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Seasonal Cycles in a Model of the Housing Market

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Abstract: The US housing market exhibits seasonal boom and bust cycles where prices *and* the speed of trade (turnover rate) rise in summers and fall in winters. We present a search model that analytically generates the observed cycles. The proposed mechanism is based on swings in market thickness rather than market tightness, the leading explanation in the literature.

Keywords: housing, search and matching, thin and thick markets, seasonality

JEL: D39, D49, D83

1 Introduction

The US housing market goes through seasonal boom and bust episodes:- in summers prices rise and trade speeds up whereas in winters prices fall, it takes much longer to sell and the number of sales slides to the annual lows. The cycles are highly predictable and repetitive, seemingly defying the no-arbitrage condition; hence difficult to explain with standard frictionless asset pricing models. Figure 1, which depicts seasonal components in purchase-price and speed of trade, illustrates these cycles using monthly US data from January 1991 to December 2011.¹

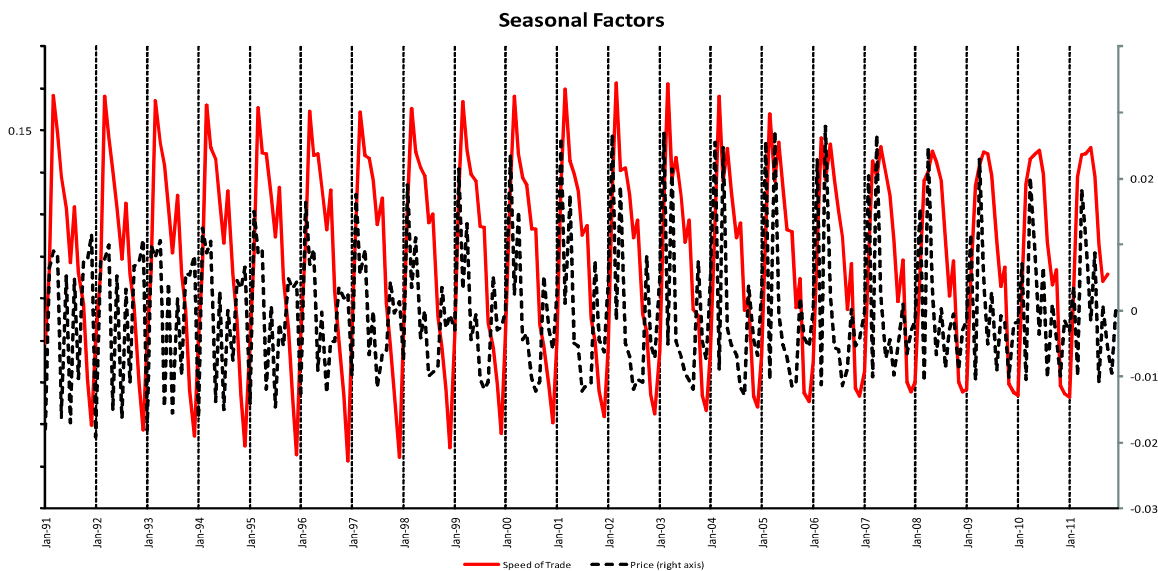


Figure 1 - Seasonal Components in Price and Speed of Trad (Turnover Rate)

In an oft cited article Novy Marx (2009) constructs a search model of the housing market to provide rationale for the observed cycles. The idea is that if a season exogenously adds more buyers to the market then the buyer-seller ratio goes up and therefore houses sell more quickly. The housing supply is assumed to be fixed so it reduces rapidly and the buyer-seller ratio increases even further,

¹The panel illustrates seasonal components in sale prices (right axis) and the speed of trade (turnover rate). The patterns show that the market systematically alternates between boom and bust episodes where in summers prices rise and trade speeds up while in winters the trend reverses. The monthly purchase price index comes from the Federal Housing Financing Agency and it is constructed by a version of the weighted-repeat sales methodology proposed by Case and Shiller (1989). The method controls for differences in the quality of the houses comprising the sample. The speed of trade, on the other hand, is proxied by the ratio of the number of new-single family houses sold at the end of the month divided by the number of houses listed as being for sale that month. The higher the ratio, the higher the speed of trade in that month. The data are obtained from the US Census Bureau and are not seasonally adjusted. We used the X-12-ARIMA procedure, developed by the Census Bureau, to obtain the seasonal factors in each data set. The procedure conducts three formal tests to assess the presence of seasonality: a parametric F-test, a non-parametric Kruskal-Wallis test and a moving seasonality test based on two way ANOVA. All tests positively indicate that identifiable seasonality is present in both series.

which, in turn, leads to higher prices.² The mechanism operates through market tightness (buyer-seller ratio) and to obtain cycles as in Figure 1 one needs to assume that the buyer-seller ratio rises every summer and falls every winter.

While it is true that there are more potential buyers in summers than in winters, the supply side is hardly fixed—in fact it exhibits the same pattern as the demand side, i.e. there are more houses on sale in summers than in winters.³ Therefore it is not clear whether or not market tightness—the key parameter of interest in Novy Marx (2009)—indeed increases in summers.

In this letter we propose an alternative mechanism that depends on market thickness (the number of market participants) instead of market tightness (ratio of participants) and is capable of producing deterministic boom and bust cycles. Market thickness refers to the fact that there are more houses on sale in the summer market than in the winter market, hence better quality matches are formed in summers. The thick summer market comes with the greatest possible choice of residence which means that buyers encounter better quality matches in such a market. People are willing to pay a premium for housing that closely matches their needs, tastes and preferences; hence prices go up in the summer. On the other hand, sellers have no means of transferring the extra value across seasons, so they have strong incentives to trade while the market is still thick. Therefore they limit the price rise to a modest amount to ensure that trade indeed speeds up. The rising prices coupled with the increased speed of trade means that the market booms in the summer. The trend reverses in the winter, so the market alternates between boom and bust episodes as seasons change.

