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Dear Professor Nemat-Nasser,

I am sending our new paper: The JKR-type adhesive contact problems for transversely isotropic elastic solids for publication in Mechanics of Materials.

(a) The paper is not concurrently submitted for publication elsewhere.
(b) As it indicated in the text of the paper the results related to pure isotropic materials are under review by JMPS.
(c) The paper has been previously submitted for possible publication only to MoM.

Yours sincerely,

Feodor Borodich

Highlights (for review)

Adhesive contact is studied for transversely isotropic materials in the framework of the JKR theory.
The theory is extended to much more general shapes of contacting axisymmetric solids,
namely the distance between the solids is
described by a monomial (power-law) function of an arbitrary degree.
It is shown that the formulae for extended JKR contact model for transversely isotropic materials have the same
mathematical form as the corresponding formulae for isotropic materials, however the effective elastic contact moduli have
different expression for different materials. The dimensionless relations between the actual force, displacements
and contact radius are given in explicit form.
Connections of the problems to nanoindentation of transversely isotropic materials are discussed.

The JKR-type adhesive contact problems for transversely isotropic elastic solids

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Abstract

The JKR (Johnson, Kendall, and Roberts) and Boussinesq-Kendall models describe adhesive frictionless contact between two isotropic elastic spheres or between a flat end punch and an elastic isotropic half-space. Here adhesive contact is studied for transversely isotropic materials in the framework of the JKR theory. The theory is extended to much more general shapes of contacting axisymmetric solids, namely the distance between the solids is described by a monomial (power-law) function of an arbitrary degree $d \geq 1$. The classic JKR and Boussinesq-Kendall models can be considered as two particular cases of these problems, when the degree of the punch d is equal to two or it goes to infinity, respectively. It is shown that the formulae for extended JKR contact model for transversely isotropic materials have the same mathematical form as the corresponding formulae for isotropic materials; however the effective elastic contact moduli have different expression for different materials. The dimensionless relations between the actual force, displacements and contact radius are given in explicit form. Connections of the problems to nanoindentation of transversely isotropic materials are discussed.

Keywords: JKR theory, adhesive contact, no-slip, power-law punches, transversely isotropic solids

1 Introduction

Molecular adhesion has been studied for a long period of time and has usually a negligible effect on surface interactions at the macro-scale, whereas it becomes increasingly significant as the contact size decreases. In particular, adhesion of surfaces plays a key role in the development of modern micro and nanotechnology. The JKR (Johnson, Kendall, and Roberts) theory of adhesive contact [1] has been widely used as a basis for modelling of various phenomena, in particular biological phenomena such as adhesion of cells, viruses, attachment devices of insects and so on ([2,3]). However, the JKR theory was originally developed for linearly elastic isotropic materials [1], while many natural and artificial materials are in fact anisotropic. Anisotropy is also a feature of many biological materials.

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The mechanics of anisotropic elastic materials is a quite developed research field (see, e.g., [4, 5]). The contact problems for anisotropic solids were studied in many papers. Willis [6, 7] studied both Hertzian and Boussinesq contact problems for anisotropic bodies and showed that the functional form of the pressure distribution between the contacting bodies can be found explicitly and therefore, the problem is reduced to finding the displacements due to a pressure distribution of this form. Later it was shown [8–11] that the Hertz-type contact problems for generally anisotropic solids have the same self-similar properties as isotropic solids [12]. Borodich presented the re-scaling formulae for the size of the contact region, the stress fields and the displacement of the bodies. Using the similarity approach, it can be easily shown [13, 14] that the displacement of any surface point of an anisotropic linear elastic half space under the influence of a point load P is proportional to the ratio, P/r, where r is the distance to the point of the application of the load. The detailed expression for the Green's function can be found in [15]. However, it is difficult to expect to obtain an analytical solution to the contact problems for generally anisotropic solids except for transversely isotropic materials [6].

Transversely isotropy is a very important case of anisotropy because many natural and artificial materials behave effectively as transversely isotropic elastic solids. For example, modern tribological coatings or layered composite materials can be often described as having transversely isotropic properties. Various problems and approaches to mechanics of transversely isotropic materials were discussed in [16]. Numerous papers were devoted to the study of various problems of mechanics for transversely isotropic materials (see., e.g., [17–21]), in particular to problems contact and indentation (see, e.g., [9, 22, 23] and references therein). Referring to Sveklo [24], Willis argued that in the case of transversely isotropic materials the Hertzian contact problem can be reduced to one of potential theory [6]. Green and Zerna also showed the similarities between isotropic and transversely isotropic frictionless two-dimensional indentation problems [25]. A similar result was presented earlier by Lekhnitskii, who computed the stresses in the interior of a transversely isotropic half-space loaded by a normal concentrated load [4, 17]. Independently of Willis, Conway at al. [26] presented an analytical expression for solution to the Hertz frictionless contact between transversely isotropic solids using the Lekhnitskii results [4, 17]).

The term "adhesion" may have rather different meanings [27]; e.g. it can be caused by chemical bonding, van der Waals forces (vdW) or by electrostatic forces. Various methods were developed in order to take the adhesive forces into account, in particular, by pointwise integration of the interaction forces between points of the bodies, whose interaction energy is proportional to r^{-6} of the distance r between the points. In 1932 Bradley [28] considered attraction between two absolutely rigid spheres by calculating pointwise the attraction of each point of one sphere to another one. Assuming additivity of the London forces, he calculated the total force of adhesion between the spheres P_c. In this paper only molecular adhesion caused by vdW forces is considered and it is assumed that one needs to know the work of adhesion, w that is equal to the energy needed to separate two dissimilar surfaces from contact to infinity to study contact problems with molecular adhesion.

In 1934 Derjaguin published a series of papers (see, e.g. [29]) where he studied the influence of adhesion on friction and contact between elastic solids. Derjaguin pointed out that to calculate adhesive interactions between particles one needs to take into account their deformations. He presented the first attempt to consider the problem of adhesion between elastic spheres or between an elastic sphere and an elastic half-space. He assumed that the deformed shape of the sphere can be calculated by solving the Hertz contact problem and suggested to calculate the adhesive interaction by using only attraction between points at the surfaces of the solids and by introduction of the work of adhesion (this is the so-called Derjaguin approximation). Unfortunately, some of his assumptions and calculations were not correct. Nevertheless, his basic *argument was correct because it equated the work done by the surface attractions against the work of deformation in the elastic spheres* (see [30], p. 183). Indeed, according to Clapeyron's theorem, the work of the external forces (both the surface tractions and the body forces) is stored in the linear-elastic body in the form of the strain energy (see, e.g. [31]). In the problem under consideration, the body forces are the forces of adhesion.

Starting from pioneering papers by Derjaguin, the mechanics of adhesive contact between isotropic elastic solids developed into a well established branch of science. The Derjaguin approximation (see [29], p. 156) can be formulated as follows: (i) instead of the pairwise summation of the the interactions between the elements of solids (as did Bradley [28]), it reduces the volume molecular attractions to surface interactions; (ii) the surface interactions are taken into account only between closest elements of the surfaces, and (iii) the interaction energy per unit area between small elements of curved surfaces is the same as this energy between two parallel infinite planar surfaces. The approximation is implicitly involved in many modern models of adhesive contact (however, these assumptions are usually used without referring to Derjaguin). One can also take into account the adhesive forces by introducing an interaction potential between points on the surfaces, for example, a Lennard-Jones potential (see, for example, [32, 33]) or by using piecewise-constant approximations of these potentials [34, 35]. Nowadays there are several well-established classic models of adhesive contract that include the JKR theory [1], the DMT (Derjaguin-Muller-Toporov) theory [36], and the Maugis transition solution between the JKR and DMT theories [37]. A detailed description of the theories is given by Maugis [34] (see also a recent discussion in [38]).

