Mellin transforms with only critical zeros: Legendre functions

Mark W. Coffey
Department of Physics
Colorado School of Mines
Golden, CO 80401
USA
mcoffey@mines.edu
Matthew C. Lettington
School of Mathematics
Cardiff University
P. O. Box 926
Cardiff CP24 4AG
UK
LettingtonMC@cf.ac.uk

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Abstract

We consider the Mellin transforms of certain Legendre functions based upon the ordinary and associated Legendre polynomials. We show that the transforms have polynomial factors whose zeros lie all on the critical line Re $s = 1/2$. The polynomials with zeros only on the critical line are identified in terms of certain ${}_3F_2(1)$ hypergeometric functions. These polynomials possess the functional equation $p_n(s) = (-1)^{\lfloor n/2 \rfloor} p_n(1-s)$. Other hypergeometric representations are presented, as well as certain Mellin transforms of fractional part and fractional part-integer part functions. The results should be of interest to special function theory, combinatorial geometry, and analytic number theory.

Key words and phrases

Mellin transformation, Legendre polynomial, associated Legendre polynomial, hypergeometric function, critical line, zeros, functional equation

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Introduction

Mellin transforms are very important in analytic number theory and asymptotic analysis. They occasionally also find application in signal and image analysis. In a series of investigations, we are determining families of polynomials arising from Mellin transformation that satisfy the Riemann hypothesis.

In particular, we are considering certain Mellin transforms comprised of classical orthogonal polynomials that yield polynomial factors with zeros only on the critical line Re $s = 1/2$ or else only on the real axis. Such polynomials have many important applications to analytic number theory, in a sense extending the Riemann hypothesis. For example, using the Mellin transforms of Hermite functions, Hermite polynomials multiplied by a Gaussian factor, Bump and Ng [8] (see also [7]) were able to generalize Riemann's second proof of the functional equation of the zeta function $\zeta(s)$, and to obtain a new representation for it. The polynomial factors turn out to be certain ${}_2F_1(2)$ Gauss hypergeometric functions, being certain shifted symmetric Meixner-Pollaczek polynomials [11].

The polynomials $\tilde{p}_n(x) = {}_2F_1(-n, -x; 1; 2) = (-1)^n {}_2F_1(-n, x+1; 1; 2)$ and $\tilde{q}_n(x) = i^n n! \tilde{p}_n(-1/2 - ix/2)$ have been studied for combinatorial and number theoretic reasons [18, 21], and they directly correspond to the Bump and Ng polynomials with $s = -x$. We note that these polynomials arise in the counting of the number of lattice points in an n-dimensional octahedron [7, 21]. In fact, combinatorial, geometrical, and coding aspects of $\tilde{p}_n(x)$ at integer argument had been noted in [16] and [20], and Lemmas 2.2 and 2.3 of [18] correspond very closely to Lemmas 2 and 3, respectively, of [20]. For the half-line Mellin transform of Laguerre functions, one may see [11].

We expect that our work will have connections with the counting of lattice points in polytopes, thus with combinatorial geometry, a polytope being a region described by a set of linear inequalities. In this context, the Ehrhart polynomial [5, 19] counts lattice points, and it has a functional equation. Along with the Ehrhart polynomial one may associate a Poincaré series, of the form $P(t) = U(t)/(1-t)^n$, with U a polynomial such that $U(1) \neq 0$. An example form of Poincaré series is

$$P(t) = \frac{\prod_{j=1}^{k}(1+t+\ldots+t^{n_j})}{(1-t)^n},$$

wherein $n_1, \ldots, n_k$ are positive integers. We expect that other various Ehrhart polynomials are of hypergeometric form, and have all of their zeros on a line.

The Riemann zeta function arises as the half-line Mellin transform of a theta function, and of many other functions. However, these are not the sole type of Mellin transform from which the zeta function may be determined. Letting $\{x\}$ denote the fractional part of x, we quickly review the representation for Re $s > 1$,

$$\int_0^1 \left\{\frac{1}{t}\right\} t^{s-1} dt = \frac{1}{s-1} - \frac{\zeta(s)}{s}.$$

For we have, with $\zeta(s, a)$ the Hurwitz zeta function,

$$\int_0^1 \left\{\frac{1}{t}\right\} t^{s-1} dt = \int_1^\infty \{v\}\, v^{-s-1} dv = \sum_{k=1}^\infty \int_k^{k+1} \{v\}\, v^{-s-1} dv$$

$$= \sum_{k=1}^\infty \int_0^1 \frac{v}{(v+k)^{s+1}} dv = \int_0^1 v\zeta(s+1, v+1) dv = \frac{1}{s-1} - \frac{\zeta(s)}{s}.$$

With Γ the Gamma function and ${}_2F_1$ the Gauss hypergeometric function, in turn we may generalize this representation to

$$\int_0^1 \left\{\frac{1}{t}\right\} \frac{t^{s-1}}{(1-t^b)^\alpha} dt = \frac{\Gamma(1-\alpha)\Gamma\left(\frac{s+b-1}{b}\right)}{\Gamma\left(\frac{s+b-\alpha b-1}{b}\right)} \frac{1}{(s-1)} - \frac{1}{s}\sum_{j=1}^\infty \frac{1}{j^s}\, {}_2F_1\left(\alpha, \frac{s}{b}; 1+\frac{s}{b}; \frac{1}{j^b}\right),$$

with $0 \leq$ Re $\alpha < 1$. This result uses the Newton series for the Beta function and the interchange of a double summation:

$$\int_0^1 \left\{\frac{1}{t}\right\} \frac{t^{s-1}}{(1-t^b)^\alpha} dt = \sum_{\ell=0}^\infty (-1)^\ell \binom{-\alpha}{\ell} \int_0^1 \left\{\frac{1}{t}\right\} t^{s+b\ell-1} dt$$

$$= \sum_{\ell=0}^\infty (-1)^\ell \binom{-\alpha}{\ell} \int_1^\infty \{v\}\, v^{-s-b\ell-1} dv = \sum_{\ell=0}^\infty (-1)^\ell \binom{-\alpha}{\ell} \sum_{k=1}^\infty \int_0^1 \frac{v}{(v+k)^{s+b\ell+1}} dv$$

$$= \sum_{\ell=0}^{\infty}(-1)^\ell \binom{-\alpha}{\ell} \int_0^1 v\zeta(s+b\ell+1, v+1)dv$$

$$= \sum_{\ell=0}^{\infty}(-1)^\ell \binom{-\alpha}{\ell} \left[\frac{1}{s+b\ell-1} - \frac{\zeta(s+b\ell)}{s+b\ell}\right]$$

$$= \frac{\Gamma(1-\alpha)\Gamma\left(\frac{s+b-1}{b}\right)}{\Gamma\left(\frac{s+b-\alpha b-1}{b}\right)} \frac{1}{(s-1)} - \sum_{\ell=0}^{\infty}(-1)^\ell \binom{-\alpha}{\ell} \frac{\zeta(s+b\ell)}{s+b\ell}$$

$$= \frac{\Gamma(1-\alpha)\Gamma\left(\frac{s+b-1}{b}\right)}{\Gamma\left(\frac{s+b-\alpha b-1}{b}\right)} \frac{1}{(s-1)} - \sum_{\ell=0}^{\infty}(-1)^\ell \binom{-\alpha}{\ell} \frac{1}{(s+b\ell)} \sum_{j=1}^{\infty} \frac{1}{j^{s+b\ell}}$$

$$= \frac{\Gamma(1-\alpha)\Gamma\left(\frac{s+b-1}{b}\right)}{\Gamma\left(\frac{s+b-\alpha b-1}{b}\right)} \frac{1}{(s-1)} - \frac{1}{s}\sum_{j=1}^{\infty} \frac{1}{j^s}\, {}_2F_1\left(\alpha, \frac{s}{b}; 1+\frac{s}{b}; \frac{1}{j^b}\right).$$

An interesting case occurs for $s=2$, $b=1$ and $\alpha \to 1$:

Corollary.

$$\int_0^1 \left\{\frac{1}{t}\right\} \frac{t}{(1-t)} dt = \gamma,$$

where γ is the Euler constant.

Proof. The $j=1$ term of the summation is first separated, and

$${}_2F_1(\alpha, 2; 3; 1) = \frac{\Gamma(3)\Gamma(1-\alpha)}{\Gamma(3-\alpha)\Gamma(1)} = \frac{2}{(2-\alpha)(1-\alpha)}$$

is used. In taking the limit, the expression

$$\frac{1}{2}\, {}_2F_1(\alpha, 2; 3; x) = \frac{1}{(1-\alpha)(2-\alpha)} \left[1 - \frac{(x-1)(\alpha x - x - 1)}{(1-x)^\alpha}\right] \frac{1}{x^2}$$

$$= -\frac{1}{x^2}[x + \ln(1-x)] + O(\alpha - 1), \quad \alpha \to 1,$$

is used, along with the sum $\sum_{j=2}^{\infty}[1/j + \ln(1-1/j)] = \gamma - 1$ and the limit

$$\lim_{\alpha\to 1}\left[\frac{1}{1-\alpha} - \frac{1}{(2-\alpha)(1-\alpha)}\right] = 1.$$

□

Of the many Mellin transforms considered in [13] (Appendix III) we mention that

$$I_j(s) \equiv \int_0^\infty \frac{x^s}{(e^x+1)^j} dx = \Gamma(s+1) \sum_{n=0}^{\infty} (-1)^n \frac{(j)_n}{n!} \frac{1}{(n+j)^{s+1}}, \qquad \text{Re } s > -1,$$

and

$$J_j(s) \equiv \int_0^\infty \frac{x^s}{(e^x-1)^j}dx = \Gamma(s+1)\sum_{n=0}^\infty \frac{(j)_n}{n!}\frac{1}{(n+j)^{s+1}}, \qquad \text{Re } s > 0,$$

where $(a)_n = \Gamma(a+n)/\Gamma(a) = (-1)^n \frac{\Gamma(1-a)}{\Gamma(1-a-n)}$ is the Pochhammer symbol, may always be written in terms of the zeta function for integers $j \geq 1$. This follows since the ratio $(j)_n/n!$ may be reduced and rewritten as a sum of powers of $n+j$. As examples we have

$$I_2(s) = \Gamma(s+1)[(1-2^{-s})\zeta(s+1) + (2^{1-s}-1)\zeta(s)],$$

and

$$I_3(s) = \Gamma(s+1)2^{-s-1}[(2^s-4)\zeta(s-1) + 3(2-2^s)\zeta(s) + 2(2^s-1)\zeta(s+1)].$$

Such integrals are useful for bounding the zeta function on the positive real axis. Other Mellin transforms of fractional part and fractional part-integer part functions are gathered below in Lemma 9.

