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Citation for final published version:

Pennig, Ulrich 2014. Twisted K-theory and obstructions against positive scalar curvature metrics. *Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology* 14 (1) , pp. 47-71. 10.1017/is014003008jkt259

Publishers page: <http://dx.doi.org/10.1017/is014003008jkt259>

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TWISTED K -THEORY AND OBSTRUCTIONS AGAINST POSITIVE SCALAR CURVATURE METRICS

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ABSTRACT. We decompose $\theta(M)$, the twisted index obstruction to a positive scalar curvature metric for closed oriented manifolds with spin universal cover, into a pairing of a twisted K -homology with a twisted K -theory class and prove that $\theta(M)$ does not vanish if M is a closed orientable enlargeable manifold with spin universal cover.

Key Words: enlargeable manifold, index theory, Dirac operator, positive scalar curvature, twisted index obstruction, twisted K -theory

Mathematics Subject Classification (2010): 19L50, 19K56

1. INTRODUCTION

In [22] Rosenberg constructed an index obstruction $\alpha(M) \in KO_n(C_{\mathbb{R}}^*(\pi_1(M)))$ for closed spin manifolds M of dimension n , which vanishes if M admits a metric of positive scalar curvature. It takes values in the K -theory of the (maximal or reduced) real group C^* -algebra associated to the fundamental group and relies on the existence of a spin structure. Gromov, Lawson and Rosenberg conjectured that $\alpha(M)$ is the only obstruction to the existence of a psc metric if $\dim(M) \geq 5$. This was proven to be true in the simply-connected case by Stolz [27], but is false in general as was shown by Schick [24].

Stolz generalized $\alpha(M)$ to the case where M itself may not be spin, but its universal cover \widetilde{M} still is [26], [23, Theorem 1.7]. The new invariant $\theta(M) \in KO(C^*\gamma)$ takes values in the K -theory of a real C^* -algebra associated to a *twisted* version of the fundamental group accounting for the missing spin structure on M . The latter is a $\mathbb{Z}/2\mathbb{Z}$ -extension $\widehat{\pi}$ of $\pi = \pi_1(M)$, therefore we will use the notation $C^*(\widehat{\pi} \rightarrow \pi)$ instead of $C^*\gamma$.

The element $\alpha(M)$ can be expressed as the pairing of the Dirac class $[D] \in KO_n(M)$ with the KO -theory class $[\mathcal{V}] \in KO_0(C(M, C^*\pi))$ of the Mishchenko-Fomenko bundle. We show that the same is true for $\theta(M)$ if one switches to the *twisted* versions of K -homology and K -theory [7, 2, 14]. For simplicity we treat the complex analogues of the real invariants and refer to [11, Section 1.4] for a nice exposition on how the different indices are related. We identify $\theta(M)$ as the pairing of the twisted fundamental K -homology class $[D^S] \in KK(C(M, \mathbb{C}\ell(M)), \mathbb{C})$ with a twisted version of the Mishchenko-Fomenko bundle representing an element $[\mathcal{V}^S] \in KK(\mathbb{C}, C(M, \mathbb{C}\ell(M)) \otimes C^*(\widehat{\pi} \rightarrow \pi))$:

$$\theta(M) = \text{ind}(D_+^{\mathcal{V}}) = [\mathcal{V}^S] \otimes_{C(M, \mathbb{C}\ell(M))} [D^S] .$$

The Dixmier-Douady class of the twist in this case is the element $W_3(M) := \beta(w_2(M)) \in H^3(M; \mathbb{Z})$, i.e. the Bockstein of the second Stiefel-Whitney class. The superscript S denotes the twisted spinor bundle over M . We will use the language of bundle gerbes developed by Murray [19] and the C^* -algebraic versions of their modules [4, 17] to obtain a geometric description of the twisted K -group with coefficients in a C^* -algebra. Most features of index theory are preserved in the twisted case: Murray and Singer proved the analogue of the Atiyah-Singer index theorem in this context [21] and Carey and Wang proved the Thom isomorphism [6] (see also [7]).

As an application of our techniques, we show that $\theta(M)$ does not vanish for closed orientable enlargeable manifolds (Definition 4.1) with spin universal cover and thereby enhance a result of Hanke and Schick [11, 12]:

Theorem 5.7. *Let M be a closed smooth orientable even-dimensional manifold with $\dim(M) \geq 3$ and \widetilde{M} spin that is enlargeable in the sense of Definition 4.1. Then we have $\theta^{\max}(M) \neq 0$.*

Extending the suspension argument from [11] it is easy to drop the assumption about even-dimensionality. Gromov and Lawson showed that closed enlargeable manifolds M do not allow a metric of positive scalar curvature. They worked with finite covers in the definition of enlargeability in [9], but later generalized to infinite ones [10]. We also allow the covers $\bar{M} \rightarrow M$ to be *non-compact* as in [12]. In this generality, transfer arguments fail, but twisted Hilbert A -module bundles provide a way to circumvent this. This result is independent of the injectivity of the twisted Baum-Connes map.

Acknowledgments. The paper is an excerpt of the author's PhD thesis under supervision of Prof. Thomas Schick to whom the author would like to express his deep gratitude. Apart from that, he would like to thank Bernhard Hanke and Andreas Thom for many stimulating discussions, Max Karoubi and the referee for pointing out valuable references about twisted K -theory and the German National Academic Foundation for financial support.

Notation. A compact manifold without boundary will be called closed. Throughout the paper, M will denote a smooth closed orientable manifold and A will be a unital C^* -algebra if not stated otherwise. Moreover, \underline{F} will denote a trivial bundle with fiber F if the base space is clear. Whenever we have a surjective submersion $Y \rightarrow M$, the notation $Y^{[n]}$ will denote the n -fold fiber product over M .

2. TWISTED HILBERT A -MODULE BUNDLES

In this section we will discuss a way to represent classes in twisted K -theory with coefficients in a C^* -algebra A by twisted Hilbert A -module bundles. These are straightforward generalizations of bundle gerbe modules [4, 17]. For an introduction to Hilbert A -modules we refer the reader to [15]. We start by reviewing the geometric description of 2-cocycles via bundle gerbes [19, 20].

