Some Mathematical Aspects of Fuzzy Systems

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Abstract

In this work, three topics which are important for the further development of fuzzy systems are chosen to be investigated.

First, the mathematical aspects of fuzzy relational equations (FREs) are explored. Solving FREs is one of the most important problems in fuzzy systems. In order to identify the algebraic information of the fuzzy space, two new tools, called fuzzy multiplicative inversion and additive inversion, are proposed. Based on these tools, the relationship among fuzzy vectors in fuzzy space is studied. Analytical expressions of maximum and mean solutions for FREs, and an optimal algorithm for calculating minimum solutions are developed.

Second, the possibility of applying functional analysis theory to Takagi-Sugeno (T-S) fuzzy systems design is investigated. Fuzzy transforms, which are based on the generalised Fourier transform in functional analysis, are proposed. It is demonstrated that, mathematically, a T-S fuzzy model is equivalent to a fuzzy transform. Hence the parameters of a T-S fuzzy system can be identified by solving equations constructed using the inner product between membership functions and a given target function. The functional point of view leads to an insight into the behaviour of a fuzzy system. It provides a theoretical basis for exploring improvements to the efficiency of T-S fuzzy modelling.

Third, the mathematical aspects of model-based fuzzy control (MBFC) are investigated. MBFC theory is not suitable for general nonlinear systems, due to an implicit linearity assumption. This assumption limits fuzzy controller design to a special case of linear time-varying systems control. To apply MBFC in general nonlinear control, a new stability criterion for general nonlinear fuzzy system is proposed.

The mathematical aspects investigated in this research, provide a systematic guidance on issues such as efficient fuzzy systems modelling, balanced "soft" and "hard" computing in fuzzy system design, and applicability of fuzzy control to general nonlinear systems. They serve as a theoretical basis for further development of fuzzy systems.

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Notation

$\|\bullet\|$ norm

$\langle \cdot, \cdot \rangle$ an inner product

$D(x, y)$ multiplicative inversion, also written as $\dfrac{x}{y}$

$D_{\text{sup}}(x, y)$ supremum multiplicative

$D_{\text{inf}}(x, y)$ infimum multiplicative

FREs   fuzzy relational equations

$\hat{x}$ maximum solution of FREs

$\bar{x}$ mean solution of FREs

$\breve{x}_i$ a minimum solution (the minimum solution is not unique) of FREs

$\underline{x}$ the lower limit for those minimum solutions of FREs

$\overline{g(x)}$ the conjugate function of $g(x)$

$M(x, y)$ additive inversion, also written as $x \ominus y$

$M_{\text{inf}}(x, y)$ infimum additive inversion

I_S S implication $I_S(x, y) = S(1-x, y) = 1 - T(x, 1-y)$

I_R R implication $I_R(x, y) = \sup\{I \in R \,|\, T(x, y) \leq y\}$

I_{QL} QL Implication $I_{QL} = S(N(x), T(x, y))$

$T(x, y)$ a triangular norm or T norm, also written as $x \otimes y$

T_M minimum T norm, $T_M(x, y) = \min(x, y)$

T_p product T norm $T_p(x, y) = xy$

T_∞ Lukasiewicz T norm $T_\infty(x, y) = \max(0, x + y - 1)$

$S(x, y)$ a triangular co-norm or T co-norm, also written as $x \oplus y$

S_M maximum T co-norm $S_M(x, y) = \max(x, y)$

S_p probabilistic sum T co-norm $S_p(x, y) = x + y - xy$

S_∞ bounded sum T co-norm $S_\infty(x, y) = \min(x + y, 1)$

$\left(\dfrac{v}{w}\right)'$ supremum projection of vector v on vector w, $\left(\dfrac{v}{w}\right)' = \inf_{i \in \{1, \cdots n\}} \left(\left(\dfrac{v_i}{w_i}\right)_{\sup}\right)$

$\left(\dfrac{v}{w}\right)''$ infimum projection of vector v on vector w, $\left(\dfrac{v}{w}\right)'' = \sup_{i \in \{1, \cdots n\}} \left(\left(\dfrac{v_i}{w_i}\right)_{\inf}\right)$

Chapter 1

Introduction

1.1   Motivation

Since Zadeh proposed the theory of fuzzy sets (Zadeh, 1965), fuzzy systems have

received considerable attention. However, despite much success, some important issues

of fuzzy systems, which are crucial for the further development, have not yet been

resolved. They are:

- Where is the bottle-neck in fuzzy system modelling? How to improve modelling

  efficiency in the case of general fuzzy systems?

- How to achieve an optimum balance between "soft" and "hard" computing in fuzzy

  system design?

- How to apply fuzzy control in general nonlinear systems?

These are closely related to the following problems:

1)   Is there an analytical solution to a fuzzy relational equation (FRE) and, if so, what

     is it?

2)   What is the mathematical principle for Takagi-Sugeno (T-S) fuzzy system design?

3)   What is the stability criterion for general nonlinear fuzzy systems?

Mathematical aspects of these problems are chosen to be further investigated in order to

help build a firm foundation for the development of the field of fuzzy control.

The solution of fuzzy relational equations (FREs) is one of the most important and widely studied problems in the field of fuzzy sets and fuzzy systems. Since it is closely related to Mamdani fuzzy system design, the efficient solution of FREs will be significant to the development of Mamdani fuzzy systems. However, without an understanding of fuzzy algebra, fuzzy relational equations can only be solved numerically; an analytical solution for FREs is not yet available. Therefore, mathematical aspects of FREs need to be investigated.

Due to a lack of understanding of the mathematical principles of T-S fuzzy system design, most T-S fuzzy systems still need to use least-squares method to identify their parameters. Inspired by a strong analogy between mathematical transforms and T-S fuzzy models, it is possible to identify parameters of a T-S fuzzy model through the inner product between membership functions and the target function. Hence, it is of interest to explore the mathematical aspects of T-S fuzzy systems from this point of view.

Modern model-based fuzzy control (MBFC) theory is not suitable for nonlinear systems control, due to the implicit linearity assumption in fuzzy controller design. Under this assumption, a fuzzy control problem is expressed in a "sector-nonlinearity" form (Tanaka et al, 2003). This limits the fuzzy controller design to a special case of linear time-varying system control. In order to introduce MBFC to general nonlinear control, a new stability criterion for general nonlinear fuzzy systems needs to be identified. Therefore, mathematical aspects of nonlinear fuzzy systems, especially those concerned with stability analysis, require investigation.

1.2 Research Objectives

In this research, the mathematical aspects of the three problems described in section 1.1 were investigated.

The overall research objectives of the study were to:

- Derive an analytical solution for the fuzzy relational equation

- Derive the mathematical basis for T-S fuzzy system design

- Develop a stability criterion for general nonlinear fuzzy systems

In order to achieve these objectives, the following were attempted:

- Completing the triangular operations under fuzzy algebra

- Deriving the maximum, mean, and lower bound of the minimum solutions for max-family of FREs

- Based on functional analysis theory, developing a functional view point for T-S fuzzy models

- Developing an exact method for T-S fuzzy model parameters identification based on fuzzy transforms

- Developing an approximate method for T-S fuzzy model parameters identification based on dual bases

- Identifying the limitations of current model-based fuzzy control

- Integrating geometric information in the stability analysis of general nonlinear fuzzy systems

1.3 Outline of the Thesis

Chapter 2 reviews the fundamentals of fuzzy algebra, fuzzy relational equations and fuzzy control. The contents of this chapter serve as background knowledge for the following chapters.

Chapter 3 focuses on the analytical solution for fuzzy relational equations. In section 3.2, the concept of inversions of general triangular operators and their connections with fuzzy implication are introduced. The formulae for fuzzy multiplicative inversion and additive inversion are then proposed. Based on the fuzzy inversions, the relationship between fuzzy vectors in fuzzy space is studied and a complete analytical solution for the max-family FREs is developed.

Chapter 4 focuses on the application of functional analysis theory to the design of T-S fuzzy systems. This study is inspired by the strong analogy between mathematical transforms and T-S fuzzy models. Functional analysis is the mathematicians' "black-box diagram" (Curtain and Pritchard, 1977). It was developed to deal with functions instead of individual numerical values. The motivation for applying functional analysis to fuzzy systems is twofold. First, as an exact mathematical method, functional analysis can handle inexact data and knowledge. Second, the simple notions in functional analysis avoid many of the complicating details in design and analysis, highlighting only the essential aspects. Functional analysis is a convenient way to examine the behaviours of various models, including fuzzy models. This chapter advocate a functional view point for fuzzy systems and the concept of fuzzy transforms

is proposed as the mathematical basis of T-S fuzzy modelling. The chapter demonstrates the parameters of a T-S fuzzy system can be identified by solving equations constructed using the inner product between membership functions and a given target function. It also introduces the application of the concepts of dual base and dual spaces to improve the efficiency of fuzzy modelling.

Since a flexible stability criterion is key to nonlinear fuzzy control theory, Chapter 5 focuses on stability analysis for nonlinear fuzzy control. The chapter investigates the limitations of the modern fuzzy control approach and a problem with the commonly adopted decomposition principle. The chapter also studies perturbation theory and propose a new stability criterion based on the geometrical information in state space.

Finally, Chapter 6 summarises the research and makes suggestions for further work.

Chapter 2

Fuzzy Algebra, Fuzzy Relational Equations and Model-Based Fuzzy Control

Fundamental ideas in fuzzy algebra and fuzzy control are reviewed in this chapter. The contents of this chapter serve as background knowledge for the following chapters.

2.1 Triangular Norms and Fuzzy Implications

2.1.1 Basic T norms and co-norms

A triangular norm (T norm) is an operation on $[0,1]$, i.e. a function

$T(x, y) : [0,1] \times [0,1] \to [0,1]$ such that

- T is associative

- T is commutative

- T is non-decreasing

- T has 1 as a neutral element

If T is a T norm, then its dual T co-norm S is given by

$S(x, y) = 1 - T(1 - x, 1 - y)$

It is obvious that a T co-norm is also non-decreasing and it satisfies

$$S(x,0) = 1 - T(1 - x, 1) = x$$

There are many different T norms and T co-norms. The basic T norms and their corresponding co-norms are:

Minimum T norm T_M and maximum T co-norm S_M

$$T_M(x,y) = \min(x,y)$$

$$S_M(x,y) = \max(x,y)$$

Product T norm T_p and probabilistic sum T co-norm S_p

$$T_p(x,y) = xy$$

$$S_p(x,y) = x + y - xy$$

Lukasiewicz T norm T_∞ and bounded sum T co-norm S_∞

$$T_\infty(x,y) = \max(0, x + y - 1)$$

$$S_\infty(x,y) = \min(x + y, 1)$$

It can be verified that T_M is the largest T norm and S_M is the smallest T co-norm. It can be written as:

$$T_M(x,y) \geq T(x,y)$$

and

$$S_M(x,y) \leq S(x,y)$$

where $T(x,y)$ and $S(x,y)$ denotes general T norm and co-norm

2.1.2 Ring and Semiring structures

Algebraically, T norms and co-norms have a semiring structure. They construct a special semigroup on the unit interval $[0,1]$ (Allenby, 1991).

A ring (in the mathematical sense) is a set S, together with two binary operators $\times$ and $+$ (commonly interpreted as multiplication and addition, respectively) satisfying the following conditions:

1). Additive associativity: For all $a, b, c \in S$

$$(a+b)+c = a+(b+c)$$

2). Additive commutativity: For all $a, b \in S$

$$a+b = b+a$$

3). Additive identity: There exists an element $0 \in S$ such that for all $a \in S$,

$$a+0 = 0+a = a$$

4). Additive inversion: For every $a \in S$ there exists $-a \in S$ such that

$$a+(-a) = (-a)+a = 0$$

5). Multiplicative associativity: For all $a, b, c \in S$

$$(a*b)*c = a*(b*c)$$

6). Left and right distributivity: For all $a, b, c \in S$

$$a*(b+c) = (a*b)+(a*c)$$

and

$$(b+c)*a = (b*a)+(c*a)$$

Since T norms and T co-norms do not satisfy the additive inversion, and the distributivity conditions, the fuzzy algebra based on T norms and T co-norms only has a semiring structure. It is a fundamental difference between fuzzy algebra and linear algebra. This issue will be further discussed in Chapter 3.

2.1.3   Fuzzy Implication

Fuzzy implication is another important fuzzy operation used in fuzzy algebra. It is possible to construct fuzzy implication by the following ways:

First, since the statement "NOT A OR B" is equivalent to the value of the implication "IF A THEN B" in Boolean logic, fuzzy implication under fuzzy logic can be written as:

$$I_S(x,y) = S(1-x, y) = 1 - T(x, 1-y)$$

where T stands for a T norm and S a T co-nom. This implication is called an S implication. The following are examples of S implication for basic triangular operations.

For minimum T norm T_M and maximum T co-norm S_M

$$I_{S\,T_M}(x,y) = \max(1-x, y)$$

For product T norm T_p and probabilistic sum T co-norm S_p

$$I_{S\,T_p}(x,y) = 1 - x + xy$$

For Lukasiewicz T norm T_∞ and bounded sum T co-norm S_∞

$$I_{S\,T_\infty}(x,y) = \min(1 - x + y, 1)$$

It is obvious that $I_s(x,y)$ is equal to $I_s(\bar{y}, \bar{x})$. Therefore, the statement "IF A THEN B" is equivalent to the statement "IF NOT B THEN NOT A" under an S implication.

Another way to define the implication for fuzzy logic is to define implication using residuation (Janowitz, 1972). This kind of implication is called an R implication, which is defined on [0, 1] × [0, 1] → [0, 1] as:

$$I_R(x,y) = \sup\{I \in R \mid T(x,y) \le y\}$$

For the three basic T norms, T_M, T_P and T_∞, the R implication is formed as:

$$I_{R\,T_M}(x,y) = \begin{cases} 1 & if\ x \le y \\ y & other \end{cases}$$

$$I_{R\,T_P}(x,y) = \min(\frac{y}{x}, 1)$$

$$I_{R\,T_\infty}(x,y) = \min(1 - x + y, 1)$$

The third type of fuzzy implication in fuzzy logic is called the QL implication. It is defined on [0, 1] × [0, 1] → [0, 1] as:

$$I_{QL} = S(N(x),\ T(x,y)) \qquad \text{where } N(x) \text{ denotes the negation}$$

QL implications for the three basic T norms, T_M, T_P and T_∞, is written as:

$$I_{QL\ T_M}(x,y) = \max(1-x, \min(x,y))$$

$$I_{QL\ T_P}(x,y) = 1-x(1-xy)$$

$$I_{QL\ T_\infty}(x,y) = \min(1-x+\max(x+y-1,0),1)$$

It should be noted that although the implications defined above are equivalent to each other under classical Boolean logic, they are quite different under fuzzy logic. This is illustrated in figures 2-1, 2-2 and 2-3. It can be seen in the figures that although they are implication models for the same T norm, their shape are quite different from one another.

2.2 Fuzzy Relational Equations

Fuzzy relational equations (FREs) have an important role in fuzzy set theory and its applications. FREs were first introduced by Sanchez (1976), who proposed an algorithm to obtain the maximum solution for Max-Min based fuzzy relational equations. The problem of FREs has been studied by many researchers since then (Tsukamoto and Terano 1977, Wang 1983, Nola 1985, Pedrycz 1990, Adamopoulos and Pappis 1993, Fang and Li 1999, etc.). Let $A = [a_{ij}]$, $a_{ij} \in [0,1]$ be an $m \times n$ -dimensional matrix and $b = (b_1 \quad \cdots \quad b_i \quad \cdots \quad b_n)^T, b_i \in [0,1]$, be an
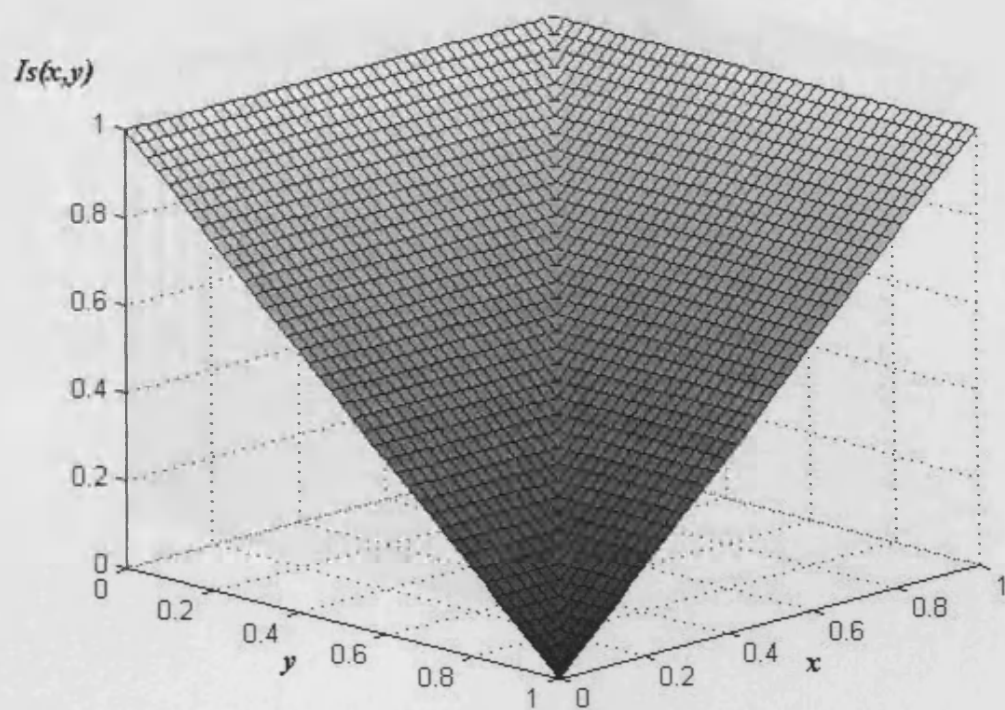
Figure 2-1 S implication based on T_M
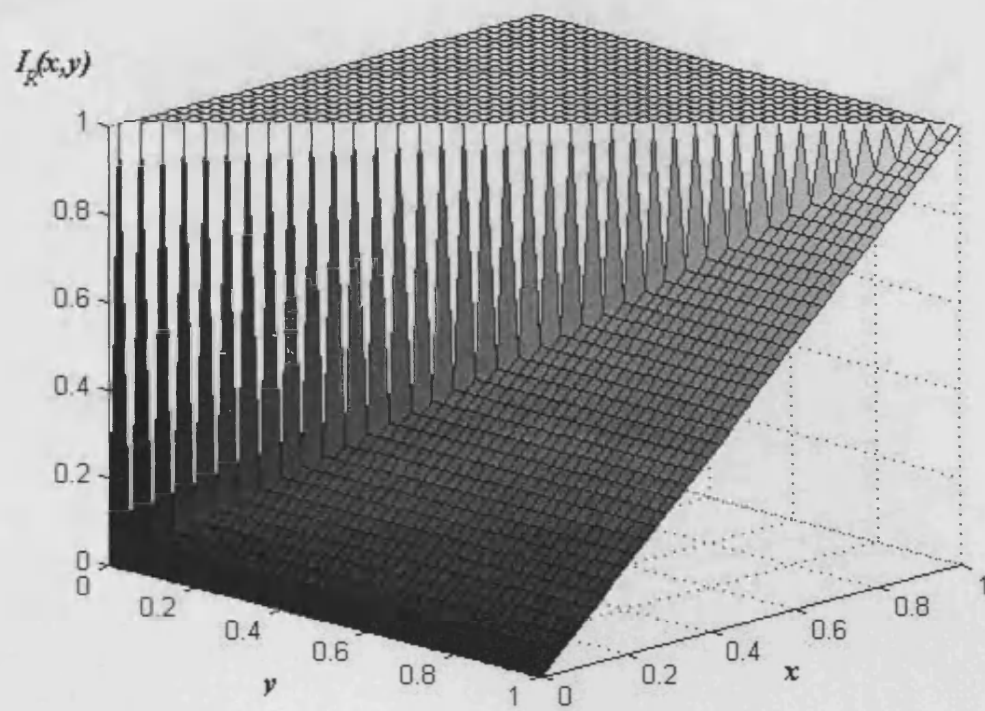
Figure 2-2 R implication based on T_M
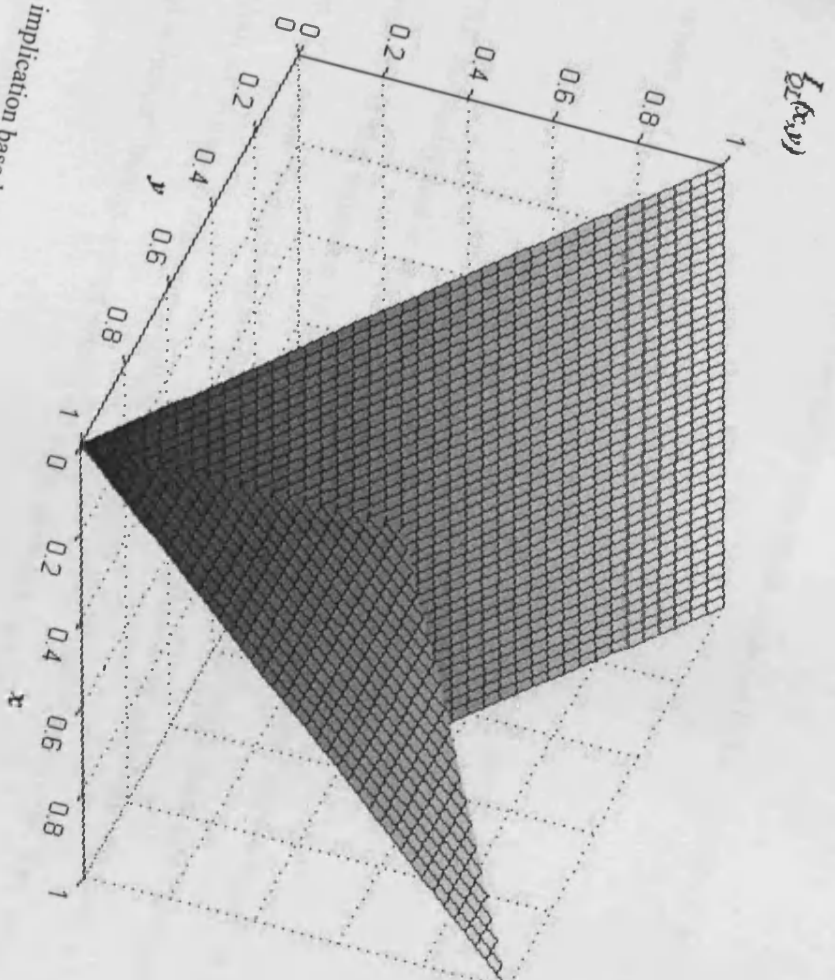
Figure 2-3 *QL* implication based on T_M

n-dimensional vector. The fuzzy relational equation (FRE) is defined as:

$$A \otimes x = b$$

where $x = \begin{pmatrix} x_1 & \cdots & x_i & \cdots & x_m \end{pmatrix}^T$ $x_i \in [0,1]$ The symbol $\otimes$ stands for a T norm.

A FRE normally has a set of solutions. There is always a maximum solution in the solution set if the equation is solvable. There can also be more than one minimum solution. The term "maximum solution" means the solution of which all elements are greater than or equal to the corresponding elements of all other solutions. The term "minimum solution" means one of which all other elements are not less than or equal to itself.

Different algorithms have been developed to solve fuzzy relational equations. Tsukamoto and Terano (1979) developed an algorithm that is able to find the maximal solution and a number of minimal solutions. This algorithm was successfully applied to fault diagnosis (Tsukamoto and Terano, 1977). The algorithm comprises the following steps:

STEP 1. Form matrix U by ω-composition, i.e.

$$u_{ij} = a_{ij} \omega b_j = \begin{cases} b_j & \text{if } a_{ij} > b_j \\ \begin{bmatrix} b_j, & 1 \end{bmatrix} & \text{if } a_{ij} = b_j \\ \phi & \text{otherwise} \end{cases}$$

u_{ij} can be a set having only one element b_j, a set with an infinite number of elements $[b_j, \quad 0]$ or an empty set ϕ.

STEP 2. Form matrix V by ϖ-composition, i.e.

$$v_{ij} = a_{ij}\varpi b_j = \begin{cases} [0, \quad b_j] & \text{if } a_{ij} > b_j \\ [0, \quad 1] & \text{otherwise} \end{cases}$$

v_{ij} is a set having an infinite number of elements. v_{ij} is either $[0, \quad b_j]$ or

$[0, \quad 1]$.

STEP 3. Construct matrix W by replacing one entry in each column of V with

the corresponding entry of U, e.g.

$$W = \begin{pmatrix} u_{11} & u_{12} & v_{13} & \cdots & v_{1n} \\ v_{21} & v_{22} & u_{23} & \cdots & v_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_{m1} & & & & u_{mn} \end{pmatrix}$$

STEP 4. Apply the intersection operation to the entries (sets) of each row of the

obtained matrix W. The result is a solution of the fuzzy relational

equation if all the resulting sets are not empty.

STEP 5. Go back to STEP 3 to form a new W matrix if there are more entries to

be replaced. Otherwise, terminate.

Another algorithm involves applying the ∝-operation to the fuzzy relation and the

given goal (Pedrycz, 1990). This algorithm has the following steps:

STEP 1. Calculate the row vector T^* by applying the ∝-operation to A and b,

the relation matrix and the given goal, respectively, i.e.

$$T^* = \min(A \propto b)^T$$

STEP 2. Substitute T^* into the fuzzy relational equation,. If T^* satisfies the

equation, then it is the maximal solution of the equation.

According to Wang (1983), Li et al, proposed another algorithm as represented in. This algorithm is described as follows Wang (1983):

STEP 1. Construct matrix U, in which

$$u_{ij} = \begin{cases} b_j & \text{if } a_{ij} > b_j \\ \phi & \text{otherwise} \end{cases}$$

where u_{ij} can either be a set having only one element b_j, or an empty set.

STEP 2. Calculate the infimum of each row of U. If all the entries in one row are ϕ, the infimum is set to be 1. The infimum of every row forms a vector, denoted. $S = \begin{pmatrix} s_1 & \cdots & s_n \end{pmatrix}$

STEP 3. Form the matrix U' by adding more b_j s of b into U.

$$u'_{ij} = \begin{cases} b_j & \text{if } a_{ij} > b_j \text{ and } b_j < s_i \\ \phi & \text{otherwise} \end{cases}$$

where u'_{ij} can either be a set having only one element b_j, or an empty set.

STEP 4. Check if every column of U', resulting from the last step, contains at least one non-empty entry. If this is the case, the fuzzy relational equation is solvable and the vector S is accepted as the maximal solution. Otherwise, terminate.

STEP 5. Form matrix W. Select one non-empty entry from each column of matrix U' and put it into the corresponding position in W. Fill other positions in the obtained W matrix with ϕ.

STEP 6. Calculate one minimal solution, $S' = \begin{pmatrix} s'_1 & s'_2 & \cdots & s'_n \end{pmatrix}$ from matrix W by applying the supremum operation to each row of W. The supremum will be 0 if all the entries in a row are empty entries.

STEP 7. Return to STEP 5 if further selections are possible, otherwise, terminate.

A good review of algorithms for FREs can be found in (Li 1999).

2.3 MBFC and Lyapunov Stability

Model-based fuzzy control (MBFC) is a new fuzzy control approach developed in the 1990s. It can be regarded as "a middle ground between conventional fuzzy control practice and established control theory " (Tanaka et al 2001). It preserves the philosophy of fuzzy sets theory, while utilising feedback control theories to improve fuzzy controller design. Its design procedure is as follows:

- First, a first-order T-S fuzzy model is constructed for the plant by linearising local dynamics in different state-space regions.

- Second, For each local linear model, a linear feedback controller is designed. The overall controller, which is nonlinear in general, is constructed as "a fuzzy blending of each individual controller"(Wang et al, 1996).

- Third, the overall stability for the entire fuzzy system is evaluated via Lyapunov's direct method.

In the first step of fuzzy controller design, expert knowledge is applied to construct a T-S fuzzy model for the target process. In the second step, control theory is applied to

design the local feedback controllers. In the third step, due to the fundamental difference between linear and nonlinear systems regarding to the local and global stability; the linear controllers designed in the second step, need to be evaluated. Lyapunov's method is applied in the evaluation.

It should be noted that a T-S fuzzy system is a nonlinear system in general. Therefore, even if all its sub-systems were stable, global stability cannot be guaranteed. This fact makes the third step the most important step in MBFC. When global stability is not achieved, local nonlinear controllers need to be redesigned. The design procedure is repeated until global requirement is achieved.

Consider a dynamic system modelled by a set of fuzzy rules:

If X is X_i Then $\dot{X}_i = AX + B_i u$

$\vdots$

The overall output of the system is:

$$\dot{X} = \sum_{i=1}^{p} h_i(X)(A_i X + B_i u)$$

with $h_i(X)$ as normalised membership functions

where $X \in R^n, u \in R^m, A_i \in R^{n \times n}, B_i \in R^{n \times m} \quad i \in \{1 \cdots p\}$

If there exists a positive definite matrix P that satisfies:

$A_i^T P + P A_i < 0$

for all $i \in \{1 \cdots p\}$, the origin of the fuzzy system is asymptotically stable.

The inequality $A_i^T P + PA_i < 0$ can be solved efficiently by the linear matrix inequality, (Boyd, 1994) and was considered the basic mathematical principle behind MBFC.

2.4 Summary

This chapter has reviewed fundamental ideas in fuzzy algebra and fuzzy control. The chapter has discussed the solution of fuzzy relational equation and described the main existing solution techniques.

Chapter 3

Fuzzy Inversions and Analytical Solutions of Fuzzy Relational Equations

3.1   Preliminaries

The solution of fuzzy relational equations (FREs) is one of the most important and widely studied problems on field of fuzzy sets and fuzzy systems.

Let $A = [a_{ij}]_{mn}$, $a_{ij} \in [0,1]$, be an $m \times n$ matrix and $b = \begin{pmatrix} b_1 & \cdots & b_i & \cdots & b_n \end{pmatrix}^T$, $b_i \in [0,1]$, be an n-dimensional vector. A fuzzy relational equation is defined as:

$$A \otimes x = b \tag{3-1}$$

where $x = \begin{pmatrix} x_1 & \cdots & x_i & \cdots & x_m \end{pmatrix}^T$ $x_i \in [0,1]$. The symbol $\otimes$ stands for a triangular (T) norm.

This formula was first introduced by Sanchez (1976) who proposed an algorithm to obtain the maximum solution for Max-Min based fuzzy relational equations. The problem of solving FREs has been studied by many researchers since then (Tsukamoto and Terano 1977, Wang 1983, Nola 1985, Pedrycz 1990, Adamopoulos and Pappis

1993, Fang and Li 1999, etc.). Various numerical methods for solving FREs have been developed. Some typical algorithms have been reviewed in chapter 2.

The solution space of a FRE is illustrated in figure 3-1, where $\hat{x}$ denotes the maximum solution, $\bar{x}$ the mean solution, $\breve{x}_i$ a minimum solution (the minimum solution is not unique) and $\underline{x}$ the lower limit for those minimum solutions.

The difference between the maximum solution and the mean solution comes from the non-strict monotonic nature of the triangular norm. Consider a triangular operation $T(x, y)$. $T(x, y)$ is not always greater than another triangular operation $T(x, z)$ even if y is greater than z.

As mentioned above, the minimum solution of FREs is not unique. This is due to coupling among column vectors of the matrix A in (3-1). (Here, coupling means that one column vector can be expressed as a fuzzy combination of other columns). A lower limit, $\underline{x}$, can be found for those minimum solutions. This characteristic was explored by Imai et al (1996). Cechlarova (1995) identified the conditions for Equation 3-1 to have a unique solvability. If those conditions are satisfied, all the vectors between the maximum solution $\hat{x}$ and the lower limit $\underline{x}$ belong to the solution space. The lower limit of the minimum solution then reduces to the mean solution $\bar{x}$.

Objectives of different ... ... relating to FREs are still open.

There are:

• It is likely to anticipate an ... tool ... ... manual for solving FREs?

• If we propose using different approaches (max-min and max-product, i.e. min-max
   min-product, etc.) Is there a method for solving them?

Sometimes these questions are crucial in practical ... ... design, especially for fuzzy
like fuzzy relational morphologies, ... all of the fine concepts state for
Morpheme ... ... at the ... such levels of ... ... to
the above ...

In the following sections, FREs will be ... comprehensively at ... ... that
that may in detail, will be addressed briefly ... ... It should be noted that max-
(min), FREs can be regarded as ... ... fuzzy spline. They constitute the
a von Neumann denoted by r 80 ... can be considered over the max operation. ...

Figure 3- 1   The solution space for FRE

$\hat{x}$ denotes the maximum solution, $\overline{x}$ the mean solution, $\check{x}_1 \cdots \check{x}_k$ minimum solutions, $\underline{x}$ the lower

limit for those minimum solutions. $y_{k+1} \cdots y_n$ denote vectors between

$$\overline{x} \text{ and } \underline{x}.$$

Therefore, structurally, yet the FREs with one of the most important satisfy the
max-simplest in ... ... and can be regarded ... ... For a wider solution to analytical
description of the ... ... should not be too difficult to obtain. Firstly, difficulties in
seeking this kind of ... ... should not be caused by the complexity of the solution
space, but rather by the lack of efficient analysis tools in fuzzy algebra.

The interpolation and solution are basic three operation tool in the ... agility. Besides
and Pedrycz [1999] suggested that in addition to max-min operation, the inversion of
the product T-norm can be used for solving max-product based FREs. The potential of

Despite the efforts of different researchers, two questions relating to FRE are still open. They are:

- Is there an analytical as opposed to numerical method for solving FREs?

- FREs formed using different triangular norms and max co-norm, e.g. max-min max-product etc., is there a unified approach for solving them?

Since these questions are closely connected to fuzzy system design, especially for tasks like fuzzy relational matrix optimisation and selection of the best triangular norm for Mamdani's fuzzy inference system. There are practical benefits in finding answers to the above questions.

In the following sections, FREs which are constructed using different triangular norms and max co-norms will be called "max family FREs". It should be noted that max family FREs can be regarded as linear equations in fuzzy space. This is because any given T norms denoted by $x \otimes y$, can be distributed over the max operation, namely:

$$\max[(a \otimes c), (b \otimes c)] = [\max(a, b)] \otimes c$$

Therefore, algebraically, all the FREs built using the max operation satisfy the aggregation principle, and can be regarded as linear. For a linear equation an analytical description of the solution should not be too difficult to obtain. Hence, difficulties in seeking this kind of description should not be caused by the complexity of the solution space, but rather by the lack of efficient analysis tools in fuzzy algebra.

The triangular norm and co-norm are two basic operators used in fuzzy space. Bourke and Fisher (1998) suggested that in addition to those basic operators, the inversion of the product T norm can be used for solving max-product based FREs. The potential of

the product inversion in fuzzy algebra was explored in their work. Based on this idea, an efficient method for solving the max-product FREs was proposed by Loetamonphong and Fung (1998).

It should be noted that since the product T norm is a one-to-one relationship, it is easy to identify the expression for its inversion. For general triangular norms and co-norms, which are likely to be many-to-one relationships. (e.g. Minimum T norm, Lukasiewicz T norm, Maximum T co-norm) how to construct the expression for their inversions is still an open problem.

In this chapter, the concept of inversion for general triangular norms and co-norms will be discussed. The connection between inversion and fuzzy implication will be explored. A complete analytical solution for the max family FREs will be developed based on inversions. The proposed method will be a unified approach for general triangular norms. It will yield a better understanding of fuzzy algebra and the solution space of FREs.

The chapter is organised as follows. The concepts of fuzzy multiplicative inversion and additive inversion are proposed in section 3.2; the relationship between fuzzy vectors in fuzzy space is explored in section 3.3; analytical expression of the maximum and mean solutions of FREs, and an optimal algorithm for deriving the minimum solutions are given in section 3.4; section 3.5 summarises the proposed method also provide an example of how to apply it to obtain analytical solution of a FRE.

3.2 Fuzzy Connectives and Fuzzy Inversions

A number of basic fuzzy logic connectives are reviewed in this section. Based on the connectives, formulae of new operators in fuzzy space are proposed.

In approximate reasoning, the value of a compound statement is determined by both the values of its constituent statements and the way those statements are connected together. According to Zadeh (1981), although the values of given statements are context-dependent, the structures of the connectives between them are invariant. In fuzzy reasoning those connectives are called *fuzzy connectives*. The basic connectives used in fuzzy logic are listed in table 3-1:

Name	Meaning	Symbol	Operation
Conjunction	$\cdots$ and $\cdots$	$\wedge$	$T(x, y)$
Disjunction	$\cdots$ or $\cdots$	$\vee$	$S(x, y)$
Implication	if $\cdots$ then $\cdots$	$\Rightarrow$	I_s I_R I_{QL}
Negation	not $\cdots$	$\neg$	$1 - x$ (etc.)

Table 3- 1 Basic fuzzy connectives

I_s I_R I_{QL} denote S implication, R implication and QL Implication respectively.

For given statements, e.g. A, B and C, it is not the values of A, B and C, but rather the rules governing the truth degrees of new statements that are the concern of fuzzy logic. Those rules are constructed from the above connectives. They are the basic components of fuzzy logic.

It might be a natural step to build fuzzy algebra operator based on the connectives in fuzzy logic. However due to the semiring structure (Allenby 1991) of fuzzy space, only the conjunction and the disjunction operation are extended into the fuzzy algebra. In order to derive the solution for FREs, two new types of operators will be proposed based on the corresponding connectives in fuzzy logic.

3.2.1 Fuzzy Implication and Fuzzy Multiplicative Inversion

The first operator proposed in this work is called fuzzy multiplicative inversion. It is designed as an inversion for the triangular norm. It should be noted that due to the semiring structure of fuzzy space, the fuzzy multiplicative inversion operation cannot be obtained directly. In this section, the inversion formula will be derived based on the fuzzy implication and the law of modus ponens.