2 Model

Time is discrete, infinite and deterministically alternates between two seasons, summer (s) and winter (w). The economy is populated by a continuum of houses and a continuum of buyers each of whom wishes to purchase a house. In summers there is a measure of h_s properties for sale and b_s buyers whereas in winters these measures are h_w and b_w . Each house is owned by a risk neutral seller, who derives no utility from the ownership. Buyers, too, are risk neutral and receive periodic housing services starting the period after the purchase and continuing forever. The measures of potential buyers and sellers are exogenous; however the number of transactions, the speed of trade and sale prices are, of course, endogenous.

²Kraimer (2001) presents an alternative model where the market fluctuates between hot and cold episodes, however the model fails to produce deterministic cycles. Indeed if the persistence parameter in Kraimer (2001) is set $\lambda = 0$ so that seasons alternate deterministically then, interestingly, one obtains the wrong cycle; the market is cold in the summer and hot in the winter. Ngai and Tenreyro (2013) present a setup generating deterministic cycles, but their results are based on quantitative simulations. See also Kaplanski and Levy (2012), Muellbauer and Murphy (1997), and Stein (1995).

³Rosen (1979), one of the most comprehensive studies on seasonality in the American housing market, presents substantial evidence documenting seasonal ups and downs in demand and supply in the residential property market and concludes that demand and supply are both high in summers and low in winters. In other words, the seasonality in housing demand coincides with the seasonality in housing supply (housing authorizations, construction of new houses and listings of existing properties). Goodman (1993), using data from separate sources confirms Rosen's findings.

The market is characterized by two types of frictions. The first is finding a counterpart, which depends on market tightness (buyer-seller ratio). Assuming an urn-ball matching function and letting $\lambda_x := b_x/h_x$ denote the buyer-seller ratio in season $x = s, w$, a seller meets a buyer with probability $1 - e^{-\lambda_x}$ whereas a buyer meets a seller with probability $(1 - e^{-\lambda_x})/\lambda_x$.

Assumption 1. *We have $\lambda_s = \lambda_w = \lambda$. Furthermore $h_s > h_w$.*

The mechanism in Novy Marx (2009) operates through market tightness, λ_x ; so, for exposition, we shut down this channel by assuming that λ_x remains constant throughout the year. The second part of the assumption is based on the aforementioned empirical findings by Rosen (1979) and Goodman (1993) and states that in summers there are more houses on the market than in winters.

The second friction deals with whether the house turns out to be a good match. After an initial inspection, the buyer realizes his valuation $v \in [0, 1]$, which is private information and a random draw via cdf $F(v, h_x) \equiv F_x(v)$. From the buyer's perspective the search process amounts to finding a high enough v . The cdf F depends on the stock of the vacant houses h_x and we assume that the larger this stock the more likely are buyers to find what they are looking for.

Assumption 2. *If $h_x > h_{\bar{x}}$ then F_x likelihood ratio dominates $F_{\bar{x}}$; that is $f_x(v)/f_{\bar{x}}(v)$ increases in v . In addition, the "Iso-Probability Curve"*

$$\Phi(v) := F_{\bar{x}}^{-1} \circ F_x(v) : [0, 1] \rightarrow [0, 1]$$

*increases and is strictly convex.*⁴ Finally we assume that the survival function $S_x = 1 - F_x$ is log-concave, that is $f_x^2(v) + f_x'(v)S_x(v) > 0$, $\forall v$ and $x = s, w$.⁵

Likelihood ratio dominance implies first order stochastic dominance (FOSD), $F_s(v) < F_w(v)$, as well as hazard rate dominance, $\eta_s(v) < \eta_w(v)$, where $\eta_x := f_x/S_x$. FOSD implies that, controlling for the probability of trade, higher quality matches are formed in summers since the house stock

⁴The strict convexity of Φ is added as an extra assumption, but under certain circumstances likelihood ratio dominance is a sufficient condition for it; for instance if $f_x' \geq 0$ and $f_{\bar{x}}' \leq 0$ with one inequality strict.

⁵Log-concavity is a mild assumption satisfied by well known distributions; see Bagnoli and Bergstrom (2005).

in summers exceeds the one in winters (recall that $h_s > h_w$).

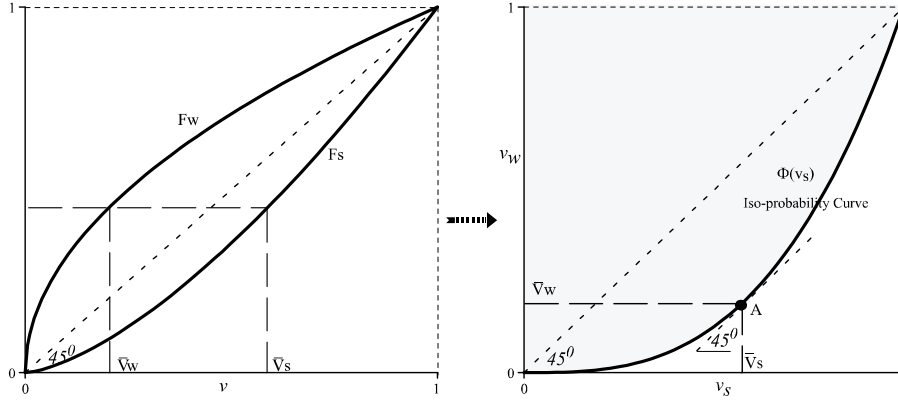


Figure 2 – CDFs and the Iso-Probability Curve

Discussion. Before proceeding further, two points are worth discussing. First, we treat the stocks of market participants exogenously, i.e. we do not seek to explain why the number of potential buyers and sellers are higher in summers than in winters. Based on the empirical studies mentioned in the Introduction we take the seasonal stocks as given and then explain how the equilibrium price and the speed of trade (turnover rate) rise in summers and fall in winters as a result. A more complete model should treat these stocks endogenously and this letter should be viewed as a first step towards that goal.

Second, Assumptions 1 and 2 do not immediately imply the results. The assumption that $h_s > h_w$ may imply that there will be more trade in summers than in winters; however the turnover rate (speed of trade) is the ratio of the number of houses sold to the number of houses on sale. Both the numerator and the denominator rise in summers; thus the change in the turnover rate is ambiguous. Similarly, a priori it is hard to predict how the equilibrium price would change across seasons since there are always λ buyers per seller in the market.