In this article, we study the Mellin transforms of certain Legendre functions, and are able to identify the resulting polynomial factors in terms of certain generalized hypergeometric functions ${}_3F_2(1)$. The key result is that these polynomials possess zeros only on the critical line.

We use standard notation, letting ${}_pF_q$ be the generalized hypergeometric function, $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ the Beta function, P_ν^μ the Legendre functions of the first kind, and $P_n = P_n^0$ the ordinary Legendre polynomials. Standard results on orthogonal polynomials may be found in [10, 22] and [2] (Chs. 5–7).

We consider Mellin transformations for functions supported on $[0,1]$,

$$(\mathcal{M}f)(s) = \int_0^1 f(x)x^s \frac{dx}{x}. \tag{1.1}$$

For properties of Mellin transforms, [9] and [6] (Ch. 4) may be consulted. A related technique is the Master Theorem of Ramanujan, (e.g., [2], section 10.12), relating a

Mellin transform to the coefficients of an alternating power series. We do not rely on this Theorem, but we illustrate it briefly in our context (see the second proof of Lemma 2).

Put, for Re $s > 0$,

$$M_n(s) \equiv \int_0^1 x^{s-1} P_n(x) \frac{dx}{\sqrt{1-x^2}}. \tag{1.2}$$

We are employing the concept of generalized Mellin transform, for functions for which the Mellin transform on all of $[0,\infty)$ does not otherwise exist [6] (section 4.3). We note that we could instead consider the Mellin transforms

$$M_n^{(-1)}(s) = \int_0^1 P_n\left(\frac{1}{x}\right) \frac{x^{-s}}{\sqrt{1-x^2}} dx,$$

as regards the polynomial factors. This is because, aside from a phase factor of i, the Gamma factors of this transform are simply the analytic continuation of those of (1.2). Indeed, the correspondence between (1.2) and $M_n^{(-1)}(s)$ is closely connected with the functional equation of the polynomial factors.

Proposition 1. For Re $s > 0$,

$$M_n(s) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{n+s+1}{2}\right)} \int_0^{\pi/2} {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; 1 - \frac{(n+s)}{2}; \cos^2\varphi\right) d\varphi. \tag{1.3}$$

Proposition 2. The polynomial factors $p_n(s)$ of (1.3) satisfy the functional equation $p_n(s) = (-1)^{\lfloor n/2 \rfloor} p_n(1-s)$. Moreover, all of their zeros are on the line Re $s = 1/2$.

In place of the ${}_3F_2$ representations for $M_n(s)$ of Lemmas 2 and 3, we have the following representation in terms of a finite sum of ${}_2F_1(-1)$ functions.

Proposition 3. Let Re $s > 0$. Then

$$M_n(s) = \frac{(n!)^2}{2^n} \Gamma(s) \sum_{k=0}^{n} \frac{1}{(k!)^2} \frac{(-1)^k}{[(n-k)!]^2} \frac{\Gamma(k+1/2)}{\Gamma(k+s+1/2)} {}_2F_1\left(\frac{1}{2}+k-n, s; \frac{1}{2}+k+s; -1\right). \tag{1.4}$$

Generalizing (1.2), for Re $s > 0$ we put

$$M_n^m(s) \equiv \int_0^1 x^{s-1} P_n^m(x) \frac{dx}{\sqrt{1-x^2}}. \tag{1.5}$$

We recall that for associated Legendre functions $P_\nu^m(x) = (-1)^m (1-x^2)^{m/2} \left(\frac{d}{dx}\right)^m P_\nu(x)$ and thus $P_n^m(x) = 0$ for $m > n$. We will take m to be nonnegative in the following. Otherwise, for negative index the following relation ([17], p. 1008) could be employed:

$$P_\nu^{-m}(x) = (-1)^m \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} P_\nu^m(x).$$

Proposition 4. The polynomial factors of $M_n^m(s)$ satisfy the functional equation $p_n^m(s) = (-1)^{\lfloor n/2 \rfloor} p_n^m(1-s)$ and moreover have all of their zeros on the critical line.

The following section of the paper contains the proof of these Propositions. Section 3 contains various supporting and reference Lemmas. Some of these Lemmas present results of special function theory that may be of interest in themselves. We note that one of us (MWC) has presented a subset of these results in an AMS special session [15]. The final Discussion section mentions a connection of the polynomial factors of the Mellin transforms with continuous Hahn polynomials.

We note that there is a possibility to physically realize these Mellin transforms in an all-optical system. A one or two dimensional Mellin transform may be carried out by making a hologram of a logarithmically scaled function, then optically Fourier transforming the reconstructed wavefront. The Fourier transform may be accomplished simply with a lens. The more challenging aspect is the logarithmic scale change, performable with a spatial light modulator. Much of the design of such processing has been given [23].

Proof of Propositions

Proposition 1. We provide two proofs for the integral representation. Method 1.

We use Lemma 3(c), along with

$$\frac{\left(\frac{1}{2}\right)_j}{(1)_j} = \frac{1}{\pi} B\left(\frac{1}{2}, j+\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(j+1/2)}{j!} = \frac{1}{\pi}\int_0^\pi \cos^{2j}\varphi \, d\varphi.$$

Then

$${}_3F_2\left(\frac{1-n}{2}, -\frac{n}{2}, \frac{1}{2}; 1, 1-\frac{(n+s)}{2}; 1\right) = \sum_{j\geq 0} \frac{\left(\frac{1-n}{2}\right)_j \left(-\frac{n}{2}\right)_j \left(\frac{1}{2}\right)_j}{\left(1-\frac{n+s}{2}\right)_j (1)_j} \frac{1}{j!}$$

$$= \frac{1}{\pi} \sum_{j\geq 0} \frac{\left(\frac{1-n}{2}\right)_j \left(-\frac{n}{2}\right)_j}{\left(1-\frac{n+s}{2}\right)_j} \frac{1}{j!} \int_0^\pi \cos^{2j}\varphi \, d\varphi.$$

Now $\int_0^\pi \cos^m \varphi \, d\varphi = 0$ for m odd. Therefore, we may write

$${}_3F_2\left(\frac{1-n}{2}, -\frac{n}{2}, \frac{1}{2}; 1, 1-\frac{(n+s)}{2}; 1\right) = \frac{1}{\pi} \sum_{j\geq 0} \frac{\left(\frac{1-n}{2}\right)_{j/2} \left(-\frac{n}{2}\right)_{j/2}}{\left(1-\frac{n+s}{2}\right)_{j/2}} \frac{1}{(j/2)!} \int_0^\pi \cos^{j}\varphi \, d\varphi$$

$$= \frac{1}{\pi} \int_0^\pi {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; 1-\frac{(n+s)}{2}; \cos^2\varphi\right) \, d\varphi.$$

From the evenness of the integrand on $[0,\pi]$, (2.6) follows. $\square$

Method 2. We may use the Beta transform [17] (p. 850)

$$\int_0^1 (1-x)^{-1/2} x^{-1/2} \, {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; 1-\frac{n+s}{2}; x\right) dx = \pi \, {}_3F_2\left(\frac{1-n}{2}, -\frac{n}{2}, \frac{1}{2}; 1, 1-\frac{(n+s)}{2}; 1\right).$$

Then making the change of variable $x = \cos^2\varphi$, we again obtain the Proposition. $\square$

Proposition 2. Up to *s-independent* factors and choice of normalization, by Proposition 1 we may take

$$p_n(s) = \frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s+\epsilon}{2}\right)} \int_0^{\pi/2} {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; 1-\frac{(n+s)}{2}; \cos^2\varphi\right) \, d\varphi,$$

where $\epsilon = 0$ for n even and $= 1$ for n odd. By [17] (p. 1043, # 9.131.2) we transform the argument of the ${}_2F_1$ function to $1-\cos^2\varphi = \sin^2\varphi$. Since one of $1/\Gamma(-n/2)$ or $1/\Gamma[(1-n)/2]$ will be zero depending upon whether n is even or odd, respectively, we have

$$p_n(s) = \frac{\Gamma\left(1-\frac{(n+s)}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} \frac{\Gamma\left(\frac{1+n-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s+\epsilon}{2}\right)} \int_0^{\pi/2} {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; \frac{(1-n+s)}{2}; \sin^2\varphi\right) \, d\varphi.$$

Since

$$\int_0^{\pi/2} \sin^{2k}\varphi \, d\varphi = \int_0^{\pi/2} \cos^{2k}\varphi \, d\varphi = \frac{1}{2} B\left(\frac{1}{2}, k + \frac{1}{2}\right),$$

and using the functional equation of the Gamma function, we may equally well write

$$p_n(s) = \frac{\pi}{\Gamma\left(1 - \frac{s}{2}\right)} \frac{\Gamma\left(\frac{1+n-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \frac{1}{\sin\pi\left(\frac{n+s}{2}\right)\Gamma\left(\frac{s+\epsilon}{2}\right)} \int_0^{\pi/2} {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; \frac{(1-n+s)}{2}; \cos^2\varphi\right) d\varphi.$$

When n is even, $\epsilon = 0$,

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(1 - \frac{s}{2}\right) = \frac{\pi}{\sin\pi(s/2)},$$

leaving the denominator factor $\Gamma\left(\frac{1-s}{2}\right)$. When n is odd, $\epsilon = 1$,

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = \frac{\pi}{\cos\pi(s/2)},$$

leaving the denominator factor $\Gamma\left(1 - \frac{s}{2}\right)$. Hence the factor $(-1)^{\lfloor n/2 \rfloor}$ emerges as $\sin(\pi s/2)/\sin[\pi(n+s)/2] = (-1)^{n/2}$ when n is even and as $\cos(\pi s/2)/\sin[\pi(n+s)/2] = (-1)^{(n-1)/2}$ when n is odd, and the functional equation of $p_n(s)$ follows.