Consider the modus ponens inference rule:

Premise 1	IF X is A THEN Y is B
Premise 2	X is A
Consequence	Y is B

In Boolean logic, the consequence "*Y is B*" is derived from the implication ("*IF X is A THEN Y is B*"), and the statement ("*X is A*") by checking a Boolean logic truth table. On the other hand, if Premise 2 ("*X is A*") and consequence ("*Y isB*") are given, the value of

the implication "*IF X is A THEN Y is B*" can be obtained by checking the same truth table. In this case, the process of finding the value of the implication acts as an inverse process of conjunction by swapping the premise and consequence.

For a generalised modus ponens reasoning procedure in fuzzy logic:

Premise 1	IF X is A, THEN Y is B
Premise 2	X is A^*
Consequence	Y is B^*

Using approximate reasoning, the value of B^* is derived as:

$$\forall\, y \in V \qquad \mu_{B^*}(y) = \sup_x T(\mu_{A^*}(x), I(\mu_A(x), \mu_B(y))) \qquad (3\text{-}2)$$

where T stands for a triangular norm. I stands for an implication. I does not need to be of a certain form. It can be any implication operation defined in Chapter 2 (e.g. I_S, I_R or I_{QL})

If two observations related to A and B, "X is A^*" and "Y is B^*" are given, the implication relationship between A and B is derived as:

Premise 1	X is A^*
Premise 2	Y is B^*
Consequence	IF X is A, THEN Y is B

The value of the implication in fuzzy logic is still derived by swapping the premise and consequence of the generalised modus ponens rule. Therefore, if a triangular operator $T(x,y)$ is given, the implication operation $I(x,y)$ can be regarded as its inversion. $I(x,y)$ should satisfy (3-2) which can be rewritten as:

$$T(\mu_{A'}(x), I(\mu_A(x), \mu_B(y))) \le \mu_{B'}(y) \qquad (3\text{-}3)$$

(3-3) is called the law of modus ponens (Lowen 1996).

It should be noted that, the law of modus ponens is not valid for all the fuzzy implication operators reviewed in chapter 2. This is illustrated by the following example.

Example 3-1

For S-implications based on triangular norms (T_M, T_P and T_∞) the law of modus ponens is only valid in the positive region of figures 3-2, 3-3 and 3-4.

Remarks

If an implication model $A \Rightarrow B$ is given, a whole family of triangular norms can be derived using the law of modus ponens. They are called *Modus Ponens Generation Functions,* as introduced by Trillas and Valverde (1985).

In contrast, if the formula for a triangular norm is given, an inversion operator can be derived based on the law of modus ponens by applying the following definition:
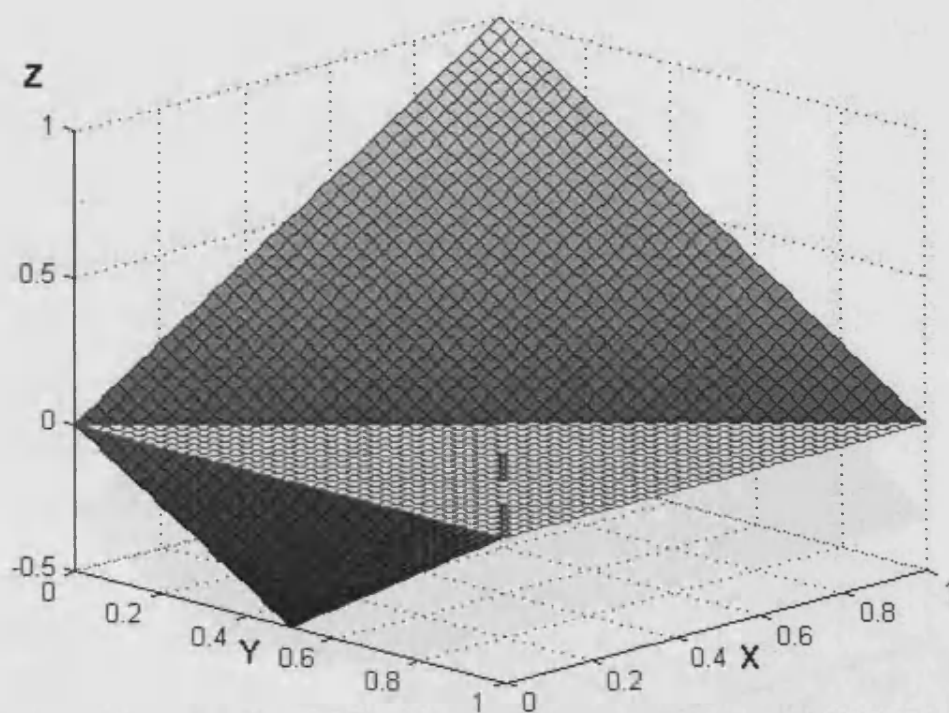
Figure 3- 2 Law of modus ponens for S-implication based on T_M, where $Z = Y - T_M(X, I_S(X,Y))$
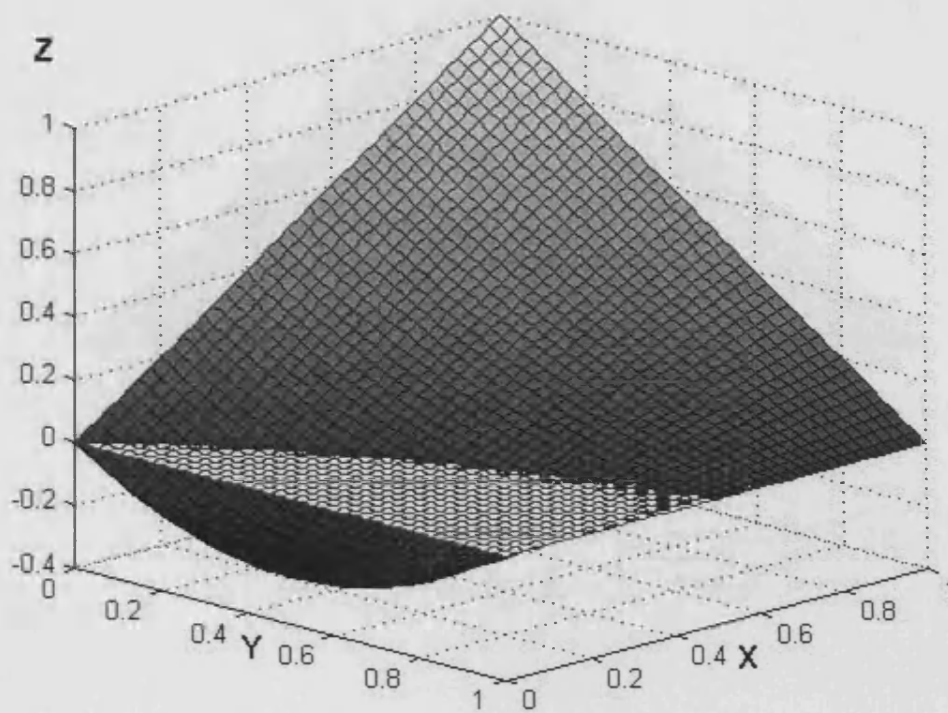
Figure 3- 3 Law of modus ponens for S-implication based on T_P where $Z = Y - T_P(X, I_S(X,Y))$
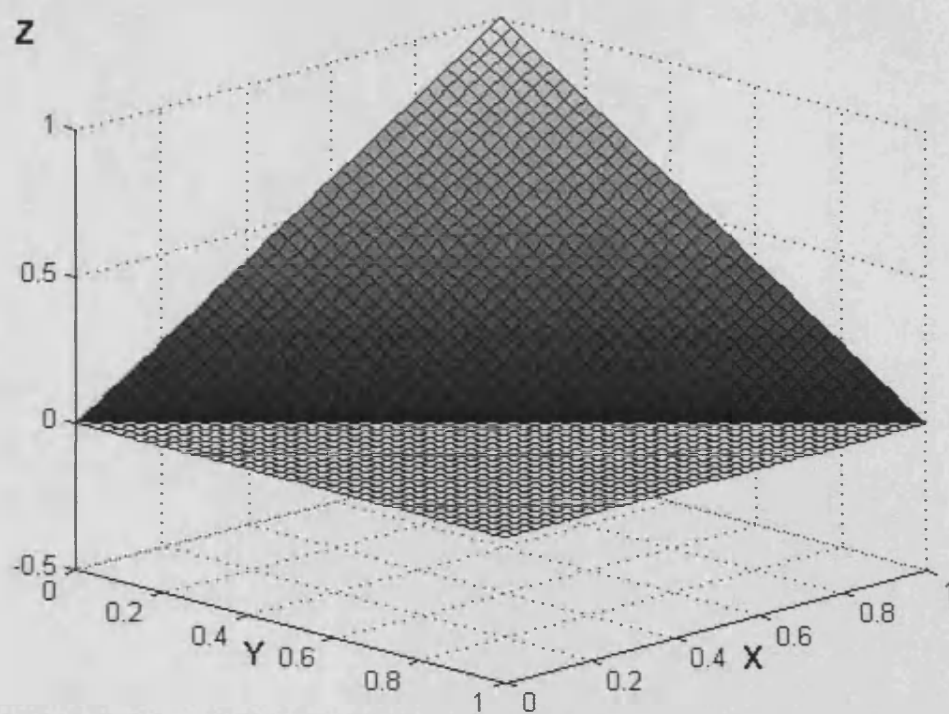
Figure 3- 4 Law of modus ponens for S-implication based on T_∞, $Z = Y - T_\infty(X, I_S(X, Y))$

Definition 3-1

For a given T norm, a multiplicative inversion $D(x,y)$ is an operation on

$[0,1] \times (0,1] \rightarrow [0,1]$, *which satisfies the equation*

$$T(y, D(x,y)) \leq x \qquad (3\text{-}4)$$

Example 3-2

$D(x,y) = \min(x,y)$ is a multiplicative inversion of T_M

$D(x,y) = \begin{cases} x/y & x < y \\ 1 & \text{otherwise} \end{cases}$ is a multiplicative inversion of T_P

$D(x,y) = \begin{cases} x - y + 1 & x < y \\ 1 & \text{otherwise} \end{cases}$ is a multiplicative inversion of T_∞

In linear algebra, an operator denoted by $\otimes$ and its corresponding inversion $\otimes^*$ should satisfy:

$$X \otimes (Y \otimes^* X) = Y$$

(e.g. " $X \times Y \div X = Y$ $(X \neq 0)$ "). However it does not hold for operators in fuzzy algebra. It is one of the fundamental differences between algebra in fuzzy space and algebra in linear space. The difference comes from inherited many-to-many relationships among conjunctions and implications, described by the inequality in the law of modus ponens.

In order to simplify the many-to-many relationships among conjunctions and implications, the supremum of multiplicative inversion is defined as follows.

Definition 3-2

For a given T norm, a supremum multiplicative inversion is an inversion that is greater than any other multiplicative inversions of T.

$$D_{\text{sup}}(x, y) = \sup \left\{ D(x, y) \mid T(y, D(x, y)) \leq x \right\}$$ (3- 5)

Example 3-3

$$D(x, y) = \begin{cases} 1 & x \geq y \\ x & otherwise \end{cases} \quad \text{is the supremum multiplicative inversion for } T_M$$

It is obvious that the R-implication I_R is the supremum multiplicative inversion for the corresponding triangular norm.

Based on the many-to-one relationship between the triangular norm and its supremum multiplicative inversion, some useful results can be derived, as shown below:

Definition 3

For a given T norm, an infimum multiplicative inversion is defined as:

$$D_{\text{inf}}(x, y) = \begin{cases} \inf E & if \ E \neq \Phi \\ 0 & otherwise \end{cases}$$ (3- 6)

where $E = \{ D(x, y) \mid T(y, D(x, y)) = x \}$

Example 3-4

$$D_{\text{inf}}(x, y) = \begin{cases} 0 & x > y \\ x & \text{otherwise} \end{cases} \quad \text{is the infimum multiplicative inversion for } T_M$$

The infimum multiplicative inversion is another limiting case of the inversions for a triangular norm. It provides an alternative means for the analysis of algebra in fuzzy space and will be extremely useful for the derivation of the mean solution for FREs.

3.2.2 Comparison and Fuzzy Additive Inversion

Inversion for the triangular norm was proposed based on the modus ponens rule and fuzzy implication in the previous section. In order to derive the formula for triangular co-norm inversion, the concept of comparison needs to be reviewed.

Given observation A and B, where $B \subset A$.

Let Diff(A,B) denote the difference between them

It is easy to verify that:

Disjunction (B , Diff(A,B)) = A

The difference between A and B can be obtained by a comparison operation e.g.

Diff(A,B) = Comparison(A,B)

If the disjunction and comparison operation are denoted by $\oplus$ and $\ominus$ respectively, the equations above becomes

B $\oplus$ Diff(A,B) = A

Diff(A,B) = A $\ominus$ B

If we consider the left hand side of the equations as premise and the right hand side of the equations as consequence, comparison operation is regarded simply an inverse of the disjunction connective by swapping between premise and consequence.

Comparison is widely used in reasoning processes. Consider the following example:

PROPOSITIONS:	TRUTH DEGREE
A=Temperature is high	u_A
B=Temperature is low	u_B

Table 3- 2 Proposition about temperature

CONCLUSION:	TRUTH DEGREE
C=Temperature is likely to be high	u_C

Table 3- 3 Final decision about temperature

Given $u_B < u_A$, u_C, which reflects "Temperature is likely to be high", can be obtained by comparing u_A and u_B.

Unlike the relationship between the fuzzy implication and conjunction, which can be explicitly described by the modus ponens rule (3-3), the relationship between fuzzy comparisons and disjunctions has not yet been identified. This aspect will be explored in subsequent sections.

In order to find the formula for the inversion of the triangular co-norm, a measure for the credibility of observations under fuzzy logic has to be defined. This measure is called the Confidence Measure in this work. It provides an estimate of the amount of credibility attached to an observation.

Definition 3-4

Let μ_A be an observation for proposition A and μ_B be an observation for proposition B. A confidence measure for μ is a function: $V(u) \in [0,1]$ with the following properties:

1) $V(T(\mu_A, \mu_B)) = T(V(\mu_A), V(\mu_B))$ (requirement of conjunction)

2) $V(S(\mu_A, \mu_B)) = S(V(\mu_A), V(\mu_B))$ (requirement of disjunction)

3) $V(\mu_A \Rightarrow \mu_B) \leq \min(V(\mu_A), V(\mu_B))$ (requirement of modus ponens)

4) $V(D(\mu_A, \mu_B)) \geq \max(V(\mu_A), V(\mu_B))$ (requirement of comparison)

where $\mu_A \Rightarrow \mu_B$ is the implication from A to B and $D(\mu_A, \mu_B)$ is an estimate of the significance of the value of μ_A and μ_B.

Remarks

1) The confidence measure for conjunction and disjunction are equal to the conjunction and disjunction of individual confidence measures.

2) An estimate of the implication relation between A and B based on observations μ_A and μ_B should not have more confidence than either of the observations. This is a requirement of the modus ponens rule.

   Recall the inverse process of generalised modus ponens

Premise 1	*X is A^**
Premise 2	*Y is B^**

   Consequence   *IF X is A   THEN Y is B*

   The information available in the above process is only contained in two observations, X is A^* and Y is B^*. There is no further information about the relationship between A and B. Therefore, no matter what the result is, the consequence should not be trusted more than either of the premises.

3) $D(\mu_A, \mu_B)$ is an estimate of the significance of the value of the observations μ_A and μ_B. Since this is **purely** about the values of μ_A and μ_B, it should be more reliable than either μ_A or μ_B. (Most of the time, one is more confident about facts such as "A is more likely than B", "A is higher than B" than facts such as

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"the possibility of A" and "the height of A". Although the former might have been derived from the latter)

Theorem 3-1 *(The modus ponens theorem)*

The following relationship holds for the confidence measure:

$$V(T(\mu_A , \mu_A \Rightarrow \mu_B)) \leq V(\mu_B) \qquad (3\text{-}7)$$

This means that a result derived from modus ponens should not be more reliable than one derived from direct observations.

Proof:

$$V(T(\mu_A, \mu_A \Rightarrow \mu_B))$$
$$\leq T(V(\mu_A), V(\mu_A \Rightarrow \mu_B))$$
$$\leq T(V(\mu_A), \min(V(\mu_A), V(\mu_B)))$$
$$\leq T(V(\mu_A), V(\mu_B))$$
$$\leq V(\mu_B)$$

Comparing (3-7) with (3-3), shows that from *confidence measure* point of view the modus ponens rule is a direct result of Theorem 3-1.

Theorem 3-2 *(Comparison theorem)*

For given observations, μ_A and μ_B the following relationship concerning disjunction and comparison holds:

$$V(S(\mu_A, D(\mu_B, \mu_A))) \geq V(\mu_B) \qquad (3\text{-}8)$$

Proof:

$$V(S(\mu_A, D(\mu_B, \mu_A)))$$

$$\geq S(V(\mu_A), V(D(\mu_B, \mu_A)))$$

$$\geq S(V(\mu_A), \max(V(\mu_A), V(\mu_B)))$$

$$\geq S(V(\mu_A), V(\mu_B))$$

$$\geq V(\mu_B)$$

Theorem 3-2 provides the theoretical basis for the definition of additive inversions. They have a similar structure as used for multiplicative inversions. The direction of the inequality relation is given by Theorem 3-2.

Definition 3-5   Additive Inversion

For a given triangular co-norm, an additive inversion $M(x, y)$ is an operation on $[0,1] \times [0,1] \rightarrow [0,1]$, which satisfies the following equation:

$$S(y, M(x, y)) \geq x \tag{3-9}$$

Example 3-5: $M(x, y) = \max(x, y)$ is an additive inversion for $S_M = \max(x, y)$

Remarks: In this case the triangular co-norm and additive inversion have the same formula.

Definition 3-6   Infimum Additive Inversion

For a given triangular co-norm, an infimum additive inversion is an inversion that is less than any of its other additive inversions.

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$$M_{\text{inf}}(x, y) = \inf \left\{ M(x, y) \mid S(y, M(x, y)) \geq x \right\}$$

For simplicity, $M_{\text{inf}}(x, y)$ will be written as $x \ominus y$ in the following section.

Example 3-6

$$M_{\text{inf}}(x, y) = x \ominus y = \begin{cases} x & x > y \\ 0 & \text{otherwise} \end{cases} \quad \text{is an infimum additive inversion for } S_M$$

It should be noted here that the many-to-many relationship between the triangular norms/co-norms and their inversions is caused by the semiring structure of fuzzy algebra. Hence, inequalities instead of equalities are used for the definition of fuzzy inversions. The directions of the inequalities are determined by the basic properties of fuzzy logic. They can be found from the modus ponens and the comparison theorem.

A comparison of the operators of linear space and fuzzy space is listed in the following table:

LINEAR SPACE	+	−	×	÷ OR /
FUZZY SPACE	$\oplus$	$\ominus$	$\otimes$	/
	T co-norm	Fuzzy Additive Inversion	T norm	Fuzzy multiplicative inversion

Table 3- 4 A comparison of the operators of linear space and fuzzy space

A complete set of tools required in fuzzy algebra analysis is defined in this section. The notations and the corresponding fuzzy connectives are listed in table 3-5.

Name	Meaning	Operation	Symbol
Conjunction	$\cdots$ and $\cdots$	T (x, y)	$x \otimes y$
Implication	if $\cdots$ then $\cdots$	D (x, y)	$\dfrac{x}{y}$
Disjunction	$\cdots$ or $\cdots$	S (x, y)	$x \oplus y$
Comparison	difference between $\cdots$ and $\cdots$	M (x, y)	$x \ominus y$
Negation	not $\cdots$	1 - x	$\neg$

Table 3- 5   Fuzzy connectives and their notations

For simplicity, the symbols $\dfrac{x}{y}$, $\left(\dfrac{x}{y}\right)_{sup}$, $\left(\dfrac{x}{y}\right)_{inf}$ and $x \ominus y$, which denote the

multiplicative inversion, supremum multiplicative inversion, infimum multiplicative

inversion, and infimum additive inversions respectively, will be used in the following

sections.

3.3   Fuzzy Vector Space

In this section, the concept of fuzzy vectors and the relationships between two fuzzy

vectors in fuzzy space will be investigated.

3.3.1   Notations

Definition 3-7

A fuzzy vector is $V = [v_i]_m$ *with* $v_i \in [0 , 1]$

A fuzzy matrix is $A = [a_{ij}]_{mn}$ *with* $a_{ij} \in [0, 1]$

Remarks:

A fuzzy vector (or matrix) is a normal vector (or matrix) with its elements belonging to [0 , 1]. The following operations can be defined for fuzzy vectors and matrices

Definition 3-8

For fuzzy matrices $A_{mn} = [a_{ij}]_{mn}$, $B_{np} = [b_{ij}]_{np}$, $C_{np} = [c_{ij}]_{np}$, the following operations are defined:

1. $B_{np} \oplus C_{np} = [b_{ij} \oplus c_{ij}]_{np}$

2. $A_{mn} \otimes B_{np} = [\sum_{k=1}^{n} a_{ik} \otimes b_{kj}]_{mp}$ where $\sum_{i=1}^{n} x_i = x_1 \oplus x_2 \oplus \cdots \oplus x_n$

3. $A^T = [a_{ji}]$ *(the transpose of A)*

4. $A^K = A^{K-1} \otimes A$ $K = (1, \cdots n)$

     and $A^1 = A$, $A^0 = I_n$ (I_n is usually but not necessary the identity matrix)

5. $B \geq C$ if and only if $b_{ij} \geq c_{ij}$ for all i and j

6. $B > C$ if and only if $b_{ij} > c_{ij}$ for all i and j

Remarks:

Definitions of operations in fuzzy space are derived from operations in linear space. The operators $\times$ and $+$ are replaced by $\otimes$ and $\oplus$. $\otimes$ and $\oplus$ denote the T norm and co-norm respectively. (See tables 3-4 and 3-5)

Example 3-7

Given min and max as T norm and co-norm, calculate $\begin{pmatrix} 0.1 & 0.3 \\ 0.2 & 0.6 \end{pmatrix} \otimes \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix} \oplus \begin{pmatrix} 0.4 \\ 0.1 \end{pmatrix}$

$$\begin{pmatrix} 0.1 & 0.3 \\ 0.2 & 0.6 \end{pmatrix} \otimes \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix} \oplus \begin{pmatrix} 0.4 \\ 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} (0.1 \otimes 0.1) \oplus (0.3 \otimes 0.2) \\ (0.2 \otimes 0.1) \oplus (0.6 \otimes 0.2) \end{pmatrix} \oplus \begin{pmatrix} 0.4 \\ 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} (0.1 \otimes 0.1) \oplus (0.3 \otimes 0.2) \oplus 0.4 \\ (0.2 \otimes 0.1) \oplus (0.6 \otimes 0.2) \oplus 0.1 \end{pmatrix}$$

Substituting $\otimes$ and $\oplus$ by min and max

$$= \begin{pmatrix} \max \left((\min(0.1, 0.1)), (\min(0.3, 0.2)), 0.4 \right) \\ \max \left((\min(0.2, 0.1)), (\min(0.6, 0.2)), 0.1 \right) \end{pmatrix}$$

$$= \begin{pmatrix} 0.4 \\ 0.2 \end{pmatrix}$$

Definition 3-9

A fuzzy space constructed by a group of vectors $\{x_1 \cdots x_i \cdots x_n\}$ is defined as:

$$Span \{x_1 \cdots x_i \cdots x_n\} = [x_1 \cdots x_i \cdots x_n] \otimes \omega$$
$$= \omega_1 \otimes x_1 \oplus \cdots \oplus \omega_i \otimes x_i \oplus \cdots \oplus \omega_n \otimes x_n$$

where $\omega = [\omega_1 \cdots \omega_i \cdots \omega_n]^T$ and $\omega_i \in [0,1]$

A vector y belongs to the space constructed by $\{x_1 \cdots x_n\}$ if and only if there exists a

vector $\omega = [\omega_1 \cdots \omega_i \cdots \omega_n]^T$ which satisfies

$$y = [x_1 \cdots x_i \cdots x_n] \otimes \omega$$

Example 3-8

Given min and max as T norm and co-norm, $x_1 = \begin{pmatrix} 0.4 \\ 0.1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}$ and $y = \begin{pmatrix} 0.4 \\ 0.2 \end{pmatrix}$

Since $y = (0.4 \otimes x_1) \oplus (0.2 \otimes x_2)$, y belongs to the space spanned by $\{x_1 \; x_2\}$.

Definition 3-10

For given vectors $v = [v_1 \; \cdots \; v_n]^T$ *and* $w = [w_1 \; \cdots \; w_n]^T$ *, the supremum*

projection of vector v on vector w is defined as

$$v' = \left(\frac{v}{w}\right)' \otimes w = \inf_{i \in \{1, \cdots n\}} \left(\left(\frac{v_i}{w_i}\right)_{\text{sup}} \right) \otimes w$$

The projection v' can be regarded as a sub-vector of v in the direction of w,

$$\text{where} \left(\frac{v}{w}\right)' = \inf_{i \in \{1, \cdots n\}} \left(\left(\frac{v_i}{w_i}\right)_{\text{sup}} \right)$$

and

$\left(\dfrac{v_i}{w_i}\right)_{\text{sup}}$ is the supremum multiplicative inversion.

Lemma 3-1

For a vector v and its supremum projection v'*, it will always be that:* $v \geq v'$

This follow directly from the modus ponens rule.

Definition 3-11

For given vectors $v = [v_1 \; \cdots \; v_n]^T$ *and* $w = [w_1 \; \cdots \; w_n]^T$ *, the infimum projection*

of v on w is defined as:

$$v'' = \left(\frac{v}{w}\right)'' \otimes w = \sup_{i \in \{1, \cdots n\}} \left(\left(\frac{v_i}{w_i}\right)_{\text{inf}}\right) \otimes w$$

Where $\left(\dfrac{v}{w}\right)'' = \sup\limits_{i \in \{1, \cdots n\}} \left(\left(\dfrac{v_i}{w_i}\right)_{\text{inf}}\right)$

and

$\left(\dfrac{v_i}{w_i}\right)_{\text{inf}}$ is the infimum multiplicative inversion.

Lemma 3-2

For given vectors $v = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T$ *and* $w = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^T$

if $\left(\dfrac{v}{w}\right)'' < \left(\dfrac{v}{w}\right)'$ then $v \geq w$

Proof:

Assume that $v \geq w$ is FALSE. There must exist an $i \in \{1 \cdots n\}$ for which $v_i < w_i$.

Therefore $\left(\dfrac{v_i}{w_i}\right)_{\text{inf}} = \left(\dfrac{v_i}{w_i}\right)_{\text{sup}}$. From definitions 3-10 and 3-11, it should be that:

$$\left(\frac{v}{w}\right)'' \geq \left(\frac{v_i}{w_i}\right)_{\text{inf}} \quad \text{and} \quad \left(\frac{v_i}{w_i}\right)_{\text{sup}} \geq \left(\frac{v}{w}\right)'$$

Then $\left(\dfrac{v}{w}\right)'' \geq \left(\dfrac{v}{w}\right)'$.

Clearly this inequality conflicts with the condition $\left(\dfrac{v}{w}\right)'' < \left(\dfrac{v}{w}\right)'$, so the assumption

must be wrong, which means $v \geq w$ is true.

End of proof.

Example 3-9

Given min and max as T norm and co-norm, $x = \begin{pmatrix} 0.4 \\ 0.2 \end{pmatrix}$ and $y = \begin{pmatrix} 0.5 \\ 0.2 \end{pmatrix}$

$$\left(\frac{y}{x}\right)' = \inf_{i \in \{1,2\}}\left(\left(\frac{y_i}{x_i}\right)_{\text{sup}}\right) = \inf_{i \in \{1,2\}}\begin{pmatrix} 1 & 1 \end{pmatrix} = 1$$

The supremum projection of vector y on vector x is:

$$y' = \left(\frac{y}{x}\right)' \otimes x = \begin{pmatrix} 0.4 \\ 0.2 \end{pmatrix}$$

$$\left(\frac{y}{x}\right)'' = \sup_{i \in \{1,2\}}\left(\left(\frac{y_i}{x_i}\right)_{\text{inf}}\right) = \sup_{i \in \{1,2\}}\begin{pmatrix} 0 & 0.2 \end{pmatrix} = 0.2$$

The infimum projection of vector y on vector x is:

$$y'' = \left(\frac{y}{x}\right)'' \otimes x = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix}$$

Since $\left(\frac{y}{x}\right)'' < \left(\frac{y}{x}\right)'$, $y \geq x$ is true.

Definitions 3-7, 3-8 and 3-9 are generalised from linear algebra. Definitions 3-10 and 3-11 are defined uniquely for fuzzy algebra. Based on these notations, some basic characteristics of vectors in fuzzy space will be explored in the following section.

3.3.2 Fuzzy Vector Algebra

Compared to vectors in the linear space, vectors in fuzzy space have a different geometrical character. This is illustrated by the following example.

Given vectors $Y = OA$, and $X = OD$ (figure 3-5).

If $X = \alpha \otimes Y$, $\alpha \in [0,1]$, in linear space, D must be located on the segment OA,

However, this is not true in fuzzy space. Depending on the triangular norm, for T_M ($T_M(x,y) = \min(x,y)$) D would be located on trajectory OB and BA. For T_∞ ($T_\infty(x,y) = \max(0, x+y-1)$) D would be located on trajectory OC and CA. Where OB is a vector inclined at $45°$ relative to the X-axis, and CA is parallel to OB.

Theorem 3-3

For given vectors $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ and $y = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}^T$, y belongs to the space constructed by x if and only if

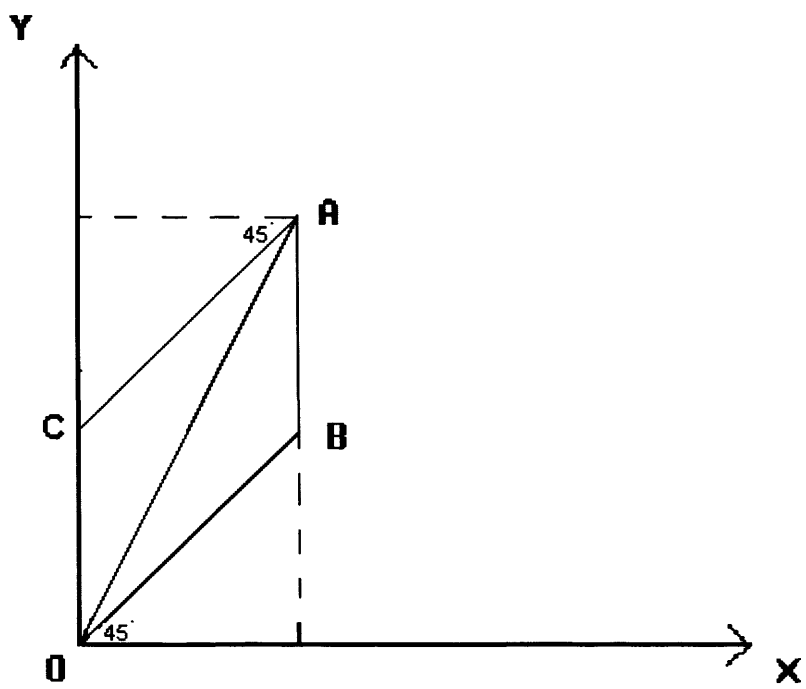$$y \leq \left(\frac{y}{x} \right)' \otimes x$$

Figure 3- 5 Relationship between Fuzzy Vectors

Proof:

Necessary condition:

If y belongs to the space constructed by x, then from Definition 3-9:

$$y = \omega \otimes x \qquad \omega \in [0,1]$$

$$\Rightarrow y = \begin{pmatrix} y_1 \\ \vdots \\ y_i \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \omega \otimes x_1 \\ \vdots \\ \omega \otimes x_i \\ \vdots \\ \omega \otimes x_n \end{pmatrix}$$

$y_i = \omega \otimes x_i$ *holds for any* $i \in \{1, \cdots, n\}$

Therefore, there exists an inversion satisfying

$$\omega = \frac{y_i}{x_i}$$

Based on the definition of the supremum inversion,

$$\omega = \frac{y_i}{x_i} \le \left(\frac{y_i}{x_i} \right)_{Sup} \quad \text{for any} \ i \in \{1, \cdots, n\}$$

Therefore,

$$y = \omega \otimes x \le \inf_{i \in \{1, \cdots, n\}} \left(\frac{y_i}{x_i} \right)_{sup} \otimes x = \left(\frac{y}{x} \right)' \otimes x$$

This can be written as:

$$y \le \left(\frac{y}{x} \right)' \otimes x$$

Sufficient condition:

To prove the sufficient condition, one needs to demonstrate that if $y \le \left(\dfrac{y}{x} \right)' \otimes x$ is

given, then there exists a value ω that satisfies $y = \omega \otimes x$.

From

$$y \le \left(\frac{y}{x}\right)' \otimes x = \begin{pmatrix} \inf\left(\frac{y_i}{x_i}\right)_{\sup} \otimes x_1 \\ \vdots \\ \inf\left(\frac{y_i}{x_i}\right)_{\sup} \otimes x_n \end{pmatrix} \qquad i \in \{1,\cdots,n\}$$

$$\le \begin{pmatrix} \left(\frac{y_i}{x_i}\right)_{\sup} \otimes x_1 \\ \vdots \\ \left(\frac{y_n}{x_n}\right)_{\sup} \otimes x_n \end{pmatrix} \le \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = y$$

Therefore $y = \omega \otimes x$ and $\omega = \left(\frac{y}{x}\right)'$

End of proof

Proposition 3-1

If y belongs to the space constructed by x , there must be $y = \left(\frac{y}{x}\right)' \otimes x$.

This directly follows from the "sufficient" part of the above proof.

Lemma 3-3

For the given vectors y and x, if y belongs to the space constructed by x then

$$\sup\left(\frac{y_j}{x_j}\right)_{\inf} \le \inf\left(\frac{y_j}{x_j}\right)_{\sup} \tag{3-10}$$

Proof:

From proposition 3-1, if y belongs to the space constructed by x then

$$y = \left(\frac{y}{x}\right)' \otimes x$$

$$\Rightarrow y_j = \left(\frac{y}{x}\right)' \otimes x_j \qquad\qquad j \in \{1, \cdots, n\}$$

From the definition of infimum multiplicative inversion

$$\left(\frac{y_j}{x_j}\right)_{\inf} \leq \left(\frac{y}{x}\right)' \qquad \text{for any } j$$

Therefore,

$$\sup\left(\frac{y_j}{x_j}\right)_{\inf} \leq \left(\frac{y}{x}\right)' = \inf\left(\frac{y_j}{x_j}\right)_{\sup}$$

End of Proof

Lemma 3-4

If the equation $Y = \omega X$ where $Y = \{y_i\}_n$ $X = \{x_i\}_n$ $\omega \in [0,1]$ is solvable, $\omega = \left(\frac{Y}{X}\right)''$

is the minimum solution.

Proof:

This can be obtained from the definition of infimum multiplicative inversion where

$\left(\frac{Y}{X}\right)''$ is the lower limit that satisfies $y_i = \left(\frac{y_i}{x_i}\right) \otimes x_i$

Lemma 3-5

If $a < \min\left(\left(\frac{Y}{X}\right)'', \left(\frac{Y}{X}\right)'\right)$ then $Y > a \otimes X$

Proof:

Assuming "$Y > a \otimes X$ is false", then there must be at least one $i \in \{1 \cdots n\}$ that satisfies

$$y_i < a \otimes x_i$$

Hence, $a > \left(\dfrac{y_i}{x_i}\right)_{sup} \geq \left(\dfrac{Y}{X}\right)'$, which conflicts with the condition

$$a < \min\left(\left(\dfrac{Y}{X}\right)'', \left(\dfrac{Y}{X}\right)'\right)$$

So the assumption is false and $Y > a \otimes X$ is true

Theorem 3-4

The solution space for equation $Y = \omega X$ can be expressed as $\omega \in \left[\left(\dfrac{Y}{X}\right)'', \left(\dfrac{Y}{X}\right)'\right]$.

This is obtained from Theorem 3-3 and Lemma 3-4.

It should be noted that the characteristics described in the theorem 3-4 for the vector case agree with the scalar case where $a = \omega \otimes b$ is only valid when $\omega \in \left[\left(\dfrac{b}{a}\right)_{inf}, \left(\dfrac{b}{a}\right)_{sup}\right]$. This is the reason that $\left(\dfrac{Y}{X}\right)''$ and $\left(\dfrac{Y}{X}\right)'$ are called infimum projection and supremum projection respectively in the Definitions 3-10 and 3-11.

In this section, the relationships between two vectors in fuzzy vector space have been identified. The notations developed here will be applied in the following section for solving FREs.

3.4 Solutions of Max Family Fuzzy Relational Equations

As mentioned previously, max family FREs are those FREs constructed using different triangular norms and max co-norms. Analytical formulae for the maximum and mean solutions of max family FREs and an optimal algorithm for deriving the minimum solutions will be developed in this section.

3.4.1 Maximum and mean solutions of FREs

Similar to Theorem 3-3, a necessary condition for the solvability of FREs can be derived as follows.

Lemma 3-6

For a group of given vectors $\{x_1 \quad \cdots \quad x_i \quad \cdots \quad x_m\}$, if a vector y belongs to the space constructed by $\{x_i\}$, it must satisfy $y \leq \sum_{i=1}^{m} \left(\frac{y}{x_i}\right)' \otimes x_i$. If the triangular co-norm is idempotent, the above condition is sufficient. A co-norm is regarded as idempotent if it satisfies $Y = Y \oplus Y \oplus \cdots \oplus Y$.

