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Motivated Sellers in the Housing Market

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Abstract

We present a search-and-matching model of the housing market where potential buyers' willingness to pay is private information and sellers may become desperate as they are unable to sell. A unique steady state equilibrium exists where desperate sellers offer sizeable price cuts and sell faster. If the number of distressed sales rises then even relaxed sellers are forced to lower their prices. Buyers, on the other hand, become more selective and search longer for better deals. The model yields a theoretical density function of the time-to-sale, which is positively skewed and may be hump-shaped. These results are consistent with recent empirical findings.

Keywords: housing, private information, random search, motivated sellers

JEL: D39, D49, D83

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1 Introduction

Selling a house involves a long and non-trivial search process where the home seller faces a trade-off between the price and the time to sale. With sufficient time and no pressure to sell immediately, a ‘relaxed’ seller can afford to wait to receive a price commensurate with the market value. However, due to factors such as bankruptcy, . some sellers become ‘desperate’ (or ‘motivated’ in real estate parlance) and want to sell urgently.

To study how this transition affects pricing and purchasing decisions of sellers and buyers and the expected time to sale we build a search-and-matching model of the housing market with two distinctive features. First, sellers may become desperate as they wait to sell, and second, buyers’ willingness to pay is private information.²

A quick description of the model is this (but see section 2). Sellers enter the market in a relaxed state and advertise take-it-or-leave-it prices to sell their houses. However as they are unable to sell they may be hit by an exogenous shock (e.g. job-loss, bankruptcy etc.) and become more impatient or desperate to sell. Buyers arrive randomly and upon inspecting house they realize their private valuation (willingness to pay) which is unobservable to sellers. Hence, sellers advertise a single price for all potential buyers, i.e., they cannot price-tailor to individual customers. Clearly a meeting may not result in trade if the house is not appealing enough for the buyer. Buyers’ purchasing decision is simple. Only if their willingness to pay exceeds an endogenous threshold they buy, otherwise they keep searching. This threshold rises with the list price; hence sellers face a trade-off between selling at a higher price vs. selling more quickly. The fraction of desperate sellers is also endogenous and buyers can raise this fraction by being choosier, i.e., by raising the aforementioned threshold.

What do we find? First we prove existence of a unique steady state equilibrium if the survival function associated with buyers’ valuations is log-concave. The equilibrium is characterized by a pair of list prices posted by relaxed and stressed sellers and threshold

²Search models are extensively used to study the housing market, e.g. see Wheaton (1990), Yavas and Yang (1995), Krainer (2001), Ngai and Tenreyro (2009). Among others Albrecht et al. (2007) is closest to our model in terms of motivation and setup; however they assume complete information.

valuations for buyers associated with those prices. In equilibrium stressed sellers post lower prices than relaxed sellers and consequently sell faster. Indeed, once hit by the shock a seller starts to discount the future more heavily, hence offers a price-cut to sell more quickly.

More importantly, if the shock starts to arrive more often, due to, say, a financial crisis or recession where home sellers are more likely to lose their jobs or face other financial difficulties, then not only more sellers become desperate, but even relaxed sellers lower their prices due to spill over effects. Buyers, on the other hand, exhibit a ‘vulture behaviour’: even though prices decrease, the rising fraction of desperate sellers induces buyers to hold off purchasing and search longer for better deals. For sellers this is indeed a triple hit: more sellers become desperate, even regular sellers lower their prices, but it is more difficult to sell.

The model yields a theoretical density function of the time on the market, which is positively skewed and may be hump-shaped. In particular the tail of the pdf slims down and the average time on the market drops as the shock starts to arrive more frequently or starts to become more severe. We also show that the expected sale price falls the longer the house stays on the market.

These results are empirically plausible. Glower et al. (1998) document that motivated sellers offer price discounts and sell faster. Campbell et al. (2009) find that forced sales have significant spill over effects on prices of unforced sales. Merlo et al. (2008) obtain the density of time-to-sale which, similar to ours, is positively skewed and hump shaped. Finally, Merlo and Ortalo-Magné (2004) document that average sale price falls with the duration.

To obtain analytically tractable results that could be compared with empirical findings in the literature we deliberately ignore bargaining in the price determination process. Indeed bargaining models with private information often yield delays and have multiple equilibria and therefore come with very limited predictive power.³ Similar concerns led us to choose a memoryless arrival rate for the shock. A state dependent arrival rate is

³See, e.g., the survey by Kennan and Wilson (1993) and the references therein.

more intuitive, however this, again, comes at the expense of tractability.

The paper proceeds as follows. Section 2 introduces the basic framework and discusses buyers' and sellers' problems. Section 3 discusses existence of steady state equilibrium and presents comparative statics. Section 4 investigates the densities of time-to-sale and prices. Section 5 concludes.

2 Model

2.1 Setup

Time is continuous and infinite. The economy consists of a continuum of agents divided into two identical sets: sellers and buyers, who meet each other at a constant Poisson rate $\alpha > 0$.⁴ Each seller has a unit of a homogenous good (a house) and each buyer seeks to purchase one. The utility to the seller from keeping the house forever is zero, in other words we assume away positive or negative flow values from owning or maintaining the house. Buyers assign different values (willingness to pay) to different houses and they differ from one another with respect to their valuations of a particular house. The suitability of a match between a house and a buyer is specific to the pair. For example, a particular house may match a buyer's needs or taste perfectly well, while at the same time being an unsatisfactory match to another buyer. Upon meeting a seller and inspecting the house, the buyer realizes his own valuation of the house $v \in [0, 1]$, which is a random draw from a distribution with cdf $F(v)$. Comparing the realized v with the price, the buyer decides whether or not to buy the house. We emphasize that buyers are identical in that their valuations are generated by the same random process, however they may differ in their valuations for any particular house which are independent draws from $F(\cdot)$.⁵ We impose log-concavity on the survival function, which is a crucial technical assumption for several

⁴What we have in mind is a Pissarides (2000) style random matching function where arrival rates are functions of the buyer-seller ratio (market tightness). Typically one allows for different number of buyers and sellers so that arrival rates for buyers and sellers differ. To avoid excessive parametrization we simply assume equal number of buyers and sellers so that traders meet each other with the same rate α .

⁵This assumption is adopted by Jovanovic (1979), Wolinsky (1988), Krainer and LeRoy (2002), Ngai and Tenreyro (2009) among others.

key results in the paper⁶.

Assumption 1. *The density function $F'(v)$ is strictly positive whereas the survival function $1 - F$ is log-concave, that is $F'^2(v) + F''(v)[1 - F(v)] > 0, \forall v$.*

It is assumed that the realization of v is the buyer's private information and unobservable to the seller. The seller only knows $F(\cdot)$ and posts a take-it-or-leave-it price p for the house, with no subsequent bargaining. If agents trade then the seller obtains payoff p , the buyer obtains $v - p$, both agents leave the market and are replaced by a buyer and a *relaxed* seller—an assumption necessary to maintain stationarity. Since the seller does not know the prospective buyer's willingness to pay, he must ask the same price regardless of the realization of v . The assumption of private information prevents the seller from tailoring the price to each individual buyer; thus we avoid the well-known Diamond Paradox (see Diamond (1971)).

A seller enters the market in a relaxed state, though, eventually as he is unable to sell, he may be hit by a shock that arrives at an exogenous Poisson rate $\mu > 0$ and become stressed. Buyers and relaxed sellers discount the future with $\delta > 0$ whereas stressed sellers are more impatient with a discount factor $\bar{\delta} > \delta$. Sellers do not exit the market until they sell, and a stressed seller remains stressed. The fraction of stressed sellers in the steady state is endogenous and denoted by θ .

In what follows we focus on a symmetric steady state equilibrium with pure strategies where identical agents follow the same strategy. In particular relaxed and stressed sellers post p_r and p_s , whereas buyers, after meeting seller $j = r, s$ and inspecting the house, purchase if their private valuation v exceeds an endogenous threshold v_j . We start by analyzing the buyer's problem.

⁶Log-concavity of the survival function is equivalent to the ratio of the density to the survival being monotone increasing. This is a common assumption in models with private information. Uniform, Normal, Cauchy and many other distributions satisfy this property. See Bagnoli and Bergstrom (2005) for more details.

2.2 The Buyer's Problem

The problem of a representative buyer has a recursive formulation. We use a dynamic programming approach letting Ω denote the value of search to a buyer. In a symmetric pure strategy equilibrium the distribution of list prices $\mathbf{p}^* = (p_r^*, p_s^*)$ is degenerate. Clearly Ω is a function of \mathbf{p}^* however we omit the argument when understood. We have

$$\delta\Omega = \alpha\theta \int_0^1 \max[v - p_s - \Omega, 0] dF(v) + \alpha(1 - \theta) \int_0^1 \max[v - p_r - \Omega, 0] dF(v).$$

A quick interpretation is this. With probability $\alpha\theta$ a buyer meets a stressed seller who asks for p_s and upon inspecting the house the buyer realizes his valuation $v \in [0, 1]$. As long as the consumer surplus $v - p_s - \Omega$ is positive the buyer purchases, otherwise he walks away. The second part of the expression is about a possible meeting with a relaxed seller and can be interpreted similarly.