According to Kendall [30], the JKR theory historically was developed in the following steps. In 1958 Johnson [39] argued that the adhesive contact problem can be considered by adding two simple stress distributions, namely the Hertz stress field to a rigid flat-ended punch tensile stress distribution. This approach is very fruitful. However, Johnson got a wrong conclusion that the infinite tension at the periphery of the contact would ensure that the spheres would peel apart when the compressive load was removed. In 1970 Kendall and Roberts discussed the experimental observations of the contact spots that were larger than expected from the Hertz equation. They found *that the answer lay in applying Derjaguins method ... to Johnsons stress distribution.* Johnson presented a mathematical realization of this idea *an evening later*; in this way they created the famous JKR theory of adhesive contact (see, [1], pp. 183-186). Here adhesive contact is studied for transversely isotropic materials in the framework of the JKR theory.

Adhesive contact problems have been already studied for transversely isotropic materials. These were mainly two-dimensional problems for cylinders (see, e.g., [40, 41]). Only recently it has been shown that the JKR and DMT models can be extended to the problem of contact between transversely isotropic spheres [42]. However, solids in contact may often have more general shapes than spherical ones; in particular the shape functions f of indenters can be well described as monomial functions of radius r.

In this paper it will be assumed that the distances between the contacting solids may be described as axisymmetric monomial functions of arbitrary degrees

$$f(r) = B_d r^d, \tag{1}$$

where d is the degree of the monomial function and B_d is the constant of the shape.

For isotropic solids, the adhesive contact problems for bodies of monomial shapes were considered earlier [33, 43, 44]. The classic JKR [1] and Boussinesq-Kendall [45] models can be considered as two particular cases of these problems when the degree of the punch d is equal to two or it goes to infinity respectively. Here the JKR theory is extended to problem of contact between axisymmetric solids such that the distance between the solids is described by a monomial (power-law) function of an arbitrary degree $d \geq 1$.

The paper is organized as follows:

In §2 we give some preliminary information concerning frictionless nonadhesive Hertz-type contact problems and the JKR-type models of adhesive contact.

In §3 the contact problems for transversely isotropic materials are considered. First the nonadhesive Hertz-type contact problems and then the Boussinesq–Kendall problem of an adhesive contact for a flat ended punch are discussed. Finally the JKR theory is extended to the case of adhesive contact between transversely isotropic material and a punch whose shape is described by monomial (power-law) function of radius of an arbitrary degree $d \geq 1$.

The depth-sensing indentation, i.e. the continuously monitoring of the $P - \delta$ diagram where P is the applied load and δ is the displacement (the approach of the distant points of the indenter and the sample) are especially important when mechanical properties of materials are studied using very small volumes of materials. The $P-\delta$ diagrams for material characterization are so important that these diagrams are often considered in the materials science community as "finger-prints" of materials. In §4 connections between the obtained results and problems of nanoindentation are discussed. The obtained equations for general monomial punches are written in dimensionless form. It is shown that the dimensionless relations between the actual force, displacements and contact radius are the same as in the case of contact between isotropic materials. The graphs of the relations are presented for some values of d, $1 \leq d \leq 3$.

2 Adhesive and Nonadhesive Indentation Problems

Let us use both the Cartesian and cylindrical coordinate frames, namely $x_1 = x, x_2 = y, x_3 = z$ and ρ, ϕ, z, where $\rho = \sqrt{x^2 + y^2}$ and $x = \rho \cos \phi, y = \rho \sin \phi$.

First the non-adhesive Hertz type contact problem will be considered. Then the conditions within the contact region for adhesive contact will be described.

2.1 Nonadhesive Hertz type contact problems

The non-adhesive Hertz theory considers three-dimensional (3D) frictionless contact of two isotropic, linear elastic solids. In a 3D Hertz contact problem, it is assumed that the punch shape is described by an elliptic paraboloid and the contact region is an ellipse. However, in the general Hertz type contact problems these assumptions are omitted. Hence, the formulation of the problem is geometrically linear, the contact region is unknown and should be found, only vertical displacements of the boundary are taken into account, and the problem has the same boundary conditions within and outside the contact region as in the original Hertz problem. It is possible to show that the problem is mathematically equivalent to the problem of contact between an indenter whose shape function f is equal to the initial distance between the surfaces, i.e. $f = f^+ + f^-$ where f^+ and f^- are the shape functions of the solids. In turn, this problem can be reduced to the problem of contact between a rigid indenter (a punch) and an elastic half-space.

It is assumed that a rigid indenter (a punch) is pressed by the force P to a boundary of the contacting solid. In a geometrically linear formulation of the contact problem, this solid can be considered as a positive half-space $x_3 \geq 0$. Initially, there is only one point of contact between the punch and the half-space. Let us put the origin (O) of Cartesian x_1, x_2, x_3 coordinates at the point of initial contact between the punch and the half-space $x_3 \geq 0$. We denote the boundary plane $x_3 = 0$ by $\mathbb{R}^2$. Hence, the equation of the surface given by a function f, can be written as $x_3 = -f(x_1, x_2), \quad f \geq 0$.

After the punch contacts with the half–space, displacements u_i and stresses σ_{ij} are generated. If material properties are time independent then the current state of the contact process can be completely characterized by the compressing force (P).

Thus, it is supposed that the shape of the punch and the external parameter of the problem P are given and one has to find the bounded region G on the boundary plane $x_3 = 0$ of the half-space at the points where the punch and the medium are in mutual contact (for axisymmetric problems, the contact radius a), displacements u_i, and stresses σ_{ij}.

In the problem the quantities sought satisfy the following equations

$$\begin{gathered}\sigma_{ji,j} = 0, \quad i, j = 1, 2, 3;\\ \sigma_{ij} = \mathcal{F}(\epsilon_{ij}), \quad \epsilon_{ij} = (u_{i,j} + u_{j,i})/2;\\ \iint_{\mathbb{R}^2} \sigma_{33}(\mathbf{x}, \mathcal{P}) d\mathbf{x} = -P,\end{gathered} \tag{2}$$

in which ϵ_{ij} are the components of the strain tensor and $\mathcal{F}$ is the operator of constitutive relations for the material. Here and henceforth, a comma before the subscript denotes the derivative with respect to the corresponding coordinate; and summation from 1 to 3 is assumed over repeated Latin subscripts.

The material behavior of the medium may be linear and non-linear, anisotropic or isotropic, depending on the form of the operator $\mathcal{F}$. For arbitrary anisotropic, linear elastic media, the constitutive relations have the form of Hooke's law

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} \quad \text{or} \quad \sigma_{ij} = c_{ijkl}u_{k,l}, \qquad c_{ijkl} = c_{jikl} = c_{klij} \tag{3}$$

where c_{ijkl} are components of the tensor of elastic constants.

The displacement vector $\mathbf{u}$ should satisfy the conditions at infinity

$$\mathbf{u}(\mathbf{x}) \to 0 \quad \text{when} \quad |\mathbf{x}| \to \infty. \tag{4}$$

Let us define the contact region G as an open region such that if $\mathbf{x} \in G$ then the gap $(u_3 - g)$ between the punch and the half-space is equal to zero and surface stresses are non-positive, while for $\mathbf{x} \in \mathbb{R}^2 \setminus G$ the gap is positive and the stresses are equal to zero. Thus, $\mathbf{u}$ and σ_{ij} should satisfy the following boundary conditions within and outside the contact region

$$\begin{aligned} &u_3(\mathbf{x}; \mathcal{P}) = g(\mathbf{x}; \mathcal{P}), \qquad \sigma_{33}(\mathbf{x}; \mathcal{P}) \leq 0, \qquad \mathbf{x} \in G(\mathcal{P}),\\ &u_3(\mathbf{x}; \mathcal{P}) > g(\mathbf{x}; \mathcal{P}), \qquad \sigma_{3i}(\mathbf{x}; \mathcal{P}) = 0, \qquad \mathbf{x} \in \mathbb{R}^2 \setminus G(\mathcal{P}),\end{aligned} \tag{5}$$

In the problem of vertical indentation of an isotropic or transversely-isotropic media by an axi-symmetric punch, the contact region is always a circle. This fact simplifies analysis of the problem. The analysis of three-dimensional contact is usually more complicated.