Proof:

Necessary conditions:

If y belongs to the space constructed by $\{x_i\}$, y can be rewritten as:

$$y = (\omega_1 \otimes x_1) \oplus (\omega_2 \otimes x_2) \oplus \cdots (\omega_n \otimes x_m)$$

$$= \left(\left(\frac{y_1}{x_1}\right)' \otimes x_1 \right) \oplus \left(\left(\frac{y_2}{x_2}\right)' \otimes x_2 \right) \oplus \cdots \oplus \left(\left(\frac{y_m}{x_m}\right)' \otimes x_m \right)$$

where $y_i = \omega_i \otimes x_i$, $i \in \{1 \cdots m\}$ and $y_i \leq y$

It is obvious that $\left(\dfrac{y_i}{x_i}\right)' \leq \left(\dfrac{y}{x_i}\right)'$

Therefore

$$y \leq \left(\left(\frac{y}{x_1}\right)' \otimes x_1 \right) \oplus \quad \cdots \quad \oplus \left(\left(\frac{y}{x_m}\right)' \otimes x_m \right) = \sum_{i=1}^{m} \left(\frac{y}{x_i}\right)' \otimes x$$

Therefore

$$y \leq \sum_{i=1}^{m} \left(\frac{y}{x_i}\right)' \otimes x_i$$

Sufficient condition:

If $y \leq \sum\limits_{i=1}^{m} \left(\dfrac{y}{x_i}\right)' \otimes x_i$ and the T co-norm is idempotent, then y belongs to the space

constructed by $\{X_i\}$.

$$y \leq \left(\left(\frac{y}{x_1}\right)' \otimes x_1 \right) \oplus \quad \cdots \quad \oplus \left(\left(\frac{y}{x_m}\right)' \otimes x_m \right)$$

$$= \begin{pmatrix} \left(\inf\limits_{i_1 \in n} (\frac{y_{i_1}}{x_{1,i_1}})_{\sup} \otimes x_{11} \right) \oplus & \cdots & \oplus \left(\inf\limits_{i_m \in n} (\frac{y_{i_m}}{x_{m,i_m}})_{\sup} \otimes x_{m1} \right) \\ \vdots & & \vdots \\ \left(\inf\limits_{i_1 \in n} (\frac{y_{i_1}}{x_{1,i_1}})_{\sup} \otimes x_{1n} \right) \oplus & \cdots & \oplus \left(\inf\limits_{i_m \in n} (\frac{y_{i_m}}{x_{m,i_m}})_{\sup} \otimes x_{mn} \right) \end{pmatrix}$$

$$\leq \begin{pmatrix} \left((\dfrac{y_1}{x_{1,1}})_{\text{sup}} \otimes x_{11} \right) \oplus \cdots \oplus \left((\dfrac{y_1}{x_{m,1}})_{\text{sup}} \otimes x_{m1} \right) \\ \vdots \qquad\qquad\qquad \vdots \\ \left((\dfrac{y_n}{x_{1,n}})_{\text{sup}} \otimes x_{1n} \right) \oplus \cdots \oplus \left((\dfrac{y_{i_m}}{x_{m,n}})_{\text{sup}} \otimes x_{mn} \right) \end{pmatrix}$$

$$\leq \begin{pmatrix} y_1 \oplus \cdots \oplus y_1 \\ \vdots \qquad\qquad \vdots \\ y_n \oplus \cdots \oplus y_n \end{pmatrix} = Y \oplus Y \oplus \cdots \oplus Y = Y$$

Therefore,

$$y = \sum_{i=1}^{m} \left(\frac{y}{x_i} \right)' \otimes x_i \quad \text{and} \quad \omega_i = \left(\frac{y}{x_i} \right)'$$

End of proof

Lemma 3-6 shows the importance of the idempotent property in fuzzy space. With an idempotent T co-norm, a necessary and sufficient condition to identify the relationship among fuzzy vectors can be derived. It is easy to verify that the max co-norm is idempotent. Considering that the max operation is a commonly used T co-norm in fuzzy logic, the following sections will focus on deriving solutions for max family FREs. For simplicity, in this work, FREs will refer to max family FREs and

$$x \oplus y = \max(x, y), \quad x \ominus y = \begin{cases} x & x > y \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 3-5

The maximum solution for $A \otimes x = b$ is $\hat{x} = (\hat{x}_1 \quad \cdots \quad \hat{x}_i \quad \cdots \quad \hat{x}_m)^T$ and $\hat{x}_i = \left(\dfrac{b}{A_i} \right)'$,

where A_i denotes the ith column of matrix A

The proof directly follows the sufficient part of Lemma 3-6

In order to derive the mean solution, the following Lemmas need to be derived first:

Lemma 3-7

$$If \ Y = \sum_{i=1}^{m} \omega_i \otimes X_i, \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad X_i = \begin{bmatrix} x_{i1} \\ \vdots \\ x_{in} \end{bmatrix}, \quad and \quad \omega_i \in [0,1] \ exists, \ the \ following$$

relationship holds:

$$Y = \sum_{\substack{i=1 \\ i \neq k}}^{m} (a_i \otimes X_i) \oplus \left(\left(\frac{Y_k}{X_k} \right)'' \otimes X_k \right) \quad and \quad Y_k = Y \ominus \sum_{\substack{i=1 \\ i \neq k}}^{m} a_i \otimes X_i \qquad (3\text{-}11)$$

where $a_i = \left(\dfrac{Y}{X_i} \right)'$

Proof:

From the condition, Y belongs to the space spanned by $\{X_i\}$, one has:

$$Y = \sum_{i=1}^{m} a_i \otimes X_i = \sum_{\substack{i=1 \\ i \neq k}}^{m} a_i \otimes X_i \oplus a_k \otimes X_k, \qquad (3\text{-}12)$$

Where $a_i = \left(\dfrac{Y}{X_i} \right)'$

without lose of generality, Y can be separated into three parts

$$Y = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \begin{matrix} \}l_1 \\ \}l_2 \\ \}l_3 \end{matrix} \quad and \quad l_1 + l_2 + l_3 = n,$$ (3- 13)

where Z_1 represents the components of vector Y, which can be derived only from

$\sum_{\substack{i=1 \\ i \neq k}}^{m} a_i \otimes X_i$, and Z_3 represents the components of vector Y, which can only be derived

from $a_k \otimes X_k$. Z_2 represents the components of vector Y, which can be derived from

either $\sum_{\substack{i=1 \\ i \neq k}}^{m} a_i \otimes X_i$ or $a_k \otimes X_k$

Hence, Equation 3-11 can be rewritten as:

$$Y = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \sum_{\substack{i=1 \\ i \neq k}}^{m} \left(a_i \otimes \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} \right) \oplus \left(a_k \otimes \begin{bmatrix} x_{k1} \\ x_{k2} \\ x_{k3} \end{bmatrix} \right)$$ (3- 14)

where

$$Z_1 = \sum_{\substack{i=1 \\ i \neq k}}^{m} a_i \otimes X_{i1} > a_k \otimes X_{k1}$$ (3- 15)

$$Z_2 = \sum_{\substack{i=1 \\ i \neq k}}^{m} a_i \otimes X_{i2} = a_k \otimes X_{k2}$$ (3-16)

$$Z_3 = a_k \otimes X_{k3} > \sum_{\substack{i=1 \\ i \neq k}}^{m} a_i \otimes X_{i3}$$ (3-17)

Let

$$V^* = \sum_{\substack{i=1 \\ i \neq k}}^{m} a_i \otimes X_{i3} < Z_3, \quad V^{**} = a_k \otimes X_{k1} < Z_1$$

and

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$$Y = Y^* \oplus Y^{**} \tag{3-18}$$

where

$$Y^* = \sum_{\substack{i=1 \\ i \neq k}}^{m} a_i \otimes X_i = \begin{bmatrix} Z_1 \\ Z_2 \\ V^* \end{bmatrix} \tag{3-19}$$

and

$$Y^{**} = a_k \otimes X_k = \begin{bmatrix} V^{**} \\ Z_2 \\ Z_3 \end{bmatrix} \tag{3-20}$$

Since

$$Y_k = Y \Theta \sum_{\substack{i=1 \\ i \neq k}}^{m} a_i \otimes X_i = Y \Theta Y^* \tag{3-21}$$

Substituting equations 3-13 and 3-19 into equation 3-21:

$$Y_k = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \Theta \begin{bmatrix} Z_1 \\ Z_2 \\ V^* \end{bmatrix} \tag{3-22}$$

From (3-17), $Z_3 > V^*$, one has

$$Y_k = \begin{bmatrix} 0 \\ 0 \\ Z_3 \end{bmatrix}$$

From Definition 3-11, the following equality holds

$$\left(\frac{Y_k}{X_k} \right)'' = \sup\left(\left(\frac{0}{X_{k1}} \right)'' , \left(\frac{0}{X_{k2}} \right)'' , \left(\frac{Z_3}{X_{k3}} \right)'' \right) = \left(\frac{Z_3}{X_{k3}} \right)'' \tag{3-23}$$

Since $Z_3 = \omega_k \otimes X_{k3}$ exists, from Lemma 3-4

$$Z_3 = \left(\frac{Y_k}{X_k} \right)'' \otimes X_{k3} \tag{3-24}$$

Since $\left(\dfrac{Z_2}{Z_3} \right) = a_k \left(\dfrac{X_{k2}}{X_{k3}} \right)$ exists, from Lemma 3-3, the inequality $\left(\dfrac{\left(\dfrac{Z_2}{Z_3} \right)}{\left(\dfrac{X_{k2}}{X_{k3}} \right)} \right)'' \leq \left(\dfrac{\left(\dfrac{Z_2}{Z_3} \right)}{\left(\dfrac{X_{k2}}{X_{k3}} \right)} \right)'$

holds. Also from Definition 3-10 and 3-11

$$\left(\frac{Z_3}{X_{k3}} \right)'' \leq \sup\left(\left(\frac{Z_2}{X_{k2}} \right)'', \left(\frac{Z_3}{X_{k3}} \right)'' \right) = \left(\frac{\left(\dfrac{Z_2}{Z_3} \right)}{\left(\dfrac{X_{k2}}{X_{k3}} \right)} \right)''$$

$$\left(\frac{Y}{X_k} \right)' = \inf\left(\frac{Z_1}{X_{k1}}, \frac{Z_2}{X_{k2}}, \frac{Z_3}{X_{k3}} \right) = \left(\frac{\left(\dfrac{Z_2}{Z_3} \right)}{\left(\dfrac{X_{k2}}{X_{k3}} \right)} \right)'$$

Therefore the following inequality holds:

$$\left(\frac{Z_3}{X_{k3}} \right)'' \leq \left(\frac{Y}{X_k} \right)' \tag{3-25}$$

Substituting Equation 3-25 into Equation 3-23 gives:

$$\left(\frac{Y_K}{X_K} \right)'' \leq \left(\frac{Y}{X_k} \right)' = a_k \tag{3-26}$$

Therefore,

$$\left(\frac{Y_k}{X_k} \right)'' \otimes X_k \leq a_k \otimes X_k \tag{3-27}$$

Let:

$$\begin{bmatrix} V_1 \\ V_2 \\ Z_3 \end{bmatrix} = \left(\frac{Y_k}{X_k} \right)'' \otimes X_k \tag{3-28}$$

Immediately one has, $Z_1 \geq V_1$ and $Z_2 \geq V_2$. Also from (3-17) $Z_3 > V^*$. Therefore:

$$\sum_{\substack{i=1 \\ i \neq k}}^{m} (a_i \otimes X_i) \oplus \left(\left(\frac{Y_k}{X_k} \right)'' \otimes X_k \right) = \begin{bmatrix} Z_1 \\ Z_2 \\ V^* \end{bmatrix} \oplus \begin{bmatrix} V_1 \\ V_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} Z_1 \oplus V_1 \\ Z_2 \oplus V_2 \\ V^* \oplus Z_3 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = Y \tag{3-29}$$

which can be written as

$$Y = \sum_{\substack{i=1 \\ i \neq k}}^{m} (a_i \otimes X_i) \oplus \left(\left(\frac{Y_k}{X_k} \right)'' \otimes X_k \right) \tag{3-30}$$

End of Proof

Example 3-10:

$$X_1 = \begin{pmatrix} 0.7 \\ 0.3 \\ 0.2 \end{pmatrix}, X_2 = \begin{pmatrix} 0.5 \\ 0.6 \\ 0.3 \end{pmatrix}, X_3 = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.4 \end{pmatrix}, Y = \begin{pmatrix} 0.5 \\ 0.3 \\ 0.4 \end{pmatrix}$$ and min max as T norm and co-norm,

then one has $a_1 = \left(\frac{Y}{X_1} \right)' = 0.5$, $a_2 = \left(\frac{Y}{X_2} \right)' = 0.3$ and $a_3 = \left(\frac{Y}{X_3} \right)' = 1$.

Since $Y \leq (a_1 \otimes X_1) \oplus (a_2 \otimes X_2) \oplus (a_3 \otimes X_3)$, Y belongs to the space spanned by

$\{X_1, X_2, X_3\}$.

Also because

$$a_1 \otimes X_1 = \begin{pmatrix} 0.5 \\ 0.3 \\ 0.2 \end{pmatrix}, a_2 \otimes X_2 = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.3 \end{pmatrix}, a_3 \otimes X_3 = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.4 \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \\ 0.4 \end{pmatrix}$$

X_1 has a unique effect on y_1

X_1 and X_2 both affect y_2

X_3 has a unique effect on y_3

Finally from (3-11)

$$Y_1 = \begin{pmatrix} 0.5 \\ 0 \\ 0 \end{pmatrix}, Y_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, Y_3 = \begin{pmatrix} 0 \\ 0 \\ 0.4 \end{pmatrix}$$

Then

$$\left(\left(\frac{Y_1}{X_1} \right)'' \otimes X_1 \right) = 0.5 \otimes \begin{pmatrix} 0.7 \\ 0.3 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \\ 0.2 \end{pmatrix}$$

$$\left(\left(\frac{Y_2}{X_2} \right)'' \otimes X_2 \right) = 0 \otimes \begin{pmatrix} 0.5 \\ 0.6 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\left(\frac{Y_3}{X_3} \right)'' \otimes X_3 \right) = 0.4 \otimes \begin{pmatrix} 0.4 \\ 0.2 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.4 \end{pmatrix}$$

It is easy to verify that Equation (3-30) is true.

Lemma 3-8

If the equation $A \otimes x = b$ is solvable, and $x = (x_1 \quad \cdots \quad x_i \quad \cdots \quad x_m)^T$ is a solution,

then $x_i \geq \left(\dfrac{B_i}{A_i} \right)''$ where

$$B_i = b \ominus \sum_{\substack{j=1 \\ j \neq i}}^{m} \left(\frac{b}{A_j} \right)' \otimes A_j \qquad (3\text{-}31)$$

and A_i denotes the column i of matrix A

Proof:

From the proof of Lemma 3-7

If the value of A_i does not have a unique effect on the value of b, then $\left(\dfrac{B_i}{A_i} \right)'' = 0$:

$$x_i \geq 0 = \left(\frac{B_i}{A_i} \right)''$$

If the value of A_i has a unique effect on the value of b, then $\left(\dfrac{B_i}{A_i} \right)'' > 0$, and from the

definition of the infimum multiplicative inversion, $\left(\dfrac{B_i}{A_i} \right)''$ is the lower boundary which

maintains the effects from A_i to b,

Therefore $x_i \geq \left(\dfrac{B_i}{A_i} \right)''$

End of proof

Theorem 3-6

If the equation $A \otimes x = b$ is solvable, then the mean solution is

$$\bar{x} = \begin{pmatrix} \bar{x}_1 & \cdots & \bar{x}_i & \cdots & \bar{x}_m \end{pmatrix}^T \text{ and } \bar{x}_i = \min\left(\left(\frac{b}{A_i} \right)', \left(\frac{b}{A_i} \right)'' \right) \qquad (3\text{-}32)$$

Proof:

Substituting Equation 3-32 into Equation 3-1yields:

$$A \otimes x = \sum_{i=1}^{m} \overline{x}_i \otimes A_i \qquad \text{(3-33)}$$

Without lose of generality, Equation 3-33 can be written as

$$\sum_{i=1}^{m_1} \left(\frac{b}{A_i}\right)'' \otimes A_i + \sum_{i=m_1+1}^{m} \left(\frac{b}{A_i}\right)' \otimes A_i$$

where $\left(\dfrac{b}{A_i}\right)'' < \left(\dfrac{b}{A_i}\right)'$ for $i \in \{1, \cdots, m_1\}$

and $\left(\dfrac{b}{A_i}\right)'' \geq \left(\dfrac{b}{A_i}\right)'$ for $i \in \{m_1 + 1, \cdots, m\}$

A_i denotes column i of matrix A

Let $b^* = \sum_{i=1}^{m_1} \left(\dfrac{b}{A_i}\right)' A_i$ and $b^{**} = \sum_{i=1}^{m_1} \left(\dfrac{b}{A_i}\right)'' A_i$

As $\left(\dfrac{b}{A_i}\right)'' < \left(\dfrac{b}{A_i}\right)'$ from Lemma 3-2, $b \geq A_i$ for $i \in \{1, \cdots, m_1\}$

Again without the loss of the generality, each of b, b^* and b^{**} can be separated into

two parts namely

$$b = \left\{\begin{matrix} b_1 \\ \overline{b_2} \end{matrix}\right\} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix}, \quad b^* = \left\{\begin{matrix} b_1^* \\ \overline{b_2^*} \end{matrix}\right\} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix} \quad \text{and} \quad b^{**} = \left\{\begin{matrix} b_1^{**} \\ \overline{b_2^{**}} \end{matrix}\right\} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix}$$

where $n_1 + n_2 = n$. These two parts satisfy the following equation

$$b_i = b_i^* = \left(\frac{b}{A_j}\right)' \otimes A_{i,j} \qquad for \ i \in \{1, \cdots, n_1\} \ \text{ and } \ j \in \{1, \cdots, m_1\} \qquad \text{(3-34)}$$

and $b_i \geq b_i^*$ for $i \in \{n_1 + 1, \cdots, n\}$

From the above relations and the definition of the infimum inversion,

$$b_i = \left(\frac{b_i}{Aij}\right)_{\text{inf}} \otimes A_{i,j} \quad \text{exists for all} \ \ i \in \{1, \cdots, n_1\} \ \ \text{and} \ \ j \in \{1, \cdots, m_1\}$$

Therefore

$$b_1^{**} = b_1^{*} = b_1$$

and

$$b_2^{**} \leq b_2^{*} < b_2$$

Because b^{*} only affects b by b_1^{*}:

$$b = \sum_{i=1}^{m} \left(\frac{b}{A_i}\right)' \otimes A_i$$

$$= b^{*} + \sum_{i=m_1+1}^{m} \left(\frac{b}{A_i}\right)' \otimes A_i$$

$$= b^{**} + \sum_{i=m_1+1}^{m} \left(\frac{b}{A_i}\right)' \otimes A_i$$

$$= \sum_{i=1}^{m_1} \left(\frac{b}{A_i}\right)'' \otimes A_i + \sum_{i=m_1+1}^{m} \left(\frac{b}{A_i}\right)' \otimes A_i$$

$$= \sum_{i=1}^{m} \bar{x}_i \otimes A_i$$

So $\bar{x}$ is the solution of Equation 3-1.

In order to prove that $\bar{x}$ is the mean solution, it is necessary to prove that any vector $x < \bar{x}$ cannot be a solution of Equation 3-1.

If $x < \bar{x}$ then $x_i < \min\left[\left(\dfrac{b}{A_i}\right)', \left(\dfrac{b}{A_i}\right)''\right]$

From Lemma 3-5, $A_i \otimes x_i < b$

Therefore,

$$A \otimes x = \sum_{i=1}^{m} A_i \otimes x_i < b$$

and $x < \bar{x}$ cannot be the solution for Equation 3-1.

End of Proof

Example 3-11:

Consider the following Max-Min based Fuzzy Relational Equations:

$$\begin{pmatrix} 0.8 & 0.8 & 0.8 & 0.15 & 0.4 & 0.7 & 0.2 & 0.5 \\ 0.3 & 0.7 & 0.7 & 0.7 & 0.7 & 0.1 & 0.6 & 0.7 \\ 0.6 & 0.6 & 0.6 & 0.6 & 0.6 & 0 & 0.6 & 0.4 \\ 0.2 & 0.7 & 0.3 & 0.55 & 0.8 & 0.8 & 0.1 & 0.3 \end{pmatrix} \otimes x = \begin{pmatrix} 0.8 \\ 0.7 \\ 0.6 \\ 0.8 \end{pmatrix} \qquad (3\text{-}35)$$

From Theorems 3-5 and 3-6, the maximum solution is $\hat{x} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$

and the mean solution is $\bar{x} = \begin{pmatrix} 0.8 & 0.8 & 0.8 & 0.7 & 0.8 & 0.8 & 0.6 & 0.7 \end{pmatrix}^T$

Any vector x, which satisfies $\bar{x} \leq x \leq \hat{x}$, is a solution for Equation 3-35.

3.4.2 Minimum solutions of FREs

As mentioned in section 3.1, the minimum solution of a FRE is not unique. In order to find all the minimum solutions, an optimal algorithm will be developed in this section.

Before developing the algorithm, the lower limit of the solution space of FREs needs to be considered first.

Theorem 3-7

1 If the equation $A \otimes x = b$ is solvable, then the lower limit of the minimum solutions

is $\underline{x} = \begin{pmatrix} \underline{x}_1 & \cdots & \underline{x}_i & \cdots & \underline{x}_m \end{pmatrix}^T$, where $\underline{x}_i = \left(\dfrac{B_i}{A_i} \right)''$ and $B_i = b \Theta \sum\limits_{\substack{j=1 \\ j \neq i}}^{m} \left(\dfrac{b}{A_j} \right)' \otimes A_j$.

2 Each minimum solution is constructed by directly combining the elements from mean solution $\bar{x}$ and $\underline{x}$.

Proof:

Lemma 3-8 shows that $\underline{x}$ is the lower limit of solutions for Equation 3-1 and Lemma 3-7 and Theorem 3-6 show that each minimum solution is constructed as a combination of the elements from $\bar{x}$ *and* $\underline{x}$

End of Proof

Lemma 3-9

If equation $A \otimes x = b$ is solvable, then it must have the same solutions as equation

$$A^* \otimes x = b \text{ where } A^* = [a_{ij}^*] \text{ and } a_{ij}^* = \begin{cases} 0 & \text{if } a_{ij} \otimes \hat{x}_j < b_i \\ a_{ij} & \text{otherwise} \end{cases}$$

Proof:

If $A \otimes x = b$ and $a_{ij} \otimes \hat{x}_j < b_i$, then a_{ij} does not affect the final result. So equation

$A^* \otimes x = b$ has the same solutions as Equation $A \otimes x = b$.

End of Proof

Lemma 3-10

If equation $A \otimes x = b$ is solvable, and there are no coupling among the columns of A, then the mean solution is equal to the minimum solution.

This can be directly derived from Lemmas 3-7 and 3-8. Here, "no coupling" means that the column vectors of A are independent from on another. None of them can be expressed as a fuzzy combination of other vectors (located in fuzzy space spanned by other columns).

From Theorem 3-7, each minimum solution is constructed by combining the elements from the mean solution $\bar{x}$ and the lower limit $\underline{x}$. Then a minimum solution can be obtained by testing all the possible combinations of $\bar{x}$ and $\underline{x}$. However, exploring all possible combinations can be time consuming because a large amount of calculations will be wasted on impossible alternatives (Wu et al 2002). The following algorithm will yield the minimal solution without exploring impossible alternatives.

Algorithm 3-1:

1. Simplify a FRE by arranging it into the form of A^*.

2. Decompose the equation by using the row with the smallest number of non-zero elements.

3. Study each decomposed sub-system. If there are no compositions between columns, go to step 4. Otherwise go to step 2 and further decompose the sub-system.

4. Find the mean solution for the sub-system. From Lemma 3-10, this will be the minimum solution for the fuzzy relational equation.

As all impossible alternatives are automatically eliminated by the decomposition procedure (steps 2 and 3), unnecessary explorations are avoided. This algorithm is an optimal method to calculate the minimum solutions. Furthermore, it is a general method for solving max family FREs; other methods developed for max-product (Wu et al 2002) or max-min. can be regarded as variants or special cases of this method.

3.5   Key Findings and A Numerical Example

3.5.1   Key Findings

Before illustrating the application of the results derived in this chapter, the key findings are first summarised as follows.

The fuzzy relational equation:

$$A \otimes x = b$$

where

$$A = \begin{pmatrix} A_1 & \cdots & A_i & \cdots & A_n \end{pmatrix}$$

$$A_i = \begin{pmatrix} a_{1,i} & \cdots & a_{j,i} & \cdots & a_{m,i} \end{pmatrix}^T, a_{j,i} \in [0,1]$$

$$x = \begin{pmatrix} x_1 & \cdots & x_i & \cdots & x_m \end{pmatrix}^T, x_i \in [0,1]$$

$$b = \begin{pmatrix} b_1 & \cdots & b_i & \cdots & b_n \end{pmatrix}^T, b_i \in [0,1]$$

is solvable if and only if:

$$b \leq \sum_{i=1}^{m} \left(\frac{b}{A_i} \right)' \otimes A_i \qquad \qquad \text{(Lemma 3-6)}$$

If the above condition is satisfied, the maximum solution is

$$\widehat{x} = \begin{pmatrix} \widehat{x}_1 & \cdots & \widehat{x}_i & \cdots & \widehat{x}_m \end{pmatrix}^T$$

where $\widehat{x}_i = \left(\dfrac{b}{A_i} \right)'$ \qquad \qquad \text{(Theorem 3-5)}

The mean solution is

$$\bar{x} = \begin{pmatrix} \bar{x}_1 & \cdots & \bar{x}_i & \cdots & \bar{x}_m \end{pmatrix}^T$$

where $\bar{x}_i = \min\left[\left(\dfrac{b}{A_i}\right)', \left(\dfrac{b}{A_i}\right)''\right]$ (Theorem 3-6)

The lower limit of the minimum solutions is

$$\underline{x} = \begin{pmatrix} \underline{x}_1 & \cdots & \underline{x}_i & \cdots & \underline{x}_m \end{pmatrix}^T$$

with $\underline{x}_i = \left(\dfrac{B_i}{A_i}\right)''$ and $B_i = b \Theta \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{m} \left(\dfrac{b}{A_j}\right)' \otimes A_j$ (Theorem 3-7)

Each of minimum solutions is constructed by combining the elements of the mean solution $\bar{x}$ and $\underline{x}$. An efficient algorithm to obtain all the minimum solutions is algorithm 3-1.

3.5.2 Example 3-12

Consider the following example from (Cechlárová 1995). The product and max are used as T norm and co-norm.

$$\begin{pmatrix} 0.8 & 0.6 & 0.2 & 0.4 & 0.2 & 0.7 & 0.7 & 0.5 \\ 0.6 & 0.3 & 0.7 & 0.6 & 0.1 & 0.3 & 0.5 & 0.3 \\ 0.5 & 0.8 & 0.7 & 0.4 & 0.7 & 0.8 & 0.3 & 0.8 \\ 0.2 & 0.4 & 0.5 & 0.1 & 0.3 & 0.5 & 0.8 & 0.4 \\ 0.6 & 0.2 & 0.5 & 0.5 & 0.1 & 0.4 & 0.7 & 0.2 \\ 0.9 & 0.9 & 0.8 & 0.2 & 0.8 & 0.6 & 0.1 & 0.4 \end{pmatrix} \otimes x = \begin{pmatrix} 0.56 \\ 0.42 \\ 0.64 \\ 0.4 \\ 0.42 \\ 0.72 \end{pmatrix}$$ (3-36)

From Lemma 3-6, the equation is solvable.

From Theorems 3-5, 3-6 and 3-7,

$$\hat{x} = \bar{x} = \begin{pmatrix} 0.7 & 0.8 & 0.6 & 0.7 & 0.9 & 0.8 & 0.5 & 0.8 \end{pmatrix}^T$$

and $\underline{x} = \begin{pmatrix} 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$

From Lemma 3-9, Equation 3-36 can be simplified to

$$
\begin{pmatrix}
0.8 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 \\
0.6 & 0 & 0.7 & 0.6 & 0 & 0 & 0 & 0 \\
0 & 0.8 & 0 & 0 & 0 & 0.8 & 0 & 0.8 \\
0 & 0 & 0 & 0 & 0 & 0.5 & 0.8 & 0 \\
0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.9 & 0 & 0 & 0.8 & 0 & 0 & 0
\end{pmatrix}
\otimes
\begin{pmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8
\end{pmatrix}
=
\begin{pmatrix}
0.56 \\ 0.42 \\ 0.64 \\ 0.4 \\ 0.42 \\ 0.72
\end{pmatrix}
\tag{3-37}
$$

Since there is only one non-zero coefficient in the 5th row of Equation 3-37, its corresponding column is separated from the equation. Equation 3-37 can be reduced to two smaller equations, which are:

$$
\begin{pmatrix}
0.8 \\ 0.6 \\ 0.6
\end{pmatrix}
\otimes x_1
=
\begin{pmatrix}
0.56 \\ 0.42 \\ 0.42
\end{pmatrix}
\tag{3-38}
$$

$$
\begin{pmatrix}
0.8 & 0 & 0 & 0 & 0.8 & 0 & 0.8 \\
0 & 0 & 0 & 0 & 0.5 & 0.8 & 0 \\
0.9 & 0 & 0 & 0.8 & 0 & 0 & 0
\end{pmatrix}
\otimes
\begin{pmatrix}
x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8
\end{pmatrix}
=
\begin{pmatrix}
0.64 \\ 0.4 \\ 0.72
\end{pmatrix}
\tag{3-39}
$$

For (3-38), it can be seen that $x_1 = 0.7$ is the minimum solution as stated on the previous page.

Equation 3-39 needs to be further decoupled in order to find its minimum solution. Since the 2nd and 3rd rows in Equation 3-39 have the lowest number of non-zero elements, the equations can be divided by either 2nd or 3rd row. In this case, it is divided

into two systems by the elements in the 2nd row. Each of the new system contains two

equations unit.

The first system is:

$$\begin{pmatrix} 0.8 \\ 0.5 \end{pmatrix} \otimes x_6 = \begin{pmatrix} 0.64 \\ 0.4 \end{pmatrix}$$

(3- 40)

and

$$(0.9 \quad 0.8) \otimes \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} = 0.72$$

(3- 41)

From Equation 3-40, $x_6 = 0.8$.

Equation 3-41 can be further reduced to

$$(0.9) \otimes x_2 = 0.72$$

and

$$(0.8) \otimes x_5 = 0.72$$

This implies $x_2 = 0.8$ and $x_5 = 0.9$

Therefore, the minimum solutions derived from the first system of equations are:

$$\breve{x}_1 = (0.7 \quad 0.8 \quad 0 \quad 0 \quad 0 \quad 0.8 \quad 0 \quad 0)^T$$

and

$$\breve{x}_2 = (0.7 \quad 0 \quad 0 \quad 0 \quad 0.9 \quad 0.8 \quad 0 \quad 0)^T$$

The second system of equations is

$$(0.8) \otimes x = 0.4$$

(3- 42)

and

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$$\begin{pmatrix} 0.8 & 0 & 0.8 \\ 0.9 & 0.8 & 0 \end{pmatrix} \otimes x = \begin{pmatrix} 0.64 \\ 0.72 \end{pmatrix} \tag{3-43}$$

As above, the minimum solutions can be found as

$$\breve{x}_3 = \begin{pmatrix} 0.7 & 0.8 & 0 & 0 & 0 & 0 & 0.5 & 0 \end{pmatrix}^T \text{ and}$$

$$\breve{x}_4 = \begin{pmatrix} 0.7 & 0 & 0 & 0 & 0.9 & 0 & 0.5 & 0.8 \end{pmatrix}^T$$

Hence the entire solution space for Equation 3-36 is:

Maximum solution $\hat{x}$	$\begin{pmatrix} 0.7 & 0.8 & 0.6 & 0.7 & 0.9 & 0.8 & 0.5 & 0.8 \end{pmatrix}^T$
Mean solution $\bar{x}$	$\begin{pmatrix} 0.7 & 0.8 & 0.6 & 0.7 & 0.9 & 0.8 & 0.5 & 0.8 \end{pmatrix}^T$
Minimum solutions $\breve{x}$	$\begin{pmatrix} 0.7 & 0.8 & 0 & 0 & 0 & 0.8 & 0 & 0 \end{pmatrix}^T$ $\begin{pmatrix} 0.7 & 0 & 0 & 0 & 0.9 & 0.8 & 0 & 0 \end{pmatrix}^T$ $\begin{pmatrix} 0.7 & 0.8 & 0 & 0 & 0 & 0 & 0.5 & 0 \end{pmatrix}^T$ $\begin{pmatrix} 0.7 & 0 & 0 & 0 & 0.9 & 0 & 0.5 & 0.8 \end{pmatrix}^T$
Lower boundary $\underline{x}$	$\begin{pmatrix} 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$

3.6 Summary

A complete analytical solution for the max family of FREs based on fuzzy inversions has been developed in this chapter. The proposed method is a unified approach for max family FREs.

Chapter 4

Fuzzy System Design And Functional Analysis

4.1   Preliminaries

This chapter focuses on the possibility of applying functional analysis theory to the design of fuzzy systems. Functional analysis is the mathematician' "black-box diagram" (Curtain and Pritchard 1977). It was developed to deal with general functions instead of specific values. The motivation for applying functional analysis to fuzzy systems is twofold. First, as an exact mathematical analytical method, functional analysis can handle inexact data and knowledge. Second, the simple notions in functional analysis avoid many of the complicating details in design and analysis, highlighting only the essential aspects. Functional analysis is a convenient way to examine the behaviour of various models, including fuzzy models. An aim of this research is to provide a new perspective for fuzzy system design, and to develop an efficient algorithm for fuzzy models from input/output data pairs.

Based on functional analysis theory, a functional point of view for fuzzy system design will be developed. From a functional point of view, fuzzy system modelling consists of two parts:

- Searching for the optimum basis (membership functions)

- Measuring the minimal distance between the given basis and the target function.

This is an iterative process. If the target function is covered by the space spanned by the given basis, the distance between the basis and the target function is zero. If this is not the case, a minimal distance can be derived from fuzzy transform action. Based on the distance, further optimisation of the basis is performed iteratively until basis that are close enough to the target function are obtained.

Fuzzy system design involves a combination of "soft-computing" and "hard-computing". The first part of fuzzy system design, "searching for optimum basis", is a soft-computing activity. Any suitable optimisation techniques, such as neural network, genetic algorithm, could be applied. The second part of fuzzy system design, "measuring the minimal distance", can be regarded as "hard-computing". The task can be completed analytically. The most commonly used method is the least-squares method.

Therefore, in order to outperform modern fuzzy system design approaches, next generation fuzzy system design needs to be enhanced in both the "soft-computing" direction and the "hard-computing" direction. Areas for improvement include:

- Reducing the search space and choosing better initial conditions

- Increasing the efficiency of the minimal distance calculation

Within the last decade, the "soft-computing" aspect of fuzzy system design has received considerable attention and various neuro-fuzzy approaches have been proposed. On the other hand, since the introduction of the least-squares method for fuzzy system design in the early 1980s, the "hard-computing" aspect of fuzzy

modelling has been almost untouched. This chapter presents a new "hard-computing" algorithm for fuzzy system design. The algorithm is based on the generalised Fourier transform. It will be demonstrated that instead of using the least-squares method, the minimal distance can be derived by fuzzy transform action. This facilitates the determination of optimum designs and the development of more efficient algorithms for the next generation fuzzy system design.

The chapter is organised as follows. The functional perspective of fuzzy systems and the analogy between fuzzy system design and functional analysis are explored in section 4.2. In section 4.3, the concept of fuzzy transforms is proposed based on the generalised Fourier transform. Fuzzy transformation is applied to T-S fuzzy system design in order to improve efficiency. In section 4.4, dual base and dual spaces are introduced for further improving the efficiency of fuzzy modelling.

4.2 Functional Perspective of Fuzzy Systems

The treatment of imprecision and vagueness can be traced back to the work of Lukasiewicz in the 1930s (Bonissone et al 1999). A multiple-valued logic was proposed by him to represent undetermined intermediate truth-values between the classical Boolean true and false values (Rescher 1965). Later the philosopher Black suggested that vague concepts could be represented by a consistency profile, while fuzziness addresses the lack of sharp boundaries between sets (Black, 1937). As mentioned previously Zadeh proposed the theory of fuzzy sets (Zadeh, 1965), which provided a systematic way to deal with ambiguous and ill-defined concepts. Based on fuzzy sets theory, fuzzy systems were developed.

In the last three decades, considerable attention has been given to research into fuzzy systems, especially into the theory of fuzzy sets. Researchers believe that fuzzy sets may lead to a better understanding of fuzzy systems and possibly more advanced fuzzy computing approaches. Thousands of papers have been published in this area and fuzzy sets theory has become a respected part of science.

As fuzzy sets theory grew in popularity, various fuzzy inference mechanisms were proposed for different application areas. Among them, Mamdani's fuzzy model and the T-S fuzzy model have been widely accepted in fuzzy system design. It is interesting to note that despite the popularity of fuzzy sets theory, only fuzzy models based on simple fuzzy sets theory have been commonly implemented. In particular for the T-S fuzzy model, little fuzzy sets theory is involved. This indicates that there is a gap between modern fuzzy sets theory and its practice in fuzzy system design. People may argue that

since fuzzy systems mimic human thinking, their applications should be straightforward and easy to understand. However, when fuzzy systems are employed for more sophisticated problems, current theory turns out to be inadequate. Innovation in fuzzy theory is required to address challenges such as the construction of fuzzy systems for complicated systems from input/output data.

During the last decade, a number of researchers have tried to explain the behaviour of fuzzy systems using input-output based mathematical descriptions. They have carried out studies with titles such as "The universal approximation ability of fuzzy system based on fuzzy basis functions", "Functional equivalent between some fuzzy systems and the Radial Basis Function Networks", and "Fuzzy PID controller is equivalent to multilevel relay and a local nonlinear proportional-integral controllers" (Ying, 1993), (Wang and Mendel, 1993), (Jang and Sun 1993), (Kosko, 1997). As opposed to the inference viewpoint commonly accepted in the fuzzy systems community, these studies were based on a functional point of view, regarding the fuzzy system as some kind of function. Their success provides new understanding of fuzzy system behaviours. However, because a complete input-output based mathematical description of the fuzzy system design is not yet available, those results of the above studies have appeared individually, and have not been applied to fuzzy system design directly.