For any given price p_j we conjecture an associated *reservation value*, or minimum willingness to pay, $v_j := p_j + \Omega$ such that the customer purchases only if $v \geq v_j$. Obviously not all meetings result in trade; for trade to occur the house must turn out to be a good match for the buyer, which happens with probability $1 - F(v_j)$. We interpret a rise in v_j as buyers becoming more selective.

Notice that there are two types of trading frictions in the model. The first is locating a vacant house and the second is whether the house, once found, is a good match. The fact that some meetings do not result in trade is in line with the empirical observation by Merlo and Ortalo-Magné (2004). Analyzing transaction histories of residential properties sold in England between 1995 and 1998 they find that about a third of all matches resolve with no agreement. This observation is in clear contradiction with most of the existing theoretical models of the housing market, e.g. Arnold (1999), Krainer (2001), Yavas and Yang (1995). Based on complete information, these models imply that all matches result in a sale. Our model, on the contrary, captures this salient feature of the housing market in a natural way by assuming private information.

Inserting the reservation values into Ω we have

$$\begin{aligned}\Omega &= \frac{\alpha\theta}{\delta} \int_{v_s}^1 (v - v_s) dF(v) + \frac{\alpha(1-\theta)}{\delta} \int_{v_r}^1 (v - v_r) dF(v) \\ &= \frac{\alpha\theta}{\delta} \int_{v_s}^1 [1 - F(v)] dv + \frac{\alpha(1-\theta)}{\delta} \int_{v_r}^1 [1 - F(v)] dv,\end{aligned}\tag{1}$$

where in the second step we use integration by parts. Notice that $\int_{v_j}^1 [1 - F(v)] dv$ is the expected surplus to the buyer from a possible meeting with seller j and Ω simply is a summation of these surpluses adjusted with appropriate meeting probabilities and discounting.

The steady state fraction of stressed sellers, denoted by θ , is endogenous and can be obtained by equating the inflow into the pool of stressed sellers $(1 - \theta)\mu$ to the outflow $\theta\alpha[1 - F(v_s)]$. We have

$$\theta = \frac{\mu}{\mu + \alpha[1 - F(v_s)]} \in (0, 1).\tag{2}$$

Notice that $\theta_{v_s} > 0$, i.e., buyers can squeeze the outflow and increase θ by becoming more selective when trading with stressed sellers. This plays a strategic role in the pricing and purchasing decisions of the agents (see Proposition 6).

Using (1) we can obtain the *indifference curves* I_r and I_s that trace combinations of prices p_j and reserve values v_j leaving a buyer indifferent between buying and searching:

$$p_j = v_j - \frac{\alpha\theta}{\delta} \int_{v_s}^1 [1 - F(v)] dv - \frac{\alpha(1-\theta)}{\delta} \int_{v_r}^1 [1 - F(v)] dv := I_j\tag{3}$$

Lemma 1 *Indifference curves I_r and I_s slope upwards in v_r and v_s , i.e., $\frac{\partial I_r}{\partial v_r} > 0$ and $\frac{\partial I_s}{\partial v_s} > 0$.*

All proofs are in the appendix.

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Figure 1a, 1b – Offer and Indifference Curves⁷

⁷We set $v_s = 0.79$ in panel *a* and $v_r = 0.82$ in panel *b*, which are equilibrium reserve valuations for the baseline parameters (see section 3).

Figure 1 illustrates the Lemma for $\alpha = 1$, $\mu = 0.5$, $\delta = 0.05$, $\bar{\delta} = 0.2$ and the uniform distribution $F(v) = v$, which are the baseline parameters that we use in subsequent figures. In panel *a* imagine a horizontal line drawn at some price p_r corresponding to the threshold valuation v_r . A buyer purchases only if the realized v happens to exceed v_r , i.e., the distance $1 - v_r$ is the probability of trade. The Lemma establishes that the higher p_r the higher the associated v_r and therefore the smaller the chance of a trade. Clearly a seller can manipulate the acceptability of his house by adjusting the list price. From the seller's point of view a price cut means a quicker sale albeit a revenue loss; hence he must strike a balance between these two effects, which we study next.

2.3 The Seller's Problem

Each seller advertises a take-it-or-leave-it price taking as given market prices and buyers' search decision. The value functions are given by

$$\begin{aligned}\bar{\delta}\Pi_s &= \alpha [1 - F(v_s)] \max(p_s - \Pi_s, 0) \\ \delta\Pi_r &= \alpha [1 - F(v_r)] \max(p_r - \Pi_r, 0) + \mu (\Pi_s - \Pi_r).\end{aligned}$$

A quick interpretation is this. A stressed seller who lists p_s meets a buyer with probability α , who purchases with probability $1 - F(v_s)$. The seller agrees to trade only if the price exceeds his continued value of search i.e. if $p_s - \Pi_s \geq 0$. The second line is similar except that a relaxed seller may become desperate with probability μ .

A stressed seller solves $\max_{p_s} \Pi_s$ subject to the constraint $v_s = p_s + \Omega$ taking Ω as given⁸. Conjecturing $p_s \geq \Pi_s$ and inserting the constraint into the value function we obtain

$$\Pi_s = \frac{\alpha [1 - F(p_s + \Omega)]}{\bar{\delta} + \alpha [1 - F(p_s + \Omega)]} p_s.$$

The expression in front of p_s is the probability of selling adjusted with the discount factor; call it $m(p_s)$ and note that $m' < 0$. From the seller's perspective raising p_s brings in a

⁸From the seller's point of view, cutting p_s directly improves the buyer's willingness to trade, but the seller fails to take into account how a drop in p_s changes the equilibrium prices and the buyer's value of search. This large market approach is used in directed search models as well, e.g. see Camera and Selcuk (2009).

larger revenue (intensive margin), but lowers the chance of a sale (extensive margin).⁹

A relaxed seller's problem is similar: $\max_{p_r} \Pi_r$ subject to $v_r = p_r + \Omega$. Conjecturing $p_r \geq \Pi_r$ and inserting the constraint, the objective function becomes

$$\Pi_r = \frac{\alpha [1 - F(p_r + \Omega)] p_r + \mu \Pi_s}{\delta + \mu + \alpha [1 - F(p_r + \Omega)]}.$$

These maximization problems are standard; hence the algebra is relegated to the appendix. The following summarizes the results.

Lemma 2 *Price posting functions for relaxed and stressed sellers are*

$$p_r = \frac{1-F(v_r)}{F'(v_r)} + \frac{\alpha[1-F(v_r)]^2}{(\mu+\delta)F'(v_r)} + \frac{\alpha\mu[1-F(v_s)]^2}{\delta(\mu+\delta)F'(v_s)}, \quad (4)$$

$$p_s = \frac{1-F(v_s)}{F'(v_s)} + \frac{\alpha}{\delta} \frac{[1-F(v_s)]^2}{F'(v_s)}. \quad (5)$$

Furthermore $\frac{\partial p_s}{\partial v_s} < \frac{\partial p_r}{\partial v_s} < 0$ and $\frac{\partial p_r}{\partial v_r} < \frac{\partial p_s}{\partial v_r} = 0$.

Expressions (4) and (5) are profit maximizing prices (which are labeled as *offer curves* O_r and O_s in Figure 1) that relaxed and stressed sellers ought to post given v_r and v_s . To see why $\frac{\partial p_r}{\partial v_r} < 0$ notice that for low values of v_r the aforementioned intensive margin effect dominates the extensive margin. Put simply, buyers are not too selective and therefore sellers can afford to post high. However as v_r rises the extensive margin starts to grow hence p_r falls. Using a similar argument one can explain why $\frac{\partial p_s}{\partial v_s} < 0$. More importantly a relaxed seller can become stressed one day; hence $\frac{\partial p_r}{\partial v_s} < 0$. Notice, however, this is an indirect effect. Indeed $\frac{\partial p_s}{\partial v_s} < \frac{\partial p_r}{\partial v_s}$, i.e., relaxed sellers are less sensitive to a change in v_s than stressed sellers. Finally $\frac{\partial p_s}{\partial v_r} = 0$ since a stressed seller never becomes relaxed again.

Simultaneous intersections of the offer and indifference curves in Figure 1a and 1b determine the equilibrium reserve valuations and list prices, which we discuss next.

3 Equilibrium: Existence and Characterization

Definition 3 *A steady-state symmetric equilibrium in pure strategies is characterized by pairs $\mathbf{v}^* = (v_r^*, v_s^*)$ and $\mathbf{p}^* = (p_r^*, p_s^*)$ that satisfy (3), (4) and (5).*

⁹It is worth noting that, although the seller can manipulate the probability of trade by adjusting the list price, search is undirected, that is the arrival rate α does not depend on the list price. Buyers decide whether to purchase or not *after* randomly meeting the seller.

