ABSOLUTELY CONTINUOUS SPECTRUM OF DIRAC OPERATORS WITH SQUARE INTEGRABLE POTENTIALS

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Abstract. We show that the absolutely continuous part of the spectral function of the one-dimensional Dirac operator on a half-line with a constant mass term and a real, square-integrable potential is strictly increasing throughout the essential spectrum $(-\infty,-1]\cup[1,\infty)$. The proof is based on estimates for the transmission coefficient for the full-line scattering problem with a truncated potential and a subsequent limiting procedure for the spectral function. Furthermore, we show that the absolutely continuous spectrum persists when an angular momentum term is added, thus establishing the result for spherically symmetric Dirac operators in higher dimensions, too.

1. Introduction

Consider the one-dimensional Schrödinger operator $\varsigma = -\frac{d^2}{dx^2}+q$. It is well known that any self-adjoint realisation of ς on $[0,\infty)$ has essential spectrum $[0,\infty)$ if q is integrable at 0 and $q(x)\to 0$ $(x\to\infty)$. Under certain conditions, e.g. if $q\in L^1([0,\infty))$, this spectrum is purely absolutely continuous [30, Thm 15.3]. In a slightly more general situation, however, the essential spectrum can be far from purely absolutely continuous; indeed Naboko [19] and Simon [27] constructed potentials such that $x|q(x)|\to\infty$ $(x\to\infty)$ arbitrarily slowly and ς has dense point spectrum in $[0,\infty)$. In these examples, dense point spectrum is overlaid with absolutely continuous spectrum. This can be seen from subsequent work focused on providing sufficient conditions on the potential to ensure the existence of (not necessarily purely) absolutely continuous spectrum. For example, Kiselev, Christ and Remling ([4], [3], [23]) have shown that for potentials obeying $|q(x)|\le C(1+|x|)^{-\frac{1}{2}-\varepsilon}$ for large x, the absolutely continuous spectral measure of ς is essentially supported on $[0,\infty)$; the examples in [19] and [27] satisfy this condition. In their celebrated paper [6], Deift and Killip discovered that an integral-type condition on the potential is more natural than a pointwise bound, proving that the absolutely continuous spectrum of the Schrödinger operator is essentially supported on $[0,\infty)$ whenever $q\in L^2([0,\infty))$. This result is optimal in terms of L^p decay, as there exist potentials belonging to L^p, for all $p>2$, such that ς has no absolutely continuous spectrum [16]. We remark that the crucial identity in [6], analogous to (3.18) below, already arises from comparing the two expressions given for the asymptotic constant c_3 in [32, §3]. More recently, Killip and Simon have given an equivalent characterisation of the spectral measures of Schrödinger operators with square-integrable potentials which includes the Deift-Killip result [15].

In the present paper we consider the relativistic counterpart of ς, the Dirac operator

$$\tau = -i\sigma_2\frac{d}{dx}+\sigma_3+q(x), \tag{1.1}$$

where σ_2,σ_3 are Pauli matrices and $q\in L^1_{\mathrm{loc}}(\mathbb{R})$. It is the Hamiltonian of a one-dimensional relativistic particle of mass 1 moving in a force field of potential q. As this formal differential expression is always in the limit point case at $\pm\infty$, it has a unique self-adjoint realisation $\widetilde{T}$ in $L^2(\mathbb{R})^2$. We are mainly interested in the self-adjoint operator T realising τ on the half-line $[0,\infty)$ with the boundary condition

$$u_1(0)\cos\alpha+u_2(0)\sin\alpha=0, \tag{1.2}$$

for fixed $\alpha\in\mathbb{R}$. The spectral analysis of $\widetilde{T}$ and T is based upon the study of the corresponding Dirac eigenvalue equation

$$\tau u(x,\lambda) = -i\sigma_2 u'(x,\lambda)+\sigma_3 u(x,\lambda)+q(x)u(x,\lambda)=\lambda u(x,\lambda), \qquad \lambda\in\mathbb{C}. \tag{1.3}$$

The Dirac operator differs from the Schödinger operator in several essential respects. Most strikingly, the spectrum of the Dirac operator is unbounded below; for example, if q is absolutely integrable, then $\widetilde{T}$ and T have purely absolutely continuous spectrum in the bands $(-\infty,-1]\cup[1,\infty)$ [30, Thm 16.7]. If one only assumes that $q(x)\to 0$ $(x\to\infty)$, the essential spectrum of T is $(-\infty,-1]\cup[1,\infty)$, but need not be purely absolutely continuous; there are examples of potentials such that $x|q(x)|\to\infty$ $(x\to\infty)$ arbitrarily slowly and the operator has a dense set of eigenvalues in the whole or part of its essential spectrum [26].

At a superficial glance, one could be inclined to think that the question about the existence of absolutely continuous spectrum of T under the assumption of square-integrability of q was settled long ago by the work on Krein systems, which are closely related to the Dirac operator (cf. [17, eq. (15)]). Indeed, Denisov's extensive reworking of Krein's ideas includes the result that the wave operators for the half-line operator

$$-i\sigma_2 \frac{d}{dx} + a(x)\,\sigma_1 + b(x)\,\sigma_3 \tag{1.4}$$

with $a, b \in L^2([0,\infty))$ relative to that with $a = b = 0$ exist [7], [8, Thm 13.3]. Thus (1.4) with square-integrable coefficients will have absolutely continuous spectrum covering the whole real axis; this had been shown directly by Martin [18] using the method of [6]. Now τ in (1.1) can be brought into the form of (1.4) by a pointwise unitary transformation; indeed, if $Q' = q$, then

$$e^{i\sigma_2 Q}\,\tau\,e^{-i\sigma_2 Q} = -i\sigma_2 \frac{d}{dx} + e^{2i\sigma_2 Q}\,\sigma_3,$$

which is (1.4) with $a = -\sin 2Q$, $b = \cos 2Q$. But then $|a|^2 + |b|^2 = 1$, so the hypothesis that both a and b are square-integrable on $[0,\infty)$ is never fulfilled. In fact, it would seem that a Dirac operator (1.4) with square-integrable a, b will arise very rarely, if ever, in physical situations.

The main result of the present paper is the following extension of [6] and [18] to Dirac operators with a mass term (1.1).

Theorem 1. *If $q \in L^2([0,\infty))$, then the absolutely continuous part of the spectral function of T is strictly increasing in $(-\infty,-1] \cup [1,\infty)$.*

Without loss of generality, we are able to restrict our attention to the case $\alpha = 0$ in the boundary condition (1.2), as the Titchmarsh-Weyl m-functions for different α are related by a Möbius transformation (see, for example, [20] Equation (4)). From this we can deduce that the absolutely continuous parts of the spectral function for two different values of α have the same essential supports.

On the basis of Theorem 1, we can also treat the case of a Dirac operator with an angular momentum term, as will arise from the rotationally symmetric two- or three-dimensional Dirac operator by separation of spherical polar coordinates [30, Appendix to Sect. 1].

Theorem 2. *Let $q \in L^2([0,\infty)) \cap L^\infty([0,\infty))$, then the absolutely continuous part of the spectral function of T_k, the self-adjoint realisation of*

$$\tau_k = -i\sigma_2 \frac{d}{dx} + \sigma_3 + \frac{k}{x}\sigma_1 + q(x) \qquad (x \in (0,\infty))$$

is strictly increasing in $(-\infty,-1] \cup [1,\infty)$.

The angular momentum quantum number k is in $\mathbb{Z} - \frac{1}{2}$ in the two-dimensional, in $\mathbb{Z} \setminus \{0\}$ in the three-dimensional case.

The paper is organised as follows. In Section 2 we prove that the spectral measure for the operator τ with potential q is the limit of the spectral measures for operators with truncated potentials, set equal to q on $[0,n]$ and to zero on $[n,\infty)$, as $n \to \infty$. Having a compactly supported potential simplifies the scattering analysis. In Section 3 we consider the transmission coefficient and prove the following crucial inequality.

Theorem 3. *Let q be a-real valued square-integrable function on $[0,\infty)$ with compact support. Then*

$$\int_{(-\infty,-1]\cup[1,\infty)} |\lambda|\sqrt{\lambda^2-1}\,\log|a(\lambda)|\,d\lambda \le \frac{\pi}{2}\int_{\mathbb{R}} q^2(x)\,dx. \tag{1.5}$$

Here the function $a(\lambda)$ is the inverse transmission coefficient. The underlying identity (3.18) appears in the expression for an asymptotic constant in [10, p. 78]. A proof of (1.5) under additional smoothness assumptions on q is given in [31]. In Section 4 we use this inequality together with the observations in Section 2 to prove Theorem 1. Finally, in Section 5 we apply subordinacy theory to extend Theorem 1 to the full line and use a perturbation theoretic approach to prove Theorem 2.

2. Compactly Supported Potentials and Convergence of Spectral Measures

For the proof of Theorem 1, we shall first prove the result for compactly supported potentials and then treat $q \in L^2(0,\infty)$ as a limit of truncated potentials as the cut-off point moves to infinity. The spectral measures of the half-line operators with truncated potentials then converge vaguely to the spectral measure of T, as our first lemma shows.

Let m, ρ be the Titchmarsh-Weyl m-function and the spectral function of T, respectively. For $n \in \mathbb{N}$, let $q_n = \chi_{[0,n]} q$ and consider the Dirac operator associated with the differential expression

$$-i\sigma_2 \frac{d}{dx} + \sigma_3 + q_n \tag{2.1}$$

on $[0, \infty)$ with boundary condition (1.2). We denote its Titchmarsh-Weyl and spectral functions by m_n and ρ_n, respectively.