In order to show that $p_n(s)$ has zeros only on the critical line, we first establish the difference equation

$$\begin{aligned}
&[n^2 + n - 1 + 4s - 2s(s+1)]\left(\frac{s+\epsilon}{2} - 1\right)\left(\frac{s+n+1}{2}\right) p_n(s) \\
&+[(s+2)(s+3) - (n^2+n-2) - 4(s+2)]\left(\frac{s+\epsilon}{2} - 1\right)\left(\frac{s+\epsilon}{2}\right) p_n(s+2) \\
&+(s-1)(s-2)\left(\frac{s+n+1}{2}\right)\left(\frac{s+n-1}{2}\right) p_n(s-2) = 0,
\end{aligned}$$

with $\epsilon = 0$ for n even and $= 1$ for n odd. We use the ordinary differential equation satisfied by Legendre polynomials,

$$(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0.$$

Putting $P_n(x) = \sqrt{1-x^2} f(x)$, we find

$$\frac{1}{\sqrt{1-x^2}}\left[(n^2 + n - 1 - (n^2+n-2)x^2)f(x) + 4x(x^2-1)f'(x) + (x^2-1)^2 f''(x)\right] = 0.$$

We then integrate the quantity in square brackets by parts and use the definition $M_n(s) = \int_0^1 x^{s-1} f(x) dx$. There results the difference equation for the Mellin transforms

$$[n^2+n-1+4s-2s(s+1)]M_n(s)+[(s+2)(s+3)-(n^2+n-2)-4(s+2)]M_n(s+2)$$

$$+(s-1)(s-2)M_n(s-2)=0.$$

The Mellin transforms are of the form

$$M_n^m(s) = \frac{\sqrt{\pi}}{2^{\lfloor n+1 \rfloor}} \frac{\Gamma\left(\frac{s+\epsilon}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{n+1}{2}\right)} p_n(s),$$

so that by repeatedly using the functional equation $\Gamma(z+1) = z\Gamma(z)$, the difference equation for $p_n(s)$ is found.

Now using shifted polynomials $q(s) = p_n(s+1/2)$ and putting $s \to s+1/2$, we have the difference equation

$$\left[n^2+n+1+4s-2\left(s+\frac{1}{2}\right)\left(s+\frac{3}{2}\right)\right]\left(\frac{s+n+1}{2}+\frac{1}{4}\right)\left(\frac{s+\epsilon}{2}+\frac{1}{4}\right)q(s)$$
$$+\left[\left(s+\frac{5}{2}\right)\left(s+\frac{7}{2}\right)-(n^2+n-2)-4\left(s+\frac{5}{2}\right)\right]\left(\frac{s+\epsilon}{2}-\frac{3}{4}\right)\left(\frac{s+\epsilon}{2}+\frac{1}{4}\right)q(s+2)$$
$$+\left(s-\frac{1}{2}\right)\left(s-\frac{3}{2}\right)\left(\frac{s+n}{2}+\frac{3}{4}\right)\left(\frac{s+n}{2}-\frac{1}{4}\right)q(s-2)=0.$$

We note the factorization

$$\left(s+\frac{5}{2}\right)\left(s+\frac{7}{2}\right)-(n^2+n-2)-4\left(s+\frac{5}{2}\right) = \left(s-n+\frac{1}{2}\right)\left(s+n+\frac{3}{2}\right).$$

It then follows that if r_k is a root of q, $q(r_k)=0$, that

$$\left(r_k+\epsilon-\frac{3}{2}\right)\left(r_k+\epsilon+\frac{1}{2}\right)\left(r_k-n+\frac{1}{2}\right)q(r_k+2)$$
$$= -\left(r_k-\frac{1}{2}\right)\left(r_k-\frac{3}{2}\right)\left(r_k+n-\frac{1}{2}\right)q(r_k-2).$$

When n is even, a factor of $r_k-3/2$ cancels on both sides, and when n is odd, a factor of $r_k-1/2$ cancels on both sides. In either case, equality of the absolute value of both sides provides a necessary condition that Re $r_i = 0$ for all the zeros of q. Hence the zeros of $p_n(s)$ lie on the critical line. $\square$

Proposition 3. We have the following generating function,

$$
{}_0F_1\left(-;1;\frac{1}{2}t(x-1)\right)\,{}_0F_1\left(-;1;\frac{1}{2}t(x+1)\right) = I_0\left(\sqrt{2}\sqrt{t(x-1)}\right) I_0\left(\sqrt{2}\sqrt{t(x+1)}\right)
$$

$$
= \sum_{n=0}^{\infty} \frac{P_n(x)t^n}{(n!)^2},
$$

giving

$$
\sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} M_n(s) = \int_0^1 \frac{x^{s-1}}{\sqrt{1-x^2}} I_0\left(\sqrt{2}\sqrt{t(x-1)}\right) I_0\left(\sqrt{2}\sqrt{t(x+1)}\right) dx.
$$

We now insert the power series for I_0 [17] (p. 961), so that

$$
\sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} M_n(s) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{t^{k+\ell}}{2^k 2^\ell (k!)^2 (\ell!)^2} \int_0^1 \frac{x^{s-1}}{\sqrt{1-x^2}} (x-1)^k (x+1)^\ell dx
$$

$$
= \sum_{m=0}^{\infty} \frac{t^m}{2^m} \sum_{k=0}^{m} \frac{1}{(k!)^2} \frac{1}{[(m-k)!]^2} \int_0^1 \frac{x^{s-1}}{\sqrt{1-x^2}} (x-1)^k (x+1)^{m-k} dx. \tag{2.1}
$$

The integral evaluates in terms of the Gauss hypergeometric function,

$$
-i\int_0^1 x^{s-1}(x+1)^{m-k-1/2}(x-1)^{k-1/2} dx = (-1)^k \frac{\Gamma(k+1/2)}{\Gamma(k+s+1/2)}\,{}_2F_1\left(\frac{1}{2}+k-m, s; \frac{1}{2}+k+s; -1\right).
$$

Then reading off the coefficient of t^n on both sides of (2.1) yields the Proposition. $\square$

Remarks. Proposition 3 is consistent with the relations

$$
{}_2F_1\left(s, \frac{1}{2}; s+\frac{1}{2}; -1\right) = \frac{\sqrt{\pi}}{2^s} \frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma^2\left(\frac{s+1}{2}\right)}, \tag{2.2a}
$$

$$
{}_2F_1\left(s, -\frac{1}{2}; s+\frac{1}{2}; -1\right) = \frac{\sqrt{\pi}}{2^s} \Gamma\left(s+\frac{1}{2}\right)\left[\frac{1}{\Gamma^2\left(\frac{s+1}{2}\right)} + \frac{s}{2}\frac{1}{\Gamma^2\left(\frac{s}{2}+1\right)}\right], \tag{2.2b}
$$

and

$$
{}_2F_1\left(s, \frac{1}{2}; s+\frac{3}{2}; -1\right) = -\frac{\sqrt{\pi}}{2^{s-1}} \Gamma\left(s+\frac{3}{2}\right)\left[\frac{2}{s}\frac{1}{\Gamma^2\left(\frac{s}{2}\right)} - \frac{1}{\Gamma^2\left(\frac{s+1}{2}\right)}\right]. \tag{2.2c}
$$

(2.2a) is Kummer's identity, and the others may be obtained with the aid of contiguous relations.

The ${}_2F_1(-1)$ function of (1.4) may be transformed to functions ${}_2F_1(1/2)$.

Proposition 4. We have ([17], p. 1015)

$$P_n^m(x) = (-1)^m \frac{(2n-1)!!}{(n-m)!}(1-x^2)^{m/2} x^{n-m} \;{}_2F_1\left(\frac{n-m}{2}, \frac{m-n+1}{2}; \frac{1}{2}-n; \frac{1}{x^2}\right),$$

giving

$$M_n^m(s) = i^{m-n}\frac{\sqrt{\pi}}{2}\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{1}{2}-n\right)\left\{\frac{\Gamma\left(\frac{s}{2}\right)\;{}_3F_2\left(\frac{m-n}{2}, \frac{m+n+1}{2}, \frac{s}{2}; \frac{1}{2}, \frac{1+m+s}{2}; 1\right)}{\Gamma\left(\frac{1-m-n}{2}\right)\Gamma\left(\frac{1+m-n}{2}\right)\Gamma\left(\frac{1+m-s}{2}\right)}\right.$$

$$\left.+\frac{2i\Gamma\left(\frac{s+1}{2}\right)\;{}_3F_2\left(\frac{1+m-n}{2}, 1+\frac{m+n}{2}, \frac{s+1}{2}; \frac{3}{2}, 1+\frac{m+s}{2}; 1\right)}{\Gamma\left(\frac{m-n}{2}\right)\Gamma\left(-\frac{(m+n)}{2}\right)\Gamma\left(1+\frac{m+s}{2}\right)}\right\}.$$

