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Woon K Wong and Laurence Copeland

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Cardiff Business School
Cardiff University
Colum Drive
Cardiff CF10 3EU
United Kingdom
t: +44 (0)29 2087 4000
f: +44 (0)29 2087 4419
www.cardiff.ac.uk/carbs

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Risk Measurement and Management in a Crisis-Prone World

Woon K. Wong[*], Laurence Copeland[†]

Investment Management Research Unit[‡]

Cardiff Business School

Aberconway Building, Colum Drive, Cardiff CF10 3EU, U.K.

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[*]Corresponding author. Email: wongwk3@cf.ac.uk

[†]Email: copelandL@cf.ac.uk

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Abstract

The current subprime crisis has prompted us to look again into the nature of risk at the tail of the distribution. In particular, we investigate the risk contribution of an asset, which has infrequent but huge losses, to a portfolio using two risk measures, namely Value-at-Risk (VaR) and Expected Shortfall (ES). While ES is found to measure the tail risk contribution effectively, VaR is consistent with intuition only if the underlying return distribution is well behaved. To facilitate the use of ES, we present a power function formula that can calculate accurately the critical values of the ES test statistic. This in turn enables us to derive a size-based multiplication factor for risk capital requirement.

Keywords: Value-at-Risk, expected shortfall, tail risk contribution, saddlepoint technique, risk capital

JEL Classification: G11, G32

1 Introduction

The 1998 failure of Long-Term Capital Management (LTCM) was said to be so severe that it posed a serious threat to the world's financial system. Barely ten years have passed, the scale of the current subprime crisis exceeds the LTCM debacle in every respect: the dollar value lost, the number of banks affected and its effects on the wider economy. Although the current problems originate in subprime mortgages whereas LTCM invested mainly in fixed income investments, both crises have one point in common: their return distributions contain rare but extremely large losses.

Jorion (2000, p. 277) described LTCM's profits as bets on extreme events, like selling options. This is supported by Agarwal and Naik (2004) who find that a large number of hedge fund strategies exhibit payoffs resembling a short position in a put option on the market index. As the Black-Scholes formula predicts, an out-of-the-money written put yields frequent small profits, but the loss incurred is large when it happens.[1]

Moreover, these extreme losses are often caused or exacerbated by high leverage. Though the present crisis will probably cause risk managers in future to restrict the size of banks' nominal exposure, the strategy of using leverage on various financial products has become indispensable in an ever more competitive financial world. Indeed, as Damodaran (2005) points out, risk management is not just about reducing risk; it may involve increasing a firm's exposure to at least some types of risk that may give the firm advantages over its competitors. A simple example is the long-short strategy which, according to an investment model, buys winners and short sell losers. While the risk of leveraging such a portfolio (so as to generate better than benchmark returns) is calculated to be acceptable, the margin for errors in the model, liquidity, counter party and other risks has also been reduced by leverage. In sum, it is essential to measure and manage infrequent risks of large losses due to leverage in modern finance.

Value-at-Risk (VaR) is the sanctioned measure of risk and its widespread popularity has raised the standard of risk management in the investment community. Both theoretical and empirical research reveal that VaR yields many meaningful results.[2] Nevertheless, Lo (2001) lists several limitations of VaR as a measure of risk in hedge funds or investments with an option-like nonlinear return structure. In this paper, we illustrate the use of Expected Shortfall (ES), a risk measure that is complementary to VaR and tells investors the average size of loss when a VaR is breached, to measure and manage aspects of the risk that underlie

[1] Indeed, rare-but-huge-loss payoff structure is not restricted to hedge funds. The world's major banks also share similar return patterns. For example, Berkowitz and O'Brien (2002) examined six large multinational banks over the period from January 1998 to March 2000, and found that they also incurred huge losses during the 1998 crisis.

[2] For instance, Dowd (1999) shows that for a "well-behaved" fat-tailed distribution, e.g. a t-distribution with five degrees of freedom, the Sharpe rule for risk management decision-making remains valid if VaR is calculated based on the appropriate t-distribution value. Another example is the recent work of Bali and Cakici (2004), which finds that VaR, together with stock size and liquidity, can explain the cross-sectional variation in expected returns.

the extraordinary financial crises mentioned earlier.

2 Risk Contribution

Given the complexity of modern financial investments, Lo (2001) highlights the need to capture the spectrum of risks involved. For example, a risk manager needs to consider the dangers of diminishing correlations for an arbitrage fund, factor exposures to the market index for a long-short strategy, optionality and inflationary pressures for a fixed-income hedge fund, and so on. In this section, we use both VaR and ES to measure how each asset or subportfolio contributes to the tail risk of a portfolio. Risk contribution analysis helps to identify the sources of risk and is a step towards managing rare but extreme risks effectively.