Lemma 1. *$\lim_{n\to\infty} \rho_n = \rho$ at all points of continuity of ρ.*

Proof. Let $z \in \mathbb{C} \setminus \mathbb{R}$, and let $v : [0, \infty) \to \mathbb{C}^2$ be the solution of the initial-value problem $\tau v = z v$, $v(0) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$. Note that the differential equations with potentials q and q_n are identical on the interval $[0, n]$. Thus, from Weyl theory (see [5, Ch. 9, Sect. 2]), it is known that the limit points $m_n(z)$ and $m(z)$ lie inside a complex circle of radius

$$r_n = \frac{1}{2\Im z \int_0^n |v|^2}.$$

Hence $|m_n(z) - m(z)| \le 2r_n(z) \to 0$ $(n \to \infty)$, as the Dirac equation is in the limit-point case at ∞ and therefore $v \notin L^2(0, \infty)$.

We deduce from the Herglotz representation of m_n (see [20, Eq. 5, 5']) and the boundedness of $(m_n(i))_{n\in\mathbb{N}}$ that

$$\int_{\mathbb{R}} \frac{d\rho_n(\lambda)}{\lambda^2 + 1} = \Im m_n(i) \le C$$

with a constant C independent of n. It follows that $|\rho_n(x)| \le C(x^2 + 1)$ $(x \in \mathbb{R})$, so by Helly's Selection Theorem, $(\rho_n)_{n\in\mathbb{N}}$ has a subsequence which converges pointwise to a non-decreasing function $\tilde\rho$. By Helly's Limit Theorem,

$$\int_{\mathbb{R}} \frac{d\tilde\rho(\lambda)}{1 + \lambda^2} \le C.$$

This allows us to relate the limit function $\tilde\rho$ to the m function for the problem with full potential

$$\frac{\Im m(\mu)}{\Im \mu} = \int_{\mathbb{R}} \frac{d\tilde\rho}{|\lambda - \mu|^2} + k, \qquad (\mu \in \mathbb{C} \setminus \mathbb{R}),$$

with some constant $k \in \mathbb{R}$. Hence we can deduce a Stieltjes inversion formula ([5, Ch. 9, Eq. 3.9]) for $\tilde\rho$ in terms of m. This, together with the Stieltjes inversion formula applied to ρ (which relates ρ to m) gives

$$\pi \int_{\mu_1}^{\mu_2} d\rho(\lambda) = \lim_{\varepsilon\to 0} \int_{\mu_1}^{\mu_2} \Im m(\nu + i\varepsilon) d\nu = \pi \int_{\mu_1}^{\mu_2} d\tilde\rho(\lambda) \qquad (\mu_1, \mu_2 \in \mathbb{R} \setminus S),$$

where S is the set of points of discontinuity of $\tilde\rho$ or ρ. Thus $\rho = \tilde\rho$ a.e., using the convention that these functions are right continuous and vanish at 0. As all subsequences of $(\rho_n)_{n\in\mathbb{N}}$ have the same limit $\tilde\rho = \rho$, it follows that $\rho_n \to \tilde\rho = \rho$ $(n \to \infty)$ on $\mathbb{R} \setminus S$. $\square$

3. The Transmission Coefficient

Throughout this section, we assume that q is a square-integrable function with compact support in $[0, \infty)$. We shall use the function

$$\omega(\lambda) = \sqrt[\star]{\lambda + 1}\sqrt{\lambda - 1} \qquad (\lambda \in \mathbb{C}), \tag{3.1}$$

where $\sqrt[\star]{\ }$ is the complex square root with branch cut along the negative real axis and $\arg \sqrt[\star]{z} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ $(z \in \mathbb{C})$, while $\sqrt{\ }$ is the standard complex square root with branch cut along the positive real axis and $\arg\sqrt{z} \in [0, \pi)$. The function ω is the relativistic substitute for the momentum variable $k = \sqrt{\lambda}$ used in scattering analysis of the Schrödinger operator. Clearly ω is analytic in $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ and satisfies $\omega(\lambda)^2 = \lambda^2 - 1$ on $\mathbb{C}$. Moreover, $\Im\omega(\lambda) > 0$ whenever $\Im\lambda > 0$, and for real λ we have

$$\omega(\lambda) = \begin{cases} -\sqrt{\lambda^2 - 1}, & \lambda \in (-\infty, -1] \\ i\sqrt{1 - \lambda^2}, & \lambda \in (-1, 1) \\ \sqrt{\lambda^2 - 1}, & \lambda \in [1, \infty). \end{cases} \tag{3.2}$$

By continuity, there is an open neighbourhood Ω of $(-1, 1)$ in which $\Im\omega > 0$.

Lemma 2. *Let $q \in L^1([a, b])$, $-\infty < a < b < \infty$. The function $\omega(\lambda)$ defined in (3.1) satisfies*

(i) $(\omega + \lambda)(\omega - \lambda) = -1$;

(ii) $\omega(z) = -\overline{\omega(-\overline{z})}$, $(z \in \overline{\mathbb{C}^+})$;

(iii) $e^{\pm\frac{i}{\omega(\omega+\lambda)}\int_x^y q} = 1 + \mathcal{O}\left(\frac{1}{|\lambda|^2}\right)$ $(\lambda \to \infty)$ *uniformly in* $x, y \in [a, b]$;

(iv) $e^{\frac{i}{\omega(\omega-\lambda)}\int_x^y q} = e^{2i\int_x^y q}\left(1 + O\left(\frac{1}{|\lambda|^2}\right)\right)$ $(\lambda \to \infty)$ *uniformly in* $x, y \in [a, b]$.

Proof. (i) and (ii) are simple calculations. For (iii) we estimate

$$\left|e^{\frac{i}{\omega(\omega+\lambda)}\int_x^y q} - 1\right| \le \sum_{j=1}^{\infty}\left|\frac{i}{\omega(\omega+\lambda)}\right|^j \frac{\|q\|_1^j}{j!} = O\left(\frac{1}{|\lambda|^2}\right) \qquad (\lambda \to \infty)$$

uniformly in $x, y \in [a, b]$. Hence (iv) follows, as

$$e^{\frac{i}{\omega(\omega-\lambda)}\int_a^b q} = e^{-i\left(1+\frac{\lambda}{\omega}\right)\int_a^b q} = e^{-2i\int_a^b q}e^{-\frac{i}{\omega(\omega+\lambda)}\int_a^b q} = e^{-2i\int_a^b q}\left(1 + O\Big(\frac{1}{|\lambda|^2}\Big)\right) \qquad (\lambda \to \infty; x, y \in [a, b])$$

$\square$

Now consider the Dirac equation (1.3) on the whole real line, extending q by 0 to the negative half-line. For $\lambda \in \mathbb{C} \setminus \{-1, 1\}$, the functions

$$u(x, \lambda) = \begin{pmatrix} -\frac{i\omega}{\lambda-1} \\ 1 \end{pmatrix} e^{i\omega x}, \qquad \tilde{u}(x, \lambda) = \begin{pmatrix} \frac{i\omega}{\lambda-1} \\ 1 \end{pmatrix} e^{-i\omega x} \qquad (x \in I)$$

(where we write briefly ω for $\omega(\lambda)$) form a fundamental system of this equation on all intervals I where q vanishes. If $\lambda \in \mathbb{R} \setminus [-1, 1]$, then $\omega(\lambda) \in \mathbb{R} \setminus \{0\}$, and hence $\tilde{u}(\cdot, \lambda) = \overline{u(\cdot, \lambda)}$.

In particular, there is a solution $y(\cdot, \lambda)$ for such λ with the property $y(x, \lambda) = u(x, \lambda)$ for all x to the right of the support of q. For x to the left of the support of q this solution can be expressed as

$$y(x, \lambda) = a(\lambda)u(x, \lambda) + b(\lambda)\tilde{u}(x, \lambda) \tag{3.3}$$

with suitable constants $a(\lambda)$ and $b(\lambda)$. Since $e^{\pm i\omega x}$ represent right- and left-traveling waves, the solution y can be interpreted as describing a scattering process of a wave of amplitude a approaching from the left and split into a transmitted wave travelling to the right of amplitude 1 and a reflected wave traveling to the left of amplitude b. Correspondingly, $t = \frac{1}{a}$ and $r = \frac{b}{a}$ are called the *transmission* and *reflection coefficients*, respectively.

For $\lambda \in \mathbb{R} \setminus [-1, 1]$, $\overline{y(\cdot, \lambda)}$ is another solution of (1.3). Evaluating the constant Wronskian of $y(\cdot, \lambda)$ and $\overline{y(\cdot, \lambda)}$ both to the right and to the left of the support of q, we find that

$$|a(\lambda)|^2 = 1 + |b(\lambda)|^2. \tag{3.4}$$

Thus $a(\lambda) \neq 0$, and $|t|^2 = 1 - |r|^2$, expressing the conservation of the probability current in the quantum mechanical scattering process.

