alpha+1}\left(1-\frac{v-A_1}{n}\right)} &= \left(1+\sum_{j=1}^{\infty}\frac{\prod_{i=1}^{j}(\alpha+i)}{j!}\left(\frac{v-A_2}{n}\right)^j\right)\times \\
&\times \quad \sum_{l=0}^{\infty}\left(\frac{v-A_1}{n}\right)^l \\
&= 1+\sum_{k=1}^{\infty}\frac{1}{n^k}\sum_{j+l=k}\theta_j(a)\,(v-A_2)^j\,(v-A_1)^l .
\end{aligned}$$

Therefore, one sees that

$$\begin{aligned}
(70) \qquad B_1 &= \left(\sum_{v=1}^{\infty}-\sum_{v=[w]}^{\infty}\right)\sum_{\Omega_v}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}}+ \\
&+\sum_{k=1}^{\infty}\frac{1}{n^k}\sum_{v=1}^{[w]}\sum_{\Omega_v}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}}\times \\
&\times \quad \sum_{j+l=k}\theta_j(a)\,(v-A_2)^j\,(v-A_1)^l .
\end{aligned}$$

The sum of (70) is finite and has been expressed in this manner in order to explain its estimation. Recall that the definition of Ω_v requires $a_r \geq 3$. Therefore $(x+1)$ for $x \geq 1$ displays the same behaviour as (a_r-1) for $a_r \geq 3$, and the first component of the series has the property that:

$$\begin{aligned}
&\sum_{v=1}^{\infty}\sum_{\Omega_v}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}} \\
(71) \qquad &\leq 2\sum_{v=1}^{\infty}\sum_{x+y=v}\frac{1}{(x+1)^{\alpha+1}}\sum_{\substack{y=a_1+\ldots+a_{r-2}\\ a_{r-2}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}} \\
&\leq 2C_1'\sum_{v=2}^{\infty}\sum_{x+y=v}\frac{1}{x^{\alpha+1}y^{\alpha+2}} \\
(72) \qquad &\leq 2C_1'\sum_{v=2}^{\infty}\frac{1}{v^{\alpha}} = 2C_1'(\zeta(\alpha)-1),
\end{aligned}$$

since Theorem 3.3 shows that $\sum_{a\in A_X}\frac{1}{\langle a_1,\ldots,a_x\rangle^{\alpha+2}} \leq \frac{C_1'}{X^{\alpha+2}}$. Likewise the second component series of (70) has asymptotic value $\mathcal{O}(w^{-(\alpha-1)})$ since:

$$\sum_{v=[w]}^{\infty}\sum_{\Omega_v}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}} \ll \sum_{v=[w]}^{\infty}\sum_{x+y=v}\frac{1}{x^{\alpha+1}}\frac{1}{y^{\alpha+2}}$$

$$(73) \qquad \ll \sum_{v=[w]}^{\infty}\frac{1}{v^{\alpha}},$$

using the inequality which leads to (72). Now, let us consider the latter summation of B_1, which will be represented as

$$(74) \qquad \sum_{k=1}^{\infty}\frac{B_k^-}{n^k},$$

where

$$\begin{aligned} B_k^- &:= \left(\sum_{v=1}^{\infty}-\sum_{v=[w]}^{\infty}\right)\sum_{\Omega_v}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}}\sum_{j+l=k}\theta_j(\alpha)\,(v-A_2)^j\,(v-A_1)^l \\ &= B_{k,1}^- - B_{k,w}^- \end{aligned}$$

Now, for fixed k and α, one sees that the quantity $B_{k,1}^-$ converges for $\alpha - k > 1 \Rightarrow k < \alpha - 1$, since:

$$\begin{aligned} B_{k,1}^- &\ll \sum_{v=1}^{\infty}v^k\sum_{\Omega_v}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}} \\ &\ll \sum_{v=1}^{\infty}v^k\sum_{x+y=v}\frac{1}{(x+1)^{\alpha+1}}\sum_{\substack{y=a_1+\ldots+a_{r-2}\\ a_{r-2}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}} \\ &\ll \sum_{v=1}^{\infty}v^k\frac{1}{v^{\alpha}}. \end{aligned}$$

With these k, the second component $B_{k,w}^-$ has the property that

$$\begin{aligned} B_{k,w}^- &\ll \sum_{v=[w]}^{\infty}v^k\sum_{\Omega_v}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}} \\ &\ll \sum_{v=[w]}^{\infty}\frac{v^k}{v^{\alpha}} \quad \text{similarly to above,} \\ &= \mathcal{O}\left(\frac{1}{w^{\alpha-k-1}}\right). \end{aligned}$$

What remains is to investigate the behaviour of the sum in (74) for the remaining k. Since $A_1, A_2 \in (0,1)$, then

$$\begin{aligned}
B_k^- &\leq \sum_{v=1}^{[w]} v^k \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}} \sum_{j\leq k} \theta_j(\alpha) \\
&\leq 2\sum_{v=1}^{[w]} v^k \sum_{x+y=v} \frac{1}{(x+1)^{\alpha+1}} \sum_{\substack{y=a_1+\ldots+a_{r-2} \\ a_{r-2}\geq 2}} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{\alpha+2}} \sum_{j\leq k} \theta_j(\alpha) \\
&\leq 2C_1' \sum_{v=1}^{[w]} v^k \sum_{x+y=v} \frac{1}{x^{\alpha+1}} \frac{1}{y^{\alpha+2}} \sum_{j\leq k} \theta_j(\alpha) \\
&\leq 2C_1' \sum_{j\leq k} \theta_j(\alpha) \int_1^w \frac{dv}{v^{\alpha-k}} \qquad (75)\\
&\leq 2C_1' \sum_{j\leq k} \theta_j(\alpha) w^{-(\alpha-k-1)},
\end{aligned}$$

using the formula $\sum_{n\geq x} \frac{1}{n^s} = \mathcal{O}\left(\frac{1}{x^{s-1}}\right)$, $(n \to \infty)$ (see for example, [1]). Therefore, when α is not integer-valued one has, for the remaining values of k (reintroducing the factor $n^{-(\alpha+2)}$ omitted for brevity earlier):

$$\begin{aligned}
\frac{1}{n^{\alpha+2}} \sum_{k>\alpha-1} \frac{B_k^-}{n^k} &\leq \frac{2C_1'}{n^{\alpha+2}} \sum_{k>\alpha-1} \frac{1}{n^k} \frac{1}{w^{\alpha-k-1}} \sum_{j\leq k} \theta_j(\alpha) \\
&\leq \frac{2C_1'}{n^{\alpha+2} w^{\alpha-1}} \sum_{k=1}^{\infty} \left(\frac{w}{n}\right)^k \sum_{j\leq k} \theta_j(\alpha).
\end{aligned}$$

Now one may apply Lemma 1.4 since $\frac{w}{n} \leq \frac{1}{2}$ to see that

$$\begin{aligned}
\frac{1}{n^{\alpha+2}} \sum_{k>\alpha-1} \frac{B_k^-}{n^k} &\leq \frac{2C_1'}{n^{\alpha+2} w^{\alpha-1}} \left(\frac{1}{\left(1-\frac{w}{n}\right)^{\alpha+2}} - 1 + \frac{\frac{w}{n}}{\frac{w}{n}-1} \right) \\
&\leq \frac{2C_1'}{n^{\alpha+2} w^{\alpha-1}} (2^{\alpha+2}-2).
\end{aligned}$$

Hence it follows that $\frac{1}{n^{\alpha+2}} \sum_{k>\alpha-1} \frac{B_k^-}{n^k} = \mathcal{O}\left(\frac{1}{n^{\alpha+2} w^{\alpha-1}}\right)$. In the case where $\alpha \in \mathbb{N} \setminus \{1\}$, then (75) implies that, in the case $k = \alpha - 1$

$$B_k^- = \mathcal{O}(\log(w)).$$

Thus giving the larger error term $\dfrac{1}{n^{\alpha+2}}\sum_{k\geq\alpha-1}\dfrac{B_k^-}{n^k}=\mathcal{O}\left(\dfrac{\log(n)}{n^{2\alpha+1}}\right)$. To give the stated result, define the constant T_α (which occurs in the limit as $n\to\infty$) as

$$T_\alpha=\sum_{v=1}^{\infty}\sum_{\Omega_v}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}},$$

and step forward the coefficients in n to return to level N of the Farey Tree. □

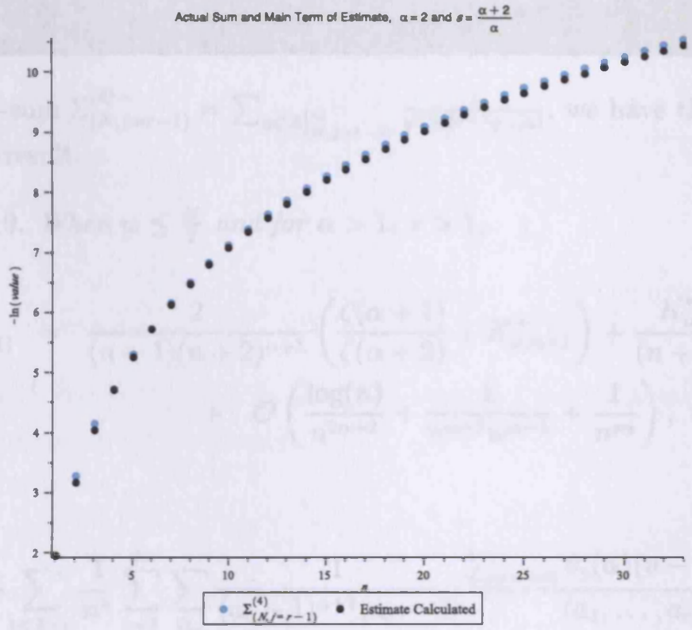
5.4.3. *Numerical Results.* Consider again the effect of the chosen value of parameter s. When $s=\frac{\alpha+2}{\alpha}$, Lemma 3.18 becomes:

$$\Sigma^{(4)+}_{(N,j=r-1)}=\frac{2}{(n+2)^{\alpha+1}}\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}+\mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right).$$

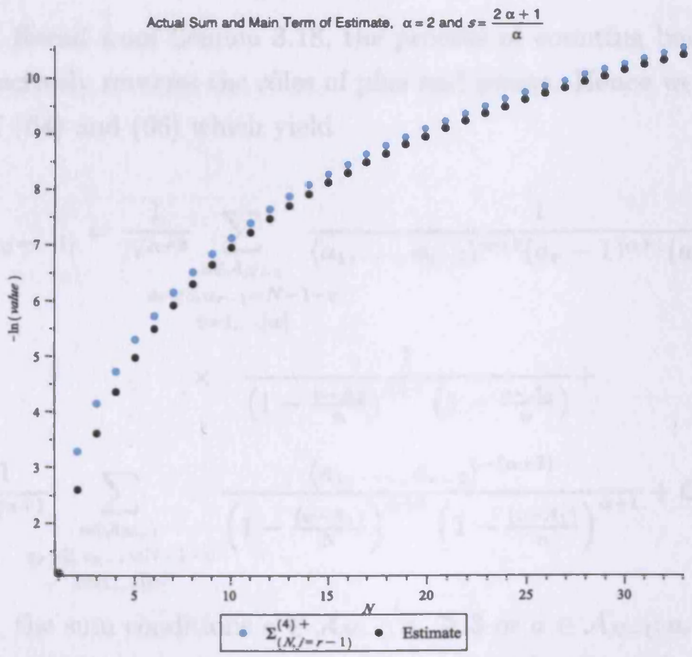
The main term of this, along with the actual value of $\Sigma^{(4)+}_{(N,j=r-1)}$ is plotted in figure 3.17(a). Conversely, with $s=\frac{2\alpha+1}{\alpha}$ one has the full main term seen in the Lemma, and for $\alpha=2$ this is:

$$\frac{2}{(n+2)^3}\frac{\zeta(3)}{\zeta(4)}+\frac{T_2}{(n+1)^4}.$$

Figure 3.17(b) plots the quantity, along with the true value of $\Sigma^{(4)+}_{(N,j=r-1)}$ using the inequality (71) and the main term of Theorem 3.3 on the sum $\displaystyle\sum_{\substack{y=a_1+\ldots+a_{r-2}\\ a_{r-2}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}}$.



(a) $\Sigma^{(4)+}_{(N,j=r-1)}$ with Main Term, $s = \frac{\alpha+2}{\alpha}$



(b) $\Sigma^{(4)+}_{(N,j=r-1)}$ with Main Term, $s = \frac{2\alpha+2}{\alpha}$

FIGURE 3.17.

5.4.4. *For* $\Sigma^{(4)-}_{(N,j=r-1)}$.

For the sub-sum $\Sigma^{(4)-}_{(N,j=r-1)} = \sum_{a\in\mathcal{A}^{(4)}_{(N,j=r-1)}} \frac{1}{q''(a)^{\alpha+1}q_-(a)}$, we have the following asymptotic result.

LEMMA 3.19. *When* $w \leq \frac{N}{2}$ *and for* $\alpha > 1$, $s > 1$,

$$\Sigma^{(4)-}_{(N,j=r-1)} = \frac{2}{(n+1)(n+2)^{\alpha+1}}\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + K^{+}_{\alpha,n+1}\right) + \frac{K^{*+}_{\alpha,n+1}}{(n+1)^{\alpha+2}} + \\ + \mathcal{O}\left(\frac{\log(n)}{n^{2\alpha+2}} + \frac{1}{n^{\alpha+3}w^{\alpha-1}} + \frac{1}{n^{s\alpha}}\right),$$

where

$$K^{*+}_{\alpha,n} = U_\alpha + \sum_{1\leq k<\alpha}\frac{1}{n^k}\sum_{v=1}^{\infty}\sum_{\Omega_v}\frac{1}{(a_r-1)^{\alpha+1}(a_r-2)}\frac{\sum_{j+l=k}\theta_j(\alpha)(v-A_2)^j(v-A_3)^l}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}},$$

and U_α *is a constant.*

PROOF. Recall from Lemma 3.18, the process of counting back a level of the Tree effectively reverses the rôles of plus and minus. Hence we require the identities of (64) and (66) which yield

$$\Sigma^{(4)+}_{(N-1,j=r-1)} = \frac{1}{N^{\alpha+2}}\sum_{\substack{a\in\mathcal{A}_{N-1}\\ a_r\geq 3,\, a_{r-1}=N-1-v\\ v=1,\ldots,[w]}}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}(a_r-2)} \\ \times\frac{1}{(1-\frac{v-A_2}{n})^{\alpha+1}(1-\frac{v-A_3}{n})}+$$

$$(76)\quad + \frac{1}{nN^{\alpha+1}}\sum_{\substack{a\in\mathcal{A}_{N-1}\\ a_r=2,\, a_{r-1}=N-1-v\\ v=1,\ldots,[w]}}\frac{\langle a_1,\ldots,a_{r-2}\rangle^{-(\alpha+2)}}{\left(1-\frac{(v-A_1)}{N}\right)^{\alpha+1}\left(1-\frac{(v-A_1)}{n}\right)^{\alpha+1}} + \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right).$$

Once again, the sum conditions $a\in\mathcal{A}_{N-1}; a_r \geq 3$ or $a\in\mathcal{A}_{N-1}; a_r = 2$ should be interpreted as the criteria that one will attain such values of a_r when performing the appropriate algorithm on the the members of $\mathcal{A}_{N-1}$ to derive set $\mathcal{A}_N$.

Let us first concentrate on the series in (76), for which one has

(77)
$$\frac{1}{nN^{\alpha+1}}\sum_{\substack{a\in A_{N-1}\\ a_r=2,a_{r-1}=N-1-v\\ v=1,\dots,[w]}}\frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{\alpha+2}}\frac{1}{\left(1-\frac{(v-A_1)}{N}\right)^{\alpha+1}\left(1-\frac{(v-A_1)}{n}\right)^{\alpha+1}}=$$
$$=\frac{2}{nN^{\alpha+1}}\sum_{v=1}^{[w]-1}\sum_{\substack{a_1+\ldots+a_{r-2}=v\\ a_{r-2}\geq 2}}\frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{\alpha+2}}\frac{1}{\left(1-\frac{(v-A_1)}{N}\right)^{\alpha+1}\left(1-\frac{(v-A_1)}{n}\right)^{\alpha+1}}.$$

In similar fashion to that proved at line (69) - one may deduce from Lemma 3.16 that the quantity at (77) is simply

$$\frac{2}{n(n+1)^{\alpha+1}}\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}+K^{+}_{\alpha,n}\right)+\mathcal{O}\left(\frac{1}{n^{s\alpha}}+\frac{\log(n)}{n^{2\alpha+3}}+\frac{1}{n^{\alpha+2}w^{\alpha+1}}\right).$$

Now let us turn our attention to the summation with coefficient $\frac{1}{N^{\alpha+2}}$. This is

$$B_2:=\sum_{v=1}^{[w]}\sum_{\Omega_v}\frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}(a_r-2)}\frac{1}{\left(1-\frac{v-A_2}{n}\right)^{\alpha+1}\left(1-\frac{v-A_3}{n}\right)}.$$

Using Lemma 1.3 we may expand the latter quantities into Taylor series as:

$$\frac{1}{\left(1-\frac{v-A_2}{n}\right)^{\alpha+1}\left(1-\frac{v-A_3}{n}\right)}=1+\sum_{k=1}^{\infty}\frac{1}{n^k}\sum_{j+l=k}\theta_j(a)\,(v-A_2)^j\,(v-A_3)^l.$$

Therefore, one sees that

$$(78)\; B_2 = \left(\sum_{v=1}^{\infty}-\sum_{v=[w]}^{\infty}\right)\sum_{\Omega_v}\frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}(a_r-2)}+$$
$$+\sum_{k=1}^{\infty}\frac{1}{n^k}\sum_{v=1}^{[w]}\sum_{\Omega_v}\frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}(a_r-2)}\times$$
$$\times\sum_{j+l=k}\theta_j(a)\,(v-A_2)^j\,(v-A_3)^l.$$

The sum of (78) is finite and has been expressed in this manner in order to explain its estimation. Recall that the definition of Ω_v requires $a_r\geq 3$. Therefore $x(x+1)^{\alpha+1}$ for $x\geq 1$ displays the same behaviour as $(a_r-2)(a_r-1)^{\alpha+1}$ for $a_r\geq 3$, and the first component of the series has the property that:

$$
\begin{aligned}
&\sum_{v=1}^{\infty}\sum_{\Omega_v}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}(a_r-2)}\\
&\le 2\sum_{v=1}^{\infty}\sum_{x+y=v}\frac{1}{x(x+1)^{\alpha+1}}\sum_{\substack{y=a_1+\ldots+a_{r-2}\\ a_{r-2}\ge 2}}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}} \qquad (79)\\
&\le 2C_1'\sum_{v=2}^{\infty}\sum_{x+y=v}\frac{1}{x^{\alpha+2}y^{\alpha+2}}\\
&\le 2C_1'\sum_{v=2}^{\infty}\frac{1}{v^{\alpha+1}}=2C_1'(\zeta(\alpha+1)-1).
\end{aligned}
$$

Likewise the second component series of (78) has asymptotic value $\mathcal{O}(w^{-\alpha})$ since:

$$
\begin{aligned}
\sum_{v=[w]}^{\infty}\sum_{\Omega_v}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}(a_r-2)} &\ll \sum_{v=[w]}^{\infty}\sum_{x+y=v}\frac{1}{x^{\alpha+2}}\frac{1}{y^{\alpha+2}}\\
&\ll \sum_{v=[w]}^{\infty}\frac{1}{v^{\alpha+1}}. \qquad (80)
\end{aligned}
$$

Now, let us consider the latter summation of B_2, which will be represented as

$$\sum_{k=1}^{\infty}\frac{B_k^+}{n^k}, \tag{81}$$

where

$$
\begin{aligned}
B_k^+ &:= \left(\sum_{v=1}^{\infty}-\sum_{v=[w]}^{\infty}\right)\sum_{\Omega_v}\frac{\sum_{j+l=k}\theta_j(\alpha)\,(v-A_2)^j\,(v-A_3)^l}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}(a_r-2)}\\
&= B_{k,1}^+-B_{k,w}^+.
\end{aligned}
$$

Now, for fixed k and α, one sees that the quantity $B_{k,1}^+$ converges for $\alpha+1-k>1\Rightarrow k<\alpha$, since:

$$B_{k,1}^{+} \ll \sum_{v=1}^{\infty} v^k \sum_{\Omega_v} \frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}(a_r-2)}$$

$$(82) \qquad \ll \sum_{v=1}^{\infty} v^k \sum_{x+y=v} \frac{1}{x(x+1)^{\alpha+1}} \sum_{\substack{y=a_1+\ldots+a_{r-2}\\ a_{r-2}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}}$$

$$(83) \qquad \ll \sum_{v=1}^{\infty} v^k \frac{1}{v^{\alpha+1}}.$$

The asymptotic at line (83) arises since the internal sums at line (82) have value $\mathcal{O}\left(\frac{1}{x^{\alpha+2}}\right)$ and $\mathcal{O}\left(\frac{1}{y^{\alpha+2}}\right)$ respectively. Now, with the stated k, the second component $B_{k,w}^{+}$ has the property that

$$\begin{aligned} B_{k,w}^{+} &\ll \sum_{v=[w]}^{\infty} v^k \sum_{\Omega_v} \frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}(a_r-2)} \\ &\ll \sum_{v=[w]}^{\infty} \frac{v^k}{v^{\alpha+1}} \quad \text{similarly to above,} \\ &= \mathcal{O}\left(\frac{1}{w^{\alpha-k}}\right). \end{aligned}$$

What remains is to investigate the behaviour of the sum in (81) for the remaining k. Since $A_2, A_3 \in (0,1)$, then

$$\begin{aligned} B_k^{+} &\leq \sum_{v=1}^{[w]} v^k \sum_{\Omega_v} \frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}(a_r-2)} \sum_{j\leq k} \theta_j(\alpha) \\ &\leq 2\sum_{v=1}^{[w]} v^k \sum_{x+y=v} \frac{1}{x(x+1)^{\alpha+1}} \sum_{\substack{y=a_1+\ldots+a_{r-2}\\ a_{r-2}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}} \sum_{j\leq k} \theta_j(\alpha) \\ &\leq 2C_1' \sum_{v=1}^{[w]} v^k \sum_{x+y=v} \frac{1}{x^{\alpha+2}} \frac{1}{y^{\alpha+2}} \sum_{j\leq k} \theta_j(\alpha) \end{aligned}$$

$$(84) \qquad \leq 2C_1' \sum_{j\leq k} \theta_j(\alpha) \int_1^w \frac{dv}{v^{\alpha+1-k}}$$

$$\leq 2C_1' \sum_{j\leq k} \theta_j(\alpha) w^{-(\alpha-k)}.$$

Therefore, when α is not integer-valued one has, for the remaining values of k (reintroducing the factor $n^{-(\alpha+2)}$):

$$\begin{aligned}\frac{1}{n^{\alpha+2}}\sum_{k>\alpha}\frac{B_k^+}{n^k} &\leq \frac{2C_1'}{n^{\alpha+2}}\sum_{k>\alpha}\frac{1}{n^k}\frac{1}{w^{\alpha-k}}\sum_{j\leq k}\theta_j(\alpha)\\ &\leq \frac{2C_1'}{n^{\alpha+2}w^\alpha}\sum_{k=1}^{\infty}\left(\frac{w}{n}\right)^k\sum_{j\leq k}\theta_j(\alpha).\end{aligned}$$

Now one may apply Lemma 1.4 since $\frac{w}{n}\leq\frac{1}{2}$ to see that

$$\begin{aligned}\frac{1}{n^{\alpha+2}}\sum_{k>\alpha}\frac{B_k^+}{n^k} &\leq \frac{2C_1'}{n^{\alpha+2}w^\alpha}\left(\frac{1}{(1-\frac{w}{n})^{\alpha+2}}-1+\frac{\frac{w}{n}}{\frac{w}{n}-1}\right)\\ &\leq \frac{2C_1'}{n^{\alpha+2}w^\alpha}(2^{\alpha+2}-2).\end{aligned}$$

Hence it follows that $\frac{1}{n^{\alpha+2}}\sum_{k>\alpha}\frac{B_k^+}{n^k}=\mathcal{O}\left(\frac{1}{n^{\alpha+2}w^\alpha}\right)$. In the case where $\alpha\in\mathbb{N}\setminus\{1\}$, then (84) implies that, in the case $k=\alpha$

$$B_k^+=\mathcal{O}(\log(w)).$$

Thus giving the larger error term $\frac{1}{n^{\alpha+2}}\sum_{k\geq\alpha}\frac{B_k^+}{n^k}=\mathcal{O}\left(\frac{\log(n)}{n^{2\alpha+2}}\right)$. To give the stated result, define constant U_α (which occurs in the limit as $n\to\infty$) as

$$U_\alpha=\sum_{v=1}^{\infty}\sum_{\Omega_v}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}(a_r-2)},$$

and step forward the coefficients in n to return to level N of the Farey Tree.