The fraction of stressed sellers θ , which is given by (2) and also implicitly part of the equilibrium, can easily be recovered from the above conditions.

In this section we first prove existence of a unique equilibrium then we discuss comparative statics. To start, define the difference functions

$$\Delta_j(v_r, v_s) := p_j - v_j + \Omega = 0 \quad (6)$$

and their locuses

$$l_j(v_r) := \{v_s \in [0, 1] \mid \Delta_j(v_r, v_s) = 0\}.$$

Clearly equilibrium \mathbf{v}^* and \mathbf{p}^* must satisfy (6). The next Lemma, which plays a key role in proving Theorem 5, establishes that l_r and l_s look as in Figure 2.

Lemma 4 *Equations in (6) define l_r and l_s as implicit and strictly decreasing functions of v_r with $\frac{dl_r}{dv_r} < \frac{dl_s}{dv_r} < 0$. Furthermore there exists some $0 < \underline{v}_s < \bar{v}_s < 1$ and $\underline{v}_r \in (0, 1)$ such that $l_s(0) = \bar{v}_s$, $l_s(1) = \underline{v}_s$ and $l_r(\underline{v}_r) = 1$. Last either there exists some $\bar{v}_r \in (\underline{v}_r, 1)$ such that $l_r(\bar{v}_r) = 0$ as in Figure 2a or there exists some $\underline{\underline{v}}_s \in (0, \underline{v}_s)$ such that $l_r(1) = \underline{\underline{v}}_s$ as in Figure 2b.*

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Figure 2a, 2b – Locuses

Figure 2, drawn for the uniform distribution and the parameter values in the left corners, is an illustration of the Lemma. The fact that l_r is steeper than l_s and the specific locations of the boundaries guarantee a unique intersection. The following Theorem states the main existence result. The proof involves showing l_r and l_s intersect once at some interior \mathbf{v}^* and then checking incentive compatibility for sellers.

Theorem 5 *There exists a unique (pooling) equilibrium characterized by some interior \mathbf{v}^* and \mathbf{p}^* . Furthermore $v_r^* > v_s^*$ and $p_r^* > p_s^*$.*

It is worth noting that the equilibrium exists for all parameter values and any cdf satisfying Assumption 1. Furthermore it is a pooling equilibrium in that both types of sellers participate by posting interior prices and buyers purchase from both types of sellers

More importantly, in equilibrium motivated sellers advertise lower prices and sell faster than regular sellers. After becoming depressed a seller starts to discount the future more heavily which induces him to offer a price cut to sell more quickly. Indeed recalling that $1 - F(v)$ is the probability of trade, $v_r^* > v_s^*$ implies that buyers are more likely to purchase when dealing with a motivated seller.

This result is empirically plausible. Glower et al. (1998) survey sellers in Columbus Ohio area to obtain information on their motivation by asking whether they have a planned date to move out or accepted a job offer elsewhere or bought another house etc., and document that motivated sellers accept lower prices and sell more quickly. Merlo and Ortalo-Magné (2004), based on home sale transaction data from England, find that 65% of sellers do not change their list price, 26% reduce only once, and 9% reduce twice or more. Based on the same data set, Merlo et al. (2008) obtain the individual list price trajectories, which are either flat or piecewise flat with typically one discontinuous jump-down at the time of the price reduction (see Figure 2.1 therein). We interpret these price drops as sellers becoming desperate and offering discounts to sell as quickly as possible. Price trajectories follow from Theorem 5: if the seller stays relaxed until he sells then the trajectory of the list price is flat at p_r^* . Otherwise it is piecewise flat with a drop from p_r^* to p_s^* at the date the seller is hit by the shock.¹⁰

The next proposition summarizes how \mathbf{v}^* and \mathbf{p}^* respond to changes in μ and $\bar{\delta}$.

Proposition 6 *In equilibrium* $\frac{dp_j^*}{d\mu} < 0$, $\frac{dv_j^*}{d\mu} > 0$, $\frac{dp_j^*}{d\bar{\delta}} < 0$, $\frac{dv_j^*}{d\bar{\delta}} < 0$.

The shock may arrive more often, for instance, during a recession or financial crisis where sellers are more likely to lose their jobs or experience other financial difficulties. Clearly as μ rises more sellers are hit by the shock and are forced to offer price discounts. In addition to this immediate effect the Proposition indicates two subsequent effects. First, even existing relaxed and stressed sellers lower their prices, $\frac{dp_j^*}{d\mu} < 0$, which is a spillover effect of the rising number of forced sales. Second, buyers exhibit a ‘vulture behavior’: even though prices decrease, buyers become more selective and search longer for better

¹⁰In our model the shock hits only once, hence the drop occurs only once.

deals, that is $\frac{dv_j^*}{d\mu} > 0$. To see why, notice that sellers' value of search declines with μ : a relaxed seller is more fearful of becoming depressed while a stressed seller faces a stiffer competition because of the rising θ (recall that $\theta_\mu > 0$); hence prices drop. Buyers' value of search, on the other hand, rises with θ : because there are so many distressed sales a buyers can afford to become more selective, hence v_j^* rises.

Campbell et al. (2009) provide some empirical evidence for the aforementioned spillover effect. Using a comprehensive data set on individual house transactions in Massachusetts they study the spillover effects of foreclosures on the prices of nearby houses and find that the rising foreclosures significantly lower the price at which a house can be sold within a 0.25 mile neighborhood.

If the shock becomes more severe, that is if $\bar{\delta}$ rises then the value of search for both types of sellers decreases; hence prices drop. Furthermore recall that $\theta_{\bar{\delta}} = 0$. From a buyer's perspective, put simply, the fraction of deals stays the same but the deals get sweeter because of the lower prices. Consequently buyers lower their threshold valuations to catch these deals. The final result is that both types of sellers reduce their prices and sell faster, i.e. $\frac{dp_j^*}{d\bar{\delta}} < 0$ and $\frac{dv_j^*}{d\bar{\delta}} < 0$. Furthermore, simulations suggest that as $\bar{\delta} \rightarrow \delta$ we have $p_s^* \rightarrow p_r^*$ and $v_s^* \rightarrow v_r^*$ and the gaps widen as $\bar{\delta}$ grows. If the shock is mild ($\bar{\delta} \approx \delta$) then there is not much difference between what relaxed and stressed sellers post. However as the shock starts to bite ($\bar{\delta} \gg \delta$), distressed sellers offer sizeable price discounts and sell considerably faster.

Since the parameters α and δ are well understood in random search models we omit their discussion.¹¹ However it is worth noting that $\frac{dv_j^*}{d\alpha} > 0$, that is if it gets easier to meet sellers (due to better technology, internet etc.) then buyers become more selective and search longer. Consequently trade not necessarily speeds up.

¹¹Following the steps outlined in the proof of Proposition 6, one can show that $\frac{dv_j^*}{d\alpha} > 0$, $\frac{dp_j^*}{d\alpha} < 0$, $\frac{dv_j^*}{d\delta} < 0$ and $\frac{dp_j^*}{d\delta} > 0$.

4 Time to Sale

Consider a seller who enters the market at time 0 (wlog). The probability that he remains relaxed without a sale until time t is given by

$$r(t) = e^{-(\mu + \alpha_r)t}, \quad (7)$$

where $\alpha_j := \alpha \left[1 - F(v_j^*) \right]$. Similarly the probability that he becomes stressed at some time $x \leq t$ while the house is still unsold at t equals

$$s(t) = \int_0^t \mu e^{-\mu x} e^{-\alpha_r x} e^{-\alpha_s(t-x)} dx. \quad (8)$$

Notice that $\mu e^{-\mu x}$ is the density (exponential pdf) of becoming stressed over $[0, x]$, $e^{-\alpha_r x}$ is the probability of no sale while the seller was relaxed and $e^{-\alpha_s(t-x)}$ is the probability of no sale between x and t during which he was stressed. Using these expressions we can obtain the density of the time-to-sale and the expected time on the market (TOM)

Proposition 7 *The density of the time on the market is given by*

$$\gamma = \frac{\mu \alpha_s e^{-\alpha_s t} - (\alpha_s - \alpha_r) (\mu + \alpha_r) e^{-(\mu + \alpha_r)t}}{\mu - \alpha_s + \alpha_r}, \quad (9)$$

which is hump shaped if $\frac{\mu}{\alpha} > \frac{[1 - F(v_r^*)]^2}{F(v_r^*) - F(v_s^*)}$ and monotone decreasing otherwise. Furthermore

$$TOM = \frac{\mu + \alpha_s}{\alpha_s (\mu + \alpha_r)}.$$

To build intuition we illustrate γ under three scenarios. $\mu = 1, \bar{\delta} = 0.2$, the benchmark; $\mu = 1, \bar{\delta} = 1$, a more severe shock; $\mu = 0.1, \bar{\delta} = 0.2$, a less frequent shock. First notice that the density is skewed to the right and can be hump shaped if the ratio μ/α is sufficiently large, i.e., if buyers are scarce and the shock is frequent. Clearly given Poisson arrivals, it is not surprising to see that γ resembles an exponential pdf, which is heavily right-skewed. The shape of γ is indeed realistic. Merlo et al. (2008), based on house sale transaction data from England, obtain the empirical distribution of times to sale, which is clearly right skewed and hump-shaped with a mean 10.27 weeks and median of 6 weeks (see figure 2.3 therein).