According to Zadeh (Zadeh 1997), there are three basic concepts in human cognition: granulation, causation and organisation. Informally, granulation involves the decomposition of a whole into parts, organisation, the integration of parts into a whole, and causation, the association of a cause with effects. In a fuzzy system, these concepts are implemented in the following way: granulation is achieved by converting crisp

values into fuzzy values; organisation is represented by the process of converting local information into global information; and causation is implemented by rules and the inference mechanism. For example, in the Mamdani fuzzy model, granulation and organisation are expressed through fuzzification and defuzzification, respectively. Causation is performed in the so-called compositional rule of inference (CRI). From the logician's point of view, this is a balanced process with all the components in a fuzzy system having their own role. Together, fuzzy values, rules and the inference mechanism, construct a "set-to-set" mapping. They successfully avoid the drawbacks of "point-to-point" mappings when dealing with ambiguity and uncertainty.

If inexact data are considered as exact signals plus random noise, mathematically, in fuzzy system modelling, the noise is eliminated by the membership functions. This is because the convex-shaped membership functions are low pass filters. The ability of fuzzy systems to deal with uncertainty is fully dependent on the size of the support of membership functions as shown later in the chapter. At the same time, membership functions also construct a function space for implementing further approximation. The membership functions become the basis of the space. The role of the remaining parts of the fuzzy system is to approximate the fuzzy system to the signals, which have been extracted from inexact data, as accurately as possible. This process can be regarded as a transform action between the function space constructed by the membership functions and the target system. Since any inexactitude and uncertainty in the input/output data pairs have been filtered by membership functions, the term "as accurately as possible" has its normal meaning in mathematical transform theory. This viewpoint is somewhat different from the inference viewpoint that is commonly accepted in fuzzy literature. Because it is very close to the concept of the transform in functional analysis, this will

be called the functional viewpoint, and the transform will be called the *fuzzy transform*. It will be demonstrated that this new point of view provides a better explanation of the input-output behaviour of fuzzy systems. From this point of view, the best approximation comes from a combination of rules, inference mechanisms and local information in the output space that gives the closest transform from the space constructed by input membership functions to the target function. The universal approximation ability of a fuzzy system is fully dependent on the completeness of the membership functional space, which is determined by its membership functions. It can be proved that if the number of membership functions is not limited, a complete space that covers any target function can be constructed. This is the principle behind the universal approximation ability of a fuzzy system.

The functional point of view leads to an insight into the behaviour of a fuzzy system. Some important results in fuzzy theory, such as the universal approximation ability of fuzzy systems and the functional equivalence between fuzzy systems and Radial Basis Function networks, become clearer from this point of view.

One important reason for the gap between modern fuzzy sets theory and fuzzy modelling practice is that the logician's viewpoint fails to realise the difference between the roles of the components in a fuzzy system. In practice, this may generate great difficulties for fuzzy modelling. For example, consider a fuzzy system with three input variables using five membership functions for each variable. If the rulebase is complete (Blyth and Janowitz, 1972), a total of 134 ($= 5^3 + 3 \times 3$) parameters have to be optimised. This is a very large search space for any kind of soft computing schema. However, from the functional point of view the optimum approximation can be

achieved by tuning the input membership functions alone, and the search space is reduced to as few as 9 parameters (for normalised triangular membership functions).

4.3 Functional Analysis for Fuzzy System Design

In the previous section, the idea of a functional viewpoint for fuzzy system design has been explained. A complete fuzzy system design theory based on functional analysis will be presented in this section.

4.3.1 Notation

1. Normed Linear Space

Let X be a vector space, any function $\|\bullet\|$ that satisfies following conditions is a norm

    i. $\|u\| \geq 0$ (Positive definiteness)

    ii. $\|\alpha \cdot u\| = |\alpha| \cdot \|u\|$ (Linearity)

    iii. $\|u + v\| \leq \|u\| + \|v\|$ (Triangle inequality)

where u and v are vectors on X, $u \in X$ $v \in X$. α is a complex-valued scalar, $\alpha \in C$.

A vector space in which a norm is defined is called a normed linear space.

For example, for a linear vector space $X[a,b]$ includes all bounded continuous functions on interval $[a,b]$, $X[a,b]$ becomes a normed space if a norm is defined on the space as

$$\|f\| = \|f\|_2 = [\int_a^b (f(t) \cdot \overline{f(t)}) \cdot dt]^{\frac{1}{2}} \tag{4-1}$$

Where $f(t)$ denotes a continuous functions on interval $[a,b]$ (It is regarded as a vector in space $X[a,b]$). $\overline{f(t)}$ is the conjugate function of $f(t)$. $\|f\|_2$ denotes the "Euclidean distance" of the vector f from the origin.

2. Linear Inner Product Space

Let V be a vector space over X. An inner product on V is a complex-valued function $\langle \cdot, \cdot \rangle : V \times V \to C$ such that for any $x, y, z \in V$ and $\alpha, \beta \in C$

   i. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$

   ii. $\langle x, y \rangle = \overline{\langle y, x \rangle}$

   iii. $\langle x, x \rangle > 0$ if $x \neq 0$

where $\overline{\langle x, y \rangle}$ is complex conjugate of $\langle x, y \rangle$

A linear vector space V with an inner product is called a linear inner product space.

Two vectors x and y in an inner product space V are said to be orthogonal if $\langle x, y \rangle = 0$ which denoted by $x \perp y$.

For given functions $f(x)$, $g(x)$ and $x \in [a,b]$, an inner product can be defined as

$$< f, g >= \int_a^b \rho(x) f(x) \overline{g(x)} dx \tag{4-2}$$

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where $\rho(x)$ is a weighting function, $\overline{g(x)}$ is the conjugate function of $g(x)$.

If $f(x) \in R, g(x) \in R$ and $\rho(x) = 1$ Equation (4-2) can be simplified to

$$< f, g >= \int_a^b f(x)g(x)dx \qquad (4\text{-}3)$$

The discrete form of (4-3) is

$$< f, g >= \sum_{i=0}^n f(i) \cdot g(i) \qquad (4\text{-}4)$$

where $n+1$ is the number of samples in $[a,b]$.

Equation (4-4) is defined based on the operator sum and product. It can be replaced by a T norm and co-norm, such as the max-product or max-min:

$$< f, g >= \sup_i [(f(i) \cdot g(i))] \qquad \text{max-product} \qquad (4\text{-}5)$$

$$< f, g >= \sup_i [\min(f(i), g(i))] \qquad \text{max-min} \qquad (4\text{-}6)$$

It is easy to verify that Equation (4-5) and (4-6) also satisfy the requirements of the inner product.

3.   Best approximation in linear inner product space.

Definition

Given $i \in \{0, \cdots, n\}$, a basis $\{\phi_i\}$ and a set of parameters $c_i^* \in C$, a function $S^* = \sum_{i=0}^n c_i^* \phi_i$

is the best approximation of f, if and only if $f - S^*$ is orthogonal to ϕ_i,

$$< f - S^* , \phi_i >= 0 , \qquad\qquad i \in \{0, \cdots, n\}$$

If $f(x)$ is best approximated by $S^* = \sum_{i=0}^{n} c_i^* \phi_i$, for any $S = \sum_{i=0}^{n} c_i \phi_i$ (it is formed by

unknown parameters $c_i \in C$ and the basis $\{\phi_i\}$), the following holds:

$$\int_a^b \rho(x)[f(x) - S^*]^2 \, dx \le \int_a^b \rho(x)[f(x) - S]^2 \, dx \qquad\qquad (4\text{-}7)$$

This means the length of $f - S$ reaches its minimum value when $S = S^*$.

$S^* = \sum_{i=0}^{n} c_i^* \phi_i$ is called the generalised Fourier transform of $f(x)$ in functional analysis.

Let:

$$I(c_0, c_1, \cdots, c_n) = \int_a^b \rho(x)[f(x) - S]^2 \, dx = \int_a^b \rho(x)[f(x) - \sum_{i=0}^{n} c_i \phi_i]^2 \, dx \qquad (4\text{-}8)$$

I in Equation (4-8) is a linear function of $(c_0, c_1, \cdots, c_n)$. To find the best approximation

of $f(x)$ in function space constructed by $\{\phi_i\}$, the minimal value for $I(c_0, c_1, \cdots, c_n)$

needs to be calculated:

From

$$\frac{\partial I}{\partial c_k} = -2 \int_a^b \rho(x)[f(x) - \sum_{i=0}^{n} c_i \phi_i] \phi_k \, dx = 0 , \quad k \in \{0, \cdots, n\} \qquad (4\text{-}9)$$

$$\sum_{i=0}^{n} (\int_a^b \rho(x) \phi_i \phi_k \, dx) c_i = \int_a^b \rho(x) f(x) \phi_k(x) \, dx , \quad k \in \{0, \cdots, n\} \qquad (4\text{-}10)$$

Equation (4-10) can be rewritten in the form of an inner product

$$\sum_{i=0}^{n} < \phi_k, \phi_i > c_i \ = < f, \phi_k >, \quad k \in \{0, \cdots, n\} \tag{4-11}$$

For a normalised orthogonal basis, $< \phi_i, \phi_j > = k\delta_{i,j}$ and $< \phi_i, \phi_i > = < \phi_j, \phi_j >$. The solution of Equation (4-11) is:

$$\begin{pmatrix} c_1 \\ \vdots \\ c_i \\ \vdots \\ c_n \end{pmatrix} = \frac{1}{< \phi_1, \phi_1 >} \begin{pmatrix} < \phi_1, f > \\ \vdots \\ < \phi_i, f > \\ \vdots \\ < \phi_n, f > \end{pmatrix} \tag{4-12}$$

or

$$c_i = \frac{< \phi_i, f >}{< \phi_i, \phi_i >} \tag{4-13}$$

Equation (4-13) is the general formula for mathematical transformation in functional analysis. Well-known transforms, such as Fourier Transform, Gabor Transform and some orthogonal wavelet transforms, can all be derived from Equation (4-13).

4.3.2 Fuzzy Modelling and Fuzzy Transform

In this section, the concept of fuzzy transform action is proposed based on the general formula for mathematical transforms. Fuzzy transform is applied to the T-S fuzzy system design in order to improve the efficiency of the development procedure.

4.3.2.1 Fuzzy Modelling for zero-order T-S model

Consider a simple zero-order T-S fuzzy system.

If Z is $A_i(Z)$ Then $y_i = a_i$

$\vdots$

and $y = \sum_{i}^{p} y_i \cdot \phi_i(Z) = \sum_{i}^{p} a_i \cdot \phi_i(Z)$

where $i \in \{1, \cdots, p\}$, p is the number of rules, $Z \in R^n$, $y \in R$, $y_i \in R$, a_i is a fuzzy

singleton and $\phi_i(Z) = \dfrac{A_i(Z)}{\sum_{j}^{p} A_j(Z)}$.

From given input/output data pairs $\{Z_j, y_j\}$, $j \in \{1 \cdots m\}$, a fuzzy system can be

constructed using the above T-S model by applying a hybrid learning method, for

examples ANFIS(Jang, 1993). The parameters $\{a_i\}$ in the fuzzy model are usually

identified by using a least-squares estimator (Takagi and Sugeno, 1985) as:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} = (A^T A)^{-1} A^T Y \qquad\qquad (4\text{-}14)$$

where:

$$A = \begin{bmatrix} \phi_1(Z_1) & \cdots & \phi_p(Z_1) \\ \vdots & & \vdots \\ \phi_1(Z_m) & \cdots & \phi_p(Z_m) \end{bmatrix}_{m \times p}$$

In order to identify the best parameters for the T-S model, the least-squares method has

to calculate $A^T A$ and the inverse of $(A^T A)$. When p and m are large, the calculation

can be very lengthy. Furthermore, in a hybrid learning process, the calculation needs to

be performed repeatedly and this can make the fuzzy modelling process very slow.

In order to solve this problem, alternative approaches are developed in this work. As shown previously, from functional analysis, the best approximation from the basis $\{\phi_i\}$ to target function $f(Z)$ is the generalised Fourier transform, $\hat{y} = \sum_i^p \hat{a}_i \cdot \phi_i(Z)$, which satisfies

$$< f(Z) - \hat{y}, \phi_i >= 0, \qquad\qquad i \in \{1 \cdots p\} \qquad\qquad (4\text{-}15)$$

where $<,>$ denotes an inner product.

Equation 4-15 indicates that the error function $f - \hat{y}$ is orthogonal to all of the basis vectors ϕ_i. Then the distance from basis $\{\phi_i\}$ to target function f is minimised.

As designing a fuzzy system is equivalent to deriving a transform from membership functions to target function, the task of fuzzy system design can be equated to that of finding a generalised Fourier transform. To distinguish the generalised Fourier transform $\hat{y} = \sum_i^p \hat{a}_i \cdot \phi_i(Z)$ for fuzzy systems from other types of transform in computational harmonic analysis, the transform is called the *fuzzy transform* in this case.

Equation (4-15) can be written as

$$\sum_{i=0}^p < \phi_k, \phi_i > \hat{a}_i =< f(Z), \phi_k >, \quad k \in \{1 \cdots p\} \qquad\qquad (4\text{-}16)$$

which is:

$$A * a = B \qquad\qquad (4\text{-}17)$$

where:

$$
A = \begin{pmatrix} <\phi_1,\phi_1> & \cdots & <\phi_1,\phi_j> & \cdots & <\phi_1,\phi_p> \\ \vdots & & \vdots & & \vdots \\ <\phi_i,\phi_1> & \cdots & <\phi_i,\phi_j> & \cdots & <\phi_i,\phi_p> \\ \vdots & & \vdots & & \vdots \\ <\phi_p,\phi_1> & \cdots & <\phi_p,\phi_j> & \cdots & <\phi_p,\phi_p> \end{pmatrix} \quad \hat{a} = \begin{pmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_i \\ \vdots \\ \hat{a}_p \end{pmatrix} \text{ and } B = \begin{pmatrix} <\phi_1,f(Z)> \\ \vdots \\ <\phi_i,f(Z)> \\ \vdots \\ <\phi_p,f(Z)> \end{pmatrix}
$$

$\{\phi_i\}$ denote normalised membership functions,

Since the matrix A represents the relationships between membership functions, it is called a membership relational matrix. The matrix B reflects the interaction between the membership functions and the target function, it is called an interaction matrix. The solution $\hat{a} = \begin{pmatrix} \hat{a}_1 & \cdots & \hat{a}_i & \cdots & \hat{a}_p \end{pmatrix}^T$ can be used to construct the fuzzy transform; $\hat{a}_i$ is called a coefficient of the fuzzy transform.

Unlike other transforms in computational harmonic analysis (such as the Fourier transform and Gabor transform), the fuzzy transform cannot be expressed explicitly due to the non-orthogonal basis. These non-orthogonal basis vectors reflect the lack of sharp boundaries between fuzzy sets in fuzzy systems. A unique property of the fuzzy transform is that the basis in the transform is not fixed but varies from one application to another. They need to be identified in some training process or using the experience of the operator. This makes the fuzzy transform a very flexible technique in function approximation.

The fuzzy transform provides a mathematical description for the principle of fuzzy system design. A fundamental problem in fuzzy system design is to find optimum basis (membership functions). When the basis have been chosen, an optimum fuzzy system

is constructed by computing fuzzy transform as the inner product of the basis and the target function.

It is easy to verify that:

- $<\phi_i, \phi_j> = <\phi_j, \phi_i>$

- $<\phi_i, \phi_j> = 0$ when ϕ_i and ϕ_j do not have common support.

Therefore, the membership relational matrix A is symmetrical and only those members close to the diagonal are not zero. This simplifies the solution of Equation (4-17)

Example 4-1

Consider the solution for equations $AX = D$

$$A = \begin{bmatrix} b_1 & c_1 & & & \\ e_1 & b_2 & c_2 & & \\ & e_2 & \ddots & & \\ & & & b_{n-1} & c_{n-1} \\ & & & e_{n-1} & b_n \end{bmatrix}, \quad D = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

The equation can be expanded into:

$$b_1 x_1 + c_1 x_2 \qquad = d_1$$
$$e_1 x_1 + b_2 x_2 + c_2 x_3 \quad = d_2$$
$$\cdots$$
$$e_{k-1} x_{k-1} + b_k x_k + c_k x_{k+1} = d_k$$
$$\cdots$$
$$e_{n-1} x_{n-1} + b_n x_n \quad = d_n$$

This gives

$$x_1 = \frac{d_1}{b_1} - \frac{c_1}{b_1} x_2 = u_1 - v_1 x_2$$

$$\vdots$$

90

$$x_k = \frac{d_k - u_{k-1}e_k}{b_k - v_{k-1}e_k} - \frac{c_k}{b_k - v_{k-1}e_k}x_{k+1} = u_k - v_k x_{k+1}$$

$$\vdots$$

$$x_n = \frac{d_n - u_{n-1}e_n}{b_n - v_{n-1}e_n} = u_n$$

Therefore the solution for $AX = D$ can be derived in two steps:

- calculate u and v, in the sequence $\begin{cases} u_1 \\ v_1 \end{cases} \rightarrow \cdots \rightarrow \begin{cases} u_k \\ v_k \end{cases} \rightarrow \cdots \rightarrow \begin{cases} u_n \\ v_n \end{cases}$

$$u_1 = \frac{d_1}{b_1} \quad , \quad v_1 = \frac{c_1}{b_1}$$

$$u_k = \frac{d_k - u_{k-1}e_k}{b_k - v_{k-1}e_k}, \quad v_k = \frac{c_k}{b_k - v_{k-1}e_k} \quad , \quad k = 2,\ 3, \cdots,\ n-1,$$

$$u_n = \frac{d_n - u_{n-1}e_n}{b_n - v_{n-1}e_n}$$

- deriving x from u and v, in the sequence $x_n \rightarrow x_{n-1} \rightarrow \cdots \rightarrow x_2 \rightarrow x_1$

$$x_n = u_n$$

$$x_k = u_k - v_k x_{k+1}, \quad k = n-1,\ n-2, \cdots,\ 2,\ 1$$

The above procedure for solving equation $AX = D$ is due to Cholesky's algorithm (Press and Flannery, 1993)

Example 4-2

For $n+1$ normalised triangular membership functions as illustrated in Figure 4-1, the membership relational matrix is:

$$A = \begin{pmatrix} <\phi_1,\phi_1> & \cdots & <\phi_1,\phi_j> & \cdots & <\phi_1,\phi_p> \\ \vdots & & \vdots & & \vdots \\ <\phi_i,\phi_1> & \cdots & <\phi_i,\phi_j> & \cdots & <\phi_i,\phi_p> \\ \vdots & & \vdots & & \vdots \\ <\phi_p,\phi_1> & \cdots & <\phi_p,\phi_j> & \cdots & <\phi_p,\phi_p> \end{pmatrix}$$

Since

$$<\phi_i,\phi_i> = \int_{C_{i-1}}^{C_i} \left(\frac{x-C_{i-1}}{C_i-C_{i-1}} \right)^2 dx + \int_{C_i}^{C_{i+1}} \left(\frac{x-C_{i+1}}{C_i-C_{i+1}} \right)^2 dx = \frac{1}{3}(C_{i+1}-C_{i-1})$$

$$<\phi_i,\phi_{i+1}> = \int_{C_i}^{C_{i+1}} \left(\frac{x-C_{i+1}}{C_i-C_{i+1}} \right) \left(\frac{x-C_i}{C_{i+1}-C_i} \right) dx = \frac{1}{6}(C_{i+1}-C_i)$$

where, C_i is the centre of the ith membership function.

This gives

$$A = \begin{bmatrix} \frac{1}{3}(C_1-C_0) & \frac{1}{6}(C_1-C_0) & 0 & & \cdots & & 0 \\ \frac{1}{6}(C_1-C_0) & \frac{1}{3}(C_2-C_0) & \frac{1}{6}(C_2-C_1) & 0 & & & \\ & \ddots & \ddots & \ddots & & & \vdots \\ & 0 & \frac{1}{6}(C_i-C_{i-1}) & \frac{1}{3}(C_{i+1}-C_{i-1}) & \frac{1}{6}(C_{i+1}-C_i) & 0 & \\ \vdots & & \ddots & \ddots & \ddots & & \\ & & & 0 & \frac{1}{6}(C_{n-1}-C_{n-2}) & \frac{1}{3}(C_n-C_{n-2}) & \frac{1}{6}(C_n-C_{n-1}) \\ 0 & & \cdots & & 0 & \frac{1}{6}(C_n-C_{n-1}) & \frac{1}{3}(C_n-C_{n-1}) \end{bmatrix}$$

<div align="right">(4-18)</div>

Example 4-2 indicates that, given the type of membership functions (triangular, Gaussian etc.), the form of the membership relational matrix can be predetermined.
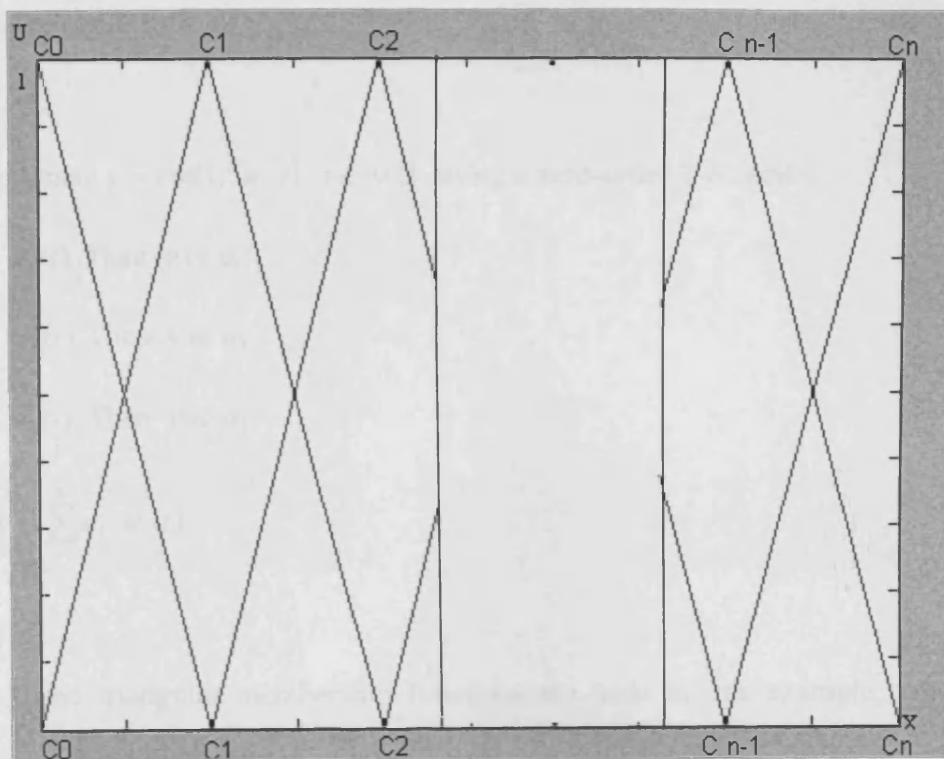
Figure 4-1 Normalised triangular membership functions.

Example 4-3

Approximate $y = \cos(1.5\pi \cdot t)$, $t \in [0,1]$, using a zero-order T-S model:

If t is $u_1(t)$ Then y is a_1

If t is $u_2(t)$ Then y is a_2

If t is $u_3(t)$ Then y is a_3

and $y = \sum_{i}^{3} a_i \cdot u_i(t)$

Normalised triangular membership functions are used in this example, which are illustrated in figure 4-2. Four parameters need to be identified in the fuzzy model. They are a_1, a_2, a_3 and the centre of middle membership function C.

When C is given, the combination of a_1, a_2, a_3 that gives the best approximation to the target function can be derived by using the fuzzy transform action (4-17). Therefore, the approximation problem is converted to an optimisation problem with only one unknown variable, C.

Based on equation 4-18, the membership relational matrix is

$$A = \begin{bmatrix} \frac{1}{3}C & \frac{1}{6}C & 0 \\ \frac{1}{6}C & \frac{1}{3} & \frac{1}{6}(1-C) \\ 0 & \frac{1}{6}(1-C) & \frac{1}{3}(1-C) \end{bmatrix}$$

The interaction matrix can be derived using the inner product operation.

$$B = \begin{bmatrix} <u_1, \cos(1.5\pi \cdot t)> \\ <u_2, \cos(1.5\pi \cdot t)> \\ <u_3, \cos(1.5\pi \cdot t)> \end{bmatrix} = \begin{bmatrix} \dfrac{-4}{9\pi^2 c}[\cos(1.5\pi\ C)-1] \\ \dfrac{-4}{9\pi^2 C(C-1)}[\cos(1.5\pi\ C)-1+C] \\ \dfrac{-4}{9\pi^2 (C-1)}[1.5\pi\ (C-1)-\cos(1.5\pi\ C)] \end{bmatrix}$$

Given the form of the membership relational matrix A and the interaction matrix B, theoretically, an optimum value of C can be obtained analytically. However, this is not the aim of fuzzy transform action. The idea behind the fuzzy transform is not to eliminate "soft-computing" activities in fuzzy system design, but to seek an optimal balance between "soft" and "hard" computing.

As mentioned in previous sections, in order to outperform modern fuzzy system design approaches, next generation fuzzy system design needs to be enhanced by reducing the search space and increasing the efficiency of the minimal distance calculation. It is clear that the fuzzy transform is a development in this direction. It provides an efficient way to calculate the minimal distance and at the same time reduces the problem of fuzzy systems modelling to the simpler problem of finding the best input membership functions.

The fuzzy modelling procedure for the above example is illustrated in figure 4.3. In order to find the best combination of parameters, the gradient descent method is applied to search for the value of C. For the initial condition $C=0.5$ and step=0.1, the following results are obtained.

Iteration	C	Error	a_1	a_2	a_3	Step
1	0.5	0.0339	1.3626	-0.8802	-0.4510	0.1
2	0.6	0.01	1.2805	-1.0966	-0.2402	0.1
3	0.7	0.095	1.1642	-1.2324	-0.0233	0.1
4	0.65	0.063	1.2261	-1.1752	-0.1319	0.05
Final Result	0.65	0.063	1.2261	-1.1752	-0.1319	

The best approximation is achieved when C is close to 0.65. This is illustrated in figure
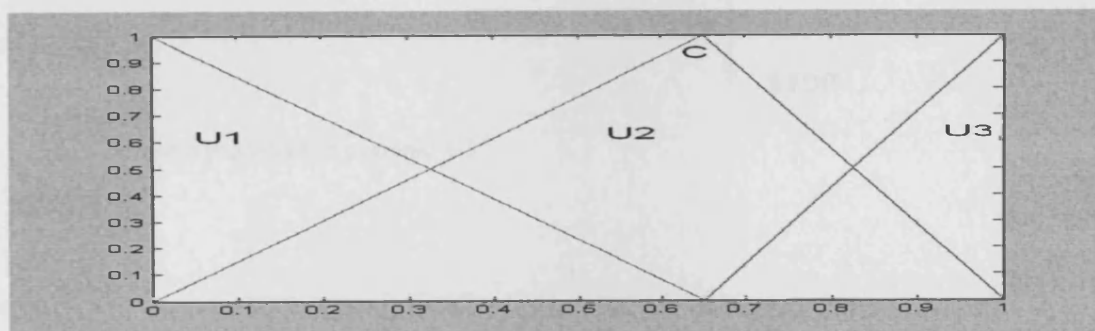
4-4.
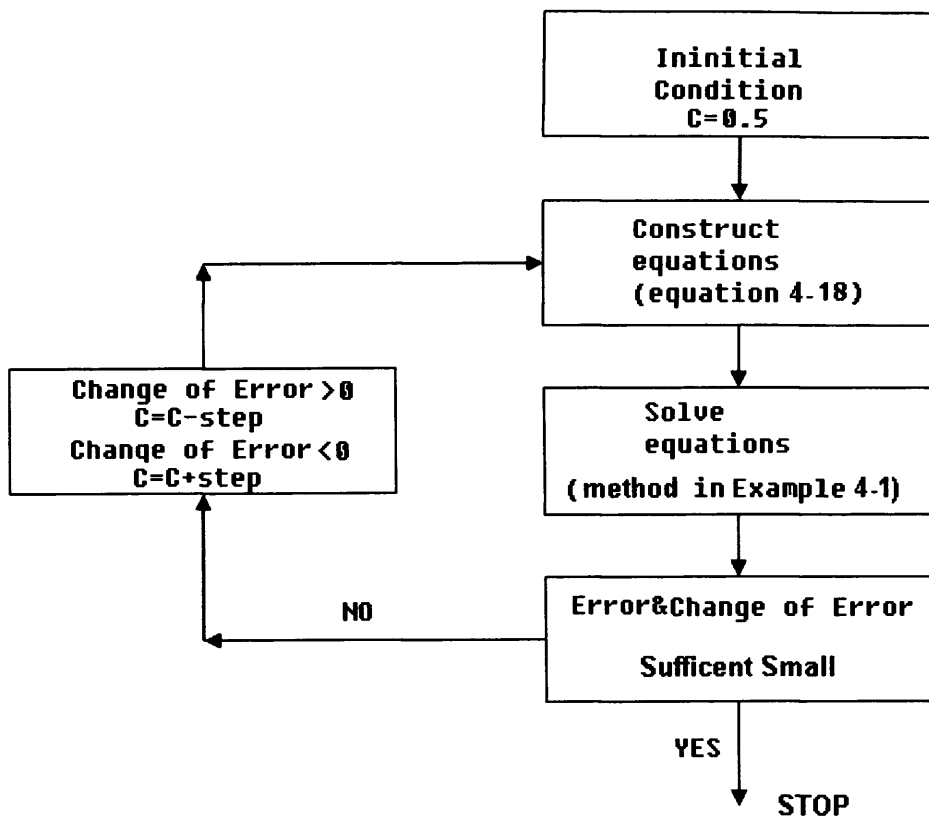

Figure 4-2 Membership functions in example 4-3

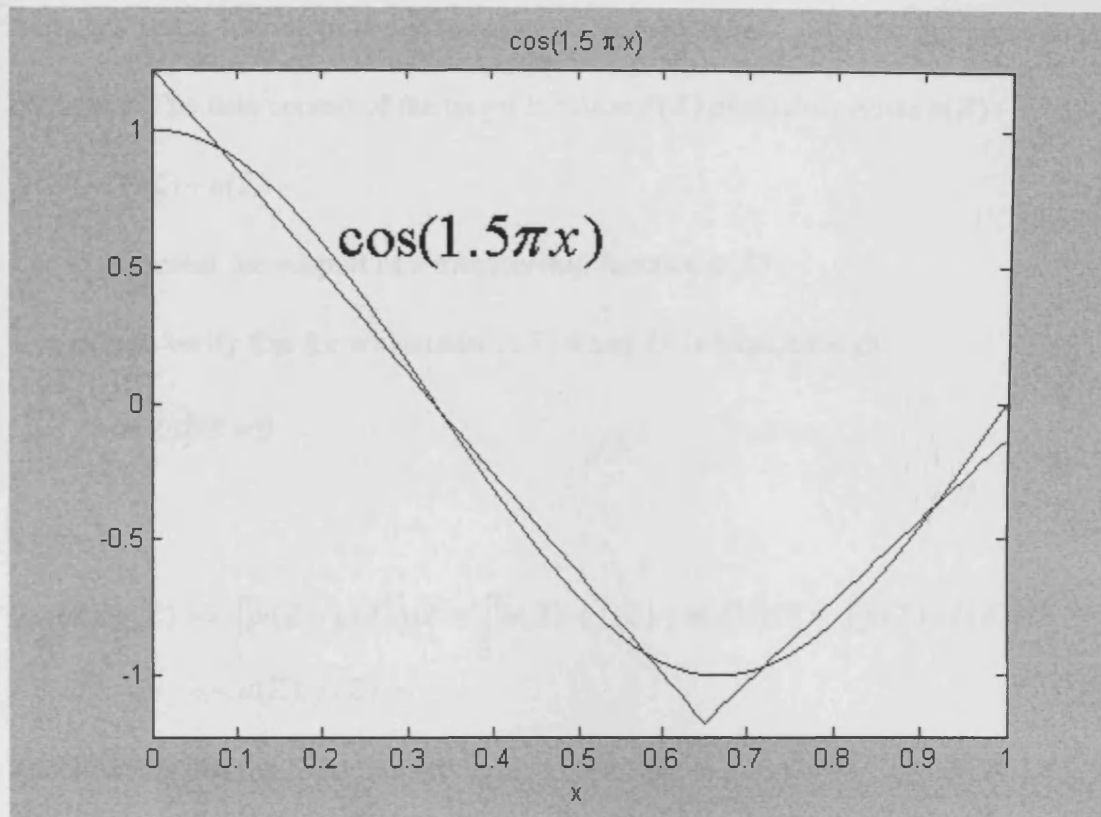Figure 4-3 Search procedure in example 4-3

Figure 4-4 Target function and the approximation result in example 4-3

It should be mentioned that when the support of membership functions is sufficiently wide, the fuzzy transform is not sensitive to noise. Let $g(Z)$ denote the input/output data pairs. The data consist of the target function $f(Z)$ plus white noise $n(Z)$:

$$g(Z) = f(Z) + n(Z).$$

Let D represent the support of a membership function $u(Z)$.

It is easy to verify that for white noise $n(Z)$ when D is large enough.

$$\int_D [u(Z) \cdot n(Z)] dZ = 0$$

Then

$$< u(Z), g(Z) >= \int_D [u(Z) \cdot g(Z)] dZ = \int_D [u(Z) \cdot (f(Z) + n(Z))] dZ = \int_D [u(Z) \cdot f(Z)] dZ$$
$$= < u(Z), f(Z) >$$

which shows that the fuzzy transform is not sensitive to noise $n(Z)$.

Therefore the fuzzy transform can be used to deal with inexactitude, if the support of the membership functions is not too small. The noise filter is illustrated in figure 4-5.

It should also be mentioned that the sum and product, used in Equation (4-17) can be replaced by other triangular operators. When different norms and co-norms are adopted in fuzzy inferencing (Max-Min, Max-Product, Sum-Product etc.), only the way in which the equations are solved is changed.
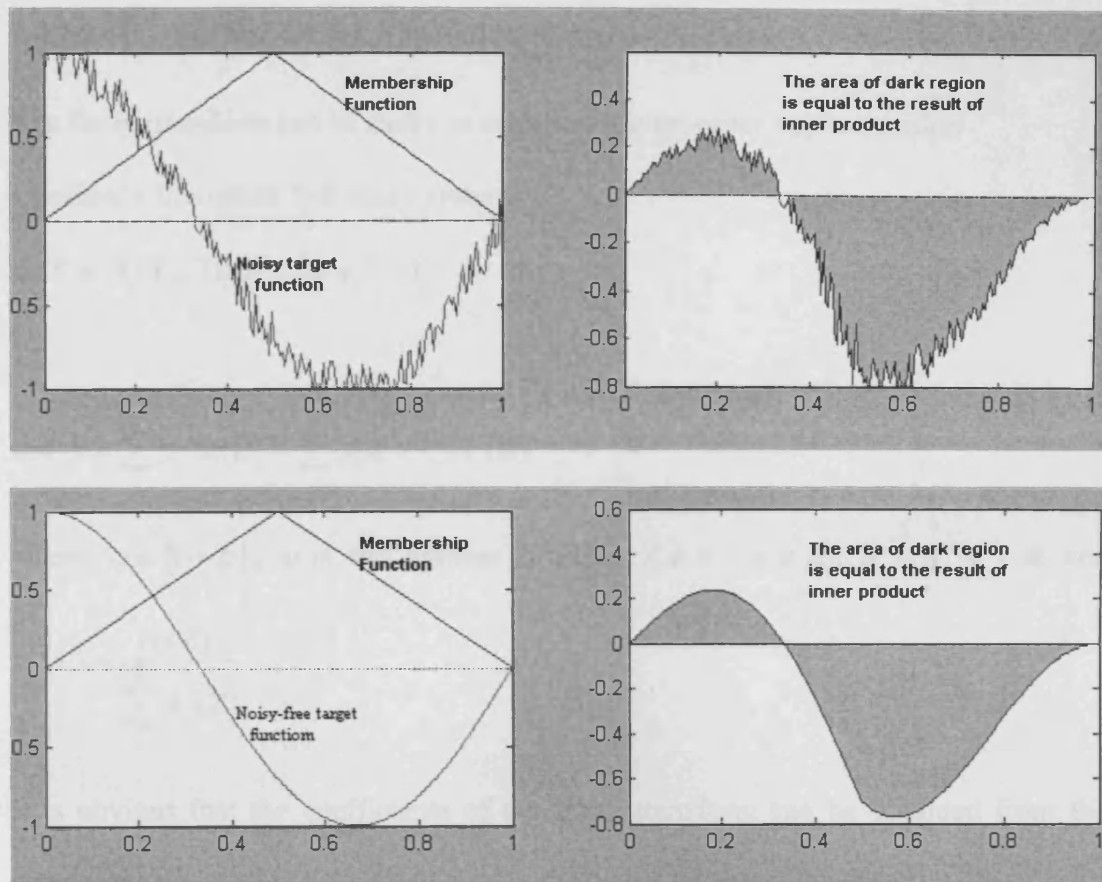
Figure 4-5 Inner product filters out the noise in the target function

4.3.2.2   Higher-Order Approximation

The fuzzy transform can be easily extended to higher-order approximation

Consider a first-order T-S fuzzy system

If Z is $A_i(Z)$ Then y_i is $q_i Z + r_i$

$\vdots$

and $y = \sum_i^p y_i \cdot u_i(Z) = \sum_i^p (q_i Z + r_i) \cdot u_i(Z)$

where $i \in \{1 \cdots p\}$, p is the number of rules, $Z \in R^n$, $y \in R$, $q_i \in R^n$ $r_i \in R$ and

$$u_i(Z) = \frac{A_i(Z) \cdot}{\sum_j^p A_j(Z)}$$

It is obvious that the coefficients of the fuzzy transform can be obtained from the equation:

$$\begin{pmatrix} <u_1,u_1> & \cdots & <u_1,u_p> & <u_1,u_1 Z> & \cdots & <u_1,u_p Z> \\ \vdots & & & & & \vdots \\ <u_p,u_1> & & <u_p,u_p> & <u_p,u_p Z> & & <u_p,u_p Z> \\ <u_1 Z,u_1> & & <u_1 Z,u_p> & <u_1 Z,u_1 Z> & & <u_1 Z,u_p Z> \\ \vdots & & & & & \vdots \\ <u_p Z,u_1> & \cdots & <u_p Z,u_p> & <u_p Z,u_1 Z> & \cdots & <u_p Z,u_p Z> \end{pmatrix} \begin{pmatrix} q_1 \\ \vdots \\ q_p \\ r_1 \\ \vdots \\ r_p \end{pmatrix} = \begin{pmatrix} <f,u_1> \\ \vdots \\ <f,u_p> \\ <f,u_1 Z> \\ \vdots \\ <f,u_p Z> \end{pmatrix}$$

(4- 19)

where f is denotes the target function.