Definition 2.1. Let M be a smooth manifold and let $Y \rightarrow M$ be a surjective submersion. A *real* line bundle $L \rightarrow Y^{[2]}$ will be called a $\mathbb{Z}/2\mathbb{Z}$ -*bundle gerbe* (or simply *bundle gerbe* for short) if there exists a multiplication over $Y^{[3]}$, i.e. an isomorphism of line bundles $\mu: \pi_{12}^*L \otimes \pi_{23}^*L \rightarrow \pi_{13}^*L$, where $\pi_{ij}: Y^{[3]} \rightarrow Y^{[2]}$ denotes the canonical projections to the fiber product of the i th and j th factor, and such that over $Y^{[4]}$ the following diagram commutes:

$$\begin{array}{ccc} (\pi_{12}^*L \otimes \pi_{23}^*L) \otimes \pi_{34}^*L & \xlongequal{\quad} & \pi_{12}^*L \otimes (\pi_{23}^*L \otimes \pi_{34}^*L) \\ \mu \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes \mu \\ \pi_{13}^*L \otimes \pi_{34}^*L & \xrightarrow{\quad \mu \quad} \pi_{14}^*L \xleftarrow{\quad \mu \quad} & \pi_{12}^*L \otimes \pi_{24}^*L \end{array}$$

The bundle gerbe $\delta Q = \pi_1^*Q \otimes \pi_2^*Q^* \rightarrow Y^{[2]}$ associated to a line bundle $Q \rightarrow Y$ will be called the *trivial bundle gerbe*. Let $L \rightarrow Y^{[2]}$ be a bundle gerbe. A choice of a line bundle $Q \rightarrow Y$ together with an isomorphism of bundle gerbes $L \rightarrow \delta Q$ will be called a *trivialization* of L .

Let $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be two bundle gerbes and let $\pi_i: Y_1^{[2]} \times_M Y_2^{[2]} \rightarrow Y_i^{[2]}$ denote the projection. The exterior tensor product $L_1 \boxtimes L_2 = \pi_1^*L_1 \otimes \pi_2^*L_2$ is again a bundle gerbe. Each bundle gerbe L has a class $dd(L) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ canonically associated to it as explained in [17]. $dd(L)$ is called the *Dixmier-Douady class* of L and we summarize its properties in the following theorem proven in [17]:

Theorem 2.2. *Let $L \rightarrow Y^{[2]}$ and $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be bundle gerbes. The Dixmier-Douady class has the following properties:*

- a) $dd(L) = 0$ if and only if L is isomorphic over $Y^{[2]}$ to a trivial bundle gerbe.
- b) $dd(L_1 \boxtimes L_2) = dd(L_1) + dd(L_2)$. □

Definition 2.3. Let L be a bundle gerbe. A covariant derivative $\nabla^L: \Omega^0(L) \rightarrow \Omega^1(L)$ on L is called a *bundle gerbe connection* if the multiplication isomorphism $\mu: \pi_{12}^*L \otimes \pi_{23}^*L \rightarrow \pi_{13}^*L$ pulls it back to the canonical connection on the tensor product, i.e.

$$(1) \quad \mu^* \pi_{13}^* \nabla^L = \pi_{12}^* \nabla^L \otimes 1 + 1 \otimes \pi_{23}^* \nabla^L .$$

Remark 2.4. The proof for the existence of such connections given in [19] works with the obvious changes for $\mathbb{Z}/2\mathbb{Z}$ -bundle gerbes as well. Since the structure group is discrete in our case, every $\mathbb{Z}/2\mathbb{Z}$ -bundle gerbe connection is automatically flat.

The main example of bundle gerbes will arise from the following construction.

Definition 2.5. Let Γ be a Lie group (possibly discrete) and let $q: \widehat{\Gamma} \rightarrow \Gamma$ be a central $\mathbb{Z}/2\mathbb{Z}$ -extension of Γ . Let $P \rightarrow M$ be a principal Γ -bundle over a manifold M . The line bundle L_P associated to the principal $\mathbb{Z}/2\mathbb{Z}$ -bundle $\widehat{L}_P \rightarrow P^{[2]}$ with $\widehat{L}_P = \{(p_1, p_2, \widehat{g}) \in P^{[2]} \times \widehat{\Gamma} \mid p_1 q(\widehat{g}) = p_2\}$ is a bundle gerbe, which will be called the *lifting bundle gerbe* associated to the extension.

For details about this construction see [20, section 6.1], [17, section 2.2]. Since the principal $\mathbb{Z}/2\mathbb{Z}$ -bundle associated to any trivialization Q of L_P is a lift of P to a principal $\widehat{\Gamma}$ -bundle, we have

Lemma 2.6 ([19]). *The class $dd(L_P)$ represents the obstruction lifting $P \rightarrow M$ to a principal $\widehat{\Gamma}$ -bundle. The isomorphism classes of trivialization of L_P are in 1 : 1-correspondence with the possible lifts of P to a principal $\widehat{\Gamma}$ -bundle.*

Definition 2.7. Let M be a smooth manifold, A a C^* -algebra and let $Y \rightarrow M$ be a surjective submersion. Let $L \rightarrow Y^{[2]}$ be a bundle gerbe. A (right) Hilbert A -module bundle $E \rightarrow Y$ together with an A -linear action $\gamma: L \otimes \pi_2^* E \rightarrow \pi_1^* E$ is called a *twisted Hilbert A -module bundle* for L if the following associativity diagram commutes:

$$\begin{array}{ccc} (\pi_{12}^* L \otimes \pi_{23}^* L) \otimes \pi_3^* E & \xlongequal{\quad} & \pi_{12}^* L \otimes (\pi_{23}^* L \otimes \pi_3^* E) \\ \mu \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes \gamma \\ \pi_{13}^* L \otimes \pi_3^* E & \xrightarrow{\quad \gamma \quad} \pi_1^* E \xleftarrow{\quad \gamma \quad} & \pi_{12}^* L \otimes \pi_2^* E \end{array}$$

E will be called *finitely generated* and *projective*, if its fibers are finitely generated and projective as Hilbert A -modules. γ will be called the *twisting* of E .