□

5.4.5. *Numerical Results.* Consider again the effect of the chosen value of parameter s. When $s=\frac{\alpha+3}{\alpha}$, Lemma 3.19 becomes:

$$\Sigma^{(4)-}_{(N,j=r-1)}=\frac{2}{(n+1)(n+2)^{\alpha+1}}\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}+\frac{U_\alpha}{(n+1)^{\alpha+2}}+\mathcal{O}\left(\frac{1}{n^{\alpha+3}}\right).$$

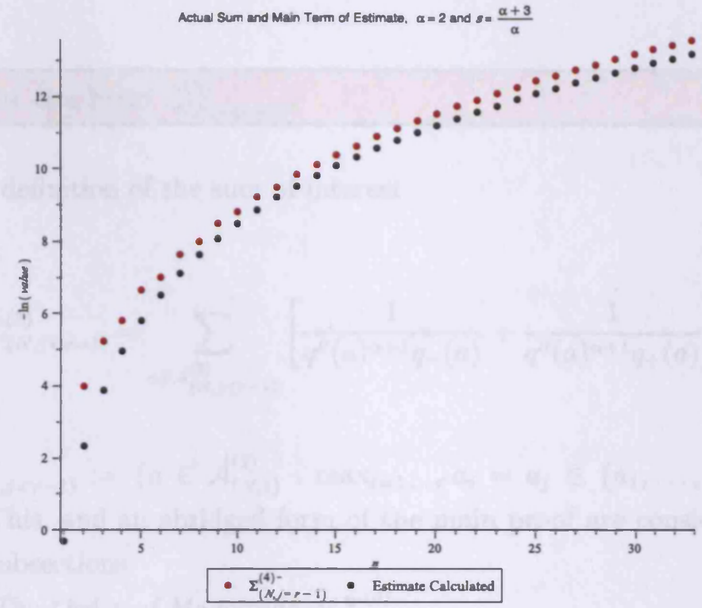
The main term of this, along with the actual value of $\Sigma^{(4)-}_{(N,j=r-1)}$ is plotted in figure 3.18(a). Conversely, with $s=\frac{2\alpha+2}{\alpha}$ one has the full main term seen in the Lemma, and for $\alpha=2$ this is:

$$\frac{2}{(n+1)(n+2)^3}\left(\frac{\zeta(3)}{\zeta(4)}+K^{+}_{2,n+1}\right)+\frac{1}{n^4}(U_2+K^{*+}_{2,n}).$$

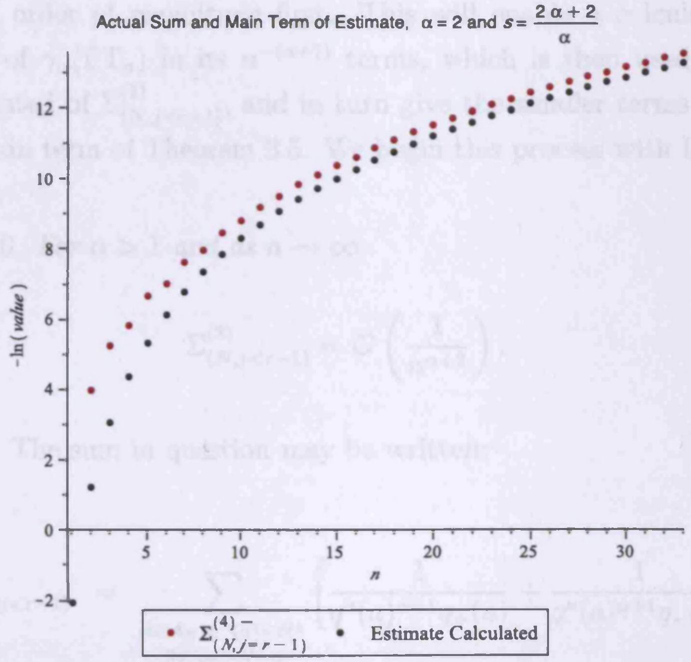
Figure 3.18(b) plots the quantity, along with the true value of $\Sigma^{(4)-}_{(N,j=r-1)}$ using the main term from the estimate:

$$\begin{aligned}
K^{*+}_{\alpha,n} &\le \sum_{1\le k<\alpha}\frac{1}{n^k}\sum_{v=1}^{\infty}v^k\sum_{j\le k}\theta_k(\alpha)\sum_{\Omega_v}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}(a_r-1)^{\alpha+1}(a_r-2)}\\
&\le \sum_{1\le k<\alpha}\frac{1}{n^k}\sum_{v=1}^{\infty}v^k\sum_{j\le k}\theta_k(\alpha)\sum_{x+y=v}\frac{1}{x(x+1)^{\alpha+1}}\sum_{\substack{y=a_1+\ldots+a_{r-2}\\ a_{r-2}\ge 2}}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}}\\
&\le \sum_{1\le k<\alpha}\frac{1}{n^k}\sum_{v=2}^{\infty}v^k\sum_{j\le k}\theta_k(\alpha)\sum_{x+y=v}\frac{1}{x^{\alpha+2}}\left(\frac{C_0'}{y^{\alpha+2}}+\mathcal{O}\left(\frac{\log^{\frac{2\alpha+4}{2\alpha+5}}(y)}{y^{\alpha+3-\frac{\alpha+5}{2\alpha+5}}}\right)\right)
\end{aligned}$$

One should note that these figures are accompanied with the caveat of having to estimate the more complicated items in the formula: U_α and $K^{*+}_{\alpha,n}$. They do still however provide useful guidelines as to the performance of the overall estimate.



(a) $\Sigma^{(4)-}_{(N,j=r-1)}$ with Main Term, $s = \frac{\alpha+2}{\alpha}$



(b) $\Sigma^{(4)-}_{(N,j=r-1)}$ with Main Term, $s = \frac{2\alpha+2}{\alpha}$

FIGURE 3.18.

5.5. On the Sum $\Sigma^{(3)}_{(N,j<r-1)}$.

Recall the definition of the sum of interest

$$\Sigma^{(3)}_{(N,j<r-1)} = \sum_{a\in\mathcal{A}^{(3)}_{(N,j<r-1)}} \left[\frac{1}{q''(a)^{\alpha+1}q_-(a)} + \frac{1}{q''(a)^{\alpha+1}q_+(a)}\right], \tag{85}$$

where $\mathcal{A}^{(3)}_{(N,j<r-1)} := \{a \in \mathcal{A}^{(2)}_{(N,1)} : \max_{i=1,\ldots,r} a_i = a_j \in \{a_1,\ldots,a_{r-2}\},\ a_j > N-w\}$. This, and an abridged form of the main proof are considered in the following subsections.

5.5.1. *The Order of Magnitude of* $\Sigma^{(3)}_{(N,j<r-1)}$.

To proceed to the full calculation of asymptotic behaviour of $\Sigma^{(3)}_{(N,j<r-1)}$ requires proof of its order of magnitude first. This will enable a calculation of the asymptotic of $\gamma_\alpha(\mathrm{FT}_n)$ in its $n^{-(\alpha+1)}$ terms, which is then used to improve those calculated of $\Sigma^{(3)}_{(N,j<r-1)}$, and in turn give the smaller terms intended to form the main term of Theorem 3.5. We begin this process with Lemma 3.20.

LEMMA 3.20. *For* $\alpha > 1$ *and as* $n \to \infty$

$$\Sigma^{(3)}_{(N,j<r-1)} = \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right).$$

PROOF. The sum in question may be written:

$$\begin{aligned}\Sigma^{(3)}_{(N,j<r-1)} &= \sum_{\substack{a\in\mathcal{A}_N,\, q''(a)<N^s\\ \exists! j:\, a_j>N-w\\ for\ j\le r-2}} \left[\frac{1}{q''(a)^{\alpha+1}q_+(a)} + \frac{1}{q''(a)^{\alpha+1}q_-(a)}\right]\\ &= \Sigma^{(3)+}_{(N,j<r-1)} + \Sigma^{(3)-}_{(N,j<r-1)},\end{aligned}$$

for which we have:

$$\Sigma^{(3)+}_{(N,j<r-1)} = \left(\sum_{\substack{a\in\mathcal{A}_N \\ \exists! j: a_j>N-w \\ for\, j\le r-2}} - \sum_{\substack{a\in\mathcal{A}_N, q''(a)\ge N^s \\ \exists! j: a_j>N-w \\ for\, j\le r-2}} \right) \frac{1}{q''(a)^{\alpha+1} q_+(a)}$$
$$= \sum_{\substack{a\in\mathcal{A}_N \\ \exists! j: a_j>N-w \\ for\, j\le r-2}} \frac{1}{q''(a)^{\alpha+1} q_+(a)} + \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right),$$

and similarly for $\Sigma^{(3)-}_{(N,j<r-1)}$ using Lemma 3.14 to give the above error term.

We proceed in a familiar fashion, setting $a_j = N - v$ for $v = 1, \ldots, [w]$ and use identity (63) on the continuants. For ease of notation let

$$\begin{aligned} E_1 &= [a_{j-1}, \ldots, a_1] + [a_{j+1}, \ldots, a_r] \\ E_2 &= [a_{j-1}, \ldots, a_1] + [a_{j+1}, \ldots, a_{r-1}] \\ E_3 &= [a_{j-1}, \ldots, a_1] + [a_{j+1}, \ldots, a_r - 1], \end{aligned}$$

meaning that

$$\frac{1}{q''(a)^{\alpha+1} q_-(a)} = \frac{1}{N^{\alpha+2} \langle a_1, \ldots, a_{j-1}\rangle^{\alpha+2} \langle a_{j+1}, \ldots, a_r\rangle^{\alpha+1} \langle a_{j+1}, \ldots, a_{r-1}\rangle} \times \tag{86}$$
$$\times \frac{1}{\left(1 - \frac{v-E_1}{N}\right)^{\alpha+1} \left(1 - \frac{v-E_2}{N}\right)}$$

$$\frac{1}{q''(a)^{\alpha+1} q_+(a)} = \frac{1}{N^{\alpha+2} \langle a_1, \ldots, a_{j-1}\rangle^{\alpha+2} \langle a_{j+1}, \ldots, a_r\rangle^{\alpha+1} \langle a_{j+1}, \ldots, a_r - 1\rangle} \times \tag{87}$$
$$\times \frac{1}{\left(1 - \frac{v-E_1}{N}\right)^{\alpha+1} \left(1 - \frac{v-E_3}{N}\right)}.$$

The familiar Taylor expansions about $\frac{v-E_1}{N}$ and $\frac{v-E_3}{N}$ are yielded, and once again $\theta_j(\alpha) = \frac{\prod_{i=1}^{j}(\alpha+i)}{j!}$. An identical result is generated when considered with $\frac{v-E_2}{N}$ present in the characteristic with the 'plus' denominator. Due to this, calculation on both items is very similar and we shall discuss both concurrently. For ease of notation in what follows, let Θ_v denote the following set:

$$\Theta_v := \left\{ \bar{a} = (a_1, \ldots, a_j, \ldots, a_r),\ a_r \geq 2 : \sum_{\substack{i=1\\ i\neq j}}^{r} a_i = v \right\}.$$

Now, for the '−' item one has, for fixed k and $\alpha > 1$

$$\begin{aligned}
\Sigma^{(3)-}_{(N,j<r-1)} &= \sum_{v=1}^{[w]} \sum_{\substack{a\in\mathcal{A}_N\\ a_1+\ldots+a_{j-1}+a_{j+1}+\ldots+a_r=v}} \frac{1}{q''(a)^{\alpha+1}q_-(a)} + \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right)\\
&= \frac{1}{N^{\alpha+2}} \sum_{v=1}^{[w]} \sum_{\Theta_v} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+1}\langle a_{j+1},\ldots,a_{r-1}\rangle} +\\
&\quad + \sum_{k=1}^{\infty} \frac{1}{N^{\alpha+2+k}} \sum_{v=1}^{[w]} \sum_{\Theta_v} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+1}\langle a_{j+1},\ldots,a_{r-1}\rangle}\\
&\qquad \times \sum_{j+l=k} \theta_j(\alpha)(v-E_1)^j(v-E_3)^l + \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right). \qquad (88)
\end{aligned}$$

The first summation in this expansion is finite and hence has order at most $\mathcal{O}\left(\frac{1}{N^{\alpha+2}}\right)$. Again, fix k and α. The summation in v at (88) is finite and therefore:

$$\begin{aligned}
&\frac{1}{N^{\alpha+2+k}} \sum_{v=1}^{[w]} \sum_{\Theta_v} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+1}\langle a_{j+1},\ldots,a_{r-1}\rangle} \times\\
&\qquad \times \sum_{j+l=k} \theta_l(\alpha)(v-E_1)^l(v-E_3)^j\\
\ll\ & \frac{1}{N^{\alpha+2+k}} \sum_{v=1}^{[w]} v^k \sum_{\Theta_v} \frac{\sum_{j\leq k}\theta_j(\alpha)}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+1}\langle a_{j+1},\ldots,a_{r-1}\rangle}\\
\ll\ & \frac{w^k}{N^{\alpha+2+k}}.
\end{aligned}$$

This means that the series takes asymptotic value $\mathcal{O}\left(\frac{w}{N^{\alpha+3}}\right)$. Moreover, the analogous quantity $\Sigma^{(3)+}_{(N,j<r-1)}$ produces a similar approximation since its expansion into a Taylor series is asymptotically identical to that of $\Sigma^{(3)-}_{(N,j<r-1)}$. Recall that $w \leq \frac{N}{2}$ which implies that, at most

$$\Sigma^{(3)}_{(N,j<r-1)} = \mathcal{O}\left(\frac{1}{N^{s\alpha}} + \frac{1}{N^{\alpha+2}}\right).$$

The prescribed value of s gives the required result. □

5.5.2. *Proof of Theorem 3.4.* Recall the partition of interest

$$\begin{aligned}\gamma_\alpha(\mathrm{FT}_n) &= \frac{1}{2}\Big(\Sigma^{(1)}_{(N,2)} + \Sigma^{(2)}_{(N,2)} + \Sigma^{(3)}_{(N,j<r-1)} + \Sigma^{(4)+}_{(N,j=r)} + \\ &+ \Sigma^{(4)-}_{(N,j=r)} + \Sigma^{(4)+}_{(N,j=r-1)} + \Sigma^{(4)-}_{(N,j=r-1)}\Big).\end{aligned}$$

It is intended that this partition is rebuilt using the terms of order $n^{-(\alpha+1)}$ only, with all those of lesser magnitude considered as error. To that end, the constituents are:

$$\Sigma^{(1)}_{(N,2)} = \mathcal{O}\left(\frac{1}{n^{s\alpha}}\right);$$

$$\Sigma^{(2)}_{(N,2)} = \mathcal{O}\left(\frac{n^2\log^{2\alpha+3}(n)}{w^{2\alpha+3}}\right);$$

$$\Sigma^{(3)}_{(N,j<r-1)} = \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right);$$

$$\Sigma^{(4)+}_{(N,j=r)} = \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right);$$

$$\Sigma^{(4)-}_{(N,j=r-1)} = \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right);$$

$$\Sigma^{(4)-}_{(N,j=r)} = \frac{2}{(n+1)^{\alpha+1}}\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + \mathcal{O}\left(\frac{1}{n^{\alpha+2}} + \frac{1}{n^{s\alpha}} + \frac{\log(n)}{n^{2\alpha+2}} + \frac{1}{n^{\alpha+1}w^{\alpha+1}}\right);$$

$$\Sigma^{(4)+}_{(N,j=r-1)} = \frac{2}{(n+2)^{\alpha+1}}\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + \mathcal{O}\left(\frac{1}{n^{\alpha+2}} + \frac{1}{n^{s\alpha}} + \frac{\log(n)}{n^{2\alpha+1}} + \frac{1}{n^{\alpha+1}w^{\alpha+1}}\right).$$

Thus $\gamma_\alpha(\mathrm{FT}_n)$ reforms as

$$\begin{aligned}\gamma_\alpha(\mathrm{FT}_n) &= \left(\frac{1}{(n+1)^{\alpha+1}} + \frac{1}{(n+2)^{\alpha+1}}\right)\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + \\ &+ \mathcal{O}\left(\frac{1}{n^{\alpha+2}} + \frac{1}{n^{s\alpha}} + \frac{\log(n)}{n^{2\alpha+1}} + \frac{1}{n^{\alpha+1}w^{\alpha+1}} + \frac{n^2\log^{2\alpha+3}(n)}{w^{2\alpha+3}}\right).\end{aligned}$$

Clearly, the largest item of error independent of the parameter w is $\frac{1}{n^{\alpha+2}}$. Therefore, setting

$$\begin{aligned}
\Sigma^{(3)+}_{(N,j<r-1)} &= \frac{1}{N^{\alpha+2}}\sum_{v=1}^{[w]}\sum_{\Theta_v}\frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle}\times \\
&\times\left(1+\sum_{k=1}^{\infty}\frac{1}{N^k}\sum_{j+l=k}\theta_j(\alpha)(v-E_1)^j(v-E_3)^l\right)+\mathcal{O}\left(\frac{1}{N^{s\alpha}}\right) \qquad (90)\\
&= \frac{1}{N^{\alpha+2}}\sum_{v=1}^{[w]}\sum_{\Theta_v}\frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle}+ \\
&+\sum_{k=1}^{\infty}\sum_{v=1}^{[w]}\sum_{\Theta_v}\frac{N^{-(k+\alpha+2)}}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle}\times \\
&\times\sum_{j+l=k}\theta_j(\alpha)(v-E_1)^j(v-E_3)^l+\mathcal{O}\left(\frac{1}{N^{s\alpha}}\right).
\end{aligned}$$

The first series in (90) is finite, and will be written as

$$\left(\sum_{v=1}^{\infty}-\sum_{v=[w]}^{\infty}\right)\sum_{\Theta_v}\frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle}, \qquad (91)$$

in order to calculate its estimate. The first of these has the property that

$$\begin{aligned}
&\sum_{v=1}^{\infty}\sum_{\Theta_v}\frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle}\\
&\leq 2\sum_{v=1}^{\infty}\sum_{x+y=v}\sum_{\substack{x=a_1+\dots+a_{j-1}\\ a_{j-1}\geq 2}}\frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}}\times\\
&\qquad\times\sum_{\substack{y=a_{j+1}+\dots+a_r\\ a_r\geq 2}}\frac{1}{\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle}.
\end{aligned}$$

Again, Theorem 3.3 implies $\displaystyle\sum_{\substack{x=a_1+\dots+a_{j-1}\\ a_{j-1}\geq 2}}\frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}}\leq\frac{C_1'}{x^{\alpha+2}}$ for constant C_1' and since

$$\langle a_1,\dots,a_r-1\rangle\leq\langle a_1,\dots,a_r\rangle$$

then $\displaystyle\frac{1}{\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle}\leq\frac{1}{\langle a_{j+1},\dots,a_r-1\rangle^{\alpha+2}}$.

Now, let $a_r' = a_r - 1$ and make the constraint $a_r' \geq 2$, yielding

$$2\sum_{v=1}^{\infty}\sum_{x+y=v}\sum_{\substack{x=a_1+\ldots+a_{j-1}\\a_{j-1}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}}\sum_{\substack{y=a_{j+1}+\ldots+a_r\\a_r\geq 2}}\frac{(\langle a_{j+1},\ldots,a_r-1\rangle)^{-1}}{\langle a_{j+1},\ldots,a_r\rangle^{\alpha+1}}$$

$$(92)\leq \quad 4\sum_{v=1}^{\infty}\sum_{x+y=v}\sum_{\substack{x=a_1+\ldots+a_{j-1}\\a_{j-1}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}}\sum_{\substack{y=a_{j+1}+\ldots+a_r'\\a_r'\geq 2}}\frac{1}{\langle a_{j+1},\ldots,a_r'\rangle^{\alpha+2}}$$

$$\leq \quad 4C_1'^2\sum_{v=2}^{\infty}\sum_{x+y=v}\frac{1}{x^{\alpha+2}}\frac{1}{y^{\alpha+2}}$$

$$\leq \quad 4C_1'^2\sum_{v=2}^{\infty}\frac{1}{v^{\alpha+1}} = 4C_1'^2(\zeta(\alpha+1)-1).$$

Likewise the second component series of (91) has asymptotic value $\mathcal{O}(w^{-\alpha})$ since:

$$\sum_{v=[w]}^{\infty}\sum_{\Theta_v}\frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}}\sum_{\substack{y=a_{j+1}+\ldots+a_r\\a_r\geq 2}}\frac{1}{\langle a_{j+1},\ldots,a_r\rangle^{\alpha+1}\langle a_{j+1},\ldots,a_r-1\rangle}$$

$$\ll \quad \sum_{v=[w]}^{\infty}\sum_{x+y=v}\sum_{\substack{x=a_1+\ldots+a_{j-1}\\a_{j-1}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}}\sum_{\substack{y=a_{j+1}+\ldots+a_r'\\a_r'\geq 2}}\frac{1}{\langle a_{j+1},\ldots,a_r'\rangle^{\alpha+2}}$$

$$\ll \quad \sum_{v=[w]}^{\infty}\sum_{x+y=v}\frac{1}{x^{\alpha+2}}\frac{1}{y^{\alpha+2}}$$

$$\ll \quad \sum_{v=[w]}^{\infty}\frac{1}{v^{\alpha+1}}.$$

Now let us turn our attention to the summation at (90) for $k = 1, \ldots, \infty$. This will be denoted

$$(93) \qquad \sum_{k=1}^{\infty}\frac{D_k^+}{N^k},$$

where

$$\begin{aligned} D_k^+ &:= \left(\sum_{v=1}^{\infty} - \sum_{v=[w]}^{\infty}\right) \sum_{\Theta_v} \frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle} \times \\ &\quad \times \sum_{j+l=k} \theta_j(\alpha)\,(v-E_1)^j\,(v-E_3)^l \\ &= D_{k,1}^+ - D_{k,w}^+. \end{aligned}$$

Now fix a natural k and $\alpha > 1$. It has been established that

$$\sum_{\substack{y=b_1+\dots+b_\varepsilon \\ b_\varepsilon \geq 2}} \frac{1}{\langle b_1,\dots,b_\varepsilon\rangle^{\alpha+1}\langle b_1,\dots,b_\varepsilon-1\rangle} = \mathcal{O}\left(\frac{1}{y^{\alpha+2}}\right),$$

and thus the quantity $D_{k,1}^+$ converges for $k < \alpha$ since

$$\begin{aligned} D_{k,1}^+ &\ll \sum_{v=1}^{\infty} v^k \sum_{\Theta_v} \frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r-1\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle} \\ &\ll \sum_{v=1}^{\infty} v^k \sum_{x+y=v} \sum_{\substack{x=a_1+\dots+a_{j-1} \\ a_{j-1}\geq 2}} \frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}} \\ &\qquad\qquad \times \sum_{\substack{y=a_{j+1}+\dots+a_r \\ a_r\geq 2}} \frac{1}{\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle} \\ &\ll \sum_{v=1}^{\infty} v^k \sum_{x+y=v} \frac{1}{x^{\alpha+2}}\frac{1}{y^{\alpha+2}} \\ &\ll \sum_{v=1}^{\infty} \frac{v^k}{v^{\alpha+1}}. \end{aligned}$$

With these k and α still fixed, the second component $D_{k,w}^+$ has the property that

$$\begin{aligned} D_{k,w}^+ &\ll \sum_{v=[w]}^{\infty} v^k \sum_{\Theta_v} \frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r-1\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle} \\ &\ll \sum_{v=[w]}^{\infty} \frac{v^k}{v^{\alpha+1}} \quad \text{similarly to above,} \\ &= \mathcal{O}\left(\frac{1}{w^{\alpha-k}}\right). \end{aligned}$$

What remains is to investigate the behaviour of the sum in (93) for the remaining k. Since $E_1, E_3 \in (0,1)$, then

$$
\begin{aligned}
D_k^+ &\leq \sum_{v=1}^{[w]} v^k \sum_{\Theta_v} \frac{\sum_{j\leq k}\theta_j(\alpha)}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r-1\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle} \\
&\leq 2\sum_{j\leq k}\theta_j(\alpha)\sum_{v=1}^{[w]} v^k \sum_{x+y=v}\sum_{\substack{x=a_1+\dots+a_{j-1}\\ a_{j-1}\geq 2}} \frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}} \times \\
&\quad \times \sum_{\substack{y=a_{j+1}+\dots+a_r\\ a_r\geq 2}} \frac{1}{\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle} \\
&\leq 2C_1'^2\sum_{j\leq k}\theta_j(\alpha)\sum_{v=1}^{[w]} v^k\sum_{x+y=v}\frac{1}{x^{\alpha+2}}\frac{1}{y^{\alpha+2}} \\
\text{(94)}\quad &\leq 2C_1'^2\sum_{j\leq k}\theta_j(\alpha)\int_1^w \frac{dv}{v^{\alpha+1-k}} \\
&\leq 2C_1'^2\sum_{j\leq k}\theta_j(\alpha)w^{-(\alpha-k)}.
\end{aligned}
$$

Therefore, for α not taking integer value one has, for the remaining values of k (reintroducing the factor $N^{-(\alpha+2)}$ omitted for brevity earlier):

$$
\begin{aligned}
\frac{1}{N^{\alpha+2}}\sum_{k>\alpha}\frac{D_k^+}{N^k} &\leq \frac{2C_1'^2}{N^{\alpha+2}}\sum_{k>\alpha}\frac{1}{N^k}\frac{1}{w^{\alpha-k}}\sum_{j\leq k}\theta_j(\alpha) \\
&\leq \frac{2C_1'^2}{N^{\alpha+2}w^\alpha}\sum_{k=1}^{\infty}\left(\frac{w}{N}\right)^k\sum_{j\leq k}\theta_j(\alpha).
\end{aligned}
$$

Now one may apply Lemma 1.4 since $\frac{w}{N}\leq\frac{1}{2}$ to see that

$$
\begin{aligned}
\frac{1}{N^{\alpha+2}}\sum_{k>\alpha}\frac{D_k^+}{N^k} &\leq \frac{2C_1'^2}{N^{\alpha+2}w^\alpha}\left(\frac{1}{(1-\frac{w}{N})^{\alpha+2}}-1+\frac{\frac{w}{N}}{\frac{w}{N}-1}\right) \\
&\leq \frac{2C_1'^2}{N^{\alpha+2}w^\alpha}(2^{\alpha+2}-2).
\end{aligned}
$$

Hence it follows that $\frac{1}{N^{\alpha+2}}\sum_{k\alpha}\frac{D_k^+}{N^k} = \mathcal{O}\left(\frac{1}{N^{\alpha+2}w^\alpha}\right)$. In the case where $\alpha \in \mathbb{N}\setminus\{1\}$, then (94) implies that, in the case $k=\alpha$

$$D_k^+ = \mathcal{O}(\log(w)).$$

Thus giving the larger error term $\dfrac{1}{N^{\alpha+2}}\sum_{k\geq\alpha}\dfrac{D_k^+}{N^k} = \mathcal{O}\left(\dfrac{\log(N)}{N^{2\alpha+2}}\right)$.