The advantage of a higher-order approximation is that it improves the approximation ability of a fuzzy system without increasing the search space. For the same number of membership functions (the same search space), a higher-order approximation adds a new basis vector into the membership functional space. The basis in membership

functional space changes from $\{u_i\}$ to $\{u_i, u_i Z\}$, and the coverage of the membership

functional space increases accordingly.

Equation 4-19 can be written as

$$
\begin{bmatrix}
<u_1,u_1> & <u_1,u_1z> & & \cdots & & <u_1,u_pz> \\
<u_1z,u_1> & <u_1z,u_1z> & & & & <u_1z,u_pz> \\
\vdots & & \ddots & & & \\
& & <u_i,u_i> & <u_i,u_iz> & & \\
& & <u_iz,u> & <u_iz,u_iz> & & \vdots \\
& & & & \ddots & \\
<u_p,u_1> & & & <u_p,u_p> & <u_p,u_pz> \\
<u_pz,u_1> & & \cdots & <u_pz,u> & <u_pz,u_pz>
\end{bmatrix}
\cdot
\begin{pmatrix}
q_1 \\
r1 \\
\vdots \\
q_i \\
r_i \\
\vdots \\
q_p \\
r_p
\end{pmatrix}
=
\begin{pmatrix}
<f,u_1> \\
<f,u_1z> \\
\vdots \\
<f,u_i> \\
<f,u_iz> \\
\vdots \\
<f,u_p> \\
<f,u_pz>
\end{pmatrix}
$$

$$(4\text{-}20)$$

where the membership relational matrix A is symmetrical and only those members

close to the diagonal are not zero. Again, this kind of equation can be solved efficiently

by Cholesky's algorithm (Press and Flannery 1993).

Thus, a complete fuzzy system design theory based on functional analysis has been

presented in this section. From the functional analysis point of view, the mathematical

principle for fuzzy system design can be described by Equation (4-15). Based on that

equation, an efficient fuzzy modelling algorithm has been developed using fuzzy

transform for T-S fuzzy model.

4.4   Dual bases and dual spaces

Non-orthogonal bases are an important feature of fuzzy transforms. However, numerically, this feature limits the ability the fuzzy transform to be applied to complicated systems. For example, for the Fourier transform, which has an orthogonal basis, a general solution for Equation (4-17) can be derived as:

$$a_k = \frac{<\phi_k, f>}{<\phi_k, \phi_k>} \qquad k \in \{1, \cdots, p\}$$

However, for the fuzzy transform, solving Equation (4-17) tends to be complicated when p is large. Furthermore, in fuzzy modelling, searching for the best partition of membership functions requires Equation (4-17) to be solved repeatedly. This further slows down the fuzzy modelling process. In order to solve this problem, a simplified approach is developed using the concept of dual bases.

For simplicity, consider a 2-D coordinate space. Any two vectors $\{\phi_1, \phi_2\}$ that are not parallel can form a basis for the space. If the angle between the two vectors is 90 degrees, they form an orthogonal basis.

Any vector A in the space can be written uniquely as a linear combination of the two basis vectors: $A = a_1\phi_1 + a_2\phi_2$. For an orthogonal basis, $<\phi_i, \phi_j> = \delta_{ij}$, and the component a_i along ϕ_i is given by the inner product:

$$< A, \phi_i > = < a_1\phi_1 + a_2\phi_2, \phi_i > = \sum_j a_j < \phi_j, \phi_i > = a_i$$

However, if the basis is not orthogonal, a_i is no longer given by the inner product

between A and ϕ_i. In order to calculate the component a_i, another set of basis vectors $\left\{\tilde{\phi_1}, \tilde{\phi_2}\right\}$, called the dual of $\left\{\phi_1, \phi_2\right\}$ is introduced. The dual basis satisfies the following relation:

$$< \tilde{\phi_i}, \phi_j >= \delta_{ij}$$

and the space spanned by the dual basis vectors is called the dual space of the original space. In terms of the dual basis, the components of a vector along the basis vector ϕ_i can be calculated as $< A, \tilde{\phi_i} >=< a_1\phi_1 + a_2\phi_2, \tilde{\phi_i} >= \sum_j a_j < \phi_j, \tilde{\phi_i} > = a_i$

So the introduction of dual basis and the dual space enables a vector to be decomposed as a linear combination of non-orthogonal basis vectors. Similarly, for a group of non-orthogonal basis vectors $\left\{\phi_i\right\}$ in a function space, their dual basis $\left\{\tilde{\phi_i}\right\}$ can be derived from:

$$< \tilde{\phi_i}, \phi_j >= \int_{-\infty}^{\infty} \tilde{\phi_i}(x)\phi_j(x)dx = \delta_{i,j} \qquad (4\text{-}21)$$

Therefore, a function $f(x)$, which is covered by the space spanned by a non-orthogonal basis $\left\{\phi_i\right\}$, can be decomposed as a combination of $\left\{\phi_i\right\}$ using a set of dual basis vectors $\left\{\tilde{\phi_i}\right\}$:

$$f(x) = \sum_i a_i\phi_i$$

$$a_k = \frac{< \tilde{\phi_k}, f(x) >}{< \tilde{\phi_k}, \phi_k >} \qquad (4\text{-}22)$$

When $\left\{\phi_i\right\}$ is orthogonal, $\phi_i = \tilde{\phi_i}$. Then Equation (4-22) is the same as Equation (4-13).

Using the concept of dual bases, a simplified solution can be derived for Equation (4-17).

It should be noted that, if target function $f(x)$ is covered by the space spanned by $\{\phi_i\}$, Equation (4-22) gives the exact solution for Equation (4-17). If it is not, Equation (4-22) gives an approximate solution for Equation (4-17).

Example 4-4: For a given normalised triangular membership function, its dual membership function is illustrated in figure 4-6.

By solving Equation (4-21), dual membership functions can be constructed. They are not unique. For example, any function that follows the "track of dual" marked in figure 4-6 is orthogonal to the neighbour membership functions, and can be regarded as a dual.
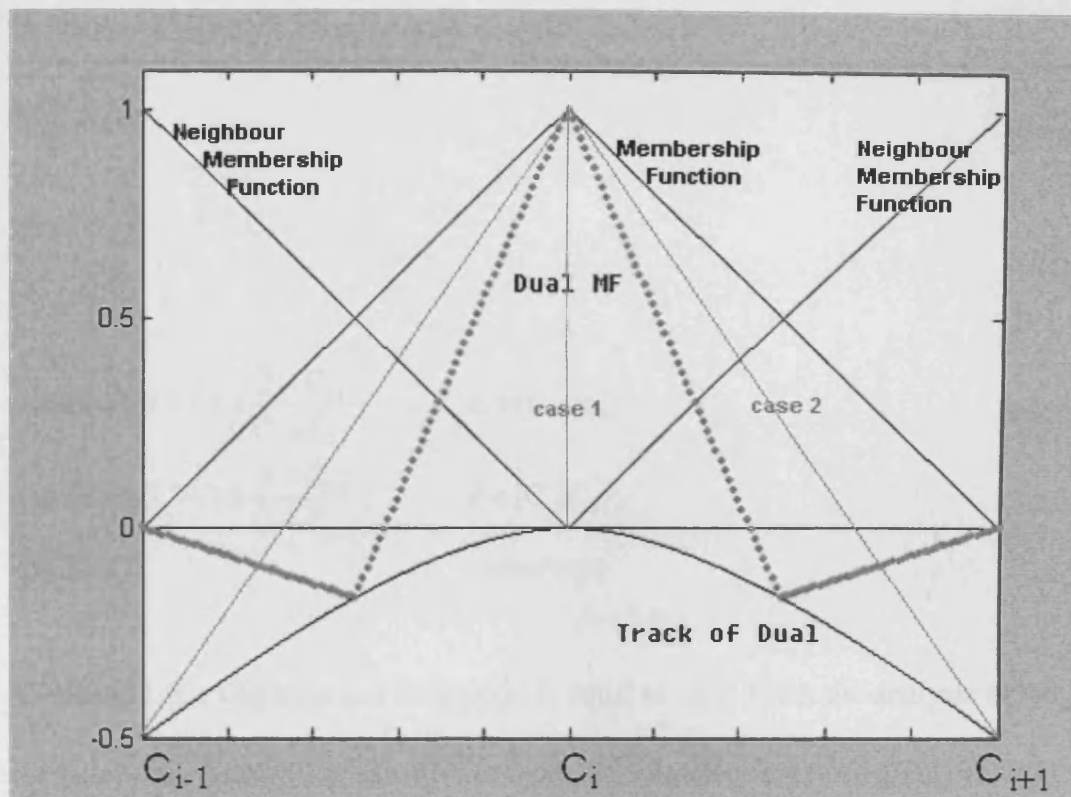
Figure 4-6 Dual membership function for normalized triangular membership function

In the above example, there are two cases of interest:

Case 1

$$\begin{cases} \tilde{u}(Z) = 1 & Z = C_i \\ \tilde{u}(Z) = 0 & Z \neq C_i \end{cases}$$

Case 2

$$\begin{cases} \tilde{u}(Z) = -0.5 + 1.5 \dfrac{Z - C_{i-1}}{C_i - C_{i-1}} & Z \in [C_{i-1}, C_i) \\ \tilde{u}(Z) = -0.5 + 1.5 \dfrac{Z - C_{i+1}}{C_i - C_{i+1}} & Z \in [C_i, C_{i+1}] \\ \tilde{u}(Z) = 0 & \textit{otherwise} \end{cases}$$

$\tilde{\mu}$ in case 1 is a singleton and its support is equal to zero. From the analysis in 4.3.2.1, the fuzzy transform will be sensitive to noise. So singletons are not a good candidate for approximation. In the following example, the dual membership function given in case 2 is applied in order to speed up the process of fuzzy modelling.

Example 4-5

Approximate $y = f(x_1, x_2) = 0.5(1 + \sin(2\pi x_1)\cos(2\pi x_2))$, $x_1 \in [0,1]$ $x_2 \in [0,1]$, using a zero-order T-S model, with following 9 rules:

If x_1 is $u_1(x_1)$ and x_2 is $v_1(x_2)$ Then y is a_{11}

$\vdots$

If x_1 is $u_i(x_1)$ and x_2 is $v_j(x_2)$ Then y is a_{ij}

$\vdots$

If x_1 is $u_3(x_1)$ and x_2 is $v_3(x_2)$ Then y is a_{33}

and $y = \sum_{i}^{3} \sum_{j}^{3} a_{ij} \cdot u_i(x_1) v_j(x_2)$

Normalised triangular membership functions are used, as illustrated in figure 4-8. There are eleven parameters to be identified in the fuzzy model. Dual membership functions (as given in case 2) are applied to speed up searching. Parameters a_{ij} obtained as:

$$a_{ij} = \frac{< u_i v_j, f(x_1 x_2) >}{< \tilde{u}_i, u_i >< \tilde{v}_j, v_j >}$$

The result is indicated in Figures 4-8 and 4-9.

For a group of given membership functions, dual membership functions are applied to identify the sub-optimum parameters $\{a_{ij}\}$. An approximate minimal distance between given membership functions and the target function can be calculated from $\{a_{ij}\}$. Instead of the exact minimal distance, an approximate minimal distance is used in this approach. It speeds up the search significantly by avoiding the procedure for solving Equation (4-17) at every iteration.

$$y = 0.5 * (1 + \sin(2\pi * x_1) * \cos(2\pi * x_2))$$
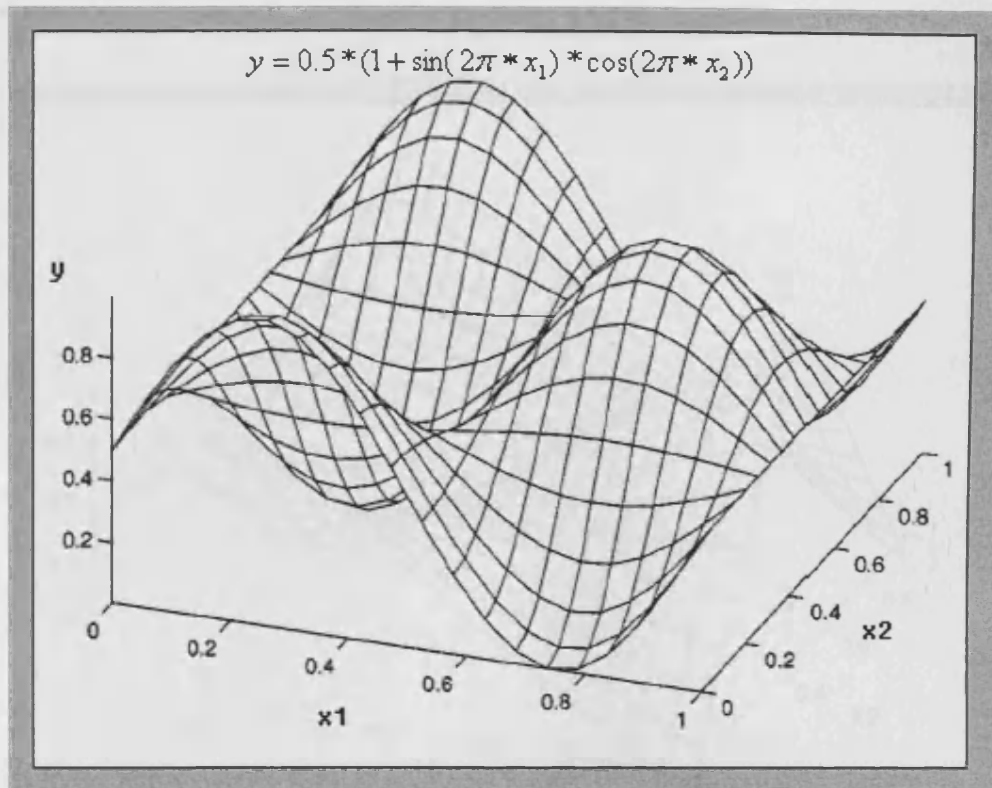
Figure 4-7 Target function in example 4-4
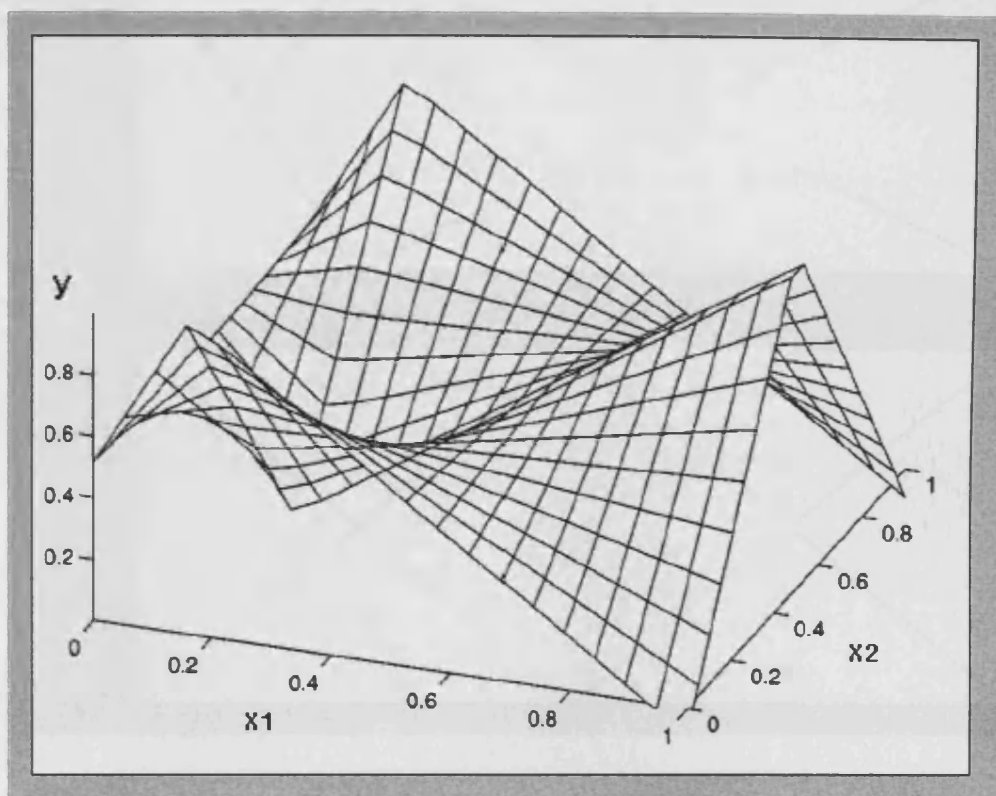
Figure 4-8 Approximation result of example 4-5

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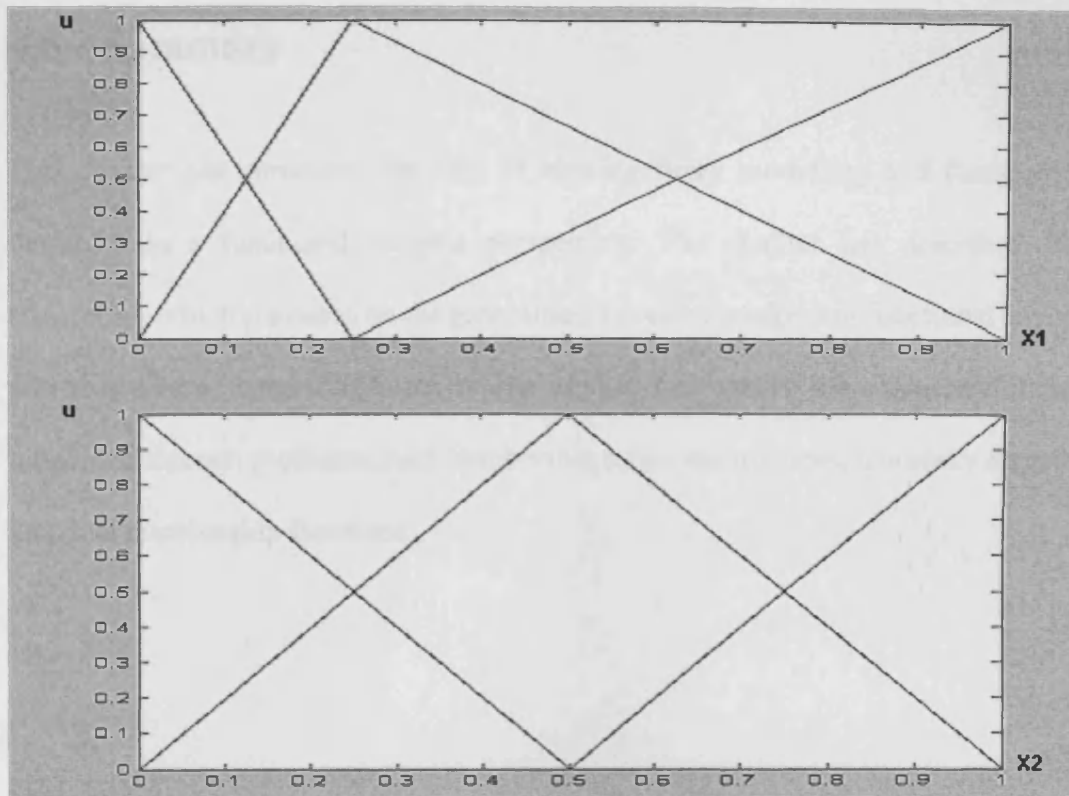
Figure 4-9 Membership functions of example 4-5

4.5 Summary

This chapter has introduced the idea of viewing fuzzy modelling and fuzzy system design from a functional analysis perspective. The chapter has described fuzzy transforms, which are based on the generalised Fourier transform in functional analysis, and shown how fuzzy transforms can be applied to improve the efficiency of fuzzy modelling through predetermined membership relational matrices, Cholesky algorithm and dual membership functions.

Chapter 5

Stability Analysis for Nonlinear Fuzzy Control

5.1 Preliminaries

Larger-scale, real life, complex systems tend to be highly nonlinear. Despite much development in nonlinear control theory, in industrial applications, the dominant approaches for solving nonlinear problems are still based on PID control theory which has been in existence for almost 100 years. This is mainly due to the following two reasons. First, nonlinear control theory based on traditional mathematical methods (e.g. differential geometry, operator theory and H-infinity) requires exact mathematical models, which are difficult to produce for real industrial plants. Artificial-Intelligence based control theory, which may avoid the use of exact mathematical models, could not satisfy important requirements of industrial applications (e.g. guaranteed stability and optimality). This issue has received considerable attention. However, apart from PID control theory, a practical nonlinear control theory for industrial applications is not yet available.

In nonlinear control research, nonlinear fuzzy control is a technique of special interest. This is because, theoretically, fuzzy control not only can avoid the use of exact

mathematical models but also may provide the performance necessary for industrial applications in terms of stability and optimality. However, in reality fuzzy control theory is still very limited in this respect. In the following sections, it will be shown that modern model-based fuzzy control is not suitable for the task of controlling general nonlinear systems. This is mainly due to the implicit linear assumption in fuzzy controller design. Under the assumption, a fuzzy control problem has to be expressed in a "sector nonlinearity" form (see 5.2.3), which as can be seen later reduces fuzzy controller design to a special case of linear time-varying system control. Due to this limitation, although fuzzy control has been widely applied to systems with simple dynamics, it is still rare to find complex industrial systems controlled by fuzzy controllers. In order to introduce fuzzy techniques to nonlinear control, a new nonlinear fuzzy control theory needs to be developed. The key issue here is to develop a stability criterion for general nonlinear fuzzy systems.

This chapter focuses on stability analysis for nonlinear fuzzy control. A new stability checking criterion for general nonlinear fuzzy systems will be described. The idea behind the new stability checking criterion is the integration of geometrical information in stability analysis. The chapter is organised as follows. The main limitations of modern fuzzy control approaches are discussed in section 5.2. The decomposition principle that has been proposed as a mean for handling nonlinear fuzzy systems is investigated in section 5.3. Perturbation theory is studied in section 5.4. Perturbation theory provides a qualitative analysis of the stability of nonlinear system, which leads to a new stability criterion in section 5.5.

5.2 Modern Fuzzy Control and its Principal Limitations

5.2.1 HBFC and MBFC

Modern fuzzy control approaches can be separated into two categories, heuristics-based fuzzy control (HBFC) and model-based fuzzy control (MBFC). Conventional fuzzy control design is heuristics-based. It implicitly assumes that there is no model for the process under investigation. Its control strategy is constructed directly from the knowledge of experienced operators, and expressed in the form of fuzzy rules. Controller performance is adjusted by tuning of membership functions. For systems with simple dynamics, this is a convenient approach. However, due to the following drawbacks (Korba, Babuska, 2003), this approach is not suitable for complicated nonlinear systems:

- "Lack of systematic and formally tractable design and tuning techniques."

- "Lack of methods for deriving basic properties such as close-loop system stability, performance and robustness. These can only be investigated via extensive tests and simulations."

Above all, the heuristic nature of HBFC limits its applications to systems with simple dynamics. For complicated nonlinear systems, which cannot be controlled by human beings directly, it is unrealistic to expect HBFC to do better than human operators.

In order to overcome the limitations of HBFC, an alternative approach, called model-based fuzzy control (MBFC) was developed in the 1980s and 1990s. The new approach can be regarded as "a middle ground between conventional fuzzy control practice and established control theory" (Tanaka and Wang, 2001). This approach

preserves the philosophy of fuzzy sets theory and, at the same time, implements of ideas of feedback control theory to improve fuzzy controller design. Its design procedure is as follows:

- First, a T-S fuzzy model is constructed for the plant in state-space by linearising local dynamics in different state-space regions.

- Second, for each local linear model, a linear feedback controller is designed. The overall controller, which is nonlinear in general, is constructed as "a fuzzy blending of each individual controller" (Wang et al, 1996).

- Third, the overall stability for the entire fuzzy system is evaluated via Lyapunov's method.

In the first step of the controller design procedure, expert knowledge is applied to construct a T-S fuzzy model for the target process. In the second step, feedback control theory is applied to design the local feedback controllers. In the third step, due to fundamental differences between linear and nonlinear systems in local and global stability, the stability of the local linear controllers designed in the second step needs to be evaluated. Lyapunov's method is applied in the evaluation.

It should be noted that a T-S fuzzy system is a nonlinear system in general. Therefore, even when all its sub-systems are stable, global stability cannot be guaranteed. This makes the third step of MBFC the most important step in the design. If the requirement for global stability is not satisfied, the local linear controllers need to be redesigned. This process is repeated until global stability is achieved.

Due to its simplicity, it is clear that MBFC is a promising technique for general nonlinear control. Considerable efforts have been channelled in this direction. A search by the author revealed, more than six hundred papers published in IEEE and ELSEVIER journals and conference proceeding on this issue between the years 2000 and 2003. Most of the authors, (e.g. Kim and Lee, 2000; Wu, Lin 2000; Tanaka and Wang, 2001; Tanaka et al, 2003; Korba and Babuska, 2003 etc) have studied systems described by the following set of rules:

If X is $U_1(X)$ Then $\dot{X} = A_1 X + B_1 u$

$\vdots$

If X is $U_i(X)$ Then $\dot{X} = A_i X + B_i u$

$\vdots$

If X is $U_p(X)$ Then $\dot{X} = A_p X + B_p u$

and

$$\dot{X} = \sum_{i=1}^{p} h_i(X)(A_i X + B_i u) \qquad (5\text{-}1)$$

where $X \in R^n, u \in R^m, A_i \in R^{nxn}, B_i \in R^{nxm} \quad i \in \{1 \cdots p\}$

and $h_i(X) = U_i(X) \Big/ \sum_j U_j(X)$

The stability criterion for the above fuzzy system has been proposed as follows:

Lemma 5-1 (Wang et al, 1996)

If there exists a positive definite matrix that satisfies:

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$$A_i^T P + PA_i < 0 \qquad\qquad\qquad (5\text{-}2)$$

for all $i \in \{1 \cdots p\}$, the origin of the fuzzy system described in Equation (5-1) is asymptotically stable.

It should be noted that the inequality (5-2) is in the form of a linear matrix inequality (LMI) (Boyd, 1994), and can be solved efficiently by a convex optimisation method, such as the interior point method.

5.2.2 Limitations of MBFC

Despite its popularity, the assumed dynamic equation described by Equation (5-1) only reflects an extremely limited case for the dynamics of general nonlinear systems. This is because the assumption implicitly considers that the linearised state-space regions for all subsystems share the same equilibrium point, which is almost impossible for practical nonlinear systems.

For a nonlinear system

$$f(\dot{X}, X, u) = 0 \qquad\qquad\qquad (5\text{-}3)$$

A first-order T-S fuzzy model derived by local linearisation in different state-space regions can be written as

$$Q_i \dot{X} + R_i X + S_i u + E_i = 0, \qquad X \in M_i \text{ and } i \in \{1 \cdots p\} \qquad (5\text{-}4)$$

M_i denotes the support for the ith rule and p denotes the number of rules in the fuzzy model. If the Q_i's are not singular, the state-space description for Equation (5-4) can be written as:

$$\dot{X} = \sum_{i=1}^{p} h_i(X)(A_i X + B_i u + D_i) \qquad i \in \{1 \cdots p\} \qquad (5\text{-}5)$$

where $h_i(X) = \dfrac{u_i(X)}{\sum_j u_j(X)}, X \in R^n, u \in R^m, A_i \in R^{nxn}, B_i \in R^{nxm}, D_i \in R^n$

It is easy to verify that unless all subsystems share the same equilibrium point, there is no coordinate transform action that can change Equation (5-5), which is for a general nonlinear system, into the form of Equation (5-1).

Due to the difference in the equilibrium points among the subsystems, the stability of the general fuzzy system described by Equation (5-5) CANNOT be guaranteed by Lemma 5-1. This is demonstrated by the following counter example.

Example 5-1

For the following three fuzzy systems

(1) Fuzzy system 1

    R1 IF X is A1 then $\dot{x} = -ax - b$

    R2 IF X is A2 then $\dot{x} = -ax + b$

    where $a > 0 \quad b > 0$

(2) Fuzzy system 2

    R1 IF X is A1 then $\dot{x} = -ax - b$

R2 IF X is A2 then $\dot{x} = -ax + b$

where $a > 0$ $b < 0$

(3) Fuzzy system 3

R1 IF X is A3 then $\dot{x} = -ax - b$

R2 IF X is A4 then $\dot{x} = -ax + b$

where $a > 0$ $b < 0$

The membership functions for A1 to A4 are illustrated in figure 5-1

All the above fuzzy systems satisfy Lemma 5-1. They are only different in the position of the equilibrium points and the partition of the membership functions. From figure 5-2 to 5-4, only fuzzy system 1 is globally asymptotically stable. Fuzzy systems 2 and 3 are not asymptotically stable.

Example 5-1 shows that the current MBFC approach, which is based on Lemma 5-1, is only suitable for a special type of fuzzy systems described by Equation (5-1). For general nonlinear systems described by Equation (5-5), Lemma 5-1 does not hold and hence a new stability checking criterion needs to be identified.

In order to distinguish fuzzy systems described by Equation (5-5) and Equation (5-1), in the following sections, Equation (5-5) is called the equation for general nonlinear fuzzy systems and Equation (5-1), that for "sector nonlinear" fuzzy system.
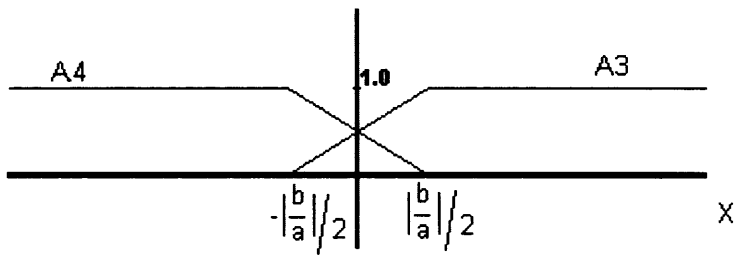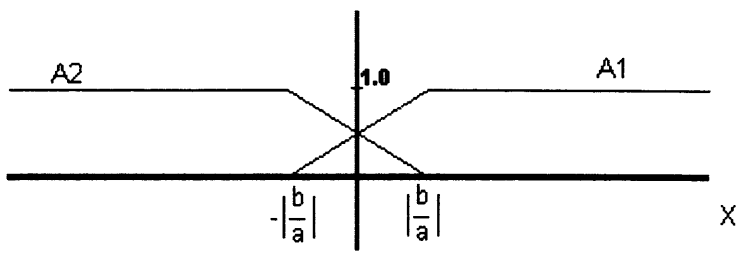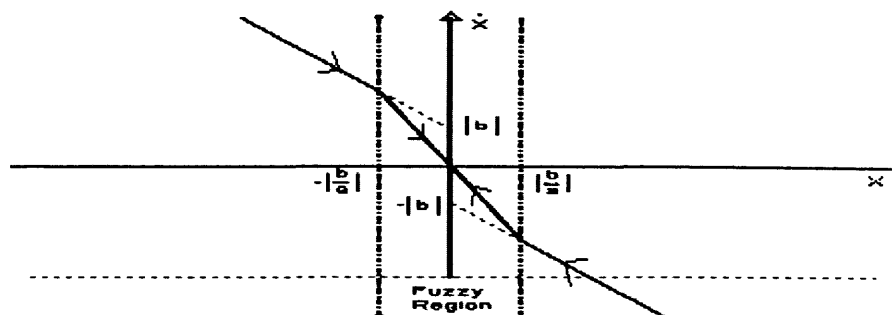
Figure 5-1 Membership functions A1-A4
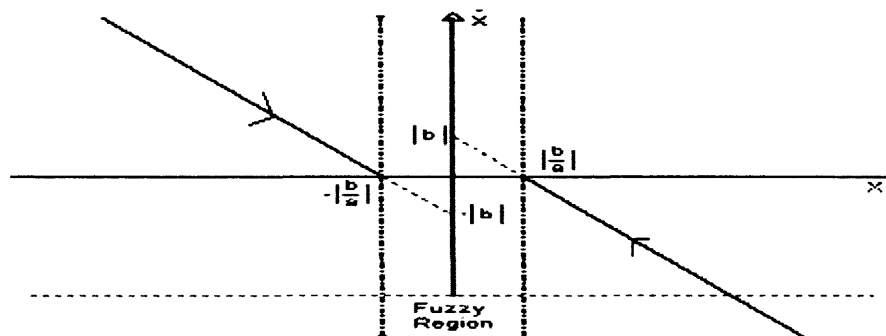
Figure 5-2 The dynamics of fuzzy system 1

Figure 5-3 The dynamics of fuzzy system 2
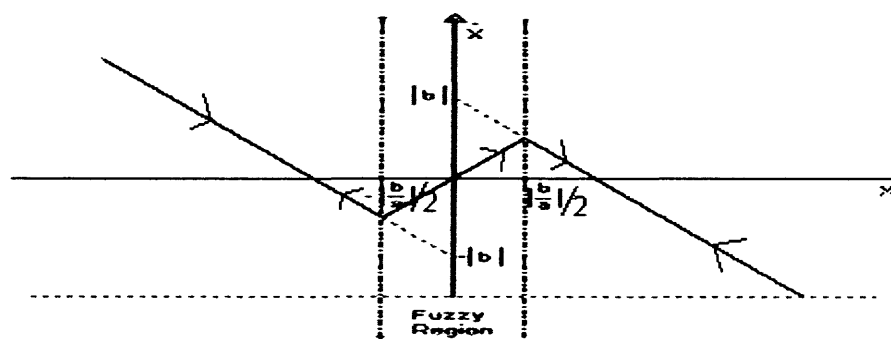
Figure 5-4 The dynamics of fuzzy system 3

5.2.3 Sector-Nonlinearity Approach

A function $\phi(q)$ is said to be in sector $[l,u]$ if for all q, $\phi(q)$ lies between lq and uq.

This is illustrated in figure 5-5. Based on the idea of sector nonlinearity, if the

nonlinearity of a given system is smooth and its mathematical model is given explicitly,

a global or semi-global T-S fuzzy model that represents the dynamics of the system can

be constructed in the form of Equation (5-1). This enables the methodology developed

in MBFC to be applied to a wider area. MBFC based on "sector nonlinearity" has been

proposed by Tanaka et al, (2001). However, in this case, although the framework of the

fuzzy control is preserved, the philosophy of fuzzy logic has been abandoned. This is

because:

- First, sector nonlinearity requires an exact mathematical description, which
  eliminates the main advantage of the application of fuzzy techniques to industrial
  problems.

- Second, the approach reduces MBFC to a special case of linear time-varying
  system control.

Consider a linear time-varying system:

$$\dot{X} = \sum_{i=1}^{p} \lambda_i(t) A_i X \qquad (5\text{-}6)$$

where the weights $\lambda_i(t)$ may vary arbitrarily with time while satisfying

$0 \le \lambda_i(t) \le 1$ and $\sum_{i=1}^{p} \lambda_i(t) = 1$.

$p = uq$

$p = lq$

q

$p = \phi(q)$
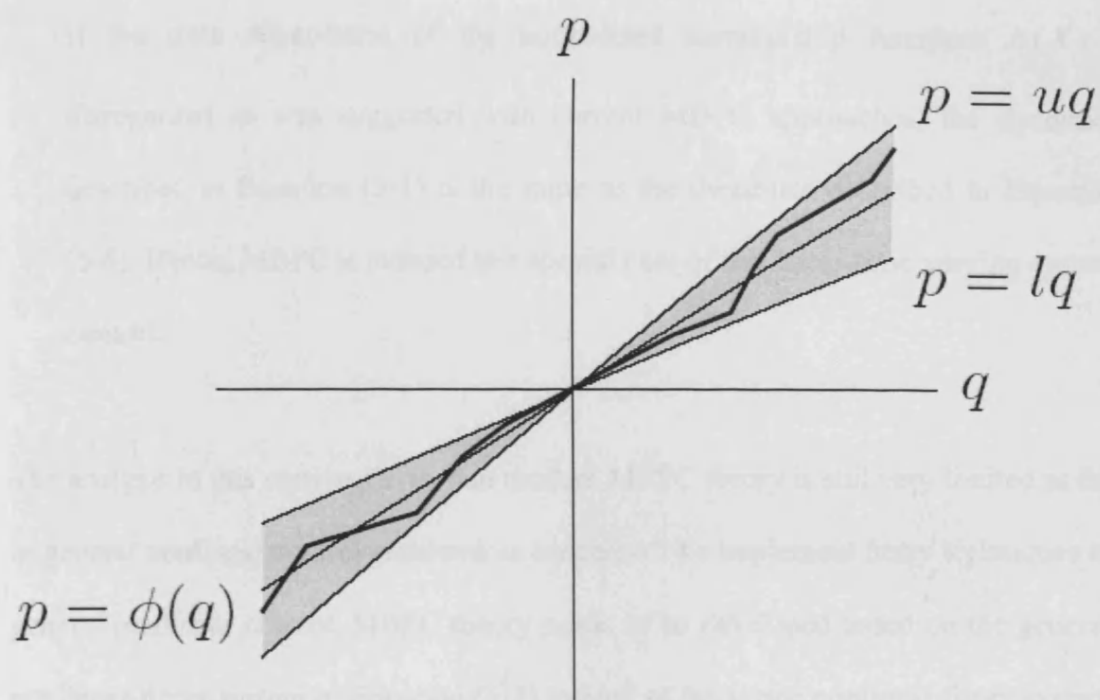
Figure 5-5 Sector nonlinearities

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If the state dependence of the normalized membership functions $h_i(X)$ is disregarded as was suggested with current MBFC approaches, the dynamics described in Equation (5-1) is the same as the dynamics described in Equation (5-6). Hence, MBFC is reduced to a special case of the linear-time varying system control.