3 Analysis

The valuation v is a buyer's private information, so the seller quotes the same take-it-or-leave-it price p_x for all customers. Letting Ω_x denote the value of search to a buyer in season $x = s, w$ we have

$$\Omega_x = \left[\left(1 - e^{-\lambda}\right) / \lambda \right] \int_0^1 \max \{v / (1 - \beta) - p_x, \beta \Omega_{\bar{x}}\} dF_x(v) + \left[1 - \left(1 - e^{-\lambda}\right) / \lambda \right] \beta \Omega_{\bar{x}}.$$

With probability $(1 - e^{-\lambda})/\lambda$ the buyer meets a seller. If he purchases he gets $v/(1 - \beta) - p_x$. If he walks away he obtains $\beta\Omega_{\bar{x}}$, which is the discounted value of search in the next season. With the complementary probability he does not encounter a seller and moves on to the next season. We have

$$\Omega_x = \theta\tau_x + \beta\theta\tau_{\bar{x}},$$

where

$$\theta := \frac{1 - e^{-\lambda}}{\lambda(1 - \beta)^2(1 + \beta)}$$

and

$$\tau_x := \int_0^1 \max\{v - (1 - \beta)(p_x + \beta\Omega_{\bar{x}}), 0\} dF_x(v).$$

For any given price p_x there is a threshold *reservation value* v_x satisfying

$$v_x = (1 - \beta)(p_x + \beta\Omega_{\bar{x}}). \quad (1)$$

For trade to occur the house must turn out to be a good match, which happens with probability $\Pr(v \geq v_x) = S_x(v_x)$. Inserting (1) into τ_x yields

$$\Omega_x = \theta \int_{v_x}^1 S_x(v) dv + \beta\theta \int_{v_{\bar{x}}}^1 S_{\bar{x}}(v) dv.$$

Substituting Ω_x into the indifference condition (1) one gets the ‘indifference curves’ I_s and I_w

$$p_x = v_x / (1 - \beta) - \beta\Omega_{\bar{x}} \equiv I_x. \quad (2)$$

The value function of a seller is given by

$$\Pi_x = \left(1 - e^{-\lambda}\right) S_x(v_x) \max\{p_x, \beta\Pi_{\bar{x}}\} + \left\{1 - \left(1 - e^{-\lambda}\right) S_x(v_x)\right\} \beta\Pi_{\bar{x}}.$$

With probability $1 - e^{-\lambda}$ the seller meets a buyer, with probability $S_x(v_x)$ the buyer agrees to purchase and the seller obtains price p_x . With the complementary probability trade does not materialize, so the seller moves to the next season. The seller quotes p_x in season x taking as given the indifference condition (1) i.e.

$$\max_{p_x} \Pi_x \text{ subject to } v_x = (1 - \beta)(p_x + \beta\Omega_{\bar{x}})$$

treating Ω_x and $\Omega_{\bar{x}}$ exogenously. The FOC is given by

$$\Pi'_x = 0 \Rightarrow p_x - \beta\Pi_{\bar{x}} = S_x(v_x) / \{f_x(v_x)(1 - \beta)\}.$$

If S_x is log concave then the second order condition holds; hence the FOC yields a maximum.⁶ Straightforward algebra yields profit maximizing prices P_s and P_w that a seller ought to post:

$$P_x = \frac{S_x(v_x)}{f_x(v_x)(1-\beta)} + \frac{\theta\lambda S_{\bar{x}}^2(v_{\bar{x}})}{f_{\bar{x}}(v_{\bar{x}})} + \frac{\beta\theta\lambda S_x^2(v_x)}{f_x(v_x)}. \quad (3)$$

Simultaneous intersections of the offer and indifference curves determine the equilibrium. More formally a steady-state, stationary and symmetric equilibrium is characterized by the pairs $\mathbf{v}^* = (v_s^*, v_w^*)$ and $\mathbf{p}^* = (p_s^*, p_w^*)$ satisfying indifference (1) and profit maximization (3)

Proposition 1 *An equilibrium exists and it is unique.*

The proof amounts to showing that there exists a unique pair $\mathbf{v}^* = (v_s^*, v_w^*) \in (0, 1)^2$ satisfying

$$\Delta_x := P_x + \beta\Omega_{\bar{x}} - v_x / (1 - \beta) = 0, \text{ for } x = s, w. \quad (4)$$

The details of the proof are relegated to the Online Appendix.

3.1 Deterministic Cycles

We now show that the equilibrium price and speed of trade together rise in summers and fall in winters. The *speed of trade* is typically proxied by the turnover rate, i.e. the number of homes sold divided by the number of homes listed; e.g. Goodman (1993), Rosen (1979). In our model season x comes with h_x houses for sale, of which $(1 - e^{-\lambda}) h_x S_x$ are sold; thus the turnover rate equals to

$$\frac{(1 - e^{-\lambda}) h_x S_x}{h_x} = (1 - e^{-\lambda}) S_x.$$

Since λ is the same in both seasons (Assumption 1) we simply focus on the probability of sale S_x to compare turnover rates. Our goal, therefore, is to show that in equilibrium $p_s^* > p_w^*$ and $S_s^* > S_w^*$.

Proposition 2 *The equilibrium price and the speed of trade are both high in the summer and low in the winter i.e. $p_s^* > p_w^*$ and $S_s^* > S_w^*$. So, we have a booming market (high prices and fast sales) in the summer and a declining market (low prices, slower sales) in the winter. In addition, sellers and buyers are strictly better off trading immediately rather than waiting for the next season.*

The proof is in the Online Appendix. The thick summer market presents the largest number of possible housing alternatives. Buyers, on average, encounter higher quality matches in such a market, so they are ready to pay more. This is why prices go up in the summer. On the other hand, sellers cannot transfer the additional value across seasons, so they wish to trade while the market is still thick. To do so, they limit the price rise to a modest amount making sure that trade

⁶Basic algebra yields $\text{sign}(\Pi_x'') = -\text{sign}(\{f_x' S_x + 2f_x^2\} / f_x)$. The expression inside the parenthesis on the rhs of the equality is positive because of log concavity; hence Π_x'' is negative.

indeed speeds up in the summer.⁷ In the winter, the scenario is reversed; so we have a setup where the market deterministically alternates between boom and bust episodes.