Let E, E' be two twisted Hilbert A -module bundles for the same bundle gerbe L and denote the twistings by γ and γ' . A right A -linear map $f: E \rightarrow E'$ will be called a *morphism of twisted Hilbert A -module bundles* or (*twisted morphism* for short) if the following diagram commutes:

$$\begin{array}{ccc} L \otimes \pi_2^* E & \xrightarrow{\quad \gamma \quad} & \pi_1^* E \\ \downarrow \text{id}_L \otimes \pi_2^* f & & \downarrow \pi_1^* f \\ L \otimes \pi_2^* E' & \xrightarrow{\quad \gamma' \quad} & \pi_1^* E' \end{array}$$

For $A = \mathbb{C}$ a twisted Hilbert A -module bundle is the same as a bundle gerbe module [4]. Let $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be two bundle gerbes with $dd(L_1) = dd(L_2)$. A trivialization of $L_1^* \boxtimes L_2$ should be seen as a generalized morphism between L_1 and L_2 [28]. In particular, it induces a push-forward from twisted bundles for L_1 to twisted bundles for L_2 [17, Proposition 2 and 3]:

Lemma 2.8 ([17]). *Let $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be two bundle gerbes with $dd(L_1) = dd(L_2)$. Let $Q \rightarrow Y_1 \times_M Y_2$ be a trivialization of $L_1^* \boxtimes L_2 \rightarrow (Y_1 \times_M Y_2)^{[2]}$. If $E \rightarrow Y_1$ is a twisted bundle for L_1 and $\rho: Y_1 \times_M Y_2 \rightarrow Y_1$ denotes the canonical projection, then $\rho^* E \otimes Q$ descends to a twisted bundle $Q(E) \rightarrow Y_2$ for L_2 . In particular if $L_2 = \underline{\mathbb{C}}$ over $M^{[2]} = M$, i.e. Q is a trivialization of L_1^* , we obtain an (untwisted) Hilbert A -module bundle $Q(E) \rightarrow M$.*

Definition 2.9. Let $L \rightarrow Y^{[2]}$ be a bundle gerbe over a surjective submersion $Y \rightarrow M$ and let A be a unital C^* -algebra. Denote by $K_{L,A}^0(M)$ the Grothendieck group of isomorphism classes of finitely generated projective twisted Hilbert A -module bundles for L with respect to \oplus .

Given a principal $PO(n)$ -bundle $P \rightarrow M$ there is a bundle of matrix algebras $\mathcal{K} \rightarrow M$ with fiber $M_n(\mathbb{C})$ associated to it via the inclusion $PO(n) \rightarrow PU(n)$ and the adjoint action of $PU(n)$. Let $L_P \rightarrow P^{[2]}$ be the corresponding lifting bundle gerbe for the extension $O(n) \rightarrow PO(n)$. We

denote a point in the fiber $(L_P)_{(p_1, p_2)}$ by $[\widehat{g}, \lambda]$, where $\widehat{g} \in O(n)$ is a lift of the element $g \in PO(n)$ with $p_1 g = p_2$, $\lambda \in \mathbb{R}$ and $(\widehat{g}, -\lambda) \sim (-\widehat{g}, \lambda)$ determines $[\widehat{g}, \lambda]$.

There is a *canonical twisted vector bundle* $S \rightarrow P$ for L_P associated to this construction as follows: Let $S = P \times \mathbb{C}^n$ (see Remark 2.17) with the twisting given by

$$(2) \quad \gamma: L_P \otimes \pi_2^* S \rightarrow \pi_1^* S \quad ; \quad \gamma([\widehat{g}, \lambda] \otimes (p_2, v)) = (p_1, \lambda \widehat{g} v)$$

Given a twisted Hilbert A -module bundle $E \rightarrow Y$ for some bundle gerbe L , note that $S^* \boxtimes E := \pi_P^* S^* \otimes \pi_Y^* E$ over $P \times_M Y$ is a twisted Hilbert A -module bundle for $L_P^* \boxtimes L$.

Theorem 2.10. *Let L be a bundle gerbe, let A be a unital C^* -algebra. Let P , L_P and \mathcal{K} be as in the above paragraph. Assume that $dd(L) = dd(L_P)$ and let Q be a trivialization of $L_P^* \boxtimes L$, then*

$$\kappa_Q: K_{L,A}^0(M) \rightarrow K_0(C(M, \mathcal{K}) \otimes A) \quad ; \quad [E] \mapsto [C(M, Q(S^* \boxtimes E))]$$

is a well-defined isomorphism.

Proof. Let V be the typical fiber of E . It is a finitely generated projective right Hilbert A -module, therefore $(\mathbb{C}^n)^* \otimes V$ is a finitely generated projective right Hilbert $M_n(\mathbb{C}) \otimes A$ -module in a canonical way. Thus, $S^* \boxtimes E$ is a bundle of finitely generated projective right Hilbert $M_n(\mathbb{C}) \otimes A$ -modules over $P \times_M Y$. When descending this bundle as in [4, Proposition 4.3], we first obtain bundles $(\mathbb{C}^n)^* \otimes E_i \rightarrow U_i$ over an open cover $U_i \subset M$. The transitions over double intersections act via the canonical action of $O(n) \subset U(n)$ on the first tensor factor and in an A -linear way on the second tensor factor. Therefore, $C(M, Q(S^* \boxtimes E))$ carries an action of $C(M, \mathcal{K} \otimes A)$ and the fiberwise $M_n(\mathbb{C}) \otimes A$ -valued inner product on $S^* \boxtimes E$ turns $C(M, Q(S^* \boxtimes E))$ into a right Hilbert $C(M, \mathcal{K} \otimes A)$ -module. Since V was finitely generated and projective, the same holds true for $C(M, Q(S^* \boxtimes E))$.

Let σ be a projection-valued section of the bundle $\mathcal{K} \otimes M_k(A) \rightarrow M$. Let $\pi: P \rightarrow M$ be the bundle projection, then $\sigma \circ \pi$ yields a section of $\pi^* \mathcal{K} \otimes M_k(A)$. This bundle has a canonical trivialization, such that $\sigma \circ \pi$ yields a continuous map $\widehat{\sigma}: P \rightarrow M_n(\mathbb{C}) \otimes M_k(A)$ with the property $\widehat{\sigma}(pg) = \widehat{g}^{-1} \sigma(p) \widehat{g}$. Let $E_\sigma = \{(p, v) \in P \times (\mathbb{C}^n \otimes A^k) \mid \widehat{\sigma}(p)v = v\}$. This is a finitely generated, projective twisted Hilbert A -module bundle for L_P . The map $[\sigma] \mapsto [E_\sigma]$ is a homomorphism $K_0(C(M, \mathcal{K}) \otimes A) \rightarrow K_{L_P, A}^0(M)$. The composition with the isomorphism $K_{L_P, A}^0(M) \rightarrow K_{L, A}^0(M)$ induced by Q^* yields the inverse of the above construction. \square