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Figure 3a, 3b¹²

Remark. *An increase in μ or $\bar{\delta}$ results in a slimmer tail and a shorter expected time on the market.*

Recall that if $\bar{\delta}$ increases then trade speeds up (Proposition 6), hence the likelihood of a long duration falls (slimmer tail). Indeed in Figure 3a the pdf drawn for $\mu = 1$, $\bar{\delta} = 1$ has a slimmer tail than the one drawn for $\mu = 1$, $\bar{\delta} = 0.2$. The expected time on the market drops as well. One can verify that

$$\frac{dTOM}{d\bar{\delta}} = \frac{\partial TOM}{\partial v_r^*} \frac{dv_r^*}{d\bar{\delta}} + \frac{\partial TOM}{\partial v_s^*} \frac{dv_s^*}{d\bar{\delta}} < 0$$

since $\frac{\partial TOM}{\partial v_j^*} > 0$ and $\frac{dv_j^*}{d\bar{\delta}} < 0$.

The effect of μ is less obvious. On the one hand if μ rises then more seller become desperate and since desperate sellers sell faster than relaxed sellers. TOM should drop. On the other hand because of the rising μ buyers become more selective (Proposition 6), hence TOM should rise. Analytically it is difficult to sign $\frac{dTOM}{d\mu}$ but simulations suggest that the former effect is dominant, that is TOM falls as the shock arrives more frequently. Indeed, in Figure 3a the pdf drawn for $\mu = 1$, $\bar{\delta} = 0.2$ has a slimmer tail than the one drawn for $\mu = 0.1$, $\bar{\delta} = 0.2$.

4.1 Expected Sale Price

Consider sales completed with a duration t . The fraction of distressed sales equals to

$$g(t) = \frac{s(t)}{r(t) + s(t)}.$$

One can easily verify that g rises in t (see the proof of Proposition 8), i.e., the longer the duration, the more likely the sellers are to be desperate. An immediate corollary is that the expected sale price falls with the duration since desperate sellers sell at a discount. To see this more precisely define the expected sale price $\bar{p}(t) = gp_s^* + (1 - g)p_r^*$ and the variance $\sigma^2(t) = g(p_s^* - \bar{p})^2 + (1 - g)(p_r^* - \bar{p})^2$.

¹²We fix $F(v) = v^{1/3}$, $\alpha = 1$, and $\delta = 0.05$.

Proposition 8 $\bar{p}(t)$ is monotone decreasing and $\sigma^2(t)$ is hump-shaped in t .

Figure 3b illustrates \bar{p} (left scale) and σ (right scale) for the benchmark parameters. Indeed if a house is sold soon after it was advertised then most likely it is a relaxed sale, that is $\lim_{t \rightarrow 0} \bar{p}(t) \rightarrow p_r^*$. However the longer it stays on the market, the more likely the seller is to become desperate and in the limit $\bar{p}(t) \rightarrow p_s^*$.

The fact that the expected sale price falls with the duration is empirically documented, e.g. see Yavas and Yang (1995), Merlo and Ortalo-Magné (2004) among others. The continuously downward slope in \bar{p} may be somewhat misleading and create an illusion that the list price continuously falls with the duration. We emphasize that an individual list price trajectory is piecewise flat with a discontinuous drop from p_r^* to p_s^* at the time the seller is hit by the shock. It is the *expected* price that falls monotonically; the actual price is either p_r^* or p_s^* . The shape of the standard deviation is also very intuitive. For very short or very long durations the sale is either relaxed or stressed, respectively, with a high probability. Only for intermediate values there is ambiguity; hence the hump shape.

5 Conclusion

We have presented a model of the housing market where buyers' valuations are private information and sellers are heterogenous in terms of their urgency to sell. The model has a number of interesting implications. Once desperate, a seller offers a price discount and consequently sells more quickly. In addition, as the number of distressed sales rises, even relaxed sellers are forced to lower their prices. To make things worse for sellers, buyers delay purchasing and search longer for better deals. Finally we obtain the distribution of the time on the market and the expected sale prices analytically. Most of our results are consistent with the empirical literature on the housing market.

Some natural extensions suggest themselves. For instance, for the sake of tractability we deliberately ignored bargaining; whereas in reality a house is hardly sold without bargaining. It would be interesting to consider, say, two rounds of negotiations à la Fudenberg and Tirole (1983) between the potential buyer and the seller. This opens up the possibility of a 'lowball' offer, which, if rejected is followed by a higher offer. Indeed,

this is what Merlo and Ortalo-Magné (2004) observe empirically.

6 References

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7 Appendix

Proof of Lemma 1. Since $I_j = v_j - \Omega$, it suffices to show $\frac{\partial \Omega}{\partial v_j} < 0$ for $j = r, s$. To start, note that

$$\frac{\partial \Omega}{\partial v_r} = -\frac{\alpha(1-\theta)}{\delta} [1 - F(v_r)],$$

which clearly is negative. Now consider

$$\frac{\partial \Omega}{\partial v_s} = \frac{\alpha\theta'}{\delta} \int_{v_s}^{v_r} [1 - F(v)] dv - \frac{\alpha\theta}{\delta} [1 - F(v_s)], \quad (10)$$

where

$$\theta' = \frac{\partial \theta}{\partial v_s} = \frac{\theta \alpha F'(v_s)}{\mu + \alpha [1 - F(v_s)]} > 0.$$

To show $\frac{\partial \Omega}{\partial v_s} < 0$ it suffices to demonstrate

$$\eta(v_s) := \int_{v_s}^1 [1 - F(v)] dv - \frac{\mu}{\alpha} \frac{1 - F(v_s)}{F'(v_s)} - \frac{[1 - F(v_s)]^2}{F'(v_s)} < 0.$$

Omitting the argument and differentiating with respect to v_s we have

$$\eta' = \frac{F'^2 + F''(1-F)}{F'^2} \left[\frac{\mu}{\alpha} + 1 - F \right]$$

which is positive under Assumption 1. Since η increases in v_s and $\eta(1) = 0$, it follows that $\eta(v_s) < 0, \forall v_s \in [0, 1)$. \square

Proof of Lemma 2. Start by analyzing the stressed seller's problem. Rearranging Π_s we have

$$\Pi_s = \frac{\alpha}{\delta} [1 - F(p_s + \Omega)] (p_s - \Pi_s). \quad (11)$$

Differentiating Π_s with respect to p_s yields (omit the argument)

$$\Pi'_s = -\frac{\alpha}{\delta} (p_s - \Pi_s) F' + \frac{\alpha}{\delta} (1 - \Pi'_s) (1 - F).$$

Notice that $\Omega' = 0$ since sellers take Ω as given. The first-order condition is given by

$$\Pi'_s = 0 \Leftrightarrow p_s - \Pi_s = \frac{1 - F(v_s)}{F'(v_s)}. \quad (12)$$

To check for the second order condition note that

$$-\frac{\delta}{\alpha} \Pi''_s = (p_s - \Pi_s) F'' + 2(1 - \Pi_s) F' + (1 - F) \Pi''_s.$$

Inserting $\Pi'_s = 0$ and using (12) we obtain $\text{sign}(\Pi''_s) = -\text{sign}(F''(1-F) + 2F'^2)$, which clearly is negative under Assumption 1.

To recover the list price p_s first insert (12) into (11) to get $\Pi_s = \frac{\alpha}{\delta} \frac{[1-F(v_s)]^2}{F'(v_s)}$. Inserting this into (12) yields the offer curve for stressed sellers, given by (5).

The problem of a relaxed seller is similar. Rearrange Π_r to obtain

$$\Pi_r = \frac{\alpha}{\delta} [1 - F(v_r)] (v_r - \Omega - \Pi_r) + \frac{\mu}{\delta} [\Pi_s - \Pi_r]. \quad (13)$$

Differentiate Π_r with respect to p_r , imposing $\Omega' = \Pi'_s = 0$ since they are taken as given, to obtain the first-order condition

$$\Pi'_r = 0 \Leftrightarrow p_r - \Pi_r = \frac{1-F(v_r)}{F'(v_r)}. \quad (14)$$

It is easy to verify that under Assumption 1 we have $\Pi''_r < 0$ (the proof is very similar to above). To obtain p_r first use (13), (14) and the expression for Π_s from above to get

$$\Pi_r = \frac{\alpha[1-F(v_r)]^2}{(\mu+\delta)F'(v_r)} + \frac{\alpha\mu[1-F(v_s)]^2}{\delta(\mu+\delta)F'(v_s)}.$$

Inserting this into (14) provides the offer curve of a relaxed seller given by (4).