This brings us to a crucial question: Why do not buyers wait until the winter to obtain better deals or why do not sellers wait until the summer to obtain better prices? The reason is that the market operates via search and matching, so an agent may not be able to meet a counter-part in the next season. Plus, even if a counter-part is found, there is no guarantee that a sale will occur as the quality of the new match may not be high enough. Therefore, assuming a suitable match is found, agents are strictly better off trading immediately.

Numerical Example. To provide further insight we run a numerical simulation based on the following parameter values:

$b_s = 1$	$h_s = 1$	$b_w = 0.8$	$h_w = 0.8$
$F_s(v) = v^2$	$F_w(v) = v$	$\lambda = 1$	$\beta = 0.9$

Table 1

The parameters yield equilibrium prices $p_s^* = 6.26$, $p_w^* = 6.17$ and probabilities of trade $S_s^* = 0.37$, $S_w^* = 0.21$.⁸ The summer market starts with $b_s = h_s = 1$ potential buyers and sellers. The measure of agents who trade and exit equals to

$$\text{summer outflow} = 1 \times (1 - e^{-\lambda}) \times S_s^* = 0.23.$$

⁷Sellers indeed do not raise prices as much as they could in summers. The following simulation (based on the parameter values in Table 1) confirms this insight

	Price	Prob. of Sale
<i>Summer</i>	6.26	0.37
<i>Winter</i>	6.17	0.21
<i>If winter never came</i>	6.36	0.33

The first two rows report equilibrium objects under the regular model. The third row is calculated under the assumption that the market remains thick throughout the entire year, i.e. if $F_w(v)$ were the same as $F_s(v)$. The equilibrium price in this imaginary scenario exceeds the summer price in the regular model. As argued above, sellers in the regular model do not raise prices sufficiently as they want to take advantage of the summer market while it lasts.

⁸Solving $\Delta_s = 0$ and $\Delta_w = 0$, where Δ_x is given by (4), yields $v_s^* = 0.796$ and $v_w^* = 0.791$, which means that $S_s^* = 0.37$ and $S_w^* = 0.21$. Substituting $v_s^* = 0.8$ and $v_w^* = 0.79$ into (3) yields prices $p_s^* = 6.27$ and $p_w^* = 6.17$.

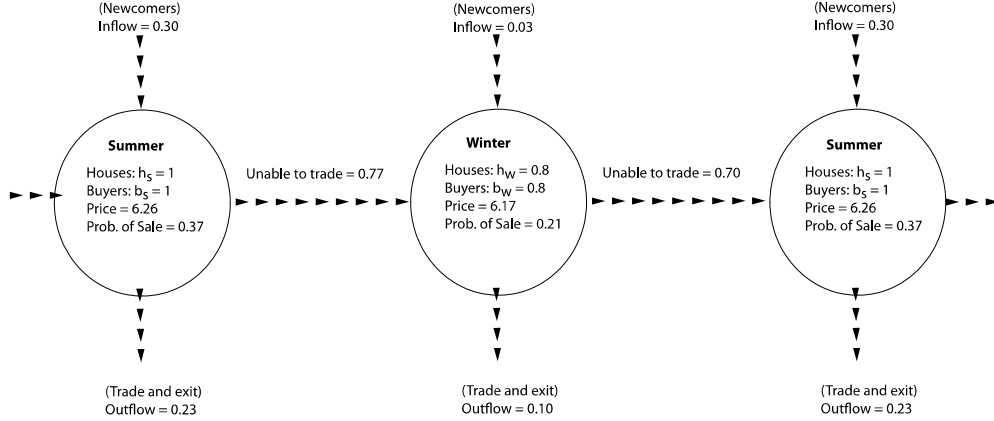


Figure 3 - Flowchart

Remaining agents, i.e. 0.77 buyers and sellers, move to the winter market. Before trading resumes, an inflow of 0.03 new agents arrive, taking the stocks of potential buyers and sellers to 0.8 (recall that $b_w = h_w = 0.8$). At the end of the winter season a measure of

$$winter\ outflow = 0.8 \times (1 - e^{-\lambda}) \times S_w^* = 0.10$$

buyers and sellers trade and exit. Remaining 0.70 buyers and sellers, move on to the next summer market. An inflow of 0.30 new agents arrive before trading resumes, thus the summer market, again, starts with $b_s = h_s = 1$. And so on. The market indeed experiences boom and bust episodes as seasons change. In summers a unit measure of houses are offered for sale, each priced at $p_s^* = 6.26$. At the end of the season 0.23 houses are sold. The turnover rate (i.e. the speed of trade), thus equals to 23%. In winters 0.8 houses are put up for sale (each priced at $p_w^* = 6.17$), of which, 0.10 are sold. Despite lower prices, the turnover rate is only 12.5%.

4 Discussion and Conclusion

We have presented a setup that generates deterministic boom and bust cycles. The proposed mechanism operates through market thickness rather than market tightness (unlike Novy Marx (2009)) and the results are analytic (unlike Ngai and Tenreyro (2013)). Finally, although the discussion so far revolved around the housing market, the model is applicable to other search and matching settings, such as the used car market or, to some extent, the labor market, that go through similar seasonal cycles

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Online Appendix – Not intended for publication

The following Lemmas are useful in proving existence and uniqueness of the equilibrium.

Lemma 1 *We have $\partial I_x / \partial v_x > 0$ and $\partial I_x / \partial v_{\bar{x}} > 0$, where I_x is given by (2).*

Proof of Lemma 1. Note that

$$\frac{\partial \Omega_x}{\partial v_x} = -\theta S_x(v_x) < 0 \quad \text{and} \quad \frac{\partial \Omega_x}{\partial v_{\bar{x}}} = -\beta \theta S_{\bar{x}}(v_{\bar{x}}) < 0.$$

Therefore

$$\frac{\partial I_x}{\partial v_x} = \frac{1}{1-\beta} + \beta^2 \theta S_x(v_x) \quad \text{and} \quad \frac{\partial I_x}{\partial v_{\bar{x}}} = \beta \theta S_{\bar{x}}(v_{\bar{x}}).$$

Clearly both expressions are positive; hence indifference curves I_s and I_w are upward sloping wrt v_s and v_w . ■