For the general case of the problem of vertical pressing, we have

$$g(\mathbf{x};\mathcal{P}) = \delta - f(x_1, x_2). \tag{6}$$

If one considers the frictionless problem, then the following two conditions hold within the contact region

$$\sigma_{31}(\mathbf{x};\mathcal{P}) = \sigma_{32}(\mathbf{x};\mathcal{P}) = 0, \quad \mathbf{x} \in G(\mathcal{P}) \subset \mathbb{R}^2. \tag{7}$$

2.2 The JKR approach to adhesive contact

The original JKR approach assumed that the problems are frictionless, i.e. the two conditions (7) hold within the contact region G. The JKR approach is based on the use of a geometrically linear formulation of the contact problem. If there were no surface forces of attraction, the radius of the contact area under a punch subjected to the external load P_0 would be a_0 and it could be found by solving the Hertz-type contact problem. However, in the presence of the forces of molecular adhesion, the equilibrium contact radius would be of value $a_1 > a_0$ under the same force P_0.

The JKR model considers the total energy of the contact system U_T as made up of three terms, the stored elastic energy U_E, the mechanical energy in the applied load U_M and the surface energy U_S. It is assumed that the contact system has come to its real state in two steps: (i) first it achieves a real contact radius a_1 and an apparent depth of indentation δ_1 under some apparent Hertz load P_1, then (ii) it is unloaded from P_1 to a real value of the external load P_0 keeping the constant contact radius a_1 (Fig. 1). The Boussinesq solution for contact between an elastic half-space and a flat punch of radius a_1 may be used on the latter step. Hence, a combination of both the Hertz contact problem for two elastic spheres and the Boussinesq relation for a flat ended cylindrical indenter is used. One can show that the Boussinesq relation for a flat ended cylindrical indenter of radius a contacting an arbitrary anisotropic elastic half-space is

$$\delta = P/[ac_{1111}F(c_{ijkl}/c_{1111})] \tag{8}$$

where F is a function of the dimensionless elastic material constants.

In this case, one can calculate U_E as the difference between the stored elastic energies $(U_E)_1$ and $(U_E)_2$ on loading and unloading branches respectively. Therefore,

$$(U_E)_1 = P_1\delta_1 - \int_0^{P_1} \delta dP. \tag{9}$$

Using the Boussinesq solution (8), we obtain for the unloading branch

$$(U_E)_2 = \int_{P_0}^{P_1} \frac{P}{c_{1111}F(c_{ijkl}/c_{1111})a_1} dP = \frac{P_1^2 - P_0^2}{2c_{1111}F(c_{ijkl}/c_{1111})a_1}. \tag{10}$$

Thus, the stored elastic energy U_E is

$$U_E = (U_E)_1 - (U_E)_2. \tag{11}$$

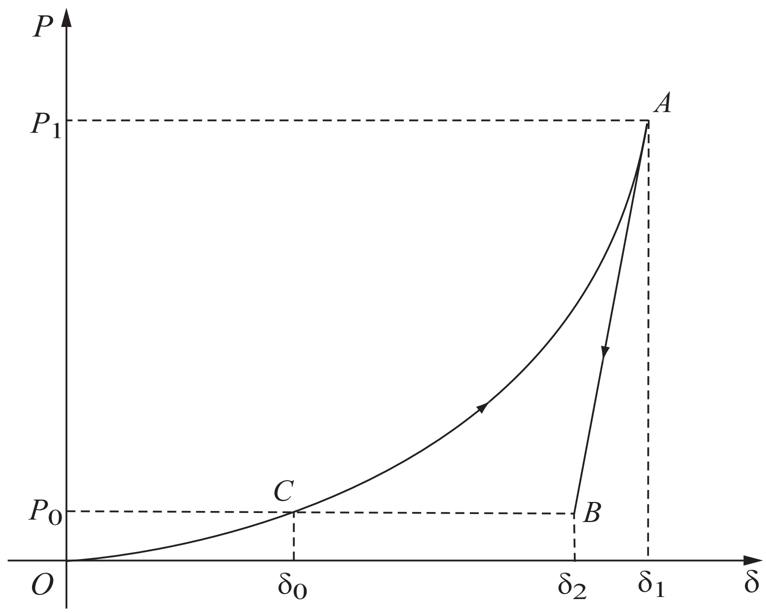

Figure 1: Loading diagram explaining the JKR model of adhesive contact. At branch OA the loading curve $P - \delta$ follows the Hertz-type non-adhesive contact relation, while the relation at the branch AB follows the linear Boussinesq relation.

The mechanical energy in the applied load

$$U_M = -P_0\delta_0 = -P_0(\delta_1 - \Delta\delta) \tag{12}$$

where $\Delta\delta = \delta_1 - \delta_2$ is the change in the depth of penetration due to unloading.

According to the Derjaguin assumptions, the adhesive interactions are reduced to the surface forces acting perpendicularly to the boundary of the half-space. The JKR theory considers only the adhesive forces acting within the contact region. The original JKR theory was developed for isotropic materials, therefore, the contact region is always a circle and the surface energy can be written as

$$U_S = -w\pi a_1^2. \tag{13}$$

The total energy U_T can be obtained by summation of (11), (12) and (13), i.e. $U_T = U_E + U_M + U_S$. It is assumed in the JKR model that the equilibrium at contact satisfies the equation

$$\frac{dU_T}{da} = 0, \quad \text{or} \quad \frac{dU_T}{dP_1} = 0. \tag{14}$$

The above was applied to the case of the initial distance between contacting solids being described by a paraboloid of revolution $z = r^2/(2R)$ (this is a very good approximation for a sphere) or contact of two spheres of the effective radius R, $1/R = 1/R^+ + 1/R^-$ where R^+ and R^- are the radii of the spheres.

3 Contact problems for transversely isotropic materials

3.1 Elastic transversely isotropic materials

If through all points of an elastic solid there pass parallel planes of elastic symmetry in which all directions are elastically equivalent, then these planes are planes of isotropy and the solid is called transversely isotropic [4]. To describe the general anisotropy of elastic solids, the tensor of elastic constants c_{ijkl} has been used, however the matrix form is more convenient for description of transversely isotropic solids. This is because the tensor of elastic constants is reduced to five elastic constants of the material a_{11}, a_{12}, a_{13}, a_{33}, and a_{44}. Let the z axis be taken normal to a plane of isotropy, i.e. z axis is the axis of rotational symmetry, then Hooke's law (3) becomes

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{23} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{11} & a_{13} & 0 & 0 & 0 \\ a_{13} & a_{13} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11}-a_{12} & 0 & \\ 0 & 0 & 0 & 0 & a_{44}/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{44}/2 \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix}. \tag{15}$$

In cylindrical coordinates Hooke's law for a transversely isotropic elastic material becomes

$$\begin{pmatrix} \epsilon_{rr} \\ \epsilon_{\phi\phi} \\ \epsilon_{zz} \\ \epsilon_{r\phi} \\ \epsilon_{rz} \\ \epsilon_{\phi z} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{11} & a_{13} & 0 & 0 & 0 \\ a_{13} & a_{13} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11}-a_{12} & 0 & \\ 0 & 0 & 0 & 0 & a_{44}/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{44}/2 \end{pmatrix} \begin{pmatrix} \sigma_{rr} \\ \sigma_{\phi\phi} \\ \sigma_{zz} \\ \sigma_{r\phi} \\ \sigma_{rz} \\ \sigma_{\phi z} \end{pmatrix} \tag{16}$$

or

$$\begin{pmatrix} \sigma_{rr} \\ \sigma_{\phi\phi} \\ \sigma_{zz} \\ \sigma_{r\phi} \\ \sigma_{rz} \\ \sigma_{\phi z} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\ A_{12} & A_{11} & A_{13} & 0 & 0 & 0 \\ A_{13} & A_{13} & A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{11}-A_{12} & 0 & \\ 0 & 0 & 0 & 0 & 2A_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2A_{44} \end{pmatrix} \begin{pmatrix} \epsilon_{rr} \\ \epsilon_{\phi\phi} \\ \epsilon_{zz} \\ \epsilon_{r\phi} \\ \epsilon_{rz} \\ \epsilon_{\phi z} \end{pmatrix} \tag{17}$$