The analysis in this section shows that modern MBFC theory is still very limited as far as general nonlinear control problems as concerned. To implement fuzzy techniques to general nonlinear control, MBFC theory needs to be developed based on the general nonlinear fuzzy system of Equation (5-5) instead of the sector nonlinear fuzzy system of Equation (5-1). Since having an efficient means of checking stability for general nonlinear fuzzy systems is the key to the general nonlinear fuzzy control, the following sections are devoted to the stability analysis of the fuzzy system of Equation (5-5).

5.3   A problem with the Decomposition Principle

As has been mentioned, the principal difference between linear and nonlinear systems is about local and global properties. A linear system has unified stability properties, which means that if the stability requirements are satisfied for any local region of a linear system, the global stability of the entire state-space is guaranteed. Therefore, there is no need to distinguish between local and global stability for a linear system.

However, the problem becomes much more complex for a general nonlinear system. On the one hand, local stability does not imply global stability; on the other hand, it is very difficult to estimate the effects of a disturbance generated from some local region on the entire system. In order to avoid this complexity, an obvious idea is to decompose a larger nonlinear system into some independent subsystems. If the stability of each independent subsystem can be examined independently, the individual solutions can be combined to yield a solution for the overall fuzzy control problem.

Based on this idea, Gao et al (1997) proposed the so-called "decomposition principle" for fuzzy system design.

Consider the discrete form of the fuzzy system of Equation (5-1):

$$X(t+1) = \sum_{i=1}^{p} h_i(X(t))(\hat{A}_i X(t) + \hat{B}_i u(t))$$ (5- 7)

where $X(t) \in R^n, u \in R^m, \hat{A}_i \in R^{n \times n}, \hat{B}_i \in R^{n \times m}$ $i \in \{1 \cdots p\}$

and $h_i(X) = u_i(X) \Big/ \sum_j u_j(X)$

The state space can be decomposed into m independent subspaces $\{S_1 \cdots S_m\}$.

For each subspace, a characteristic function is defined by

$$\eta_l = \begin{cases} 1 & (X \in S_l) \\ 0 & (X \notin S_l) \end{cases} \qquad l \in \{1 \cdots m\} \tag{5-8}$$

Then Equation (5-7) can be rewritten as:

$$X(t+1) = \sum_{l=1}^{m} \sum_{i=1}^{p} \eta_l h_i(X(t))(\hat{A}_i X(t) + \hat{B}_i u(t)) \tag{5-9}$$

For the lth subspace, the dynamics of fuzzy system is denoted by

$$X(t+1) = \sum_{i=1}^{p} \eta_l h_i(X(t))(\hat{A}_i X(t) + \hat{B}_i u(t))$$

It is obvious that

$$\eta_l h_i(X) \geq 0$$

if and only if the support of the corresponding membership function is located inside the subspace l.

Therefore, if $X \in S_l$, Equation (5-9) is equivalent to

$$X(t+1) = \sum_{j=1}^{m_l} \left[h_j(X(t))(\hat{A}_j X(t) + \hat{B}_j u) \right] \tag{5-10}$$

$j \in \{1 \cdots m_l\}$ denotes the fired rules in S_i

Since the number of rules fired inside subspace l can be far less than the total number of rules in a fuzzy system, it will be much simpler to develop a stability checking criterion for the decomposed system instead of the original system.

Lemma 5-2 (Cao et al, 1997)

The fuzzy system of Equation (5-7) is asymptotically stable if the m subsystems described by Equation (5-10) are asymptotically stable or there exist a set of Lyapunov functions $(V_1 \quad \cdots \quad V_m)$,

$$V_l = Z_l^T P_l Z_l, \text{ and } Z_l = \eta_l X$$

such that $\Delta V_l \leq 0$.

P_l, $l \in [1 \cdots m]$, is a positive definite matrix.

The above Lemma intends to build a theoretical background for decomposition in fuzzy space. The idea is very appealing at first sight. However, it will be demonstrated that the Lemma is not correct.

Consider the proof from (Cao, 1997), given in Appendix B.

In the proof, the authors demonstrated that, for each subsystem it has,

$$\|Z_l(t)\| \leq c\sigma^{t-\bar{t}} \|Z_l(\bar{t})\| \qquad\qquad l = \{1 \cdots m\}$$

where

$$c = \max_l \left[\frac{\lambda_{\max}(P_l)^{1/2}}{\lambda_{\min}(P_l)^{1/2}} \right] \tag{5-11}$$

$$\sigma = \max_l \left[1 - \frac{\lambda_{\min}(Q_l)}{\lambda_{\max}(P_l)} \right]^{1/2} \qquad (5\text{-}12)$$

P_l is a positive definite matrix, Q_l is a positive definite matrix. $Q_l = -\hat{A}_l^T P \hat{A}_l + P$

Since $Z_l = \eta_l X$

$$\|X(t)\| \le c \|X(0)\| \sigma^t \quad t \in [0, \tau)$$

is true for subsystem l

Suppose $X(t)$ enters another subspace S_j at time instant τ_1.

$$\|X(\tau_1)\| \le c \|X(0)\| \sigma^t \qquad (5\text{-}13)$$

$$\|X(t)\| \le c \|X(\tau_1)\| \sigma^{t-\tau_1} = c^2 \|X(0)\| \sigma^t \qquad (5\text{-}14)$$

Hence from the continuity of $X(t)$, the result can be generalised into all the regions as

$$\|X(t)\| \le c^T \|X(0)\| \sigma^t \qquad (5\text{-}15)$$

where T denotes the number of times that trajectory $X(t)$ switches between subspaces.

Global asymptotical stability requires $\lim_{t \to \infty} X(t) = 0$ to be satisfied for any trajectory. This is satisfied if and only if $\sigma < 1$

which requires:

$$\lambda_{\min}(Q_l) < \lambda_{\max}(P_l) \qquad (5\text{-}16)$$

From

$$Q_l = -\hat{A}_l^T P \hat{A}_l + P \qquad \text{for the discrete case} \qquad (5\text{-}17)$$

$$Q_l = -A_l^T P - PA_l \qquad \text{for the continuous case} \qquad \text{(5-18)}$$

and $Q_l > 0$, it is easy to verify that Equation (5-16) is not true in general. Therefore the proof given by Cao et al (1997) is wrong.

An interesting issue regarding the decomposition principle is that whether the stability of each independent subsystem can be examined independently. The answer is negative. This is because although the dynamics in fuzzy system of Equation (5-1) or Equation (5-5) is autonomous, where t does not appear explicitly in the dynamic equations, the trajectories space $\{St \subset R^n \times R^+ \mid where \; X(t) \in St\}$ is still a $n+1$ dimensional space. Therefore, the subsystems in state-space are not truly independent of one another on trajectories space.

Also, suppose a relationship similar to Equation (5-15) is true for all trajectories in state-space, there must be a global Lyapunov function (such as $V = \|X(t)\|$ in this case) for the entire state-space. From the point of view of solving a linear matrix inequality, there is no need to derive the Lyapunov function for each individual subspace.

The error of the decomposition principle for fuzzy system design has been investigated in this section. In the following section, the dynamics of fuzzy systems will be further explored by examining the so-called perturbation theory.

5.4   Stability under Perturbations

This section focuses on the stability of a dynamic system under perturbation. Consider an open-loop dynamic system

$$\dot{x} = Ax \qquad\qquad (5\text{-}19)$$

Two types of perturbation can be found in fuzzy system design. They are:

1)   Structural perturbation

$$\dot{x} = Ax + Bx \qquad\qquad (5\text{-}20)$$

where Bx is the source of the perturbation.

Since the perturbation is proportional to the state-space variables, this kind of perturbation is called structural perturbation

2)   Positional perturbation

$$\dot{x} = Ax + g(x) \qquad\qquad (5\text{-}21)$$

where $g(x)$ is the source of the perturbation.

Since it is position-related, this kind of perturbation is called positional perturbation.

Before perturbation theory is further studied, a summary of useful results is given below:

Lemma 5-3 (Curtain, 1977)

$(\lambda I - T)u = v$ has a unique solution on $X = C[a,b]$ for all $\lambda \neq 0$, the solution is given

as

$$u = \lambda \sum_{n}^{\infty} \lambda^{-n} T^n v \qquad\qquad (5\text{-}22)$$

Lemma 5-4 (Gronwall's Inequality Lemma) (Curtain 1977)

Let $a \in L_1(\tau,T)$ $a(t) \geq 0$, let b be continuous on $[\tau,T]$

If $x(t) \leq b(t) + \int_{\tau}^{t} a(s)x(s)ds$

then

$$x(t) \leq b(t)\exp(\int_{\tau}^{t} a(s)ds) + \int_{\tau}^{t} b^{'}(s)\exp(\int_{s}^{t} a(\rho)d\rho)ds \qquad (5\text{-}23)$$

Lemma 5-5

If dynamic system $\dot{x} = Ax$ is asymptotically stable, then any given positive definite

matrix C forms a Lyapunov function $V = x^T P x$

Where

$$P = \int_{0}^{\infty} e^{A^T t} C e^{At} dt \qquad\qquad (5\text{-}24)$$

Proof:

It is obvious that $P > 0$

Since the dynamic system is asymptotically stable, a positive value M can always be found to satisfy $\left\| e^{At} \right\| = \left\| e^{A^T t} \right\| \le M e^{-\omega t}$

$$\|P\| = \left\| \int_0^\infty e^{A^T t} C e^{At} dt \right\| \le \|C\| \int_0^\infty M^2 e^{-2\omega t} dt = \|C\| \frac{M^2}{2\omega}$$

Therefore P is well defined

From Equation (5-24)

$$A^T P + PA$$

$$= \int_0^\infty (A^T e^{A^T t} C e^{At} + e^{A^T t} C e^{At} A) dt$$

$$= \int_0^\infty \frac{d}{dt} (e^{A^T t} C e^{At}) dt$$

$$= -C$$

the derivative of V is

$$\dot{V} = \frac{d\left(x^T P x\right)}{dt}$$

$$= \dot{x}^T P x + x^T P \dot{x}$$

$$= (Ax)^T P x + x^T P (Ax)$$

$$= x^T (A^T P + PA) x$$

$$= -x^T C x < 0$$

Therefore, the dynamic system is stable.

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5.4.1 Stability under Structural Perturbation

Consider the dynamic system of Equation (5-20). If the system is asymptotically stable when there is no perturbation, then

$$\left\| e^{At} \right\| \leq M e^{-\omega t}; \quad \omega > 0 \text{ and } t \geq 0$$

From Lemma 5-5, it is known that exist P and C satisfy

$$A^T P + PA + C = 0$$

Let $V = x^T P x$

Then

$$\dot{V} = -x^T C x + x^T (B^T P + PB)x$$

If $\left\| B \right\| < K$ is given,

then an estimate of $\dot{V}$ can be given by setting the value of C.

For example, if $C = I$ then

$$P = \int_0^{\infty} e^{A^T t} e^{At} dt$$

$$\left\| P \right\| \leq \frac{M^2}{2\omega}$$

Under this assumption, the derivative of the Lyapunov function satisfies:

$$\dot{V} = -\left\| x \right\|^2 (1 - \frac{KM^2}{\omega})$$

In this case the stability of the entire system requires

$$\omega > KM^2$$

or

$$K < \frac{\omega}{M^2}$$

A better estimation of $\dot{V}$ for the dynamic system of Equation (5-20) can be obtained from the following lemma:

Lemma 5-6 Stability under structural perturbation

The dynamic system of Equation (5-20) is stable if there exist $M > 0$ and $\omega > 0$ such that

$$e^{At} \leq Me^{-\omega t} \tag{5-25}$$

and

$$\|B\| < \omega / M \tag{5-26}$$

Proof:

Equation (5-25) indicates that the unperturbed system $\dot{x} = Ax$ is asymptotically stable. Therefore for

$$\dot{x} = Ax + Bx$$

$x(t)$ is given by

$$x(t) = \exp[(A+B)t]\bar{x}, \qquad \bar{x} \text{ is the initial condition}$$

From the solution of a differential delay equation, this is equivalent to:

$$x(t) = \exp[At]\bar{x} + \int_0^t \exp[A(t-s)Bx(s)ds$$

Thus

$$\exp[(A+B)t] = \exp[At] + \int_0^t \exp[A(t-s)] \cdot B \cdot \exp[(A+B)s]ds$$

This is a second kind of volterra equation (Press, Flannery 1993).

From Lemma 5-3 and the property of the volterra equation (Press, Flannery 1993), this equation has a unique solution:

$$\exp[(A+B)t] = \sum_{n=0}^{\infty} \mu_n(t)$$

where

$$u_n(t) = \int_0^t \exp[A(t-s)] \cdot B \cdot u_{n-1}(s)ds$$

$$u_0 = \exp[At]$$

From Equation (5-25)

$$\left\| \exp[(A+B)t] \right\| \leq \left\| \exp[At] \right\| + \int_0^t \left\| \exp[A(t-s)] \right\| \cdot \left\| B \right\| \cdot \left\| \exp[(A+B)s] \right\| ds$$

$$\left\| \exp[(A+B)t] \right\| \leq M \exp(-\omega t) + \int_0^t M \exp[-\omega(t-s)] \cdot \left\| B \right\| \cdot \exp[(A+B)s] \right\| ds$$

$$\left\| \exp(\omega t) \cdot \exp[(A+B)t] \right\| \leq M + M \int_0^t \left\| \exp[\omega s] \right\| \cdot \left\| B \right\| \cdot \exp[(A+B)s] \right\| ds$$

$$\left\| \exp(\omega t) \cdot \exp[(A+B)t] \right\| \leq M + M \int_0^t \left\| B \right\| \cdot \left\| \exp[\omega s] \cdot \exp[(A+B)s] \right\| ds$$

From Lemma 5-4

$$\left\| \exp[\omega t] \cdot \exp[(A+B)s] \right\| \leq M \cdot \exp(M \left\| B \right\| t)$$

Then

$$\left\| \exp[(A+B)s] \right\| \leq M \cdot \exp[(-\omega + M \left\| B \right\|)t]$$

Therefore dynamic system 5-20 is stable if

$$\left\| B \right\| < \omega / M$$

End of Proof

5.4.2 Stability under Positional Perturbation

The effects of position perturbation can be deduced by the following Lemma.

Lemma 5-7 Stability under Position Perturbation

The dynamic system of Equation (5-21) is stable if there exist M and ω such that

$$e^{At} \leq Me^{-\omega t}$$

and

$$\|g(x)\| \leq \frac{\|x\|\omega}{M^2} \tag{5-27}$$

Proof:

$e^{At} \leq Me^{-\omega t}$ indicates that the unperturbed system $\dot{x} = Ax$ is stable. Therefore there exists

$$A^T P + PA + I = 0$$

for

$$V = x^T P x$$

$$\dot{V} = x^T \left(A^T P + PA \right) x + x^T P g(x) + g(x)^T P x$$

Hence

$$\dot{V} \leq -\|x\|^2 + 2\|x\|\|P\|\|g(x)\|$$

$$= \|x\| \left(2\|P\|\|g(x)\| - \|x\| \right)$$

Stability requires

$$\|g(x)\| < \frac{\|x\|}{2\|P\|}$$

since $\|P\| \le \dfrac{M^2}{2\omega}$

$\|g(x)\| \le \dfrac{\|x\|\omega}{M^2}$

End of Proof

Remarks:

Lemma 5-7 indicates that a dynamic system is more sensitive to positional perturbations occurring closer to the origin.

A qualitative analysis of the stability of dynamic systems has been performed in this section. In the following section, a quantitative analysis of the stability of nonlinear fuzzy control system will be undertaken.

5.5 New Stability Criterion for General Nonlinear Fuzzy Systems

Consider again the fuzzy system described by Equation (5-5)

If X is $U_1(X)$ Then $\dot{X} = A_1 X + B_1 u + D_1$ $X \in M_1$

$\vdots$

If X is $U_i(X)$ Then $\dot{X} = A_i X + B_i u + D_i$ $X \in M_i$

$\vdots$

If X is $U_p(X)$ Then $\dot{X} = A_p X + B_p u + D_p$ $X \in M_p$

and

$$\dot{X} = \sum_{i=1}^{p} h_i(X)(A_i X + B_i u + D_i)$$

where $X \in R^n$, $u \in R^m$, $A_i \in R^{n \times n}$, $B_i \in R^{n \times m}$ $i \in \{1 \cdots p\}$ h_i's are normalised membership functions and M_i's denote the support of the corresponding membership functions.

Given Lyapunov function

$$V = X^T P X$$

the derivative of the Lyapunov function can be obtained as

$$\dot{V} = \dot{X}^T P X + X^T P \dot{X}$$

$$\dot{V} = \sum_{i=1}^{p} h_i(X)\left[X^T \left(A_i^T P + P A_i \right) X + D^T P X + X^T P D \right]$$

As mentioned previously, the stability of above fuzzy system requires $\dot{V} < 0$. However, since the expression of $\dot{V}$ is non-convex, it cannot be solved by convex optimisation

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techniques such as LMI. In order to avoid this problem, the geometrical information in the fuzzy system needs to be considered.

Definition 5-1 Characteristic Ellipsoid

Consider the fuzzy system of Equation (5-5). If the local dynamics of the rule i can be described as

$$\dot{X} = A_i X + D_i,$$

then for a given positive definite matrix P, a characteristic ellipsoid of P is defined as

$$(X - r_i)^T Q_i (X - r_i) = r_i^T Q_i r_i \qquad (5\text{-}28)$$

where, $Q_i = -\left(A_i^T P + P A_i\right)$ and $r_i = Q_i^{-1} P D_i$

Theorem 5-1

If there exists a positive definite matrix satisfying:

$$A_i^T P + P A_i < 0 \qquad (5\text{-}29)$$

and the support of the ith rule M_i is located outside its characteristic ellipsoid for all $i \in \{1 \cdots p\}$, then the origin of the fuzzy system described in Equation (5-5) is globally asymptotically stable.

Proof:

Given Lyapunov function

$$V = X^T P X$$

the derivative of the Lyapunov function can be obtained as

$$\dot{V} = \dot{X}^T P X + X^T P \dot{X}$$

$$\dot{V} = \sum_{i=1}^{p} h_i(X) \left[X^T \left(A_i^T P + P A_i \right) X + D^T P X + X^T P D \right]$$

Substituting $Q_i = -\left(A_i^T P + P A_i \right)$ into $\dot{V}$ gives

$$\dot{V} = \sum_{i=1}^{p} h_i(X) \left[X^T (-Q_i) X + (PD)^T X + X^T P D \right]$$

$$\dot{V} = \sum_{i=1}^{p} h_i(X) \left[X^T (-Q_i) X + (Q_i^{-1} P D)^T Q_i X + X^T Q_i (Q_i^{-1} P D) \right]$$

$$\dot{V} = \sum_{i=1}^{p} h_i(X) \left[(X - Q_i^{-1} P D)^T (-Q_i)(X - Q_i^{-1} P D) - (Q_i^{-1} P D)^T (-Q_i)(Q_i^{-1} P D) \right]$$

Substituting $r_i = Q_i^{-1} P D_i$ into the above equation gives:

$$\dot{V} = \sum_{i=1}^{p} h_i(X) \left[(X - r_i)^T (-Q_i)(X - r_i) - r_i^T (-Q_i) r_i \right] \qquad (5\text{-}30)$$

From

$$-Q_i = A_i^T P + P A_i < 0$$

It is known that the Q_i is a positive definite matrix. Therefore the curve

$$(X - r_i)^T Q_i (X - r_i) = r^T Q_i r$$

$$r_i = Q_i^{-1} P D_i$$

is an ellipsoid. The following relation hold:

$$\begin{cases} (X - r_i)^T Q_i (X - r_i) - r^T Q_i r < 0 & \text{when } X \text{ is inside the ellipsoid} \\ (X - r_i)^T Q_i (X - r_i) - r^T Q_i r > 0 & \text{when } X \text{ is outside the ellipsoid} \end{cases}$$

Hence if $Q_i > 0$ and the support of the ith rule is outside its characteristic ellipsoid for

all $i \in \{1 \cdots p\}$,

$$\dot{V} < 0 \quad \text{for } X \in R^n$$

Therefore the origin of the fuzzy system described in Equation (5-5) is asymptotically stable in the entire state-space.

Remarks

- The first part of the condition for global asymptotical stability, namely

$$A_i^T P + P A_i < 0 \qquad \text{for all } i \in \{1 \cdots p\}$$

is a structure-related requirement.

The second part of the condition for global asymptotical stability, namely, that the support of the ith rule locates outside its characteristic ellipsoid for all $i \in \{1 \cdots p\}$, is a position-related requirement.

- The origin $X = 0$ and the point $X = -A_i^{-1} D_i$ are two special points on the characteristic ellipsoid. This can be verified by

$$(0 - r_i)^T Q_i (0 - r_i) = r_i^T Q_i r_i$$

and

$$(-A_i^{-1} D_i - r_i)^T Q_i (-A_i^{-1} D_i - r_i) = r_i^T Q_i r_i$$

Therefore, the positions of the characteristic ellipsoids are close to the origin.

- If the structural requirement of Equation (5-29) is satisfied, the stability of a general nonlinear fuzzy system is more sensitive to those fuzzy rules fired near the origin. This is because the characteristic ellipsoids are close to the origin. The supports of fuzzy rules that are fired near the origin are more likely to intersect the characteristic ellipsoids. This result is consistent with the result derived from positional perturbation in section 5.4.2.

- Lemma 5-1 is a special case of Theorem 5-1 applied to sector-nonlinear fuzzy systems. Consider a sector-nonlinear system described by Equation (5-1), which has $D_i = 0$ for all $i \in \{1 \cdots p\}$

Then

$$r_i = Q_i^{-1} P D_i = 0$$

Hence, the characteristic ellipsoids reduce to the origin and the second condition of Theorem 5-1 can be satisfied automatically.

- To satisfy the stability condition in Theorem 5-1, if the support of a fuzzy rule covers the origin, its corresponding characteristic ellipsoid must has zero diameter. This implies $D_i = 0$. If the origin is covered by several rules at the same time, the constraint is not necessary. This is demonstrated as follows.

Consider nonlinear fuzzy system of Equation (5-5), with trapezoidal or triangular membership functions,

If the function $\sum_{j=1}^{p} h_j(X) D_j$ is smooth inside a subspace M_0, and M_0 covers the origin of the state-space. the following transform action of Equation (5-5) can be carried out:

Let

$$h_j D_j = \left[l_{j0} + \begin{pmatrix} l_{j1} & \cdots & l_{jn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right] \begin{pmatrix} d_{j1} \\ \vdots \\ d_{jn} \end{pmatrix} \qquad (5\text{-}31)$$

$$= \overline{D}_j X + \overline{G}_j$$

where

$$\overline{D}_j = \begin{bmatrix} l_{j1}d_{j1} & \cdots & l_{jk}d_{j1} & \cdots & l_{jn}d_{j1} \\ \vdots & \ddots & & & \vdots \\ l_{j1}d_{jk} & & l_{jk}d_{jk} & & l_{jn}d_{jk} \\ \vdots & & & \ddots & \vdots \\ l_{j1}d_{jn} & \cdots & l_{jk}d_{jn} & \cdots & l_{jn}d_{jn} \end{bmatrix}$$

(5- 32)

and

$$\overline{G}_j = \begin{bmatrix} l_{j0}d_{j1} \\ \vdots \\ l_{j0}d_{jn} \end{bmatrix}$$

(5- 33)

Hence, near the origin Equation 5-5 can be written as:

$$\dot{X} = \sum_{j=1}^{p} \left[h_j(X)(A_j X + B_j u) \right] + \overline{D}X + \overline{G} \qquad X \in M_0$$

(5- 34)

where

$$\overline{D} = \sum_{j=1}^{m_i} \overline{D}_j$$

and

$$\overline{G} = \sum_{j=1}^{p} \overline{G}_j$$

From the fact that $X = 0$ is the equilibrium point of the state-space, $\overline{G} = 0$.

Therefore the fuzzy system described by Equation (5-5) can be written as

If X is $U_1(X)$ Then $\dot{X} = A_1 X + B_1 u + D_1$ $X \in M_1^{'} = M_1 - M_0$

$\vdots$

If X is $U_i(X)$ Then $\dot{X} = A_i X + B_i u + D_i$ $X \in M_i^{'} = M_i - M_0$

$\vdots$

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If X is $U_p(X)$ Then $\dot{X} = A_p X + B_p u + D_p$ $\qquad X \in M'_p = M_p - M_0$

and

$$\dot{X} = \sum_{i=1}^{p} h_i(X)(A_i X + B_i u + D_i) \qquad \text{for} \qquad X \notin M_0 \qquad (5\text{-}35)$$

and

$$\dot{X} = \sum_{i=1}^{p} \left[h_i(X)(A_i X + B_i u) \right] + \overline{D} X \qquad \text{for} \qquad X \in M_0 \qquad (5\text{-}36)$$

where $X \in R^n$, $u \in R^m$, $A_i \in R^{nxn}$, $B_i \in R^{nxm}$ $\quad i \in \{1 \cdots p\}$ and h_i's are the normalised membership function.

Immediately following theorem 5-1, a stability condition for the fuzzy system above can be derived as follows.

Proposition 5-1

If there exists a positive definite matrix P satisfying:

$$A_i^T P + P A_i < 0 \qquad (5\text{-}37)$$

$$D^T P + P D \leq 0 \qquad (5\text{-}38)$$

and the support of the ith rule M_i' is located outside its characteristic ellipsoid for all $i \in \{1 \cdots p\}$, then the origin of the fuzzy system described in Equation (5-5) is globally asymptotically stable.

The proof is similar to that for Theorem 5-1.

As mentioned above, the location of the support can be determined by checking whether $(X - r_i)^T Q_i (X - r_i) - r^T Q_i r > 0$ is true for $X \in M_i'$.

Remarks:

- Proposition 5-1 is a modified version of the Theorem 5-1, extended to deal with fuzzy rules close to the origin.

- For the Lyapunov function's derivative to be negative, only requires either $A_i^T P + P A_i$ or $D^T P + PD$ to be a strictly negative definite function. Because for other regions, it is also necessary for

$$A_i^T P + P A_i < 0$$

in order to guarantee stability near the origin, the matrix $D^T P + PD$ only needs to be non-strictly negative, i.e.

$$D^T P + PD \leq 0$$

This relaxed requirement is especially useful when D is singular.

In this section a new stability checking criterion for general nonlinear fuzzy system has been developed by the integration of geometrical information in the stability analysis.

Example 5-2

Recall the fuzzy systems in example 5-1:

(1) Fuzzy system 1

R1 IF X is A1 then $\dot{x} = -ax - b$

R2 IF X is A2 then $\dot{x} = -ax + b$

where $a > 0$ $b > 0$

(2) Fuzzy system 2

    R1 IF X is A1 then $\dot{x} = -ax - b$

    R2 IF X is A2 then $\dot{x} = -ax + b$

    where $a > 0$ $b < 0$

The membership functions for A1 and A2 are:

$$h_1(x) = \begin{cases} 1 & x > c \\ \dfrac{1}{2c}(c + x) & -c \le x \le c \\ 0 & x < -c \end{cases}$$

and

$$h_2(x) = \begin{cases} 0 & x > c \\ \dfrac{1}{2c}(c - x) & -c \le x \le c \\ 1 & x < -c \end{cases}$$

where $c = \left| \dfrac{b}{a} \right|$

Since the supports M_1 and M_2 of both rules cover the origin, the fuzzy systems need to

be written in the form of Equation (5-35) and Equation (5-36). The new supports M_0,

$M'_1 = M_1 - M_0$ and $M'_2 = M_2 - M_0$ are illustrated in figure 5-6.

From $A_1 = A_2 = -a$ and $D = 0$,

$$A_i^T P + P A_i < 0$$

$$D^T P + P D \le 0$$

for all $P > 0$

From Equation (5-28),

$$r_1 = -\frac{b}{2a}$$

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$$r_2 = \frac{b}{2a}$$

Hence, for fuzzy system 1 with $b > 0$, there is no intersection between the characteristic ellipsoids and the support. The origin is asymptotically stable in the entire state-space.

For fuzzy system 2 with $b < 0$, there are intersections between characteristic ellipsoids and the support. In this case the origin is not stable.
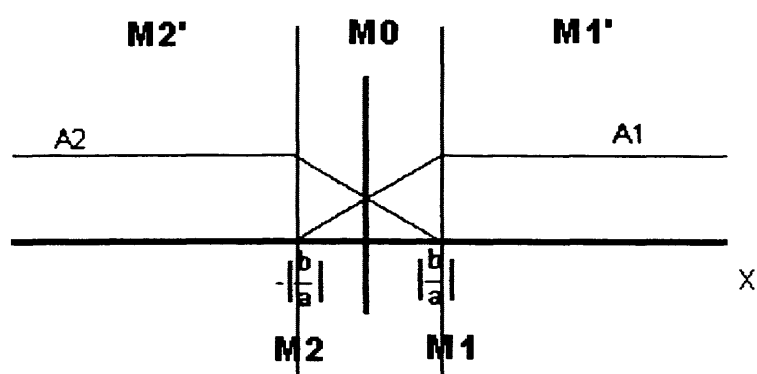
Figure 5-6 The supports in example 5-2

Example 5-3

Consider the fuzzy system described by the following rules:

R1 IF X is M_1 then $\dot{X} = A_1 X - D_1$

R2 IF X is M_2 then $\dot{X} = A_2 X$

R3 IF X is M_3 then $\dot{X} = A_3 X - D_3$

where

$$A_1 = \begin{bmatrix} -10 & -11 \\ 10 & 2 \end{bmatrix}, \ D_1 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1 & -2 \\ 2 & -10 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -10 & -10 \\ 10 & -4 \end{bmatrix}, \ D_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

The membership function for M_1, M_2 and M_3 are plotted in Figure 5-7.

From Theorem 5-1 stability requires:

$$\begin{cases} A_1^T P + A_1 P < 0 \\ A_2^T P + A_2 P < 0 \\ A_2^T P + A_2 P < 0 \\ P > 0 \end{cases} \tag{5-39}$$

The Linear matrix inequality (5-37) is feasible for

$$P = \begin{bmatrix} 0.0623 & 0.0174 \\ 0.0174 & 0.0657 \end{bmatrix}$$

The trajectories for the system are plotted in figure 5-8.

The characteristic ellipsoids of P for R1 and R3 are illustrated in figure 5-9. Since the characteristic ellipsoids do not intersect with the supports of R1 and R3, the fuzzy system in this example is stable.

5.6 Summary

This chapter has examined the problem of nonlinear fuzzy control, focusing on the analysis of the stability of fuzzy nonlinear systems. The chapter has proposed a new criterion for checking stability. This criterion integrates geometric information in the analysis of stability and can be applied to general nonlinear fuzzy systems. The new stability criterion is the theoretical basis for the development of future nonlinear fuzzy control approaches.
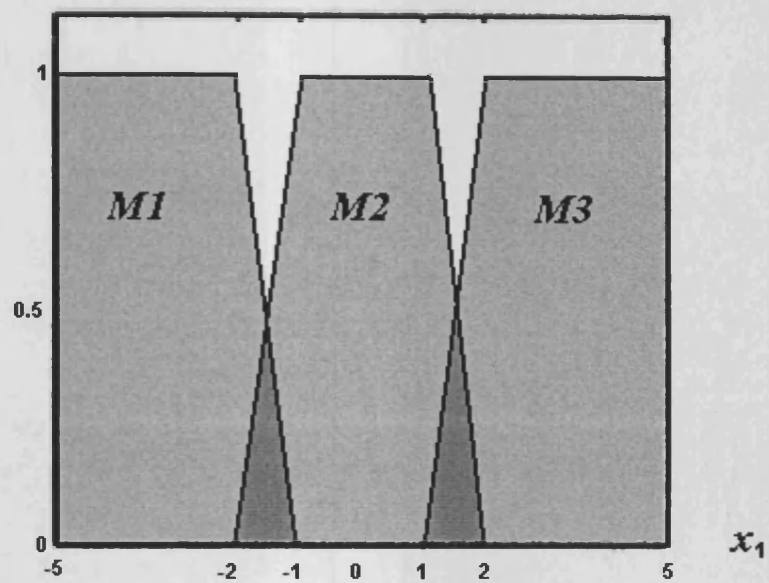
Figure 5-7 The membership functions for example 5-3
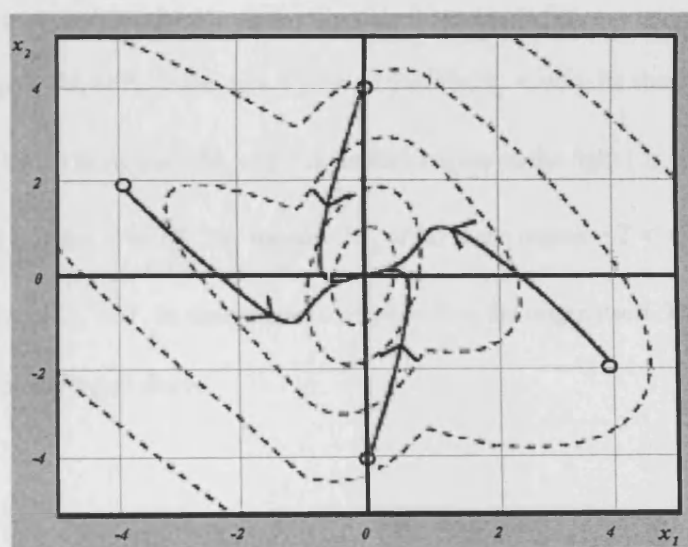
Figure 5-8 The trajectories of the fuzzy system in example 5-3
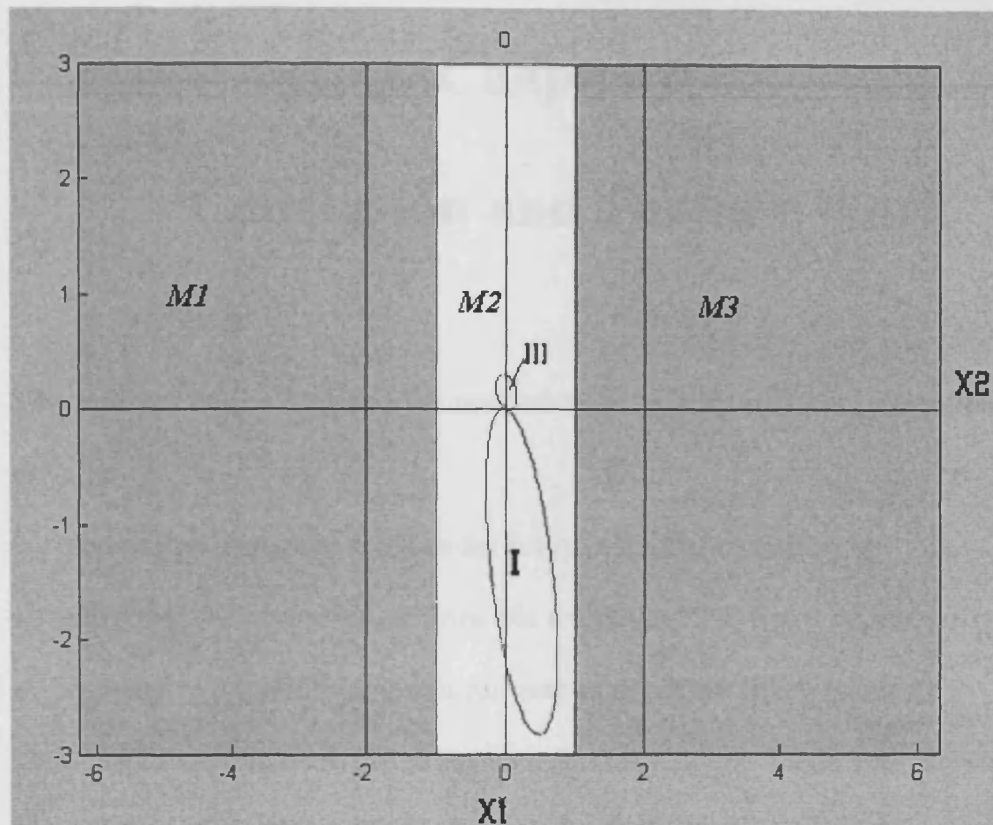
Figure 5-9 The supports and the characteristic ellipsoids for example 5-3.

The support M_1 of R1 is the dark region on the left ($x_1 < -1$). Its characteristic ellipsoid is

denoted by I. The support M_3 of R3 is the dark region on the right ($x_1 > 1$). Its characteristic

ellipsoid is denoted by III. The support M_2 of R2 is the region $-2 < x_1 < 2$, which covers

origin. Since $D_2 = 0$, its characteristic ellipsoid II is the origin itself. Therefore, it is not

marked on the figure above.

Chapter 6

Conclusion and Further Work

This research has investigated the mathematical aspects of fuzzy systems with a view to:

- deriving an analytical solution for fuzzy relational equations

- clarifying the mathematical principle underlying T-S fuzzy system design

- developing a stability criterion for general nonlinear fuzzy systems

This chapter summarises the main contributions of the work and the conclusions reached. It also suggests possible research topics for further study.

6.1   Contributions

This research has

1)   proposed new triangular operations in fuzzy algebra. Inversions for triangular norms and co-norms were developed based on the modus ponens theorem and the comparison theorem.