Lemma 2 *We have $dP_x/dv_x < dP_{\bar{x}}/dv_x < 0$, for $x = s, w$ where P_x is given by (3).*

Proof of Lemma 2. Observe that

$$\frac{dP_x}{dv_x} = -\frac{M_x(v_x)}{1-\beta} - \beta \theta S_x(v_x) \{1 + M_x(v_x)\} < 0 \quad \text{and} \quad \frac{dP_x}{dv_{\bar{x}}} = -\theta S_{\bar{x}}(v_{\bar{x}}) \{1 + M_{\bar{x}}(v_{\bar{x}})\} < 0,$$

where $M_x(v_x) = 1 + f'_x(v_x) S_x(v_x) / f_x^2(v_x)$, which is positive because of log concavity (Assumption 2); thus both derivatives are negative. Furthermore

$$\frac{d\{P_x - P_{\bar{x}}\}}{dv_x} = \frac{1}{1-\beta^2} \times \left\{ -M_x \left\{ 1 + \beta - (1 - e^{-\lambda}) S_x \right\} + (1 - e^{-\lambda}) S_x \right\} < 0,$$

which is negative because M_x is positive. ■

Proof of Proposition 1. The proof amounts to showing that there exists a unique pair $\mathbf{v}^* = (v_s^*, v_w^*) \in (0, 1)^2$ satisfying

$$\Delta_x := P_x + \beta \Omega_{\bar{x}} - v_x / (1 - \beta) = 0, \quad \text{for } x = s, w. \quad (5)$$

In equilibrium the difference function $D := \Delta_s - \Delta_w$ must equal to zero as well. Note that

$$D = \underbrace{P_s - P_w}_{T_1} + \underbrace{\beta (\Omega_w - \Omega_s) - (v_s - v_w) / (1 - \beta)}_{T_2}. \quad (6)$$

For expositional purposes we will focus on Δ_w and D . Below we show that the locus of Δ_w is downward sloping whereas the locus of D is upward sloping. The equilibrium \mathbf{v}^* lies at their intersection. In what follows we omit the superscript * when understood.

Claim 1. The function Δ_w decreases in both arguments v_s and v_w while its locus, denoted by l_Δ , is downward sloping wrt v_s . The function D , on the other hand, decreases in v_s and increases in v_w , whereas its locus, denoted by l_D , is upward sloping wrt v_s .

Proof of Claim 1. Let $l_\Delta(v_s) := \{v_w : \Delta_w(v_s, v_w) = 0\}$ be the locus of $\Delta_w = 0$. Its slope wrt v_s is given by (Implicit Function Theorem)

$$\frac{dl_\Delta}{dv_s} = -\frac{\partial\Delta_w}{\partial v_s} / \frac{\partial\Delta_w}{\partial v_w} < 0.$$

Observe that

$$\frac{\partial\Delta_w}{\partial v_s} = \frac{\partial P_w}{\partial v_s} + \beta \frac{\partial\Omega_s}{\partial v_s} < 0,$$

which is negative because $\partial P_w / \partial v_s < 0$ (Lemma 2) and $\partial\Omega_s / \partial v_s = -\theta S_s(v_s) < 0$. Similarly one can show that $\partial\Delta_w / \partial v_w < 0$; hence $dl_\Delta / dv_s < 0$.

Now turn to the difference function D . Recall that $D = T_1 + T_2$, where T_1 and T_2 are defined in (6). Substitute for P_s , P_w , Ω_s and Ω_w and simplify to obtain

$$T_1(v_s, v_w) = \frac{1}{1-\beta} \times \left\{ \frac{1}{\eta_s(v_s)} - \frac{1}{\eta_w(v_w)} + \frac{1-e^{-\lambda}}{1-\beta} \left[\frac{S_w(v_w)}{\eta_w(v_w)} - \frac{S_s(v_s)}{\eta_s(v_s)} \right] \right\} \quad \text{and} \quad (7)$$

$$T_2(v_s, v_w) = \frac{(1-e^{-\lambda})\beta}{(1-\beta^2)\lambda} \times \left\{ \int_{v_s}^1 F_s(v) dv - \int_{v_w}^1 F_w(v) dv \right\} - \frac{1+\beta \left(1 - \frac{1-e^{-\lambda}}{\lambda}\right)}{1-\beta^2} (v_s - v_w). \quad (8)$$

Let $l_D(v_s) := \{v_w : D(v_s, v_w) = 0\}$ be the locus of $D = 0$. Its slope wrt v_s is given by

$$\frac{dl_D}{dv_s} = -\frac{\partial\{T_1+T_2\}}{\partial v_s} / \frac{\partial\{T_1+T_2\}}{\partial v_w} > 0.$$

Observe that

$$\frac{\partial T_1}{\partial v_s} = \frac{\partial P_s - P_w}{\partial v_s} < 0 \quad ; \quad \frac{\partial T_1}{\partial v_w} = \frac{\partial P_s - P_w}{\partial v_w} > 0 \quad (\text{Lemma 2}) \quad (9)$$

$$\frac{\partial T_2}{\partial v_s} = -\frac{1}{1-\beta} \times \left\{ 1 - \frac{\beta(1-e^{-\lambda})}{(1+\beta)\lambda} S_s(v_s) \right\} < 0 \quad \text{and}$$

$$\frac{\partial T_2}{\partial v_w} = \frac{1}{1-\beta} \times \left\{ 1 - \frac{\beta(1-e^{-\lambda})}{(1+\beta)\lambda} S_w(v_w) \right\} > 0.$$

The signs of $\partial T_1 / \partial v_s$, $\partial T_1 / \partial v_w$, $\partial T_2 / \partial v_s$ and $\partial T_2 / \partial v_w$ imply that dl_D / dv_s is indeed positive.

Q.E.D.