where A_{11}, A_{12}, A_{13}, A_{33}, and A_{44} are five elastic constants of the material

$$A_{11} = \frac{a_{11}a_{33} - a_{13}^2}{(a_{11}-a_{12})m}, \qquad A_{12} = \frac{a_{13}^2 - a_{12}a_{33}}{(a_{11}-a_{12})m}, \qquad A_{13} = \frac{-a_{13}}{m}, \tag{18}$$

$$A_{33} = \frac{a_{11}+a_{12}}{m}, \qquad A_{44} = \frac{1}{a_{44}} \qquad A_{11} - A_{12}, = \frac{1}{a_{11}-a_{12}} \tag{19}$$

and

$$m = (a_{11}+a_{12})a_{33} + 2a_{13}^2.$$

3.2 Non-adhesive contact problems for transversely isotropic materials

Using the above expressions and results by Lekhnitskii [4, 26], one can show that the solution of the Boussinesq problem for a concentrated load P is

$$u_3(r,0) = \frac{P}{\pi E_{TI} r}$$

where the coefficient $(E_{TI})^{-1}$ is

$$(E_{TI})^{-1} = -\frac{S_1 + S_2}{2D^{1/2}(AC - D)}[(D - 2BD + AC)a_{11} - (2D - BD - AC)a_{12}] \tag{20}$$

and

$$A = \frac{a_{13}(a_{11} - a_{12})}{a_{11}a_{33} - a_{13}^2}, \quad B = \frac{a_{13}(a_{13} + a_{44}) - a_{12}a_{33}}{a_{11}a_{33} - a_{13}^2},$$

$$C = \frac{a_{13}(a_{11} - a_{12}) + a_{11}a_{44}}{a_{11}a_{33} - a_{13}^2}, \quad D = \frac{a_{11}^2 - a_{12}^2}{a_{11}a_{33} - a_{13}^2},$$

$$2DS_{1,2}^2 = A + C \pm [(A + C)^2 - 4D]^{1/2},$$

$$S_{1,2} = \sqrt{\frac{A + C}{2D} \pm \left[\left(\frac{A + C}{2D}\right)^2 - \frac{1}{D}\right]^{1/2}}.$$

For isotropic materials one has $E_{TI} = E^*$ where E^* is the effective elastic contact modulus

$$\frac{1}{E^*} = \frac{1 - (\nu^+)^2}{E^+} + \frac{1 - (\nu^-)^2}{E^-}.$$

Here E^+ and ν^+ and E^- and ν^- are the elastic modulus and the Poisson ratio of the first and the second solid respectively. Further in this paper only rigid axisymmetric indenters are considered and, therefore, $E^- = \infty$ and $E^* = E/(1 - \nu^2)$ where E and ν are the elastic modulus and the Poisson ratio of the half-space, respectively.

3.3 The Boussinesq-Kendall problem

Consider an axisymmetric flat ended punch of radius a_1 that is vertically pressed into an elastic half-space. For isotropic elastic materials, this problem was considered by Boussinesq (see, e.g. [47]), and frictionless contact with molecular adhesion was studied by Kendall [45]. Let us consider the Boussinesq-Kendall problem for transversely isotropic materials using the Kendall arguments.

It follows from the Boussinesq solution (8) and (20) that the transversely isotropic elastic material deforms as

$$\delta = \frac{P}{2E_{TI}a_1}. \tag{21}$$

The surface energy is given as above by (13). Using (21), one obtains that the stored elastic energy U_E and the mechanical energy of the applied load U_M are respectively

$$U_E = \int P d\delta = \frac{P^2}{4E_{TI}a_1} + A, \quad U_M = -P\delta + B = -\frac{P^2}{2E_{TI}a_1} + B \tag{22}$$

where A and B are arbitrary constants.

The total energy U_T can be obtained by summation of all components given by (13) and (22)

$$U_T = -w\pi a_1^2 - \frac{P^2}{4E_{TI}a_1} + A + B. \tag{23}$$

From the equilibrium equation (14), one has

$$\frac{dU_T}{da_1} = 0 = -2w\pi a_1 + \frac{P_c^2}{4E_{TI}a_1^2} \tag{24}$$

and, hence, one may obtain the adherence force (the pull-off force) of a flat ended circular punch of radius a_1

$$P_c = \sqrt{8\pi w E_{TI} a_1^3}. \tag{25}$$

One can see from (25) that the adherence force is proportional neither to the energy of adhesion nor to the area of the contact [34].

3.4 Adhesive contact between a monomial indenter and a transversely isotropic elastic material

Let us generalize the JKR model of contact with molecular adhesion and consider the case of the distances between the contacting solids being described as axisymmetric monomial functions (1) of arbitrary degrees d.

The non-adhesive frictionless Hertz type contact problem for an isotropic elastic material and punches described by (1) was given by Galin [46,47]. It follows from this solution and the above discussion that for transversely isotropic elastic materials, the contact radius a_0 under the external load P_0 is given by

$$a_0 = \left(\frac{P_0}{C(d)E_{TI}B_d}\right)^{1/(d+1)}, \quad C(d) = \frac{d^2}{d+1}2^{d-1}\frac{[\Gamma(d/2)]^2}{\Gamma(d)}. \tag{26}$$

The contact radius a_1 and depth of indentation δ_1 under some apparent Hertz load P_1, are given by

$$a_1 = \left(\frac{P_1}{C(d)E_{TI}B_d}\right)^{1/(d+1)}, \qquad \delta_1 = \left[\frac{C(d)B_d}{(E_{TI})^d}\right]^{\frac{1}{d+1}} \left(\frac{d+1}{2d}\right) P_1^{d/(d+1)}. \tag{27}$$

Substituting (27) into (9) and (10) , we obtain

$$(U_E)_1 = \frac{d+1}{2(2d+1)} P_1^{(2d+1)/(d+1)} \left[\frac{C(d)B_d}{(E_{TI})^d}\right]^{\frac{1}{d+1}}.$$

$$(U_E)_2 = \frac{[C(d)B_d]^{1/(d+1)}}{4(E_{TI})^{d/(d+1)}} \left(P_1^{(2d+1)/(d+1)} - P_0^2 P_1^{-1/(d+1)} \right).$$

Using the above expressions and (11), and substituting (27) into (12) and (13), we obtain the following expressions for the components of energy

$$U_E = \frac{1}{4} \left[\frac{C(d)B_d}{(E_{TI})^d} \right]^{1/(d+1)} \left(\frac{1}{2d+1} P_1^{(2d+1)/(d+1)} + P_0^2 P_1^{-1/(d+1)} \right), \tag{28}$$

$$U_M = -P_0 \frac{d+1}{2d} \frac{[C(d)B_d]^{1/(d+1)}}{(E_{TI})^{d/(d+1)}} \left[\frac{P_1^{d/(d+1)}}{d+1} + \frac{P_0 P_1^{-1/(d+1)} d}{d+1} \right]. \tag{29}$$

and

$$U_S = -w\pi \left(\frac{P_1}{C(d)E_{TI}B_d} \right)^{2/(d+1)}. \tag{30}$$

Thus, the total energy U_T can be obtained by summation of (28), (29) and (30)

$$U_T = \frac{1}{4} \left[\frac{C(d)B_d}{(E_{TI})^d} \right]^{\frac{1}{d+1}} \left[\frac{1}{2d+1} P_1^{\frac{2d+1}{d+1}} - P_0^2 P_1^{\frac{-1}{d+1}} - \frac{2}{d} P_0 P_1^{\frac{d}{d+1}} \right] - w\pi \left(\frac{P_1}{C(d)E_{TI}B_d} \right)^{\frac{2}{d+1}}. \tag{31}$$