There are many further advantages in carrying out a risk contribution analysis using VaR and ES. Firstly, correlation and linear models are the traditional tools used to attribute sources of risk. However, Embrechts et al. (1999) point out that dependence measured by linear correlation does not hold in a non-elliptical world, which is often characterized by skewed and fat-tailed returns. Unlike linear correlation, a VaR or ES contribution focuses on the dependence between the portfolio and its components at the tail of the distribution, which is useful for managing extreme risks.

Secondly, Embrechts et al. (1999) introduce for measurement of nonlinear dependence the concepts of comonotonicity and copula, which may be difficult for general investors to understand. VaR is a risk measure that condenses a usually complex distribution of returns into a single number that can be easily understood by investors and risk managers as well as regulators. As measures of how much an asset or subportfolio is contributing towards a portfolio's tail risk, both VaR and ES have the attraction of being simple enough to be easily understood.

2.1 VaR and ES Contributions

VaR can be defined as the maximum loss on a portfolio over a specified period (typically 10 working days) with a given confidence level, say $1-\alpha$. Let R be the portfolio return with distribution function F. If F is continuous,[3]

$$VaR_\alpha(R) = -F^{-1}(\alpha).$$

Note that in this paper, a negative R means a loss whereas the risk measures, e.g. VaR and ES, are represented by positive numbers. Hence, ES of the portfolio, defined as the expected loss given that a loss exceeds VaR, can be written as

$$ES_\alpha(R) = E[-R| - R > VaR_\alpha(R)].$$

Where there is no ambiguity, the subscript α and argument R will be dropped for simplicity.

Former definitions provided by Tasche (2002) for attributing VaR and ES risk contributions to portfolio components are now stated as follows. Suppose there are d assets in the portfolio. The portfolio return R can be written as

$$R = \sum_{i=1}^{d} \omega_i R_i,$$

where R_i and ω_i are respectively the return and weight for the i-th asset. Under certain smoothness assumptions on the joint distribution of $(R_1, ..., R_n)$, the contribution of $\omega_i R_i$ to the portfolio's VaR is defined as

$$VaRC_i = E\left[-R_i| - R = VaR\right] \cdot \omega_i.$$

[3] For the sake of simplicity without affecting the result of our research, all distributions considered in this paper, unless stated otherwise, are absolutely continuous.

Similarly, the ES-contribution of i-th asset to the portfolio's ES is given by

$$ESC_i = E\left[-R_i \middle| -R > VaR\right] \cdot \omega_i.$$

Note that in both cases the sum of risk contributions of each asset is equal to that of the portfolio. That is, $\sum_{i=1}^{d} VaRC_i = VaR$ and $\sum_{i=1}^{d} ESC_i = ES$. If it is to be a satisfactory guide to risk management, $VaRC_i$ or ESC_i should provide some indication of how great is the contribution of each asset to the tail risk of the portfolio as a whole. In doing so, it should change monotonically as risky assets are added to a portfolio. Appendix A.1 at the end of this paper provides details on the numerical methods used to approximate the risk measures.

2.2 A Portfolio with Derivative Assets

We consider a portfolio made up of an investment of $1 divided between three assets: (1) the S&P 500 stocks, (2) an index of corporate bonds, (3) an index of 7-10 year Treasury bonds, plus (4) a written put on the S&P500 with varying degrees of moneyness, $k = -5\%, ..., +5\%$.[4] The results reported here are based on a sample of daily data for the period 3 August 1998 to 27 May 2008, a total of 2470 observations. The characteristics of the dataset are summarised by the statistics given in Table 1.

< Insert Table 1: Basic Statistics >

Thoughout the analysis, the dollar investments in the first three assets remain fixed: $0.143 in S&P 500, $0.571 in corporate bonds and $0.286 in Treasury bonds.[5] We focus here

[4] The moneyness, k, is defined as (strike - current price)/current price.

[5] Since experience shows that the standard deviation of stock returns is about four times as great as that on corporate debt, these proportions mean that the value of the investment in stocks and bonds have approximately equal volatility.

on the impact on the portfolio as a whole when there are changes in the moneyness of the written put. Table 2 provides the results based on daily returns.

2.2.1 The Case of No Leverage

First, consider the case of no leverage in Panel A where the value ω_4 of the short position in the option remains at \$0.107. As the written put is increasingly out-of-the-money, its volatility diminishes. Though the portfolio's skewness and kurtosis indicate a larger downside risk, decreasing values of VaR, ES and worst loss reveal a net effect of lower risk profile for the portfolio as a whole. As the basic principle of finance dictates a lower return for lower risk, the portfolio's average daily return decreases from $k = 0\%$ onwards. Applying the Sharpe rule of Dowd (1999) based on the ratio of average return to VaR, the portfolio is optimized at $k \approx -1\%$.