Now consider the sum with minus denominator. This gives

$$\Sigma^{(3)-}_{(N,j<r-1)} = \frac{1}{N^{\alpha+2}}\sum_{v=1}^{[w]}\sum_{\Theta_v}\frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_{r-1}\rangle}\times$$

$$(95)\qquad \times\left(1+\sum_{k=1}^{\infty}\frac{1}{N^k}\sum_{j+l=k}\theta_j(\alpha)(v-E_1)^j(v-E_2)^l\right)+\mathcal{O}\left(\frac{1}{N^{s\alpha}}\right)$$

$$= \frac{1}{N^{\alpha+2}}\sum_{v=1}^{[w]}\sum_{\Theta_v}\frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_{r-1}\rangle}+$$

$$+\sum_{k=1}^{\infty}\sum_{v=1}^{[w]}\sum_{\Theta_v}\frac{N^{-(k+\alpha+2)}}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_{r-1}\rangle}\times$$

$$\times\sum_{j+l=k}\theta_j(\alpha)(v-E_1)^j(v-E_2)^l+\mathcal{O}\left(\frac{1}{N^{s\alpha}}\right).$$

The first series in (95) is finite, and will be written as

$$(96)\qquad \left(\sum_{v=1}^{\infty}-\sum_{v=[w]}^{\infty}\right)\sum_{\Theta_v}\frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_{r-1}\rangle},$$

in order to calculate its estimate. The first of these has has the property that

$$\sum_{v=1}^{\infty}\sum_{\Theta_v}\frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_{r-1}\rangle}$$

$$\leq 2\sum_{v=1}^{\infty}\sum_{x+y=v}\sum_{\substack{x=a_1+\ldots+a_{j-1}\\ a_{j-1}\geq 2}}\frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}}\times$$

$$\times\sum_{\substack{y=a_{j+1}+\ldots+a_r\\ a_r\geq 2}}\frac{1}{\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_{r-1}\rangle}.$$

We have by Lemma 3.17 and Theorem 3.4 that

$$\sum_{a \in A_X} \frac{1}{\langle a_1, \ldots a_x \rangle^{\alpha+1} \langle a_1, \ldots a_{x-1} \rangle} = \mathcal{O}\left(\frac{1}{X^{\alpha+1}}\right),$$

(as $X \to \infty$). Therefore, this means that there exists another constant C_2' such that

$$\sum_{\substack{y=a_{j+1}+\ldots+a_r \\ a_r \geq 2}} \frac{1}{\langle a_{j+1}, \ldots, a_r \rangle^{\alpha+1} \langle a_{j+1}, \ldots, a_{r-1} \rangle} \leq \frac{C_2'}{y^{\alpha+1}}.$$

Therefore:

(97)

$$\begin{aligned}
2 \sum_{v=1}^{\infty} \sum_{x+y=v} \sum_{\substack{x=a_1+\ldots+a_{j-1} \\ a_{j-1} \geq 2}} & \frac{1}{\langle a_1, \ldots, a_{j-1} \rangle^{\alpha+2}} \sum_{\substack{y=a_{j+1}+\ldots+a_r \\ a_r \geq 2}} \frac{(\langle a_{j+1}, \ldots, a_{r-1} \rangle)^{-1}}{\langle a_{j+1}, \ldots, a_r \rangle^{\alpha+1}} \\
\leq \quad & 2C_1'C_2' \sum_{v=2}^{\infty} \sum_{x+y=v} \frac{1}{x^{\alpha+2}} \frac{1}{y^{\alpha+1}} \\
\leq \quad & 2C_1'C_2' \sum_{v=2}^{\infty} \frac{1}{v^{\alpha}} = 2C_1'C_2'(\zeta(\alpha) - 1)
\end{aligned}$$

(and for fixed $\alpha > 1$, this upper bound is finite). Likewise the second component series of (96) has asymptotic value $\mathcal{O}(w^{-(\alpha-1)})$ since:

$$\begin{aligned}
& \sum_{v=[w]}^{\infty} \sum_{\Theta_v} \frac{1}{\langle a_1, \ldots, a_{j-1} \rangle^{\alpha+2}} \sum_{\substack{y=a_{j+1}+\ldots+a_r \\ a_r \geq 2}} \frac{1}{\langle a_{j+1}, \ldots, a_r \rangle^{\alpha+1} \langle a_{j+1}, \ldots, a_{r-1} \rangle} \\
\ll \quad & \sum_{v=[w]}^{\infty} \sum_{x+y=v} \frac{1}{x^{\alpha+2}} \frac{1}{y^{\alpha+1}} \\
\ll \quad & \sum_{v=[w]}^{\infty} \frac{1}{v^{\alpha}}.
\end{aligned}$$

The summation at (95) for $k = 1, \ldots, \infty$ will be denoted

(98)
$$\sum_{k=1}^{\infty} \frac{D_k^-}{N^k},$$

where

$$\begin{aligned} D_k^- &:= \left(\sum_{v=1}^{\infty} - \sum_{v=[w]}^{\infty}\right) \sum_{\Theta_v} \frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_{r-1}\rangle} \times \\ &\quad \times \sum_{j+l=k} \theta_j(\alpha)\,(v-E_1)^j\,(v-E_2)^l \\ &= D_{k,1}^- - D_{k,w}^-. \end{aligned}$$

Now fix a natural k and $\alpha > 1$. It has been established that

$$\sum_{\substack{y=c_1+\dots+c_\varepsilon \\ c_\varepsilon \geq 2}} \frac{1}{\langle c_1,\dots,c_\varepsilon\rangle^{\alpha+1}\langle c_1,\dots,c_{\varepsilon-1}\rangle} = \mathcal{O}\left(\frac{1}{y^{\alpha+1}}\right), \tag{99}$$

and thus the quantity $D_{k,1}^-$ converges for $k < \alpha - 1$ since

$$\begin{aligned} D_{k,1}^- &\ll \sum_{v=1}^{\infty} v^k \sum_{\Theta_v} \frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r - 1\rangle^{\alpha+1}\langle a_{j+1},\dots,a_{r-1}\rangle} \\ &\ll \sum_{v=1}^{\infty} v^k \sum_{x+y=v} \sum_{\substack{x=a_1+\dots+a_{j-1} \\ a_{j-1}\geq 2}} \frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}} \times \\ &\qquad\qquad \times \sum_{\substack{y=a_{j+1}+\dots+a_r \\ a_r \geq 2}} \frac{1}{\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_{r-1}\rangle} \\ &\ll \sum_{v=1}^{\infty} v^k \sum_{x+y=v} \frac{1}{x^{\alpha+2}}\frac{1}{y^{\alpha+1}} \\ &\ll \sum_{v=1}^{\infty} \frac{v^k}{v^\alpha}. \end{aligned}$$

With these k and α still fixed, the second component $D_{k,w}^-$ has the property that

$$\begin{aligned} D_{k,w}^- &\ll \sum_{v=[w]}^{\infty} v^k \sum_{\Theta_v} \frac{1}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r - 1\rangle^{\alpha+1}\langle a_{j+1},\dots,a_{r-1}\rangle} \\ &\ll \sum_{v=[w]}^{\infty} \frac{v^k}{v^\alpha} \quad \text{similarly to above,} \\ &= \mathcal{O}\left(\frac{1}{w^{\alpha-k-1}}\right). \end{aligned}$$

What remains is to investigate the behaviour of the sum in (98) for the remaining k. Since $E_1, E_2 \in (0,1)$, then

$$\begin{aligned}
D_k^- &\leq \sum_{v=1}^{[w]} v^k \sum_{\Theta_v} \frac{\sum_{j\leq k}\theta_j(\alpha)}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r-1\rangle^{\alpha+1}\langle a_{j+1},\ldots,a_{r-1}\rangle} \\
&\leq 2\sum_{j\leq k}\theta_j(\alpha)\sum_{v=1}^{[w]} v^k \sum_{x+y=v}\ \sum_{\substack{x=a_1+\ldots+a_{j-1}\\ a_{j-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}} \times \\
&\qquad \times \sum_{\substack{y=a_{j+1}+\ldots+a_r\\ a_r\geq 2}} \frac{1}{\langle a_{j+1},\ldots,a_r\rangle^{\alpha+1}\langle a_{j+1},\ldots,a_{r-1}\rangle} \\
&\leq 2C_1'C_2'\sum_{j\leq k}\theta_j(\alpha)\sum_{v=1}^{[w]} v^k \sum_{x+y=v}\frac{1}{x^{\alpha+2}}\frac{1}{y^{\alpha+1}} \qquad (100)\\
&\leq 2C_1'C_2'\sum_{j\leq k}\theta_j(\alpha)\int_1^w \frac{dv}{v^{\alpha-k}} \qquad (101)\\
&\leq 2C_1'C_2'\sum_{j\leq k}\theta_j(\alpha) w^{-(\alpha-k-1)},
\end{aligned}$$

where line (100) follows by line (99). Therefore, for α not taking integer value one has, for the remaining values of k (reintroducing the factor $N^{-(\alpha+2)}$ omitted for brevity earlier):

$$\begin{aligned}
\frac{1}{N^{\alpha+2}}\sum_{k>\alpha-1}\frac{D_k^-}{N^k} &\leq \frac{2C_1'C_2'}{N^{\alpha+2}}\sum_{k>\alpha-1}\frac{1}{N^k}\frac{1}{w^{\alpha-k-1}}\sum_{j\leq k}\theta_j(\alpha) \\
&\leq \frac{2C_1'C_2'}{N^{\alpha+2}w^{\alpha-1}}\sum_{k=1}^{\infty}\left(\frac{w}{N}\right)^k\sum_{j\leq k}\theta_j(\alpha).
\end{aligned}$$

Now one may apply Lemma 1.4 since $\frac{w}{N}\leq\frac{1}{2}$ to see that

$$\begin{aligned}
\frac{1}{N^{\alpha+2}}\sum_{k>\alpha-1}\frac{D_k^-}{N^k} &\leq \frac{2C_1'C_2'}{N^{\alpha+2}w^{\alpha-1}}\left(\frac{1}{\left(1-\frac{w}{N}\right)^{\alpha+2}}-1+\frac{\frac{w}{N}}{\frac{w}{N}-1}\right) \\
&\leq \frac{2C_1'C_2'}{N^{\alpha+2}w^{\alpha-1}}(2^{\alpha+2}-2).
\end{aligned}$$

Hence it follows that $\frac{1}{N^{\alpha+2}}\sum_{k>\alpha-1}\frac{D_k^-}{N^k} = \mathcal{O}\left(\frac{1}{N^{\alpha+2}w^{\alpha-1}}\right)$. Where $\alpha\in\mathbb{N}\setminus\{1\}$, then (101) implies that, in the case $k=\alpha-1$

$$D_k^- = \mathcal{O}(\log(w)).$$

Thus giving the larger error term $\frac{1}{N^{\alpha+2}} \sum_{k\geq\alpha-1} \frac{D_k^-}{N^k} = \mathcal{O}\left(\frac{\log(N)}{N^{2\alpha+1}}\right)$.

To give the stated result, define constant V_α (which occurs in the limit as $n \to \infty$) as

$$\begin{aligned} V_\alpha &= \sum_{v=1}^{\infty}\sum_{\Theta_v} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r-1\rangle^{\alpha+1}\langle a_{j+1},\ldots,a_r-1\rangle} \\ &+\sum_{v=1}^{\infty}\sum_{\Theta_v} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+1}\langle a_{j+1},\ldots,a_{r-1}\rangle}, \end{aligned}$$

and define $K_{n,\alpha}^{**}$ as in the statement of the Lemma. One should note that the extra term given by the sum $\Sigma_{(N,j<r-1)}^{(3)+}$ must be absorbed by the error term since it is of smaller magnitude than the error. □

5.5.4. *Numerical Results.* Consider once again, the effect of the chosen value of parameter s on the result. With $s = \frac{\alpha+3}{\alpha}$, Lemma 3.21 becomes:

$$\Sigma_{(N,j<r-1)}^{(3)} = \frac{V_\alpha}{(n+1)^{\alpha+2}} + \mathcal{O}\left(\frac{1}{n^{\alpha+3}}\right),$$

One may again use the main term of Theorem 3.3 to estimate the value of V_α using inequalities (92) on page 121 and (97) on page 125. However the combined effect of estimating both sub-sums of V_α is somewhat problematic for the illustration due to the combined result being a great overestimate. Thus, figure 3.19 plots actual values of $\Sigma_{(N,j<r-1)}^{(3)}$ along with the term $\frac{1}{N^{\alpha+2}}$, for $\alpha = 2$. This illustrates the trend of the characteristic of interest, with respect to its order of magnitude.

Remark: should the choice $s = \frac{2\alpha+1}{\alpha}$ be made, one will observe the full main term seen in the Lemma, though with $\alpha = 2$, the quantity $K_{\alpha,n}^{**}$ is undefined.

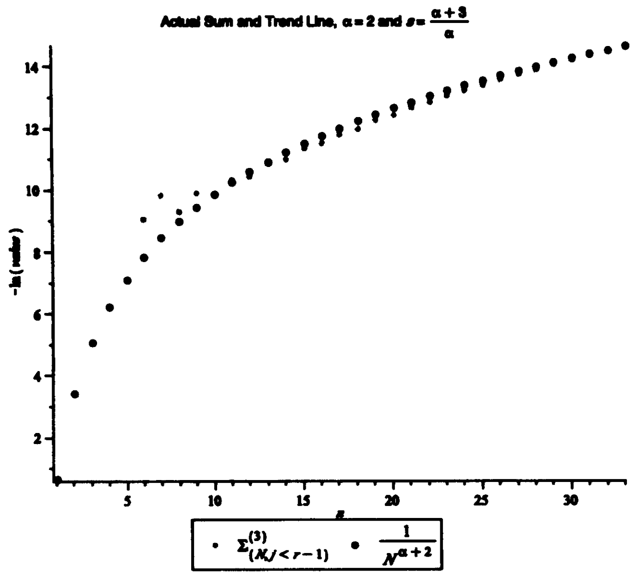


FIGURE 3.19. $\Sigma^{(3)}_{(N,j<r-1)}$ with trend $\frac{1}{N^{\alpha+2}}$ for $s = \frac{\alpha+3}{\alpha}$

6. Final Proof of Theorem 3.5

6.1. Reconstruction. Recall the partition

$$\begin{aligned}\gamma_\alpha(\mathrm{FT}_n) &= \frac{1}{2}\left(\Sigma^{(1)}_{(N,2)} + \Sigma^{(2)}_{(N,2)} + \Sigma^{(3)}_{(N,j<r-1)} + \Sigma^{(4)+}_{(N,j=r)} + \right.\\ &\quad + \left. \Sigma^{(4)-}_{(N,j=r)} + \Sigma^{(4)+}_{(N,j=r-1)} + \Sigma^{(4)-}_{(N,j=r-1)}\right).\end{aligned}$$

Collecting together terms from each of Lemmas 3.14, 3.15, 3.16, 3.17, 3.18, 3.19 and 3.21 one observes an error term of:

$$\mathcal{O}\left(\frac{1}{n^{s\alpha}} + \frac{n^2\log^{2\alpha+3}(n)}{w^{2\alpha+3}} + \frac{\log(n)}{n^{2\alpha+1}} + \frac{1}{n^{\alpha+3}w^{\alpha-2}}\right).$$

The reformation process therefore leaves items in the main term of lesser order than the error; these arise in the entities $K^-_{\alpha,n}$, $K^+_{\alpha,n}$, $K^{+*}_{\alpha,n}$ and $K^{**}_{\alpha,n}$. Letting the terms of lesser order of magnitude than $\frac{1}{n^{2\alpha}}$ form part of the error term allows us to reconstruct as

$$\frac{2}{n^{\alpha+1}}\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + \widehat{K}^-_{\alpha,n}\right) + \frac{1}{n^{\alpha+2}}\left(D_\alpha + \widehat{K}_{\alpha,n}\right),$$

where

(102) $$D_\alpha = T_\alpha + U_\alpha + V_\alpha + \frac{\zeta(\alpha+1)}{\zeta(\alpha+2)},$$

(103) $$\widehat{K}^-_{\alpha,n} = \sum_{1\le k<\alpha} \frac{1}{n^k}\left(\sum_{v=1}^{\infty} \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\ge 2}} \frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}(v-A)^k\theta_k(\alpha)\right),$$

and $\widehat{K}_{\alpha,n}$ is the sum $K^+_{\alpha,n} + K^{+*}_{\alpha,n} + K^{-*}_{\alpha,n} + K^{**}_{\alpha,n}$;

$$\begin{aligned}\widehat{K}_{\alpha,n} = \sum_{1\le k<\alpha-1} \frac{1}{n^k} & \left(\sum_{v=1}^{\infty}\sum_{\Theta_v} \frac{\sum_{j+l=k}\theta_j(\alpha)(v-E_1)^j(v-E_3)^l}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r-1\rangle^{\alpha+1}\langle a_{j+1},\ldots,a_r-1\rangle}\right. \\ & + \sum_{v=1}^{\infty}\sum_{\Theta_v} \frac{\sum_{j+l=k}\theta_j(\alpha)(v-E_1)^j(v-E_2)^l}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+1}\langle a_{j+1},\ldots,a_{r-1}\rangle} \\ & + \sum_{v=1}^{\infty} \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\ge 2}} \frac{(v-A)^k}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}\sum_{j\le k}\theta_j(\alpha) \\ & + \sum_{v=1}^{\infty}\sum_{\Omega_v} \frac{1}{(a_r-1)^{\alpha+1}} \frac{\sum_{j+l=k}\theta_j(\alpha)(v-A_2)^j(v-A_1)^l}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}} \\ & \left. + \sum_{v=1}^{\infty}\sum_{\Omega_v} \frac{1}{(a_r-1)^{\alpha+1}(a_r-2)} \frac{\sum_{j+l=k}\theta_j(\alpha)(v-A_2)^j(v-A_3)^l}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}}\right).\end{aligned} \quad (104)$$

We shall write $\mathcal{G}_{k,\alpha}$ and $\tilde{\mathcal{G}}_{k,\alpha}$ as the large terms within parentheses in (103) and (104); that is

$$\begin{aligned}\widehat{K}^-_{\alpha,n} &= \sum_{1\le k<\alpha} \frac{\mathcal{G}_{k,\alpha}}{n^k} \\ \widehat{K}_{\alpha,n} &= \sum_{1\le k<\alpha-1} \frac{\tilde{\mathcal{G}}_{k,\alpha}}{n^k}.\end{aligned}$$

The expressing of the quantity $K^+_{\alpha,n}$ to have denominator $\frac{1}{n^k}$ yields an error term of lesser order that the result of the calculation which gives the final error in this theorem. It is thus omitted for brevity. With the simplification of the additional terms of orders greater than or equal to $n^{-2\alpha}$ the remaining task is to deduce appropriate values for the parameters s and w in the error term:

$$\mathcal{O}\left(\frac{1}{n^{s\alpha}}+\frac{n^2\log^{2\alpha+3}(n)}{w^{2\alpha+3}}+\frac{\log(n)}{n^{2\alpha+1}}+\frac{1}{n^{\alpha+3}w^{\alpha-2}}\right).$$

The entity of least magnitude in the main term has order $\frac{1}{n^{2\alpha}}$, and largest in the error term (independent of w) is $\frac{\log(n)}{n^{2\alpha+1}}$. Hence it is obvious that one can fix an appropriate value of s to be:

$$s=\frac{2\alpha+1}{\alpha}.$$

However setting w to a value such that the term $\frac{n^2\log^{2\alpha+3}(n)}{w^{2\alpha+3}}$ is of equal order to the other dominant term in the error is unsatisfactory, since the result will be of magnitude greater than n. With $w=\frac{n}{2}$ one sees that the error term takes form

$$\mathcal{O}\left(\frac{\log^{2\alpha+3}(n)}{n^{2\alpha+1}}\right).$$

This gives, as $n\to\infty$ and for $D_\alpha=T_\alpha+U_\alpha+V_\alpha+\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}$,

$$\begin{aligned}\gamma_\alpha(\mathrm{FT}_n) &= \frac{2}{n^{\alpha+1}}\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)}+K^-_{\alpha,n}\right)+\frac{1}{n^{\alpha+2}}\left(D_\alpha+\widehat{K}_{\alpha,n}\right)+\\ &\quad+\ \mathcal{O}\left(\frac{\log^{2\alpha+3}(n)}{n^{2\alpha+1}}\right).\end{aligned}$$

■

Remark: In the case where $w=n^{\frac{2\alpha+5/2}{2\alpha+3}}\log^{\frac{2\alpha+2}{2\alpha+3}}(n)$ is the lesser quantity, then this improves to

$$\mathcal{O}\left(\frac{\log(n)}{n^{\frac{1}{2}(4\alpha+1)}}\right). \tag{105}$$

6.2. Numerical Evidence. Due to the presence of the infinite sum in v within $K^-_{\alpha,n}$, this quantity is not easy to calculate directly. However, using the main term of Theorem 3.3 obtains the following familiar estimate:

$$K^-_{\alpha,n}\sim C_0'\sum_{1\le k<\alpha}\frac{1}{n^k}\frac{(\zeta(\alpha+2-k)-1)}{k!}\prod_{i=1}^{k}(\alpha+i). \tag{106}$$

The behaviour of this upper bound is illustrated in figure 3.20.

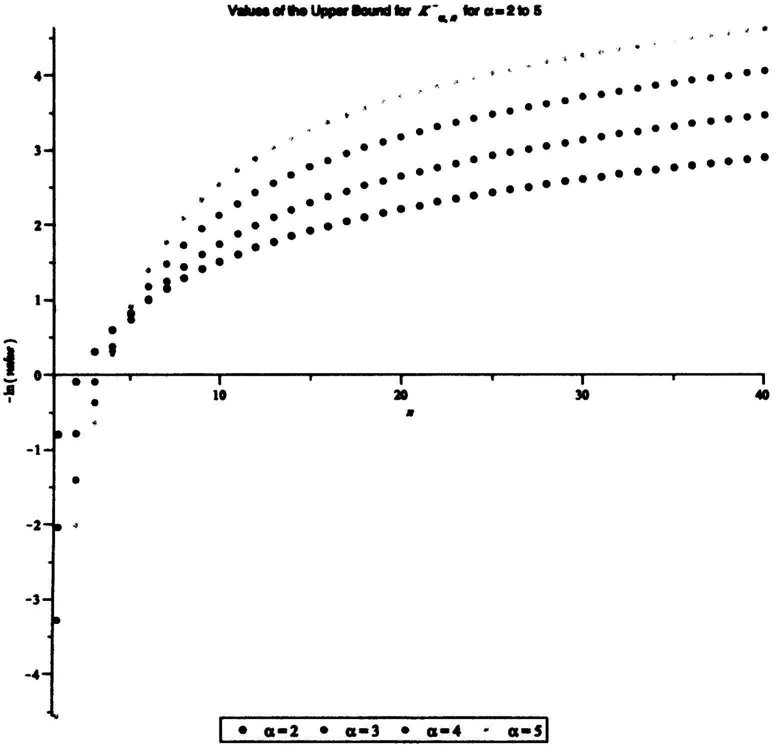


FIGURE 3.20. The behaviour of the upper bound for the entity $K_{\alpha,n}^-$

Now, in the case where $\alpha = 2$ the entity $\widehat{K}_{\alpha,n}$ is zero and one has that

$$(107) \qquad \gamma_\alpha(\mathrm{FT}_n) = \frac{2}{n^{\alpha+1}}\left(\frac{\zeta(\alpha+1)}{\zeta(\alpha+2)} + K_{\alpha,n}^-\right) + \frac{D_\alpha}{n^{\alpha+2}} + \mathcal{O}\left(\frac{\log^{2\alpha+3}(n)}{n^{2\alpha+1}}\right).$$

- An estimate of quantity (107) (using estimate (106)) is plotted, along with $\gamma_\alpha(\mathrm{FT}_n)$ and the lower bound of Lemma 3.6 in figure 3.21.

- Figure 3.22 illustrates the normalised differences between the main term of (107) and the actual value of $\gamma_\alpha(\mathrm{FT}_n)$ for $\alpha = 2$. The normalisation used is that of the improved error term suggested in (105).