To show the second part of the Lemma differentiate (4) and (5) to get

$$\frac{\partial p_r}{\partial v_r} = -\frac{F'_r{}^2 + F''_r(1-F_r)}{F'^2_r} - \frac{\alpha(1-F_r)}{(\mu+\delta)} \left[\frac{2F'_r{}^2 + F''_r(1-F_r)}{F'^2_r} \right] < 0, \quad (15)$$

$$\frac{\partial p_r}{\partial v_s} = -\frac{\alpha\mu(1-F_s)}{\delta(\mu+\delta)} \left[\frac{2F'_s{}^2 + F''_s(1-F_s)}{F'^2_s} \right] < 0, \quad \frac{\partial p_s}{\partial v_r} = 0, \quad (16)$$

$$\frac{\partial p_s}{\partial v_s} = -\frac{F'_s{}^2 + F''_s(1-F_s)}{F'^2_s} - \frac{\alpha(1-F_s)}{\delta} \left[\frac{2F'_s{}^2 + F''_s(1-F_s)}{F'^2_s} \right] < 0, \quad (17)$$

where $F_j := F(v_j)$. Given Assumption 1, $\frac{\partial p_r}{\partial v_r}$, $\frac{\partial p_r}{\partial v_s}$ and $\frac{\partial p_s}{\partial v_s}$ are obviously negative. $\frac{\partial p_s}{\partial v_s} < \frac{\partial p_r}{\partial v_s}$ is also obvious after comparing (16) and (17) term by term. \square

Proof of Lemma 4. We will first demonstrate that $\frac{dl_r}{dv_r} < \frac{dl_s}{dv_r}$ and then we will focus on the existence of boundaries $\underline{v}_j, \bar{v}_j$. To start, recall that

$$\Delta_r(v_r, v_s) = p_r - v_r + \Omega \quad \text{and} \quad \Delta_s(v_r, v_s) = p_s - v_s + \Omega$$

where Ω , p_r , and p_s are respectively given by (1), (4) and (5). Notice that

$$\frac{\partial \Delta_r}{\partial v_r} = \frac{\partial p_r}{\partial v_r} - 1 + \frac{\partial \Omega}{\partial v_r} < \frac{\partial \Delta_s}{\partial v_r} = \frac{\partial p_s}{\partial v_r} + \frac{\partial \Omega}{\partial v_r} < 0, \quad (18)$$

$$\frac{\partial \Delta_s}{\partial v_s} = \frac{\partial p_s}{\partial v_s} - 1 + \frac{\partial \Omega}{\partial v_s} < \frac{\partial \Delta_r}{\partial v_s} = \frac{\partial p_r}{\partial v_s} + \frac{\partial \Omega}{\partial v_s} < 0. \quad (19)$$

These inequalities follow from the facts that $\frac{\partial \Omega}{\partial v_j} < 0$, $\frac{\partial p_s}{\partial v_s} < \frac{\partial p_r}{\partial v_s} < 0$ and $\frac{\partial p_r}{\partial v_r} < \frac{\partial p_s}{\partial v_r} = 0$ (see Lemma 1 and Lemma 2). Therefore $\Delta_j(v_r, v_s) = 0$ defines $v_s = l_j(v_r)$ as an implicit function of v_r (Implicit Function Theorem) with

$$\frac{dl_j}{dv_r} = -\frac{\partial \Delta_j / \partial v_r}{\partial \Delta_j / \partial v_s} < 0.$$

Since $\frac{\partial \Delta_r}{\partial v_r} < \frac{\partial \Delta_s}{\partial v_r} < 0$ and $\frac{\partial \Delta_s}{\partial v_s} < \frac{\partial \Delta_r}{\partial v_s} < 0$ it is obvious that $\frac{dl_r}{dv_r} < \frac{dl_s}{dv_r} < 0$.

Boundaries. Start by evaluating $\Delta_s(v_r, v_s)$ at end points. It is straightforward to show that $\Delta_s(0, 0) > \Delta_s(1, 0) > 0$ and $\Delta_s(0, 1) = \Delta_s(1, 1) = -1 < 0$. Since $\Delta_s(1, 0) > 0$ and $\Delta_s(1, 1) < 0$ and Δ_s decreases in v_s the Intermediate Value Theorem guarantees existence of some $\underline{v}_s \in (0, 1)$ such that $\Delta_s(1, \underline{v}_s) = 0$, i.e., $l_s(1) = \underline{v}_s$. Similarly $\Delta_s(0, 0) > 0$ and $\Delta_s(0, 1) < 0$ implies existence of some $\bar{v}_s \in (0, 1)$ such that $\Delta_s(0, \bar{v}_s) = 0$, i.e., $l_s(0) = \bar{v}_s$. Note that $l_s(1) < l_s(0)$ and since l_s decreases in v_r clearly $\underline{v}_s < \bar{v}_s$.

Now evaluate $\Delta_r(v_r, v_s)$ at end points. We have $\Delta_r(0, 0) > \Delta_r(0, 1) > 0$ and $\Delta_r(1, 1) = -1 < 0$. However the expression

$$\Delta_r(1, 0) = \frac{\alpha\mu}{\bar{\delta}(\mu+\delta)F'(0)} - 1 + \frac{\alpha\theta}{\delta} \int_0^1 [1 - F(v)] dv$$

can be positive or negative. To show existence of $\underline{v}_r \in (0, 1)$ notice that $\Delta_r(0, 1) > 0$ and $\Delta_r(1, 1) < 0$ and since Δ_r decreases in v_r the Intermediate Value Theorem guarantees existence of some $\underline{v}_r \in (0, 1)$ such that $\Delta_r(\underline{v}_r, 1) = 0$ which is equivalent to $l_r(\underline{v}_r) = 1$. Existence of \bar{v}_r or \underline{v}_s hinges on the sign of $\Delta_r(1, 0)$ as we study below.

- *Suppose $\Delta_r(1, 0) < 0$* : Since $\Delta_r(0, 0) > 0$ there exists some $\bar{v}_r \in (0, 1)$ such that $\Delta_r(\bar{v}_r, 0) = 0$ or equivalently $l_r(\bar{v}_r) = 0$, and since l_r is a decreasing function of v_r we have $\underline{v}_r < \bar{v}_r$.

- *Suppose $\Delta_r(1, 0) > 0$* : First we will show that $\Delta_r(1, \underline{v}_s) < 0$. Notice that

$$\Delta_s(1, \underline{v}_s) - \Delta_r(1, \underline{v}_s) = \frac{1-F(\underline{v}_s)}{F'(\underline{v}_s)} + \frac{\alpha\delta}{\bar{\delta}(\mu+\delta)} \frac{[1-F(\underline{v}_s)]^2}{F'(\underline{v}_s)} + 1 - \underline{v}_s > 0,$$

and since $\Delta_s(1, \underline{v}_s) = 0$ it is clear that $\Delta_r(1, \underline{v}_s) < 0$. Now since $\Delta_r(1, 0) > 0$ there exists some $\underline{v}_s \in (0, \underline{v}_s)$ such that $\Delta_r(1, \underline{v}_s) = 0$ or equivalently $l_r(1) = \underline{v}_s$. \square

Proof of Theorem 5. First we will demonstrate l_r and l_s intersect once in the unit interval. Define $z := l_r - l_s$ and notice that it decreases in v_r since $\frac{dz}{dv_r} = \frac{dl_r}{dv_r} - \frac{dl_s}{dv_r} < 0$ (Lemma 4). Furthermore it is easy to verify $z(\underline{v}_r) > 0$ since and $z(1) < 0$. Indeed $z(\underline{v}_r) = l_r(\underline{v}_r) - l_s(\underline{v}_r) = 1 - l_s(\underline{v}_r) > 0$ since $l_s(\underline{v}_r) < l_s(0) = \bar{v}_s < 1$ and similarly $z(1) = l_r(1) - l_s(1) = l_r(1) - \underline{v}_s < 0$ since $l_r(1)$ is either negative or equals \underline{v}_s both of which are smaller than \underline{v}_s (see the proof of Lemma 4). Consequently the Intermediate Value Theorem guarantees existence of a unique $v_r^* \in (\underline{v}_r, 1)$ such that $l_r(v_r^*) = l_s(v_r^*) = v_s^*$.