2.2.2 The Case of Leverage

In highly efficient financial markets, opportunities for abnormal profit can often be eroded due to competition, so that trading may not be feasible at the optimum, but only at lower k, say -4%. Since the return is now far lower, leverage is used to pursue higher payoffs. Panel B shows the results when leverage is only applied to the option position, with value of investment ω_4 increasing from \$0.107 to \$0.795.[6] Several interesting observations can be made. First, the portfolio volatility remains almost constant (with a very small decline from $k = 1\%$ onwards). Second, conflicting messages are signalled by VaR and ES. VaR peaks at $k = -3\%$ and then decreases when the option is further out-of-the-money, indicating the tail risk of portfolio decreases from that point onwards. By contrast, the increase in ES is monotonic. That ES provides a more correct picture of extreme risk is supported by the same monotonic increases of worst loss, size of negative skewness, and kurtosis.

These examples are consistent with the findings of Basak and Shapiro (2001), who show

[6] In our simplified case, borrowing costs are ignored and leverage is represented by the increase in ω_4 from \$0.107 to \$0.795.

theoretically that, if investors optimized their portfolios with VaR constant, larger losses would be incurred during adverse market downturns.

2.2.3 Risk contributions

VaR and ES risk contributions are summarised in the rest of Panel B in Table 2, and graphically in Figure 1 and 2. For ease of comparison, each VaR (ES) contribution is divided by the corresponding portfolio VaR (ES) so that the reported figures are in percentages that sum to 100. In general, the message is that VaR behaves sensibly if the (possibly fat-tailed) return distribution is relatively well behaved,[7] and that ES proves a more reliable measure, especially for highly nonlinear payoffs.

In the first place, when the put is deeper in-the-money ($k \geq 3\%$), there is little variation in risk contributions as measured by either measure, but considerable disagreement between the two risk measures with respect to the three risky assets, namely equity (asset 1), corporate bonds (asset 2) and option (asset 4). Next, over near-the-money range ($2\% \geq k \geq -2\%$), when there is little nonlinearity in the option's payoff, both VaR and ES detect increasing risk contributions from the larger investments in the short put position. There are corresponding drops in risk contributions for corporate bonds and Treasury over this near-the-money range.

Finally, beyond $k = -0.02$ when the option is further out-of-the-money (and thus the payoff structure is highly nonlinear), VaR counter-intuitively indicates a smaller risk for higher leverage in the written put. Also, equally perverse is the corresponding sharp rise in risk attributable to the corporate debt. Unlike VaR, the ES contribution of the option increases monotonically with smaller k, reflecting rising risk due to leverage and the nonlinear payoff structure in the written put position. It is noteworthy that with larger leverage, the option replaces the corporate bond as the main contributor of risk to the portfolio.

< Insert Table 2: Risk Contributions >

[7] Our results are consistent with the conclusions of Dowd (1999); see footnote 2.

3 Nonlinearities

Nonlinear payoff structure is another reason given by Lo (2001) for not using VaR as a risk measure in hedge funds. The risk contribution analysis given in the previous section illustrates the case: VaR misleads investors into a false sense of security for deep out-of-the-money puts with highly nonlinear payoffs. The main reason for this lies in the fact that VaR is a quantile measure which ignores sizes of losses beyond the VaR boundary; Yamai and Yoshiba (2005) describes this as the tail risk of VaR. Lessons from the LTCM and current subprime crises suggest that the nonlinear payoff structure could be modeled as regime switches between normal and disaster states. Base on a regime-switching payoff structure of this kind, we provide analytical illustrations of why it is crucial for a risk measure to take into account the sizes of losses beyond VaR.

Consider a portfolio whose return has a standard Gaussian distribution, Φ, during normal times, but with a small probability p of a disaster bringing a huge loss L. Then the distribution of the portfolio return can be written as

$$F(R) = \begin{cases} p + (1-p)\Phi(R) & \text{if } R \geq -L, \\ (1-p)\Phi(R) & \text{if } R < -L. \end{cases}$$

Since disaster is rare, it is reasonable to assume that $p < \alpha$, the tail probability at which VaR is calculated. As it is often the case that $L > VaR$, Appendix A.2 shows that the portfolio VaR can be obtained as

$$VaR = -\Phi^{-1}\left((\alpha - p)/(1-p)\right),$$

and the associated portfolio ES is

$$ES = \alpha^{-1}\left[pL + \frac{(1-p)\exp\left[-(VaR)^2/2\right]}{\sqrt{2\pi}}\right].$$

For market risk, Basle II set $\alpha = 0.01$. If $p = 0.005$, $VaR = -2.574$. Note that in this case VaR is independent of L, whereas ES increases with the size of extreme loss. For instance, if leverage causes L to increase from 5 to 10, ES also increases from 3.945 to 6.445 whereas VaR remains constant.