With specific reference to figure 3.21, one should note that estimates have been used to calculate the values of such terms as D_α. However, despite this one may clearly see a strong pattern of convergence to the desired quantity in the illustration

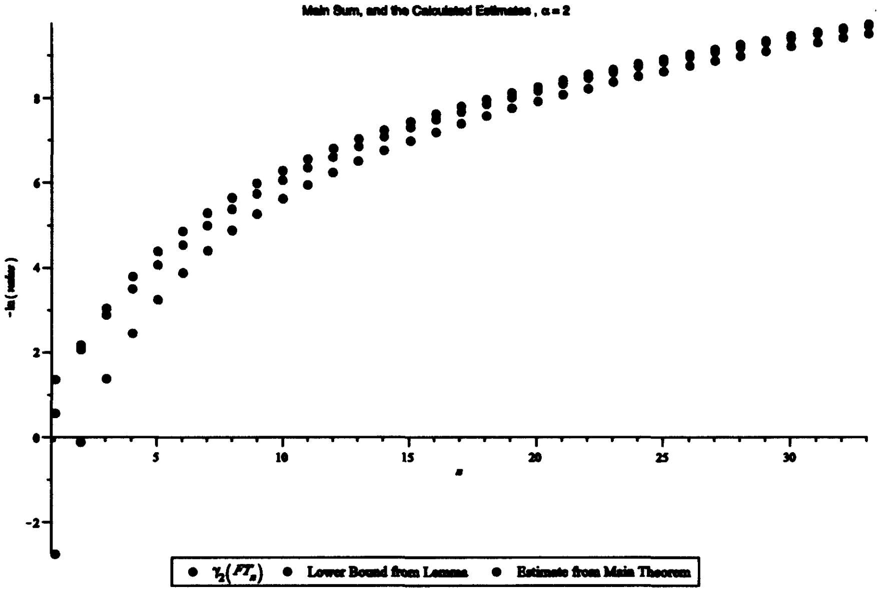


FIGURE 3.21. Behaviour of the estimates from Lemma 3.6 and Theorem 3.5

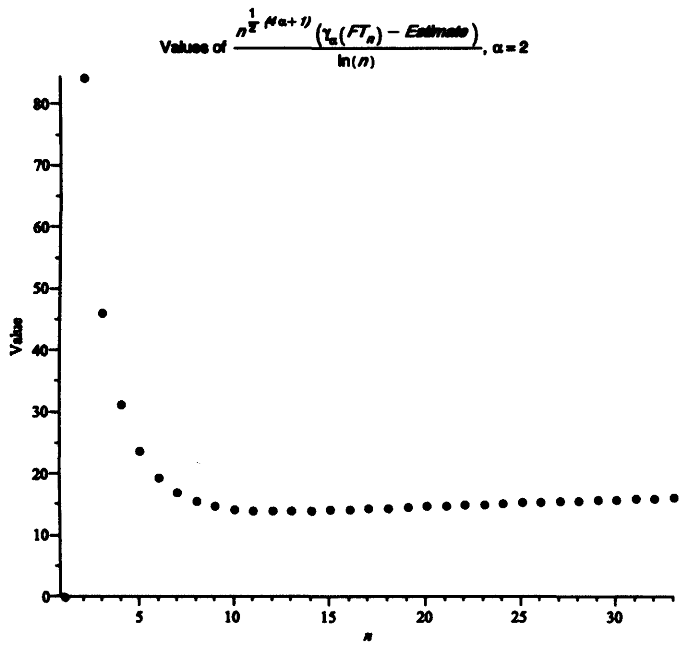


FIGURE 3.22. The normalised differences

CHAPTER 4

On Previous Results Concerning the ρ-Metric

1. Introduction

The aim of the following is to apply the algorithm of Chapter 3 to the original results compiled in [**31**] and [**12**]. This will act both as a form of verification of the proposed method, and will introduce terms to the main term that are thus far unseen.

Recall the hierarchy (33) whose notation we shall retain and the resultant partition applied to $\sigma_\beta(\mathrm{FT}_n)$ for the ρ- metric. This means that similar operations are performed on the sum $\sigma_\beta(\mathrm{FT}_n)$ as on $\gamma_\alpha(\mathrm{FT}_n)$, whereby

$$\sigma_\beta(\mathrm{FT}_n) = \frac{1}{2} \sum_{a \in \mathcal{A}_{n+1}} \frac{1}{(q(a)q'(a))^\beta}.$$

Let this sum be represented

$$\sigma_\beta(\mathrm{FT}_n) = \frac{1}{2}\left(\Sigma^{(1)}_{(N,2)} + \Sigma^{(2)}_{(N,2)} + \Sigma^{(3)}_{(N,j<r-1)} + \Sigma^{(4)}_{(N,j=r-1)} + \Sigma^{(4)}_{(N,j=r)}\right), \tag{108}$$

where $N = n+1$ and $\Sigma^{(j)}_{(N,i)} := \sum_{a \in \mathcal{A}^{(j)}_{(N,i)}} \frac{1}{(q(a)q'(a))^\beta}$. Note now that this notation is retained for consistency with the previous work of [**31**] and [**12**] and that these sums are not the same as those used in Chapter 3.

The sums $\Sigma^{(1)}_{(N,2)}$, $\Sigma^{(2)}_{(N,2)}$ and $\Sigma^{(4)}_{(N,j=r)}$ are discussed in [**31**] and [**12**]; the new methodology proposed yields the following Lemmas.

LEMMA 4.1. *For $\beta > 1$, $w \leq \frac{n}{2}$ and $s > 1$:*

$$\Sigma^{(4)}_{(n,j=r-1)} = \mathcal{O}\left(\frac{1}{n^{\beta}}\right),$$

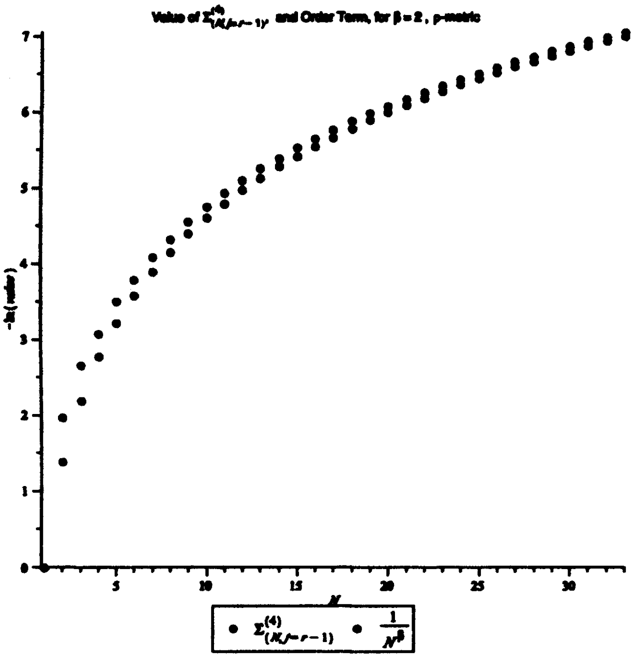
as $n \to \infty$.

LEMMA 4.2. *For $\beta > 1$ and as $n \to \infty$,*

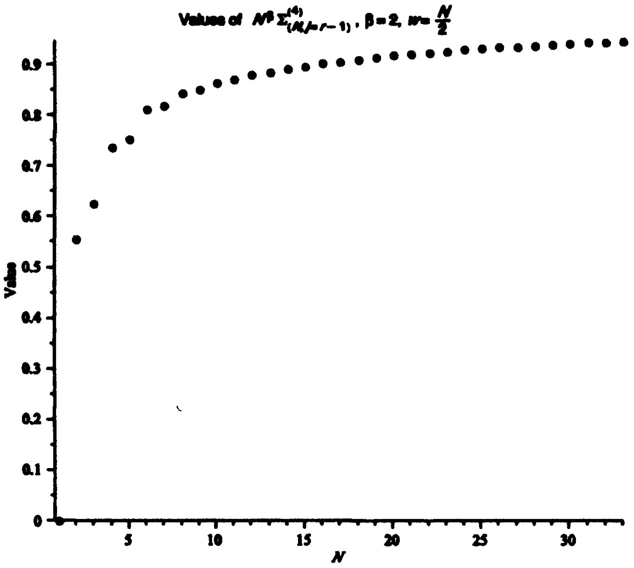
$$\Sigma^{(3)}_{(n,j<r-1)} = \mathcal{O}\left(\frac{1}{n^{2\beta}}\right),$$

as $n \to \infty$.

The behaviour of sum $\Sigma^{(4)}_{(N,j=r-1)}$ is illustrated in figures 4.1(a) and 4.1(b) overleaf. As in the previous chapter, these serve to illustrate a behaviour similar to that of $\Sigma^{(4)}_{(N,j=r)}$ which contains a term of of order $\frac{1}{n^{2\beta}}$.



(a) $\Sigma^{(4)}_{(n,j=r-1)}$ with the quantities $\frac{1}{n^\beta}$, $\beta = 2$



(b) Normalised sum $n^\beta \Sigma^{(4)}_{(n,j=r-1)}$ for $\beta = 2$

FIGURE 4.1.

2. On the Sum $\Sigma^{(4)}_{(N,j=r-1)}$

Recall the shorthand $q = q(a)$, $q_- = q_-(a)$ and $q_+ = q_+(a)$. The sum $\Sigma^{(4)}_{(N,j=r-1)}$ may be expressed as follows:

$$\begin{aligned}\Sigma^{(4)}_{(N,j=r-1)} &= \sum_{\substack{a_1+\ldots+a_r=N \\ a_r>1,\, q(a)<N^s \\ a_{r-1}>N-w}} \left[\frac{1}{(qq_+)^\beta} + \frac{1}{(qq_-)^\beta}\right] \\ &= \Sigma^{(4)+}_{(N,j=r-1)} + \Sigma^{(4)-}_{(N,j=r-1)}.\end{aligned}$$

For the sub-sum with '+' denominator we have

$$\begin{aligned}\Sigma^{(4)+}_{(N,j=r-1)} &= \left(\sum_{\substack{a_1+\ldots+a_r=N \\ a_r>1 \\ a_{r-1}>N-w}} - \sum_{\substack{a_1+\ldots+a_r=N \\ a_r>1,\, q\geq n^s \\ a_{r-1}>N-w}}\right)\frac{1}{(qq_+)^\beta} \\ &= \sum_{\substack{a_1+\ldots+a_r=N \\ a_r>1 \\ a_{r-1}>N-w}} \frac{1}{(qq_+)^\beta} + \mathcal{O}\left(\frac{1}{n^{(\beta-1)(2s-1)}}\right), \qquad (109)\end{aligned}$$

where the latter term first appearing at (109) is the result of Lemma 11 of [**12**]. An obvious analogue holds for the version containing the '$-$' denominator. Let us consider the 'plus' and 'minus' sums above in turn and proceed as Lemmas 3.18 and 3.19. Take $q(a)$ as the denominator of an element from $\mathcal{Q}_N$ such that $a = (a_1, \ldots, a_r)$, then the denominator $q(\tilde{a})$ from the 'previous' level $N-1$ (where $\tilde{a} = (a_1, \ldots, a_r - 1)$) yields

$$\begin{aligned}q(\tilde{a}') &= \langle a_1, \ldots, a_{r-1}, 1\rangle = \langle a_1, \ldots, a_{r-1}+1\rangle \\ q_+(\tilde{a}') &= \langle a_1, \ldots, a_{r-1}\rangle \\ q_-(\tilde{a}') &= \langle a_1, \ldots, a_{r-2}\rangle\end{aligned}$$

if $a_r = 2$, and

$$\begin{aligned}
q(\tilde{a}) &= \langle a_1, \ldots, a_r - 1\rangle \\
q_+(\tilde{a}) &= \langle a_1, \ldots, a_r - 2\rangle \\
q_-(\tilde{a}) &= \langle a_1, \ldots, a_{r-1}\rangle
\end{aligned}$$

for $a_r > 2$. Set $a_{r-1} = N - 1 - v$ for $v = 1, \ldots, [w]$ (where one would have $w \leq \frac{N-1}{2}$) and recall the identity (63). Using this, for $a_r > 2$ one has

$$\begin{aligned}
\frac{1}{(q(\tilde{a})q_+(\tilde{a}))^\beta} &= \frac{1}{\langle a_1, \ldots, a_r - 1\rangle^\beta \langle a_1, \ldots, a_r - 2\rangle^\beta} \\
&= \frac{(a_r - 1)^{-\beta}(a_r - 2)^{-\beta}}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}} \frac{1}{(a_{r-1} + A_2)^\beta} \frac{1}{(a_{r-1} + A_3)^\beta} \\
&= \frac{1}{n^{2\beta}} \frac{(a_r - 1)^{-\beta}(a_r - 2)^{-\beta}}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}} \frac{1}{\left(1 - \frac{v-A_2}{n}\right)^\beta} \frac{1}{\left(1 - \frac{v-A_3}{n}\right)^\beta}.
\end{aligned} \tag{110}$$

The shorthand

$$\begin{aligned}
A_1 &= [a_{r-2}, \ldots, a_1] \\
A_2 &= [a_r - 1] + [a_{r-2}, \ldots, a_1] \\
A_3 &= [a_r - 2] + [a_{r-2}, \ldots, a_1]
\end{aligned}$$

(where $A_1 \in (0, \frac{1}{2})$ $A_2 \in (0, 1)$, $A_3 \in (0, \frac{3}{2})$ under conditions $a_r, a_{r-2} \geq 2$) has been reintroduced for brevity. Furthermore, we have for $a_r = 2$,

$$\begin{aligned}
\frac{1}{(q(\tilde{a}')q_+(\tilde{a}'))^\beta} &= \frac{1}{\langle a_1, \ldots, a_{r-1} + 1\rangle^\beta \langle a_1, \ldots, a_{r-1}\rangle^\beta} \\
&= \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}} \frac{1}{(a_{r-1} + 1 + A_1)^\beta} \frac{1}{(a_{r-1} + A_1)^\beta} \\
&= \frac{1}{N^\beta n^\beta} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}} \frac{1}{\left(1 - \frac{v-A_1}{N}\right)^\beta} \frac{1}{\left(1 - \frac{v-A_1}{n}\right)^\beta}.
\end{aligned} \tag{111}$$

For the quantity involving the denominator q_-, and for $a_r > 2$, one also has that

$$
\begin{aligned}
\frac{1}{(q(\tilde a)q_-(\tilde a))^\beta} &= \frac{1}{\langle a_1,\dots,a_r-1\rangle^\beta \langle a_1,\dots,a_{r-1}\rangle^\beta} \\
&= \frac{(a_r-1)^{-\beta}}{\langle a_1,\dots,a_{r-2}\rangle^{2\beta}} \frac{1}{(a_{r-1}+A_1)^\beta} \frac{1}{(a_{r-1}+A_2)^\beta} \\
&= \frac{1}{n^{2\beta}} \frac{(a_r-1)^{-\beta}}{\langle a_1,\dots,a_{r-2}\rangle^{2\beta}} \frac{1}{\left(1-\frac{v-A_1}{n}\right)^\beta} \frac{1}{\left(1-\frac{v-A_2}{n}\right)^\beta}. \qquad (112)
\end{aligned}
$$

Now let us consider the last possibility, that is the quantity involving denominator q_-, with $a_r = 2$;

$$
\begin{aligned}
\frac{1}{(q(\tilde a')q_-(\tilde a'))^\beta} &= \frac{1}{\langle a_1,\dots,a_{r-1}+1\rangle^\beta \langle a_1,\dots,a_{r-2}\rangle^\beta} \\
&= \frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{2\beta}} \frac{1}{(a_{r-1}+1+A_1)^\beta} \\
&= \frac{1}{N^\beta} \frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{2\beta}} \frac{1}{\left(1-\frac{v-A_1}{N}\right)^\beta}. \qquad (113)
\end{aligned}
$$

The quantities $\frac{v-A_i}{n}$ are all less that one in modulus by construction:

$$\frac{v-A_i}{n} \le \frac{v}{n} \le \frac{1}{2}, \quad \text{since } w \le \frac{n}{2}.$$

As in Lemma 1.3, where $|X| < 1$, one has

$$\frac{1}{(1-X)^\beta} = 1 + \sum_{k=1}^{\infty} \frac{X^k}{k!} \prod_{i=1}^{k} (\beta + i - 1),$$

and likewise, for $|Y| < 1$ letting $\eta_k(\beta) := \frac{\prod_{i=1}^{k}(\beta+i-1)}{k!}$ for brevity:

$$
\begin{aligned}
\frac{1}{(1-Y)^\beta (1-X)^\beta} &= \left(1 + \sum_{l=1}^{\infty} \frac{X^l}{l!} \prod_{i=1}^{l} (\beta+i-1)\right) \times \\
&\times \left(1 + \sum_{j=1}^{\infty} \frac{Y^j}{j!} \prod_{i=1}^{j} (\beta+i-1)\right) \\
&= 1 + \sum_{k=1}^{\infty} \sum_{j+l=k} Y^j X^l \eta_j(\beta)\eta_l(\beta). \qquad (114)
\end{aligned}
$$

It follows that by taking a Taylor expansion of (110),(111) and (112) with respect to these $\frac{v-A_i}{n}$ that each of the items at (110),(111) and (112) are $\mathcal{O}(N^{-2\beta})$.

We shall consider each of these terms in turn, however performing a similar calculation on (113) yields

$$\begin{aligned}\frac{1}{(q(\tilde{a}')q_-(\tilde{a}'))^\beta} &= \frac{1}{N^\beta}\frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{2\beta}}\frac{1}{\left(1-\frac{v-A_1}{N}\right)^\beta}\\ &= \frac{1}{N^\beta}\frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{2\beta}}\left(1+\sum_{k=1}^{\infty}\frac{(v-A_1)^k}{N^k}\frac{\prod_{i=1}^k(\beta+i-1)}{k!}\right).\end{aligned}$$

Let us discuss the sum $\Sigma^{(4)+}_{(N-1,j=r-1)}$ (the predecessor of $\Sigma^{(4)-}_{(N,j=r-1)}$) first, splitting into components yielded by the values $a_r = 2$ and $a_r > 2$ as below. One should interpret the conditions $a \in \mathcal{A}_{N-1}; a_r \geq 3$ or $a \in \mathcal{A}_{N-1}; a_r = 2$ as the criteria that one will attain such values of a_r when performing the appropriate algorithm on the the members of $\mathcal{A}_{N-1}$ to derive set $\mathcal{A}_N$:

$$\begin{aligned}\Sigma^{(4)+}_{(N-1,j=r-1)} &= \sum_{v=1}^{[w]}\left(\sum_{\substack{a\in\mathcal{A}_{N-1}\\ a_r\geq 3,\, a_{r-1}=N-1-v}} + \sum_{\substack{a\in\mathcal{A}_{N-1}\\ a_r=2,\, a_{r-1}=N-1-v}}\right)\frac{1}{(q(\tilde{a})q_-(\tilde{a}))^\beta}\\ &\quad + \mathcal{O}\left(\frac{1}{n^{(\beta-1)(2s-1)}}\right)\\ &= \frac{1}{n^{2\beta}}\sum_{\substack{a\in\mathcal{A}_{N-1}\\ a_r\geq 3,\, a_{r-1}=N-1-v\\ v=1,\dots,[w]}}\frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{2\beta}(a_r-1)^\beta}\frac{1}{\left(1-\frac{v-A_1}{n}\right)^\beta\left(1-\frac{v-A_2}{n}\right)^\beta}+\\ &\quad +\frac{1}{N^\beta}\sum_{\substack{a\in\mathcal{A}_{N-1}\\ a_r=2,\, a_{r-1}=N-1-v\\ v=1,\dots,[w]}}\frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{2\beta}}\frac{1}{\left(1-\frac{(v-A_1)}{N}\right)^\beta}+ \qquad (115)\\ &\quad + \mathcal{O}\left(\frac{1}{n^{(\beta-1)(2s-1)}}\right).\end{aligned}$$

The sum at (115) represents the component which is of order $\mathcal{O}\left(\frac{1}{N^\beta}\right)$: this is discussed in Lemma 15 of [**12**]. Let us derive an approximation of the term above of order $\mathcal{O}\left(\frac{1}{N^{2\beta}}\right)$. Recalling definition of Ω_v from Lemma 3.18, this sum is

$$J_1 := \sum_{v=1}^{[w]}\sum_{\Omega_v}\frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{2\beta}(a_r-1)^\beta}\frac{1}{\left(1-\frac{v-A_1}{n}\right)^\beta\left(1-\frac{v-A_2}{n}\right)^\beta}.$$

From (114), with $X = \frac{v-A_1}{n}$, $Y = \frac{v-A_2}{n}$, this gives

$$(116) \qquad J_1 = \left(\sum_{v=1}^{\infty} - \sum_{v=[w]}^{\infty}\right) \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}(a_r - 1)^{\beta}} + $$
$$+ \sum_{k=1}^{\infty} \frac{1}{n^k} \sum_{v=1}^{[w]} \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}(a_r - 1)^{\beta}} \times$$
$$\times \sum_{j+l=k} \eta_j(\beta)\,(v - A_1)^j\, \eta_j(\beta)\,(v - A_2)^l .$$

The finite sum for $v = 1, \ldots, [w]$ at (116) is expressed in this way to explain its estimation. Its first component has constant upper bound since:

$$\begin{aligned}
&\sum_{v=1}^{\infty} \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}(a_r - 1)^{\beta}} \\
&\leq 2\sum_{v=1}^{\infty} \sum_{x+y=v} \frac{1}{(x+1)^{\beta}} \sum_{\substack{y=a_1+\ldots+a_{r-2} \\ a_{r-2}\geq 2}} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}} \\
&\leq 4C_0(\beta) \sum_{v=1}^{\infty} \sum_{x+y=v} \frac{1}{x^{\beta} y^{2\beta}} \\
&\leq 4C_0(\beta) \sum_{v=1}^{\infty} \frac{1}{v^{\beta-1}} = 4C_0(\beta)\zeta(\beta),
\end{aligned}$$

using Lemma 9 of [**12**] and where $C_0(\beta)$ is the original quantity defined $C_0(\beta) = \frac{\zeta(2\beta-1)}{\zeta(2\beta)} + 2\left(\frac{\zeta(2\beta-1)}{\zeta(2\beta)}\right)^2$. Notice again the introduction of the factor 2, which compensates for the imposition $a_{r-2} \geq 2$. The second factor 2 arises from the same Lemma from [**12**]. Similarly

$$\sum_{v=[w]}^{\infty} \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}(a_r - 1)^{\beta}} \ll \frac{1}{w^{\beta-2}}.$$

The second component of J_1 will be expressed

$$\sum_{k=1}^{\infty} \frac{F_k^-}{n^{2\beta+k}},$$

where

$$\begin{aligned} F_k^- &:= \left(\sum_{v=1}^{\infty} - \sum_{v=[w]}^{\infty}\right) \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}(a_r-1)^{\beta}} \times \\ &\qquad \times \sum_{j+l=k} \eta_j(\beta)\,(v-A_1)^j\,\eta_l(\beta)\,(v-A_2)^l \\ &= F_{k,1}^- - F_{k,w}^- \end{aligned}$$

For fixed k and $\beta > 1$, $\eta_k(\beta)$ is constant and since $A_1, A_2 \in (0,1)$, then

$$\begin{aligned} F_{k,1}^- &\ll \sum_{v=1}^{\infty} v^k \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}(a_r-1)^{\beta}} \\ &\ll \sum_{v=1}^{\infty} v^k \sum_{x+y=v} \frac{1}{x^{\beta}} \sum_{\substack{y=a_1+\ldots+a_{r-2} \\ a_{r-2}\geq 2}} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}} \\ &\ll \sum_{v=1}^{\infty} v^k \frac{1}{v^{\beta-1}}. \end{aligned}$$

This is convergent for $k < \beta - 2$, and similarly

$$F_{k,w}^- \ll \sum_{v=[w]}^{\infty} v^k \frac{1}{v^{\beta-1}} = \mathcal{O}\left(\frac{1}{w^{\beta-2-k}}\right).$$

What remains is to investigate the behaviour of F_k^- for the remaining k. We have the following inequality:

$$\begin{aligned} F_k^- &\leq \sum_{v=1}^{[w]} v^k \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}(a_r-1)^{\beta}} \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) \\ &\leq 2\sum_{v=1}^{[w]} v^k \sum_{x+y=v} \frac{1}{(x+1)^{\beta}} \sum_{\substack{y=a_1+\ldots+a_{r-2} \\ a_{r-2}\geq 2}} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}} \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) \\ &\leq 4C_0 \sum_{v=1}^{[w]} v^k \sum_{x+y=v} \frac{1}{x^{\beta}} \frac{1}{y^{2\beta}} \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) \\ &\leq 4C_0 \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) \int_1^w \frac{dv}{v^{\beta-1-k}} \qquad (117) \\ &\leq 4C_0 \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) w^{-(\beta-k-2)}, \end{aligned}$$

In the case where β is integer valued, specifically where $k = \beta - 2$, then (117) implies that $F_k^- = \mathcal{O}(\log(w))$. Furthermore, for $\beta > 1$, $\beta \notin \mathbb{N}$

$$\begin{aligned}\frac{1}{n^{2\beta}} \sum_{k>\beta-2} \frac{F_k^-}{n^k} &\leq \frac{4C_0}{n^{2\beta}} \sum_{k>\beta-2} \frac{1}{N^k} \frac{1}{w^{\beta-k-2}} \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) \\ &\leq \frac{4C_0}{n^{2\beta} w^{\beta-2}} \sum_{k=1}^{\infty} \left(\frac{w}{n}\right)^k \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) \\ &\leq \frac{4C_0}{n^{2\beta} w^{\beta-2}} \frac{1}{\left(1 - \frac{w}{n}\right)^{2\beta}}.\end{aligned}$$

Therefore, in this case

$$\sum_{k>\beta-2} \frac{F_k^-}{n^{2\beta+k}} = \mathcal{O}\left(\frac{1}{N^{2\beta} w^{\beta-2}}\right),$$

and when $\beta \in \mathbb{N} \setminus \{1\}$, specifically $\beta = k + 2$,

$$\sum_{k\geq\beta-2} \frac{F_k^-}{n^{2\beta+k}} = \mathcal{O}\left(\frac{\log(N)}{N^{2\beta} w^{\beta-2}}\right).$$

Now to complete this section of the calculation, set W_β as the entity

$$W_\beta = \sum_{v=1}^{[w]} \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta} (a_r - 1)^\beta}.$$