The first line on the right side of this expression gives M_n^m for $m-n$ even, and the second line for $m-n$ odd. The key step in demonstrating the functional equation is transforming the ${}_3F_2$ functions so that a numerator parameter is twice a denominator parameter. Then the Beta transformation

$$\int_0^1 (1-x)^{\beta-1}x^{\beta-1}\;{}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; 1-\frac{(n+s)}{2}; x\right)dx$$

$$= 2^{1-2\beta}\frac{\sqrt{\pi}\Gamma(\beta)}{\Gamma(\beta+1/2)}\;{}_3F_2\left(\beta, \frac{1-n}{2}, -\frac{n}{2}; 2\beta, 1-\frac{(n+s)}{2}; 1\right),$$

together with a transformation of the argument of the ${}_2F_1$ function from x to $1-x$ may be used to verify the functional equation. Each of the three transformations given in the Appendix may used to put the ${}_3F_2$ functions of $M_n^m(s)$ in suitable form. We illustrate this procedure using (A.3) and the ${}_3F_2$ function for $m-n$ even,

$${}_3F_2\left(\frac{m-n}{2}, \frac{m+n+1}{2}, \frac{s}{2}; \frac{1}{2}, \frac{1+m+s}{2}; 1\right) = \frac{\Gamma\left(\frac{1+m-n}{2}\right)\Gamma\left(1+\frac{m-n}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1+m+s}{2}\right)}{\sqrt{\pi}\Gamma(1+m-n)\Gamma\left(\frac{1+m-n+s}{2}\right)\Gamma\left(\frac{1+n+s}{2}\right)}$$

$$\times\;{}_3F_2\left(\frac{m+1}{2}, \frac{m-n}{2}\frac{1+m-n}{2}; m+1, \frac{1+m+s}{2}; 1\right).$$

There is only one ${}_3F_2$ function on the right side as the other has a vanishing $1/\Gamma[(m-n)/2]$ prefactor. The functional equation then follows.

The demonstration that all of the zeros of $p_n^m(s)$ lie on the critical line follows very closely that corresponding to the proof of Proposition 2. The ordinary differential equation satisfied by associated Legendre polynomials,

$$(1-x^2)P_n^{m''}(x) - 2xP_n^{m'}(x) + \left[n(n+1) - \frac{m^2}{1-x^2}\right] P_n^m(x) = 0,$$

leads to the difference equation of the Mellin transforms,

$$[n^2-m^2+n-1+4s-2s(s+1)]M_n^m(s)+[(s+2)(s+3)-(n^2+n-2)-4(s+2)]M_n^m(s+2)$$

$$+(s-1)(s-2)M_n^m(s-2) = 0.$$

So in the difference equation for $p_n^m(s)$ and the corresponding shifted polynomial $q(s) = p_n^m(s+1/2)$, only the coefficient of $p_n^m(s)$ or $q(s)$ is modified, thus having no effect on the rest of the proof. It follows once again that the zeros of $q(s)$ are pure imaginary and thus the zeros of $p_n^m(s)$ all lie on the critical line. $\square$

Lemmas

Lemma 1. Let $n \geq 1$ and Re $s > 0$. Then

$$M_n(s) = \frac{1}{n}[(2n-1)M_{n-1}(s+1) - (n-1)M_{n-2}(s)], \tag{3.1}$$

and

$$M_0(s) = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}, \qquad M_1(s) = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}.$$

Notice that with $M_{-1}(s)$ arbitrary but finite, the recursion (3.1) properly degenerates to $M_1(s) = M_0(s+1)$. It is easily seen that $M_0(s) = B(s/2, 1/2)/2$, where B is the Beta function.

Proof. The mixed recurrence (3.1) follows from

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

The expressions for M_0 and M_1 follow from the integral representation

$$\int_0^1 x^{a-1}(1-x^2)^{b-1}dx = \frac{1}{2}B\left(\frac{a}{2}, b\right), \qquad \text{Re } a > 0, \quad \text{Re } b > 0.$$

$\square$

There are a great many transformations of ${}_3F_2(z)$ and ${}_3F_2(1)$ functions, including Thomae's. Rather than list the result of several of these applied to the Mellin transforms $M_n(s)$, we present the result of obtaining these transforms directly through integration.

Lemma 2. Let $(2n+1)!! \equiv (2n+1)(2n-1)\cdots 3$. The following representations for $M_n(s)$ hold for Re $s>0$ when n is even and for Re $s>-1$ when n is odd. (a)

$$M_{2n+1}(s) = \frac{(-1)^n}{2}\frac{\left(\frac{2-s}{2}\right)_n}{\left(\frac{s+1}{2}\right)_{n+1}}\ {}_3F_2\left(\frac{1}{2}, \frac{s+1}{2}, \frac{s}{2}; \frac{s}{2}-n, \frac{s}{2}+n+\frac{3}{2}; 1\right),$$

(b)

$$M_{2n}(s) = \frac{(-1)^n}{2}\frac{\left(\frac{1-s}{2}\right)_n}{\left(\frac{s}{2}\right)_{n+1}}\ {}_3F_2\left(\frac{1}{2}, \frac{s+1}{2}, \frac{s}{2}; \frac{s}{2}-n+\frac{1}{2}, \frac{s}{2}+n+1; 1\right),$$

(c)

$$M_{2n}(s) = (-1)^n\frac{(2n-1)!!}{2^{n+1}n!}\frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}\ {}_3F_2\left(-n, n+\frac{1}{2}, \frac{s}{2}; \frac{1}{2}, \frac{s+1}{2}; 1\right),$$

(d)

$$M_{2n+1}(s) = (-1)^n\frac{(2n+1)!!}{2^n n!}\frac{\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)}{s\Gamma\left(\frac{s}{2}\right)}\ {}_3F_2\left(-n, n+\frac{3}{2}, \frac{s+1}{2}; \frac{3}{2}, \frac{s}{2}+1; 1\right),$$

and (e)

$$M_n(s) = (-1)^n\frac{(2n-1)!!}{2^{n+1}n!}\frac{\pi\Gamma\left(\frac{s-n}{2}\right)}{\Gamma\left(\frac{1-n}{2}\right)\Gamma\left(\frac{s-n+1}{2}\right)}\ {}_3F_2\left(-n, \frac{1}{2}, \frac{s-n}{2}; \frac{1-n}{2}, \frac{s-n+1}{2}; 1\right).$$

Proof. (a) and (b) follow by binomially expanding $(1-x^2)^{-1/2} = \sum_{\ell=0}^{\infty}\binom{-1/2}{\ell}(-1)^\ell x^{2\ell}$ in the integrand of $M_n(s)$ and performing term-by-term integration. For the rest we recall the representation [17] (p. 850) for Re $s>0$ and Re $\nu>1$

$$\int_0^1 x^{s-1}(1-x^2)^\nu\ {}_2F_1(-n, a; b; x^2)dx = \frac{1}{2}B\left(\nu+1, \frac{s}{2}\right)\ {}_3F_2\left(-n, a, \frac{s}{2}; b, \nu+1+\frac{s}{2}; 1\right).$$

Then we employ various ${}_2F_1$ expressions for $P_n(x)$ from [17] (p. 1025). In particular, for (e), we use

$$P_n(x) = \frac{(2n-1)!!}{n!}x^n\ {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2}-n; \frac{1}{x^2}\right),$$

along with a simple change of variable. $\square$

We also present as an example use of Ramanujan's Master Theorem an alternative proof of part (c), using the hypergeometric expression

$$P_{2n}(x) = (-1)^n \frac{(2n-1)!!}{2^n n!} \, {}_2F_1\left(-n, n+\frac{1}{2}; \frac{1}{2}; x^2\right).$$

We then obtain the power series

$$\frac{P_{2n}(x)}{\sqrt{1-x^2}} = (-1)^n \frac{(2n-1)!!}{2^n n!} \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{(-n)_j (n+1/2)_j}{(1/2)_j} \frac{(-1)^{m-j}}{j!} \binom{-1/2}{m-j} x^{2m}.$$

Writing $\Phi(x^2) = \sum_{k=0}^{\infty} (-1)^k \phi(k) x^{2k}$, we have

$$\phi(m) = (-1)^n \frac{(2n-1)!!}{2^n n!} \sum_{j=0}^{m} \frac{(-n)_j (n+1/2)_j}{(1/2)_j} \frac{(-1)^j}{j!} \binom{-1/2}{m-j},$$

giving

$$\frac{\pi}{\sin \pi s} \phi(-s) = (-1)^n \frac{(2n-1)!!}{2^n n!} \frac{\pi}{\sin \pi s} \frac{\sqrt{\pi} \, {}_3F_2\left(-n, n+\frac{1}{2}, s; \frac{1}{2}, s+\frac{1}{2}; 1\right)}{\Gamma(1-s)\Gamma(s+1/2)}$$

$$= (-1)^n \frac{(2n-1)!!}{2^n n!} \sqrt{\pi} \frac{\Gamma(s)}{\Gamma(s+1/2)} \, {}_3F_2\left(-n, n+\frac{1}{2}, s; \frac{1}{2}, s+\frac{1}{2}; 1\right).$$

Taking into account the change of variable $\int_0^1 u^{s-1} \Phi(u) du = 2 \int_0^1 x^{2s-1} \Phi(x^2) dx$, so that replacing $s \to s/2$ just above, we obtain agreement with $M_{2n}(s)$ in part (c). $\square$

Lemma 3. Alternative hypergeometric representation for $M_n(s)$. (a)

$$M_n(s) = \frac{1}{2^{n+1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} B\left(\frac{s+n}{2} - k, \frac{1}{2}\right),$$

(b)

$$M_n(s) = 2^{n-1} \frac{\Gamma\left(n+\frac{1}{2}\right) \Gamma\left(\frac{n+s}{2}\right)}{n! \Gamma\left(\frac{n+s+1}{2}\right)} \, {}_3F_2\left(\frac{1-n}{2}, -\frac{n}{2}, \frac{1-n-s}{2}; \frac{1}{2} - n, 1 - \frac{n}{2} - \frac{s}{2}; 1\right),$$

and (c)

$$M_n(s) = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{n+s+1}{2}\right)} \, {}_3F_2\left(\frac{1-n}{2}, -\frac{n}{2}, \frac{1}{2}; 1, 1 - \frac{(n+s)}{2}; 1\right).$$

The forms (b) and (c) are convenient for presenting the truncation at degree $\lfloor n/2 \rfloor$, due to the presence of both the numerator parameters $-n/2$ and $(1-n)/2$.