This provdes insights for the counterintuitive performance of VaR in the risk contribution analysis. As the put becomes further out-of-the-money, nonlinearity in its return structure increases until rare but large losses exist far in the tail of distribution.

4 Backtesting Rare Events

Backtesting plays a very important role in risk management. It enables risk managers to find out if the model at hand is appropriate and gives regulators a means to punish poor (or reward good) risk management practices. Further, the process of backtesting leads risk professionals to search for better models, which in turn allows them to understand better the nature of risk, thereby raising the standard of risk management.

Backtesting VaR is a simple matter of counting exceptions and computing their failure rate. If the model is adequate, the rate of exceedance should be 5% or 1%, depending on the confidence level set for VaR. This is the basis of the Basle, Kupiec (1995) and Christoffersen (1998) tests. By contrast, since VaR exceptions are by definition rare in practice, there are too few observations to derive a meaningful confidence interval to backtest the ES estimate.[8] This is one frequently-cited reason for preferring VaR to ES.

[8] If $\alpha = 1\%$, there are by definition only 2.5 breaches of VaR in 250 trading days, the sample size stipulated by the Basle II over which backtesting is carried out.

4.1 Power Function Formula

Recently, Wong (2008) showed that it is possible to use the saddlepoint technique to overcome the problem in backtesting ES. Specifically, a portfolio return R is distributed as standard Gaussian, the p-value of sample ES statistics, calculated as average of VaR exceedances (see Appendix A.1), can be approximated very accurately by the small sample asymptotic technique.

Nevertheless, applying the saddlepoint backtest is a rather complicated exercise, since it requires us to solve numerically for a saddle-point and then evaluate derivatives of higher-order cumulants around the point. In this paper, we present a straightfoward way of deriving critical values of the test distribution by evaluating the following power function:

$$\pi_\beta(n) = -\mu_x - \frac{\sigma_x}{\sqrt{n}}\left[z_\beta + \frac{a}{(1+1000\cdot n/b)^c}\right] \tag{1}$$

where n is the number of exceptions in the sample, z_β is the standard normal β-quantile (the chosen confidence level), and $-\mu_x = 2.6652$ and $\sigma_x^2 = 0.09685$ are respectively the expectation and variance of $ES_{0.01}(R)$ under the null hypothesis. Consider the probability calculated using the saddlepoint technique:

$$P\left(\overline{X} < -\pi^*_\beta(n)\right) = \beta$$

where $\overline{X}$ is the random sample mean of (minus) ES and the critical value of the power function, $\pi^*_\beta(n)$, is obtained by numerical methods. Then minimizing the sum of squared differences between $\pi_\beta(n)$ and $\pi^*_\beta(n)$ for $n = 1, ..., 200$ makes it possible to solve for the coefficients a, b and c. The resulting values are given in the top half of Table 3.

< Insert Table 3: Power function formula >

The associated critical values for the 5% tail are given in the bottom half of the table. The gap between the analytical and simulated critical values is highest for only two exceedances, but even in this case, the gap is only 0.0002. It is half as great for $n = 5, 10, 20$. Beyond this level, the gap is zero up to the fourth decimal place. Indeed, when n approaches infinity, $-\sqrt{n}\sigma_x^{-1}(\pi_\beta(n) + \mu_x) \longrightarrow z_\beta$.

4.2 Risk Capital

This analysis also has implications for the appropriate multiplication factor, M, to be used in computing required risk capital. Kerkhof and Melenberg (2004) suggested a new approach to implementing rewards and punishment by proposing to calculate M based on statistical backtesting of ES. Their proposal is that the multiplication factor should be proportional to the amount by which the ES under the null needs to be increased to make it no longer significantly smaller than the sample ES. However, since their functional delta approach to backtesting may be inaccurate for small samples, we derive a saddlepoint version of the multiplication factor formula:[9]

$$M = \min\left\{3 \times \max\left[1, 1 + \frac{\sigma_x}{\sqrt{n}\mu_x}\left(\overline{z} - z_\beta - \frac{a}{(1 + 1000n/b)^c}\right)\right], 4\right\}, \tag{2}$$

where if $\overline{x}$ is the sample mean of $-$ES, $\overline{z} = n\sigma_x^{-1}(\overline{x} - \mu_x)$. Comparing (2) with equation (31) in Kerkhof and Melenberg (2004), two differences are apparent: firstly, the new formulation depends on the number of exceptions, but not on sample size, and secondly, the introduction of three additional parameters, a, b and c, makes this formula for M accurate for any $n > 0$. Finally, we remark that for large n, equations (2) above and (31) in Kerkhof and Melenberg (2004) converge to the same number.

[9] Proof of (2) is provided in Appendix A.3.