Lemma 2.11. *Consider principal $PO(n_i)$ -bundles P_i for $i \in \{1, 2\}$ and let \mathcal{K}_i be the associated bundles of complex matrix algebras. We have $dd(L_{P_1}) = dd(L_{P_2})$ if and only if $\mathcal{K}_1 \otimes \mathcal{K}_2 \cong \text{End}(\mathcal{V}) \otimes \mathbb{C}$ for a real vector bundle $\mathcal{V} \rightarrow M$. The isomorphism classes of trivializations of $L_{P_1}^* \boxtimes L_{P_2}$ are in 1 : 1-correspondence with the isomorphism classes of possible \mathcal{V} 's.*

Proof. Let $O(n_1) \otimes O(n_2)$ be the quotient of the product $O(n_1) \times O(n_2)$ by the diagonal $Z/2\mathbb{Z}$ -action. If $dd(L_{P_1}) = dd(L_{P_2})$, then $P_1 \times_M P_2$ lifts to a principal $O(n_1) \otimes O(n_2)$ -bundle P and $\mathcal{V} = P \times_\rho (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ satisfies the condition, where ρ is the standard representation. The isomorphism

class of \mathcal{V} only depends on the one of P and the choices of possible P 's are in 1 : 1-correspondence with the trivializations of $L_{P_1}^* \boxtimes L_{P_2}$.

On the other hand, if \mathcal{V} exists, let P be its principal $O(n_1 n_2)$ -bundle. $P/(\mathbb{Z}/2\mathbb{Z})$ reduces to the principal $PO(n_1) \times PO(n_2)$ -bundle $P_1 \times_M P_2$ and the pullback of $P \rightarrow P/(\mathbb{Z}/2\mathbb{Z})$ via $P_1 \times_M P_2 \rightarrow P/(\mathbb{Z}/2\mathbb{Z})$ is a lift of $P_1 \times_M P_2$ to a principal $O(n_1) \otimes O(n_2)$ -bundle, which shows that $dd(L_{P_1}) = dd(L_{P_2})$. \square

Remark 2.12. Note that $C(M, \mathcal{V} \otimes \mathbb{C}) \otimes A$ provides a Morita equivalence bimodule between $C(M, \mathcal{K}_1) \otimes A$ and $C(M, \mathcal{K}_2) \otimes A$ and $\psi_{\mathcal{V}}: K_0(C(M, \mathcal{K}_1) \otimes A) \rightarrow K_0(C(M, \mathcal{K}_2) \otimes A)$ with $\psi_{\mathcal{V}}([X]) = [X \otimes_{C(M, \mathcal{K}_1) \otimes A} C(M, \mathcal{V} \otimes \mathbb{C}) \otimes A]$ is the corresponding isomorphism on K -theory.

Let L be a bundle gerbe. Let \mathcal{K}_i, P_i for $i \in \{1, 2\}$ be as in Lemma 2.11. Suppose that $dd(L) = dd(L_{P_1}) = dd(L_{P_2})$. Trivializations Q_i of $L^* \boxtimes L_{P_i}$ induce a trivialization of $L_{P_1}^* \boxtimes L_{P_2}$, since $Q_1^* \boxtimes Q_2$ over $P_1 \times_M P \times_M P \times_M P_2$ descends to a line bundle Q_{12} over $P_1 \times_M P_2$ in a way compatible with the actions of L_{P_i} . The trivialization Q_{12} corresponds to a real vector bundle \mathcal{V}_{12} as described in Lemma 2.11. Let $\kappa_{Q_i}([E]) = [C(M, Q_i(S_i^* \boxtimes E))]$ be the homomorphism from Theorem 2.10 associated to the data Q_i and the canonical bundle S_i . A bundle gerbe L provides a canonical trivialization of $L^* \boxtimes L$ (in particular, if P is a principal $PO(n)$ -bundle, then the principal $PO(n) \times PO(n)$ -bundle $P^{[2]}$ has a canonical lift to a $O(n) \otimes O(n)$ -bundle). Let $\kappa_{L_{P_i}}$ be the associated isomorphism from Theorem 2.10. Then the the following diagram of isomorphisms commutes:

$$\begin{array}{ccccc}
& & K_0(C(M, \mathcal{K}_1) \otimes A) & \xleftarrow{\kappa_{L_{P_1}}} & K_{L_{P_1}, A}^0(M) \\
& \nearrow^{\kappa_{Q_1}} & \downarrow \psi_{\mathcal{V}} & & \downarrow Q_{12} \\
K_{L, A}^0(M) & & K_0(C(M, \mathcal{K}_2) \otimes A) & \xleftarrow{\kappa_{L_{P_2}}} & K_{L_{P_2}, A}^0(M) \\
& \searrow_{\kappa_{Q_2}} & & &
\end{array}$$

2.1. The twisted Mishchenko-Fomenko bundle. The obstruction $\alpha(M)$ is built out of the Dirac operator on M and a certain bundle of Hilbert $C^*(\pi_1(M))$ -modules called the Mishchenko-Fomenko bundle. In the twisted case the algebra as well as the bundle itself have to be replaced by twisted versions.

Definition 2.13. Let $1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \hat{\pi} \rightarrow \pi \rightarrow 1$ be a central $\mathbb{Z}/2\mathbb{Z}$ -extension of a discrete group π . Let $C_r^*(\hat{\pi})$ be the reduced group C^* -algebra associated to $\hat{\pi}$, likewise let $C_{\max}^*(\hat{\pi})$ be its maximal group C^* -algebra. Let $e \in \mathbb{Z}/2\mathbb{Z}$ be the non-trivial element and observe that $q = \frac{1}{2}(1 - e) \in C_{r/\max}^*(\hat{\pi})$ is a central projection. Let $C_r^*(\hat{\pi} \rightarrow \pi) = q C_r^*(\hat{\pi}) = q C_r^*(\hat{\pi}) q$ and define $C_{\max}^*(\hat{\pi} \rightarrow \pi)$ analogously. It is straightforward to check that these are in fact C^* -algebras, which we will call the *reduced*, respectively *maximal twisted group C^* -algebra* [26, Definition 8.1].