Let us now discuss the value of $\Sigma^{(4)+}_{(N,j=r-1)}$, which as convention in notation has $\Sigma^{(4)-}_{(N-1,j=r-1)}$ as predecessor. Once again, the sum conditions $a \in \mathcal{A}_{N-1}; a_r \geq 3$ or $a \in \mathcal{A}_{N-1}; a_r = 2$ should be interpreted as the criteria that one will attain such values of a_r when performing the appropriate algorithm on the the members of $\mathcal{A}_{N-1}$ to derive set $\mathcal{A}_N$:

$$\Sigma^{(4)-}_{(N-1,j=r-1)} = \sum_{v=1}^{[w]}\left(\sum_{\substack{a\in\mathcal{A}_{N-1}\\ a_r=2,\, a_{r-1}=N-1-v}} + \sum_{\substack{a\in\mathcal{A}_{N-1}\\ a_r\geq 3,\, a_{r-1}=N-1-v}}\right)\frac{1}{(q(\tilde{a})q_+(\tilde{a}))^\beta} + \mathcal{O}\left(\frac{1}{n^{(\beta-1)(2s-1)}}\right)$$

$$= +\frac{1}{N^\beta n^\beta}\sum_{\substack{a\in\mathcal{A}_{N-1}\\ a_r=2,\, a_{r-1}=N-1-v\\ v=1,\dots,[w]}}\frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{2\beta}}\frac{1}{\left(1-\frac{(v-A_1)}{N}\right)^\beta\left(1-\frac{(v-A_1)}{n}\right)^\beta}+ \tag{118}$$

$$+\frac{1}{n^{2\beta}}\sum_{\substack{a\in\mathcal{A}_{N-1}\\ a_r\geq 3,\, a_{r-1}=N-1-v\\ v=1,\dots,[w]}}\frac{(a_r-1)^{-\beta}(a_r-2)^{-\beta}}{\langle a_1,\dots,a_{r-2}\rangle^{2\beta}}\frac{1}{\left(1-\frac{v-A_2}{n}\right)^\beta\left(1-\frac{v-A_3}{n}\right)^\beta} \tag{119}$$

$$+ \mathcal{O}\left(\frac{1}{n^{(\beta-1)(2s-1)}}\right).$$

Let's deal with the term at (118) first. This calculation is very similar to that of Lemma 3.16; let $X = \frac{(v-A_1)}{N}$ and $Y = \frac{(v-A_1)}{n}$ in (114), which gives

$$\begin{aligned}\frac{1}{(1-X)^\beta(1-Y)^\beta} &= 1+\sum_{k=1}^{\infty}\sum_{j+l=k}\frac{\eta_j(\beta)\eta_l(\beta)(v-A_1)^j(v-A_1)^l}{N^j n^l}\\ &= 1+\sum_{k=1}^{\infty}(v-A_1)^k\sum_{j+l=k}\frac{\eta_j(\beta)\eta_l(\beta)}{N^j n^l}.\end{aligned}$$

Therefore, the sum at (118) is

$$\begin{aligned}&\frac{2}{N^\beta n^\beta}\sum_{v=1}^{[w]}\sum_{\substack{a_1+\dots+a_{r-2}=v\\ a_{r-2}\geq 2}}\frac{1}{\langle a_1,\dots,a_{r-2}\rangle^{2\beta}}+\\ &+\frac{2}{N^\beta n^\beta}\sum_{k=1}^{\infty}\sum_{v=1}^{[w]}\sum_{\substack{a_1+\dots+a_{r-2}=v\\ a_{r-2}\geq 2}}\frac{(v-A_1)^k}{\langle a_1,\dots,a_{r-2}\rangle^{2\beta}}\sum_{j+l=k}\frac{\eta_l(\beta)\eta_j(\beta)}{N^l n^j},\end{aligned} \tag{120}$$

for which the first sum has property

$$\begin{aligned}\left(\sum_{v=1}^{\infty}-\sum_{v=[w]}^{\infty}\right)\sum_{\substack{a_1+\ldots+a_{r-2}=v\\ a_{r-2}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{2\beta}} &= \frac{\zeta(2\beta-1)}{\zeta(2\beta)}+\sum_{v=[w]}^{\infty}\frac{C_0(\beta)}{v^{2\beta}}\\ &= \frac{\zeta(2\beta-1)}{\zeta(2\beta)}+\mathcal{O}\left(\frac{1}{w^{2\beta-1}}\right).\end{aligned}$$

The second sum of (120) will now be denoted

$$\sum_{k=1}^{\infty}\frac{F^+_{k,n}}{N^\beta n^\beta},$$

where

$$\begin{aligned}F^+_{k,n} &:= \left(\sum_{v=1}^{\infty}-\sum_{v=[w]}^{\infty}\right)\sum_{\substack{a_1+\ldots+a_{r-2}=v-1\\ a_{r-2}\geq 2}}\frac{(v-A_1)^k}{\langle a_1,\ldots,a_{r-2}\rangle^{2\beta}}\sum_{j+l=k}\frac{\eta_j(\beta)\eta_l(\beta)}{N^j n^l}\\ &= F^+_{k,n,1}-F^+_{k,n,w}.\end{aligned}$$

The quantity $F^+_{k,n,1}$ is convergent for $k<2\beta-1$ since (noting for fixed k and $\beta>1$, $\eta_k(\beta)$ is constant),

$$\begin{aligned}F^+_{k,n,1} &= \sum_{v=1}^{\infty}\sum_{\substack{a_1+\ldots+a_{r-2}=v-1\\ a_{r-2}\geq 2}}\frac{(v-A)^k}{\langle a_1,\ldots,a_{r-2}\rangle^{2\beta}}\sum_{j+l=k}\frac{\eta_j(\beta)\eta_l(\beta)}{n^l N^j}\\ &\ll \sum_{v=1}^{\infty}\frac{v^k}{N^k}\sum_{\substack{a_1+\ldots+a_{r-2}=v\\ a_{r-2}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{2\beta}}\\ &\ll \frac{1}{N^k}\sum_{v=1}^{\infty}\frac{1}{v^{2\beta-k}}.\end{aligned}$$

Similarly, the remainder $F^+_{k,n,w}$ may be estimated as

$$\begin{aligned}F^+_{k,n,w} &\ll \sum_{v=[w]}^{\infty}\frac{v^k}{N^k}\frac{1}{v^{2\beta}}\\ &= \mathcal{O}\left(\frac{1}{N^k w^{2\beta-k-1}}\right).\end{aligned}$$

What remains is to determine the behaviour of $F^+_{k,n}$ for the remaining k. An upper bound may be calculated as

$$
\begin{aligned}
F^+_{k,n} &\leq \frac{1}{n^k}\sum_{v=1}^{[w]} v^k \sum_{\substack{a_1+\ldots+a_{r-2}=v-1\\ a_{r-2}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}} \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta)\\
&\leq \frac{2C_0(\beta)}{n^k}\left(\sum_{j+l=k}\eta_j(\beta)\eta_l(\beta)\right)\sum_{v=1}^{[w]}\frac{v^k}{v^{2\beta}}\\
&\leq \frac{2C_0(\beta)}{n^k}\left(\sum_{j+l=k}\eta_j(\beta)\eta_l(\beta)\right)\int_1^{[w]}\frac{dv}{v^{2\beta-k}} \qquad (121)\\
&\leq \frac{2C_0(\beta)}{n^k}\left(\sum_{j+l=k}\eta_j(\beta)\eta_l(\beta)\right)\frac{1}{w^{2\beta-1-k}}.
\end{aligned}
$$

In particular, when 2β is integer-valued and $k = 2\beta - 1$, then we see that

$$F^+_{k,n} = \mathcal{O}\left(\frac{\log(w)}{N^k}\right).$$

When this is not the case, one sees that

$$
\begin{aligned}
\frac{2}{N^\beta n^\beta}\sum_{k>2\beta-1} F^+_{k,n} &\leq \frac{4C_0}{N^\beta n^\beta}\sum_{k>2\beta-1}\frac{1}{n^k}\frac{1}{w^{2\beta-1-k}}\sum_{j+l=k}\eta_j(\beta)\eta_l(\beta)\\
&\leq \frac{4C_0}{N^\beta n^\beta w^{2\beta-1}}\sum_{k=1}^{\infty}\left(\frac{w}{n}\right)^k\sum_{j+l=k}\eta_j(\beta)\eta_l(\beta)\\
&\leq \frac{4C_0}{N^\beta n^\beta w^{2\beta-1}}\frac{1}{(1-\frac{w}{n})^{2\beta}}.
\end{aligned}
$$

Since $w \leq \frac{n}{2}$ by construction,

$$\frac{2}{N^\beta n^\beta}\sum_{k>2\beta-1} F^+_{k,n} = \mathcal{O}\left(\frac{1}{N^{2\beta}w^{2\beta-1}}\right).$$

When 2β is integer-valued and $k = 2\beta - 1$ then

$$\frac{2}{N^\beta n^\beta}\sum_{k\geq 2\beta-1} F^+_{k,n} = \mathcal{O}\left(\frac{\log(N)}{N^{4\beta-1}}\right).$$

Therefore the sum at (118) has form

$$\frac{2}{N^\beta n^\beta}\left(\frac{\zeta(2\beta-1)}{\zeta(2\beta)} + \sum_{1\leq k<2\beta-1} F^+_{k,n,1}\right) + \mathcal{O}\left(\frac{\log(N)}{N^{4\beta-1}} + \frac{1}{N^{2\beta}w^{2\beta-1}}\right),$$

Finally, let us consider the sum at (119). From (114), with $X = \frac{v-A_2}{n}$, $Y = \frac{v-A_3}{n}$, this gives (omitting coefficient $n^{-2\beta}$ for brevity)

$$(122) \quad J_2 = \left(\sum_{v=1}^{\infty} - \sum_{v=[w]}^{\infty}\right) \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}(a_r-1)^{\beta}(a_r-2)^{\beta}} + $$
$$+ \sum_{k=1}^{\infty} \frac{1}{n^k} \sum_{v=1}^{[w]} \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}(a_r-1)^{\beta}(a_r-2)^{\beta}} \times$$
$$\times \sum_{j+l=k} \eta_j(\beta)\,(v-A_2)^j\, \eta_j(\beta)\,(v-A_3)^l.$$

The finite sum at (122) is again expressed as this difference of sums with upper limit infinity to determine an appropriate estimated value. The first component has the upper bound determined by:

$$\sum_{v=1}^{\infty} \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}(a_r-1)^{\beta}(a_r-2)^{\beta}}$$
$$\leq 2\sum_{v=1}^{\infty} \sum_{x+y=v} \frac{1}{x^{\beta}(x+1)^{\beta}} \sum_{\substack{y=a_1+\ldots+a_{r-2} \\ a_{r-2}\geq 2}} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}}$$
$$\leq 4C_0(\beta) \sum_{v=1}^{\infty} \sum_{x+y=v} \frac{1}{x^{2\beta}y^{2\beta}}$$
$$\leq 4C_0(\beta) \sum_{v=1}^{\infty} \frac{1}{v^{2\beta-1}} = 4C_0\zeta(2\beta-1).$$

Similarly

$$\sum_{v=[w]}^{\infty} \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}(a_r-1)^{\beta}(a_r-2)^{\beta}} \ll \frac{1}{w^{2\beta-2}}.$$

The second component of J_2 will be expressed

$$\sum_{k=1}^{\infty} \frac{F_k^+}{n^{2\beta+k}},$$

where

$$\begin{aligned} F_k^+ &:= \left(\sum_{v=1}^{\infty} - \sum_{v=[w]}^{\infty}\right) \sum_{\Omega_v} \frac{1}{\langle a_1, \dots, a_{r-2}\rangle^{2\beta}(a_r-1)^{\beta}(a_r-2)^{\beta}} \times \\ &\qquad \times \sum_{j+l=k} \eta_j(\beta)\,(v-A_2)^j\,\eta_l(\beta)\,(v-A_3)^l \\ &= F_{k,1}^+ - F_{k,w}^+. \end{aligned}$$

The first term $F_{k,1}^+$ is convergent for $k < 2\beta - 2$ since $A_2, A_3 \in (0,1)$ and for each fixed k, $\eta_k(\beta)$ is constant - this means that:

$$\begin{aligned} F_{k,1}^+ &\ll \sum_{v=1}^{\infty} v^k \sum_{\Omega_v} \frac{1}{\langle a_1, \dots, a_{r-2}\rangle^{2\beta}(a_r-1)^{\beta}(a_r-2)^{\beta}} \\ &\ll \sum_{v=1}^{\infty} v^k \sum_{x+y=v} \frac{1}{x^{2\beta}} \sum_{\substack{y=a_1+\dots+a_{r-2}\\ a_{r-2}\geq 2}} \frac{1}{\langle a_1, \dots, a_{r-2}\rangle^{2\beta}} \\ &\ll \sum_{v=1}^{\infty} v^k \sum_{x+y=v} \frac{1}{x^{2\beta}} \frac{1}{y^{2\beta}} \\ &\ll \sum_{v=1}^{\infty} v^k \frac{1}{v^{2\beta-1}}. \end{aligned}$$

This is convergent for $k < 2\beta - 2$, and similarly with these k:

$$F_{k,w}^+ \ll \sum_{v=[w]}^{\infty} v^k \frac{1}{v^{2\beta-1}} = \mathcal{O}\left(\frac{1}{w^{2\beta-2-k}}\right).$$

The final task is to demonstrate the behaviour of F_k^+ for the remaining values of k. The following upper bound follows;

$$\begin{aligned}
F_k^+ &\leq \sum_{v=1}^{[w]} v^k \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}(a_r-1)^\beta(a_r-2)^\beta} \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) \\
&\leq 2\sum_{v=1}^{[w]} v^k \sum_{x+y=v} \frac{1}{x^\beta(x+1)^\beta} \sum_{\substack{y=a_1+\ldots+a_{r-2}\\ a_{r-2}\geq 2}} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}} \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) \\
&\leq 4C_0 \sum_{v=1}^{[w]} v^k \sum_{x+y=v} \frac{1}{x^{2\beta}}\frac{1}{y^{2\beta}} \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) \\
(123) &\leq 4C_0 \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) \int_1^w \frac{dv}{v^{2\beta-1-k}} \\
&\leq 4C_0 \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) w^{-(2\beta-k-2)},
\end{aligned}$$

In the case where 2β is integer-valued, specifically $k = 2\beta - 2$, then (123) implies that $F_k^+ = \mathcal{O}(\log(w))$. Furthermore, for $\beta > 1$, $\beta \notin \mathbb{N}$

$$\begin{aligned}
\sum_{k>2\beta-2} \frac{F_k^+}{n^{2\beta+k}} &\leq \frac{4C_0}{n^{2\beta}} \sum_{k>2\beta-2} \frac{1}{n^k}\frac{1}{w^{2\beta-k-2}} \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) \\
&\leq \frac{4C_0}{n^{2\beta}w^{2\beta-2}} \sum_{k=1}^{\infty} \left(\frac{w}{n}\right)^k \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta) \\
&\leq \frac{4C_0}{n^{2\beta}w^{2\beta-2}} \frac{1}{\left(1-\frac{w}{n}\right)^{2\beta}}.
\end{aligned}$$

Therefore, in this case (since $w \leq \frac{n}{2}$),

$$\sum_{k>2\beta-2} \frac{F_k^+}{n^{2\beta+k}} = \mathcal{O}\left(\frac{1}{N^{2\beta}w^{2\beta-2}}\right),$$

and when $\beta \in \mathbb{N} \setminus \{1\}$, specifically where $k = 2\beta - 2$,

$$\sum_{k\geq 2\beta-2} \frac{F_k^+}{n^{2\beta+k}} = \mathcal{O}\left(\frac{\log(N)}{N^{4\beta-2}}\right).$$

Now to complete this final part of the calculation, set U_β as the entity

$$U_\beta = \frac{\zeta(2\beta-1)}{\zeta(2\beta)} + \sum_{v=1}^{[w]} \sum_{\Omega_v} \frac{1}{\langle a_1, \ldots, a_{r-2}\rangle^{2\beta}(a_r-1)^\beta(a_r-2)^\beta},$$

which we may place inside the reconstructed quantity

$$\Sigma^{(4)-}_{(N-1,j=r-1)} = \frac{2}{N^{2\beta}}\left(U_\beta + \sum_{1\le k<2\beta-2}\frac{F^+_{k,1}}{N^k} + \sum_{1\le k<2\beta-2} F^+_{k,n}\right) + + \mathcal{O}\left(\frac{\log(N)}{N^{4\beta-2}} + \frac{1}{N^{2\beta+1}w^{2\beta-3}}\right).$$

as $n \to \infty$.

3. Lemma 4.2

We recall the statement and proof of Lemma 13 from [**12**]. This is the estimate

$$\Sigma^{(3)}_{(N,2)} = \sum_{0\le k<\beta-2}\frac{B_k}{N^{2\beta+k}} + \mathcal{O}\left(\frac{1}{N^{(\beta-1)(2s-1)}} + \frac{1}{N^{2\beta}w^{\beta-2}} + \frac{\log(N)}{N^{3\beta-2}}\right),$$

where the B_k are some constants. Note that, the framework proposed in this thesis implies that the set $\mathcal{A}^{(4)}_{(N,j=r-1)}$ is contained within the original $\mathcal{A}^{(3)}_{(N,2)}$, and it thus proposed that the previous lemma of Dushistova now describes the behaviour of the sum $\Sigma^{(3)}_{(N,j<r-1)}$ only. The reasoning for this is reasonably simple and arises from the discussion of Lemma 4.1. Since the restriction $a_j > n - w$ now applies to any of the first $r-2$ partial quotients in the expansion

$$q(a) = \langle a_1, \dots a_r\rangle,$$

we have that the characteristic is present in both of the denominators $q_+(a)$ and $q_-(a)$.

4. Reconstruction

Recall the partition (108). Using Lemmas 11, 12, 14 and 15 of [**12**] and Lemmas 4.1 and 4.2 of this chapter, one sees that

$$\begin{aligned}
\sigma_\beta(\mathrm{FT}_n) &= \frac{1}{n^\beta}\left(\frac{\zeta(2\beta-1)}{\zeta(2\beta)} + \sum_{1\le k<2\beta-1}\frac{E_k}{n^k}\right) + \\
&+ \frac{1}{(n+1)^\beta}\left(\frac{\zeta(2\beta-1)}{\zeta(2\beta)} + \sum_{1\le k<2\beta-1}\frac{E_k}{(n+1)^k}\right) \\
&+\frac{1}{(n+1)^{2\beta}}\left(\frac{\zeta(2\beta-1)}{\zeta(2\beta)} + \sum_{1\le k<2\beta-1}\frac{D_k}{(n+1)^k} + \sum_{0\le k<\beta-2}\frac{B_k}{(n+1)^k}\right) \\
&+\frac{1}{n^{2\beta}}\left(U_\beta + W_\beta + \sum_{1\le k<2\beta-2}\frac{F^+_{k,1}}{n^k} + \sum_{1\le k<\beta-2}\frac{F^-_{k,1}}{n^k}\right) \\
&+\frac{1}{n^\beta(n+1)^\beta}\left(\frac{\zeta(2\beta-1)}{\zeta(2\beta)} + \sum_{1\le k<2\beta-2} F^+_{k,n,1}\right) \\
&+\mathcal{O}\left(\frac{1}{n^{(\beta-1)(2s-1)}} + \frac{n^2\log^{3\beta}(n)}{w^{3\beta}} + \frac{\log(n)}{n^{3\beta-2}} + \frac{\log(n)}{n^{4\beta-1}} + \frac{1}{n^{2\beta}w^{\beta-2}}\right).
\end{aligned}$$

Let us first consider the error term. The largest items are the same as those that appear in the original result of [**12**] and thus select parameters in a similar manner: $s = \frac{3\beta-2}{2(\beta-1)} + \frac{1}{2}$ and $w = \frac{n}{2}$.

Consider also the term

$$F^+_{k,n,1} := \sum_{v=1}^{\infty} \sum_{\substack{a_1+\ldots+a_{r-2}=v-1\\ a_{r-2}\ge 2}} \frac{(v-A_1)^k}{\langle a_1,\ldots,a_{r-2}\rangle^{2\beta}} \sum_{j+l=k}\frac{\eta_j(\beta)\eta_l(\beta)}{N^j n^l},$$

and define analogously

$$\tilde{F}_{k,1} = \sum_{v=1}^{\infty}\frac{1}{n^k} \sum_{\substack{a_1+\ldots+a_{r-2}=v-1\\ a_{r-2}\ge 2}} \frac{(v-A_1)^k}{\langle a_1,\ldots,a_{r-2}\rangle^{2\beta}} \sum_{j+l=k}\eta_j(\beta)\eta_l(\beta).$$

The difference between these quantities is of lesser magnitude than the largest error term, and thus the overall result rearranges as:

$$\begin{aligned}
\sigma_\beta(\mathrm{FT}_n) &= \frac{2}{n^\beta}\left(\frac{\zeta(2\beta-1)}{\zeta(2\beta)} + \sum_{1\le k<2\beta-2}\frac{E_k}{n^k}\right) + \frac{1}{n^{2\beta}}\left(U^*_\beta + \sum_{0\le k<\beta-2}\frac{B^*_k}{n^k}\right) \\
&+\mathcal{O}\left(\frac{\log^{3\beta}(n)}{n^{3\beta-2}}\right),
\end{aligned}$$

where

$$U_\beta^* = \frac{\zeta(2\beta-1)}{\zeta(2\beta)} + U_\beta + W_\beta,$$

$$B_k^* = B_k + F_{k,1}^- + F_{k,1}^+ + \tilde{F}_{k,1} + D_k$$

as $n \to \infty$. The original sum $\sum_{1\le k<2\beta-2} \frac{F_k^*}{n^{k+2\beta}}$ is of lesser magnitude than the error term, its latter 2β terms are thus placed in the error term.

The algorithm produces a result similar to the original formulation. Of lesser importance is that the error term of Dushistova is unchanged; this is due to the dominance of the large error whose origin is the sum $\Sigma_{(N,2)}^{(2)}$. However, the new method proposed here suggests an improvement in the constants with coefficient $\frac{1}{n^{2\beta}}$. The constants U_β and W_β are present for $\beta > 2$ along with the terms

$$F_{k,1}^- + F_{k,1}^+ + \tilde{F}_{k,1};$$

these are the new items yielded by the suggested algorithm. When $\beta = 2$ one has that

$$\sigma_\beta(\mathrm{FT}_n) = \frac{2}{n^2}\left(\frac{\zeta(3)}{\zeta(4)} + \frac{E_1}{n}\right) + \mathcal{O}\left(\frac{\log^6(n)}{n^4}\right),$$

which is evidently identical to the result of Dushistova with the same β.

5. Conclusion

During the course of this work, we have observed and discussed a formulation of the integral of the ρ'-metric over the unit interval partitioned by the Farey Tree points. Partitioning the sum yielded by this integral according to the scheme suggested by figure 3.3 yielded a number of Lemmas, which were each considered in term, contributing to the final proof of Theorem 3.5. These results aggregated to form a final term of order $\mathcal{O}\left(\frac{1}{n^{\alpha+1}}\right)$, which matches the original estimate (as a lower bound) for the quantity $\gamma_\alpha(\mathrm{FT}_n)$ suggested by Lemma 3.6.

Furthermore, the figures supplied in Chapter Three (and furthermore in Appendix B) confirm numerically (within the bounds of running the appropriate calculations in a timely manner) that the suggested approximations to the sums of the partition, and indeed the aggregate value $\gamma_\alpha(\mathrm{FT}_n)$ behave in a manner which is of the same order of magnitude. Moreover, we have seen that the proposed method produces a similar, though improved (in terms of introducing further constants into the main term) result for the ρ-metric, which was first considered in **[31]** and **[12]**.

APPENDIX A

More on the Sum with Single Denominator

In Theorem 3.3, it was proved that

$$\sum_{a\in\mathcal{A}_X}\frac{1}{q(a)^{\alpha+2}}=\mathcal{O}\left(\frac{1}{X^{\alpha+2}}\right).$$

However, one can easily see that the given error in that result may be improved. It is in this appendix that such an improvement is offered. We work to a scheme very similar to that presented by Dushistova in her Theorem 1 of [**12**].