4.3 Responsive Risk Management

As pointed out in Wong (2008), the fact that this approach to backtesting ES is so reliable even with only one or two exceptions in the data sample means that this risk measure can be used to monitor the impact of extreme events rapidly, allowing management to react promptly to crises.

< Insert Table 4: Responsive Risk Management >

Table 4 above tabulates the results of applying the power function formula in backtesting and risk capital determination for the portfolio return, R, when moneyness $k = -5\%$. As our aim is to illustrate the use of equation (1) for responsive risk management, in-sample VaRs are estimated based on assumptions that R is distributed as iid normal and iid skewed Student, which is found by Giot and Laurent (2003) to model well the US stock index returns.

Panel A gives the backtest results for the entire sample. As in Wong (2008) and Kerkhof and Melenberg (2004), the portfolio returns in the iid skewed Student case are transformed into standard normal before they are backtested. The idea is that, if R is fatter-tailed than the presumed distribution, the transformed data will also have fatter-tails than a standard normal. It can be seen that normality fails to match the fat-tails in the portfolio returns, both in terms of frequency and size of exceedances.

Following the above backtest results, Panel B calculates the first ten associated critical values and risk capital multiplication factors for the estimated VaRs, based on the standard normal null hypothesis. For comparison, the multiplication factor m based on the Basle rules is provided. When a loss first exceeds VaR, the power function formula finds it significant at 5% level and the associated M adjusts to 3.19 accordingly. After less than a month, another exceedance occurs, and the sample ES is found to be significantly large at the 1% level so that M is now raised to 3.78. For all subsequent exceptions, ES backtest results remain significant at the 1% level and M is constant at the maximum value of 4. By contrast, m based on the Basle rules, only starts to increase at the fifth exception, and reaches its maximum at the tenth exception.

4.4 VaR, ES and Stress Testing

In fairness, it has to be conceded that VaR was never intended to be the sole guide to risk management. Being a quantile measure, a one-day $VaR_{0.01}(R)$ tells investors the worst possible outcome in 99% of trading days. As for the question of how much worse the outcomes could be beyond the VaR boundary, it is left to the role of stress testing.

However, there is no standard way to carry out stress testing and no standard set of scenarios to consider. As Linsmeier and Pearson (2000) note, *"the process depends crucially on the judgement and experience of the risk manager."* In this sense, stress testing could be regarded as more like an art than a science: a risk manager needs to find a middle way between the extremes of being too cautious on the one hand and too indifferent to risk on the other. Excessive risk avoidance could mean no business ever being written because provision has to be made for every conceivable extreme event; the opposite extreme might result in no effective risk reduction measures being taken after lots of pointless discussions about the plausibility of particular scenarios.

These considerations are not meant to imply that stress testing is unhelpful. Rather, we regard the analysis as a demonstration of the benefit of being able to use ES backtesting in a disciplined manner. In particular, since there is often no formal probability estimation for stress scenarios, statistically backtesting the sizes of extreme losses provides information which can be a useful complement to the usual VaR and stress testing analyses.

5 Conclusions

VaR has a number of attractive features, the most obvious from a management point of view being that it is easy to justify focussing on the maximum loss on a portfolio over a specified period with a given confidence level. Moreover, it is easy to backtest a VaR model, as no distribution is assumed.

However, given the complexity and widespread use of leverage in modern financial investments, it now becomes important to measure and manage the sizes of extreme losses in a disciplined manner. In this regard, ES tells investors the expected value of the loss when we observe an extreme outcome, and to that extent can be viewed as supplementing rather than replacing VaR.

In this paper, we carry out a risk contribution analysis to demonstrate the importance of identifying the source of extreme risks. Being a quantile measure, VaR could give a false sense of security if there is excessive nonlinearity in the payoff structure. On the other hand, ES measures the average size of losses beyond VaR and thus reflects more realistically the risk of rare but huge losses. We also show how the saddlepoint approximation method can be used in a straightfoward manner to solve the problem commonly cited in the literature of backtesting ES. Based on the saddlepoint approximation, a responsive risk management practice involving dynamic adjusment of risk capital is illustrated.

A Appendix

A.1 Estimation of Risk Measures

Consider a random sample $(R_1, ..., R_T)$. Let $R_{(i)}$ be the order statistics such that $R_{(i)} \leq R_{(i+1)}$. Then VaR at confidence level $1-\alpha$ is estimated using order statistics as in Dowd (2001). Specifically, if $[T\alpha] = n$ denotes the integral part of $T\alpha$ and $r = T\alpha - n$, then