Remark 2.14. An extension in the above definition can be classified by a group 2-cocycle $c_{\hat{\pi}} \in H_{\text{gr}}^2(\pi, \mathbb{Z}/2\mathbb{Z})$, which yields an alternative description of $C_r^*(\hat{\pi} \rightarrow \pi)$ using completions of the

twisted group ring $\mathbb{C}[\pi, c_{\widehat{\pi}}]$ [5, Definition 2.1 for $A = \mathbb{C}$]. The algebra $C_{\max}^*(\widehat{\pi} \rightarrow \pi)$ has the universal property that any projective representation $\rho: \pi \rightarrow U(H)$ for the cocycle $c_{\widehat{\pi}}$ extends to a $*$ -homomorphism $C_{\max}^*(\widehat{\pi} \rightarrow \pi) \rightarrow B(H)$.

Let M be a closed smooth oriented manifold with $\dim(M) \geq 3$, such that its universal cover \widetilde{M} carries a spin structure. Denote by P_{SO} the oriented frame bundle of M and let $\pi = \pi_1(M)$. Let $P_{SO(\widetilde{M})}$ be the frame bundle of the universal cover. Consider the exact sequence

$$(3) \quad \pi_2(\widetilde{M}) \rightarrow \pi_1(SO(n)) \rightarrow \pi_1(P_{SO(\widetilde{M})}) \rightarrow 1$$

The map $\pi_2(\widetilde{M}) \rightarrow \pi_1(SO(n))$ sends $f: S^2 \rightarrow \widetilde{M}$ to the transition map $\varphi_f: S^1 \rightarrow SO(n)$ obtained from the pullback $f^*P_{SO(\widetilde{M})}$. If \widetilde{M} carries a spin structure, then φ_f factors through $S^1 \rightarrow \text{Spin}(n)$ and is therefore nullhomotopic. Comparing (3) with the exact sequence

$$\pi_2(M) \rightarrow \pi_1(SO(n)) \rightarrow \pi_1(P_{SO}) \rightarrow \pi_1(M) \rightarrow 1$$

for the bundle $P_{SO} \rightarrow M$, a diagram chase shows that $\pi_2(M) \rightarrow \pi_1(SO(n))$ is also trivial in this case and we obtain a central $\mathbb{Z}/2\mathbb{Z}$ -extension [26]:

$$(4) \quad 1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_1(P_{SO}) \rightarrow \pi \rightarrow 1 .$$

Definition 2.15. The lifting bundle gerbe $L_{\widehat{\pi}} \rightarrow \widetilde{M}^{[2]}$ associated to the above central extension (4) is called the *Mishchenko-Fomenko bundle gerbe*.

Note that a trivialization of $L_{\widehat{\pi}} \rightarrow \widetilde{M}^{[2]}$ as a line bundle is given by a split $\pi \rightarrow \pi_1(P_{SO}) = \widehat{\pi}$. After applying this trivialization, the bundle gerbe multiplication over points $\tilde{m}_1, \tilde{m}_2, \tilde{m}_3$ with $\tilde{m}_2 = \tilde{m}_1 g_1$, $\tilde{m}_3 = \tilde{m}_2 g_2$ for some $g_1, g_2 \in \pi$ is transformed to $\mu((\tilde{m}_1, \tilde{m}_2, z_1), (\tilde{m}_2, \tilde{m}_3, z_2)) = (\tilde{m}_1, \tilde{m}_3, z_1 z_2 c_{\widehat{\pi}}(g_1, g_2))$ for some cocycle $c_{\widehat{\pi}}: \pi \times \pi \rightarrow \mathbb{Z}/2\mathbb{Z}$ representing the extension in the group $H^2(\pi, \mathbb{Z}/2\mathbb{Z})$ and $z_i \in \mathbb{R}$.

Definition 2.16. Let $\widehat{\pi} = \pi_1(P_{SO})$ and let $1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \widehat{\pi} \rightarrow \pi \rightarrow 1$ be the extension described above. The canonical twisted Hilbert A -module bundle for $A = C_{\max}^*(\widehat{\pi} \rightarrow \pi)$ given by $\mathcal{V}_{\max} = \widetilde{M} \times C_{\max}^*(\widehat{\pi} \rightarrow \pi)$ is called the (*maximal*) *twisted Mishchenko-Fomenko bundle*. Similarly, we define $\mathcal{V}_{\text{red}} = \widetilde{M} \times C_r^*(\widehat{\pi} \rightarrow \pi)$. In both cases the twisting is given as in (2).

Remark 2.17. Although \mathcal{V}_{red} is trivial as a bundle over \widetilde{M} , the twisting provides extra structure that replaces the associated bundle construction, which does not make sense in the twisted case. This is analogous to viewing a vector bundle E with fiber V and frame bundle P_{SO} as the equivariant vector bundle $P_{SO} \times V$ over P_{SO} and keeping track of the action.

Let $E \rightarrow \widetilde{M}$ be a twisted Hilbert A -module bundle for $L_{\widehat{\pi}}$. Trivializing the bundle gerbe $L_{\widehat{\pi}}$ (and thereby fixing a cocycle $c_{\widehat{\pi}}$) turns the twisting into a collection of bundle maps that are on the fibers given by $\gamma^g: E_{\tilde{m}} \rightarrow E_{\tilde{m}g^{-1}}$ with the property that $\gamma^g \circ \gamma^h = c_{\widehat{\pi}}(g, h)\gamma^{gh}$, i.e. it behaves like a projective representation.

Now suppose that E is smooth and comes equipped with a twisted connection ∇ . Fix $\tilde{m} \in \widetilde{M}$, let m be its image in M and let $c: I \rightarrow \widetilde{M}$ be a smooth curve in \widetilde{M} starting at \tilde{m} . Let $\mathcal{P}_c: I \times E_{\tilde{m}} \rightarrow E$ be the parallel transport with respect to ∇ . Let $\hat{g} \in \hat{\pi}$, let $g \in \pi$ be its image under the projection. Observe that \mathcal{P}_c is equivariant in the sense that $\mathcal{P}_{cg^{-1}}(t, \hat{g} \cdot v) = \hat{g} \cdot \mathcal{P}_c(t, v)$ for $t \in I$, $v \in E$. Let $\tau: S^1 \rightarrow M$ be a smooth loop in M at m representing an element $h \in \pi = \pi_1(M)$, let $\bar{\tau}: I \rightarrow \widetilde{M}$ be the lift of τ to \tilde{m} . Note that the endpoint of $\bar{\tau}$ is $\tilde{m}h^{-1}$. Define $\mathcal{P}(\tilde{m}, \tau)(v) := \mathcal{P}_{\bar{\tau}}(1, v)$ and let $\text{hol}(\tau, \tilde{m}): E_{\tilde{m}} \rightarrow E_{\tilde{m}}$ be given by $\text{hol}(\tau, \tilde{m}) = \gamma^{h^{-1}} \circ \mathcal{P}(\tilde{m}, \tau)$. It satisfies