Now we will show $v_r^* > v_s^*$ and $p_r^* > p_s^*$. Note that (v_r^*, v_s^*) satisfies $\Delta_r(v_r^*, v_s^*) - \Delta_s(v_r^*, v_s^*) = 0$, where

$$\frac{\partial(\Delta_r - \Delta_s)}{\partial v_r} = \frac{\partial p_r}{\partial v_r} - 1 - \frac{\partial p_s}{\partial v_r} < 0$$

because $\frac{\partial p_r}{\partial v_r} < 0$ and $\frac{\partial p_s}{\partial v_r} = 0$ (see (15) and (16)). As a contradiction suppose $v_r^* = v_s^* = v$ and notice that

$$\Delta_r(v, v) - \Delta_s(v, v) = \frac{\alpha(\bar{\delta} - \delta)}{\bar{\delta}(\mu + \delta)} \frac{[1 - F(v)]^2}{F'(v)} > 0.$$

It follows that if $\Delta_r(v_r^*, v_s^*) - \Delta_s(v_r^*, v_s^*) = 0$ then $v_r^* > v_s^*$. Furthermore $p_r^* > p_s^*$ is immediate since in equilibrium $p_r^* - p_s^* = v_r^* - v_s^* > 0$. For future reference, plugging in for p_r^* and p_s^* we have

$$p_r^* - p_s^* = \frac{1 - F(v_r^*)}{F'(v_r^*)} - \frac{1 - F(v_s^*)}{F'(v_s^*)} + \frac{\alpha}{\mu + \delta} \left[\frac{[1 - F(v_r^*)]^2}{F'(v_r^*)} - \frac{\delta}{\bar{\delta}} \frac{[1 - F(v_s^*)]^2}{F'(v_s^*)} \right] > 0. \quad (20)$$

Finally we check incentive compatibility for the sellers. A type j seller sells if $p_j^* > \Pi_j$. It is obvious from (12) and (14) that $p_j^* - \Pi_j = \left[1 - F(v_j^*) \right] / F'(v_j^*) > 0$, since $v_j^* \in (0, 1)$. \square

Proof of Proposition 6. Recall that v_r^* and v_s^* satisfy $\Delta_r(v_r^*, v_s^*) = 0$ and $\Delta_s(v_r^*, v_s^*) = 0$ simultaneously. Omit the superscript and note that (General Implicit Function Theorem) $\frac{dv_j}{dx} = \frac{\det B_j(x)}{\det A}$ for any $x = \bar{\delta}, \mu$ and $j = r, s$ where

$$B_r(x) = \begin{bmatrix} -\frac{\partial \Delta_r}{\partial x} & \frac{\partial \Delta_r}{\partial v_s} \\ -\frac{\partial \Delta_s}{\partial x} & \frac{\partial \Delta_s}{\partial v_s} \end{bmatrix}, \quad B_s(x) = \begin{bmatrix} \frac{\partial \Delta_r}{\partial v_r} & -\frac{\partial \Delta_r}{\partial x} \\ \frac{\partial \Delta_s}{\partial v_r} & -\frac{\partial \Delta_s}{\partial x} \end{bmatrix}, \quad A = \begin{bmatrix} \frac{\partial \Delta_r}{\partial v_r} & \frac{\partial \Delta_r}{\partial v_s} \\ \frac{\partial \Delta_s}{\partial v_r} & \frac{\partial \Delta_s}{\partial v_s} \end{bmatrix}.$$

Note that $\det A = \frac{\partial \Delta_r}{\partial v_r} \frac{\partial \Delta_s}{\partial v_s} - \frac{\partial \Delta_s}{\partial v_r} \frac{\partial \Delta_r}{\partial v_s} > 0$ since $\frac{\partial \Delta_r}{\partial v_r} < \frac{\partial \Delta_s}{\partial v_r} < 0$ and $\frac{\partial \Delta_s}{\partial v_s} < \frac{\partial \Delta_r}{\partial v_s} < 0$ (see (18) and (19)). It follows that $\text{sign} \left(\frac{\partial v_j}{\partial x} \right) = \text{sign} (\det B_j (x))$.

Partial Derivatives. Realizing $\frac{\partial \Delta_j}{\partial x} = \frac{\partial p_j}{\partial x} + \frac{\partial \Omega}{\partial x}$, we need the partial derivatives of Ω , p_r , and p_s with respect to $x = \bar{\delta}, \mu$. Starting with (1) it is easy to verify that

$$\frac{\partial \Omega}{\partial \bar{\delta}} = 0 \quad \text{and} \quad \frac{\partial \Omega}{\partial \mu} = \frac{\theta^2 \alpha [1-F_s]}{\mu^2} \int_{v_s}^{v_r} [1-F(v)] dv > 0.$$

Notice that Ω_μ is positive since $v_r^* > v_s^*$ (Theorem 5). Now differentiate (4) and (5) to get

$$\begin{aligned} \frac{\partial p_s}{\partial \bar{\delta}} &= -\frac{\alpha}{\bar{\delta}^2} \frac{[1-F(v_s)]^2}{F'(v_s)} < 0, & \frac{\partial p_r}{\partial \bar{\delta}} &= \frac{\mu}{(\mu+\bar{\delta})} \frac{\partial p_s}{\partial \bar{\delta}} < 0 \\ \frac{\partial p_s}{\partial \mu} &= 0, & \frac{\partial p_r}{\partial \mu} &= -\frac{\alpha}{(\mu+\bar{\delta})^2} \left[\frac{(1-F_r)^2}{F'_r} - \frac{\bar{\delta}}{\bar{\delta}} \frac{(1-F_s)^2}{F'_s} \right] < 0. \end{aligned}$$

The signs of the first three expressions are obvious whereas $\frac{\partial p_r}{\partial \mu}$ is negative if

$$\frac{(1-F_r)^2}{F'_r} - \frac{\bar{\delta}}{\bar{\delta}} \frac{(1-F_s)^2}{F'_s} > 0. \quad (21)$$

To show that (21) holds, focus on (20) and note that $\frac{1-F(v)}{F'(v)}$ decreases in v ; thus the term $\frac{1-F_r}{F'_r} - \frac{1-F_s}{F'_s}$ is negative, since $v_r^* > v_s^*$. Since (20) is positive, (21) must be also positive.

Reserve Values. Now we analyze $\text{sign} (\det B_j (x))$ for $x = \bar{\delta}, \mu$ and $j = r, s$.

– Since $\frac{\partial \Omega}{\partial \bar{\delta}} = 0$ we have

$$\det B_s (\bar{\delta}) = \frac{\partial \Delta_s}{\partial v_r} \frac{\partial p_r}{\partial \bar{\delta}} - \frac{\partial \Delta_r}{\partial v_r} \frac{\partial p_s}{\partial \bar{\delta}}.$$

Furthermore since $\frac{\partial p_s}{\partial \bar{\delta}} < \frac{\partial p_r}{\partial \bar{\delta}} < 0$ and $\frac{\partial \Delta_r}{\partial v_r} < \frac{\partial \Delta_s}{\partial v_r} < 0$ (see (18)) it follows that $\det B_s (\bar{\delta}) < 0$, hence $\frac{dv_s}{d\bar{\delta}} < 0$.

– Because $\frac{\partial p_s}{\partial \mu} = 0$ we have

$$\det B_s (\mu) = \frac{\partial \Delta_s}{\partial v_r} \frac{\partial p_r}{\partial \mu} + \frac{\partial \Omega}{\partial \mu} \left[\frac{\partial \Delta_s}{\partial v_r} - \frac{\partial \Delta_r}{\partial v_r} \right].$$

Since $\frac{\partial p_r}{\partial \mu} < 0$, $\frac{\partial \Omega}{\partial \mu} > 0$ and $\frac{\partial \Delta_r}{\partial v_r} < \frac{\partial \Delta_s}{\partial v_r} < 0$ it follows that $\det B_s (\mu)$ is positive and therefore $\frac{dv_s}{d\mu} > 0$.

– Recalling $\frac{\partial p_r}{\partial \bar{\delta}} = \frac{\mu}{\mu+\bar{\delta}} \frac{\partial p_s}{\partial \bar{\delta}} < 0$ and $\frac{\partial \Omega}{\partial \bar{\delta}} = 0$ it is easy to verify that

$$\det B_r (\bar{\delta}) = \frac{\partial p_s}{\partial \bar{\delta}} \left[\frac{\partial p_r}{\partial v_s} - \frac{\mu}{\mu+\bar{\delta}} \frac{\partial p_s}{\partial v_s} + \frac{\bar{\delta}}{\mu+\bar{\delta}} \frac{\partial \Omega}{\partial v_s} + \frac{\mu}{\mu+\bar{\delta}} \right].$$

Using (17), (16), (10) and (2) one can show that the expression inside the square brackets equals to

$$\frac{\mu}{\mu+\delta} \frac{F_s'^2 + F_s''(1-F_s)}{F_s'^2} + \frac{\alpha\theta'}{\mu+\delta} \int_{v_s}^{v_r} [1 - F(v)] dv + \frac{\mu\theta}{\mu+\delta} > 0.$$

Note that the first term is positive under Assumption 1 and the second term is positive since $\theta' > 0$ (see the proof of Lemma 1) and $v_r^* > v_s^*$. It follows that $\det B_r(\bar{\delta}) < 0$, hence $\frac{dv_r}{d\bar{\delta}} < 0$.