$$\widehat{VaR} = -(1-r)R_{(n)} - rR_{(n+1)}.$$

The associated ES estimator is given by

$$\widehat{ES} = -\frac{1}{n}\sum_{i=1}^{n} R_{(i)}.$$

For the VaR-contribution, it is not easy to numerically estimate the conditional expectation, $E[-R|-R = VaR]$, as the condition $-R = VaR$ hardly exists in practice. So we consider a subset around VaR, $(R_{([T\alpha-\varepsilon])}, ..., R_{([T\alpha+\varepsilon])})$ that conprise τ_ε observations such that as T tends to infinity, τ_ε approaches to infinity but $T^{-1}\tau_\varepsilon$ diminishes to zero. Let R_ε^j $(1 \leq j \leq \tau_\varepsilon)$ denote such τ_ε portfolio returns, and $R_{\varepsilon i}^j$ $(1 \leq i \leq n, 1 \leq j \leq \tau_\varepsilon)$ refers to the associated returns of i-th asset. Then VaR-contribution can be numerically estimated as

$$\widehat{VaRC}_i = -\frac{\omega_i}{\tau_\varepsilon}\sum_{j=1}^{\tau_\varepsilon} R_{\varepsilon i}^j$$

.

For the ES-contribution of i-th asset, the numerical estimator is

$$\widehat{ESC}_i = -\frac{\omega_i}{\tau_\alpha}\sum_{j=1}^{\tau_\alpha} R_{\alpha i}^j,$$

where if R_α^j $(1 \leq j \leq \tau_\alpha)$ refer to the τ_α portfolio losses that are larger than the portfolio VaR $(-R \geq VaR_\alpha(R))$, $R_{\alpha i}^j$ are the associated returns of the i-th asset.

For numerical analysis in Section 2, $\tau_\varepsilon = \tau_\alpha \approx \alpha T$, where $\alpha = 0.01$ and $T = 2470$.

A.2 Nonlinearity

Since $p < \alpha$ and $L > VaR_\alpha(R)$, we can write the Stieltjes integral, $\int_{-\infty}^{-VaR} dF(r) = \alpha$, as

$$(1-p)\int_{-\infty}^{-VaR}\phi(r)dr=\alpha,$$

from which the VaR formula follows. For ES,

$$\begin{aligned}
ES_\alpha(R) &= E[-R|-R>VaR] \\
&= \frac{-\int_{-\infty}^{-VaR} rdF(r)}{P\left[-R>VaR\right]} \\
&= \alpha^{-1}\left[pL+(1-p)\int_{-\infty}^{-VaR} r\phi(r)dr\right] \\
&= \alpha^{-1}\left[pL+\frac{(1-p)\exp\left[-VaR^2/2\right]}{\sqrt{2\pi}}\right].
\end{aligned}$$

A.3 Derivation of (2)

Let X_i, $i=1,...,n$, refer to the n returns that breach VaR. The associated mean is denoted by $\overline{X}$ with its realization by $\overline{x}$. Under the null hypothesis, $E[\overline{X}]=\mu_x$. Let M be such that the ES under the null hypothesis should be increased so that it is no longer significantly more negative than the sample ES:

$$P(\overline{X}-\mu_x<\overline{x}-M\mu_x)=P(\overline{X}-\mu_x<-\pi_\beta(n)-\mu_x)=\beta$$

where β may be set at 0.05, the conventional significance level used for hypothesis testing. Substituting $\pi_\beta(n)$, using (1) and equating the arguments in the probability functions, we have:

$$\overline{x}-M\mu_x=\frac{\sigma_x}{\sqrt{n}}\left[z_\beta+\frac{a}{(1+1000n/b)^c}\right]$$

Substituting $\overline{x} = \sigma_x n^{-1/2}\overline{z} + \mu_x$ and rearranging the terms, the following expression is obtained

$$M = 1 + \frac{\sigma_x}{\sqrt{n}\mu_x}\left[\overline{z} - z_\beta - \frac{a}{(1+1000n/b)^c}\right].$$

Since the multiplication factor, M, ranges from 3 to 4, we have:

$$M = \min\left\{3 \times \max\left[1, 1 + \frac{\sigma_x}{\sqrt{n}\mu_x}\left(\overline{z} - z_\beta - \frac{a}{(1+1000n/b)^c}\right)\right]\right\}.$$

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Table 1. Basic statistics

	Asset 1	Asset 2	Asset 3	Asset 4		
		Corporate			Short put	
	S&P 500	Bond	Treasury	$k = 0.05$	$k = 0.00$	$k = -0.05$
Mean*	0.0089	-0.0014	0.0025	0.0110	0.0380	0.0035
Stdev	1.158	0.282	0.402	1.069	0.622	0.144
Skew	-0.007	1.530	-0.227	-0.179	-1.114	-5.078
Kurtosis	5.520	65.284	4.929	4.713	7.389	70.185
Min	-7.044	-2.563	-2.080	-6.230	-4.815	-2.561
Q1	-0.622	-0.079	-0.226	-0.591	-0.245	-0.001
Median	0.044	0.000	0.010	0.051	0.117	0.003
Q3	0.619	0.088	0.240	0.615	0.406	0.037
Max	5.574	5.199	2.048	4.010	2.142	0.684