For points x to the left of the support of q, we obtain from (3.3) that

$$y_1(x, \lambda)\tilde{u}_2(x, \lambda) - y_2(x, \lambda)\tilde{u}_1(x, \lambda) = a(\lambda)(u_1(x, \lambda)\tilde{u}_2(x, \lambda) - u_2(x, \lambda)\tilde{u}_1(x, \lambda)) = \frac{2i\omega(\lambda)}{1-\lambda}\,a(\lambda),$$

so

$$a(\lambda) = \frac{1-\lambda}{2i\omega(\lambda)}(y_1(x, \lambda)\tilde{u}_2(x, \lambda) - y_2(x, \lambda)\tilde{u}_1(x, \lambda)) \tag{3.5}$$

with arbitrary $x < 0$. From this formula it is apparent that a is well defined for $\lambda \in \mathbb{C} \setminus \{-1, 1\}$, continuous for $\lambda \in (\overline{\mathbb{C}^+} \cup \Omega) \setminus \{-1, 1\}$ and analytic in $\mathbb{C}^+ \cup \Omega$. Furthermore, it is immediate from (3.3) that

$$\begin{pmatrix} -\frac{i\omega}{\lambda-1} \\ 1 \end{pmatrix} a(\lambda) = \lim_{x \to -\infty} e^{-i\omega x}y(x, \lambda) \qquad (\lambda \in \mathbb{C}^+ \cup \Omega).$$

The zeros of a in $\mathbb{C}^+ \cup \Omega$ are exactly the eigenvalues of the full-line Dirac operator $\widetilde{T}$ and hence are all real. Indeed, for such λ, $y(\cdot, \lambda)$ is square integrable at ∞, and $\tilde{u}(\cdot, \lambda)$ is square integrable at $-\infty$ whilst $u(\cdot, \lambda)$ is not. Thus $y(\cdot, \lambda) \in L^2(\mathbb{R}^2)$ if and only if $a(\lambda) = 0$. Further, (3.4) implies that $a(\lambda)$ has no zeros for $\lambda \in \mathbb{R} \setminus [-1, 1]$. Moreover, we have the following information about the zeros of a.

Lemma 3. *The number of zeros of* a *in* $\mathbb{C}^+ \cup \Omega$ *is finite. On* $(-1, 1)$*,* a *is real-valued and all its zeros are simple.*

Proof. For the first statement, see [13, Cor. 3.2], bearing in mind that q has compact support. If $\lambda \in (-1, 1)$, then $\omega(\lambda) = i\sqrt{1-\lambda^2}$, so

$$y(x, \lambda) = \begin{pmatrix} \frac{\sqrt{1-\lambda^2}}{\lambda-1} \\ 1 \end{pmatrix} e^{-\sqrt{1-\lambda^2}x} \in \mathbb{R}^2$$

for x to the right of the support of q. As all coefficients of the Dirac equation (1.3) are real, $y(\cdot, \lambda)$ is real-valued throughout, in particular

$$a(\lambda) \begin{pmatrix} \frac{\sqrt{1-\lambda^2}}{\lambda-1} \\ 1 \end{pmatrix} + b(\lambda) \begin{pmatrix} -\frac{\sqrt{1-\lambda^2}}{\lambda-1} \\ 1 \end{pmatrix} = y(0, \lambda) \in \mathbb{R}^2,$$

which implies that $a(\lambda), b(\lambda) \in \mathbb{R}$. The last statement can be proved as [28, Lemma 2.12]. $\square$

In the proof of Theorem 3, we shall on several occasions use the following observation about the function $\sin(2R|x|)/|x|$, which is not absolutely integrable and has an R-independent envelope, but nevertheless turns out to generate an asymptotically diagonal integral kernel in a weak sense as $R \to \infty$. The square integral of q on the right-hand side of the inequality (1.5) arises in this way.

Lemma 4. *For compactly supported $q \in L^2(\mathbb{R})$,*

$$\lim_{R\to\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sin(2R|x-y|)}{\pi|x-y|} q(x)dx\, \overline{q(y)}\, dy = \int_{\mathbb{R}} |q|^2.$$

Proof. Set $q_R := q(\frac{\pi}{2R}\cdot)$ and $f(t) = \frac{\sin \pi t}{\pi t}$ $(t \in \mathbb{R} \setminus \{0\})$. Then $\hat{q_R}(\lambda) = \frac{2R}{\pi} \hat{q}\left(\frac{2R\lambda}{\pi}\right)$ and $\hat{f} = \chi_{[-\frac{1}{2},\frac{1}{2}]}$. Hence

$$\begin{aligned} \frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sin(2R(y-x))}{(y-x)} q(x)\, dx\, \overline{q(y)}\, dy &= \frac{\pi}{2R}(f * q_R, q_R) = \frac{\pi}{2R}(\hat{f} \cdot \hat{q_R}, \hat{q_R}) \\ &= \frac{\pi}{2R} \int_{\mathbb{R}} \hat{f}(\lambda)|\hat{q_R}(\lambda)|^2 d\lambda = \int_{\mathbb{R}} \hat{f}\left(\frac{\pi\zeta}{2R}\right) |\hat{q}(\zeta)|^2 d\zeta = \int_{-\frac{R}{\pi}}^{\frac{R}{\pi}} |\hat{q}(\zeta)|^2 d\zeta \to \|\hat{q}\|^2 = \|q\|^2 \end{aligned}$$

as $R \to \infty$. $\square$

We now proceed to prove Theorem 3. Consider $\lambda \in \mathbb{C}^+ \cup \Omega$; then $\Im\omega > 0$, where we write ω briefly for $\omega(\lambda)$. Let y be as in (3.3), and define associated functions $a(x, \lambda)$, $b(x, \lambda)$ $(x \in \mathbb{R}, \lambda \in \mathbb{C} \setminus \{-1, 1\})$ by setting

$$y(x, \lambda) = a(x, \lambda)u(x, \lambda) + b(x, \lambda)\tilde{u}(x, \lambda)$$

for all $x \in \mathbb{R}$. By comparison with (3.3), $a(x, \lambda) = a(\lambda)$ and $b(x, \lambda) = b(\lambda)$ to the left of the support of q, while $a(x, \lambda) = 1$ and $b(x, \lambda) = 0$ to the right of the support of q. We shall now derive an integral equation for the function a.

The function $w(x, \lambda) := e^{-i\int_x^\infty q(t)dt - i\omega(\lambda)x} y(x, \lambda)$ satisfies the differential equation

$$w' = (\sigma_1 + i\sigma_2\lambda + i\omega)w + iq(1 - \sigma_2)w.$$

Treating the potential term as a perturbation, we note that the equation in which the term involving q is dropped has the fundamental system

$$\varphi(x, \lambda) = e^{-i\omega x}(u(x, \lambda), \widetilde{u}(x, \lambda)) = \begin{pmatrix} -\frac{i\omega(\lambda)}{\lambda-1} & \frac{i\omega(\lambda)}{\lambda-1} e^{-2i\omega(\lambda)x} \\ 1 & e^{-2i\omega(\lambda)x} \end{pmatrix} \qquad (x \in \mathbb{R}, \lambda \in \mathbb{C} \setminus \{-1, 1\}).$$

Writing $w = \varphi A$, we find that $A(x, \lambda) = e^{-i\int_x^\infty q} \begin{pmatrix} a(x, \lambda) \\ b(x, \lambda) \end{pmatrix}$. In particular, $A(x, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for x to the right of the support of q, and $A' = \varphi'(1 - \sigma_2)iq\varphi A$. This yields the integral equation

$$A(x, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{\omega(\lambda)} \int_x^\infty q(t)\Phi(t)A(t, \lambda)dt$$

with

$$\Phi(t) := \begin{pmatrix} i(\omega - \lambda) & ie^{-2i\omega t} \\ -ie^{2i\omega t} & i(\omega + \lambda) \end{pmatrix}.$$

Iterating this equation twice, we obtain the following identity for the top entry of A,

$$e^{-i\int_x^\infty q}a(x,\lambda) = A_1(x,\lambda) = 1 + \frac{i}{\omega(\lambda+\omega)}\int_x^\infty q\,dt + \frac{1}{\omega^2}\int_x^\infty\int_t^\infty q(t)q(s)\left\{e^{2i\omega(s-t)} - \frac{1}{(\lambda+\omega)^2}\right\}ds\,dt$$
$$-\frac{1}{\omega^3}\int_x^\infty\int_t^\infty\int_s^\infty q(t)q(s)q(r)e^{-i\int_r^\infty q}\Bigg\{\left[\frac{i}{(\omega+\lambda)^3} - \frac{ie^{2i\omega(s-t)}}{\omega+\lambda} - \frac{ie^{2i\omega(r-s)}}{\omega+\lambda} - \frac{ie^{2i\omega(r-t)}}{\omega-\lambda}\right]a(r,\lambda)$$
$$+\left[ie^{2i\omega(s-t-r)} - \frac{ie^{2i\omega r}}{(\lambda+\omega)^2} + ie^{-2i\omega s} - \frac{ie^{-2i\omega t}}{(\omega-\lambda)^2}\right]b(r,\lambda)\Bigg\}dr\,ds\,dt$$

Now, from the differential equation for A, we see that

$$A_1'(x,\lambda) = -\frac{iq(x)A_1(x,\lambda)}{\omega(\lambda)(\omega(\lambda)+\lambda)} + \frac{iq(x)e^{-2i\omega(\lambda)x}A_2(x,\lambda)}{\omega(\lambda)}$$
$$A_2'(x,\lambda) = -\frac{iq(x)e^{2i\omega(\lambda)x}A_1(x,\lambda)}{\omega(\lambda)} - \frac{iq(x)A_2(x,\lambda)}{\omega(\lambda)(\omega(\lambda)-\lambda)}$$

and so, solving each as a first order differential equation,

$$A_1(x,\lambda) = e^{\frac{i}{\omega(\lambda)(\omega(\lambda)+\lambda)}\int_x^\infty q} - \int_x^\infty \frac{iqe^{-2i\omega(\lambda)t}}{\omega(\lambda)}e^{\frac{i}{\omega(\lambda)(\omega(\lambda)+\lambda)}\int_x^t q}A_2(t,\lambda)dt$$
$$A_2(x,\lambda) = \int_x^\infty \frac{iq(t)e^{2i\omega(\lambda)t}}{\omega(\lambda)}e^{\frac{i}{\omega(\lambda)(\omega(\lambda)-\lambda)}\int_x^t q}A_1(t,\lambda)dt.$$