$$\begin{aligned} c_{\hat{\pi}}(g, h)^{-1} \text{hol}(\bar{m}, \sigma * \tau) &= c_{\hat{\pi}}(g, h)^{-1} \gamma^{(gh)^{-1}} \mathcal{P}(\bar{m}, \sigma * \tau) = \gamma^{h^{-1}} \circ \gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}g^{-1}, \tau) \circ \mathcal{P}(\bar{m}, \sigma) \\ &= \gamma^{h^{-1}} \circ \left(\gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}g^{-1}, \tau) \circ \gamma^g \right) \circ \gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}, \sigma) \\ &= \gamma^{h^{-1}} \circ \mathcal{P}(\bar{m}, \tau) \circ \gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}, \sigma) = \text{hol}(\bar{m}, \tau) \circ \text{hol}(\bar{m}, \sigma), \end{aligned}$$

which proves that it is a projective representation of π on $E_{\tilde{m}}$ with respect to $c_{\hat{\pi}}$.

Definition 2.18. We will call $\text{hol}(\cdot, \tilde{m})$ the *projective holonomy of ∇ at \tilde{m}* .

3. INDEX THEORY ON TWISTED BUNDLES

3.1. The Chern Character. Twisted versions of the Chern character have been developed in [4, Section 6], [21, Section 3.2]. Crucial in both cases is the idea that the endomorphism bundle $\text{End}(F) \rightarrow Y$ of a twisted vector bundle F for a bundle gerbe L descends to a vector bundle $\text{end}(F)$ over M . Let ∇^F be a twisted connection on F . Since we deal with the case of real bundle gerbes, where curving forms do not matter, the curvature descends to a form $\Omega_F \in \Omega^2(M, \text{end}(F))$.

Definition 3.1. The *twisted Chern character* $\text{ch}(F)$ is the class

$$(5) \quad \text{ch}(F) = \text{tr} \left(\exp \left(\frac{i\Omega_F}{2\pi} \right) \right) \in H^{\text{even}}(M, \mathbb{R}).$$

It induces a group homomorphism $\text{ch}: K_{L, \mathbb{C}}^0(M) \rightarrow H^{\text{even}}(M, \mathbb{R})$. If L_1, L_2 are two bundle gerbes with $dd(L_1) = dd(L_2)$ and Q is a trivialization of $L_1^* \boxtimes L_2$, then

$$\begin{array}{ccc} K_{L_1, \mathbb{C}}^0(M) & \xrightarrow{\text{ch}} & H^{\text{even}}(M, \mathbb{R}) \\ \downarrow Q & & \uparrow \\ K_{L_2, \mathbb{C}}^0(M) & \xrightarrow{\text{ch}} & H^{\text{even}}(M, \mathbb{R}) \end{array}$$

commutes, since Q is a flat line bundle.

Choose a lifting bundle gerbe L_P with $dd(L) = dd(L_P)$. The Künneth theorem and the canonical isomorphism $K_{L_P, \mathbb{C}}^0(M) \rightarrow K^0(C(M, \mathcal{K}))$ allows us to extend the above definition to a $K_0(A) \otimes \mathbb{R}$ -valued Chern character via

$$\text{ch}: K_{L, A}^0(M) \rightarrow K_{L_P, \mathbb{C}}^0(M) \otimes K_0(A) \otimes \mathbb{R} \rightarrow H^{\text{even}}(M, K_0(A) \otimes \mathbb{R}).$$

The first map is obtained from Theorem 2.10 using a trivialization Q of $L^* \boxtimes L_P$. As we have seen in Remark 2.12, the result of ch does not depend on the choice of P and L_P .

Lemma 3.2. *Let $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be bundle gerbes. The following diagram commutes:*

$$\begin{array}{ccc} K_{L_1, A}^0(M) \times K_{L_2, \mathbb{C}}^0(M) & \xrightarrow{\boxtimes} & K_{L_1 \boxtimes L_2, A}^0(M) \\ \downarrow \text{ch} \times \text{ch} & & \downarrow \text{ch} \\ H^{\text{even}}(M, K_0(A) \otimes \mathbb{R}) \times H^{\text{even}}(M, \mathbb{R}) & \xrightarrow{\wedge} & H^{\text{even}}(M, K_0(A) \otimes \mathbb{R}) \end{array}$$

Proof. For $A = \mathbb{C}$ the proof is analogous to the untwisted case. For arbitrary A , choose a principal $PO(n)$ -bundle P and a trivialization Q of $L_1^* \boxtimes L_P$ as in Theorem 2.10. Note that Q and the canonical trivialization of $L_2^* \boxtimes L_2$ induce a trivialization of $(L_1 \boxtimes L_2)^* \boxtimes (L_P \boxtimes L_2)$. The statement is a consequence of the following commutative diagram

$$\begin{array}{ccc} K_{L_1, A}^0(M) \times K_{L_2, \mathbb{C}}^0(M) & \longrightarrow & K_{L_P, \mathbb{C}}^0(M) \otimes K_0(A) \otimes \mathbb{R} \times K_{L_2, \mathbb{C}}^0(M) \\ \downarrow & & \downarrow \\ K_{L_1 \boxtimes L_2, A}^0(M) & \longrightarrow & K_{L_P \boxtimes L_2, \mathbb{C}}^0(M) \otimes K_0(A) \otimes \mathbb{R} \end{array}$$

in which the vertical maps are given by tensor products and the horizontal ones are the decomposition maps from the Künneth theorem. \square

3.1.1. Chern character and traces. If A comes equipped with a trace τ , which we will assume for the rest of this section, there is a more direct approach to the Chern character, which was explored in the untwisted case in [25, Section 4]. If V is a finitely generated projective Hilbert A -module, then we have $\text{End}(V) = \mathcal{K}(V, V) \cong V \otimes_A \mathcal{K}(V, A)$, where $\mathcal{K}(V, W)$ denotes the compact adjointable A -linear operators. Setting $\tau_V(v \otimes T) = \tau(T(v))$ on elementary tensors, extends the trace to $\text{End}(V)$.