Proof. Parts (a) and (b) follow from a series in [17] (p. 1025),

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}.$$

Relations such as

$$\frac{\Gamma\left(\frac{s+n+1}{2}\right)}{\Gamma\left(\frac{s+n+1}{2} - k\right)} = (-1)^k \left(\frac{1-n-s}{2}\right)_k,$$

and

$$\frac{\Gamma\left(\frac{s+n}{2}\right)}{\Gamma\left(\frac{s+n}{2} - k\right)} = (-1)^k \left(1 - \frac{(n+s)}{2}\right)_k,$$

are then used. Part (c) follows from using Laplace's integral (3.4). $\square$

Remarks. Various transformation corollaries follow from the equalities of the expressions in Lemmas 2 and 3.

For $s = 1$, the ${}_3F_2$ function in Lemma 3(b) reduces to particular ${}_2F_1$ values,

$${}_2F_1\left(-\frac{n}{2}, -\frac{n}{2}; \frac{1}{2} - n; 1\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2} - n\right)}{\Gamma^2\left(\frac{1-n}{2}\right)}.$$

This gives the special values

$$M_n(1) = \frac{(-1)^n \pi^2}{2\Gamma^2\left(\frac{1-n}{2}\right)\Gamma^2\left(\frac{n}{2} + 1\right)}.$$

Here is another way to calculate the transforms $M_n(s)$. Use the Fourier transform

$$P_n(x) = \frac{(-i)^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{-1/2} J_{n+1/2}(t) e^{itx} dt, \qquad |x| < 1,$$

where J_n is the Bessel function of the first kind of order n. Then interchange integrations.

Here is $M_n(s)$ written as a 1/2-line transform:

$$M_n(s) = \int_0^{\infty} \tanh^{s-1} u \frac{P_n(\tanh u)}{\cosh u} du,$$

wherein the weight function $1/\cosh u$ is self-reciprocal, up to scaling, on the full line, under the (exponential) Fourier transform.

We also have

$$M_n(s) = \int_0^{\pi/2} \cos^{s-1}\theta P_n(\cos\theta) d\theta.$$

Using [17] (p. 1027) allows alternative ${}_3F_2$ forms to be obtained, including

$$M_n(s) = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{n+s+1}{2}\right)} {}_3F_2\left(\frac{1}{2}, \frac{1-n}{2}, -\frac{n}{2}; 1, 1 - \frac{(n+s)}{2}; 1\right),$$

in agreement with Lemma 3(c).

Lemma 4. (Generating function of Mellin transforms.) For Re $s > 0$,

$$G(t,s) \equiv \sum_{k=0}^\infty M_k(s) t^k = \int_0^1 \frac{1}{(1-x^2)^{1/2}} \frac{x^{s-1}}{(1-2tx+t^2)^{1/2}} dx$$

$$= \frac{\sqrt{\pi}}{(1+t^2)^{1/2}} \left\{ \frac{\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+1}{2}\right)} {}_3F_2\left[\frac{1}{4}, \frac{3}{4}, \frac{s}{2}; \frac{1}{2}, \frac{s+1}{2}; \frac{4t^2}{(1+t^2)^2}\right] \right.$$

$$\left. + \frac{t}{(1+t^2)} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} {}_3F_2\left[\frac{3}{4}, \frac{5}{4}, \frac{s+1}{2}; \frac{3}{2}, \frac{s}{2}+1; \frac{4t^2}{(1+t^2)^2}\right] \right\}.$$

The first line of the right member yields $M_{2k}(s)$ and the second line, $M_{2k+1}(s)$.

Proof. A generating function of Legendre polynomials is

$$\sum_{k=0}^\infty t^k P_k(x) = \frac{1}{\sqrt{1-2tx+t^2}}.$$

Then binomially expanding we have

$$\int_0^1 \frac{1}{(1-x^2)^{1/2}} \frac{x^{s-1}}{(1-2tx+t^2)^{1/2}} dx = \sum_{\ell=0}^\infty \binom{-\frac{1}{2}}{\ell} (-1)^\ell \int_0^1 \frac{x^{2\ell+s-1}}{(1-2tx+t^2)^{1/2}} dx$$

$$= \frac{1}{(1+t^2)^{1/2}} \sum_{\ell=0}^\infty \binom{-\frac{1}{2}}{\ell} \frac{(-1)^\ell}{(2\ell+s)} {}_2F_1\left(\frac{1}{2}, 2\ell+s; 1+2\ell+s; \frac{2t}{1+t^2}\right).$$

We then interchange sums, and separate terms of even and odd summation index. $\square$

Remarks. Another generating function relation is

$$\sum_{k=0}^\infty \frac{t^k}{k!} M_k(s) = \int_0^1 e^{xt} x^{s-1} \frac{J_0(t\sqrt{1-x^2})}{\sqrt{1-x^2}} dx$$

$$= \int_0^1 e^{\sqrt{1-u^2}t}(1-u^2)^{s/2-1} J_0(tu)du.$$

The first equality in

$$\sum_{n=0}^{\infty} \frac{P_n(x)}{n!} r^n = e^{xr} J_0(r\sqrt{1-x^2}) = e^{xr} e^{-r^2/4} \sum_{n=0}^{\infty} \frac{r^{2n}}{4^n n!} L_n(1-x^2)$$

follows from Laplace's integral for $P_n(x)$. Here L_n is the Laguerre polynomial of degree n.

Lemma 5.

The polynomials p_n satisfy the following recursion relation, with $p_0 = p_1 = 1$.

For n even,

$$p_n(s) = \frac{2}{n}[(2n-1)sp_{n-1}(s+1) - (n-1)(s+n-1)p_{n-2}(s)],$$

and for n odd,

$$p_n(s) = \frac{1}{n}[(2n-1)p_{n-1}(s+1) - 2(n-1)(s+n-1)p_{n-2}(s)].$$

Proof. By Proposition 1 or the proof of Proposition 2, the Mellin transforms are of the form

$$M_n(s) = \frac{\sqrt{\pi}}{2^n} \frac{p_n(s)\Gamma\left(\frac{s+\epsilon}{2}\right)}{\Gamma\left(\frac{s}{2} + \frac{n+1}{2}\right)},$$

where $\epsilon = 0$ for n even and $= 1$ for n odd. (The 2^{-n} normalization makes $p_n(s)$ of leading coefficient $2^{\lfloor n/2 \rfloor}$.) Then using Lemma 1 and the functional equation of the Gamma function gives the result. □

Lemma 6. Let Re $s > 0$. Then

$$M_n^m(s) = \frac{1}{(n-m)}[(2n-1)M_{n-1}^m(s+1) - (n+m-1)M_{n-2}^m(s)],$$

and

$$M_0^m(s) = \delta_{m0} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}, \qquad M_1^m(s) = \delta_{m0} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} - \delta_{m1}\frac{1}{s},$$

where δ_{nm} is the Kronecker symbol. Furthermore, $M^m_{m+1}(s) = (2m+1)M^m_m(s+1)$ and

$$M^m_m(s) = (-1)^m(2m-1)!!\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(\frac{s+m+1}{2}\right)}.$$

Proof. This follows from

$$(2\nu+1)xP^m_\nu(x) = (\nu-m+1)P^m_{\nu+1}(x) + (\nu+m)P^m_{\nu-1}(x),$$

$P^m_0(x) = \delta_{0m}$, $P^m_1(x) = \delta_{0m}x - \delta_{1m}\sqrt{1-x^2}$, $P^m_m(x) = (-1)^m(2m-1)!!(1-x^2)^{m/2}$, and $P^m_{m+1}(x) = (2m+1)xP^m_m(x)$. $\square$

Remarks. A way to calculate these Mellin transforms, especially for odd values of m, is via the relation (e.g., [17], p. 1008)

$$P^m_\nu(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\nu(x).$$

Using $P_n(1) = 1$ and [1] (p. 338), integrating by parts we have

$$M^1_n(s) = \frac{\sqrt{\pi}}{2^{s-1}}\frac{\Gamma(s)}{\Gamma\left(\frac{s-n}{2}\right)\Gamma\left(\frac{s+n+1}{2}\right)} - 1. \tag{3.2a}$$

As $M^1_0 = 0$, it is evident that this expression recovers Legendre's duplication formula when $n = 0$. I.e., alternatively we may write

$$M^1_n(s) = \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s-n}{2}\right)\Gamma\left(\frac{s+n+1}{2}\right)} - 1. \tag{3.2b}$$

In so far as

$$\Gamma\left(\frac{s-n}{2}\right)\Gamma\left(\frac{s+n+1}{2}\right) = \sqrt{\pi}2^{n-s}\Gamma(s-n)(s-n+1)\left(\frac{3+s-n}{2}\right)_{n-1},$$

$M^1_n(s)$ is a rational function of s. More generally, when m is odd, $M^m_n(s)$ is a rational function of s with simple poles. We note that for m odd, $P^m_n(x)$ is of the form $\sqrt{1-x^2}$ times a polynomial in x.

Lemma 7. For m an even integer, for n even,

$$p^m_n(s) = \frac{2}{n-m}[(2n-1)sp^m_{n-1}(s+1) - (n+m-1)(s+n-1)p^m_{n-2}(s)],$$

and for n odd,

$$p_n^m(s) = \frac{1}{n-m}[(2n-1)p_{n-1}^m(s+1) - 2(n+m-1)(s+n-1)p_{n-2}^m(s)].$$

Further, $p_m^m = (-1)^m(2m-1)!!(m-1)!!$ and $p_{m+1}^m = (2m+1)p_m^m$.