Hence eliminating A_2,

$$A_1(x,\lambda) = e^{\frac{i}{\omega(\omega+\lambda)}\int_x^\infty q} + \frac{1}{\omega^2}\int_x^\infty\int_t^\infty q(t)q(s)e^{2i\omega(s-t)}e^{\frac{i}{\omega(\omega+\lambda)}\int_x^t q}e^{\frac{i}{\omega(\omega-\lambda)}\int_t^s q}A_1(s,\lambda)ds\,dt. \tag{3.6}$$

We now assume that $|\lambda|$, and hence $|\omega|$, is large enough so that $\left|e^{\frac{i}{\omega(\lambda)(\omega(\lambda)+\lambda)}\int_\cdot^\infty q}\right| \le 2$ and

$$\frac{1}{\omega(\lambda)^2}\int_x^\infty\int_t^\infty |q(t)|\,|q(s)|\,\left|e^{\frac{i}{\omega(\lambda)(\omega(\lambda)+\lambda)}\int_x^t q}\right|\,\left|e^{\frac{i}{\omega(\lambda)(\omega(\lambda)-\lambda)}\int_t^s q}\right|\,ds\,dt < \frac{1}{2};$$

this can be achieved in view of Lemma 2 (iii) and (iv), respectively. Hence, noting that $|a| = |A_1|$, we obtain from (3.6) that

$$\|a(\cdot,\lambda)\|_\infty \le 2 + \frac{\|a(\cdot,\lambda)\|_\infty}{2},$$

which implies that $\|a(\cdot,\lambda)\|_\infty \le 4$, for such values of λ. Thus

$$e^{-i\int_x^\infty q}a(x,\lambda) = 1 + \frac{i}{\omega(\lambda+\omega)}\int_x^\infty q\,dt + \frac{1}{\omega^2}\int_x^\infty\int_t^\infty q(t)q(s)\left\{e^{2i\omega(s-t)} - \frac{1}{(\lambda+\omega)^2}\right\}ds\,dt$$
$$-\frac{1}{\omega^3}\int_x^\infty\int_t^\infty\int_s^\infty q(t)q(s)q(r)\left[\frac{i}{(\omega+\lambda)^3} - \frac{ie^{2i\omega(s-t)}}{\omega+\lambda} - \frac{ie^{2i\omega(r-s)}}{\omega+\lambda} - \frac{ie^{2i\omega(r-t)}}{\omega-\lambda}\right]$$
$$\times\left\{e^{\frac{i}{\omega(\omega+\lambda)}\int_r^\infty q} + \frac{1}{\omega^2}\int_r^\infty\int_p^\infty q(r)q(p)e^{2i\omega(u-p)}e^{\frac{i}{\omega(\omega+\lambda)}\int_r^p q}e^{\frac{i}{\omega(\omega-\lambda)}\int_p^u q}A_1(u,\lambda)du\,dp\right\}dr\,ds\,dt$$
$$-\frac{1}{\omega^3}\int_x^\infty\int_t^\infty\int_s^\infty q(t)q(s)q(r)\left[ie^{2i\omega(s-t-r)} - \frac{ie^{2i\omega r}}{(\lambda+\omega)^2} + ie^{-2i\omega s} - \frac{ie^{-2i\omega t}}{(\omega-\lambda)^2}\right]$$
$$\times\left\{\int_r^\infty \frac{iq(p)e^{2i\omega p}}{\omega}e^{\frac{i}{\omega(\omega-\lambda)}\int_r^p q}A_1(p,\lambda)dp\right\}dr\,ds\,dt$$
$$= 1 + \frac{i}{\omega(\lambda+\omega)}\int_x^\infty q\,dt + \frac{1}{\omega^2}\int_x^\infty\int_t^\infty q(t)q(s)e^{2i\omega(s-t)}ds\,dt$$
$$-\frac{i(\omega+\lambda)}{\omega^3}\int_x^\infty\int_t^\infty\int_s^\infty q(t)q(s)q(r)e^{2i\omega(r-t)}e^{\frac{i}{\omega(\omega+\lambda)}\int_r^\infty q}dr\,ds\,dt$$
$$-\frac{(\omega+\lambda)^2}{\omega^4}\int_x^\infty\int_t^\infty\int_s^\infty\int_r^\infty q(t)q(s)q(r)q(p)e^{2i\omega(p-t)}e^{\frac{i}{\omega(\omega-\lambda)}\int_r^p q}A_1(p,\lambda)dp\,dr\,ds\,dt$$
$$+O\left(\frac{1}{|\lambda|^4}\right) \qquad (|\lambda|\to\infty).$$

We now take the limit $x \to -\infty$. Furthermore, using equation (3.6) we can substitute for $A_1(x,\lambda)$ in the above equation. The 6-fold integral which arises can be moved directly into the asymptotic term. Using Lemma 2 (iii) to handle the first term from (3.6) we obtain:

$$\begin{aligned}
e^{-i\int_{-\infty}^{\infty} q} a(\lambda) &= \lim_{x\to-\infty} e^{-i\int_x^{\infty} q} a(x,\lambda) \\
&= 1 + \frac{i}{\omega(\lambda+\omega)} \int_{\mathbb{R}} q dt + \frac{1}{\omega^2} \int_{-\infty}^{\infty} \int_t^{\infty} q(t)q(s) e^{2i\omega(s-t)} ds\, dt \\
&\quad - \frac{i(\omega+\lambda)}{\omega^3} \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} q(t)q(s)q(r) e^{2i\omega(r-t)} dr\, ds\, dt \\
&\quad - \frac{(\omega+\lambda)^2}{\omega^4} \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} \int_r^{\infty} q(t)q(s)q(r)q(p) e^{2i\omega(p-t)} e^{\frac{i}{\omega(\omega-\lambda)}\int_r^p q} dp\, dr\, ds\, dt \\
&\quad + O\left(\frac{1}{|\lambda|^4}\right) \qquad (|\lambda| \to \infty).
\end{aligned} \tag{3.7}$$

Consider the anticlockwise contour in the complex upper half-plane γ_R parametrised by $\lambda(\theta) = \sqrt{R^2 e^{2i\theta} + 1}$ ($\theta \in [0,\pi]$), with $R > 1$. This contour is chosen so that $\omega(\lambda(\theta)) = Re^{i\theta}$; it follows that $d\lambda = i\frac{\omega(\lambda)^2}{\lambda}\, d\theta = \frac{\omega}{\lambda}\, d\omega$. Then

$$\begin{aligned}
\int_{\gamma_R} \lambda\omega(\lambda) \log\left[e^{i\int_{-\infty}^{\infty} q} a(\lambda)\right] d\lambda &= \int_{\gamma_R} \bigg\{ \frac{i\lambda}{\lambda+\omega} \int_{\mathbb{R}} q dt + \frac{\lambda}{\omega} \int_{-\infty}^{\infty} \int_t^{\infty} q(t)q(s) e^{2i\omega(s-t)}\, ds\, dt \\
&\quad + \frac{i\lambda}{\omega^2(\omega-\lambda)} \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} q(t)q(s)q(r) e^{2i\omega(r-t)}\, dr\, ds\, dt \\
&\quad - \frac{\lambda}{\omega^3(\omega-\lambda)^2} \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} \int_r^{\infty} q(t)q(s)q(r)q(p) e^{2i\omega(p-t)} e^{\frac{i}{\omega(\omega-\lambda)}\int_r^p q}\, dp\, dr\, ds\, dt \\
&\quad + O\left(\frac{1}{|\lambda|^2}\right) \bigg\}\, d\lambda \qquad (|\lambda| \to \infty).
\end{aligned} \tag{3.8}$$

We now consider each integral term on the right-hand side in turn. The first one evaluates to

$$\int_{\gamma_R} \left(\frac{i\lambda}{\lambda+\omega} \int_{\mathbb{R}} q\, dt \right) d\lambda = i \left(\int_{\mathbb{R}} q\, dt \right) \int_{\gamma_R} \lambda(\lambda - \sqrt{\lambda^2 - 1})\, d\lambda = \left[\frac{2iR^3}{3} - \frac{2i\sqrt{R^2+1}}{3} \right] \int_{\mathbb{R}} q\, dt,$$

which is purely imaginary. To treat the second term, we apply the symmetrisation rule which states that

$$F \in L^1(\mathbb{R}^2), \quad F(x,y) = F(y,x) \quad ((x,y) \in \mathbb{R}^2) \quad \Longrightarrow \quad \int_{-\infty}^{\infty} \int_x^{\infty} F(x,y)\, dy\, dx = \frac{1}{2} \int_{\mathbb{R}^2} F. \tag{3.9}$$