– Recalling $\frac{\partial p_s}{\partial \mu} = 0$ we obtain

$$\det B_r(\mu) = \frac{\partial \Omega}{\partial \mu} \left[\frac{\partial \Delta_r}{\partial v_s} - \frac{\partial \Delta_s}{\partial v_s} \right] + \frac{\partial \Delta_s}{\partial v_s} \frac{\partial p_r}{\partial \mu}.$$

The first term is positive since $\frac{\partial \Delta_s}{\partial v_s} < \frac{\partial \Delta_r}{\partial v_s} < 0$ (see (19)) and $\frac{\partial \Omega}{\partial \mu} > 0$. The second term is also positive since $\frac{\partial \Delta_s}{\partial v_s} < 0$ and $\frac{\partial p_r}{\partial \mu} < 0$. It follows that $\det B_r(\mu) > 0$, thus $\frac{dv_r}{d\mu} > 0$.

Prices. Totally differentiating p_j with respect to μ one obtains

$$\frac{dp_j}{d\mu} = \frac{\partial p_j}{\partial \mu} + \frac{\partial p_j}{\partial v_r} \frac{dv_r}{d\mu} + \frac{\partial p_j}{\partial v_s} \frac{dv_s}{d\mu}.$$

Recall that $\frac{\partial p_r}{\partial \mu} < \frac{\partial p_s}{\partial \mu} = 0$, $\frac{dp_j}{dv_r} \leq 0$, $\frac{dp_j}{dv_s} < 0$ and $\frac{dv_j}{d\mu} > 0$, hence $\frac{dp_j}{d\mu} < 0$.

To show $\frac{dp_j}{d\bar{\delta}} < 0$, recall that $p_j = v_j - \Omega$ in equilibrium. Differentiation with respect to $\bar{\delta}$ yields

$$\frac{dp_j}{d\bar{\delta}} = \frac{dv_j}{d\bar{\delta}} - \frac{\partial \Omega}{\partial v_r} \frac{dv_r}{d\bar{\delta}} - \frac{\partial \Omega}{\partial v_s} \frac{dv_s}{d\bar{\delta}},$$

which is negative since $\frac{dv_j}{d\bar{\delta}} < 0$ and $\frac{\partial \Omega}{\partial v_j} < 0$. \square

Proof of Proposition 7. Evaluating the integral in (8) we have

$$s(t) = \frac{\mu e^{-\alpha_s t} - \mu r(t)}{\mu - \alpha_s + \alpha_r}.$$

The expected time on the market TOM equals to $\int_0^\infty (r + s) dt$ whereas the density function γ is given by $-\frac{d(r+s)}{dt}$. Basic algebra reveals that TOM and γ are given by the expressions on display in Proposition 7. It is easy to verify that γ is positive and that $\int_0^\infty \gamma dt = -[r(t) + s(t)]|_0^\infty = 1$. To analyze the shape of γ note that

$$\gamma' = \frac{-\mu \alpha_s^2 e^{-\alpha_s t} + (\alpha_s - \alpha_r)(\mu + \alpha_r)^2 e^{-(\mu + \alpha_r)t}}{\mu - \alpha_s + \alpha_r},$$

where $\alpha_s - \alpha_r = \alpha [F(v_r^*) - F(v_s^*)] > 0$ since $v_r^* > v_s^*$. Notice that the denominator could be either positive or negative. It follows that

$$\begin{aligned} \text{If } \mu > \alpha_s - \alpha_r \text{ then } \gamma'(t) > 0 &\Leftrightarrow \frac{(\alpha_s - \alpha_r)(\mu + \alpha_r)^2}{\mu \alpha_s^2} > e^{(\mu + \alpha_r - \alpha_s)t}, \\ \text{If } \mu < \alpha_s - \alpha_r \text{ then } \gamma'(t) > 0 &\Leftrightarrow \frac{(\alpha_s - \alpha_r)(\mu + \alpha_r)^2}{\mu \alpha_s^2} < e^{(\mu + \alpha_r - \alpha_s)t}. \end{aligned}$$

First note that $\lim_{t \rightarrow \infty} \gamma' < 0$, i.e., γ is monotone decreasing for t large. Now evaluate $\lim_{t \rightarrow 0} \gamma$. Note that in the first line the exponential term is minimum when $t = 0$ whereas in the second line it is maximum when $t = 0$. Based on this observation one can demonstrate that $\gamma'(0) > 0$ if $\frac{\mu}{\alpha} > \frac{[1 - F(v_r^*)]^2}{F(v_r^*) - F(v_s^*)}$. Clearly if $\gamma'(0) > 0$ then γ first rises and then falls (hump-shape). Otherwise if $\gamma'(0) < 0$ it falls monotonically. \square

Proof of Proposition 8. Notice that $\frac{d\bar{p}(t)}{dt} = -\frac{dg(t)}{dt} (p_r^* - p_s^*)$. One can verify that

$$\frac{dg(t)}{dt} \propto \mu e^{-(\alpha_s + \alpha_r + \mu)t} > 0.$$

It follows that $\bar{p}' < 0$ since $p_r^* > p_s^*$. Finally note that

$$\frac{d\sigma^2}{dt} = (p_r^* - p_s^*) [g'(2\bar{p} - p_r^* - p_s^*) + 2g\bar{p}'].$$

Clearly $\frac{d\sigma^2}{dt}$ shares the sign of the expression in the square brackets, since $p_r^* > p_s^*$. One can verify that $\lim_{t \rightarrow 0} g(t) = 0$ and $\lim_{t \rightarrow \infty} g(t) = 1$ so that $\lim_{t \rightarrow 0} \bar{p}(t) = p_r^*$ and $\lim_{t \rightarrow \infty} \bar{p}(t) = p_s^*$. It follows that $\frac{d\sigma^2}{dt}$ is positive for t small and negative for t large because $g' > 0$ and $\bar{p}' < 0$. In other words σ^2 first rises and subsequently falls with t . \square