Proof. For m even, the Mellin transforms are of the form

$$M_n^m(s) = \frac{\sqrt{\pi}}{2^{\lfloor n-m/2+1 \rfloor}} \frac{p_n^m(s)\Gamma\left(\frac{s+\varepsilon}{2}\right)}{\Gamma\left(\frac{s}{2}+\frac{n+1}{2}\right)},$$

where $\epsilon = 0$ for n even and $= 1$ for n odd. For $n < m$, they vanish. Then by using Lemma 6 and the functional equation of the Gamma function gives the recurrences.

□

Lemma 8. Let $p \equiv (n+1)(n+2)\cdots(n+m)$. Then

$$M_n^m(s) = \frac{p\pi}{2\Gamma\left(\frac{n+s+1}{2}\right)} \left\{ \sqrt{\pi}\Gamma\left(\frac{n+s}{2}\right) \frac{{}_4F_3\left(\frac{1}{2}, 1, \frac{1-n}{2}, -\frac{n}{2}; 1-\frac{m}{2}, 1+\frac{m}{2}, 1-\frac{(n+s)}{2}; 1\right)}{\Gamma\left(\frac{1-m}{2}\right)\Gamma(1-m)\Gamma\left(\frac{1+m}{2}\right)m!} \right.$$

$$\left. -2in\,\Gamma\left(\frac{n+s-1}{2}\right) \frac{{}_4F_3\left(1, 1, \frac{1-n}{2}, 1-\frac{n}{2}; \frac{3-m}{2}, \frac{3+m}{2}, \frac{3-(n+s)}{2}; 1\right)}{\Gamma(2-m)\Gamma\left(-\frac{m}{2}\right)\Gamma\left(\frac{m}{2}\right)(m+1)!} \right\}. \qquad (3.3)$$

Proof. Laplace's integral for P_n^m is [17] (p. 1001)

$$P_n^m(x) = \frac{1}{\pi}(n+1)(n+2)\cdots(n+m)\int_0^\pi [x+\sqrt{x^2-1}\cos\theta]^n \cos m\theta \; d\theta. \qquad (3.4)$$

We binomially expand the integrand to develop $M_n^m(s)$,

$$M_n^m(s) = -p\sum_{\ell=0}^n \binom{n}{\ell} i \int_0^\pi \int_0^1 x^{n-\ell+s-1}(x^2-1)^{(\ell-1)/2}\cos m\theta \cos^\ell \theta dx d\theta$$

$$= \frac{p}{2}\sum_{\ell=0}^n \binom{n}{\ell} i^\ell \frac{\Gamma\left(\frac{\ell+1}{2}\right)\Gamma\left(\frac{n+s-\ell}{2}\right)}{\Gamma\left(\frac{n+s+1}{2}\right)} \int_0^\pi \cos m\theta \cos^\ell \theta \; d\theta$$

$$= \frac{p}{2}\sum_{\ell=0}^n \binom{n}{\ell} i^\ell \frac{\Gamma\left(\frac{\ell+1}{2}\right)\Gamma\left(\frac{n+s-\ell}{2}\right)}{\Gamma\left(\frac{n+s+1}{2}\right)} \frac{2^\ell \pi^2 (-1)^\ell \ell!}{(\ell-m)!(\ell+m)!\Gamma\left(\frac{1-\ell-m}{2}\right)\Gamma\left(\frac{1-\ell+m}{2}\right)}.$$

The trigonometric integral here is given in [17] (p. 374). □

Remarks. (3.3) may be written in several ways using functional equations of the Gamma function. It includes many special cases, such as for $n = m$, and $m = 1$. For the latter case, there is reduction to a ${}_2F_1(1)$ function that gives (3.2).

Following up the example Mellin transforms of the Introduction, we have the following. We let $[x]$ denote the integer part of x.

Lemma 9. (a) For Re $s > 1$,

$$\int_0^1 \left\{\frac{1}{t}\right\}\left[\frac{1}{t}\right] t^{s-1}dt = \frac{1}{s(s-1)}\left\{(s-1)\zeta(s) + (2-s)\zeta(s-1)\right\},$$

with the value at $s = 2$ being $[\zeta(2) - 1]/2$, (b) for Re $s > 2$,

$$\int_0^1 \left\{\frac{1}{t}\right\}\left[\frac{1}{t}\right]^2 t^{s-1}dt = \frac{1}{s(s-1)}\left[(3-s)\zeta(s-2) + (2s-3)\zeta(s-1) + (1-s)\zeta(s)\right],$$

with the value at $s = 3$ being $\zeta(2)/2 - \zeta(3)/3 - 1/6$, and (c) for $n \geq 1$ an integer and Re $s > n$,

$$\int_0^1 \left\{\frac{1}{t}\right\}\left[\frac{1}{t}\right]^n t^{s-1}dt = \frac{1}{s(s-1)}\left\{-\sum_{\ell=0}^{n-1}(-1)^{n-\ell}\left[\binom{n}{\ell} - \binom{n}{\ell+1}(s-1)\right]\zeta(s-\ell-1)\right.$$
$$\left. +(-1)^{n+1}(s-1)\zeta(s)\right\}.$$

For this last integral, the value at $s = n+1$ is given by

$$\int_0^1 \left\{\frac{1}{t}\right\}\left[\frac{1}{t}\right]^n t^n dt = \frac{1}{n(n+1)}\left\{-\sum_{\ell=0}^{n-2}(-1)^{n-\ell}\left[\binom{n}{\ell} - \binom{n}{\ell+1}n\right]\zeta(n-\ell) - 1\right.$$
$$\left. +(-1)^{n+1}n\zeta(n+1)\right\}.$$

(d) For Re $s > n+1$,

$$\int_0^1 \left\{\frac{1}{t}\right\}^2\left[\frac{1}{t}\right]^n t^{s-1}dt = -\frac{1}{s(s-1)(s-2)}\left\{\sum_{\ell=0}^{n-1}(-1)^{n-\ell}\binom{n}{\ell}\left[2\zeta(s-\ell-2)\right.\right.$$
$$\left.\left. +2(s-2)\zeta(s-\ell-1) + (s-1)(s-2)\zeta(s-\ell)\right] + 2(s-2)\zeta(s-n-1) + (s-1)(s-2)\zeta(s-n)\right\}.$$

The value for $s \to n+2$ is given by

$$-\frac{1}{n(n+1)(n+2)}\left\{\sum_{\ell=0}^{n-2}(-1)^{n-\ell}\binom{n}{\ell}\left[2\zeta(n-\ell) + 2n\zeta(n-\ell+1) + n(n+1)\zeta(n-\ell+2)\right]\right.$$

$$+2-n(n-1)\zeta(2)-n^2(n+1)\zeta(3)\Big\}\,.$$

(e) Let $H_n=\sum_{k=1}^n \frac{1}{k}$ be the nth harmonic number and $\psi^{(j)}(z)$ be the polygamma function (e.g. [1, 17]). Let $s>1$ be an integer. Then

$$\int_0^1 \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\} x^{s-1}dx=\sum_{\ell=1}^s \frac{(-1)^\ell}{\ell!}\psi^{(\ell-1)}(1)+\frac{1}{2}-H_s+\frac{1-s2^{s-1}}{s2^s}$$

$$+\frac{1}{s(s-1)}\sum_{k=1}^\infty\left[\frac{k^s}{(k+1)^{s-3}}+\frac{[s-(k+1)^2]}{(k+2)^s}(k+1)^{s-1}\right],$$

where $\psi^{(\ell-1)}(1)=(-1)^\ell(\ell-1)!\zeta(\ell)$. For Re $ks>1$, (f)

$$\int_0^1\left\{\frac{1}{\sqrt[k]{t}}\right\}t^{s-1}dt=\frac{k}{ks-1}-\frac{\zeta(ks)}{s},\qquad \int_0^1\left[\frac{1}{\sqrt[k]{t}}\right]t^{s-1}dt=\frac{\zeta(ks)}{s},$$

$$\int_0^1\left\{\frac{1}{\sqrt[k]{t}}\right\}\left[\frac{1}{\sqrt[k]{t}}\right]t^{s-1}dt=\frac{1}{s}\zeta(ks)+\frac{(2-ks)}{s(ks-1)}\zeta(ks-1),$$

$$\int_0^1\left\{\frac{1}{\sqrt[k]{t}}\right\}\left[\frac{1}{\sqrt[k]{t}}\right]^n t^{s-1}dt=-\frac{1}{s(ks-1)}\left\{\sum_{j=0}^{n-1}(-1)^{n-j}\left[\binom{n}{j}-\binom{n}{j+1}(ks-1)\right]\zeta(ks-j-1)\right.$$

$$\left.+(-1)^n(ks-1)\zeta(ks)\right\},\qquad \text{Re } ks>n,$$

when also $k>1$, (g)

$$\int_0^1\left\{\frac{1}{k\sqrt[k]{t}}\right\}t^{s-1}dt=\frac{1}{ks-1}-\frac{\zeta(ks)}{k^{ks}s},\qquad \int_0^1\left[\frac{1}{k\sqrt[k]{t}}\right]t^{s-1}dt=\frac{\zeta(ks)}{k^{ks}s},$$

$$\int_0^1\left\{\frac{1}{k\sqrt[k]{t}}\right\}\left[\frac{1}{k\sqrt[k]{t}}\right]t^{s-1}dt=k^{-ks}\left[\frac{1}{s}\zeta(ks)+\frac{(2-ks)}{s(ks-1)}\zeta(ks-1)\right],$$

$$\int_0^1\left\{\frac{1}{k\sqrt[k]{t}}\right\}\left[\frac{1}{k\sqrt[k]{t}}\right]^n t^{s-1}dt=-\frac{k^{-ks}}{s(ks-1)}\left\{\sum_{j=0}^{n-1}(-1)^{n-j}\left[\binom{n}{j}-\binom{n}{j+1}(ks-1)\right]\zeta(ks-j-1)\right.$$