We also make a change of variables in the contour integral, using ω to denote the transformed variable by a slight abuse of notation. The transformed contour is $\gamma_R^\omega := \omega(\gamma_R)$, in fact a simple semicircle. Since $\int_{\gamma_R} \frac{\lambda}{\omega(\lambda)} e^{2i\omega(\lambda)|s-t|}\, d\lambda = \int_{\gamma_R^\omega} e^{2i\omega|s-t|}\, d\omega = -\sin(2R|x-y|)/|x-y|$,

$$\int_{\gamma_R} \frac{\lambda}{\omega} \int_{-\infty}^{\infty} \int_t^{\infty} q(t)q(s) e^{2i\omega(s-t)} ds\, dt\, d\lambda = \int_{\gamma_R} \frac{\lambda}{2\omega} \int_{\mathbb{R}} \int_{\mathbb{R}} q(t)q(s) e^{2i\omega|s-t|} ds\, dt\ d\lambda \to -\frac{\pi}{2} \int_{\mathbb{R}} q^2 \quad (R \to \infty)$$

by (3.9) and Lemma 4. For the third integral in (3.8),

$$\begin{aligned}
&\int_{\gamma_R} \frac{i\lambda}{\omega^2(\omega-\lambda)} \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} q(t)q(s)q(r) e^{2i\omega(r-t)}\, dr\, ds\, dt\, d\lambda \\
&= \int_{\gamma_R^\omega} i \left(-2 - \frac{1}{\omega(\omega+\lambda)} \right) \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} q(t)q(s)q(r) e^{2i\omega(r-t)}\, dr\, ds\, dt\, d\omega \\
&= -2i \int_{-\infty}^{\infty} \int_t^{\infty} \int_s^{\infty} \frac{\sin(2R(r-t))}{(r-t)} q(t)q(s)q(r)\, dr\, ds\, dt + O\left(\frac{1}{R}\right) \qquad (R \to \infty)
\end{aligned}$$

noting that the length of the contour γ_R^ω is $O(R)$. This is purely imaginary up to the error term. For the final integral term in (3.8), we have

$$
\begin{aligned}
&\int_{\gamma_R} \frac{\lambda}{\omega^3(\omega-\lambda)^2} \int_{-\infty}^{\infty}\int_t^{\infty}\int_s^{\infty}\int_r^{\infty} q(t)q(s)q(r)q(p)e^{2i\omega(p-t)}e^{\frac{i}{\omega(\omega-\lambda)}\int_r^p q}dp\,dr\,ds\,dt\,d\lambda \\
&=\int_{\gamma_R} \frac{\lambda\left[4\omega^2+4(\lambda-\omega)\omega+(\lambda-\omega)^2\right]}{\omega^3} \int_{-\infty}^{\infty}\int_t^{\infty}\int_s^{\infty}\int_r^{\infty} q(t)q(s)q(r)q(p)e^{2i\omega(p-t)}e^{\frac{i}{\omega(\omega-\lambda)}\int_r^p q}dp\,dr\,ds\,dt\,d\lambda \\
&=4\int_{\gamma_R^\omega}\int_{-\infty}^{\infty}\int_t^{\infty}\int_s^{\infty}\int_r^{\infty} q(t)q(s)q(r)q(p)e^{2i\omega(p-t)}e^{-2i\int_r^p q}dp\,dr\,ds\,dt\,d\omega + O\left(\frac{1}{R}\right) \qquad (R\to\infty)
\end{aligned}
$$

where we used Lemma 2 (iv) in the last step. By an integration by parts,

$$
\int_r^{\infty} q(p)e^{2i\omega(p-t)}e^{-2i\int_r^p q}dp = -\frac{i}{2}e^{2i\omega(r-t)} + \omega\int_r^{\infty} e^{-2i\int_r^p q}e^{2i\omega(p-t)}dp.
$$

Thus

$$
\begin{aligned}
&4\int_{\gamma_R^\omega}\int_{-\infty}^{\infty}\int_t^{\infty}\int_s^{\infty}\int_r^{\infty} q(t)q(s)q(r)q(p)e^{2i\omega(p-t)}e^{-2i\int_r^p q}\,dp\,dr\,ds\,dt\,d\omega \\
&=-2i\int_{\gamma_R^\omega}\int_{-\infty}^{\infty}\int_r^{\infty}\int_s^{\infty} q(t)q(s)q(r)e^{2i\omega(r-t)}\,dr\,ds\,dt\,d\omega \\
&\qquad + \int_{\gamma_R^\omega} 4\omega \int_{-\infty}^{\infty}\int_t^{\infty}\int_s^{\infty} q(t)q(s)q(r)\left[\int_r^{\infty} e^{-2i\int_r^p q}e^{2i\omega(p-t)}dp\right]\,dr\,ds\,dt\,d\omega. \qquad (3.10)
\end{aligned}
$$

This leaves us with two integrals to consider. Performing the contour integral first, we see that the first term is purely imaginary. The remaining integral can be resolved by repeated integrations by parts, starting from the innermost integral. Observe that for any $z\in\mathbb{R}$ and $x\geq v$

$$
\int_z^{\infty} q(x)e^{-2i\int_x^{\infty} q}\left[\int_x^{\infty} e^{2i\int_y^{\infty} q}e^{2i\omega(y-v)}dy\right]dx = \frac{i}{2}e^{-2i\int_z^{\infty} q}\int_z^{\infty} e^{2i\int_y^{\infty} q}e^{2i\omega(y-v)}dy + \frac{1}{4\,\omega}\,e^{2i\omega(z-v)}. \quad (3.11)
$$

Thus, by an integration by parts, the last term in (3.10) equals

$$
\begin{aligned}
&\int_{\gamma_R^\omega}\int_{-\infty}^{\infty}\int_t^{\infty} q(t)q(s)e^{2i\omega(s-t)}\,ds\,dt\,d\omega \\
&\qquad + \int_{\gamma_R^\omega} 2i\omega\int_{-\infty}^{\infty}\int_t^{\infty} q(t)q(s)e^{-2i\int_s^{\infty} q}\left[\int_s^{\infty} e^{2i\int_p^{\infty} q}e^{2i\omega(p-t)}dp\right]ds\,dt\,d\omega. \qquad (3.12)
\end{aligned}
$$

Again we have two integrals to consider. After symmetrisation, the first term in (3.12) tends to $-\frac{\pi}{2}\int_{\mathbb{R}} q^2$ as $R\to\infty$ by Lemma 4. We can again apply (3.11) to the second integral in (3.12), then integrate by parts in the innermost integral, giving

$$
\begin{aligned}
&\int_{\gamma_R^\omega} 2i\omega\int_{-\infty}^{\infty} q(t)\int_t^{\infty} q(s)e^{-2i\int_s^{\infty} q}\left[\int_s^{\infty} e^{2i\int_p^{\infty} q}e^{2i\omega(p-t)}dp\right]ds\,dt\,d\omega \\
&=-\int_{\gamma_R^\omega}\omega\int_{\mathbb{R}} q(t)e^{-2i\int_t^{\infty} q}\left[\int_t^{\infty} e^{2i\int_p^{\infty} q}e^{2i\omega(p-t)}\,dp\right]dt\,d\omega + \frac{i}{2}\int_{-\infty}^{\infty} q\int_{\gamma_R^\omega} d\omega \\
&=-\int_{\gamma_R^\omega}\int_{-\infty}^{\infty} q(t)\,e^{-2i\int_t^{\infty} q}\int_t^{\infty} q(p)\,e^{2i\int_p^{\infty} q}\,e^{2i\omega(p-t)}\,dp\,dt\,d\omega \\
&=\int_{-\infty}^{\infty}\int_t^{\infty} q(t)\,q(p)\,e^{2i\int_p^t q}\,\frac{\sin(2R(p-t))}{(p-t)}\,dp\,dt.
\end{aligned}
$$

Taking the real part, symmetrising and applying Lemma 4 twice, we find that

$$\begin{aligned}
&\Re\left(\int_{\gamma_R^\omega} 2i\omega \int_{-\infty}^{\infty} q(t) \int_t^{\infty} q(s) e^{-2i\int_s^\infty q} \left[\int_s^\infty e^{2i\int_p^\infty q} e^{2i\omega(p-t)} dp\right] ds\, dt\, d\omega\right)\\
&= \int_{-\infty}^{\infty}\int_t^{\infty} q(t)q(p)\cos\left(2\int_p^t q\right)\frac{\sin(2R(p-t))}{(p-t)}dp\,dt\\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(t)q(p)\left(\cos\left(2\int_p^\infty q\right)\cos\left(2\int_t^\infty q\right)+\sin\left(2\int_p^\infty q\right)\sin\left(2\int_t^\infty q\right)\right)\frac{\sin(2R|p-t|)}{|p-t|}dp\,dt\\
&\to \frac{\pi}{2}\int_{-\infty}^{\infty} q^2(t)\left[\cos^2\left(2\int_t^\infty q\right)+\sin^2\left(2\int_t^\infty q\right)\right]dt = \frac{\pi}{2}\int_{-\infty}^{\infty} q^2(t)dt
\end{aligned}$$

as $R\to\infty$. This cancels out the first term of (3.12). In summary, (3.8) comes down to

$$\lim_{R\to\infty}\Re\int_{\gamma_R}\lambda\omega(\lambda)\,\log\left[e^{-i\int_{-\infty}^\infty q}a(\lambda)\right]d\lambda = -\frac{\pi}{2}\int_{\mathbb{R}} q^2(x)dx. \tag{3.13}$$

Let $0<\varepsilon<1$ and consider the closed contour $\widetilde{\Gamma}_\varepsilon^R = \Gamma_{R,\varepsilon}\cup\Gamma_\varepsilon\cup\gamma_{R,\varepsilon}$ where $\gamma_{R,\varepsilon}=\gamma_R\cap\{\lambda:\varepsilon\le\Im\lambda\}$, $\Gamma_\varepsilon=[-1+i\varepsilon,1+i\varepsilon]$ and $\Gamma_{R,\varepsilon}=[\varkappa_-,-1+i\varepsilon]\cup[1+i\varepsilon,\varkappa_+]$; here $\varkappa_\pm$ are the points where the contour γ_R intersects the line $\Im\lambda=\varepsilon$. In addition, we consider the two-component contour $\gamma_{R,\varepsilon}^c=\gamma_R\setminus\gamma_{R,\varepsilon}$. Recalling that $\lambda\omega(\lambda)\,\log\left[e^{-i\int_{-\infty}^\infty q}a(\lambda)\right]=O(1)$ $(|\lambda|\to\infty)$ from (3.7), we see that

$$\Re\int_{\gamma_{R,\varepsilon}^c}\lambda\omega(\lambda)\,\log\left[e^{-i\int_{-\infty}^\infty q}a(\lambda)\right]d\lambda = O(\varepsilon)\qquad(\varepsilon\to 0).$$