Let L be a bundle gerbe and E be a finitely generated projective twisted Hilbert A -module bundle for L . As was already noted above, $\text{End}(E)$ descends to a bundle of C^* -algebras $\text{end}(E)$ over M . Extending τ to the fibers yields a linear map $\Omega^n(M, \text{end}(E)) \rightarrow \Omega^n(M)$ on $\text{end}(E)$ -valued forms, which we will again denote by τ .

Definition 3.3. Let M be a closed smooth manifold, L be a bundle gerbe and E be a twisted Hilbert A -module bundle for L equipped with a twisted connection ∇^E that has curvature $\Omega_E \in \Omega^2(M, \text{end}(E))$. As in [25, Lemma 4.2] it follows that

$$\tau \left(\exp \left(\frac{i\Omega_E}{2\pi} \right) \right) \in \Omega^{\text{even}}(M)$$

is a closed form, which is independent of the choice of twisted connection. Its cohomology class $\text{ch}_\tau(E)$ is called the τ -Chern character of E .

To explain the connection between ch from Definition 3.1 and ch_τ we need the following concept.

Definition 3.4. Let V be a finitely generated projective right Hilbert A -module. Its *dimension* is defined to be $\dim_\tau(V) = \tau_V(\text{id}_V)$, where τ_V . If $V \cong tA^n$ for some projection $t \in M_n(A) = M_n(\mathbb{C}) \otimes A$, then $\dim_\tau(V) = (\text{tr} \otimes \tau)(t)$. \dim_τ yields a well-defined map $K_0(A) \rightarrow \mathbb{R}$.

Theorem 3.5. Let L be a bundle gerbe and let E be a twisted Hilbert A -module bundle for L . Then we have

$$(6) \quad \dim_\tau(\text{ch}(E)) = \text{ch}_\tau(E) \in H^{\text{even}}(M, \mathbb{R}) .$$

Proof. First observe that ch_τ is still multiplicative, in the sense that for a twisted Hilbert A -module bundle E and a twisted vector bundle F we have $\text{ch}_\tau(E \boxtimes F) = \text{ch}_\tau(E) \cup \text{ch}_\tau(F)$. In fact, this property rests only on the fact that the canonical tensor product connection $\nabla^{E \boxtimes F}$ has curvature $\Omega_{E \boxtimes F} = \Omega_E \otimes \text{id} + \text{id} \otimes \Omega_F \in \Omega^2(M, \text{end}(E) \otimes \text{end}(F))$ together with the calculation

$$\tau \left(\exp \left(\frac{i \Omega_{E \boxtimes F}}{2\pi} \right) \right) = \tau \left(\exp \left(\frac{i \Omega_E}{2\pi} \right) \right) \wedge \tau \left(\exp \left(\frac{i \Omega_F}{2\pi} \right) \right)$$

which is proven as in the untwisted case.

Choose P and a trivialization Q as in Theorem 2.10 and let F be a twisted vector bundle for L_P . Let V be a finitely generated projective Hilbert A -module. Observe that for the trivial bundle \underline{V} we have $\text{ch}_\tau(\underline{V}) = \dim_\tau(V)$. Thus,

$$\text{ch}_\tau(F \otimes V) = \text{ch}_\tau(F \boxtimes \underline{V}) = \text{ch}_\tau(F) \cup \text{ch}_\tau(\underline{V}) = \dim_\tau(V) \text{ch}_\tau(F) .$$

Likewise we have $\text{ch}(F \otimes V) = \text{ch}(F) \otimes [V] \in H^{\text{even}}(M, K_0(A) \otimes \mathbb{R})$. Therefore the following diagram commutes

$$\begin{array}{ccccc} K_{L_P, \mathbb{C}}^0(M) \otimes K_0(A) & \longrightarrow & K_0(C(M, \mathcal{K} \otimes A)) & \xrightarrow{\rho_Q^{-1}} & K_{L, A}^0(M) \\ & \searrow \text{ch} \otimes \dim_\tau & & \swarrow \text{ch}_\tau & \\ & & H^{\text{even}}(M, \mathbb{R}) & & \end{array}$$

where the first horizontal map is given by the canonical identification $K_{L_P, \mathbb{C}}^0(M) \cong K_0(C(M, \mathcal{K}))$ and the tensor product of modules. The statement follows if ch_τ vanishes on the second summand in the Künneth decomposition. We can identify $K_{L_P, \mathbb{C}}^1(M)$ with $K_{\pi^* L_P, \mathbb{C}}^0(\mathbb{R} \times M)$. Likewise we can represent classes in $K_1(A)$ as compactly supported virtual Hilbert A -module bundles over \mathbb{R} . If $F = (F_+, F_-, \varphi)$ represents an element in $K_{\pi^* L_P, \mathbb{C}}^0(\mathbb{R} \times M)$ and $W = (W_+, W_-, \psi)$ an element in $K_0(C_0(\mathbb{R}, A))$, then the graded tensor product $F \widehat{\boxtimes} W$ yields an element in $K_{\pi^* L_P, A}^0(\mathbb{R}^2 \times M)$ and Bott periodicity maps it to $b([F \widehat{\boxtimes} W]) \in K_{L_P, A}^0(M)$. It is a consequence of the multiplicativity that $\text{ch}_\tau(X) = \text{ch}_\tau(b(X)) \cup e$, where $e \in H_c^2(\mathbb{R}^2, \mathbb{R}) \cong \mathbb{R}$ denotes a generator and X is a triple

representing a class in $K_{\pi^*L_P,A}^0(\mathbb{R}^2 \times M)$. Thus,

$$\mathrm{ch}_\tau(b(F \boxtimes W)) \cup e = \mathrm{ch}_\tau(F \boxtimes W) = \mathrm{ch}(F) \cup \mathrm{ch}_\tau(W) = 0$$

since $\mathrm{ch}_\tau(W) \in H_c^{\mathrm{even}}(\mathbb{R}, \mathbb{R}) = 0$. \square

3.2. Local C^* -algebras and twisted bundles. A *local C^* -algebra* is a $*$ -algebra B equipped with a pre- C^* -norm such that $M_n(B)$ is closed under holomorphic functional calculus for all $n \in \mathbb{N}$. If $\mathcal{B} \rightarrow M$ is a locally trivial bundle with fiber B and structure group $\mathrm{Aut}(B)$ equipped with the point norm topology, then the section algebra $C(M, \mathcal{B})$ is again a local C^* -algebra as can be checked directly from the definitions using the naturality of holomorphic functional calculus.