$$\left.+(-1)^n(ks-1)\zeta(ks)\right\},\qquad \text{Re } ks>n,$$

and (h)

$$\int_0^1\left\{\frac{k}{\sqrt[k]{t}}\right\}t^{s-1}dt=\frac{k}{s(ks-1)}-\frac{k^{ks}}{s}\zeta(ks,k+1),\qquad \int_0^1\left[\frac{k}{\sqrt[k]{t}}\right]t^{s-1}dt=\frac{k}{s}+\frac{k^{ks}}{s}\zeta(ks,k+1),$$

$$\int_0^1\left\{\frac{k}{\sqrt[k]{t}}\right\}\left[\frac{k}{\sqrt[k]{t}}\right]t^{s-1}dt=\frac{k^2}{s(ks-1)}+\frac{k^{ks}}{s(ks-1)}[(2-ks)\zeta(ks-1,k+1)+(ks-1)\zeta(ks,k+1)],$$

$$\int_0^1 \left\{\frac{k}{\sqrt[k]{t}}\right\} \left[\frac{k}{\sqrt[k]{t}}\right]^n t^{s-1}dt = \frac{k^{n+1}}{s(ks-1)}$$

$$-\frac{k^{ks}}{s(ks-1)} \left\{\sum_{j=0}^{n-1}(-1)^{n-j}\left[\binom{n}{j} - \binom{n}{j+1}(ks-1)\right]\zeta(ks-j-1,k+1)\right.$$

$$\left. +(-1)^n(ks-1)\zeta(ks,k+1)\right\}, \qquad \text{Re } ks > n.$$

Proof. We first consider convergence, of the more general integrals

$$\int_0^1 \left\{\frac{1}{t}\right\}^\alpha \left[\frac{1}{t}\right]^\beta t^{s-1}dt = \int_1^\infty \{v\}^\alpha [v]^\beta v^{-s-1}dv$$

$$= \sum_{k=1}^\infty k^\beta \int_k^{k+1} \frac{(v-k)^\alpha}{v^{s+1}}dv = \sum_{k=1}^\infty k^\beta \int_0^1 \frac{u^\alpha}{(u+k)^{s+1}}du.$$

Noting that

$$\frac{1}{(k+1)^{s+1}} < \frac{1}{(u+k)^{s+1}} < \frac{1}{k^{s+1}},$$

we have

$$\frac{1}{(\alpha+1)}\sum_{k=1}^\infty \frac{k^\beta}{(k+1)^{s+1}} < \int_0^1 \left\{\frac{1}{t}\right\}^\alpha \left[\frac{1}{t}\right]^\beta t^{s-1}dt < \frac{1}{(\alpha+1)}\sum_{k=1}^\infty \frac{k^\beta}{k^{s+1}}.$$

Therefore, for convergence of the integral we require Re $s > \beta$. This is regardless of the value of α, as long as Re $\alpha > -1$.

Next, we will repeatedly use the relation $\sum_{k=1}^\infty \frac{(k+1)^p}{(k+1)^s} = \zeta(s-p)-1$ for Re $s > p+1$.

(a) We write

$$\int_0^1 \left\{\frac{1}{t}\right\} \left[\frac{1}{t}\right] t^{s-1}dt = \int_1^\infty \{v\}[v]v^{s-1}dv = \sum_{k=1}^\infty \int_k^{k+1} \{v\}[v]v^{s-1}dv$$

$$= \sum_{k=1}^\infty k \int_0^1 \frac{u}{(u+k)^{s+1}}du$$

$$= \frac{1}{s(s-1)}\sum_{k=1}^\infty \left[\frac{1}{k^{s-2}} - \frac{k(s+k)}{(k+1)^s}\right]$$

$$= \frac{1}{s(s-1)}\left\{[1-\zeta(s-1)](s-1) + (s-1)[\zeta(s)-1] + \zeta(s-1)\right\}$$

$$= \frac{1}{s(s-1)}\left\{s-1-s\zeta(s-1) + 2\zeta(s-1) + (s-1)[\zeta(s)-1]\right\}.$$

For the case of $s \to 2$, we note that

$$-\lim_{s\to 2}(s-2)\zeta(s-1) = -\lim_{s\to 1}(s-1)\zeta(s) = -1.$$

(b) goes similarly. For (c) we use

$$\int_0^1 \left\{\frac{1}{t}\right\}\left[\frac{1}{t}\right]^n t^{s-1}dt = \frac{1}{s(s-1)}\sum_{k=1}^\infty \left[\frac{1}{k^{s-n-1}} - \frac{k^n(k+s)}{(k+1)^s}\right]$$

$$= \frac{1}{s(s-1)}\left[\zeta(s-n-1) - \sum_{k=1}^\infty \frac{\sum_{\ell=0}^n (-1)^{n-\ell}\binom{n}{\ell}(k+1)^\ell[(k+1)+(s-1)]}{(k+1)^s}\right]$$

$$= \frac{1}{s(s-1)}\left\{\zeta(s-n-1) - \sum_{\ell=0}^n (-1)^{n-\ell}\binom{n}{\ell}\left[[\zeta(s-\ell-1)-1] + (s-1)[\zeta(s-\ell)-1]\right]\right\}.$$

Herein the $\zeta(s-n-1)$ term is cancelled by a $\ell = n$ term of the sum. Since $\sum_{\ell=0}^n (-1)^{n-\ell}\binom{n}{\ell} = (1-1)^n = 0$, with a shift of index in the summation with the $(s-1)$ factor, the stated result for the integral follows. For the value at $s = n+1$, the sum has $(n-s+1)\zeta(s-n)$ for the $\ell = n-1$ term. Since $\lim_{s\to n+1}(n-s+1)\zeta(s-n) = -\lim_{s\to 1}(s-1)\zeta(s) = -1$ the value for $s = n+1$ obtains. (d) goes similarly as (c), with

$$\int_0^1 \left\{\frac{1}{t}\right\}^2\left[\frac{1}{t}\right]^n t^{s-1}dt = \frac{1}{s(s-1)(s-2)}\sum_{k=1}^\infty k^n\left[\frac{2}{k^{s-2}} - \frac{(2k^2+2ks+s^2-s)}{(k+1)^s}\right].$$

For (e) we employ a sublemma. In addition, at times we use the functional equation of the polygamma function,

$$\psi^{(\ell)}(u+2) = \psi^{(\ell)}(u+1) + \frac{(-1)^\ell \ell!}{(u+1)^\ell}.$$

Sublemma. For $n \geq 1$ an integer, let

$$\mathcal{S}_n(u) \equiv \sum_{k=2}^\infty \frac{1}{(u+k)^{n+1}}\frac{1}{(u+k-1)}.$$

Then

$$\mathcal{S}_j(u) = \sum_{\ell=0}^j \frac{(-1)^\ell}{\ell!}\psi^{(\ell)}(u+2) - \psi(u+1)$$

$$= \sum_{\ell=1}^{j} \frac{(-1)^\ell}{\ell!} \left[\psi^{(\ell)}(u+1) + \frac{(-1)^\ell \ell!}{(u+1)^{\ell+1}}\right] + \frac{1}{u+1},$$

wherein the functional equation of $\psi^{(\ell)}(u+2)$ has been used.

Proof. Starting with the sum

$$\sum_{k=2}^{\infty} \frac{1}{(u+k)} \frac{1}{(w+k-1)} = \frac{1}{(u-w+1)} \left[\psi(u+2) - \psi(w+1)\right],$$

we differentiate j times with respect to u, resulting in

$$(-1)^j j! \sum_{k=2}^{\infty} \frac{1}{(u+k)^{j+1}} \frac{1}{(w+k-1)} = \sum_{\ell=0}^{j} \frac{(-1)^{j-\ell}(j-\ell)!\binom{\ell}{j}}{(u-w+1)^{\ell-j+1}} \psi^{(\ell)}(u+2) - \frac{(-1)^j j!}{(u-w+1)^{j+1}} \psi(w+1).$$

We then cancel a factor of $(-1)^j$ on both sides, put w to u, and rearrange. □

Now

$$\int_0^1 \left\{\frac{1}{x}\right\} \left\{\frac{1}{1-x}\right\} x^{s-1} dx = \int_1^\infty \frac{\{t\}}{t^2} t^{1-s} \left\{\frac{t}{t-1}\right\} dt$$
$$= \int_1^2 \frac{\{t\}}{t^{s+1}} \left\{\frac{t}{t-1}\right\} dt + \sum_{k=2}^{\infty} \int_k^{k+1} \frac{(t-k)}{t^{s+1}} \left\{\frac{t}{t-1}\right\} dt. \tag{3.5}$$

The latter sum becomes

$$\sum_{k=2}^{\infty} \int_k^{k+1} \frac{(t-k)}{t^{s+1}} \left\{\frac{t}{t-1}\right\} dt = \sum_{k=2}^{\infty} \int_k^{k+1} \frac{(t-k)}{t^{s+1}} \left(\frac{t}{t-1} - 1\right) dt$$
$$= \sum_{k=2}^{\infty} \int_k^{k+1} \frac{(t-k)}{t^{s+1}} \frac{dt}{(t-1)}$$
$$= \sum_{k=2}^{\infty} \int_0^1 \frac{u}{(u+k)^{s+1}} \frac{du}{(u+k-1)}.$$