*Note that the corporate bond and treasury returns are based on price indices

Table 2. Risk contribution

					Panel A: No leverage; $\omega_4 = 0.107$						
k	0.05	0.04	0.03	0.02	0.01	0.00	-0.01	-0.02	-0.03	-0.04	-0.05
VaR	0.856	0.854	0.849	0.853	0.847	0.836	0.830	0.793	0.737	0.656	0.573
ES	1.157	1.154	1.150	1.146	1.140	1.130	1.112	1.079	1.028	0.956	0.858
Mean (%)	0.236	0.317	0.426	0.559	0.687	0.762	0.744	0.642	0.503	0.365	0.244
Stdev	0.331	0.330	0.328	0.324	0.318	0.309	0.295	0.277	0.253	0.226	0.196
Skew	0.221	0.195	0.159	0.110	0.047	-0.032	-0.127	-0.242	-0.386	-0.571	-0.804
Kurtosis	11.999	11.940	11.909	11.952	12.141	12.578	13.374	14.635	16.489	19.132	22.807
Min	-2.384	-2.382	-2.384	-2.391	-2.408	-2.434	-2.464	-2.484	-2.478	-2.427	-2.315
					Panel B: Leverage; increasing ω_4						
ω_4	0.107	0.112	0.119	0.132	0.152	0.184	0.233	0.307	0.414	0.570	0.795
VaR	0.856	0.857	0.859	0.872	0.882	0.893	0.925	0.936	0.941	0.931	0.930
ES	1.157	1.159	1.163	1.171	1.186	1.208	1.239	1.274	1.314	1.355	1.392
Mean (%)	0.236	0.318	0.431	0.571	0.715	0.815	0.829	0.758	0.643	0.517	0.396
Stdev	0.331	0.331	0.331	0.331	0.331	0.330	0.329	0.327	0.324	0.321	0.317
Skew	0.221	0.195	0.159	0.110	0.047	-0.032	-0.127	-0.242	-0.386	-0.571	-0.804
Kurtosis	11.999	11.940	11.909	11.952	12.141	12.578	13.374	14.635	16.489	19.132	22.807
Min	-2.384	-2.393	-2.410	-2.445	-2.505	-2.602	-2.744	-2.932	-3.165	-3.441	-3.755
$VaRC_1$	0.407	0.406	0.405	0.385	0.409	0.428	0.417	0.425	0.354	0.305	0.278
$VaRC_2$	0.270	0.269	0.268	0.290	0.252	0.237	0.262	0.250	0.322	0.424	0.492
$VaRC_3$	0.038	0.038	0.038	0.036	0.020	-0.025	-0.052	-0.081	-0.022	-0.018	-0.018
$VaRC_4$	0.284	0.286	0.289	0.289	0.320	0.360	0.373	0.406	0.346	0.289	0.248
ESC_1	0.322	0.322	0.321	0.327	0.335	0.340	0.338	0.336	0.333	0.323	0.312
ESC_2	0.466	0.465	0.463	0.456	0.437	0.425	0.407	0.393	0.359	0.348	0.338
ESC_3	-0.009	-0.009	-0.009	-0.023	-0.030	-0.047	-0.051	-0.061	-0.056	-0.054	-0.051
ESC_4	0.221	0.223	0.226	0.240	0.258	0.283	0.306	0.332	0.363	0.383	0.401

Table 3. Power function formula

Panel A: Coefficients of power function

	β level			
	0.005	0.01	0.025	0.05
z_β	-2.5758	-2.3263	-1.9600	-1.6449
a	-15.7925	-14.4907	-13.1094	-12.6446
b	6.2965	4.6150	2.2280	0.6994
c	0.4817	0.4832	0.4828	0.4758
ξ	0.0003	0.0002	0.0003	0.0002

Panel B: Examples of critical values ($\beta = 0.05$)

n	$\pi^*_\beta(n)$	$\pi_\beta(n)$
1	-3.3012	-3.3012
2	-3.0901	-3.0903
5	-2.9200	-2.9199
10	-2.8403	-2.8402
20	-2.7864	-2.7863
50	-2.7403	-2.7403
100	-2.7178	-2.7178
200	-2.7021	-2.7021

ξ is to the maximum absolute error between $\pi^*_\beta(n)$ and $\pi_\beta(n)$.