2)   produced an analytical solution for max-family FREs. Formulae for the maximum, mean, and lower bound of the minimum solutions of FREs have been derived.

3) identified fuzzy transform action as the mathematical principle underlying T-S fuzzy system design. Based on the fuzzy transform, the optimum parameters c_i of a fuzzy T-S model can be derived exactly by solving the following equation

$$\begin{pmatrix} <\phi_1,\phi_1> & \cdots & <\phi_1,\phi_j> & \cdots & <\phi_1,\phi_n> \\ \vdots & & \vdots & & \vdots \\ <\phi_i,\phi_1> & \cdots & <\phi_i,\phi_j> & \cdots & <\phi_i,\phi_n> \\ \vdots & & \vdots & & \vdots \\ <\phi_n,\phi_1> & \cdots & <\phi_n,\phi_j> & \cdots & <\phi_n,\phi_n> \end{pmatrix} * \begin{pmatrix} c_1 \\ \vdots \\ c_i \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} <\phi_1,f> \\ \vdots \\ <\phi_i,f> \\ \vdots \\ <\phi_n,f> \end{pmatrix}$$

where ϕ_i is a membership function, f is the target function and $<x,y>$ denotes the inner product of x and y.

4) developed a new stability criterion for checking the stability of general nonlinear fuzzy systems.

6.2   Conclusions

The following are the main conclusions from the research:

1) The solution space of max-family fuzzy relational equations is linear under fuzzy algebra. Difficulties in solving FREs are not caused by the complexity of the solution space, but by the lack of efficient fuzzy algebra analysis tools. The new tools produced in this work enable an analytical solution to be obtained directly.

2) Due to fuzzy transform action, the efficiency of T-S fuzzy model design can be improved by using dual bases.

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3) Mathematically, a T-S fuzzy model eliminates the effects of noise via the support of its membership functions; therefore a T-S model is less sensitive to noise when its membership functions have wider support.

4) Modern model-based fuzzy control (MBFC) theory is not suitable for nonlinear control, due to the implicit linearity assumption in fuzzy controller design. Under this assumption, a fuzzy control problem has to be expressed in a sector-nonlinear form. This limits fuzzy controller design to being a special case of linear time-varying system control.

6.3   Further work

1) Although the max operation is a commonly used triangular co-norm in the Mamdani fuzzy model, it would be of interest to investigate the solution of fuzzy relational equations based on other types of triangular co-norms, such as the probabilistic sum and bounded sum.

2) The strong analogy between mathematical transforms and the T-S fuzzy model indicates the potential of applying techniques such as wavelets and Splines in T-S fuzzy modelling.

3) The Lyapunov function for nonlinear fuzzy systems is in the form of a bilinear matrix inequality (BMI). Therefore, it is possible to develop a BMI based, non-convex optimisation technique for stability analysis in order to achieve a less conservative checking criterion.

Appendix A

Fuzzy Transform and Reverse Engineering

The possibility of implementing the fuzzy transform in reverse engineering is explained here to demonstrated the applicability of the algorithm.

Reverse engineering is defined as the production of new parts, products or tooling from existing physical models or components. One of the main applications for reverse engineering is to generate CAD models from physical objects. The first step in reverse engineering is part digitisation. This is the process of acquiring point coordinates from the surface of a part. In order to model the part surface in CAD, the surface feature of the cloud of points acquired in digitisation is identified by surface fitting techniques. The most popular fitting technique is least-square fitting with Non-Uniform Rational B-Splines or NURBS (Piegl 1992).

The motivation of this application is arose from the similarity between the NURBS and a zero-order T-S fuzzy model. If a NURBS can be treated a special case of a zero-order T-S fuzzy model, the efficiency of surface fitting in reverse engineering can be improved by applying fuzzy system design theory.

NURBS curves and surfaces

NURBS curves and surfaces are widely used in CAD for the representation of free-form curves and surfaces. A NURBS curve is a vector-valued piecewise rational polynomial and is defined as

$$C(u) = \frac{\displaystyle\sum_{i=0}^{n} w_i P_i N_{i,p}(u)}{\displaystyle\sum_{i=0}^{n} w_i N_{i,p}(u)} \qquad\qquad (A-1)$$

where the w_i's are the so-called weights, the P_i's are the control points, and

the $N_{i,p}(u)$'s are the normalised B-Spline functions of degree p defined as:

$$N_{i,0}(u) = \begin{cases} 1 & \text{if } u \in [u_i, u_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u) \qquad\qquad (A-2)$$

where u_i's are knots.

In a similar way, a NURBS surface is defined as

$$S(u,v) = \frac{\displaystyle\sum_{i=0}^{n}\sum_{j=0}^{m} w_{i,j} P_{i,j} N_{i,p}(u) N_{j,q}(v)}{\displaystyle\sum_{i=0}^{n}\sum_{j=0}^{m} w_{i,j} N_{i,p}(u) N_{j,q}(v)} \qquad\qquad (A-3)$$

where the $w_{i,j}$'s are the weights, the $P_{i,j}$'s are the control points, $N_{i,p}(u)$ and

$N_{j,q}(v)$ are the normalised B-Spline functions of degrees p and q in the u and v

directions.

By comparison with zero-order T-S fuzzy model in 4.3.2.1, it is interesting to note

that if the B-Splines in Equations (A-2) and (A-3) were replaced by certain

membership functions, the NURBS and the zero order T-S model have the same

mathematical description. Furthermore, if the B-Splines satisfy the requirements of

membership functions, NURBS can be regarded as a special case of zero-order fuzzy system. It has been proved in (Hsu and Wang, 1995) that B-Splines can be regarded as a special type of membership function. Then, Equation (A-2) can be re-written in the fuzzy form as:

If u is $A_i(u)$ Then y_i is P_i

$\vdots$

and $y = \sum\limits_{i}^{n} P_i \cdot \phi_i(u)$

where $\phi_i(Z) = \dfrac{A_i(Z) \cdot}{\sum\limits_{j}^{p} A_j(Z)}$, $i \in \{1 \cdots n\}$, n is the number of rules, $A_i(u) = \omega_i N_{i,p}(u)$ and

control points in NURBS can be regarded as the consequent parts in the T-S model.

Since NURBS can be regarded as a special case of zero-order T-S models. The fuzzy system design techniques can be applied to reverse engineering directly. The advantages of applying fuzzy system design techniques in reverse engineering are twofold:

- First, fuzzy theory provides an alternative explanation to NURBS surface fitting, which is useful for choosing the initial condition.

- Second, in order to improve the efficiency of surface fitting, the fuzzy system design technique developed in section 4-3 can be used to replace the least-squares method

CAD modelling of a free-form surface from a point cloud with NURBS can be formulated as the creation of a NURBS surface in Equation (A-3) that approximates a cloud of m measured points within a given tolerance. The surface parameters to be

determined from the points are the B-Spline functions $N_p(u)$ and $N_q(v)$, uniquely

defined by their order p, q and knots u_i, v_j respectively. For surface fitting in reverse

engineering, the number of measurement points is much larger than the number of

control points. Therefore, the least-squares method is commonly used to minimise the

error between the NURBS surface and the m measured points.

$$error = \sum_{s=1}^{m} \left[\frac{\sum_{i=0}^{n} \sum_{j=0}^{m} w_{i,j} P_{i,j} N_{i,p}(u_s) N_{j,q}(v_s)}{\sum_{i=0}^{n} \sum_{j=0}^{m} w_{i,j} N_{i,p}(u_s) N_{j,q}(v_s)} - P_s \right]^2$$

Instead of using the least-squares method, the algorithm developed in 4.3 can be

applied here to improve the efficiency of curve and surface fitting.

Appendix B

Proof of Decomposition Principle

The following proof is cited from "Analysis and design for a class of complex control systems part II", Automatica, vol 33, no. 6, pp 1029-1039, by Cao etc (1997).
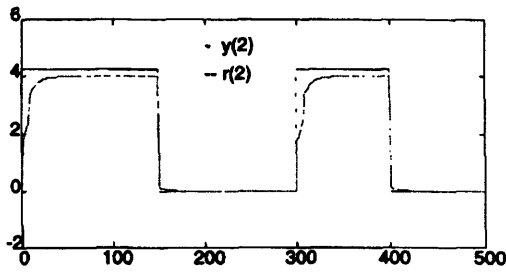
Fig. 5. The second output response of the system (30).

All the identification and controller design methods proposed in these two papers have been implemented using the MATLAB computer language, and form an efficient complex control system design software package (CASCADE). Using CASCADE and the design procedure proposed in this paper, we can easily obtain the fuzzy controller for a complex system.

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APPENDIX—PROOFS

In the proofs of the theorems, the following preliminary results will be used.

Lemma A.1. (Garcia *et al.* (1994)). Let A_0 and E_1 be matrices of appropriate dimensions, and let P be a positive-definite symmetric matrix satisfying

$$\frac{1}{\varepsilon}I - P > 0, \quad \varepsilon > 0.$$

Then

$$A_0^T P E_1 + E_1^T P A_0 + E_1^T P E_1 \le A_0^T P\left(\frac{1}{\varepsilon}I - P\right)^{-1} P A_0 + \frac{1}{\varepsilon}E_1^T E_1.$$

Lemma A.2. (Petersen (1987)). Let X, Y and Z be given $n \times n$ symmetric matrices such that $X \ge 0$, $Y \ge 0$ and $Z \ge 0$. Furthermore, assume that

$$(\xi^T Y \xi)^2 - 4(\xi^T X \xi \xi^T Z \xi) > 0 \qquad \text{(A.1)}$$

for all $\xi \in \mathbf{R}^n$ with $\xi \ne 0$. Then there exists a constant $\lambda > 0$ such that

$$\lambda^2 X + \lambda Y + Z < 0. \qquad \text{(A.2)}$$

Lemma A.3. Given matrices $\Delta A(\mu)$ and $\Delta B(\mu)$ satisfying

$$[\Delta A(\mu)]^T[\Delta A(\mu)] \le E^T E \quad \forall \mu \in M, \qquad \text{(A.3a)}$$

$$[\Delta A(\mu^*)]^T[\Delta A(\mu^*)] = E^T E, \quad \mu^* \in M, \qquad \text{(A.3b)}$$

and given A_0 and $\bar{Q}$, there exists a positive-definite symmetric matrix P such that

$$[A_0 + \Delta A(\mu)]^T P[A_0 + \Delta A(\mu)] + \bar{Q} < 0 \qquad \text{(A.4)}$$

if and only if there exists a scalar $\varepsilon > 0$ such that the following conditions hold:

$$\frac{1}{\varepsilon}I - P > 0, \qquad \text{(A.5a)}$$

$$A_0^T P A_0 + A_0^T P\left(\frac{1}{\varepsilon}I - P\right)^{-1} P A_0 + \frac{1}{\varepsilon}E^T E + \bar{Q} < 0. \quad \text{(A.5b)}$$

Proof. Suppose that (A.5) holds. Then, using (A.3) and Lemma A.1, we have

$$[A_0 + \Delta A(\mu)]^T P[A_0 + \Delta A(\mu)] + \bar{Q}$$

$$= A_0^T P A_0 + \Delta A(\mu)^T P A_0 + A_0^T P \Delta A(\mu)$$

$$+ \Delta A(\mu)^T P \Delta A(\mu) + \bar{Q}$$

$$\le A_0^T P A_0 + A_0^T P\left(\frac{1}{\varepsilon}I - P\right)^{-1} P A_0 + \frac{1}{\varepsilon}E^T E + \bar{Q} < 0.$$

In order to prove necessity, suppose that (A.4) holds. Then, from the standard Schur complement result, we note that (A.4) is equivalent to

$$\begin{bmatrix} -P^{-1} & A_0 + \Delta A(\mu) \\ [A_0 + \Delta A(\mu)]^T & \bar{Q} \end{bmatrix} < 0.$$

Now let

$$Y = \begin{bmatrix} -P^{-1} & A_0 \\ A_0^T & \bar{Q} \end{bmatrix}, \quad N(\mu) = \begin{bmatrix} 0 & \Delta A(\mu) \\ \Delta A(\mu)^T & 0 \end{bmatrix}.$$

We have

$$x^T Y x < -\left\{\max_\mu x^T N(\mu)x, \mu \in M\right\} < 0$$

for any $x \in \mathbf{R}^n$, $x \ne 0$. Hence, letting $x^T = [x_1^T \ x_2^T]$, the above implies that

$$(x^T Y x)^2 > 4\max_{\mu \in M}[x_1^T \Delta A(\mu)x_2]^2$$

Using the Schwarz inequality, it follows that

$$[x_1^T \Delta A(\mu)x_2]^2 = x_1^T x_1 x_2^T \Delta A(\mu)^T \Delta A(\mu)x_2$$
$$\le x_1^T x_1 x_2^T E^T E x_2$$

Because there exists a μ^* such that (A.3b) holds, it follows that

$$(x^T Y x)^2 > 4 \max_{\mu \in M} [x_1^T \Delta A(\mu)x_2]^2 = 4x_1^T x_1 x_2^T E^T E x_2.$$

Thus

$$(x^T Y x)^2 > 4x_1^T x_1 x_2^T E^T E x_2 = 4x^T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x x^T \begin{bmatrix} 0 & 0 \\ 0 & E^T E \end{bmatrix} x.$$

Thus, by Lemma A.2, it follows that there exists an $\varepsilon > 0$ such that

$$\varepsilon^2 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} -P^{-1} & A_0 \\ A_0^T & \bar{Q} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & E^T E \end{bmatrix} < 0;$$

that is, $-P^{-1} + \varepsilon I < 0$ and

$$\begin{bmatrix} -P^{-1} + \varepsilon I & A_0 \\ A_0^T & \frac{1}{\varepsilon} E^T E + \bar{Q} \end{bmatrix} < 0,$$

which are equivalent to (A.5). $\square$

Proof of Theorem 3.1. Let the lth Lyapunov function be

$$V_l(z_l) = z_l^T P_l z_l.$$

We need to define the difference of $V_l(z_l)$ at the boundary of the subspaces. Suppose that $x(t) = z_l(\tau_i) \in S_l$ and $x(\tau_i + 1) = z_j(\tau_i + 1) \in S_j$; that is, $x(t)$ jumps into the subspace S_j from the subspace S_l at traversing time instant τ_i. In this case, the difference of V_l is defined by

$$\Delta V_l = V_l(z_j(\tau_i + 1)) - V_l(z_l(\tau_i)).$$

Thus the difference of $V_l(z_l)$ at the boundary of the subspaces is well defined.

Suppose that there exists a set of positive-definite symmetric matrices $(P_1, P_2, \ldots, P_m)$ such that

$$\tilde{A}_l(\mu)^T P_l \tilde{A}_l(\mu) - P_l < 0, \quad l = 1, 2, \ldots, m. \quad (A.6)$$

It can be seen that the lth Lyapunov function satisfies the inequalities

$$\lambda_{\min}(P_l) \|z_l\|^2 \le V_l(z_l) \le \lambda_{\max}(P_l) \|z_l\|^2. \quad (A.7)$$

It follows from (A.6) that there exists a positive-definite symmetric matrix Q_l such that

$$\Delta V_l \le -\lambda_{\min}(Q_l) \|z_l\|^2. \quad (A.8)$$

From (A.7) and (A.8), we get

$$V_l(z_l(t)) \le \left[1 - \frac{\lambda_{\min}(Q_l)}{\lambda_{\max}(P_l)} \right]^{t-\bar{t}} V_l(z_l(\bar{t})), \quad (A.9)$$

where $t > \bar{t} \ge 0$. Let

$$\sigma = \max_l \left\{ \left[1 - \frac{\lambda_{\min}(Q_l)}{\lambda_{\max}(P_l)} \right]^{1/2} \right\} < 1, \quad c = \max_l \left[\frac{\lambda_{\max}(P_l)^{1/2}}{\lambda_{\min}(P_l)^{1/2}} \right].$$

Then, using (A.7) and (A.9), we obtain

$$\|z_l(t)\| \le c\sigma^{t-\bar{t}} \|z_l(\bar{t})\|, \quad l = 1, 2, \ldots, m. \quad (A.10)$$

Suppose that $x(0) \in S_l$, then it follows from (A.10) that

$$\|x(t)\| \le c \|x(0)\| \sigma^t, \quad 0 \le t \le \tau_1. \quad (A.11)$$

Suppose that $x(\tau_1 + 1)$ enters S_j at traversing time instant τ_1. Then, from (A.10) and (A.11), we obtain

$$\|x(t)\| \le c \|x(\tau_1)\| \sigma^{(t-\tau_1)} \le c^2 \|x(0)\| \sigma^t, \quad 0 \le t \le \tau_2.$$

Thus we have that, for any $t > 0$,

$$\|x(t)\| \le C \|x(0)\| \sigma^t, \quad C = c^{\bar{t}} \quad (A.12)$$

where $C < \infty$ since $\bar{t} < \infty$. This proves the theorem. $\square$

Proof of Theorem 3.2. Let the lth Lyapunov function be

$$V_l(z_l) = z_l^T P_l z_l \quad (A.13)$$

and let the state-feedback control law be (15). Suppose that there exists a fuzzy control law (15) that can quadratically stabilize the m extreme subsystems in (9) and that there exists a set of membership functions denoted by $(\mu_1^*, \mu_2^*, \ldots, \mu_m^*)$ such that the upper bound (8) can be achieved. Then the lth Lyapunov difference along the trajectory of the lth extreme subsystem in (9) is given by

$$\Delta \bar{V}_l = z_l^T \{ [A_l + B_l K(\mu) + (E_{l1} + E_{l2} K(\mu))]^T P_l$$
$$\times [A_l + B_l K(\mu) + (E_{l1} + E_{l2} K(\mu))] - P_l \} z_l$$
$$< 0$$

It follows by Lemma (A.3) that

$$A_{0l}^T P_l A_{0l} - P_l + A_{0l}^T P_l \left(\frac{1}{\varepsilon_l} I - P_l \right)^{-1} P_l A_{0l} + \frac{1}{\varepsilon_l} E_{0l}^T E_{0l} < 0,$$
$$\frac{1}{\varepsilon_l} I - P_l > 0, \quad (A.14)$$

where

$$A_{0l} = A_l + B_l K(\mu), \quad E_{0l} = E_{l1} + E_{l2} K(\mu).$$

Using the same Lyapunov function (A.13), the corresponding lth Lyapunov difference along the trajectory of the lth subsystem in (7) is given by

$$\Delta V_l = z_l^T \{ [A_l + \Delta A_l(\mu) + (B_l + \Delta B_l(\mu))K(\mu))]^T$$
$$\times P_l [A_l + \Delta A_l(\mu) + (B_l + \Delta B_l(\mu))K(\mu))] - P_l \} z_l$$
$$= z_l^T \{ [A_l + B_l K(\mu) + (\Delta A_l(\mu) + \Delta B_l(\mu)K(\mu))]^T P_l$$
$$\times [A_l + B_l K(\mu) + (\Delta A_l(\mu) + \Delta B_l(\mu)K(\mu))] - P_l \} z_l.$$

It follows by Lemma A.1 and (A.14), that

$$\Delta V_l \le z_l^T \left[A_{0l}^T P_l A_{0l} - P_l + A_{0l}^T P_l \left(\frac{1}{\varepsilon_l} I - P_l \right)^{-1} P_l A_{0l} + \frac{1}{\varepsilon_l} E_{0l}^T E_{0l} \right] z_l$$
$$< 0,$$
$$\frac{1}{\varepsilon_l} I - P_l > 0, \quad l = 1, 2, \ldots, m.$$

It follows by Theorem 3.1 that the fuzzy system (4) is quadratically stabilizable.

Conversely, if the fuzzy system (4) is quadratically stabilizable, let $\mu = \mu_l^*$, $l = 1, 2, \ldots, m$, in the m subsystems (7). Then the m extreme subsystems in (9) are quadratically stabilizable. $\square$

Proof of Theorem 3.3. Let $A_0 = A_l + B_l K(\mu)$ and $E = E_{l1} + E_{l2} K(\mu)$. Then the claimed result follows immediately from Lemma A.3. $\square$

Proof of Theorem 4.1. By results from linear system theory (Kailath, 1980), it follows that the control law (20) can stabilize the m extreme subsystems.

(i) If accurate upper bounds are used in the extreme subsystems then it follows from Theorem 3.2 that the fuzzy system (4) is quadratically stabilizable.

(ii) If approximate upper bounds are used in the extreme subsystems then it follows by Theorem 3.3 that the control law (20) can stabilize the m extreme subsystems. Then, by Theorem 3.2, the fuzzy system (4) is quadratically stabilizable. $\square$

Proof of Theorem 4.2. This is similar to that of Theorem 4.1.

Appendix C

The Lyapunov method for stability analysis

Definition: Lyapunov function

A function V is a Lyapunov function if

(a) $V(0) = 0$ $V(x) > 0$ for $X \neq 0$

(b) V is continuous

(c) The time derivative of V is not positive, that is $\dot{V} \leq 0$

For a dynamic system, the existence of a Lyapunov function V with $\dot{V} = 0$ for the origin and $\dot{V} < 0$ for the rest of the state-space is a sufficient condition for the origin to be asymptotically stable. For a linear system $\dot{X} = AX$, the condition above becomes necessary and sufficient (Curtain, 1977).

Consider the Lyapunov function

$V = X^T P X$

where P is a symmetric positive definite matrix

The time derivative of V is

$\dot{V} = \dot{X}^T P X + X^T P \dot{X}$

$= (AX)^T P X + X^T P (AX)$

$= X^T (A^T P + PA) X$

Which is negative if

$A^T P + PA < 0$

Hence, the condition $A^T P + PA < 0$ becomes the stability criterion for a linear dynamic system.

The stability criterion above is developed based on the Lyapunov method. The basic idea of Lyapunov method is to search for a positive definite function of the state space whose time derivative is negative definite.

For a fuzzy system, the time derivative of the above Lyapunov function is

$$\dot{V} = \sum_{i=1}^{p} X^T \left(A_i^T P + PA_i \right) X$$

Therefore, if there exists a positive definite matrix that satisfies:

$$A_i^T P + PA_i < 0$$

for all $i \in \{1 \cdots p\}$, the origin of the fuzzy system is asymptotically stable.

Appendix D

Model-Based Fuzzy Control and Bilinear Matrix

Inequality

The model based fuzzy control, was developed in the 1990s. It can be regarded as "a middle ground between conventional fuzzy control practice and established control theory". It preserved the philosophy of fuzzy sets theory, meanwhile, the idea of feedback control theories are implemented to improve the fuzzy controller design. Its design procedure is as follow:

- First, a first order T-S fuzzy model is constructed for the plant by linearising local dynamics in different state space regions.

- Second, for each local linear model, a linear feedback controller is designed. And the overall controller, which is nonlinear in general, is constructed as "a fuzzy blending of each individual controller".

- Third, the overall stability for the entire fuzzy system is evaluated via Liapunov's direct method.

Expert knowledge is applied to construct a T-S fuzzy model for the target process in the first step of fuzzy controller design. In the second step, the control theory is applied to design the local feedback controllers. In the third step, because of the fundamental difference between the linear and nonlinear systems in the local and global stability, the linear controllers, which are designed in the second step, need to be evaluated. Lyapunov method is applied in the evaluation.

It should be noted that the T-S fuzzy system is a nonlinear system in general. Therefore, even all its sub-systems were stable, the global stability can not be guaranteed. This fact makes the third step of the MBFC the most important step in the model-based fuzzy control. If the global stability can not be satisfied, local nonlinear controllers need to be redesigned until global requirement is achieved.

Considerable amount of papers have been published based on this idea. According to our survey, there are more than six hundred papers published in the journals or conferences of IEEE and ELSEVIER about this issue since year 2000. Most of the publications, assume the dynamics of a fuzzy system is expressed as

$$\dot{X} = \sum_{i=1}^{p} h_i(X)(A_i X + B_i u) \tag{D-1}$$

where $X \in R^n, u \in R^m, A_i \in R^{n \times n}, B_i \in R^{n \times m}$ $i \in \{1 \cdots p\}$

and $h_i(X) = \dfrac{u_i(X)}{\sum_j u_j(X)}$

And the stability checking criterion is developed for Equation (1) as:

Lemma 1

If there exists a positive definite matrix that satisfies:

$$A_i^T P + P A_i < 0 \tag{D-2}$$

for all $i \in \{1 \cdots p\}$, the fuzzy system described in Equation (D-1) is asymptotically stable.

The inequality (D-2) can be solved efficiently by convex optimization approach , and was considered as a basic mathematical principle behind MBFC.

Despite the popularity of the assumption, Equation (D-1) only reflects an extremely limited case for the dynamics of general non-linear system. This is because the assumption implicitly assumes that the linearised state space regions for all subsystems share a same equilibrium point, which is almost impossible for practical nonlinear system. It can be demonstrated as follows.

For a nonlinear system denoted by

$$f(\dot{X}, X, u) = 0 \tag{D-3}$$

A T-S fuzzy model is derived by local linearisation in different state space regions, which can be written as

$$Q_i\dot{X} + R_iX + S_iu + E_i = 0, \qquad X \in M_i \text{ and } i \in \{1 \cdots p\} \tag{D-4}$$

M_i denote the support for the ith rule, p denotes the number of rules in fuzzy model.

If the Q_i are not singular, the state space description for (D-4) can be written as:

$$\dot{X} = \sum_{i=1}^{p} h_i(X)(A_iX + B_iu + D_i) \qquad i \in \{1 \cdots p\} \tag{D-5}$$

where $h_i(X) = \dfrac{u_i(X)}{\sum_j u_j(X)}, X \in R^n, u \in R^m, A_i \in R^{n \times n}, B_i \in R^{n \times m}, D_i \in R^n$

It is easy to verify that unless all subsystems share a same equilibrium point, there is no coordination transform that can transform Equation (D-5), which is for a general

nonlinear system, into the form of Equation (D-1). Because of the difference in the equilibrium points among the sub-systems, the stability of the fuzzy system can not be guaranteed by lemma 1.

From the descriptions above, it is important to note that the highly anticipated modern MBFC approach, which is based on Lemma 1, is still not suitable for the tasks of control nonlinear systems. As it has been mentioned, the principle difference between linear and nonlinear systems is the local and global properties. A Linear system has unified stability properties. This means that for a linear dynamic system, $\dot{X} = AX + Bu$, only the transit matrix A contributes to the stability of the system. The geometrical position of the membership function, does not affect the stability of the system. If lemma 1 were satisfied for a local region, the global stability in the entire state space can be guaranteed. It is the reason that most of modern MBFC literatures do not take the partition of the membership functions into account in the controller design. From this point of view, it is possible to view the modern MBFC a generalisation of existing linear system design techniques. In order to apply the fuzzy techniques to the field of nonlinear control, nonlinear system design techniques need to be developed. The limitation of MBFC has been realised recently by some researchers. However, since those approaches are still based on the linear system design techniques, they can only provide very conservative results and can not satisfy the requirements of nonlinear fuzzy control.

The latest development in the MEC of Cardiff University proves that the stability problem of MBFC is equal to a feasible problem of the following bilinear matrix inequalities:

$$F(y,z) = \sum_i y_i F_{i,0} + \sum_{i=1}^{r} \sum_{j=1}^{n} y_i z_j F_{i,j} < 0 \qquad \text{(D-6)}$$

$$L_0 + \sum_{i=1}^{m} y_i L_i < 0 \qquad \text{(D-7)}$$

$$Z \in \overline{M} \qquad \text{(D-8)}$$

Where $F_{i,j}$, L_i are constants derived from the parameters of T-S fuzzy model and

$M = [m_1 \cdots m_n]^T$ is a hyper-rectangle satisfies $m_i \in [0,1]$.

Consider the affine fuzzy system (D-5), which is derived from local linearisation, the geometrical constraint of the ith rule is denoted by $X \in M_i$, where M_i is a hyper-rectangle express the support of the corresponding membership function.

For simplicity, the hyper rectangles are mapped into $\overline{M} = [\overline{m}_1, \cdots, \overline{m}_n]$ where $\overline{m}_i \in [0,1]$ by the following transform:

Let $Z = [z_1 \cdots z_n]^T$ and $Z \in \overline{M}$;

$$G_i = \begin{bmatrix} g_{i,1} & & 0 \\ & \ddots & \\ 0 & & g_{i,n} \end{bmatrix}$$

And

$$H_i = \begin{bmatrix} h_{i,1} & & 0 \\ & \ddots & \\ 0 & & h_{i,n} \end{bmatrix}$$

be diagonal matrix for the ith rule.

It is obvious that, for each $i \in \{1 \cdots p\}$ and $j \in \{1 \cdots n\}$, the transform can be constructed in the following form:

$$x_j = \left. h_{i,j} \middle/ \left(z_j + g_{i,j} \right) \right.$$

If the equilibrium point of ith sub-system is denoted by: $Z_{i0} = [z_{i0,1} \cdots z_{i0,n}]^T$

where $A_i Z_{i0} + D_i = 0$

And it is written in the diagonal form as:

$$\overline{Z}_{i0} = \begin{bmatrix} z_{i0,1} & & 0 \\ & \ddots & \\ 0 & & z_{i0,n} \end{bmatrix}$$

It has the following theorem

Theorem D-1

If there exists a positive definite matrix $P > 0$ that satisfies

$$\overline{A}_i^{\mathrm{T}} P + P\overline{A}_i + Z^T B_i^{\mathrm{T}} P + PB_i Z < 0 \quad and \quad Z \in \overline{M} \tag{D-9}$$

for all $i \in \{1 \cdots p\}$, the affine fuzzy system described in (D-5) is asymptotically stable.

Where $\overline{A}_i = A_i (I - G_i \overline{Z}_{i0} H_i^{-1})$, $B_i = -A_i \overline{Z}_{i0} H_i^{-1}$

Proof

The dynamics of the affine fuzzy system (D-5) can be written as:

$$\dot{X} = A(X)X + D(X) \tag{D-10}$$

where $A(X) = \sum_{i=1}^{p} h_i(X)A_i$ and $D(X) = \sum_{i=1}^{p} h_i(X)D_i$

Given a positive definite matrix P, a Lyapunov function can be constructed as

$$V = <X, PX>$$

where $<,>$ denotes a inner product.

Hence:

$$\dot{V} = <\dot{X}, PX> + <X, P\dot{X}>$$

$$= <A(X)X + D(X), PX> + <X, P(A(X)X + D(X))>$$

$$= \sum_{i=1}^{p} h_i(X)[<A_i X + D_i, PX> + <X, P(A_i X + D_i)>]$$

$$= \sum_{i=1}^{p} h_i(X)[<A_i(X - Z_{i0}), PX> + <X, P(A_i(X - Z_{i0}))>] \tag{D-11}$$

Substitute $\overline{A}_i = A_i(I - G_i \overline{Z}_{i0} H_i^{-1})$, $B_i = -A_i \overline{Z}_{i0} H_i^{-1}$, $Z = [z_1 \cdots z_n]^T$ and $Z \in \overline{M}$ into

equation(D-13), it becomes

$$\dot{V} = \sum_{i=1}^{p} h_i(X)\left[<X, \left(A_i^{\mathrm{T}} P + PA_i + Z^T B_i^{\mathrm{T}} P + PB_i Z\right)X>\right]$$

So, a sufficient condition for $\dot{V} < 0$ is:

$$A_i^{\mathrm{T}} P + PA_i + Z^T B_i^{\mathrm{T}} P + PB_i Z < 0 \quad \text{and} \quad Z \in \overline{M}$$

Therefore, Lyapunov function: $V \geq 0$ and $\dot{V} < 0$ satisfies for all $X \in R^n$ except

origin, the affine fuzzy system described in (D-5) is asymptotically stable

End of proof.

Immediately after theorem D-1, it has the following propositions:

Proposition D-1

The matrix inequality (D-9) can be written as the following form:

$$F^k(y,z) = \sum_i^r y_i F_{i,0}^k + \sum_{i=1}^r \sum_{j=1}^n y_i z_j F_{i,j}^k < 0 \qquad \text{(D-12)}$$

$$k \in \{1 \cdots p\}$$

$$L_0 + \sum_{i=1}^m y_i L_i < 0 \qquad \text{(D-13)}$$

$$Z \in \overline{M} \qquad \text{(D-14)}$$

Remarks:

- (D-12) is a group of bilinear matrix inequalities, each of them corresponding to a constrain derived from one fuzzy rule.

- (D-13) is come from the constrain of the positive definite matrix $P > 0$, y_i denotes the elements in P.

- Since P is symmetrical, $r = \dfrac{n(n+1)}{2}$.

Proposition D-2

Bilinear matrix inequalities (14) can be expressed by the following inequality

$$F(y,z) = \sum_i^r y_i F_{i,0} + \sum_{i=1}^r \sum_{j=1}^n y_i z_j F_{i,j} < 0 \qquad \text{(D-15)}$$

Where $F_{i,j} = \begin{bmatrix} F_{i,j}^1 & & & & 0 \\ & \ddots & & & \\ & & F_{i,j}^k & & \\ & & & \ddots & \\ 0 & & & & F_{i,j}^p \end{bmatrix}$

Therefore, a nonlinear fuzzy control approach can be developed based on non-convex optimisation techniques, e.g. the bilinear matrix inequalities (BMIs) technique. The BMIs technique is a non-convex optimisation technique extended from LMIs techniques. It was proposed by Goh and Safonov in 1995. The feasible problem described by inequalities (D-6)-(D-8) can be solved by Branch and Cut algorithm.