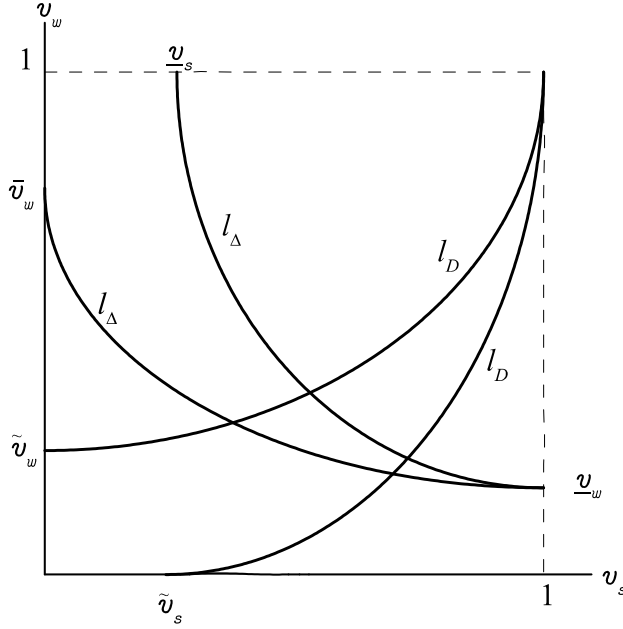


Figure 4 – Locuses (all possible scenarios)

Claim 2. There exists a unique $\underline{v}_w \in (0, 1)$ satisfying $l_\Delta(1) = \underline{v}_w$. In addition either there exists $\underline{v}_s \in (0, 1)$ s.t. $l_\Delta(\underline{v}_s) = 1$ or there exists $\bar{v}_w \in (0, 1)$ s.t. $l_\Delta(0) = \bar{v}_w$.

Proof of Claim 2. Recall that Δ_w decreases in v_w . In addition we have

$$\Delta_w(1, 1) = -\frac{1}{1-\beta} < 0 \quad \text{and} \quad \Delta_w(1, 0) = P_s(1, 0) + \beta\Omega_w(1, 0) > 0.$$

Therefore there exists a unique $\underline{v}_w \in (0, 1)$ satisfying $\Delta_w(1, \underline{v}_w) = 0$ i.e. $l_\Delta(1) = \underline{v}_w$. The rest of the claim depends on whether $\Delta_w(0, 1)$ is positive or negative:-

- Suppose $\Delta_w(0, 1) \geq 0$. Since (i) $\Delta_w(1, 1) < 0$ and (ii) Δ_w decreases in v_s there exists $\underline{v}_s \in (0, 1)$ such that $\Delta_w(\underline{v}_s, 1) = 0$ i.e. $l_\Delta(\underline{v}_s) = 1$.
- Suppose $\Delta_w(0, 1) < 0$. Since $\Delta_w(0, 0) > 0$ and (ii) Δ_w decreases in v_w there exists $\bar{v}_w \in (0, 1)$ such that $\Delta_w(0, \bar{v}_w) = 0$ i.e. $l_\Delta(0) = \bar{v}_w$. Recall that l_Δ is a decreasing function of v_s ; therefore $l_\Delta(0) > l_\Delta(1)$; hence $\bar{v}_w > \underline{v}_w$. *Q.E.D.*

Claim 3. We have $l_D(1) = 1$. In addition either there exists $\tilde{v}_w \in (0, 1)$ s.t. $l_D(0) = \tilde{v}_w$ or there exists $\tilde{v}_s \in (0, 1)$ s.t. $l_D(\tilde{v}_s) = 0$. Finally $\tilde{v}_w < \bar{v}_w$.

Proof of Claim 3. One can immediately verify that $D(1, 1) = 0$; hence $l_D(1) = 1$. The rest of the arguments depend on the sign of $D(0, 0)$:-

- Suppose that $D(0, 0) \leq 0$. The fact that $D(1, 1) = 0$ implies that $D(0, 1)$ is positive since D decreases in v_s . Hence there exists some $\tilde{v}_w \in (0, 1)$ s.t. $D(0, \tilde{v}_w) = 0$ i.e. $l_D(0) = \tilde{v}_w$.
- Suppose that $D(0, 0) > 0$. The fact that $D(1, 1) = 0$ implies that $D(1, 0)$ is negative since D increases in v_w . Hence there exists some $\tilde{v}_s \in (0, 1)$ s.t. $D(\tilde{v}_s, 0) = 0$ i.e. $l_D(\tilde{v}_s) = 0$.

To see why $\tilde{v}_w < \bar{v}_w$ note that

$$\Delta_s(0, \tilde{v}_w) = P_s(0, \tilde{v}_w) + \beta\Omega_w(0, \tilde{v}_w) > 0.$$

Recall that $D = \Delta_s - \Delta_w$. Since $D(0, \tilde{v}_w) = 0$ it follows that $\Delta_w(0, \tilde{v}_w)$ equals to $\Delta_s(0, \tilde{v}_w)$, and therefore, positive. On the other hand, recall that $\Delta_w(0, \bar{v}_w) = 0$. Since Δ_w decreases in v_w , it follows that $\tilde{v}_w < \bar{v}_w$. *Q.E.D.*

Based on Claims 2 and 3 we draw l_D and l_Δ in Figure 4. A visual inspection reveals that under all scenarios the curves intersect once in the unit interval, hence the equilibrium exists and it is unique. For the precision minded reader, below we make this argument clear.

Define $\xi(v_s) := l_D(v_s) - l_\Delta(v_s)$ as an increasing function of v_s (recall that l_D increases whereas l_Δ decreases in v_s). Note that $\xi(1) = l_D(1) - l_\Delta(1)$ is positive since $l_D(1) = 1$ and $l_\Delta(1) = \underline{v}_w < 1$. At the lower end, however, there are two cases depending on the sign of $D(0, 0)$:-

- If $D(0, 0) > 0$ then $l_D(\tilde{v}_s) = 0$ for some $\tilde{v}_s \in (0, 1)$. Note that $l_\Delta(\tilde{v}_s)$ is positive because $l_\Delta(1)$ is positive, l_Δ is a decreasing function and $\tilde{v}_s < 1$. Hence $\xi(\tilde{v}_s) = l_D(\tilde{v}_s) - l_\Delta(\tilde{v}_s)$ is negative. In addition since $\xi(1)$ is positive there exists some $v_s^* \in (\tilde{v}_s, 1)$ satisfying $\xi(v_s^*) = 0$.
- If $D(0, 0) \leq 0$ then $l_D(0) = \tilde{v}_w$. Recall that $l_\Delta(0) = \bar{v}_w$; hence $\xi(0) = \tilde{v}_w - \bar{v}_w < 0$, which is negative since $\tilde{v}_w < \bar{v}_w$ (see above). In addition since $\xi(1)$ is positive there exists some $v_s^* \in (0, 1)$ satisfying $\xi(v_s^*) = 0$.