One may obtain from (31) and (14)

$$P_0^2 - 2P_0(C(d)B_d E_{TI}) a_1^{d+1} + (C(d)B_d E_{TI})^2 a_1^{2(d+1)} - 8w\pi E_{TI} a_1^3 = 0.$$

Solving this equation and taking the stable solution, one obtains an exact formula giving the relation between the real load P_0 and the real radius of the contact region a_1

$$P_0 = P_1 - \sqrt{8\pi w E_{TI} a_1^3} = C(d)B_d E_{TI} a_1^{d+1} - \sqrt{8\pi w E_{TI} a_1^3}. \tag{32}$$

The real displacement of the punch is $\delta_2 = (\delta_1 - \Delta\delta)$

$$\delta_2 = B_d C(d) \frac{d+1}{2d} a_1^d - \left(\frac{2\pi w a_1}{E_{TI}} \right)^{1/2} \tag{33}$$

It is convenient to write the formula for the real displacement δ_2 in the case of the frictionless boundary condition as

$$\delta_2 = \frac{B_d C(d)}{2d} a_1^d \left[1 + d\frac{P_0}{P_1} \right]. \tag{34}$$

One can see that (34) does not depend on the elastic properties of materials. In the case $d = 2$ one has $C(2) = 8/3$, $B_2 = 1/(2R)$ and (34) coincides exactly with (4.111) presented by Maugis (see [34] p. 274)

$$\delta_2 = \frac{a_1^2}{3R} \left[1 + 2\frac{P_0}{P_1} \right].$$

4 The depth-sensing indentation by non-ideal shaped indenters

The depth-sensing indentation (DSI) is the continuously monitoring of the $P - \delta$ diagram where P is the applied load and δ is the displacement (the approach of the distant points of the indenter and the sample). The DSI techniques are especially important when mechanical properties of materials are studied using very small volumes of materials (see reviews in [48,49]. The DSI testing of materials was proposed in the pioneering paper by Kalei [50] where he noted that adhesion to a solid surface (substrate) may affect the measurements obtained by DSI.

4.1 Nonadhesive indentation and the BG method

In 1975 it was suggested to use the slope S of the experimental DSI unloading curves for extraction of the effective contact modulus of materials [51] (the so-called BASh formula and its modifications)

$$S = \frac{dP}{d\delta} = C\frac{2\sqrt{A}}{\sqrt{\pi}}E^*. \tag{35}$$

Here A is the area of the contact region and C is a constant depending on the boundary conditions of the contact [52]. In fact, (35) is a semi-empirical approximation for an exact expression $S = dP/d\delta = 2CE^*a$ where a is the contact radius under an external load P. The BASh approach was also applied to to anisotropic materials [15, 22].

In practical papers devoted to applications of the DSI techniques, it is usually assumed that the indenter is a sharp pyramid or a cone. However, the nominally sharp indenters are in fact not ideal. It was shown that the indenter shape at the shallow depths may be well described by a power-law function (1) of degree d, $1 \le d \le 2$ [11, 53]. The limiting cases of this range are conical and spherical indenters. Second-order asymptotic models for canonical indenters penetrating a transversely isotropic elastic layer were presented in [23].

Although the above mentioned approach for determination of E^* is very popular for small material samples, it has a number of drawbacks. In particular, the plastic deformations of contact surfaces and the existence of the residual stresses are ignored in (35). On the other hand, the work of adhesion is usually determined by direct measurements of pull-off force of an adhesive sphere. These measurements are unstable due to instability of the load-displacement diagrams at tension, and they can be greatly affected by roughness of contacting solids. Recently a new experimental method (the BG method) have been introduced that allow the researchers to quantify the work of adhesion w and elastic contact modulus E^* of isotropic materials. The BG method is a non-direct method [54] based on an inverse analysis of all points at a bounded stable interval of the experimental force-distance curve. The application of the BG method showed that the obtained estimations of the elastic contact modulus and work of adhesion have very small error even for rather contaminated data [55, 56].

It follows from the above discussion and the results presented in [42] that the BG method can be used to quantify the work of adhesion w and elastic contact modulus E_{TI} of transversely isotropic materials.

4.2 Frictionless adhesive indentation

Let us apply the JKR type approach to problems of nanoindentation when the indenter shape near the tip has some deviation from its nominal shape. It will be assumed further that the indenter shape function can be approximated by a monomial function of radius. It follows from (32) that the radius a_1 of the contact region at $P_0 = 0$ is

$$a_1(0) = \left[\frac{8\pi w}{E_{TI} C^2(d) B_d^2}\right]^{1/(2d-1)}.$$

This value can be used as a characteristic size of the contact region in order to write dimensionless parameters. Let us write the characteristic parameters of the adhesive contact problems as

$$a^* = a_1(0), \quad P^* = \left\{\frac{(8\pi w)^{d+1}(E_{TI})^{d-2}}{[C(d)B_d]^3}\right\}^{\frac{1}{2d-1}}, \quad \delta^* = \left[\frac{2^{d+1}}{C(d)B_d}\left(\frac{\pi w}{E_{TI}}\right)^d\right]^{\frac{1}{2d-1}}. \tag{36}$$

Then (32) and (33) have the following form

$$P_0/P^* = (a_1/a^*)^{d+1} - (a_1/a^*)^{3/2} \tag{37}$$

and

$$\frac{\delta_2}{\delta^*} = \frac{d+1}{d}\left(\frac{a_1}{a^*}\right)^d - \left(\frac{a_1}{a^*}\right)^{1/2}. \tag{38}$$

4.3 Dimensionless relations for adhesive indentation

Let us denote $\bar{P} = P_0/P^*$, $\bar{a} = a_1/a^*$ and $\bar{\delta} = \delta_2/\delta^*$. Then (37) and (38) can be written as the following dimensionless relations

$$\bar{P} = \bar{a}^{d+1} - \bar{a}^{3/2} \tag{39}$$

and

$$\bar{\delta} = \frac{d+1}{d}(\bar{a})^d - (\bar{a})^{1/2} \tag{40}$$

that are valid for arbitrary axisymmetric monomial punch of degree $d \geq 1$.

The graphs of the dimensionless relations (39) and (40) for several values of the degree d of the indenter shape monoms are shown respectively in Figure 2 and Figure 3.

The instability point of $\bar{P} - \bar{\delta}$ curve is at the point where $d\bar{P}/d\bar{\delta} = 0$. Taken into account that $d\bar{P}/d\bar{\delta} = d\bar{P}/d\bar{a} \cdot d\bar{a}/d\bar{\delta}$, one obtains from (39) at the instability point

$$d\bar{P}/d\bar{a} = (d+1)\bar{a}^d - (3/2)\bar{a}^{1/2} = 0.$$

Solving this equation, one obtains for a dimensionless critical contact radius

$$\bar{a}_c = \left[\frac{3}{2(d+1)}\right]^{\frac{2}{2d-1}}. \tag{41}$$

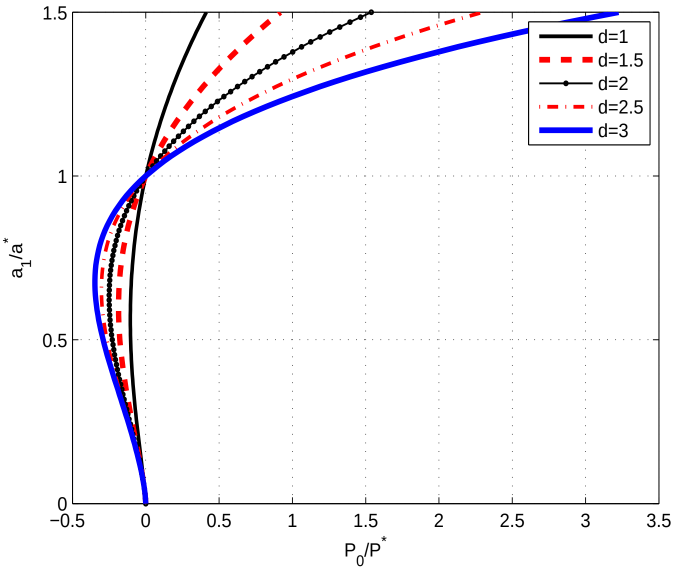

Figure 2: The dimensionless JKR $\bar{P}$ -$\bar{a}$ relation for monomial indenters.