Appendix E

Fuzzy Transform example Source Code

2 Rules 1 Dimension 0 order

```
maximum=200;
ratio=0.01;

for i=1:1:maximum
    t(i)=(i-1)*ratio;
    y(i)=(sin(pi*(sqrt(t(i))*t(i)))*(sqrt(t(i)))));
end
plot (t,y)
hold

a11=quad(inline('(1-0.5*x).*(1-0.5*x)'),0,2,0.01);
a12=quad(inline('(1-0.5*x).*(0.5*x)'),0,2,0.01);
a22=quad(inline('(0.5*x).*(0.5*x)'),0,2,0.01);

A=[a11 a12
   a12 a22]

b1=quad(inline('(1-0.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,2,0.01);
b2=quad(inline('(0.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,2,0.01);

b=[b1,b2]';

W=A\b

F=inline('(w1*(1-0.5*x)+w2*(0.5*x))')

for i=1:1:maximum
    t(i)=(i-1)*ratio;
    y1(i)=feval(F,W(1),W(2),t(i));
end

plot(t,y1)
```

2 Rules 1 Dimension 1st order

```
maximum=200;
ratio=0.01;

accuracy=0.0005;
```

```
for i=1:1:maximum
   t(i)=(i-1)*ratio;
   y(i)=(sin(pi*(sqrt(t(i))*t(i)))*(sqrt(t(i))));
end
plot (t,y)
hold


a11=quad(inline('(x.*(1-0.5*x)).*(x.*(1-0.5*x))'),0,2,accuracy);
a12=quad(inline('(x.*(1-0.5*x)).*(0.5*x.*x)'),0,2,accuracy);
a22=quad(inline('(0.5*x.*x).*(0.5*x.*x)'),0,2,accuracy);
A1=[a11 a12
   a12 a22 ]

a11=quad(inline('(x.*(1-0.5*x)).*((1-0.5*x))'),0,2,accuracy);
a12=quad(inline('(x.*(1-0.5*x)).*(0.5*x)'),0,2,accuracy);
a22=quad(inline('(0.5*x.*x).*(0.5*x)'),0,2,accuracy);
A2=[a11 a12
   a12 a22 ]

a11=quad(inline('((1-0.5*x)).*((1-0.5*x))'),0,2,accuracy);
a12=quad(inline('((1-0.5*x)).*(0.5*x)'),0,2,accuracy);
a22=quad(inline('(0.5*x).*(0.5*x)'),0,2,accuracy);
A3=[a11 a12
   a12 a22 ]



A=[A1 A2
   A2' A3]

b1=quad(inline('(x.*(1-0.5*x)).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,2,accuracy);
b2=quad(inline('(0.5*x.*(x)).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,2,accuracy);
b3=quad(inline('(1-0.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,2,accuracy);
b4=quad(inline('(0.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,2,accuracy);

b=[b1,b2,b3,b4,]';

W=A\b

F=inline('((p1*x+q1)*(1-0.5*x)+(p2*x+q2)*(0.5*x))')

p1=W(1);
p2=W(2);
q1=W(3);
q2=W(4);

maximum=200;
ratio=0.01;
```

```
for i=1:1:maximum
   t(i)=(i-1)*ratio;
   y1(i)=feval(F,p1,p2,q1,q2,t(i));
end


plot(t,y1)
```

2 Rules 1 Dimension 0 order

```
maximum=200;
ratio=0.01;

for i=1:1:maximum
   t(i)=(i-1)*ratio;
   y(i)=(sin(pi*(sqrt(t(i))*t(i)))*(sqrt(t(i))));
end
plot (t,y)
%hold


a11=quad(inline('(1-x).*(1-x)'),0,1,0.01);
a12=quad(inline('(1-x).*(x)'),0,1,0.01);
a13=0
a22=quad(inline('(x).*(x)'),0,1,0.01)+quad(inline('(2-x).*(2-x)'),1,2,0.01);
a23=quad(inline('(2-x).*(x-1)'),1,2,0.01);
a33=quad(inline('(x-1).*(x-1)'),1,2,0.01);


A=[a11 a12 a13
   a12 a22 a23
   a13 a23 a33]

b1=quad(inline('(1-x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,1,0.01);
b2=quad(inline('(x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,1,0.01)+quad(inline('(2-
x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),1,2,0.01);
b3=quad(inline('(x-1).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),1,2,0.01);

b=[b1,b2,b3]';

W=A\b

F1=inline('(w1*(1-x)+w2*(x))')

F2=inline('(w1*(2-x)+w2*(x-1))')

w1=W(1);
w2=W(2);
```

```
maximum=200;
ratio=0.01;

for i=1:1:maximum/2
    t(i)=(i-1)*ratio;
    y1(i)=feval(F1,w1,w2,t(i));
end

w1=W(2);
w2=W(3);

for i=maximum/2+1:1:maximum
    t(i)=(i-1)*ratio;
    y1(i)=feval(F2,w1,w2,t(i));
end

plot(t,y1)
```

2 Rules 1 Dimension 1st order

```
maximum=200;
ratio=0.01;

accuracy=0.1;

for i=1:1:maximum
    t(i)=(i-1)*ratio;
    y(i)=(sin(pi*(sqrt(t(i))*t(i)))*(sqrt(t(i))));
end
plot (t,y)
%hold


a11=quad(inline('(x.*(1-x)).*(x.*(1-x))'),0,1,accuracy);
a12=quad(inline('(x.*(1-x)).*(x.*x)'),0,1,accuracy);
a13=0
a22=quad(inline('(x.*x).*(x.*x)'),0,1,0.01)+quad(inline('(x.*(2-x)).*(x.*(2-
x))'),1,2,accuracy);
a23=quad(inline('(x.*(2-x)).*(x.*(x-1))'),1,2,accuracy);
a33=quad(inline('(x.*(x-1)).*(x.*(x-1))'),1,2,accuracy);


A1=[a11 a12 a13
    a12 a22 a23
    a13 a23 a33]

a11=quad(inline('(x.*(1-x)).*((1-x))'),0,1,accuracy);
```

```
a12=quad(inline('(x.*(1-x)).*(x)'),0,1,accuracy);
a13=0
a22=quad(inline('(x.*x).*(x)'),0,1,0.01)+quad(inline('(x.*(2-x)).*((2-x))'),1,2,accuracy);
a23=quad(inline('(x.*(2-x)).*((x-1))'),1,2,accuracy);
a33=quad(inline('(x.*(x-1)).*((x-1))'),1,2,accuracy);


A2=[a11 a12 a13
    a12 a22 a23
    a13 a23 a33]


a11=quad(inline('(1-x).*(1-x)'),0,1,accuracy);
a12=quad(inline('(1-x).*(x)'),0,1,accuracy);
a13=0
a22=quad(inline('(x).*(x)'),0,1,0.01)+quad(inline('(2-x).*(2-x)'),1,2,accuracy);
a23=quad(inline('(2-x).*(x-1)'),1,2,accuracy);
a33=quad(inline('(x-1).*(x-1)'),1,2,accuracy);


A3=[a11 a12 a13
    a12 a22 a23
    a13 a23 a33]


A=[A1 A2
   A2' A3]

b1=quad(inline('(x.*(1-x)).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,1,accuracy);
b2=quad(inline('(x.*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,1,accuracy)+quad(inline('(x.*(2-x)).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),1,2,accuracy);
b3=quad(inline('(x.*(x-1)).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),1,2,accuracy);
b4=quad(inline('(1-x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,1,accuracy);
b5=quad(inline('(x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,1,accuracy)+quad(inline('(2-x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),1,2,accuracy);
b6=quad(inline('(x-1).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),1,2,accuracy);

b=[b1,b2,b3,b4,b5,b6]';

W=A\b

F1=inline('((p1*x+q1)*(1-x)+(p2*x+q2)*(x))')

F2=inline('((p1*x+q1)*(2-x)+(p2*x+q2)*(x-1))')

p1=W(1);
p2=W(2);
q1=W(4);
q2=W(5);
```

```
maximum=200;
ratio=0.01;

for i=1:1:maximum/2
    t(i)=(i-1)*ratio;
    y1(i)=feval(F1,p1,p2,q1,q2,t(i));
end

p1=W(2);
p2=W(3);
q1=W(4);
q2=W(5);

for i=maximum/2+1:1:maximum
    t(i)=(i-1)*ratio;
    y1(i)=feval(F2,p1,p2,q1,q2,t(i));
end

plot(t,y1)
```

4 Rules 2 Dimension 0 order

```
clear
maxnum=51;
width=10;
ratio=maxnum/width;
accuracy=0.0001

for i=1:1:maxnum
    t1(i)=i/ratio-5;
    for j=1:1:maxnum
        t2(j)=j/ratio-5;
        x(i,j)=((t1(i)*t1(i)-t2(j)*t2(j))*sin(t1(i))*sqrt(abs(t2(i))));
    end
end
surf(t1,t2,x)

hold on

%f=inline('((x.*x-y.*y).*sin(0.5*x)*sqrt(abs(y)))');
%A1=inline('(0.5-0.1*x)')  A2=inline('(0.5+0.1*x)');
%B1=inline('(0.5-0.1*y)')  B2=inline('(0.5+0.1*y)'); (0.5-0.1*x).*(0.5-0.1*y)*).*

%AB11=inline('((0.5-0.1*x).*(0.5-0.1*y))')
%AB12=inline('((0.5-0.1*x).*(0.5+0.1*y))')
%AB21=inline('((0.5+0.1*x).*(0.5-0.1*y))')
%AB22=inline('((0.5+0.1*x).*(0.5+0.1*y))')
```

```
f=inline('((((0.5-0.1*x).*(0.5-0.1*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b1=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((((0.5-0.1*x).*(0.5+0.1*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b2=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((((0.5+0.1*x).*(0.5-0.1*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b3=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((((0.5+0.1*x).*(0.5+0.1*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b4=dblquad(f,-5,5,-5,5,accuracy);
b=[b1,b2,b3,b4]'

f=inline('((0.5-0.1*x).*(0.5-0.1*y)).*((0.5-0.1*x).*(0.5-0.1*y))');
a11=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((0.5-0.1*x).*(0.5-0.1*y)).*((0.5-0.1*x).*(0.5+0.1*y))');
a12=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((0.5-0.1*x).*(0.5-0.1*y)).*((0.5+0.1*x).*(0.5-0.1*y))');
a13=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((0.5-0.1*x).*(0.5-0.1*y)).*((0.5+0.1*x).*(0.5+0.1*y))');
a14=dblquad(f,-5,5,-5,5,accuracy);

f=inline('((0.5-0.1*x).*(0.5+0.1*y)).*((0.5-0.1*x).*(0.5+0.1*y))');
a22=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((0.5-0.1*x).*(0.5+0.1*y)).*((0.5+0.1*x).*(0.5-0.1*y))');
a23=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((0.5-0.1*x).*(0.5+0.1*y)).*((0.5+0.1*x).*(0.5+0.1*y))');
a24=dblquad(f,-5,5,-5,5,accuracy);

f=inline('((0.5+0.1*x).*(0.5-0.1*y)).*((0.5+0.1*x).*(0.5-0.1*y))');
a33=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((0.5+0.1*x).*(0.5-0.1*y)).*((0.5+0.1*x).*(0.5+0.1*y))');
a34=dblquad(f,-5,5,-5,5,accuracy);

f=inline('((0.5+0.1*x).*(0.5+0.1*y)).*((0.5+0.1*x).*(0.5+0.1*y))');
a44=dblquad(f,-5,5,-5,5,accuracy);


A=[a11 a12 a13 a14
   a12 a22 a23 a24
   a13 a23 a33 a34
   a14 a24 a34 a44]



W=A\b

f=inline('w1*((0.5-0.1*x).*(0.5-0.1*y))+w2*((0.5-
0.1*x).*(0.5+0.1*y))+w3*((0.5+0.1*x).*(0.5-
0.1*y))+w4*((0.5+0.1*x).*(0.5+0.1*y))');

w1=W(1);
```

```
w2=W(2);
w3=W(3);
w4=W(4);

for i=1:1:maxnum
   t1(i)=i/ratio-5;
   for j=1:1:maxnum
      t2(j)=j/ratio-5;
      x1(i,j)=feval(f,w1,w2,w3,w4,t1(i),t2(j));
   end
end
surf(t1,t2,x1)

hold off
```

4 Rules 1 Dimension 1st order

```
clear
maxnum=51;
width=10;
ratio=maxnum/width;
accuracy=0.0001

for i=1:1:maxnum
   t1(i)=i/ratio-5;
   for j=1:1:maxnum
      t2(j)=j/ratio-5;
      x(i,j)=((t1(i)*t1(i)-t2(j)*t2(j))*sin(t1(i))*sqrt(abs(t2(i))));
   end
end
surf(t1,t2,x)

hold on

%f=inline('((x.*x-y.*y).*sin(0.5*x)*sqrt(abs(y)))');
%A1=inline('(0.5-0.1*x)')  A2=inline('(0.5+0.1*x)');
%B1=inline('(0.5-0.1*y)')  B2=inline('(0.5+0.1*y)'); (0.5-0.1*x).*(0.5-0.1*y)*).*

%AB11=inline('((0.5-0.1*x).*(0.5-0.1*y))')
%AB12=inline('((0.5-0.1*x).*(0.5+0.1*y))')
%AB21=inline('((0.5+0.1*x).*(0.5-0.1*y))')
%AB22=inline('((0.5+0.1*x).*(0.5+0.1*y))')


f=inline('(((0.5-0.1*x).*(0.5-0.1*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b1=dblquad(f,-5,5,-5,5,accuracy);
f=inline('(((0.5-0.1*x).*(0.5+0.1*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b2=dblquad(f,-5,5,-5,5,accuracy);
f=inline('(((0.5+0.1*x).*(0.5-0.1*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b3=dblquad(f,-5,5,-5,5,accuracy);
```

```
f=inline('((((0.5+0.1*x).*(0.5+0.1*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y)))))');
b4=dblquad(f,-5,5,-5,5,accuracy);
b=[b1,b2,b3,b4]'

f=inline('(((0.5-0.1*x).*(0.5-0.1*y)).*((0.5-0.1*x).*(0.5-0.1*y)))');
a11=dblquad(f,-5,5,-5,5,accuracy);
f=inline('(((0.5-0.1*x).*(0.5-0.1*y)).*((0.5-0.1*x).*(0.5+0.1*y)))');
a12=dblquad(f,-5,5,-5,5,accuracy);
f=inline('(((0.5-0.1*x).*(0.5-0.1*y)).*((0.5+0.1*x).*(0.5-0.1*y)))');
a13=dblquad(f,-5,5,-5,5,accuracy);
f=inline('(((0.5-0.1*x).*(0.5-0.1*y)).*((0.5+0.1*x).*(0.5+0.1*y)))');
a14=dblquad(f,-5,5,-5,5,accuracy);

f=inline('(((0.5-0.1*x).*(0.5+0.1*y)).*((0.5-0.1*x).*(0.5+0.1*y)))');
a22=dblquad(f,-5,5,-5,5,accuracy);
f=inline('(((0.5-0.1*x).*(0.5+0.1*y)).*((0.5+0.1*x).*(0.5-0.1*y)))');
a23=dblquad(f,-5,5,-5,5,accuracy);
f=inline('(((0.5-0.1*x).*(0.5+0.1*y)).*((0.5+0.1*x).*(0.5+0.1*y)))');
a24=dblquad(f,-5,5,-5,5,accuracy);

f=inline('(((0.5+0.1*x).*(0.5-0.1*y)).*((0.5+0.1*x).*(0.5-0.1*y)))');
a33=dblquad(f,-5,5,-5,5,accuracy);
f=inline('(((0.5+0.1*x).*(0.5-0.1*y)).*((0.5+0.1*x).*(0.5+0.1*y)))');
a34=dblquad(f,-5,5,-5,5,accuracy);

f=inline('(((0.5+0.1*x).*(0.5+0.1*y)).*((0.5+0.1*x).*(0.5+0.1*y)))');
a44=dblquad(f,-5,5,-5,5,accuracy);


A=[a11 a12 a13 a14
   a12 a22 a23 a24
   a13 a23 a33 a34
   a14 a24 a34 a44]


W=A\b

f=inline('w1*((0.5-0.1*x).*(0.5-0.1*y))+w2*((0.5-
0.1*x).*(0.5+0.1*y))+w3*((0.5+0.1*x).*(0.5-
0.1*y))+w4*((0.5+0.1*x).*(0.5+0.1*y))');

w1=W(1);
w2=W(2);
w3=W(3);
w4=W(4);

for i=1:1:maxnum
    t1(i)=i/ratio-5;
    for j=1:1:maxnum
```

```
            t2(j)=j/ratio-5;
            x1(i,j)=feval(f,w1,w2,w3,w4,t1(i),t2(j));
    end
end
surf(t1,t2,x1)

hold off
```

6 Rules 1 Dimension 0 order

```
maximum=200;
ratio=1/100;

for i=1:1:maximum
    t(i)=(i-1)*ratio;
    y(i)=(sin(pi*(sqrt(t(i))*t(i)))*(sqrt(t(i))));
end
plot (t,y)
%hold
```

```
a11=quad(inline('(1-2.5*x).*(1-2.5*x)'),0,2/5,0.01);
a12=quad(inline('(1-2.5*x).*(2.5*x)'),0,2/5,0.01);
a22=quad(inline('(2.5*x).*(2.5*x)'),0,2/5,0.01)+quad(inline('(2-2.5*x).*(2-
2.5*x)'),2/5,4/5,0.01);
```

```
A=[a11 a12 0   0   0   0
   a12 a22 a12 0   0   0
   0   a12 a22 a12 0   0
   0   0   a12 a22 a12 0
   0   0   0   a12 a22 a12
   0   0   0   0   a12 a11]
```

```
b1=quad(inline('(1-2.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,2/5,0.01);
b2=quad(inline('(2.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,2/5,0.01)+quad(inline('(2-
2.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),2/5,4/5,0.01);
b3=quad(inline('(2.5*x-
1).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),2/5,4/5,0.01)+quad(inline('(3-
2.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),4/5,6/5,0.01);
b4=quad(inline('(2.5*x-
2).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),4/5,6/5,0.01)+quad(inline('(4-
2.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),6/5,8/5,0.01);
b5=quad(inline('(2.5*x-
3).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),6/5,8/5,0.01)+quad(inline('(5-
2.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),8/5,2,0.01);
b6=quad(inline('(2.5*x-4).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),8/5, 2, 0.01);
```

```
b=[b1,b2,b3,b4,b5,b6]';
```

```
W=A\b

F1=inline('(w1*(1-2.5*x)+w2*(2.5*x))')
F2=inline('(w1*(2-2.5*x)+w2*(2.5*x-1))')
F3=inline('(w1*(3-2.5*x)+w2*(2.5*x-2))')
F4=inline('(w1*(4-2.5*x)+w2*(2.5*x-3))')
F5=inline('(w1*(5-2.5*x)+w2*(2.5*x-4))')




step=maximum/5

w1=W(1);
w2=W(2);
for i=1:1:step
    t(i)=(i-1)*ratio;
    y1(i)=feval(F1,w1,w2,t(i));
end

w1=W(2);
w2=W(3);
for i=step+1:1:2*step
    t(i)=(i-1)*ratio;
    y1(i)=feval(F2,w1,w2,t(i));
end

w1=W(3);
w2=W(4);
for i=2*step+1:1:3*step
    t(i)=(i-1)*ratio;
    y1(i)=feval(F3,w1,w2,t(i));
end
w1=W(4);
w2=W(5);

for i=3*step+1:1:4*step
    t(i)=(i-1)*ratio;
    y1(i)=feval(F4,w1,w2,t(i));
end
w1=W(5);
w2=W(6);

for i=4*step+1:1:5*step
    t(i)=(i-1)*ratio;
    y1(i)=feval(F5,w1,w2,t(i));
end

plot(t,y1)
```

9 Rules 2 Dimension 0 order

```
clear
maxnum=51;
width=10;
ratio=maxnum/width;
accuracy=0.0001

for i=1:1:maxnum
    t1(i)=i/ratio-5;
    for j=1:1:maxnum
        t2(j)=j/ratio-5;
        x(i,j)=((t1(i)*t1(i)-t2(j)*t2(j))*sin(t1(i))*sqrt(abs(t2(i))));
    end
end
surf(t1,t2,x)

hold on

%f=inline('((x.*x-y.*y).*sin(0.5*x)*sqrt(abs(y)))');
%A1=inline('(-0.2*x)')  A2=inline('(1+0.2*x)')  A2=inline('(1-0.2*x)')
A3=inline('(0.2*x)') ;
%A1=inline('(-0.2*y)')  A2=inline('(1+0.2*y)')  A2=inline('(1-0.2*y)')
A3=inline('(0.2*y)') ;

%AB11=inline('((-0.2*x).*(-0.2*y))')
%AB12=inline('((-0.2*x).*(1+0.2*y))') +  inline('((-0.2*x).*(1-0.2*y))')
%AB13=inline('((-0.2*x).*(0.2*y))')

%AB21=inline('((1+0.2*x).*(-0.2*y))') + inline('((1-0.2*x).*(-0.2*y))')
%AB22=inline('((1+0.2*x).*(1+0.2*y))') + inline('((1+0.2*x).*(1-0.2*y))') +
inline('((1-0.2*x).*(1+0.2*y))') + inline('((1-0.2*x).*(1-0.2*y))')
%AB23=inline('((1+0.2*x).*(0.2*y))') + inline('((1-0.2*x).*(0.2*y))')

%AB31=inline('((0.2*x).*(-0.2*y))')
%AB32=inline('((0.2*x).*(1+0.2*y))') +  inline('((-0.2*x).*(1-0.2*y))')
%AB33=inline('((0.2*x).*(0.2*y))')

f=inline('(((-0.2*x).*(-0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b1=dblquad(f,-5,0,-5,0,accuracy);
f=inline('(((-0.2*x).*(1+0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b2=dblquad(f,-5,0,-5,0,accuracy);
f=inline('(((-0.2*x).*(1-0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b2=b2+dblquad(f,-5,0,0,5,accuracy);
f=inline('(((-0.2*x).*(0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b3=dblquad(f,-5,0,0,5,accuracy);

%AB21=inline('((1+0.2*x).*(-0.2*y))') + inline('((1-0.2*x).*(-0.2*y))')
f=inline('(((1+0.2*x).*(-0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b4=dblquad(f,-5,0,-5,0,accuracy);
f=inline('(((1-0.2*x).*(-0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
```

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```
b4=b4+dblquad(f,0,5,-5,0,accuracy);

%AB22=inline('((1+0.2*x).*(1+0.2*y))') + inline('((1+0.2*x).*(1-0.2*y))') +
inline('((1-0.2*x).*(1+0.2*y))') + inline('((1-0.2*x).*(1-0.2*y))')
f=inline('(((1+0.2*x).*(1+0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b5=dblquad(f,-5,0,-5,0,accuracy);
f=inline('(((1+0.2*x).*(1-0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b5=b5+dblquad(f,-5,0,0,5,accuracy);
f=inline('(((1-0.2*x).*(1+0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b5=b5+dblquad(f,0,5,-5,0,accuracy);
f=inline('(((1-0.2*x).*(1-0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b5=b5+dblquad(f,0,5,0,5,accuracy);

%AB23=inline('((1+0.2*x).*(0.2*y))') + inline('((1-0.2*x).*(0.2*y))')
f=inline('(((1+0.2*x).*(0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b6=dblquad(f,-5,0,0,5,accuracy);
f=inline('(((1-0.2*x).*(0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b6=b6+dblquad(f,0,5,0,5,accuracy);

%AB31=inline('((0.2*x).*(-0.2*y))')
%AB32=inline('((0.2*x).*(1+0.2*y))') +  inline('((0.2*x).*(1-0.2*y))')
%AB33=inline('((0.2*x).*(0.2*y))')
f=inline('(((0.2*x).*(-0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b7=dblquad(f,0,5,-5,0,accuracy);
f=inline('(((0.2*x).*(1+0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b8=dblquad(f,0,5,-5,0,accuracy);
f=inline('(((0.2*x).*(1-0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b8=b8+dblquad(f,0,5,0,5,accuracy);
f=inline('(((0.2*x).*(0.2*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b9=dblquad(f,0,5,0,5,accuracy);

b=[b1,b2,b3,b4,b5,b6,b7,b8,b9]'
% %
% %
% %
% %
% % f=inline('(((-0.2*x).*(-0.2*y)).*((-0.2*x).*(-0.2*y)))');
% % a11=dblquad(f,-5,0,-5,0,accuracy);
% % f=inline('(((-0.2*x).*(-0.2*y)).*((-0.2*x).*(1+0.2*y)))');
% % a12=dblquad(f,-5,0,-5,0,accuracy);
% % a13=0
% % f=inline('(((-0.2*x).*(-0.2*y)).*((1+0.2*x).*(-0.2*y)))');
% % a14=dblquad(f,-5,0,-5,0,accuracy);
% % f=inline('(((-0.2*x).*(-0.2*y)).*((1+0.2*x).*(1+0.2*y)))');
% % a15=dblquad(f,-5,0,-5,0,accuracy);
% % a16=0
% % a17=0
% % a18=0
% % a19=0
% %
```

```
% %
% % f=inline('((((-0.2*x).*(1+0.2*y)).*((-0.2*x).*(1+0.2*y))))');
% % a22=dblquad(f,-5,0,-5,0,accuracy);
% % f=inline('((((-0.2*x).*(1-0.2*y)).*((-0.2*x).*(1-0.2*y))))');
% % a22=a22+dblquad(f,-5,0,0,5,accuracy);
% % f=inline('((((-0.2*x).*(1-0.2*y)).*((-0.2*x).*(0.2*y))))');
% % a23=dblquad(f,-5,0,0,5,accuracy);
% % f=inline('((((-0.2*x).*(1+0.2*y)).*((1+0.2*x).*(-0.2*y))))');
% % a24=dblquad(f,-5,0,-5,0,accuracy);
% % f=inline('((((-0.2*x).*(1+0.2*y)).*((1+0.2*x).*(1+0.2*y))))');
% % a25=dblquad(f,-5,0,-5,0,accuracy);
% % f=inline('((((-0.2*x).*(1-0.2*y)).*((1+0.2*x).*(1-0.2*y))))');
% % a25=a25+dblquad(f,-5,0,0,5,accuracy);
% % f=inline('((((-0.2*x).*(1-0.2*y)).*((1+0.2*x).*(0.2*y))))');
% % a26=dblquad(f,-5,0,0,5,accuracy);
% % a27=0
% % a28=0
% % a29=0
% %
% % f=inline('((((-0.2*x).*(0.2*y)).*((-0.2*x).*(0.2*y))))');
% % a33=dblquad(f,-5,0,0,5,accuracy);
% % a34=0;
% % f=inline('((((-0.2*x).*(0.2*y)).*((1+0.2*x).*(1-0.2*y))))');
% % a35=dblquad(f,-5,0,0,5,accuracy);
% % f=inline('((((-0.2*x).*(0.2*y)).*((1+0.2*x).*(0.2*y))))');
% % a36=dblquad(f,-5,0,0,5,accuracy);
% % a37=0
% % a38=0
% % a39=0
% %
% %
% %
% % f=inline('((((1+0.2*x).*(-0.2*y)).*((1+0.2*x).*(-0.2*y))))');
% % a44=dblquad(f,-5,0,-5,0,accuracy);
% % f=inline('((((1-0.2*x).*(-0.2*y)).*((1-0.2*x).*(-0.2*y))))');
% % a44=a44+dblquad(f,0,5,-5,0,accuracy);
% %
% % f=inline('((((1+0.2*x).*(-0.2*y)).*((1+0.2*x).*(1+0.2*y))))');
% % a45=dblquad(f,-5,0,-5,0,accuracy);
% % f=inline('((((1-0.2*x).*(-0.2*y)).*((1-0.2*x).*(1+0.2*y))))');
% % a45=a45+dblquad(f,0,5,-5,0,accuracy);
% %
% % a46=0
% %
% %
% % f=inline('((((1-0.2*x).*(-0.2*y)).*((0.2*x).*(-0.2*y))))');
% % a47=dblquad(f,0,5,-5,0,accuracy);
% % f=inline('((((1-0.2*x).*(-0.2*y)).*((0.2*x).*(1+0.2*y))))');
% % a48=dblquad(f,0,5,-5,0,accuracy);
% % a49=0
```

```
% %
% % %A1=inline('(-0.2*x)')  A2=inline('(1+0.2*x)')  A2=inline('(1-0.2*x)')
A3=inline('(0.2*x)') ;
% % %A1=inline('(-0.2*y)')  A2=inline('(1+0.2*y)')  A2=inline('(1-0.2*y)')
A3=inline('(0.2*y)') ;
% %
% % %1  AB11=inline('((-0.2*x).*(-0.2*y))')
% % %2  AB12=inline('((-0.2*x).*(1+0.2*y))') +  inline('((-0.2*x).*(1-0.2*y))')
% % %3  AB13=inline('((-0.2*x).*(0.2*y))')
% %
% % %4  AB21=inline('((1+0.2*x).*(-0.2*y))') + inline('((1-0.2*x).*(-0.2*y))')
% % %5  AB22=inline('((1+0.2*x).*(1+0.2*y))') + inline('((1+0.2*x).*(1-0.2*y))') +
inline('((1-0.2*x).*(1+0.2*y))') + inline('((1-0.2*x).*(1-0.2*y))')
% % %6  AB23=inline('((1+0.2*x).*(0.2*y))') + inline('((1-0.2*x).*(0.2*y))')
% %
% % %7  AB31=inline('((0.2*x).*(-0.2*y))')
% % %8  AB32=inline('((0.2*x).*(1+0.2*y))') +  inline('((-0.2*x).*(1-0.2*y))')
% % %9  AB33=inline('((0.2*x).*(0.2*y))')
% %
% % f=inline('(((1-0.2*x).*(-0.2*y)).*((0.2*x).*(-0.2*y)))');
% % a47=dblquad(f,0,5,-5,0,accuracy);
% %
% % a56
% % a57
% % a58
% % a59
%
% f=inline('(((-0.2*x).*(-0.2*y)).*((-0.2*x).*(-0.2*y)))');
% a11=dblquad(f,-5,0,-5,0,accuracy);
% c=a11;
%
% a11=c;
% a12=c/2;
% a13=0;
% a14=c/2;
% a15=c/4;
% a16=0
% a17=0;
% a18=0
% a19=0;
%
% a22=2*c;
% a23=c/2;
% a24=c/4
% a25=

f=inline('(((-0.2*x).*(-0.2*y)).*((-0.2*x).*(-0.2*y)))');
k=dblquad(f,-5,0,-5,0,accuracy);

A=[ 1   0.5   0  0.5   0.25  0    0  0    0
```

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```
    0.5 2    0.5 0.25  1    0.25  0  0    0
    0   0.5  1   0     0.25 0.5   0  0    0
    0.5 0.25 0   2     1    0     0.5 0.25 0
    0.25 1   0.25 1    4    1     0.25 1   0.25
    0   0.25 0.5  0    1    2     0  0.25 0.5
    0 0 0 0.5 0.25 0 1 0.5 0
    0 0 0 0.25 1 0.25 0.5 2 0.5
    0 0 0 0 0.25 0.5 0 0.5 1]

A=k*A

W=A\b


%f=inline('((x.*x-y.*y).*sin(0.5*x)*sqrt(abs(y)))');
%A1=inline('(-0.2*x)') A2=inline('(1+0.2*x)') A2=inline('(1-0.2*x)')
A3=inline('(0.2*x)') ;
%A1=inline('(-0.2*y)') A2=inline('(1+0.2*y)') A2=inline('(1-0.2*y)')
A3=inline('(0.2*y)') ;

%AB11=inline('((-0.2*x).*(-0.2*y))')
%AB12=inline('((-0.2*x).*(1+0.2*y))') + inline('((-0.2*x).*(1-0.2*y))')
%AB13=inline('((-0.2*x).*(0.2*y))')

%AB21=inline('((1+0.2*x).*(-0.2*y))') + inline('((1-0.2*x).*(-0.2*y))')
%AB22=inline('((1+0.2*x).*(1+0.2*y))') + inline('((1+0.2*x).*(1-0.2*y))') +
inline('((1-0.2*x).*(1+0.2*y))') + inline('((1-0.2*x).*(1-0.2*y))')
%AB23=inline('((1+0.2*x).*(0.2*y))') + inline('((1-0.2*x).*(0.2*y))')

%AB31=inline('((0.2*x).*(-0.2*y))')
%AB32=inline('((0.2*x).*(1+0.2*y))') + inline('((-0.2*x).*(1-0.2*y))')
%AB33=inline('((0.2*x).*(0.2*y))')

fA1=inline('max(-0.2*x,0)');
fA2=inline('max(min(1+0.2*x,1-0.2*x),0)');
fA3=inline('max(0.2*x,0)');


M=[W(1) W(2) W(3)
   W(4) W(5) W(6)
   W(7) W(8) W(9)];

for i=1:1:maxnum
   for j=1:1:maxnum
      x1(i,j)=[feval(fA1,t1(i)), feval(fA2,t1(i)),feval(fA3,t1(i))]*M*[feval(fA1,t2(j)),
feval(fA2,t2(j)),feval(fA3,t2(j))]';
   end
end
surf(t1,t2,x1)
```

hold off

10 Rules 1 Dimension 0 order

```
maximum=180;
ratio=1/90;

for i=1:1:maximum
   t(i)=(i-1)*ratio;
   y(i)=(sin(pi*(sqrt(t(i))*t(i)))*(sqrt(t(i))));
end
plot (t,y)
%hold
```

```
a11=quad(inline('(1-4.5*x).*(1-4.5*x)'),0,2/9,0.01);
a12=quad(inline('(1-4.5*x).*(4.5*x)'),0,2/9,0.01);
a22=quad(inline('(4.5*x).*(4.5*x)'),0,2/9,0.01)+quad(inline('(2-4.5*x).*(2-
4.5*x)'),2/9,4/9,0.01);
```

```
A=[a11 a12 0  0  0  0  0  0   0 0
   a12 a22 a12 0  0  0  0  0   0 0
   0  a12 a22 a12 0  0  0  0   0 0
   0  0  a12 a22 a12 0  0  0   0 0
   0  0  0  a12 a22 a12 0  0   0 0
   0  0  0  0  a12 a22 a12 0   0 0
   0  0  0  0  0  a12 a22 a12  0 0
   0  0  0  0  0  0  a12 a22 a12 0
   0  0  0  0  0  0  0  a12 a22 a12
   0  0  0  0  0  0  0  0  a12 a11]
```

```
b1=quad(inline('(1-4.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,2/9,0.01);
b2=quad(inline('(4.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),0,2/9,0.01)+quad(inline('(2-
4.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),2/9,4/9,0.01);
b3=quad(inline('(4.5*x-
1).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),2/9,4/9,0.01)+quad(inline('(3-
4.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),4/9,6/9,0.01);
b4=quad(inline('(4.5*x-
2).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),4/9,6/9,0.01)+quad(inline('(4-
4.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),6/9,8/9,0.01);
b5=quad(inline('(4.5*x-
3).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),6/9,8/9,0.01)+quad(inline('(5-
4.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),8/9,10/9,0.01);
b6=quad(inline('(4.5*x-
4).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),8/9,10/9,0.01)+quad(inline('(6-
4.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),10/9,12/9,0.01);
```

b7=quad(inline('(4.5*x-5).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),10/9,12/9,0.01)+quad(inline('(7-4.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),12/9,14/9,0.01);
b8=quad(inline('(4.5*x-6).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),12/9,14/9,0.01)+quad(inline('(8-4.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),14/9,16/9,0.01);
b9=quad(inline('(4.5*x-7).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),14/9,16/9,0.01)+quad(inline('(9-4.5*x).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),16/9,2,0.01);
b10=quad(inline('(4.5*x-8).*(sqrt(x).*sin(pi.*x.*sqrt(x)))'),16/9, 2, 0.01);

b=[b1,b2,b3,b4,b5,b6,b7,b8,b9,b10]';

W=A\b

F1=inline('(w1*(1-4.5*x)+w2*(4.5*x))')
F2=inline('(w1*(2-4.5*x)+w2*(4.5*x-1))')
F3=inline('(w1*(3-4.5*x)+w2*(4.5*x-2))')
F4=inline('(w1*(4-4.5*x)+w2*(4.5*x-3))')
F5=inline('(w1*(5-4.5*x)+w2*(4.5*x-4))')
F6=inline('(w1*(6-4.5*x)+w2*(4.5*x-5))')
F7=inline('(w1*(7-4.5*x)+w2*(4.5*x-6))')
F8=inline('(w1*(8-4.5*x)+w2*(4.5*x-7))')
F9=inline('(w1*(9-4.5*x)+w2*(4.5*x-8))')

step=(maximum/9)

w1=W(1);
w2=W(2);
for i=1:1:step
   t(i)=(i-1)*ratio;
   y1(i)=feval(F1,w1,w2,t(i));
end

w1=W(2);
w2=W(3);
for i=step+1:1:2*step
   t(i)=(i-1)*ratio;
   y1(i)=feval(F2,w1,w2,t(i));
end

w1=W(3);
w2=W(4);
for i=2*step+1:1:3*step
   t(i)=(i-1)*ratio;
   y1(i)=feval(F3,w1,w2,t(i));
end

```
w1=W(4);
w2=W(5);

for i=3*step+1:1:4*step
    t(i)=(i-1)*ratio;
    y1(i)=feval(F4,w1,w2,t(i));
end
w1=W(5);
w2=W(6);

for i=4*step+1:1:5*step
    t(i)=(i-1)*ratio;
    y1(i)=feval(F5,w1,w2,t(i));
end
w1=W(6);
w2=W(7);

for i=5*step+1:1:6*step
    t(i)=(i-1)*ratio;
    y1(i)=feval(F6,w1,w2,t(i));
end
w1=W(7);
w2=W(8);

for i=6*step+1:1:7*step
    t(i)=(i-1)*ratio;
    y1(i)=feval(F7,w1,w2,t(i));
end
w1=W(8);
w2=W(9);

for i=7*step+1:1:8*step
    t(i)=(i-1)*ratio;
    y1(i)=feval(F8,w1,w2,t(i));
end
w1=W(9);
w2=W(10);

for i=8*step+1:1:9*step
    t(i)=(i-1)*ratio;
    y1(i)=feval(F9,w1,w2,t(i));
end
plot(t,y1)
```

16 Rules 2 Dimension 0 order

```
clear
maxnum=51;
width=10;
```

```
ratio=maxnum/width;
accuracy=0.0001

for i=1:1:maxnum
    t1(i)=i/ratio-5;
    for j=1:1:maxnum
        t2(j)=j/ratio-5;
        x(i,j)=((t1(i)*t1(i)-t2(j)*t2(j))*sin(t1(i))*sqrt(abs(t2(i))));
    end
end
surf(t1,t2,x)

hold on

%f=inline('((x.*x-y.*y).*sin(0.5*x)*sqrt(abs(y)))');
%A1=inline('(0.5-0.1*x)')  A2=inline('(0.5+0.1*x)');
%B1=inline('(0.5-0.1*y)')  B2=inline('(0.5+0.1*y)'); (0.5-0.1*x).*(0.5-0.1*y)*).*

%AB11=inline('((0.5-0.1*x).*(0.5-0.1*y))')
%AB12=inline('((0.5-0.1*x).*(0.5+0.1*y))')
%AB21=inline('((0.5+0.1*x).*(0.5-0.1*y))')
%AB22=inline('((0.5+0.1*x).*(0.5+0.1*y))')


f=inline('(((0.5-0.1*x).*(0.5-0.1*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b1=dblquad(f,-5,5,-5,5,accuracy);
f=inline('(((0.5-0.1*x).*(0.5+0.1*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b2=dblquad(f,-5,5,-5,5,accuracy);
f=inline('(((0.5+0.1*x).*(0.5-0.1*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b3=dblquad(f,-5,5,-5,5,accuracy);
f=inline('(((0.5+0.1*x).*(0.5+0.1*y)).*((x.*x-y.*y).*sin(x)*sqrt(abs(y))))');
b4=dblquad(f,-5,5,-5,5,accuracy);
b=[b1,b2,b3,b4]'

f=inline('((0.5-0.1*x).*(0.5-0.1*y)).*((0.5-0.1*x).*(0.5-0.1*y))');
a11=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((0.5-0.1*x).*(0.5-0.1*y)).*((0.5-0.1*x).*(0.5+0.1*y))');
a12=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((0.5-0.1*x).*(0.5-0.1*y)).*((0.5+0.1*x).*(0.5-0.1*y))');
a13=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((0.5-0.1*x).*(0.5-0.1*y)).*((0.5+0.1*x).*(0.5+0.1*y))');
a14=dblquad(f,-5,5,-5,5,accuracy);

f=inline('((0.5-0.1*x).*(0.5+0.1*y)).*((0.5-0.1*x).*(0.5+0.1*y))');
a22=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((0.5-0.1*x).*(0.5+0.1*y)).*((0.5+0.1*x).*(0.5-0.1*y))');
a23=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((0.5-0.1*x).*(0.5+0.1*y)).*((0.5+0.1*x).*(0.5+0.1*y))');
a24=dblquad(f,-5,5,-5,5,accuracy);
```

```
f=inline('((0.5+0.1*x).*(0.5-0.1*y)).*((0.5+0.1*x).*(0.5-0.1*y))');
a33=dblquad(f,-5,5,-5,5,accuracy);
f=inline('((0.5+0.1*x).*(0.5-0.1*y)).*((0.5+0.1*x).*(0.5+0.1*y))');
a34=dblquad(f,-5,5,-5,5,accuracy);

f=inline('((0.5+0.1*x).*(0.5+0.1*y)).*((0.5+0.1*x).*(0.5+0.1*y))');
a44=dblquad(f,-5,5,-5,5,accuracy);


A=[a11 a12 a13 a14
   a12 a22 a23 a24
   a13 a23 a33 a34
   a14 a24 a34 a44]



W=A\b

f=inline('w1*((0.5-0.1*x).*(0.5-0.1*y))+w2*((0.5-
0.1*x).*(0.5+0.1*y))+w3*((0.5+0.1*x).*(0.5-
0.1*y))+w4*((0.5+0.1*x).*(0.5+0.1*y))');

w1=W(1);
w2=W(2);
w3=W(3);
w4=W(4);

for i=1:1:maxnum
   t1(i)=i/ratio-5;
   for j=1:1:maxnum
      t2(j)=j/ratio-5;
      x1(i,j)=feval(f,w1,w2,w3,w4,t1(i),t2(j));
   end
end
surf(t1,t2,x1)

hold off
```

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