LEMMA A.1. *For $\alpha>1$, and as $N\to\infty$,*

$$\sum_{a\in\mathcal{A}_N}\frac{1}{q(a)^{\alpha+2}}=\frac{1}{N^{\alpha+2}}\left(C_0'(\alpha)+\sum_{1\leq k<\alpha}\frac{L_{k,\alpha}}{N^k}\right)+\mathcal{O}\left(\frac{\log^{2\alpha+4}(N)}{N^{2\alpha+2}}\right),$$

where

$$L_{k,\alpha}=\sum_{v=1}^{\infty}\sum_{x+y=v}\sum_{\substack{x=a_1+\ldots+a_{j-1}\\ y=a_{j+1}+\ldots+a_r,\ a_r\geq 2}}\frac{(v-([a_{j-1},\ldots,a_1]+[a_{j+1},\ldots,a_r]))^k}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}}\frac{\prod_{i=1}^k(\alpha+i+1)}{k!}.$$

PROOF. Recall that, from Lemma 3.13 that

$$\begin{aligned}\tilde{\Sigma}^{(2)}_{(N,1)} &= \sum_{\substack{a\in\mathcal{A}_N\\ a_j=N-v\\ for\ v=1,\ldots,[w]}}\frac{1}{N^{\alpha+2}}\frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}}\\ &\quad\times\left(1+\sum_{k=1}^{\infty}\frac{\prod_{i=1}^k(\alpha+i+1)}{k!}\frac{(v-A)^k}{N^k}\right)+\mathcal{O}\left(\frac{1}{N^{\frac{\alpha}{2}(2s-1)}}\right),\end{aligned}$$

where $A=[a_{j-1},\ldots,a_1]+[a_{j+1},\ldots,a_r]$. Let $\kappa_k(\alpha)=\frac{\prod_{i=1}^k(\alpha+i+1)}{k!}$, $x=a_1+\ldots+a_{j-1}$ and $y=a_{j+1}+\ldots+a_r$. Then the following rearrangement is possible:

$$\tilde{\Sigma}^{(2)}_{(N,1)} = \sum_{v=1}^{[w]} \frac{1}{N^{\alpha+2}} \sum_{x+y=v} \sum_{\substack{x=a_1+\ldots+a_{j-1} \\ y=a_{j+1}+\ldots+a_r,\ a_r\geq 2}} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}} +$$

$$+\sum_{k=1}^{\infty}\sum_{v=1}^{[w]} \frac{1}{N^{k+\alpha+2}} \sum_{x+y=v} \sum_{\substack{x=a_1+\ldots+a_{j-1} \\ y=a_{j+1}+\ldots+a_r,\ a_r\geq 2}} \frac{\kappa_k(\alpha)(v-A)^k}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}}$$

$$+\mathcal{O}\left(\frac{1}{N^{\frac{\alpha}{2}(2s-1)}}\right).$$

The first of these terms gives

$$\sum_{x+y\leq w} \sum_{x=a_1+\ldots+a_{j-1}} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}} \sum_{y=a_{j+1}+\ldots+a_r} \frac{1}{\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}}$$
$$= \frac{C_0'}{N^{\alpha+2}} + \mathcal{O}\left(\frac{1}{N^{\alpha+2}w^{\alpha}}\right).$$

Now let us investigate the properties of the remaining series:

$$\sum_{k=1}^{\infty} \frac{R_k}{N^k},$$

where

$$R_k = \left(\sum_{v=1}^{\infty} - \sum_{v=[w]}^{\infty}\right) \sum_{x+y=v} \sum_{\substack{x=a_1+\ldots+a_{j-1} \\ y=a_{j+1}+\ldots+a_r,\ a_r\geq 2}} \frac{\kappa_k(\alpha)(v-A)^k}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}}.$$

The first of these infinite series gives

$$\sum_{v=1}^{\infty} \sum_{x+y=v} \sum_{\substack{x=a_1+\ldots+a_{j-1} \\ y=a_{j+1}+\ldots+a_r,\ a_r\geq 2}} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}} \kappa_k(\alpha)(v-A)^k$$

$$\ll \sum_{v=1}^{\infty} v^k \sum_{x+y=v} \sum_{\substack{x=a_1+\ldots+a_{j-1} \\ y=a_{j+1}+\ldots+a_r,\ a_r\geq 2}} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}}$$

$$\ll \sum_{v=1}^{\infty} v^k \sum_{x+y=v} \sum_{\substack{x=a_1+\ldots+a_{j-1} \\ a_{j-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}} \sum_{\substack{y=a_{j+1}+\ldots+a_r \\ a_r\geq 2}} \frac{1}{\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}}.$$

Moreover, since it is already established that $\sum_{\substack{x=a_1+\ldots+a_{j-1}\\ a_{j-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}} =$ $\mathcal{O}\left(\frac{1}{x^{\alpha+2}}\right)$, and similarly for the sum in y where one has that

$$\sum_{v=1}^{\infty}\sum_{x+y=v}\sum_{\substack{x=a_1+\ldots+a_{j-1}\\ y=a_{j+1}+\ldots+a_r,\ a_r\geq 2}} \frac{\kappa_k(\alpha)(v-A)^k}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}} \ll \sum_{v=1}^{\infty}\frac{v^k}{v^{\alpha+1}}.$$

This is a convergent sum for all those k such that $\alpha+1-k>1 \Leftrightarrow k<\alpha$, and so with these k let us estimate the remaining terms in the summation, whence

$$\begin{aligned}
&\sum_{v=[w]}^{\infty}\sum_{x+y=v}\sum_{\substack{x=a_1+\ldots+a_{j-1}\\ y=a_{j+1}+\ldots+a_r,\ a_r\geq 2}} \frac{\kappa_k(\alpha)(v-A)^k}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}}\\
&\quad\ll \sum_{v=[w]}^{\infty} v^k \sum_{x+y=v}\sum_{\substack{x=a_1+\ldots+a_{j-1}\\ a_{j-1}\geq 2}} \frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}} \sum_{\substack{y=a_{j+1}+\ldots+a_r\\ a_r\geq 2}} \frac{1}{\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}}\\
&\quad\ll \sum_{v=[w]}^{\infty}\frac{v^k}{v^{\alpha+1}} \ll \int_w^{\infty}\frac{dv}{v^{\alpha+1-k}} = \mathcal{O}\left(\frac{1}{w^{\alpha-k}}\right).
\end{aligned}$$

To summarise at this point one has, for $k<\alpha$

$$R_k = L_{k,\alpha} + \mathcal{O}\left(\frac{1}{w^{\alpha-k}}\right),$$

where $L_{k,\alpha} = \sum_{v=1}^{\infty}\sum_{x+y=v}\sum_{\substack{x=a_1+\ldots+a_{j-1}\\ y=a_{j+1}+\ldots+a_r,\ a_r\geq 2}} \frac{(v-A)^k\kappa_k(\alpha)}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}}$ is a fixed quantity for each k. Thus

$$\sum_{1\leq k<\alpha}\frac{R_k}{N^{k+\alpha+2}} = \sum_{1\leq k<\alpha}\frac{L_{k,\alpha}}{N^{k+\alpha+2}} + \mathcal{O}\left(\frac{1}{N^{\alpha+3}w^{\alpha-1}}\right).$$

Now, let's investigate the behaviour of the sum for the remaining k. Using the fact that, for constant C_1' Theorem 3.3 implies that

$$\sum_{a\in A}\frac{1}{\langle a_1,\ldots,a_r\rangle^{\alpha+2}} \leq \frac{C_1'}{X^{\alpha+2}},$$

one may derive the following upper bound for R_k:

$$\begin{aligned}
R_k &\leq \kappa_k(\alpha)\sum_{v=1}^{\infty}\sum_{x+y=v}\sum_{\substack{x=a_1+\ldots+a_{j-1}\\ y=a_{j+1}+\ldots+a_r,\ a_r\geq 2}}\frac{(v-A)^k\kappa_k(\alpha)}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}}\\
&\leq 2\kappa_k(\alpha)\sum_{v=1}^{\infty}v^k\sum_{x+y=v}\left(\sum_{\substack{x=a_1+\ldots+a_{j-1}\\ a_{j-1}\geq 2}}\frac{1}{\langle a_1,\ldots,a_{j-1}\rangle^{\alpha+2}}\times\right.\\
&\qquad\qquad\left.\times\sum_{\substack{y=a_{j+1}+\ldots+a_r\\ a_{j-1}\geq 2}}\frac{1}{\langle a_{j+1},\ldots,a_r\rangle^{\alpha+2}}\right)\\
&\leq 2\kappa_k(\alpha)\sum_{v=1}^{\infty}v^k\sum_{x+y=v}\frac{C_1'^2}{x^{\alpha+2}y^{\alpha+2}}
\end{aligned}$$

$$(124)\quad \leq 2\kappa_k(\alpha)C_1'^2\int_1^w\frac{dv}{v^{\alpha+1-k}}\leq 2\kappa_k(\alpha)C_1'^2w^{k-\alpha},$$

which follows by the integral test. Therefore, one will see that:

$$\begin{aligned}
\sum_{k>\alpha}\frac{R_k}{N^{k+\alpha+2}} &\leq 2C_1'^2\sum_{k>\alpha}\kappa_k(\alpha)\frac{w^{k-\alpha}}{N^{k+\alpha+2}}\\
&\leq \frac{2C_1'^2}{N^{\alpha+2}w^{\alpha}}\sum_{k=1}^{\infty}\kappa_k(\alpha)\left(\frac{w}{N}\right)^k\\
&= \frac{2C_1'^2}{N^{\alpha+2}w^{\alpha}}\left(\frac{1}{1-\frac{w}{N}}\right)^{\alpha+2}=\mathcal{O}\left(\frac{1}{N^{\alpha+2}w^{\alpha}}\right).
\end{aligned}$$

The final equality above holds since the quantity $\frac{1}{1-\frac{w}{N}}$ does not exceed 2. The particular case where α is integer valued, from (124) and for fixed $\alpha = k$, gives

$$R_k = \mathcal{O}(\log(w)).$$

Therefore, one would have

$$\sum_{k\geq\alpha}\frac{R_k}{N^{k+\alpha+2}}=\mathcal{O}\left(\frac{\log(N)}{N^{2(\alpha+1)}}\right).$$

Thus, reassembling the original quantity $\tilde{\Sigma}^{(2)}_{(N,1)}$, we now see that

$$\begin{aligned}\tilde{\Sigma}^{(2)}_{(N,1)} &= \frac{C_0'}{N^{\alpha+2}} + \frac{1}{N^{\alpha+2}} \sum_{1\le k<\alpha} \frac{\kappa_k(\alpha)}{N^k} \sum_{v=1}^{\infty} \sum_{\substack{a\in\mathcal{A}_N \\ a_+..+a_{j-1}+a_{j+1}+..+a_r=v}} \frac{(v-A)^k \langle a_{j+1},\dots,a_r\rangle^{-(\alpha+2)}}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}} + \\ &+ \mathcal{O}\left(\frac{1}{N^{\frac{\alpha}{2}(2s-1)}} + \frac{1}{N^{\alpha+3}w^{\alpha-1}} + \frac{\log(N)}{N^{2(\alpha+1)}}\right),\end{aligned}$$

taking into account that this is the larger of the two possible error terms (yielded when α is an integer). We can therefore reassemble the main sum, using the previous proof of Theorem 3.3 in the following manner:

$$\begin{aligned}\sum_{a\in\mathcal{A}_N} \frac{1}{q(a)^{\alpha+2}} &= \frac{1}{N^{\alpha+2}}\left(C_0' + \sum_{1\le k<\alpha} \frac{L_{k,\alpha}}{N^k}\right) + \\ &+ \mathcal{O}\left(\frac{1}{N^{\frac{\alpha}{2}(2s-1)}} + \frac{1}{N^{\alpha+3}w^{\alpha-1}} + \frac{\log(N)}{N^{2(\alpha+1)}} + \frac{N^2\log^{2(\alpha+2)}(N)}{w^{2(\alpha+2)}}\right).\end{aligned}$$

Recall the definition of $L_{k,\alpha}$: here we see that the entity of least order of magnitude in the main term is $\mathcal{O}(N^{-(2\alpha+1)})$. Therefore one should select s such that

$$\begin{aligned}\frac{\alpha}{2}(2s-1) &= 2\alpha+2 \\ \Leftrightarrow s &= \frac{5\alpha+4}{2\alpha}.\end{aligned}$$

Let us now consider an appropriate value for the parameter w. It is easy to see that with $w = \frac{N}{2}$ the final error is $\mathcal{O}\left(\frac{\log^{2\alpha+4}(N)}{N^{2\alpha+2}}\right)$; this then gives the stated result of the Lemma.

□

Remark: Fixing the parameter such that the terms $\frac{N^2\log^{2(\alpha+2)}(N)}{w^{2(\alpha+2)}}$ and $\frac{\log(N)}{N^{2(\alpha+1)}}$ are equal yields w of order greater than N. One may obtain a satisfactory result by setting

$$w = \min\left\{\frac{N}{2}, N^{\frac{2\alpha+7/2}{2\alpha+4}} \log^{\frac{2\alpha+3}{2\alpha+4}}(N)\right\},$$

which gives final error $\mathcal{O}\left(\frac{\log(N)}{N^{2\alpha+\frac{3}{2}}}\right)$.

APPENDIX B

Further Data Plots Relating to the Main Theorem

In this appendix some illustrations supporting the statements and calculations of the main sections are presented. It is arranged so that there are several sections of explanatory text, followed by a cache of referenced figures.

1. Histograms

The histograms of figures B.1, B.2 and B.3 (on page 164) add further illustration as to the behaviour of the metric $\rho'(x)$ on the Farey partition. The end points along the width of each bar are the Farey neighbours $\left\{\frac{p}{q}, \frac{p'}{q'}\right\}$. This means that the bar widths are the values $p_{i,n} = \frac{1}{qq'}$: these are plotted against the heights $h'_{i,n} = \frac{1}{q+q'}$ yielded by the ρ'-metric.

2. Scatterplots

The scatterplots provided aim to illustrate the correlations between the main entities of interest as n increases - in essence these figures were produced as a form of sense checking. To aid readability a $-\log$ transform is again made of the data on both x and y axes. The first plot, figure B.4 on page 167, shows the relationship between the sums of largest contribution - the major sums - and shows that their contribution is indeed of similar magnitude. The second scatterplot, figure B.5, illustrates the relationship between the aggregated 'large' and 'small' sums. That is, the relationship between (for $\alpha = 2$)

$$S_{big} = \Sigma^{(4)+}_{(n,j=r-1)} + \Sigma^{(4)-}_{(n,j=r)}$$

and

$$S_{small} = \Sigma^{(4)+}_{(n,j=r)} + \Sigma^{(4)-}_{(n,j=r-1)}.$$

The third, figure B.6 shows a similar relationship between $\sigma_2(\mathrm{FT}_n)$ and $\gamma_2(\mathrm{FT}_n)$.

3. Pie Graphs

Figures B.7 (see page 169), and B.8 illustrate perhaps more intuitively the behaviour of the items within the partition. These are pie graphs whose sections represent the proportion yielded by each sum from that partition for $\alpha = 2$, and with parameter $s = \frac{2\alpha+1}{\alpha}$. These figures are designed to supplement the $-log$ plots provided earlier in this thesis (figures 3.4 and 3.5 on pages 57 and 58) and are based on true values (not log-scaled) as a proportion of the total value $\gamma_2(\mathrm{FT}_n)$. Moreover, the same colouring convention as is utilised throughout this thesis is retained when mapping the pie-slice sections. A key is included in figure B.8, which serves as a reminder to the reader of this convention.

The figures serve to highlight the dominance of the terms $\Sigma^{(4)-}_{(n,j=r)}$ and $\Sigma^{(4)+}_{(n,j=r-1)}$ over the others: for only the small n illustrated here the contribution of the other sums becomes of little relative significance very early in the sequence. This message is repeated in figure B.9, the analogous illustrations for $\alpha = 5$; the dominance of the terms of largest order is established very early in the sequence; the figures for small n are thus of most observable interest.

4. Evidence Supporting the Introduction of the New Method

Original consideration of the quantity $\gamma_\alpha(\mathrm{FT}_n)$ involved a grouping derived from the original work of [**31**]. In particular the analogous quantity of Lemma 5 in [**31**],

$$\Sigma^{(2)}_{(N,2)} + \Sigma^{(3)}_{(N,2)},$$

was considered. Figure B.10 (on page 172) illustrates the behaviour of this within the original hierarchy. Clearly the figure shows that these sums represent a major contribution to the overall result, which was not the case in the original result. The individual items $\Sigma^{(2)}_{(N,2)}$ and $\Sigma^{(3)}_{(N,2)}$ are illustrated in figures 3.14 and B.11 respectively. The first highlights a lesser order of magnitude. The second of these figures suggests that the component sum with partial quotient a_{r-1} as the largest, with the overall result being a significant contributor the overall main term of $\gamma_\alpha(\mathrm{FT}_n)$ (indeed this is calculated in Lemmas 3.18 and 3.19 to be $\mathcal{O}(n^{-(\alpha+1)})$).

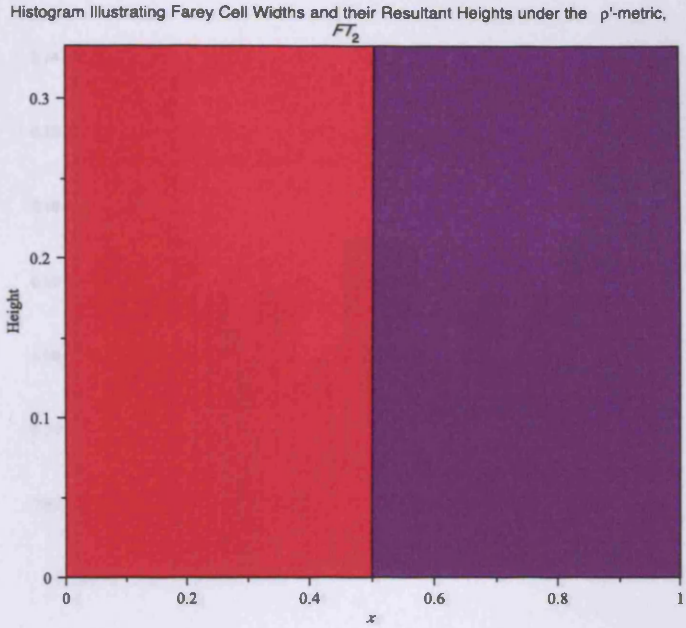
5. A Plot with Larger Value for s

The illustration of figure B.12 on page 173 demonstrates the effect of choosing a larger value for parameter s; the reader is directed to figures 3.4 and 3.5 on pages 57 and 58 respectively by which to draw comparison. Clearly the quantity represented by $\Sigma^{(1)}_{(N,2)}$ has greatest order of decrease when s is particularly large, although the comparative orders of magnitude calculated in the main lemmas of this thesis are still seen.

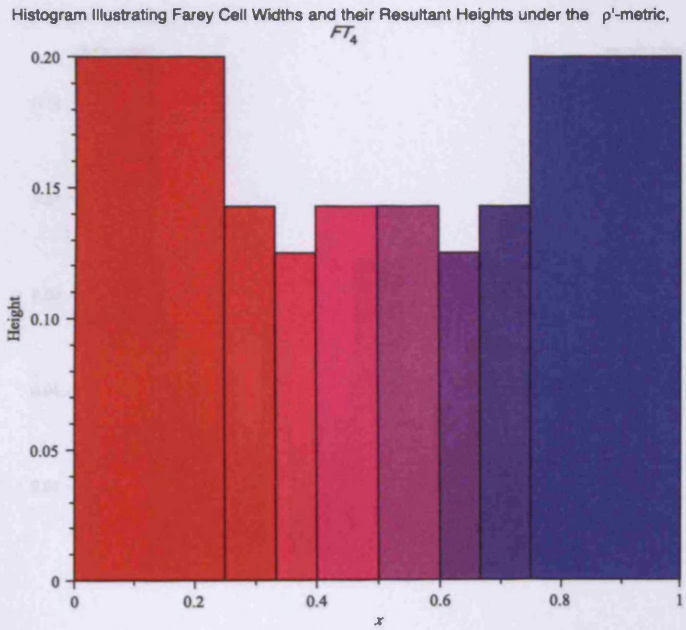
6. Plots for $\alpha = 5$

Finally, a number of figures in support of the main Theorem for $\alpha = 5$ are provided. Figures B.13 and B.14 (see page 174) illustrate the split (33) for this α, for $s = \frac{\alpha+3}{\alpha}$ and $s = \frac{2\alpha+3}{\alpha}$. One should note the main difference in these figures, where the smallest terms become of lesser significance, as would be expected under this construction.

The appendix is concluded with Figures B.15(a) and B.15(b) (see page 176). These illustrate the behaviour of the main terms in Lemmas 3.16 and 3.17 for $\alpha = 5$. These display a clear decrease in the error between the estimate and the actual values of the relevant sums. Figures for Lemmas 3.19, 3.18 and 3.21 are omitted due to their added dependence on the further estimates highlighted in those Lemmas.

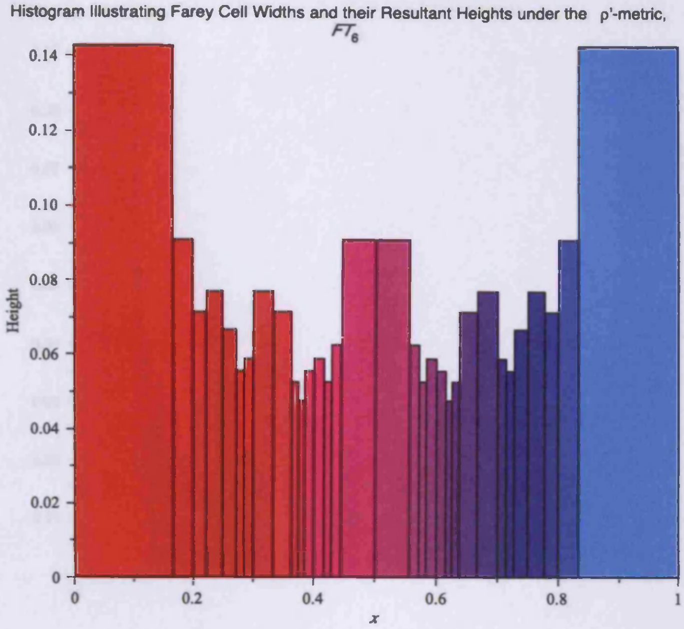


(a) $n = 2$

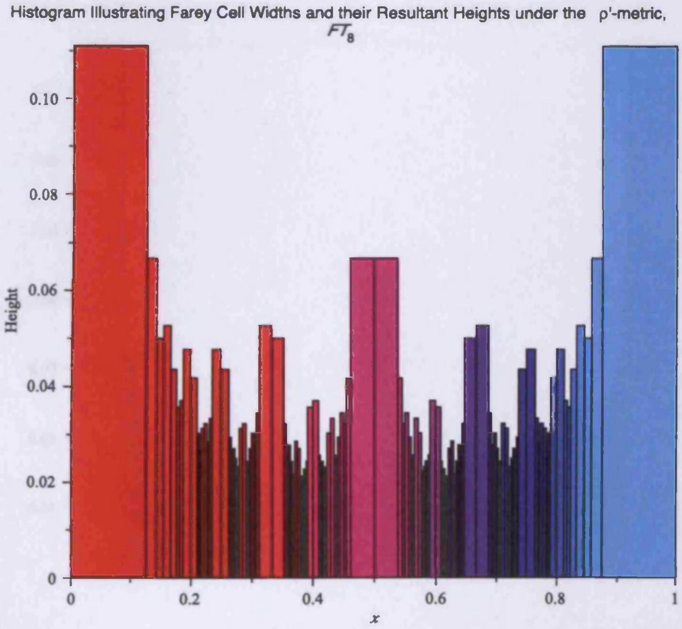


(b) $n = 4$

FIGURE B.1. Histograms of Cell Widths and Heights Under $\rho'_n(x)$

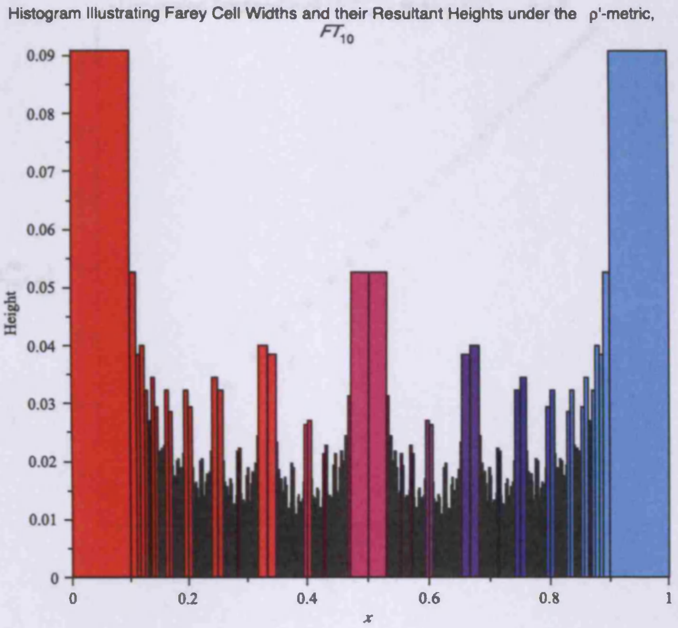


(a) $n = 6$

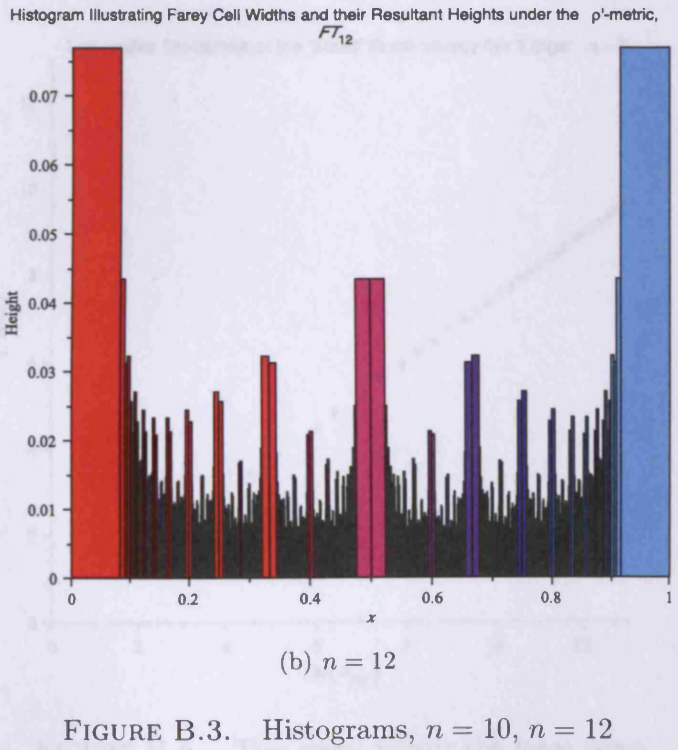


(b) $n = 8$

FIGURE B.2. Histograms, $n = 6$, $n = 8$



(a) $n = 10$



(b) $n = 12$

FIGURE B.3. Histograms, $n = 10$, $n = 12$

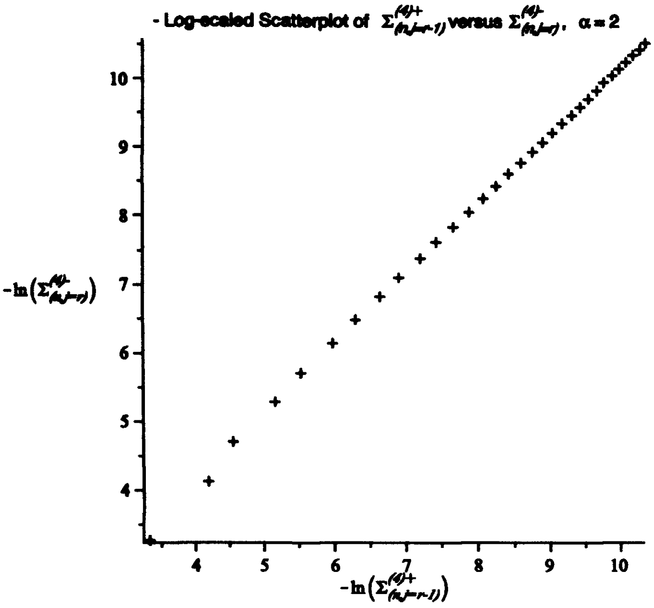


FIGURE B.4. The major sums, $\alpha = 2$

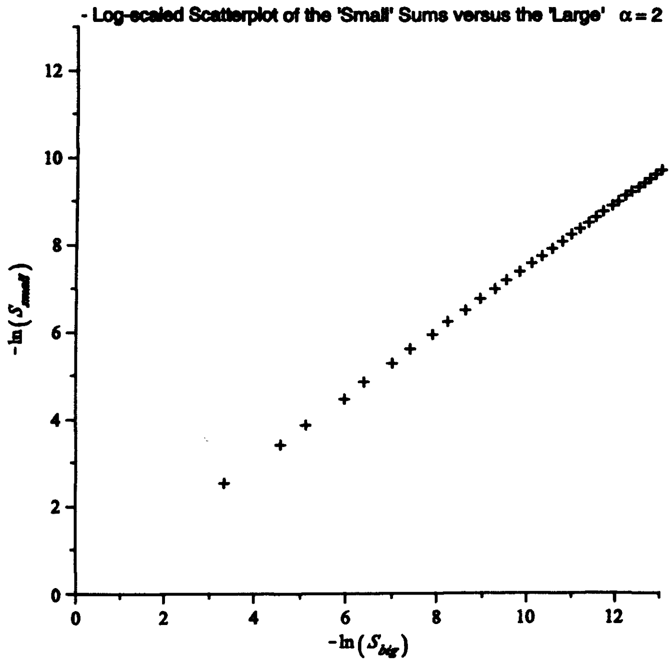


FIGURE B.5. The *small* versus the *large* sums

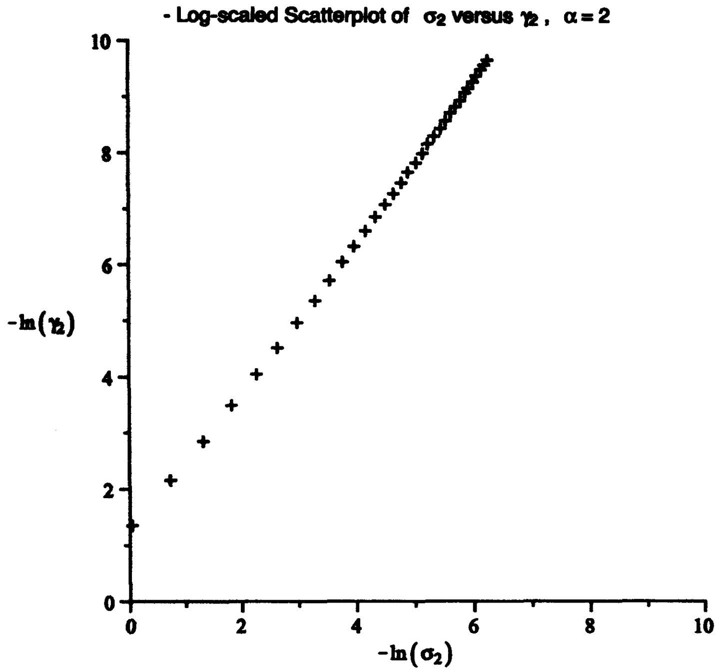


FIGURE B.6. ρ versus ρ'