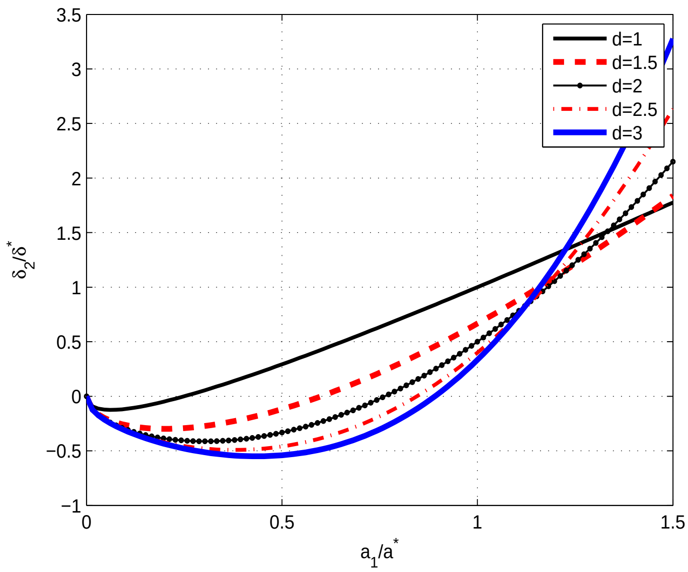

Figure 3: The dimensionless JKR $\bar{\delta}$ -$\bar{a}$ relation for monomial indenters.

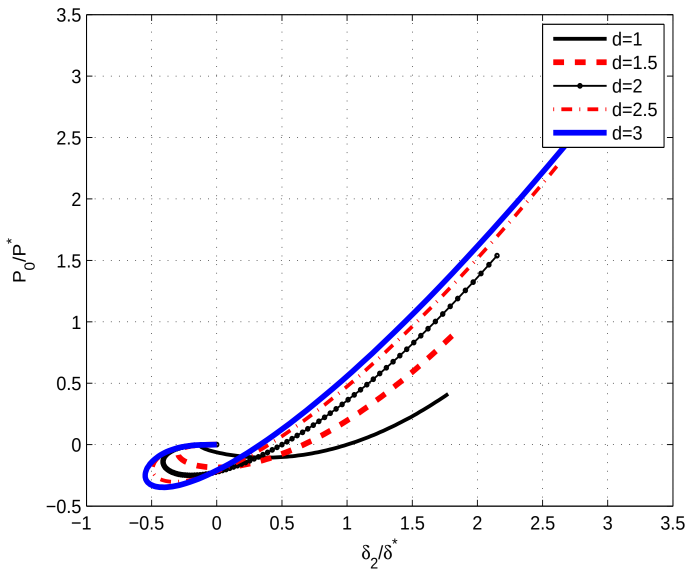

Figure 4: The dimensionless JKR $\bar{P}$ -$\bar{\delta}$ relation for monomial indenters within the $1 \leq d \leq 3$ range.

Substituting this expression into (39), one obtains the expression for the critical load $\bar{P}_c$ (the adherence force at fixed load)

$$\bar{P}_c = \left[\frac{3}{2(d+1)}\right]^{\frac{2(d+1)}{2d-1}} - \left[\frac{3}{2(d+1)}\right]^{\frac{3}{2d-1}}. \tag{42}$$

The formulae (37) -(42) do not depend on any elastic property of material and therefore, they are absolutely the same as for isotropic materials. The graphs of the dimensionless $\bar{P}$ -$\bar{\delta}$ relation for monomial indenters whose degree d are within the $1 \leq d \leq 3$ range are shown in Figure 4.

A detailed description of the dimensionless $\bar{P}$ -$\bar{\delta}$ relation for indenters within the $1 \leq d \leq 2$ range are shown in Figure 5.

Using the above general solution for monomial punches, one can consider analytically the adhesive contact for spherical and conical indenters. For a sphere of radius R, one has $d = 2$, $B_2 = 1/(2R)$, $C(2) = 8/3$, $f(r) = B_2 r^2$ and the expression (32) coincides with the classic JKR formula. Further one has $\bar{P}_c = -1/4$. In dimensional form one has $P^* = 6\pi R w$ and obtains respectively the classic JKR value $P_c = -(1/4)P^* = -(3/2)\pi R w$.

For a conical indenter, one can get the dimensional form from (36). This gives the following values

$$P^* = 512 w^2/(\pi E_{TI} B_1^3) \quad \text{and} \quad P_c = -54 w^2/(\pi E_{TI} B_1^3). \tag{43}$$

The contact radius and displacement under zero load are respectively

$$a_1(0) = a^* = 32 w/(\pi E_{TI} B_1^2) \quad \text{and} \quad \delta_2(0) = \delta^* = 8 w/(E_{TI} B_1).$$

For isotropic materials, these expressions coincide with the formulae presented by Maugis except that the formulae (4.253) for $\delta_2(0)$ in [34] has a wrong coefficient 24 (see also a discussion in [57]).

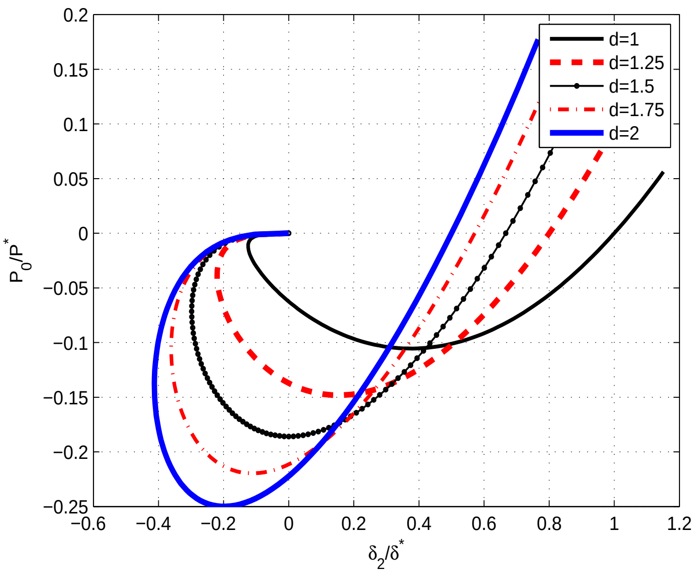

Figure 5: A detailed description of the $\bar{P}$ -$\bar{\delta}$ relation for monomial indenters within the $1 \leq d \leq 2$ range.

5 Discussion and Conclusion

The derivation of the main formulae of the above considered JKR and Boussinesq-Kendall models have been based on the assumption that the material points within the contact region can move along the punch surface without any friction. However, it is more natural to assume that a material point that came into contact with the punch sticks to its surface. Hence, the no-slip boundary conditions [52, 58–61] are physically more appropriate for modeling contact problems with molecular adhesion than the frictionless contact model. The no-slip boundary conditions have been already used for studying mechanics of adhesive contact [40,41,57,62,63].

It is known that the frictionless JKR results can be obtained by the use of linear fracture mechanics concepts [34,37,64]. In the frictionless case equilibrium is given by $G = w$ where G is the energy release rate at the edge of the contact. In the no-slip case, one has to be careful in the use of the fracture mechanics formalism due to the oscillations of the stress fields near the edge of the contact region. Various issues related to the use of the fracture mechanics concepts in application to mechanics of adhesive contact between isotropic elastic materials were discussed (see, e.g. [62,64,65]). In particular, it has been argued that the adhesion energy is not a material constant independent of the local failure mode but rather is a function of the mode mixity [62, 65]. However, the authors prefer to use Johnson's interpretation [64]: the work of adhesion w is a material constant, while the critical energy release rate G_c is $G_c = w[1 + \alpha(K_{II}^2/K_I^2)$ where the parameter α can vary from 0 to 1.0 and K_I, and K_{II} are mode I and II stress intensity, respectively. Note that the size of the ring within the contact region where the above mentioned oscillations occur is very small [66]. These oscillations have no physical meaning and the assumptions of the linear elasticity are violated in the ring, therefore Rvachev and Protsenko [67] referred to the corresponding strains as fictitious strains. In the present work the authors have followed the original JKR approach as closely as possible. Hence, it is assumed that the contribution to the system energy by adhesive forces can be calculated by (13) even in the case of no-slip contact. In this case, using the known approaches [68, 69], one could extend the above results to the case of no-slip adhesive contact with transversely isotropic materials. However, the recent analysis [57, 63] shows that for compressible materials, the critical radius of the contact region and the corresponding critical load in the case of no-slip contact are just slightly less than the values predicted by the frictionless JKR-type models. Therefore, no-slip boundary conditions have not been considered.