So, in either case there exists a unique v_s^* in the unit interval, or within a subset of the unit interval, satisfying $l_\Delta(v_s^*) = l_D(v_s^*) = v_s^*$. This completes the proof. ■

The next Lemma says that the locus of $T_1 = P_s - P_w = 0$, denoted by Γ (the "iso-price curve") looks as in Figure 5, which in turn will be useful in proving the main result of the paper, Proposition 2.

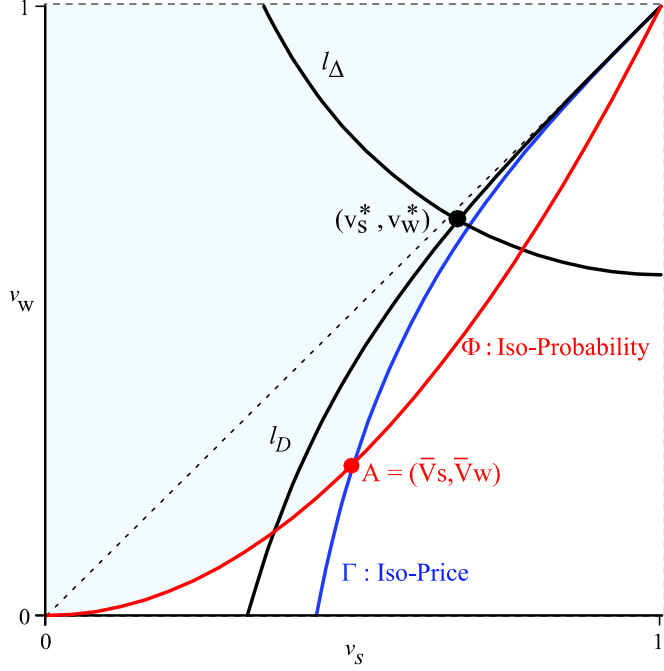


Figure 5 - Iso-Probability and Iso-Price Curves

Lemma 3 *There exists a unique point $A = (\bar{V}_s, \bar{V}_w)$ on the iso-probability curve Φ such that $f_s(\bar{V}_s) = f_w(\bar{V}_w)$, $S_s(\bar{V}_s) = S_w(\bar{V}_w)$ and therefore $\eta_s(\bar{V}_s) = \eta_w(\bar{V}_w)$. The iso-price curve Γ monotonically increases in v_s and intersects with curve Φ at point A . In addition Γ lies underneath l_D , while point A lies underneath both l_D and l_Δ .*

Proof of Lemma 3. The iso-probability curve Φ is obtained by drawing horizontal lines across the cdfs and tracing combinations of v_s and v_w satisfying $F_s(v_s) = F_w(v_w)$; see Figure 2 in the main text. For clarity, the points on the border of Φ are denoted with capital letters. The slope of Φ equals to $f_s(V_s)/f_w(V_w)$. Strict convexity of Φ (Assumption 2) ensures that there exists a unique point $A = (\bar{V}_s, \bar{V}_w) \in (0, 1)^2$ on Φ such that $f_s(\bar{V}_s) = f_w(\bar{V}_w)$.⁹ Since $A = (\bar{V}_s, \bar{V}_w)$ lies on Φ , we have $F_s(\bar{V}_s) = F_w(\bar{V}_w)$ and therefore $S_s(\bar{V}_s) = S_w(\bar{V}_w)$. In addition since $f_s(\bar{V}_s) = f_w(\bar{V}_w)$, we have $\eta_s(\bar{V}_s) = \eta_w(\bar{V}_w)$, where $\eta = f/S$ is the hazard rate.

Recall that $T_1 = P_s - P_w$. The locus of $T_1(v_s, v_w) = 0$ is the iso-price curve and it is given by

$$\Gamma(v_s) = \{v_w : T_1(v_s, v_w) = 0\}.$$

⁹To see this note that $dF_w^{-1} \circ F_s(V_s)/dV_s = f_s(V_s)/f_w(V_w)$. The function is convex and lies underneath the 45^o line cutting it at the origin and at (1, 1). The Intermediate Value Theorem implies existence of a unique point $A = (\bar{V}_s, \bar{V}_w)$ such that $f_s(\bar{V}_s) = f_w(\bar{V}_w)$. Strict convexity of Φ ensures that \bar{V}_s and \bar{V}_w are strictly between zero and one.

The Implicit Function Theorem asserts that

$$\frac{d\Gamma}{dv_s} = -\frac{\partial T_1}{\partial v_s} / \frac{\partial T_1}{\partial v_w}.$$

Recall that $\partial T_1 / \partial v_s$ is negative while $\partial T_1 / \partial v_w$ is positive (see (9)). Hence $d\Gamma / dv_s$ is positive. To see that Γ intersect with the iso-probability curve Φ at point A recall that at A we have $f_s(\bar{V}_s) = f_w(\bar{V}_w)$ and $S_s(\bar{V}_s) = S_w(\bar{V}_w)$; therefore $\eta_s(\bar{V}_s) = \eta_w(\bar{V}_w)$. Substituting these equalities into (7) yields $T_1(\bar{V}_s, \bar{V}_w) = 0$; hence A belongs to Γ .

Now we argue that Γ lies to the right of the locus of $D = 0$ (the curve l_D). In Figure 5 fix some v_w and imagine a horizontal line going through v_w cutting the curve Γ at v'_s and the curve l_D at v''_s . Formally $T_1(v'_s, v_w) = 0$ and $D(v''_s, v_w) = 0$. We will show that $v'_s > v''_s$.