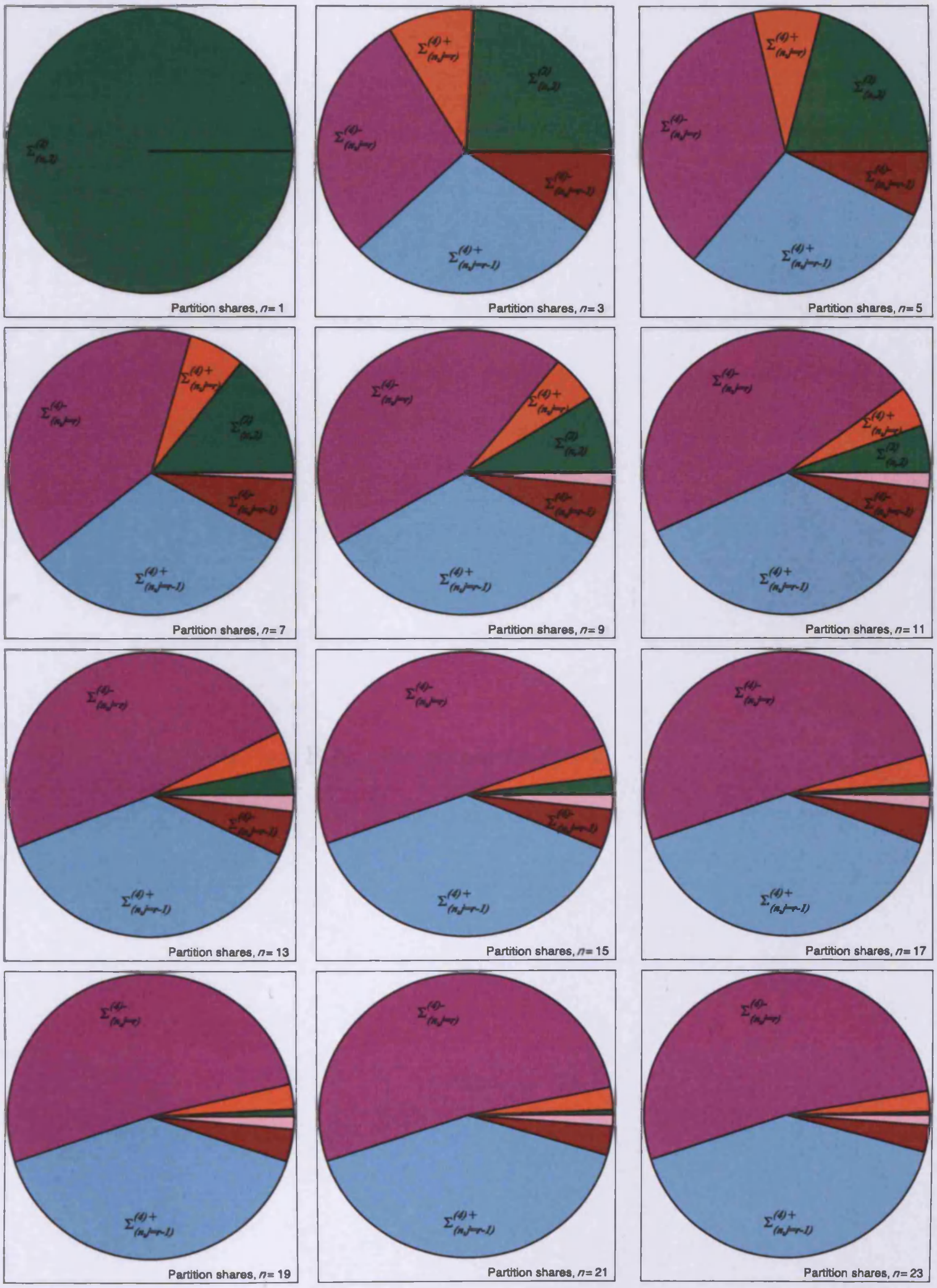


FIGURE B.7. Pie graphs for $\alpha = 2$

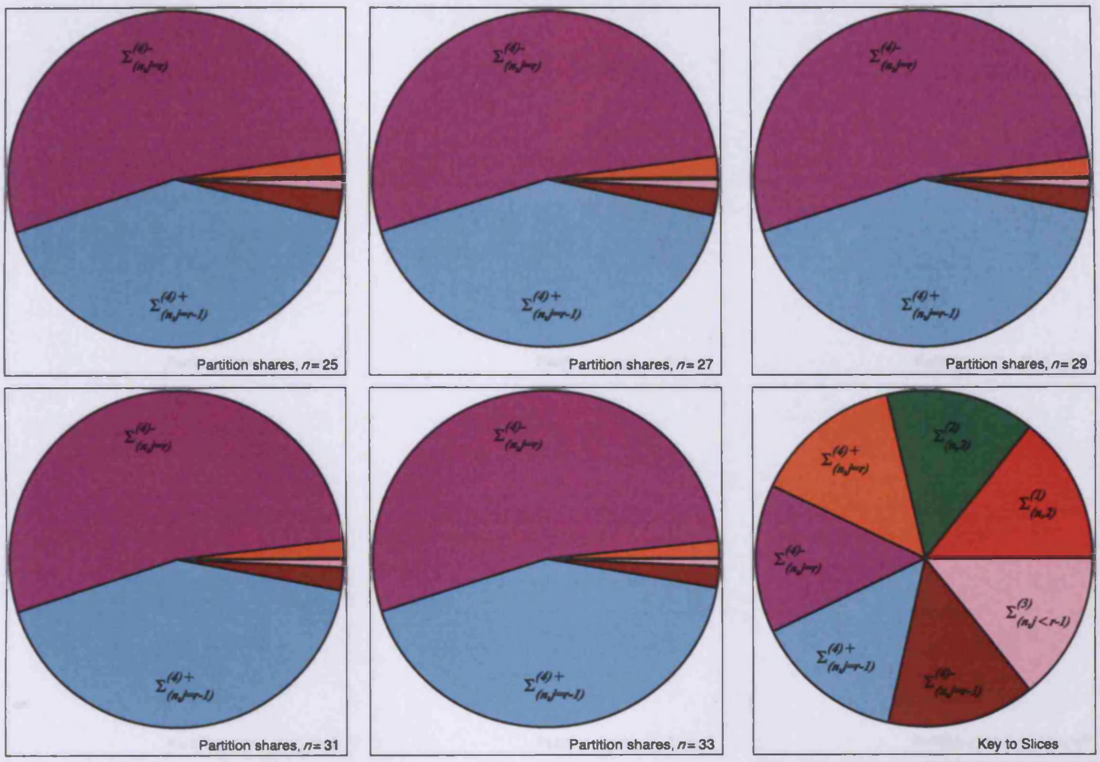


FIGURE B.8. Pie graphs for $\alpha = 2$

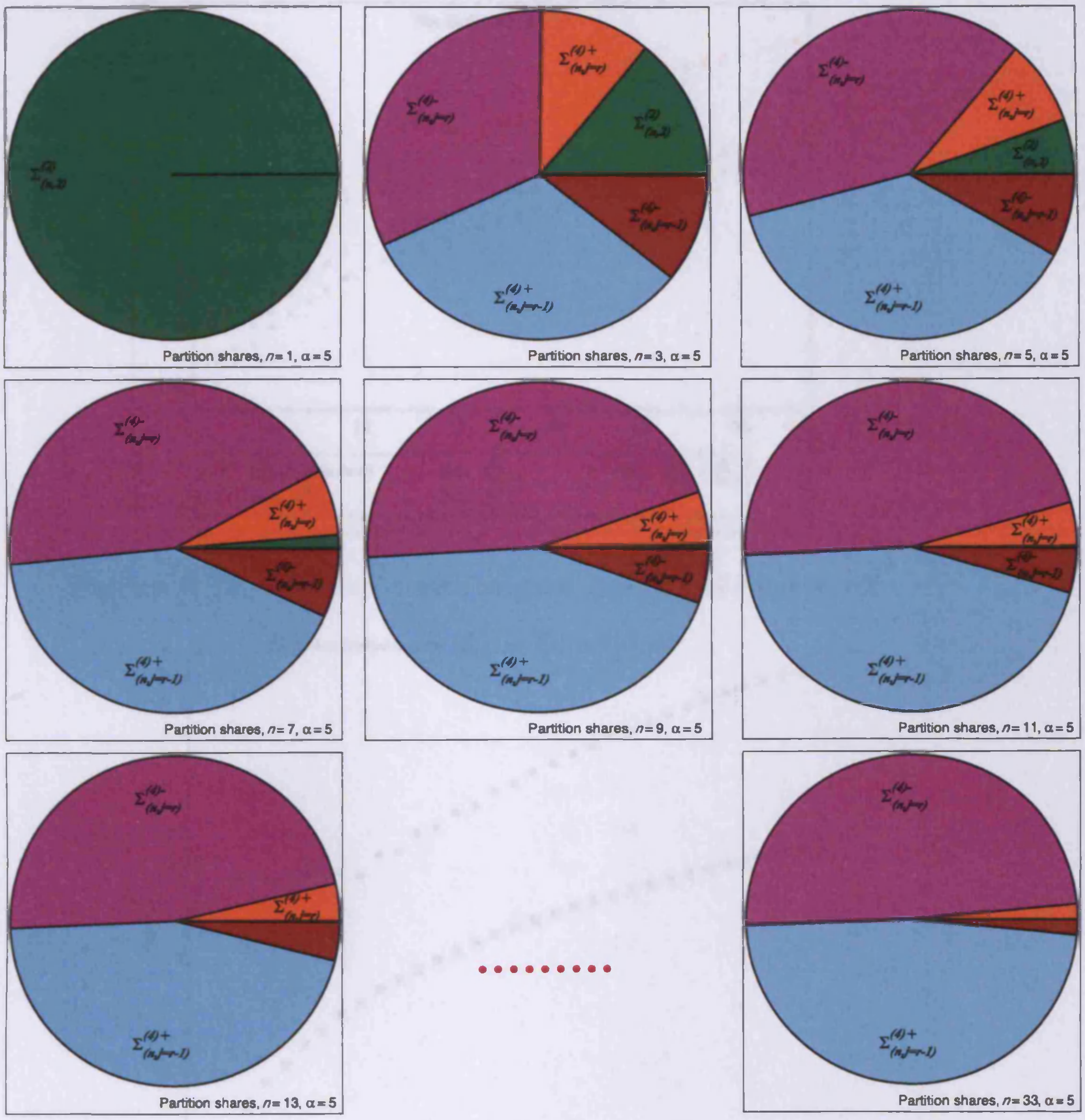


FIGURE B.9. Pie graphs for $\alpha = 5$

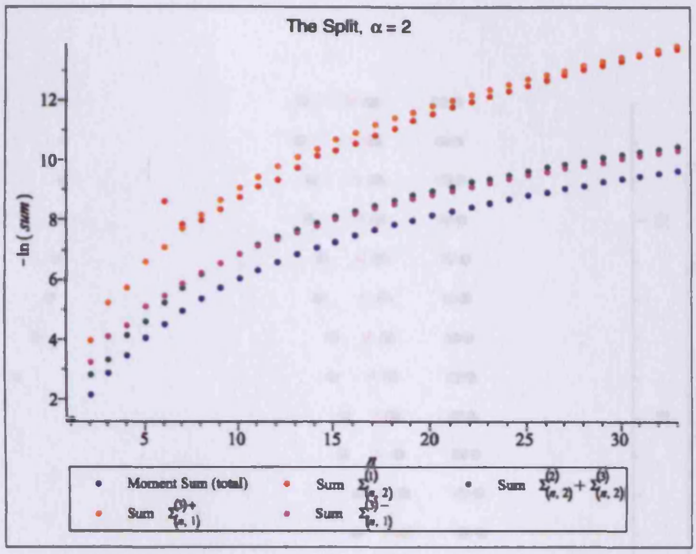


FIGURE B.10. The split with original grouping of sums $\alpha = 2$

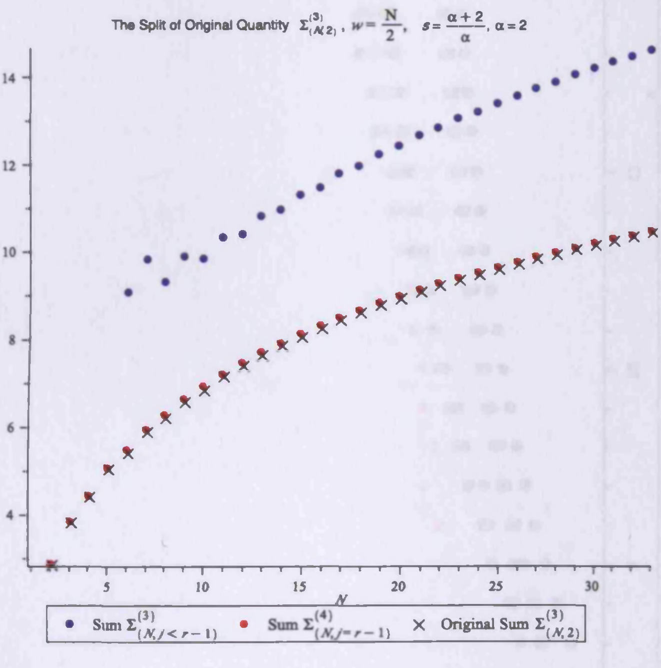


FIGURE B.11. The split of the original quantity

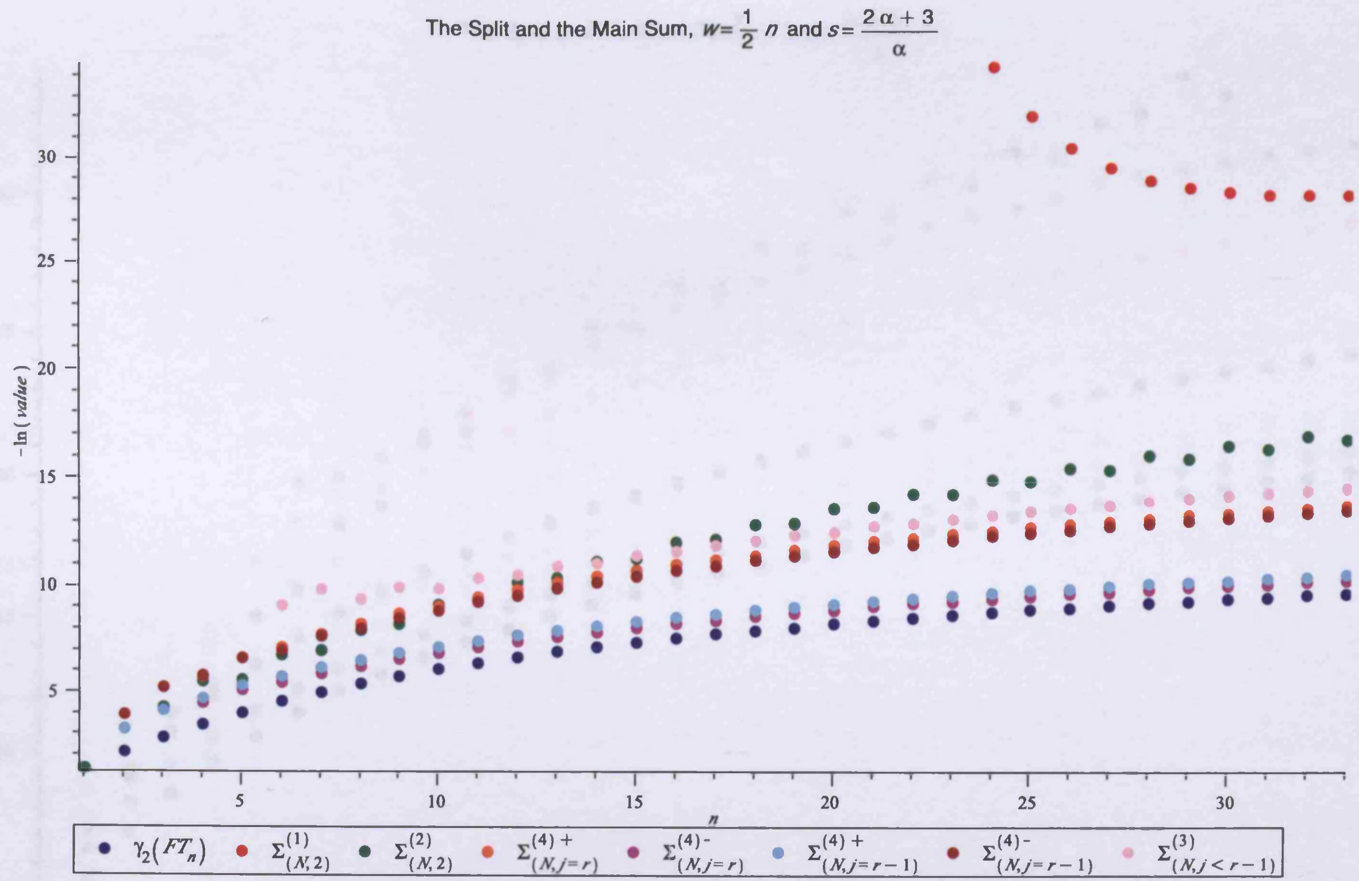


FIGURE B.12. The partition, $-\log(\cdot)$-scale. $\alpha = 2$ and $s = \frac{2\alpha+3}{\alpha}$

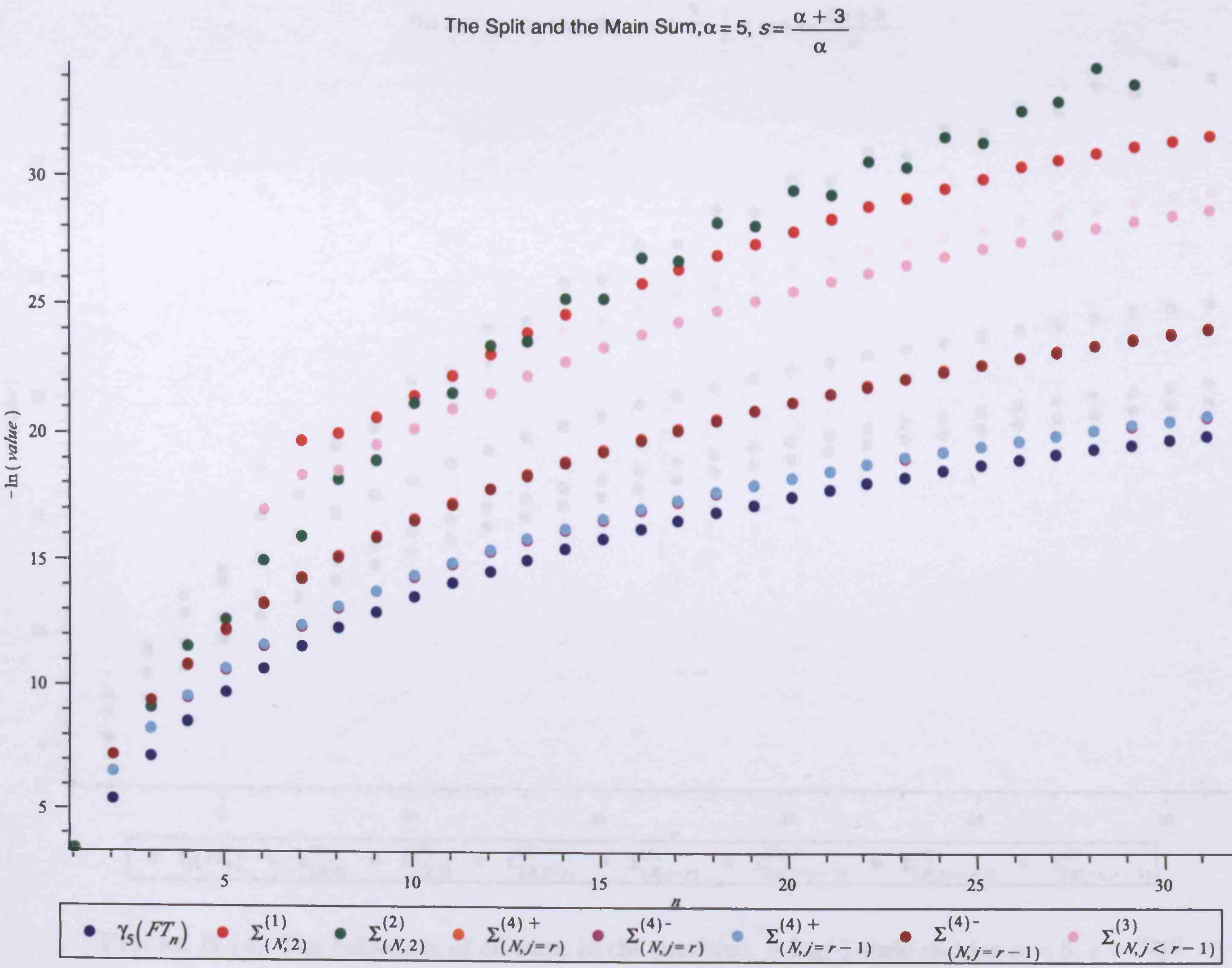


FIGURE B.13. The behaviour of all sums in the partition, $-\log(\cdot)$ scale and for $\alpha = 5$,