The original JKR theory was restricted to a very important case of contact between isotropic spheres. Using the Galin solution [46] in Borodich and Keer [48] representation, one can show that the JKR theory for transversely isotropic materials leads to the following expressions for an arbitrary convex body of revolution $f(r)$, $f(0) = 0$

$$P_1 = P_0 + \sqrt{8\pi w E_{TI} a_1^3}, \quad \delta_2 = \delta_1 - \sqrt{\frac{2\pi w a_1}{E_{TI}}}$$

or

$$P_0 = P_1 - \sqrt{8\pi w E_{TI} a_1^3} = 2E_{TI} \int_0^{a_1} \frac{r^2 f'(r) dr}{\sqrt{a_1^2 - r^2}} - \sqrt{8\pi w E_{TI} a_1^3}$$

and

$$\delta_2 = \int_0^{a_1} \frac{f'(r)}{\sqrt{1 - r^2/a_1^2}} dr - \left(\frac{2\pi w a_1}{E_{TI}}\right)^{1/2}.$$

However, we concentrated here on the solution to the JKR problems for monomial solids, while a discussion of problems for arbitrary solids of revolution is out the scope of the paper. In this paper the classic JKR approach has been generalized to transversely isotropic materials and punches whose shape function is an arbitrary monomial (power-law) function of degree $d \geq 1$. The Boussinesq-Kendall model of adhesive contact for a flat ended punch has been also discussed. The JKR and Boussinesq-Kendall models can be considered as two particular cases of contact problems with molecular adhesion for monomial punches, when the degree of the punch d is equal to two or it goes to infinity respectively. It has been shown that the solution to the JKR-type adhesive contact problems reduced to the same dimensionless relations between the actual force, displacements and contact radius as for isotropic materials. The results obtained are applied to problems of nanoindentation when the indenter shape near the tip has some deviation from its nominal shape and the shape function can be approximated by a monomial function of radius.

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References

[1] Johnson, K.L., Kendall, K. and Roberts, A.D. (1971) Surface energy and the contact of elastic solids. *Proc. R.. Soc. Lond. A*, **324**, 301–313.

[2] Scherge, M. and Gorb S.N. (2001) Biological Micro- and Nanotribology: Nature's Solutions. Springer-Verlag, Berlin.

[3] Kendall, K., Kendall, M., Rehfeldt, F. (2011) *Adhesion of Cells, Viruses and Nanoparticles.* Springer, Dordrecht.

[4] Lekhnitskii, S. G. (1981) *Theory of Elasticity of an Anisotropic Body.* Moscow: Mir Publishers.

[5] Ting, T.C.T. (1996) *Anisotropic Elasticity.* Oxford University Press, New York.

[6] Willis, J.R. (1966) Hertzian contact of anisotropic bodies. *J. Mech. Phys. Solids*, **14**, 163–176.

[7] Willis, J.R. (1967) Boussinesq problems for an anisotropic half-space. *J. Mech. Phys. Solids*, **15**, 331–339.

[8] Borodich, F.M. (1990) Solution of contact problems of elasticity theory for an anisotropic body by the method of similarity. *Soviet Appl. Mech.*, **26**(7), 631–636.

[9] Borodich, F.M. (1989) Hertz contact problems for an anisotropic physically nonlinear elastic medium. *Strength of Materials*, **21**, 1668–1676.

[10] Borodich, F.M. (1993) The Hertz frictional contact between nonlinear elastic anisotropic bodies (the similarity approach). *Int. J. Solids Structures*, **30**, 1513–1526.

[11] Borodich, F.M., Keer, L.M. and Korach, C.S. (2003) Analytical study of fundamental nanoindentation test relations for indenters of non-ideal shapes. *Nanotechnology*, **14**, 803–808.

[12] Galanov, B.A. (1981) Approximate solution to some problems of elastic contact of two bodies. *Mechanics of Solids*, **16**, 61–67.

[13] Borodich, F.M. (1990) Three-dimensional problems of contact of blunt bodies with continuous media. *DSc Thesis (Dr. Phys.-Math. Sci.)*, Moscow State University, 1–283. (Russian)

[14] Borodich, F.M. (1998) Similarity methods in Hertz contact problems and their relations with the Meyer hardness test. *Technical Report TR/MAT/FMB/98-98*, Glasgow Caledonian University, Glasgow, 1–45.

[15] Vlassak, J.J. and Nix, W.D. (1994) Measuring the elastic properties of anisotropic materials by means of indentation experiments. *J. Mech. Phys. Solids*, **42**, 1223–1245.

[16] Ding, H., Chen, W. and Zhang, L. (2006) *Elasticity of Transversely Isotropic Materials.* Springer

[17] Lekhnitskii, S.G. (1940) Symmetrical deformation and torsion of a body of revolution with a special kind of anisotropy. *PMM J. Appl. Math, Mech.*, **4**, 43–60. (Russian)

[18] Elliott, H.A. (1948) Three-dimensional stress distributions in hexagonal aeolotropic crystals. *Proc. Cambridge Phil. Soc.*, **44**, 522–533.

[19] Lin, W. and Keer, L.M. (1989) Three-dimensional analysis of cracks in layered transversely isotropic media. *Proc. R. Soc. Ser. A*, **424**, 307–322.

[20] Kuo, C.H. and Keer, L.M. (1992) Contact stress analysis of a layered transversely isotropic half-space. *Journal of Tribology*, **114**, 253–262.

[21] Kaplunov, J.D., Kossovich, L.Y., and Rogerson, G.A. (2000) Direct asymptotic integration of the equations of transversely isotropic elasticity for a plate near cut-off frequencies, *Q. J. Mech. Appl. Math.*, **53**, 323–341.

[22] Delafargue, A. and Ulm, F.J. (2004) Explicit approximations of the indentation modulus of elastically orthotropic solids for conical indenters. *Int. J. Solids Structures*, **41**, 7351–7360 .

[23] Argatov, I.I. (2011) Depth-sensing indentation of a transversely isotropic elastic layer: Second-order asymptotic models for canonical indenters. *Int. J. Solids Structures*, **48**, 3444–3452.

[24] Sveklo V.A. (1964) Boussinesq type problems for the anisotropic half-space. *PMM J. Appl. Math, Mech.*, **28**, 1099–1105.

[25] Green, A.E. and Zerna, W. (1968) *Theoretical Elasticity.* Dover, New York.

[26] Conway, H.D., Farnham, K.A. and Ku, T.C. (1967) The indentation of a transversely isotropic half space by a rigid sphere. *J. Appl. Mech. Trans. ASME, Ser. E*, **34**, 491–492.

[27] Deryagin, B.V., Krotova, N.A. and Smilga, V.P. (1978) *Adhesion of Solids.* Consultants Bureau, New York, London.

[28] Bradley, R.S. (1932) The cohesive force between solid surfaces and the surface energy of solids. *Phil. Mag*, **13**, 853–862.

[29] Derjaguin, B. (1934) Untersuchungen über die Reibung und Adhäsion, IV. Theorie des Anhaftens kleiner Teilchen. *Kolloid Zeitschrift*, **69**, 155–164.

[30] Kendall, K. (2001) *Molecular Adhesion and Its Applications.* Kluwer Academic/Plenum Publishers, New York.