On the other hand, we find using Cauchy's Integral Theorem that

$$\begin{aligned}
&-\Re\int_{\gamma_{R,\varepsilon}}\lambda\omega(\lambda)\,\log\left[e^{-i\int_{-\infty}^\infty q}a(\lambda)\right]d\lambda\\
&\quad=\Re\Big(\int_{\Gamma_{R,\varepsilon}}\lambda\omega(\lambda)\,\log a(\lambda)\,d\lambda+\int_{\Gamma_\varepsilon}\lambda\omega(\lambda)\,\log a(\lambda)\,d\lambda+\log\left[e^{-i\int_{-\infty}^\infty q}\right]\int_{\Gamma_{R,\varepsilon}\cup\Gamma_\varepsilon}\lambda\omega(\lambda)\,d\lambda\Big).
\end{aligned}\tag{3.14}$$

By Lemma 2 (ii), it is clear that $\Re\omega(-\mu+i\varepsilon)=-\Re\omega(\mu+i\varepsilon)$ and $\Im\omega(-\mu+i\varepsilon)=\Im\omega(\mu+i\varepsilon)$. Hence the imaginary part of the integrand of the last integral in (3.14) is odd, and the logarithmic factor is purely imaginary. Thus the real part of the last term in (3.14) vanishes.

The first integral in (3.14) can be rewritten as

$$\int_{\Gamma_{R,\varepsilon}}\lambda\omega(\lambda)\,\log a(\lambda)\,d\lambda=\int_{\Gamma_{R,\varepsilon}}\lambda\omega(\lambda)\,\log|a(\lambda)|\,d\lambda+i\int_{\Gamma_{R,\varepsilon}}\lambda\omega(\lambda)\,\arg a(\lambda)\,d\lambda.$$

Now it is clear that

$$\lim_{\varepsilon\to 0}\Im\int_{\Gamma_{R,\varepsilon}}\lambda\omega(\lambda)\,\arg a(\lambda)\,d\lambda=0,$$

as $\arg a(\lambda)$ is real and bounded and $\Im(\lambda\omega(\lambda))\to 0$ uniformly. Thus we need only consider

$$\lim_{\varepsilon\to 0}\int_{\Gamma_{R,\varepsilon}}\lambda\,\omega(\lambda)\,\log|a(\lambda)|\,d\lambda=\lim_{\varepsilon\to 0}\int_{(-\sqrt{R^2+1},-1]\cup[1,\sqrt{R^2+1})}(t+i\varepsilon)\,\omega(t+i\varepsilon)\,\log|a(t+i\varepsilon)|\,dt.$$

In view of (3.5),

$$\lim_{\lambda\to\pm 1}\lambda\,\omega(\lambda)\,\log|a(\lambda)|=0$$

for real λ; also a is continuous and has no zeros in $\mathbb{R}\setminus[-1,1]$. Thus $(t+i\varepsilon)\,\omega(t+i\varepsilon)\,\log|a(t+i\varepsilon)|$ is bounded uniformly in ε on $(-\sqrt{R^2+1},-1]\cup[1,\sqrt{R^2+1})$, and by dominated convergence

$$\lim_{\varepsilon\to 0}\Re\int_{\Gamma_{R,\varepsilon}}\lambda\omega(\lambda)\,\log a(\lambda)\,d\lambda=\int_{(-\sqrt{R^2+1},-1]\cup[1,\sqrt{R^2+1})}\lambda\omega(\lambda)\,\log|a(\lambda)|\,d\lambda. \tag{3.15}$$

Finally we consider the second integral in (3.14),

$$\int_{\Gamma_\varepsilon}\lambda\omega(\lambda)\,\log a(\lambda)\,d\lambda=\int_{-1}^{1}(t+i\varepsilon)\,\omega(t+i\varepsilon)\,\log|a(t+i\varepsilon)|\,dt+i\int_{-1}^{1}(t+i\varepsilon)\,\omega(t+i\varepsilon)\,\arg a(t+i\varepsilon)\,dt. \tag{3.16}$$

For the first of these integrals, we note that a has a finite number of distinct zeros in the interval $(-1,1)$, which we label $\beta_1,\dots,\beta_M$ in increasing order. The (real) logarithm function is integrable at zero and so, by dominated convergence and (3.2), this integral tends to the purely imaginary limit

$$\int_{-1}^{1} t\,\omega(t)\,\log|a(t)|\,dt = \int_{-1}^{1} it\sqrt{1-t^2}\,\log|a(t)|\,dt$$

as $\varepsilon\to 0$. Concerning the second integral in (3.16), we note that, by Lemma 3, $a(\lambda)$ is real for $\lambda\in(-1,1)$. Therefore, between any two zeros of a on $(-1,1)$, the argument of a is constant. Thus we need only consider the argument of a at a zero β_j. We write

$$a(\lambda) = (\lambda-\beta_j)\,b(\lambda),$$

where b is analytic and non-zero in some neighbourhood Ω of β_j. The respective arguments satisfy

$$\arg a(\lambda) = \arg(\lambda-\beta_j) + \arg b(\lambda)$$

and $\arg b(\lambda)$ is continuous at β_j. Therefore, if we consider $a(\lambda)$ on the intersection of $\{\lambda : \Im\lambda = \varepsilon\}$ with $B_{\sqrt{\varepsilon}}(\beta_j)$, the ball of radius $\sqrt{\varepsilon}$ and centre β_j, the argument of b is almost constant and thus the change in the argument of a between the left and right ends of this interval is $\sim -2\arccos\sqrt{\varepsilon}$, which tends to $-\pi$ in the limit $\varepsilon\to 0$. The limiting values of the argument of a thus have the form

$$\arg a(\lambda) = \arg a(-1+0) - \pi\sum_{m=1}^{M}\chi_{(\beta_m,1)}(\lambda) \qquad (\lambda\in(-1,1)\setminus\{\beta_i \mid i\in\{1\dots M\}\}).$$

Thus, bearing in mind (3.2),

$$\begin{aligned} &i\lim_{\varepsilon\to 0}\int_{-1}^{1}(\lambda+i\varepsilon)\,\omega(\lambda+i\varepsilon)\,\arg a(\lambda+i\varepsilon)\,d\lambda \\ &\quad= -\arg a(-1+0)\int_{-1}^{1}\lambda\sqrt{1-\lambda^2}\,d\lambda + \pi\sum_{m=1}^{M}\int_{\beta_m}^{1}\lambda\sqrt{1-\lambda^2}\,d\lambda \\ &\quad= -\frac{\pi}{3}\sum_m (1-\beta_m^2)^{\frac{3}{2}}. \end{aligned} \tag{3.17}$$

Hence, by (3.13), (3.14), (3.15), (3.17) and (3.2),

$$\begin{aligned} \frac{\pi}{2}\int_{\mathbb{R}} q^2 &= -\lim_{R\to\infty}\lim_{\varepsilon\to 0}\Re\int_{\gamma_{R,\varepsilon}}\lambda\,\omega(\lambda)\,\log\left[e^{i\int_{\mathbb{R}} q}\,a(\lambda)\right]d\lambda \\ &= \int_{(-\infty,-1]\cup[1,\infty)}|\lambda|\sqrt{\lambda^2-1}\log|a(\lambda)|\,d\lambda + \frac{\pi}{3}\sum_m(1-\beta_m^2)^{\frac{3}{2}}. \end{aligned} \tag{3.18}$$

This completes the proof of Theorem 3.

4. The Spectral Function

We now proceed to prove Theorem 1. We shall show that for all compact subsets $K\in\mathbb{R}\setminus[-1,1]$ of positive Lebesgue measure, $\rho(K)>0$. This implies the statement of Theorem 1; indeed, assume $\lambda\in\mathbb{R}\setminus[-1,1]$ is not a growth point of the absolutely continuous part of the spectral function, ρ_{ac}. Then there is $\varepsilon>0$ such that $\rho_{ac}([\lambda-\varepsilon,\lambda+\varepsilon])=0$. Let $B\subset\mathbb{R}$ be an open set of Lebesgue measure $<\varepsilon$ such that $\rho_{sing}(\mathbb{R}\setminus B)=0$. Then $K:=[\lambda-\varepsilon,\lambda+\varepsilon]\setminus B$ is compact and has positive Lebesgue measure, so by the above

$$0<\rho(K)=\rho_{ac}(K)\le\rho_{ac}([\lambda-\varepsilon,\lambda+\varepsilon])=0,$$

a contradiction.

The proof of the above statement is based on the following estimate (cf. [6]).

Lemma 5. *Let $A\subset\mathbb{R}$ be open and let $w\in L^1_{loc}(A)$, $w>0$. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of absolutely continuous non-decreasing functions which converge to a non-decreasing function ρ at its points of continuity. Let $K\subset A$ be compact and of positive Lebesgue measure. Then*

$$\limsup_{n\to\infty}\int_K \log\left(\frac{\rho_n'}{w}\right)\frac{w}{w(K)} \le \log\left(\frac{\rho(K)}{w(K)}\right), \tag{4.1}$$

where $w(K)=\int_K w$.