FIGURES

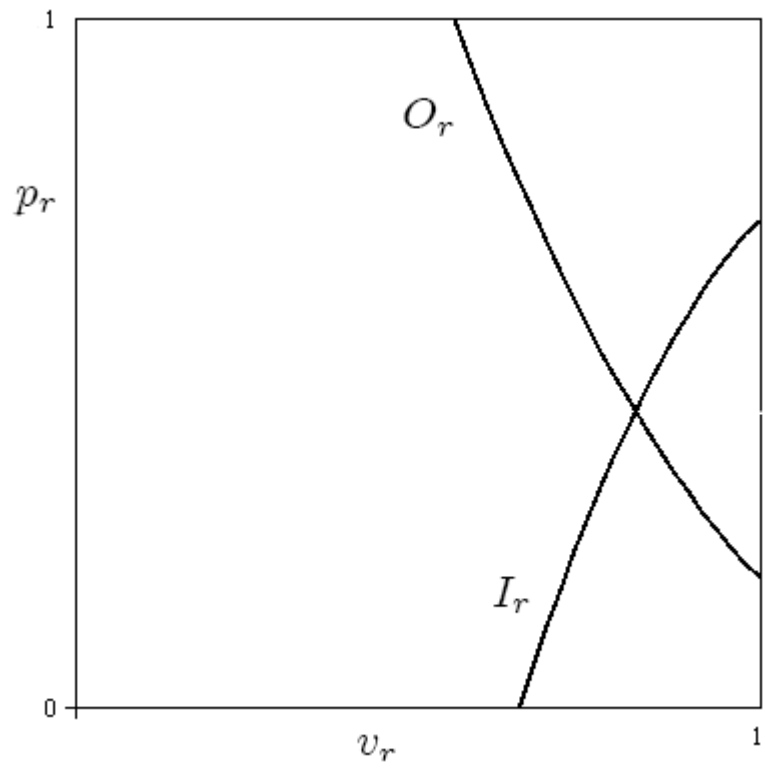


Figure 1a

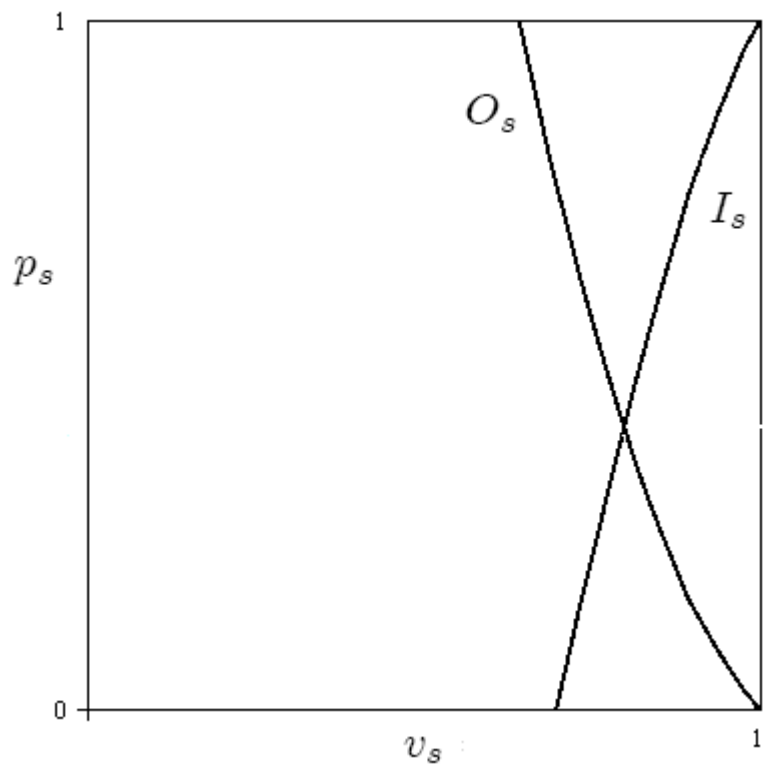


Figure 1b

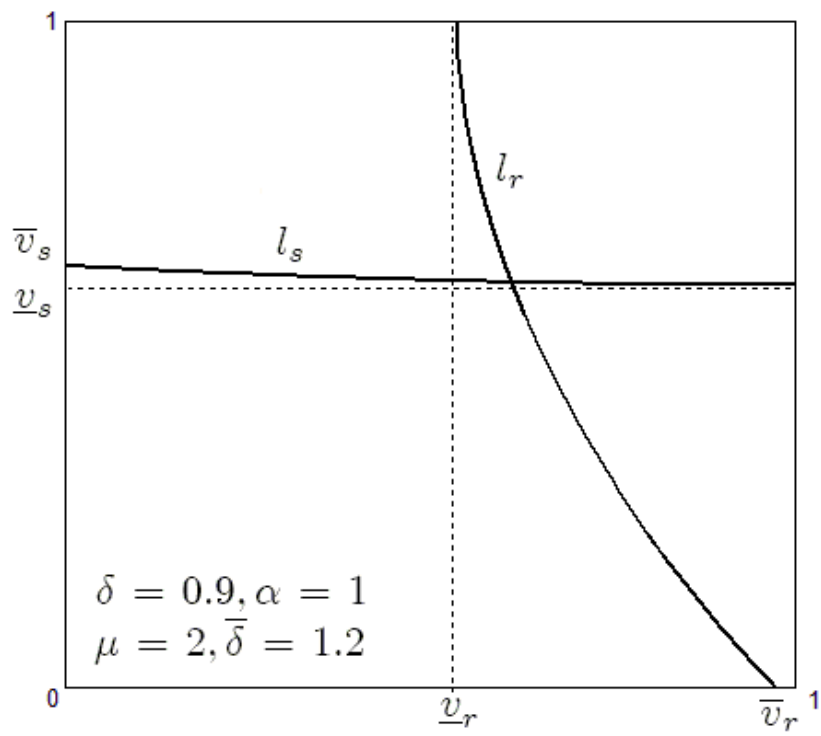


Figure 2a

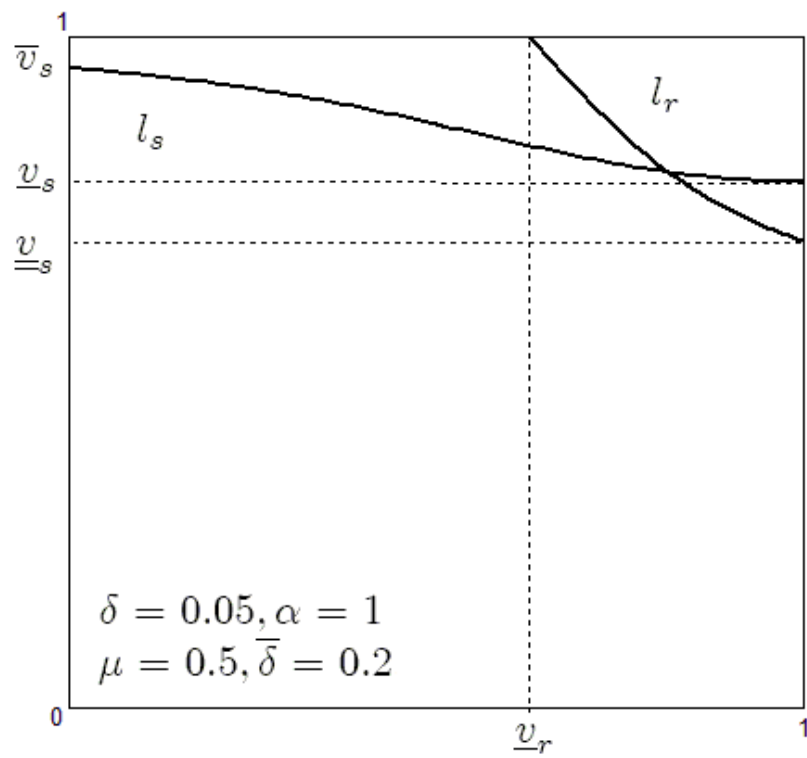


Figure 2b

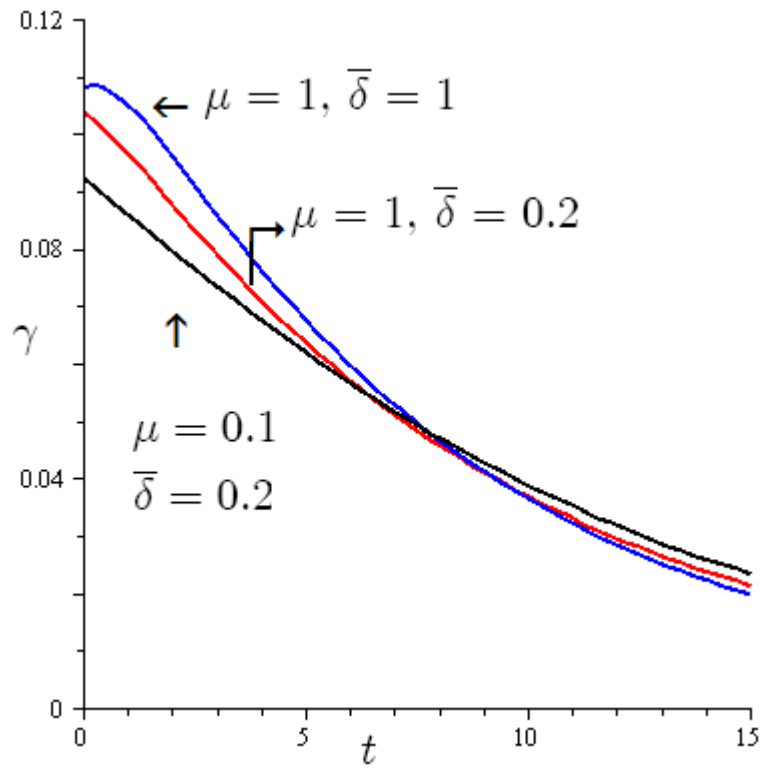


Figure 3a

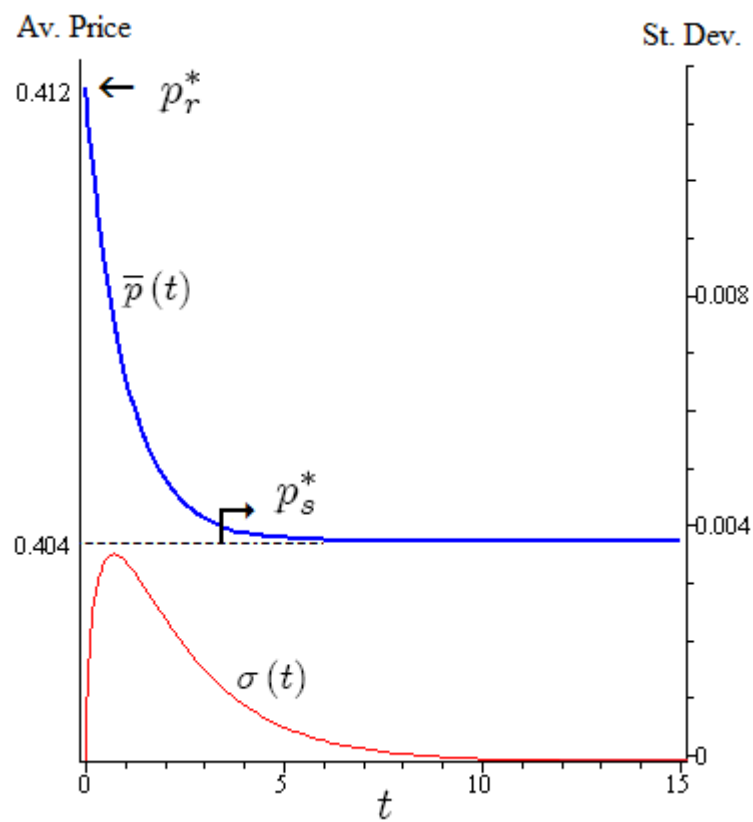


Figure 3b