[31] Lurie, A.I. (2005) *Theory of Elasticity*, Springer-Verlag, Berlin.

[32] Muller, V.M., Yushchenko, V.S., and Derjaguin, B.V. (1980) On the influence of molecular forces on the deformation of an elastic sphere and its sticking to a rigid plane. *J. Colloid Interface Sci.*, **77**, 91–101.

[33] Borodich, F.M. and Galanov, B.A. (2004) Molecular adhesive contact for indenters of nonideal shapes. In: *ICTAM04, Abstracts book and CD-Rom Proceedings.* Warsaw, IPPT PAN.

[34] Maugis, D. (2000) *Contact, Adhesion and Rupture of Elastic Solids.* Springer-Verlag, Berlin.

[35] Goryacheva, I.G, and Makhovskaya, Yu.Yu. (2001) Adhesive interaction of elastic bodies. *PMM J. Appl. Math, Mech.*, **65**, 273–282.

[36] Derjaguin, B.V., Muller, V.M. and Toporov, Y.P. (1975) Effect of contact deformations on adhesion of particles. *J. Colloid Interface Sci.*, **53**, 314–326.

[37] Maugis, D. (1992) Adhesion of spheres: The JKR-DMT transition using a Dugdale model. *J. Colloid Interface Sci.*, **150**, 243–269.

[38] Barthel, E. (2008) Adhesive elastic contacts: JKR and more. *Phys. D: Appl. Phys.*, **41**, 163001.

[39] Johnson, K.L. (1958) A note on the adhesion of elastic solids. *Brit. J. Appl. Phys.*, **9**, 199–200.

[40] Chen, S. and Gao, H. (2007) Bio-inspired mechanics of reversible adhesion: Orientation-dependent adhesion strength for non-slipping adhesive contact with transversely isotropic elastic materials. *JMPS*, **55**, 1001–1015.

[41] Guo, X. and Jin, F. (2009) A generalized JKR-model for two-dimensional adhesive contact of transversely isotropic piezoelectric half-space. *Int. J. Solids Structures*, **46**, 3607–3619.

[42] Espinasse, L., Keer, L., Borodich, F., Yu, H. and Wang, J. (2010) A note on JKR and DMT theories of contact on a transversely isotropic half-space. *Mechanics of Materials*, **42**, 477–480.

[43] Galanov, B.A. (1993) Development of analytical and numerical methods for studying models of materials. Report for the project 7.06.00/001-92, 7.06.00/015-92. Kiev, Institute of Problems in Materials Science. (Ukrainian)

[44] Borodich, F.M. (2008) Hertz type contact problems for power-law shaped bodies. In: *Galin, L.A. Contact Problems. The Legacy of L.A. Galin, Ed. G.M.L. Gladwell*, Springer, 261–292.

[45] Kendall, K. (1971) The adhesion and surface energy of elastic solids. *J. Phys. D: Appl. Phys.*, **4**, 1186–1195.

[46] Galin, L.A. (1946) Spatial contact problems of the theory of elasticity for punches of circular shape in planar projection. *J. Appl. Math. Mech. (PMM)*, **10**, 425–448. (Russian)

[47] Galin, L.A. (1961) *Contact Problems in the Theory of Elasticity*, Ed. I.N. Sneddon, North Carolina State College, Departments of Mathematics and Engineering Research, NSF Grant No. G16447, 1961.

[48] Borodich, F.M. and Keer, L.M. (2004) Contact problems and depth-sensing nanoindentation for frictionless and frictional boundary conditions. *Int. J. Solids Struct.*, **41**, 2479–2499.

[49] Borodich, F.M. (2011) Contact problems at nano/microscale and depth sensing indentation techniques. *Materials Science Forum*, **662**, 53–76.

[50] Kalei, G.N. (1968) Some results of microhardness test using the depth of impression. *Mashinovedenie* **4**(3), 105–107. (Russian)

[51] Bulychev, S.I., Alekhin, V.P., Shorshorov, M.K., Ternovskii, A.P. and Shnyrev G.D. (1975) Determination of Young's modulus according to indentation diagram. *Industrial Lab.*, **41**, 1409–1412.

[52] Borodich, F.M., and Keer, L.M. (2004) Evaluation of elastic modulus of materials by adhesive (no-slip) nanoindentation *Proc. R. Soc. Ser. A*, **460**, 507–514.

[53] Choi, J.H. and Korach, C.S. (2011) Tip bluntness transition measured with atomic force microscopy and the effect on hardness variation with depth in silicon dioxide nanoindentation. *Int. J. Precision Engin. Manufact.*, **12**, 345–354.

[54] Borodich, F.M. and Galanov, B.A. (2008) Non-direct estimations of adhesive and elastic properties of materials by depth-sensing indentation. *Proc. R. Soc. Ser. A*, **464**, 2759–2776.

[55] Borodich, F.M., Galanov, B.A., Gorb, S.N., Prostov, M.Y., Prostov, Y.I., Suarez-Alvarez, M.M. (2012) Evaluation of adhesive and elastic properties of materials by depth-sensing indentation of spheres. *J. Applied Physics A: Materials Science & Processing*, **108**, 13–18.

[56] Borodich, F.M., Galanov, B.A., Gorb, S.N., Prostov, M.Y., Prostov, Y.I., Suarez-Alvarez, M.M. (2012) An inverse problem for adhesive contact and non-direct evaluation of material properties for nanomechanics applications. *Nanoscale Systems: Mathematical Modeling, Theory and Applications*, **1**, 80–92.

[57] Borodich, F.M., Galanov, B.A., Prostov, Y.I., and Suarez-Alvarez, M.M. (2012) Influence of the no-slip boundary condition on indentation of an elastic half-space by a rigid cone in presence of molecular adhesion. *PMM J. Appl. Math. Mech.*, **76**, 744–753.

[58] Mossakovskii, V.I. (1954) The fundamental mixed problem of the theory of elasticity for for a half-space with a circular line separating the boundary conditions. *PMM J. Appl. Math. Mech.*, **18**, 187–196. (Russian)

[59] Mossakovskii, V.I. (1963) Compression of elastic bodies under conditions of adhesion (axisymmetric case). *PMM J. Appl. Math. Mech.*, **27**, 630–643.

[60] Spence, D.A. (1968) Self similar solutions to adhesive contact problems with incremental loading. *Proc. R. Soc. London A*, **305**, 55–80.

[61] Zhupanska, O.I. (2009) Axisymmetric contact with friction of a rigid sphere with an elastic half-space. *Proc. R. Soc. A*, **465**, 2565–2588.

[62] Waters, J.F. and Guduru, P.R. (2010) Mode-mixity-dependent adhesive contact of a sphere on a plane surface. *Proc. R. Soc. Ser. A*, **466**, 1303–1325.

[63] Borodich, F.M., Galanov, B.A., and Suarez-Alvarez, M.M. (2013) The JKR-type adhesive contact problems for power-law shaped axisymmetric punches. (submitted)

[64] Johnson, K.L. (1996) Continuum mechanics modeling of adhesion and friction. *Langmuir*, **12**, 4510–4513.

[65] Chen, B., Wu, P. and Gao, H. (2009) Pre-tension generates strongly reversible adhesion of a spatula pad on substrate. *J. R. Soc. Interface*, **6**, 529-537.

[66] Muskhelishvili, N.I. (1949) *Some Basic Problems of the Mathematical Theory of Elasticity.* Moscow. (English transl. by J.R.M. Radok, Noordhoff International Publishing, Leyden. 1977)

[67] Rvachev, V.L. and Protsenko, V.S. (1977) *Contact Problems of the Theory of Elasticity for Non-Classical Regions.* Kiev: Naukova Dumka.

[68] Fabrikant, V. I. (1991) *Mixed boundary problems of potential theory and their application in mechanics.* Kluwer Academic Publishers, Dordrecht

[69] Rogowski, B. and Kalinski, W. (2007) The adhesive contact problem for a piezoelectric half-space. *Int. J. Pressure Vessels Piping Structures*, **84**, 502–511.