Thus, we require the integral

$$\sum_{k=2}^{\infty} \int_0^1 \frac{u}{(u+k)^{s+1}} \frac{du}{(u+k-1)} = \int_0^1 u\mathcal{S}_s(u) du,$$

to be evaluated according to the Sublemma. Using two elementary integrals performed by integrating by parts, $\int_0^1 \frac{u}{u+1} du = 1 - \ln 2$ and $\int_0^1 \frac{u}{(u+1)^2} du = -\frac{1}{2} + \ln 2$, and further integrating by parts, we find

$$\int_0^1 u\mathcal{S}_j(u) du = \sum_{\ell=1}^{j} \frac{(-1)^\ell}{\ell!} \left[\psi^{(\ell-2)}(1) - \psi^{(\ell-2)}(2) + \psi^{(\ell-1)}(1)\right] + \frac{1}{2}$$

$$+\sum_{\ell=2}^{j} 2^{-\ell}\frac{(2^\ell-\ell-1)}{\ell(\ell-1)}. \tag{3.6}$$

The latter sum is given by

$$\sum_{\ell=2}^{j} 2^{-\ell}\frac{(2^\ell-\ell-1)}{\ell(\ell-1)} = \frac{j-1}{j} - \sum_{\ell=2}^{j}\frac{\ell+1}{2^\ell\ell(\ell-1)}$$

$$= \frac{j-1}{j} - \sum_{\ell=2}^{j}\frac{1}{2^\ell}\left(\frac{2}{\ell-1}-\frac{1}{\ell}\right) = \frac{j-1}{j} + \frac{(1-j2^{j-1})}{j2^j}.$$

For the other contribution in (3.5), we have

$$\int_1^2 \frac{\{t\}}{t^{s+1}}\left\{\frac{t}{t-1}\right\}dt = \int_0^1 \frac{u}{(u+1)^{s+1}}\left\{\frac{u+1}{u}\right\}du = \int_0^1 \frac{u}{(u+1)^{s+1}}\left\{\frac{1}{u}\right\}du$$

$$= \int_1^\infty \frac{\{t\}}{t^3}\frac{t^{s+1}}{(1+t)^{s+1}}dt = \sum_{k=1}^\infty \int_k^{k+1}\frac{(t-k)t^{s-2}}{(1+t)^{s+1}}dt$$

$$= \sum_{k=1}^\infty \int_0^1 \frac{u(u+k)^{s-2}}{(u+k+1)^{s+1}}du$$

$$= \frac{1}{s(s-1)}\sum_{k=1}^\infty \left[\frac{k^s}{(k+1)^{s-3}} + \frac{[s-(k+1)^2]}{(k+2)^s}(k+1)^{s-1}\right].$$

Combining this sum with (3.6), using the functional equation of the polygamma function and the definition of the harmonic numbers gives the final result.

(f) we omit the proof of this part, but show how (g) follows from it. We have

$$\int_0^1 \left\{\frac{1}{k\sqrt[k]{t}}\right\}t^{s-1}dt = k^{1-ks}\int_{1/k}^\infty \frac{\{y\}}{y^{sk+1}}dy$$

$$= k^{1-ks}\left(\int_{1/k}^1 \frac{dy}{y^{sk}} + \int_1^\infty \frac{\{y\}}{y^{sk+1}}dy\right).$$

Evaluating the elementary integral on the right side and using (f) for the fractional-part integral there gives the result. The other two integrals are performed similarly.

For one of the integrals in (h),

$$\int_0^1 \left[\frac{k}{\sqrt[k]{t}}\right]t^{s-1}dt = k^{ks+1}\int_k^\infty \frac{[y]}{y^{ks+1}}dy = k^{ks+1}\sum_{\ell=k}^\infty \int_\ell^{\ell+1}\frac{\ell}{y^{ks+1}}dy$$

$$= \frac{k^{ks}}{s}\sum_{\ell=k}^\infty \ell\left(\frac{1}{\ell^{ks}} - \frac{1}{(\ell+1)^{ks}}\right)$$

$$= \frac{k^{ks}}{s}[\zeta(ks-1,k) - \zeta(ks-1,k+1) + \zeta(ks,k+1)]$$

$$= \frac{k^{ks}}{s}[k^{-ks+1} + \zeta(ks,ks+1)],$$

wherein we used the functional equation $\zeta(s,a+1) - \zeta(s,a) = -a^{-s}$. The other integrals proceed similarly. □

Remarks. When evaluating several of the integrals in (f)-(h) when $ks \to 2$, the following relation may be used,

$$\lim_{ks\to 2} \frac{(2-ks)}{(ks-1)}\zeta(ks-1,a) = -1.$$

The Sublemma may also be proved using the following approach. We observe that

$$\mathcal{S}_n(u) = \sum_{k=2}^{\infty}\left[\frac{1}{(u+k)^n}\frac{1}{(u+k-1)} - \frac{1}{(u+k)^{n+1}}\right].$$

Hence $\mathcal{S}_n = \mathcal{S}_{n-1} - \zeta(n+1,u+2)$, where $\zeta(n+1,x) = (-1)^{n+1}\psi^{(n)}(x)/n!$. It follows that

$$\mathcal{S}_n(u) = f(u) - \sum_{k=0}^{n-1}\zeta(k+2,u+2),$$

with

$$\mathcal{S}_0(u) = \frac{1}{1+u}, \qquad \mathcal{S}_1(u) = \frac{1}{1+u} - \psi'(u+2) = \frac{2+u}{(1+u)^2} - \psi'(u+1).$$

Thus $\mathcal{S}_0(u)$ suffices to determine $f(u)$, and

$$\mathcal{S}_n(u) = \frac{1}{1+u} - \sum_{k=0}^{n-1}\zeta(k+2,u+2).$$

For a given integer s value in (e), the infinite sum also evaluates in terms of $\zeta(n)$ values.

For $s=1$, the sum and integral below (3.6) are

$$\sum_{k=1}^{\infty}\int_0^1 \frac{u(u+k)^{-1}}{(u+k+1)^2}du = \sum_{k=1}^{\infty}\int_0^1\left[\frac{k+1}{(u+k+1)^2} + \frac{k}{u+k+1} - \frac{k}{u+k}\right]du$$

$$= \sum_{k=1}^{\infty} \frac{1}{(k+2)}[1 + k(k+2)(\ln(k+2) - 2\ln(k+1) + \ln k)]$$

$$= -\frac{1}{2} + \gamma.$$

With $-\psi(1) = \gamma$ and $1/2 - H_1 = -1/2$, it follows that $\int_0^1 \left\{\frac{1}{x}\right\} \left\{\frac{1}{1-x}\right\} dx = 2\gamma - 1$.

Discussion

Within the Askey scheme of hypergeometric polynomials, the continuous Hahn polynomials occur at the ${}_3F_2(1)$ level. Our Mellin transforms are closely connected with instances of these polynomials, which are given by (e.g., [2] p. 331, [3])

$$p_n(x; a, b, c, d) = i^n \frac{(a+c)_n (a+d)_n}{n!} \, {}_3F_2 \left(-n, n+a+b+c+d-1, a+ix; a+c, a+d; 1\right).$$

For example, for the transform $M_{2n}(s)$, its polynomial factors are proportional to $p_n\left(-\frac{is}{2}; -n, 0, \frac{1}{2}, \frac{1}{2} - n\right)$. The continuous Hahn polynomials are orthogonal on the line with respect to the measure

$$\frac{1}{2\pi}\Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix)dx.$$

Due to the Parseval relation for the Mellin transform,

$$\int_0^\infty f(x)g^*(x)dx = \frac{1}{2\pi i}\int_{(0)} (\mathcal{M}f)(s)(\mathcal{M}g)^*(s)ds,$$

the polynomial factors $p_n(1/2+it)$ form an orthogonal family with respect to a suitable measure with Γ factors. Since orthogonal polynomials have real zeros, this approach provides another way of showing that $p_n(s)$ has zeros only on the critical line.

The Gegenbauer polynomials $C_n^\lambda(x)$ are orthogonal on $[-1, 1]$ with weight function $(1-x^2)^{\lambda-1/2}$. The associated Legendre polynomials are related to them via

$$C_{n-m}^{m+1/2}(x) = \frac{1}{(2m-1)!!}\frac{d^m P_n(x)}{dx^m} = (-1)^m 2^m \frac{m!}{(2m)!}(1-x^2)^{-m/2} P_n^m(x).$$

Elsewhere [14] we develop a suitable generalization for the Mellin transforms of Gegenbauer functions. This would provide another approach for proving Proposition 4.

Appendix: Selected transformations of ${}_3F_2(1)$

The following three transformations [4] are valuable in the proof of Proposition 4.

$$
{}_3F_2(a,b,c;d,e;1) = \frac{\Gamma(e-a-b)\Gamma(e)}{\Gamma(e-a)\Gamma(e-b)}\,{}_3F_2(a,b,d-c;d,1+a+b-e;1)
$$

$$
-\frac{\Gamma(a+b-e)\Gamma(d)\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(a)\Gamma(b)\Gamma(d-c)\Gamma(d+e-a-b)}\,{}_3F_2(e-a,e-b,d+e-a-b-c;1+e-a-b,d+e-a-b;1). \tag{A.1}
$$

$$
{}_3F_2(a,b,c;d,e;1) = \frac{\Gamma(1+a-d)\Gamma(1+b-d)\Gamma(1+c-d)\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(1+e-d,2-d)}
$$

$$
\times\,{}_3F_2(1+a-d,1+b-d,1+c-d;1+e-d,2-d;1)
$$

$$
+\frac{\Gamma(1+a-d)\Gamma(1+c-d)}{\Gamma(1-d)\Gamma(1+a+c-d)}\,{}_3F_2(a,c,e-b;1+a+c-d,e;1). \tag{A.2}
$$

$$
{}_3F_2(a,b,c;d,e;1) = \frac{\Gamma(1+a-d)\Gamma(1+b-d)\Gamma(1+c-d)\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(1+e-d,2-d)}
$$

$$
\times\,{}_3F_2(1+a-d,1+b-d,1+c-d;1+e-d,2-d;1)
$$

$$
+\frac{\Gamma(1+a-d)\Gamma(1+b-d)\Gamma(1+c-d)\Gamma(e)}{\Gamma(1-d)\Gamma(1+a+b-d)\Gamma(1+a+c-d)\Gamma(e-a)}
$$

$$
\times\,{}_3F_2(a,1+a-d,1+a+b+c-d-e;1+a+b-d,1+a+c-d;1). \tag{A.3}
$$

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