Proof. Let

$$\phi_n(x) = \max\{0, 1 - n \operatorname{dist}(x, K)\}, \quad (x \in \mathbb{R},\, n \in \mathbb{N}).$$

Then $\operatorname{supp}(\phi_n(x)) \subset [\inf K - 1, \sup K + 1]$. Further, $(\phi_n)_{n\in\mathbb{N}}$ is a non-increasing sequence converging to χ_K pointwise as $n \to \infty$, the characteristic function of K. Thus

$$\rho(K) = \int_{\mathbb{R}} \chi_K(x)\, d\rho = \lim_{m\to\infty} \int_{\mathbb{R}} \phi_m(x)\, d\rho = \lim_{m\to\infty} \lim_{n\to\infty} \int_{\mathbb{R}} \phi_m(x)\, d\rho_n \geq \limsup_{n\to\infty} \int_K \rho_n',$$

where the second equality follows from the monotone convergence theorem, the third by Helly's integration theorem and the inequality from the fact that $\phi_n \geq \chi_K$. Thus

$$\log\left(\frac{\rho(K)}{w(K)}\right) \geq \limsup_{n\to\infty} \log \int_K \frac{\rho_n'}{w} \frac{w}{w(K)} \geq \limsup_{n\to\infty} \int_K \log\left(\frac{\rho_n'}{w}\right) \frac{w}{w(K)},$$

where the last inequality follows from Jensen's Inequality [24, Theorem 3.3]. □

From Lemma 1 and Lemma 5 we see that it is sufficient to prove that

$$\int_K \left(-\log\left(\frac{\rho_n'}{w}\right) w\right) = \int_K \left(-\log\left[\frac{\Im m_n(\lambda + i0)}{\pi w(\lambda)}\right] w(\lambda)\right) d\lambda$$

is bounded above uniformly in n for some positive weight function w. As q_n is square integrable with compact support, the Titchmarsh-Weyl m-function for the Dirac equation associated with (2.1) with boundary condition (1.2) can be expressed in terms of the solution y of (3.3),

$$m_n(\lambda) = \frac{y_{n,2}(0,\lambda)}{y_{n,1}(0,\lambda)} = i\,\frac{\lambda - 1}{\omega(\lambda)}\,\frac{a_n(\lambda) + b_n(\lambda)}{a_n(\lambda) - b_n(\lambda)} = i\,\frac{\lambda - 1}{\omega(\lambda)}\,\frac{1 + r_n(\lambda)}{1 - r_n(\lambda)},$$

denoting by a_n and b_n the coefficients of y_n, and by r_n the corresponding reflection coefficient. Conversely $r_n(\lambda) = \frac{m_n(\lambda) - i(\lambda-1)/\omega(\lambda)}{m_n(\lambda) + i(\lambda-1)/\omega(\lambda)}$. Thus

$$|t_n(\lambda)|^2 = 1 - |r_n(\lambda)|^2 = \frac{4\Re\left(\overline{m_n(\lambda + i0)}\,\frac{i(\lambda-1)}{\omega(\lambda)}\right)}{\left|m_n(\lambda + i0) + \frac{i(\lambda-1)}{\omega(\lambda)}\right|^2}$$

for a.e. $\lambda \in \mathbb{R} \setminus [-1, 1]$. Now the spectrum is purely absolutely continuous in this set because the potential has compact support, and so (see [12]) $0 < \lim_{\varepsilon\to 0} m_n(\lambda + i\varepsilon) < \infty$ and $(\lambda - 1)/\omega(\lambda) > 0$. Consequently

$$\lim_{\varepsilon\to 0}\left|m_n(\lambda + i\varepsilon) + \frac{i(\lambda + i\varepsilon - 1)}{\omega(\lambda + i\varepsilon)}\right|^2 = \left(\frac{\lambda - 1}{\omega(\lambda)} + \Im \lim_{\varepsilon\to 0} m_n(\lambda + i\varepsilon)\right)^2 + (\Re \lim_{\varepsilon\to 0} m_n(\lambda + i\varepsilon))^2 \geq \left(\frac{\lambda - 1}{\omega(\lambda)}\right)^2$$

$(\lambda \in \mathbb{R} \setminus [-1, 1])$. Thus we can estimate

$$\frac{1}{|a_n(\lambda)|^2} = |t_n(\lambda)|^2 \leq \frac{4\,\omega(\lambda)}{\lambda - 1}\,\Im \lim_{\varepsilon\to 0} m_n(\lambda + i\varepsilon).$$

Now let $\delta > 0$ and apply Lemma 5 with $A := \mathbb{R} \setminus [-1 - \delta, 1]$ and $w(\lambda) := |\lambda - 1|/(4\pi\sqrt{\lambda^2 - 1})$ $(\lambda \in A)$. For any compact set $K \subset A$ of positive Lebesgue measure, we find using Theorem 3 and the facts that $|a_n| \geq 1$ and

$$\frac{|\lambda - 1|}{\sqrt{\lambda^2 - 1}} = \frac{\sqrt{\lambda^2 - 1}}{|\lambda + 1|} \leq \frac{\sqrt{\lambda^2 - 1}}{\delta} \qquad (\lambda \in A)$$

that

$$\begin{aligned}
-\int_K \log\left(\frac{\rho_n'}{w}\right) w &= -\int_K \log\left(\frac{\Im \lim_{\varepsilon\to 0} m_n(\lambda + i\varepsilon)}{\pi w(\lambda)}\right) w(\lambda)\, d\lambda \\
&= -\frac{1}{4\pi}\int_K \log\left(\frac{4\Im \lim_{\varepsilon\to 0} m_n(\lambda + i\varepsilon)\omega(\lambda)}{\lambda - 1}\right) \frac{|\lambda - 1|}{\sqrt{\lambda^2 - 1}}\, d\lambda \\
&\leq \frac{1}{2\pi}\int_K \log|a_n(\lambda)|\,\frac{|\lambda|}{\delta}\sqrt{\lambda^2 - 1}\, d\lambda \leq \frac{1}{2\pi\delta}\int_{\mathbb{R}\setminus(-1,1)} \log|a_n(\lambda)|\,|\lambda|\sqrt{\lambda^2 - 1}\, d\lambda \qquad (4.2) \\
&\leq \frac{1}{4\delta}\int_{\mathbb{R}} q_n^2 \leq \frac{1}{4\delta}\int_{\mathbb{R}} q^2.
\end{aligned}$$

Thus the integral in (4.1) is bounded below independently of $n \in \mathbb{N}$, and so $\rho(K) > 0$. This concludes the proof of Theorem 1.

Remark. The first inequality in (4.2) is a rather bad estimate for large values of λ; indeed, the bounded factor $|\lambda-1|/\sqrt{\lambda^2-1}$ is replaced with the upper bound $\sqrt{\lambda^2-1}\,|\lambda|/\delta$, which grows as λ^2 for $\lambda\to\pm\infty$, in order to fit the estimate (1.5).

In fact, the assertion of Theorem 1 will already follow if

$$\int_{(-\infty,-1]\cup[1,\infty)} \frac{\lambda\omega}{\lambda^2+1} \log|a_n(\lambda)|\,d\lambda$$

is bounded above. Estimating this integral by the method of Section 3 turns out to be easier due to the better decay properties of the integrand, and gives, instead of (1.5),

$$\begin{aligned}\int_{(-\infty,-1]\cup[1,\infty)} \frac{\lambda\omega}{\lambda^2+1} \log(a_n(\lambda))\,d\lambda &= -\sqrt{2}\pi\log|a_n(i)| - \pi\sum_{m=1}^{M}\int_{\beta_m}^{1}\frac{\lambda\sqrt{1-\lambda^2}}{\lambda^2+1}\\ &\le -\sqrt{2}\pi\log|a_n(i)|,\end{aligned}$$

where the β_m, $m\in\{1,2,...,M\}$, are the zeros of a as before. More generally,

$$\int_{(-\infty,-1]\cup[1,\infty)} \frac{\lambda\omega}{\lambda^2+\alpha^2} \log|a_n(\lambda)|\,d\lambda \le -\sqrt{2}\pi\log|a_n(i\alpha)|$$

for any $\alpha>0$. This means that, in order to obtain an equivalent result to Theorem 1, one only needs to show that there exists an $\alpha>0$ such that $|a_n(i\alpha)|\not\to 0$ as the cut off point of the potential tends to infinity. This seems to be a very weak condition and its relation to the L^2 condition in Theorem 1 is somewhat obscure. Note that if we consider a constant potential, for which the assertion of Theorem 1 clearly does not hold, then $|a_n(i\alpha)|\to 0$ $(n\to\infty)$ for all $\alpha>0$.

5. Angular Momentum

In practice the one-dimensional Dirac operator most commonly arises from the three-dimensional Dirac operator with a spherically symmetric potential by separation of variables in spherical polar coordinates (cf. [30, Appendix to Ch. 1]). It then takes the form

$$-i\sigma_2\frac{d}{dx}+\sigma_3+\frac{k}{x}\sigma_1+q(x) \qquad (x\in(0,\infty)),$$

where σ_1 is the third Pauli matrix and $k\in\mathbb{Z}\setminus\{0\}$. (In the case of a rotationally symmetric two-dimensional Dirac operator, $k\in\mathbb{Z}-\frac{1}{2}$.) The additional angular momentum term $\frac{k}{x}\sigma_1$ introduces a singularity at 0. This singular end-point is in the limit-point case if $|k|\ge\frac{1}{2}$ and q is less singular at 0; indeed $q\in L^1([0,*])$ (which follows from $q\in L^2([0,\infty))$) is sufficient to ensure limit-point case at zero [9]. As the operator is always in the limit-point case at ∞ (see [30, Thm 6.8]), this means that it has a unique self-adjoint realisation T_k.