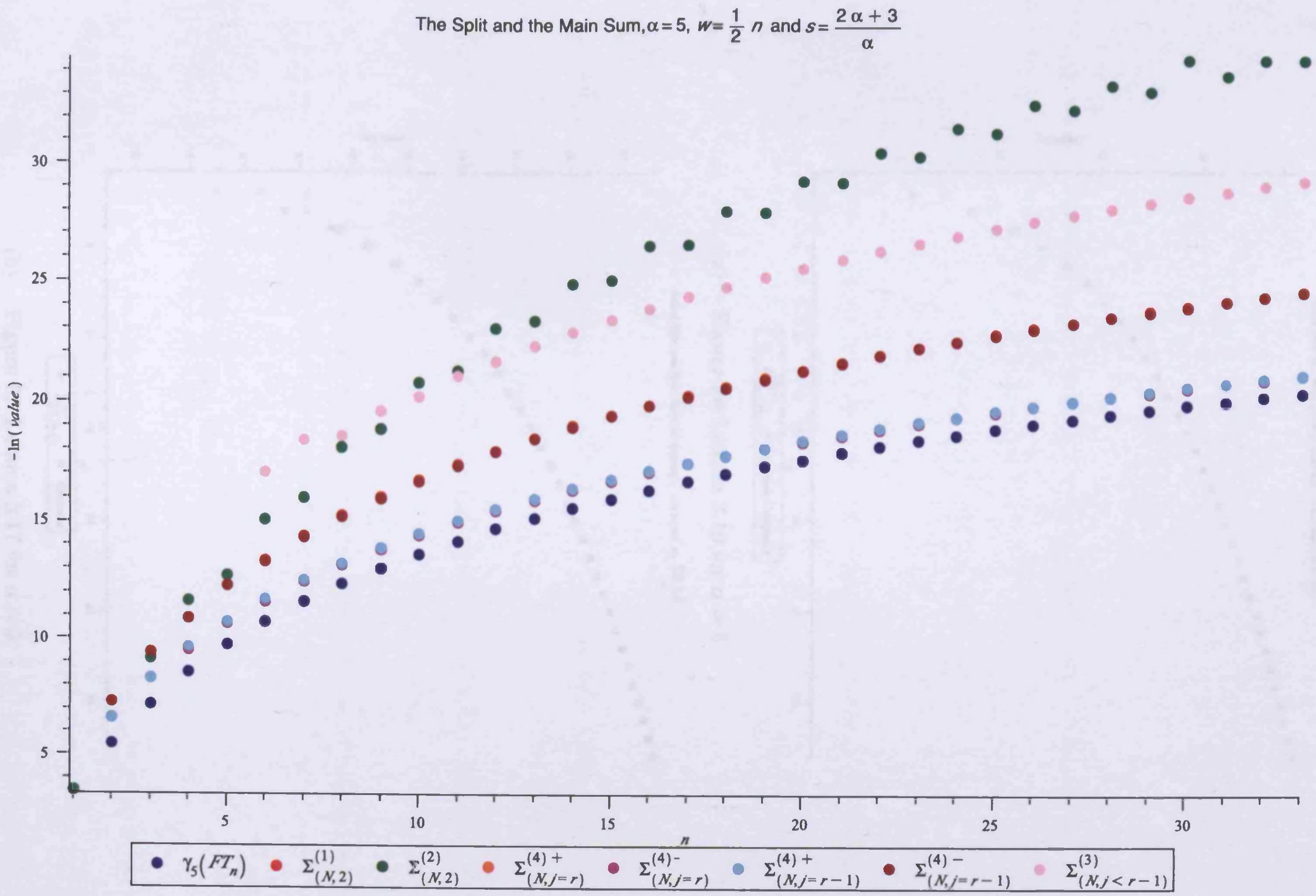
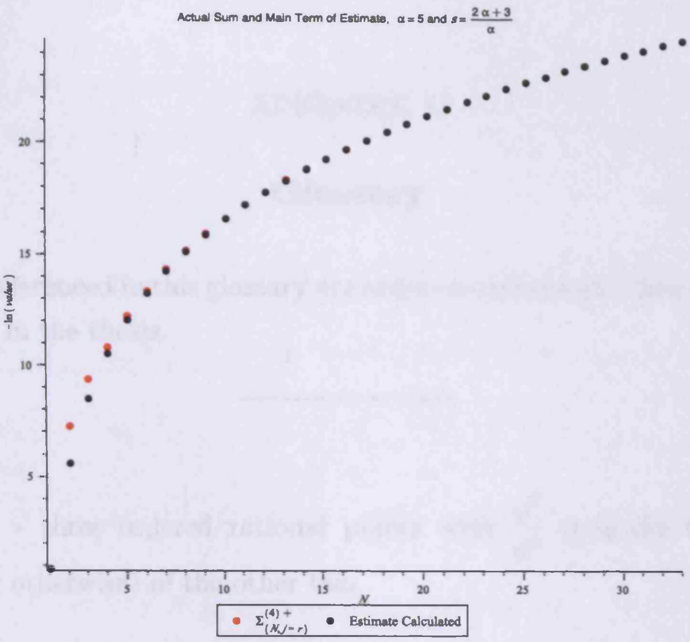
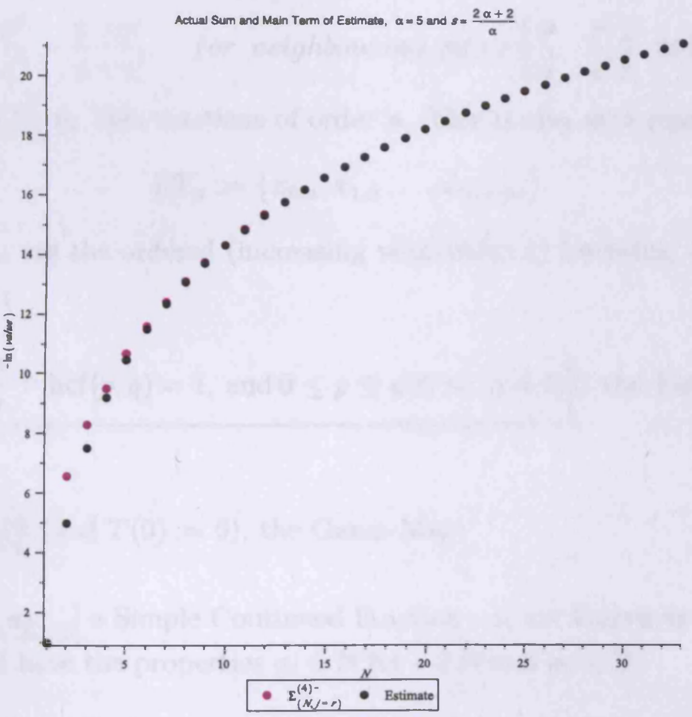


FIGURE B.14. The behaviour of all sums in the partition, $-\log(\cdot)$ scale and for $\alpha = 5$, $s = \frac{2\alpha+3}{\alpha}$



(a) Figure for Lemma 3.16 for $\alpha = 5$



(b) Figure for Lemma 3.17 for $\alpha = 5$

FIGURE B.15.

APPENDIX C

Glossary

The terms referenced in this glossary are ordered according to their first notable appearance in the thesis.

- $\frac{p}{q}, \frac{p''}{q''}, \frac{p'}{q'}$ - three ordered rational points with $\frac{p''}{q''}$ typically the mediant (Farey or otherwise) of the other two;

- $\mathrm{FT}_n = \mathrm{FT}_{n-1} \cup \mathcal{Q}_n$, where

  $$\mathcal{Q}_n := \left\{ \frac{p''}{q''} = \frac{p+p'}{q+q'},\ \ for\ neighbouring\ pairs \left\{ \frac{p}{q}, \frac{p'}{q'} \right\}\ in\,\mathrm{FT}_{n-1} \right\}:$$

  the set of Farey Tree fractions of order n. This is also seen represented as

  $$\mathrm{FT}_n := \left\{ x_{0,n}, x_{1,n} \dots, x_{N(n),n} \right\}$$

  where $x_{i,n}$ are the ordered (increasing with index i) fractions, and $N(n) = |\mathrm{FT}_n| - 1$;

- $\mathcal{F}_n := \left\{ \frac{p}{q}\ :\ \mathrm{hcf}(p,q) = 1,\ \mathrm{and}\, 0 \le p \le q \le n,\ \ q \ne 0 \right\}$, the Farey Series of order n;

- $T(x) = \left\{ \frac{1}{x} \right\}$ (and $T(0) := 0$), the Gauss Map;

- $[a_0; a_1, a_2, a_3 \dots]$ a Simple Continued Fraction - a_i are known as *partial quotients* and have the properties $a_i \in \mathbb{N}$ for $i \in \mathbb{N}$ and $a_0 \in \mathbb{Z}$;

- $\langle a_1, \dots, a_n \rangle$, a continuant polynomial (see page 11 for full definition);

- $\mathbf{F}(n)$ - the n^{th} Fibonacci number;

- $U(x)$ the Farey map, with branches $\frac{x}{1-x}$, for $x \in \left[0, \frac{1}{2}\right]$ and $\frac{1-x}{x}$ for $x \in \left[\frac{1}{2}, 1\right]$. The inverse branches of the Farey map are denoted by the functions Φ_1 and Φ_2;

- $z^{\overline{n}} = z(z+1)\dots(z+n-1)$, the 'climbing factorial' function;

- $z^{\underline{n}} = z(z-1)\dots(z-n+1)$, the 'falling factorial' function;

- $F\left(\begin{matrix} a_1, & a_2, & \dots & a_m \\ b_1, & b_2, & \dots & b_n \end{matrix} \middle| z\right) = \sum_{k=0}^{\infty} \frac{a_1^{\overline{k}}.a_2^{\overline{k}}\dots a_m^{\overline{k}}}{b_1^{\overline{k}}.b_2^{\overline{k}}\dots b_n^{\overline{k}}} \frac{z^k}{k!}$, a hypergeometric series (the left hand side is used as a convenient shorthand in calculations);

- A Farey Cell, defined as $I_i := [x_{i,n}, x_{i+1,n})$, where the $x_{j,n}$ are points from the set FT_n, generically $x_{j,n} = \frac{p}{q}$, $x_{j+1,n} = \frac{p'}{q'}$. The Farey cells form the *Farey Partition* $\mathcal{P}_n$ on interval $[0, 1)$;

- $p_{i,n} := x_{i+1,n} - x_{i,n} = \frac{1}{qq'}$, the standard length of a Farey Cell;

- $\rho(x, \mathrm{FT}_n) := \rho_n(x) = \min_{\frac{p}{q} \in \mathrm{FT}_n} \left| x - \frac{p}{q} \right|$, the standard (non-normalised) distance of a given x to its nearest neighbour from the Farey Tree set of order n;

- $\rho'(x, \mathrm{FT}_n) := \rho'_n(x) = \min_{\frac{p}{q} \in \mathrm{FT}_n} q \left| x - \frac{p}{q} \right|$, the normalised distance of a given x to its nearest neighbour from the Farey Tree set of order n. The normalisation used is the denominator of the nearest Farey Tree neighbour;

- $m^{(\cdot)}$ - the midpoint of a Farey Cell, as given by the metric ρ (seen as $m_{i,n}$) or ρ' (seen as $m'_{i,n}$). The generic identifier omits the subindex notation for brevity;

- $h^{(\cdot)}$ - the height of a Farey Cell, as given by the metric ρ (seen as $h_{i,n}$) or ρ' (seen as $h'_{i,n}$). The generic identifier omits the subindex notation for brevity;

- $\mu'_n = \sum_{i=1}^{N(n)} P_{i,n}\delta(t - h'_{i,n})$ - the measure assigning the quantities $P_{i,n} = \frac{p_{i,n}}{h'_{i,n}}$ to each of the points $h'_{i,n}$;

- $\Psi_n(t) = \text{meas}\{x \in [0,1] : \rho'_n(x) \leq \tau\}$, where 'meas' is the Lebesgue measure. Figure 2.3 illustrates its construction;

- $\int_0^1 \rho_n(x)^\delta = \frac{2^{-\delta}}{\delta+1} \sum_{(q,q')} \frac{1}{(qq')^{\delta+1}}$, the integral of function $\rho_n(x)$ for $x \in [0,1]$ (and thus over all Farey Cells partitioning that interval);

- $\sigma_\beta(\text{FT}_n) = \sum_{(q,q')} \frac{1}{(qq')^\beta}$, the $\rho_n(x)$ moment sum, defined for $\beta > 1$;

- $\int_0^1 \rho'_n(x)^\alpha = \frac{1}{\alpha+1} \sum_{(q,q')} \frac{1}{qq'(q+q')^\alpha}$, the integral of function $\rho'_n(x)$ for $x \in [0,1]$ (and thus over all Farey Cells partitioning that interval);

- $\gamma_\alpha(\text{FT}_n) = \sum_{(q,q'',q')} \frac{1}{qq'(q'')^\alpha}$, where $q'' = q + q'$, the $\rho'_n(x)$ moment sum, defined for $\alpha > 1$;

- $N = n + 1$ - for a given level n of the Farey Tree sets, N is used to denote the set given by one further iteration of the algorithm which generates the Farey Tree fractions;

- $\mathcal{A}_N := \left\{ (a_1, a_2, \ldots, a_r) : a_r \geq 2, \sum_{i=1}^{r} a_i = n + 1 = N \right\}$, the set of vectors whose members contain the partial quotients from each of the elements of $\mathcal{Q}_N$;

- $\frac{p_-(a)}{q_-(a)} = [a_1, \ldots, a_{r-1}]$, the 'minus' neighbour of fraction $\frac{p(a)}{q(a)} \in \text{FT}_N$. This fraction has origin FT_m, for $m < N - 1$;

- $\frac{p_+(a)}{q_+(a)} = [a_1, \ldots, a_r - 1]$, the 'plus' neighbour of fraction $\frac{p(a)}{q(a)} \in \text{FT}_N$. This fraction has origin FT_{N-1};

- $\sum_{a \in \mathcal{A}_N} \frac{1}{q''(a)^{\alpha+2}}$, the sum over all items $\frac{1}{q(a)^{\alpha+2}}$ with $a = (a_1, \ldots, a_r) \in \mathcal{A}_N$;

- $\mathcal{A}^{(1)}_{(N,1)} := \{a \in \mathcal{A}_N : \langle a_1, \ldots, a_r \rangle < N^s\}$, a subset of $\mathcal{A}_N$, with $1 \leq s < \frac{\log(F(N+1))}{\log(N)}$;
- $\mathcal{A}^{(1)}_{(N,2)} := \{a \in \mathcal{A}_N : \langle a_1, \ldots, a_r \rangle \geq N^s\}$, a subset of $\mathcal{A}_N$
- $\mathcal{A}^{(2)}_{(N,1)} := \{a \in \mathcal{A}^{(1)}_{(N,1)} : \max_{j=1,\ldots,r} a_j > N - w\}$, a subset of $\mathcal{A}_N$, with $1 \leq w \leq N$;
- $\mathcal{A}^{(2)}_{(N,2)} := \{a \in \mathcal{A}^{(1)}_{(N,1)} : \max_{j=1,\ldots,r} a_j \leq N - w\}$, a subset of $\mathcal{A}_N$
- $\mathcal{A}^{(3)}_{(N,j\geq r-1)} := \{a \in \mathcal{A}^{(2)}_{(N,1)} : \max_{i=1,\ldots,r} a_i = a_j \in \{a_{r-1}, a_r\},\ a_j > N - w\}$, a subset of $\mathcal{A}_N$
- $\mathcal{A}^{(3)}_{(N,j<r-1)} := \{a \in \mathcal{A}^{(2)}_{(N,1)} : \max_{i=1,\ldots,r} a_i = a_j \in \{a_1, \ldots, a_{r-2}\},\ a_j > N - w\}$, a subset of $\mathcal{A}_N$
- $\mathcal{A}^{(4)}_{(N,j=r-1)} := \{a \in \mathcal{A}^{(3)}_{(N,j\geq r-1)} : a_j = a_{r-1} > N - w\}$, a subset of $\mathcal{A}_N$
- $\mathcal{A}^{(4)}_{(N,j=r)} := \{a \in \mathcal{A}^{(3)}_{(N,j\geq r-1)} : a_j = a_r > N - w\}$, a subset of $\mathcal{A}_N$;

- $\Sigma^{(j)}_{(N,i)} := \sum_{a \in \mathcal{A}^{(j)}_{(N,i)}} \left(\frac{1}{q_+(q'')^{\alpha+1}} + \frac{1}{q_-(q'')^{\alpha+1}}\right)$ a sum over each of the sets of vectors defined above, with i and j used to denote which of the sets is in use;

- $\Sigma^{(j)+}_{(N,i)}$ - as above with only the summands which have the 'plus' denominators. For example $\Sigma^{(4)+}_{(N,j=r)} = \sum_{a \in \mathcal{A}^{(4)}_{(N,j=r)}} \frac{1}{q_+(q'')^{\alpha+1}}$;

- $\Sigma^{(j)-}_{(N,i)}$ - the 'minus' analogue of the previous definition;

- $\tilde{\Sigma}^{(j)}_{(N,i)} = \sum_{a \in \mathcal{A}^{(j)}_{(N,i)}} \frac{1}{q(a)^{\alpha+2}}$ - analogous sub-sums for the sum with single denominator-terms in its summands;

- $\Sigma^{(1)}_{(N,2)} = \mathcal{O}\left(\frac{1}{N^{s\alpha}}\right)$ - the error term of Lemma 3.14, which appears extensively throughout the other associated Lemmas of Theorem 3.5;

- $\theta_k(\alpha) = \frac{1}{k!}\prod_{i=1}^{k}(\alpha + i)$ - a binomial coefficient which appears in the main terms of the formulae calculated in Lemmas 3.16, 3.17, 3.18, 3.19 and 3.21;

- $K^{+}_{\alpha,n} = \sum_{1\le k<\alpha+1} \sum_{v=1}^{\infty} \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\ge 2}} \frac{(v-A)^k}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}} \sum_{j+l=k} \frac{\theta_j(\alpha)}{n^l(n+1)^j}$ - the item from the main term of Lemma 3.16 yielded by Taylor expansions and associated calculations on the sum $\Sigma^{(4)+}_{(N,j=r)}$;

- $K^{-}_{\alpha,n} = \sum_{1\le k<\alpha+1} \frac{1}{(n+1)^k} \sum_{v=1}^{\infty} \sum_{\substack{a_1+\ldots+a_{r-1}=v \\ a_{r-1}\ge 2}} \frac{1}{\langle a_1,\ldots,a_{r-1}\rangle^{\alpha+2}}(v-A)^k\theta_k(\alpha)$ - the item from the main term of Lemma 3.17 yielded by Taylor expansions and associated calculations on the sum $\Sigma^{(4)-}_{(N,j=r)}$;

- $q''(\tilde{a}) = \begin{cases} \langle a_1,\ldots,a_r-1\rangle = q_+(a), & \text{if } a_r > 2 \\ \langle a_1,\ldots,a_{r-1}+1\rangle =: q''(\tilde{a}'), & \text{if } a_r = 2. \end{cases}$ These are the partial quotients derived from $q(a) = \langle a_1,\ldots,a_r\rangle$ as being the possible 'parent' nodes from the *previous* level under Farey arithmetic;

- $\Omega_v = \{\bar{a} = (a_1,\ldots,a_{r-2},a_r) \,:\, a_1+\ldots+a_{r-2}+a_r = v,\ a_r \ge 3\}$ - a set of integer vectors with the partial quotient a_{r-1} removed, as defined first in Lemma 3.18;

- $K^{*-}_{\alpha,n} = T_\alpha + \sum_{1\le k<\alpha-1} \frac{1}{n^k} \sum_{v=1}^{\infty} \sum_{\Omega_v} \frac{(a_r-1)^{-(\alpha+1)}}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}} \sum_{j+l=k} \theta_j(\alpha)(v-A_2)^j(v-A_1)^l$, - the item from the main term of Lemma 3.18 yielded by Taylor expansions and associated calculations on the sum $\Sigma^{(4)+}_{(N,j=r-1)}$. The item T_α is a constant and A_1 and A_2 are used to denote the continued fractions on page 100 for brevity;

- $K^{*+}_{\alpha,n} = U_\alpha + \sum_{1\le k<\alpha} \frac{1}{n^k} \sum_{v=1}^{\infty} \sum_{\Omega_v} \frac{(a_r-1)^{-(\alpha+1)}}{(a_r-2)} \frac{\sum_{j+l=k}\theta_j(\alpha)(v-A_2)^j(v-A_3)^l}{\langle a_1,\ldots,a_{r-2}\rangle^{\alpha+2}}$ - the item from the main term of Lemma 3.19 yielded by Taylor expansions and associated calculations on the sum $\Sigma^{(4)-}_{(N,j=r-1)}$. The item U_α is a constant and A_2 and A_3 are used to denote the continued fractions on page 100 for brevity;

- $K^{**}_{\alpha,n} = \sum_{1\le k<\alpha-1} n^{-k} \left(\begin{array}{l} \sum_{v=1}^{\infty}\sum_{\Theta_v} \frac{\sum_{j+l=k}\theta_j(\alpha)(v-E_1)^j(v-E_3)^l\langle a_1,\dots,a_{j-1}\rangle^{-(\alpha+2)}}{\langle a_{j+1},\dots,a_r-1\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle} \\ +\sum_{v=1}^{\infty}\sum_{\Theta_v} \frac{\sum_{j+l=k}\theta_j(\alpha)(v-E_1)^j(v-E_2)^l\langle a_1,\dots,a_{j-1}\rangle^{-(\alpha+2)}}{\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_{r-1}\rangle} \end{array} \right)$ - the item from the main term of Lemma 3.21 yielded by Taylor expansions and associated calculations on the sum $\Sigma^{(3)}_{(N,j<r-1)}$. The item U_α is a constant and E_1, E_2 and E_3 are used to denote the continued fractions on page 116 for brevity;

- $\mathcal{G}_{k,\alpha} = \sum_{v=1}^{\infty} \sum_{\substack{a_1+\dots+a_{r-1}=v \\ a_{r-1}\ge 2}} \frac{1}{\langle a_1,\dots,a_{r-1}\rangle^{\alpha+2}}(v-A)^k\theta_k(\alpha)$ - this is the term associated with the quantity $\widehat{K}^-_{\alpha,n}$ which appears in the final main term of Theorem 3.5;

-

$$\begin{aligned}
\tilde{\mathcal{G}}_{k,\alpha} = & \sum_{v=1}^{\infty}\sum_{\Theta_v} \frac{\sum_{j+l=k}\theta_j(\alpha)(v-E_1)^j(v-E_3)^l}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r-1\rangle^{\alpha+1}\langle a_{j+1},\dots,a_r-1\rangle} \\
& + \sum_{v=1}^{\infty}\sum_{\Theta_v} \frac{\sum_{j+l=k}\theta_j(\alpha)(v-E_1)^j(v-E_2)^l}{\langle a_1,\dots,a_{j-1}\rangle^{\alpha+2}\langle a_{j+1},\dots,a_r\rangle^{\alpha+1}\langle a_{j+1},\dots,a_{r-1}\rangle} \\
& + \sum_{v=1}^{\infty} \sum_{\substack{a_1+\dots+a_{r-1}=v \\ a_{r-1}\ge 2}} \frac{(v-A)^k}{\langle a_1,\dots,a_{r-1}\rangle^{\alpha+2}} \sum_{j\le k}\theta_j(\alpha) \\
& + \sum_{v=1}^{\infty}\sum_{\Omega_v} \frac{1}{(a_r-1)^{\alpha+1}} \frac{\sum_{j+l=k}\theta_j(\alpha)(v-A_2)^j(v-A_1)^l}{\langle a_1,\dots,a_{r-2}\rangle^{\alpha+2}} \\
& + \sum_{v=1}^{\infty}\sum_{\Omega_v} \frac{1}{(a_r-1)^{\alpha+1}(a_r-2)} \frac{\sum_{j+l=k}\theta_j(\alpha)(v-A_2)^j(v-A_3)^l}{\langle a_1,\dots,a_{r-2}\rangle^{\alpha+2}}.
\end{aligned}$$

This is the analogue to the previous definition for the term associated with the quantity $\widehat{K}_{\alpha,n}$

- $\Sigma^{(j)}_{(N,i)} := \sum_{a\in\mathcal{A}^{(j)}_{(N,i)}} \frac{1}{(q(a)q'(a))^\beta}$ - for consistency of notation with previous work, we match these definitions onto the equivalents from the previous work of

[31] and [12]. It should be noted that in Chapter 4, this definition overwrites that stated in Chapter 3;

- E_k, D_k, B_k - these refer to items lifted directly from Lemmas 11, 12, 14 and 15 of [12] and are reintroduced under reassembly of the characteristic $\sigma_\beta(\mathrm{FT}_n)$ in Chapter 4;

- $\eta_k(\beta) := \frac{\prod_{i=1}^{k}(\beta+i-1)}{k!}$ - a binomial coefficient that arises in the calculations of Lemma 4.1, Chapter 4;

- $\tilde{F}_{k,1} = \sum_{v=1}^{\infty} \frac{1}{n^k} \sum_{\substack{a_1+\ldots+a_{r-2}=v-1 \\ a_{r-2}\geq 2}} \frac{(v-A_1)^k}{\langle a_1,\ldots,a_{r-2}\rangle^{2\beta}} \sum_{j+l=k} \eta_j(\beta)\eta_l(\beta)$ - this is an entity from Lemma 4.1 which appears in the final main term of the reconstructed quantity $\sigma_\beta(\mathrm{FT}_n)$ in Chapter 4;

- $F^-_{k,1} = \sum_{v=1}^{\infty}\sum_{\Omega_v} \frac{1}{\langle a_1,\ldots,a_{r-2}\rangle^{2\beta}(a_r-1)^\beta} \sum_{j+l=k} \eta_j(\beta)\,(v-A_1)^j\,\eta_l(\beta)(v-A_2)^l$ - this is an entity from Lemma 4.1 which appears in the final main term of the reconstructed quantity $\sigma_\beta(\mathrm{FT}_n)$ in Chapter 4. The entities A_1 and A_2 are continued fractions defined on page 139;

- $F^+_{k,1} = \sum_{v=1}^{\infty}\sum_{\Omega_v} \frac{(a_r-1)^{-\beta}(a_r-2)^{-\beta}}{\langle a_1,\ldots,a_{r-2}\rangle^{2\beta}} \sum_{j+l=k} \eta_j(\beta)\,(v-A_2)^j\,\eta_l(\beta)\,(v-A_3)^l$ - this is an entity from Lemma 4.1 which appears in the final main term of the reconstructed quantity $\sigma_\beta(\mathrm{FT}_n)$ in Chapter 4. The entity A_3 is a continued fraction defined on page 139;

- $\Sigma^{(3)}_{(N,2)} = \sum_{0\leq k<\beta-2} \frac{B_k}{N^{2\beta+k}} + \mathcal{O}\left(\frac{1}{N^{(\beta-1)(2s-1)}} + \frac{1}{N^{2\beta}w^{\beta-2}} + \frac{\log(N)}{N^{3\beta-2}}\right)$ - this is a previous result of [12], used in this work. It is notable here due to the change of notation to sum $\Sigma^{(3)}_{(N,j<r-1)}$ (under associated calculations) that is made.

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