Table 4. Responsive risk management

Panel A: Backtesting

Null hypothesis	n	a	p	$\widehat{ES}$	$\pi_{0.05}(n)$	$\pi_{0.01}(n)$
Normal	48	0.0194	0.000	3.492	2.742	2.777
Skewed Student	27	0.0109	0.647	2.668	2.769	2.818

Panel B: Multiplication factor

Date	n	m	M	$\widehat{ES}$	$\pi_{0.05}(n)$	$\pi_{0.01}(n)$
19980804	1	3.00	3.19	3.472	3.301	3.724
19980827	2	3.00	3.78	3.783	3.090	3.347
19980828	3	3.00	4.00	4.019	3.003	3.197
19980831	4	3.00	4.00	5.975	2.953	3.113
19981007	5	3.40	4.00	5.269	2.920	3.058
19981008	6	3.50	4.00	4.983	2.896	3.018
19990212	7	3.65	4.00	4.622	2.877	2.988
19990323	8	3.75	4.00	4.342	2.862	2.965
19990514	9	3.85	4.00	4.156	2.850	2.945
19990527	10	4.00	4.00	3.997	2.840	2.929

a and p in Panel A are failure rate and p-value respectively.

Figure 1: VaR contribution when leverage is applied

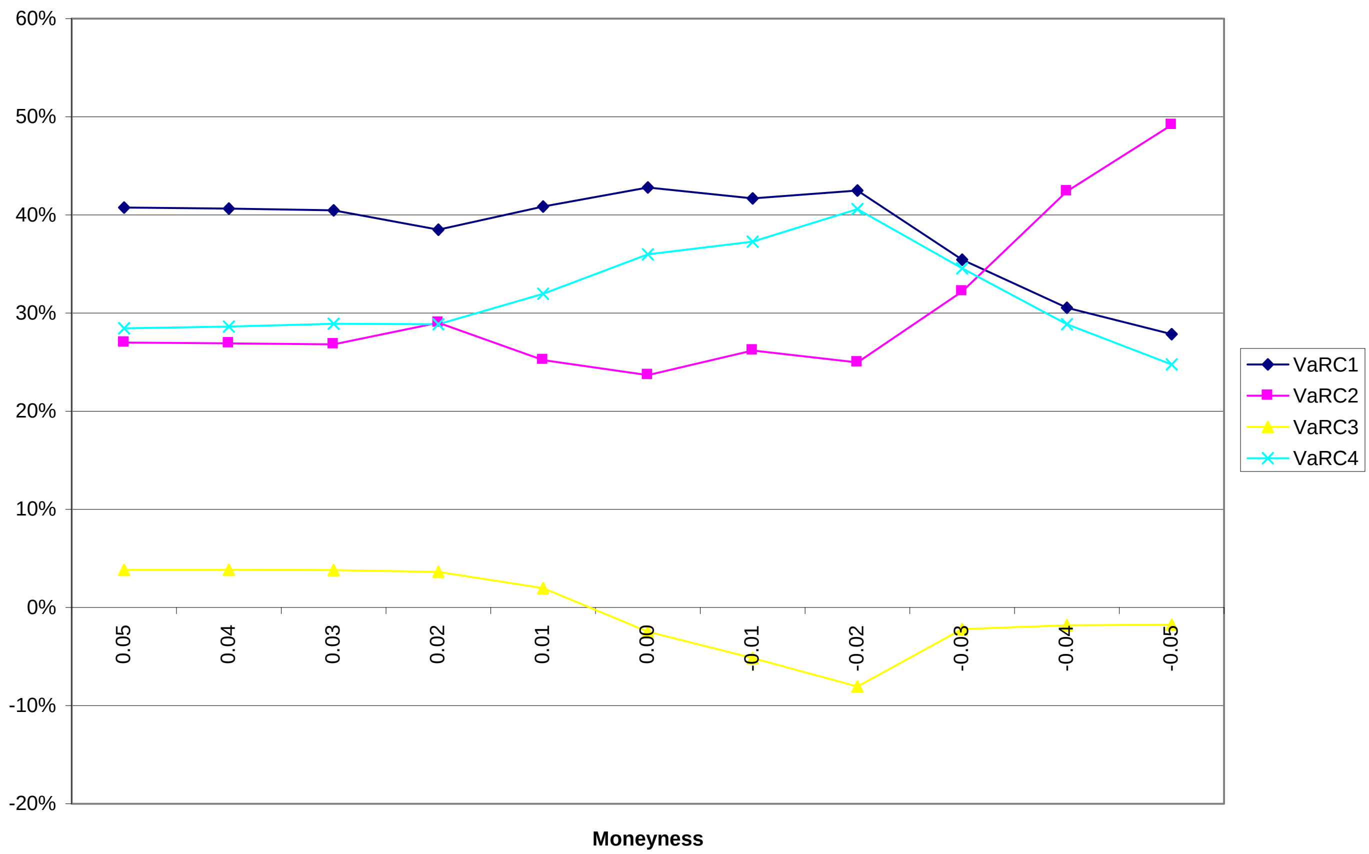

Figure 2: ES contribution when leverage is applied