To start, note that v'_s must exceed v_w . To see why substitute $v_s = v_w = v$ into the expression for T_1 , given by (7), to obtain

$$T_1(v, v) = \frac{1+\beta-(1-e^{-\lambda})S_s(v)}{\eta_s(v)} - \frac{1+\beta-(1-e^{-\lambda})S_w(v)}{\eta_w(v)} > 0,$$

which is positive because $\eta_s(v) < \eta_w(v)$ and $S_s(v) > S_w(v)$ for all v . The former relationship is the hazard rate dominance and the latter is the FOSD. Recall that $\partial T_1 / \partial v_s < 0$. So, if $T_1(v'_s, v_w) = 0$ then v'_s must indeed exceed v_w . Since v'_s exceeds v_w we have $T_2(v'_s, v_w) < 0$ (one can immediately verify this from (8)).

Now, since $T_1(v'_s, v_w) = 0$ and $T_2(v'_s, v_w) < 0$, their sum $D(v'_s, v_w)$ is, therefore, negative. Recall that, on the other hand, $D(v''_s, v_w) = 0$. Since D decreases in v_s we have $v''_s < v'_s$, which means that Γ lies to the right of l_D .

Finally, we show that point A lies underneath l_D and l_Δ . The first relationship immediately follows from the facts that A belongs to Γ and that Γ lies below l_D . The claim that A lies below l_Δ is also easy to verify. Substitute $A = (\bar{V}_s, \bar{V}_w)$ into Δ_w , given by (5), and simplify the expression using the fact that $f_s(\bar{V}_s) = f_w(\bar{V}_w)$ and $S_s(\bar{V}_s) = S_w(\bar{V}_w)$ to obtain $\Delta_w(\bar{V}_s, \bar{V}_w) > 0$. This implies that A lies underneath l_Δ since the function Δ_w is positive at any point underneath its locus l_Δ . ■

Proof of the Proposition 2. The arguments below are best understood with the aid of Figure 5. First we show that $p_s^* > p_w^*$. Note that at any point (v_s, v_w) above the iso-price curve Γ we have $P_s(v_s, v_w) > P_w(v_s, v_w)$ ¹⁰, so we want to show that the equilibrium point \mathbf{v}^* falls above Γ . Recall that \mathbf{v}^* lies at the intersection of l_D and l_Δ ; so, by definition, \mathbf{v}^* belongs to curve l_D . Recall also that l_D lies above the iso-price curve Γ (Lemma 3). Therefore $P_s(v_s^*, v_w^*) > P_w(v_s^*, v_w^*)$.

Now we turn to the claim that $S_s^* > S_w^*$. Our objective is to show that \mathbf{v}^* falls inside the iso-probability curve Φ . The curve l_Δ is downward sloping whereas l_D is upward sloping and point A lies beneath both (Lemma 3). This means that point \mathbf{v}^* must lie above point A , that is $v_w^* > \bar{V}_w$;

¹⁰This claim follows from the fact that the function $T_1 = P_s - P_w$ decreases in v_s and increases in v_w (see the proof of Proposition 1). More precisely, fix some (\hat{v}_w, \hat{v}_s) on Γ and note that $T_1(v_s, \hat{v}_w) > 0 \Leftrightarrow P_s(v_s, \hat{v}_w) > P_w(v_s, \hat{v}_w)$ for any $v_s < \hat{v}_s$ since $\partial T_1 / \partial v_s < 0$. Similarly $T_1(\hat{v}_s, v_w) > 0$ for any $v_w > \hat{v}_w$ since $\partial T_1 / \partial v_w > 0$.

so the region below \bar{V}_w can be dismissed as it cannot contain the equilibrium.

We now claim that along the border of Φ lying above \bar{V}_w the function $D = T_1 + T_2$ is negative. Recall that at point $A = (\bar{V}_s, \bar{V}_w)$ we have $f_s(\bar{V}_s) = f_w(\bar{V}_w)$. The strict convexity of Φ (Assumption 2) also ensures that $f_s(V_s) > f_w(V_w)$ for all (V_s, V_w) on Φ lying above A . Along such points the function T_1 is negative. To see why substitute (V_s, V_w) into (7) and use the fact that $S_s(V_s) = S_w(V_w)$ to obtain

$$T_1(V_s, V_w) = \left\{1 + \beta - (1 - e^{-\lambda}) S_s(V_s)\right\} S_s(V_s) \left\{\frac{1}{f_s(V_s)} - \frac{1}{f_w(V_w)}\right\}.$$

The expression is negative because $f_s(V_s) > f_w(V_w)$. Now focus on T_2 , given by (8), and note that $T_2(V_s, V_w) < 0$ because $V_s > V_w$ and FOSD. Since both T_1 and T_2 are negative, so is D .

Since $D(V_s, V_w) < 0$ along the border of Φ the equilibrium point \mathbf{v}^* must be inside Φ (recall that $D(v_s^*, v_w^*) = 0$). To see why fix V_w and note that $D(v_s, V_w) = 0$ for some $v_s < V_s$ since D decreases in v_s . Alternatively fix V_s and note the $D(V_s, v_w) = 0$ for some $v_w > V_w$ since D increases in v_w . Finally, since (v_s^*, v_w^*) lies inside Φ we have $S_s(v_s^*) > S_w(v_w^*)$.

Finally we turn to the last claim in the proposition—that agents prefer trading immediately rather than waiting. Start with buyers. Suppose a buyer encounters a house in season x and the realized valuation v exceeds the threshold v_x . If he buys he obtains the payoff $v/(1 - \beta) - p_x$. If he waits he obtains $\beta\Omega_{\tilde{x}}$ which is the present value of being a buyer in the next season. Now, rearrange the indifference condition (1) in to obtain

$$v_x/(1 - \beta) - p_x = \beta\Omega_{\tilde{x}}.$$

Since $v > v_x$ it follows that $v/(1 - \beta) - p_x$ exceeds $\beta\Omega_{\tilde{x}}$ and therefore the buyer is strictly better off by purchasing right away.

Now turn to sellers. If a seller decides to sell in the current season x , he obtains price p_x . If he waits he obtains the the present value remaining on the market in the next season $\beta\Pi_{\tilde{x}}$. Recall that

$$p_x - \beta\Pi_{\tilde{x}} = S_x(v_x) / \{f_x(v_x)(1 - \beta)\}.$$

The expression on the right hand side is positive for all v_x , hence $p_x > \beta\Pi_{\tilde{x}}$. Said differently, sellers, too, are better off trading immediately. This completes the proof. ■