In the following, we denote by S_1 the space of trace-class operators and by S_2 the space of Hilbert-Schmidt operators. We shall use the following corollary to the Kato-Rosenblum perturbation theorem.

Theorem 4 ([14, Thm 4.12])**.** *Let H_1 and H_2 be self-adjoint operators in a Hilbert space such that*

$$(H_2-z)^{-1}-(H_1-z)^{-1}\in S_1$$

for some non-real z. Then the wave operators $W_\pm(H_2,H_1)$ exist and are complete. In particular, the absolutely continuous parts of H_1 and H_2 are unitarily equivalent.

From the Gilbert-Pearson theory of subordinacy ([12], [11]), as well as its extension to Dirac operators ([2], [1]), it is known that a minimal support of the absolutely continuous spectral measure of a self-adjoint Dirac operator L on (α,β) is given by

$$\mathcal{M}_{ac}(L)=\{\lambda\in\mathbb{R} : \text{no solution of } Lu=\lambda u \text{ is subordinate at } \beta\}$$

if α is a regular, β a singular end-point, and

$$\begin{aligned}\mathcal{M}_{ac}(L)=&\{\lambda\in\mathbb{R} : \text{no solution of } Lu=\lambda u \text{ is subordinate at } \beta\}\\ &\cup\{\lambda\in\mathbb{R} : \text{no solution of } Lu=\lambda u \text{ is subordinate at } \alpha\}\end{aligned} \tag{5.1}$$

if both end-points are singular. Here a subset S of $\mathbb{R}$ is said to be a *minimal support* of a measure ν if $\nu(\mathbb{R}\setminus S)=0$ and $\nu(S_0)=0\Rightarrow \operatorname{mes}S_0=0$ $(S_0\subset S)$, where mes denotes the Lebesgue measure. Also, the *essential closure* of a set $\Sigma\subset\mathbb{R}$ is defined as

$$\overline{\Sigma}^{\text{ess}}=\{\lambda\in\mathbb{R} : \operatorname{mes}((\lambda-\varepsilon,\lambda+\varepsilon)\cap\Sigma)>0\ (\varepsilon>0)\}.$$

It follows immediately that if $\Sigma_1 \subset \Sigma_2$, then $\overline{\Sigma_1}^{\text{ess}} \subset \overline{\Sigma_2}^{\text{ess}}$. The relationship between the absolutely continuous spectrum and the minimal support $\mathcal{M}_{ac}$ of the absolutely continuous part of the spectral measure is expressed in the following lemma.

Lemma 6. *The set of growth points of ρ_{ac} is equal to $\overline{\mathcal{M}_{ac}}^{ess}$.*

Proof. Let λ be a growth point of ρ_{ac}. Then $0 < \rho_{ac}((\lambda-\varepsilon, \lambda+\varepsilon) \cap \mathcal{M}_{ac})$ for all $\varepsilon > 0$, by the defining property of a minimal support. As ρ_{ac} is absolutely continuous with respect to the Lebesgue measure, this implies that $\operatorname{mes}((\lambda-\varepsilon, \lambda+\varepsilon) \cap \mathcal{M}_{ac}) > 0$, and hence $\lambda \in \overline{\mathcal{M}_{ac}}^{\text{ess}}$.

Conversely, let $\lambda \in \overline{\mathcal{M}_{ac}}^{\text{ess}}$. Then for all $\varepsilon > 0$, $\operatorname{mes}((\lambda-\varepsilon, \lambda+\varepsilon) \cap \mathcal{M}_{ac}) > 0$. Hence $0 < \rho_{ac}((\lambda-\varepsilon, \lambda+\varepsilon) \cap \mathcal{M}_{ac}) = \rho_{ac}((\lambda-\varepsilon, \lambda+\varepsilon))$ and so λ is a growth point of (ρ_{ac}). ☐

In particular, (5.1) implies that

$$\sigma_{ac}(H) = \sigma_{ac}(H_c^\alpha) \cup \sigma_{ac}(H_c^\beta),$$

where $c \in (\alpha, \beta)$ and H_c^β is the operator restricted to $[c, \beta)$ and H_c^β that to $(\alpha, c]$ with some boundary condition at c [11]. Thus we can draw the following conclusion from Theorem 1.

Corollary 1. *Consider the self-adjoint Dirac operator on $\mathbb{R}$*

$$\tilde{T} = -i\sigma_2 \frac{d}{dx} + \sigma_3 + q \qquad (x \in \mathbb{R}).$$

If $q \in L^2(\mathbb{R})$, then the absolutely continuous part of the spectral function of $\tilde{T}$ is strictly increasing in $(-\infty, -1] \cup [1, \infty)$.

We now consider

$$T_k = -i\sigma_2 \frac{d}{dx} + \sigma_3 + \frac{k}{x}\sigma_1 + q(x)$$

in $L^2((0,\infty))$, where $|k| \geq \frac{1}{2}$ and $q \in L^2([0,\infty)) \cap L^\infty([0,\infty))$. Then, by [25, Lemma 3], the operators T_k and

$$\tilde{T_k} = -i\sigma_2 \frac{d}{dx} + \sigma_3 + \mu(x)\sigma_3 + \tilde{q}(x),$$

with $\mu(x) = \sqrt{1 + \frac{k^2}{x^2}} - 1$ and $\tilde{q}(x) = q(x) + \frac{k}{2(x^2+k^2)}$ $(x > 0)$, are unitarily equivalent. Obviously $\tilde{q} \in L^2([0,\infty)) \cap L^\infty([0,\infty))$ and $\mu \in L^1((c,\infty)) \cap L^2((c,\infty))$, fixing $c > 0$ arbitrarily.

Consider also the operator on $\mathbb{R}$,

$$H = -i\sigma_2 \frac{d}{dx} + \sigma_3 + \hat{\mu}(x)\sigma_3 + \hat{q}(x) \qquad (x \in \mathbb{R}),$$

where $\hat{q}$ is the even extension of $\tilde{q}$ to the whole real line and $\hat{\mu}$ is the even extension of $\chi_{[c,\infty)}\mu$ to the whole real line. The transformation $u(x) = \sigma_3 v(-x)$ then turns $Hu = \lambda u$ into $Hv = \lambda v$. Because of this symmetry, the sets

$$\{\lambda \in \mathbb{R} : \text{no solution of } Hu = \lambda u \text{ is subordinate at } \infty\}$$

and

$$\{\lambda \in \mathbb{R} : \text{no solution of } Hu = \lambda u \text{ is subordinate at } -\infty\}$$

coincide. As the differential expressions for H and $\tilde{T_k}$ are the same near $+\infty$, (5.1) then implies that

$$\mathcal{M}_{ac}(H) \subset \mathcal{M}_{ac}(\tilde{T}). \tag{5.2}$$

Define two further operators on $\mathbb{R}$, namely

$$H_0 = -i\sigma_2 \frac{d}{dx} + \sigma_3 + \hat{q}(x), \qquad H_{00} = -i\sigma_2 \frac{d}{dx} + \sigma_3.$$

As both H and H_0 have the form $H_{00} + F$, where F is a bounded perturbation, all three operators have the same domain. From [21, Thm XI.20] and a simple modification of its proof, we obtain the following statement.

Lemma 7. *Let $\varphi \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then*

$$\varphi(H_{00} - \lambda)^{-1} \in \mathcal{S}_2, \qquad (H_{00} - \lambda)^{-1}\varphi \in \mathcal{S}_2.$$

Thus, taking $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and using the Second Resolvent Identity [29, Theorem 5.1], we find

$$
\begin{aligned}
&(H-\lambda)^{-1} - (H_0-\lambda)^{-1} = (H-\lambda)^{-1}(-\hat{\mu}\sigma_3)(H_0-\lambda)^{-1} \\
&= (H-\lambda)^{-1}(-\hat{\mu}\sigma_3)(H_{00}-\lambda)^{-1} - (H-\lambda)^{-1}(-\hat{\mu}\sigma_3)(H_{00}-\lambda)^{-1}\hat{q}(H_0-\lambda)^{-1} \\
&= (H_{00}-\lambda)^{-1}(-\sqrt{\hat{\mu}}\,\sigma_3\sqrt{\hat{\mu}})(H_{00}-\lambda)^{-1} + (H-\lambda)^{-1}(\hat{\mu}\sigma_3+\hat{q})(H_{00}-\lambda)^{-1}(-\hat{\mu}\sigma_3)(H_{00}-\lambda)^{-1} \\
&\quad + (H-\lambda)^{-1}\hat{\mu}\sigma_3(H_{00}-\lambda)^{-1}\hat{q}(H_{00}-\lambda)^{-1} + (H-\lambda)^{-1}\hat{\mu}\sigma_3(H_{00}-\lambda)^{-1}\hat{q}(H_{00}-\lambda)^{-1}\hat{q}(H_0-\lambda)^{-1} \in \mathcal{S}_1.
\end{aligned}
$$

Here we used Lemma 7 together with the facts that $S_2S_2 \subset S_1$ and that S_1 and S_2 are invariant under multiplication with bounded operators [22, Section VI.6]. Thus, by Theorem 4, the absolutely continuous parts of H and H_0 are unitarily equivalent. By Corollary 1, this implies that H has absolutely continuous spectrum on $(-\infty,-1]\cup[1,\infty)$. Thus (5.2) and Lemma 6 give

$$(-\infty,-1]\cup[1,\infty) \subset \sigma_{ac}(H) = \overline{\mathcal{M}}_{ac}^{\mathrm{ess}}(H) \subset \overline{\mathcal{M}}_{ac}^{\mathrm{ess}}(\tilde{T}_k) = \sigma_{ac}(\tilde{T}_k),$$

and Theorem 2 follows.

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