A right B -module is called *finitely generated* and *projective* if it is isomorphic to one of the form tB^n for a projection $t \in M_n(B)$. The groups $K_i(B)$, $i \in \{0, 1\}$ are now defined as in [3]. If A is the completion of B , then $K_i(B) \cong K_i(A)$. Twisted B -module bundles are defined just as in 2.7 and yield analogous groups $K_{L,B}^0(M)$ and $K_{L,B}^1(M) := K_{\pi_M^*L,B}^0(\mathbb{R}^2 \times M)$. Given a twisted B -module bundle \mathcal{E} , the fiberwise inner product $\mathcal{E} \otimes_B A$ is a twisted Hilbert A -module bundle. Thus we get a homomorphism $K_{L,B}^0(M) \rightarrow K_{L,A}^0(M)$

Let S be as in Theorem 2.10. The exterior tensor product $S^* \boxtimes \mathcal{E}$ involves no completion. Therefore the proof of that theorem still works and gives $K_{L,B}^0(M) \cong K_0(C(M, \mathcal{K} \otimes B))$, which also proves $K_{L,B}^0(M) \cong K_{L,A}^0(M)$. The Bott map $b: K_{L,B}^0(M) \rightarrow K_{\pi_M^*L,B}^0(\mathbb{R}^2 \times M)$ is also still well-defined and a comparison with the corresponding map for its completion shows that b still is an isomorphism. Likewise, there is a Künneth homomorphism

$$K_{L_P,\mathbb{C}}^0(M) \otimes K_0(B) \otimes \mathbb{R} \oplus K_{L_P,\mathbb{C}}^1(M) \otimes K_1(B) \otimes \mathbb{R} \rightarrow K_0(C(M, \mathcal{K} \otimes B)) \otimes \mathbb{R}$$

that just involves algebraic tensor products in the fibers. Comparison with the completion reveals this to be an isomorphism as well. This enables us to define $\mathrm{ch}: K_{L,B}^0(M) \rightarrow H^{\mathrm{even}}(M, K_0(B) \otimes \mathbb{R})$.

Let B be a unital local C^* -algebra carrying a trace τ . If \mathcal{V} is a finitely generated and projective B -module, then $\mathrm{End}(\mathcal{V})$ coincides with the finite rank operators and we have $\mathrm{End}(\mathcal{V}) \cong \mathcal{V} \otimes_B \mathrm{Hom}(\mathcal{V}, B)$, where the tensor product is a quotient of the algebraic tensor product. Repeating the constructions in Section 3.1.1 we can define $\mathrm{ch}_\tau: K_{L,B}^0(M) \rightarrow H^{\mathrm{even}}(M, \mathbb{R})$ and $\mathrm{dim}_\tau: K_0(B) \rightarrow \mathbb{R}$. The proof of Theorem 3.5 shows that $\mathrm{dim}_\tau(\mathrm{ch}(\mathcal{E})) = \mathrm{ch}_\tau(\mathcal{E})$.

3.3. The projective Dirac operator. Let (M, g) be a closed oriented Riemannian manifold of even dimension $n > 3$. The oriented frame bundle P_{SO} together with the defining central $\mathbb{Z}/2\mathbb{Z}$ -extension $\mathrm{Spin}(n) \rightarrow SO(n)$ gives a lifting bundle gerbe $L_{\mathrm{spin}} \rightarrow P_{SO}^{[2]}$. Since we assume M to be even-dimensional, the complex spinor module $\Delta_{\mathbb{C}}$ is $\mathbb{Z}/2\mathbb{Z}$ -graded, i.e. $\Delta_{\mathbb{C}} = \Delta_+ \oplus \Delta_-$. There is a $\mathbb{Z}/2\mathbb{Z}$ -graded twisted vector bundle $S = P_{SO} \times \Delta_{\mathbb{C}}$ with $S_{\pm} = P_{SO} \times \Delta_{\pm}$ for L_{spin} , where the action is given by $\gamma([\hat{g}, \lambda] \otimes v) = \lambda \hat{g} \cdot v$ for $\hat{g} \in \mathrm{Spin}(n)$, $\lambda \in \mathbb{C}$ and $v \in \Delta_{\mathbb{C}}$. If we fix a connection form $\eta \in \Omega^1(P_{SO}, \mathfrak{so}(n))$, then there is a canonical twisted connection ∇^S on S , which acts like $\nabla^S(\sigma) = d\sigma + \rho_*(\eta) \cdot \sigma$ for a section $\sigma \in C(P_{SO}, S)$, where ρ is induced by $O(n) \rightarrow U(n)$.

Let $E \rightarrow Y$ be a smooth twisted Hilbert A -module bundle for a bundle gerbe L with $dd(L) = -dd(L_{\text{spin}}) = dd(L_{\text{spin}})$ equipped with a twisted connection ∇^E . Choose a trivialization Q of $L_{\text{spin}} \boxtimes L$, then $Q(S \boxtimes E)$ is a Hilbert A -module bundle over M . Fixing a (flat) connection on Q , $\nabla^{S \boxtimes E} = \nabla^S \otimes \text{id} + \text{id} \otimes \nabla^E$ descends to a connection on $Q(S \boxtimes E)$. The latter bundle carries an action of the complex Clifford bundle $\mathbb{C}\ell(M)$. We have $\nabla_X^{S \boxtimes E}(c \cdot \sigma) = \nabla_X^{\mathbb{C}\ell(M)}(c) \cdot \sigma + c \cdot \nabla_X^{S \boxtimes E}(\sigma)$ for $X \in C^\infty(M, TM)$, $c \in C^\infty(M, \mathbb{C}\ell(M))$, $\sigma \in C^\infty(M, Q(S \boxtimes E))$.

Definition 3.6. Let S, E and Q be as above, then the elliptic first order differential operator $D^E: C^\infty(M, Q(S \boxtimes E)) \rightarrow C^\infty(M, TM \otimes Q(S \boxtimes E)) \rightarrow C^\infty(M, Q(S \boxtimes E))$ where the first map is induced by the connection $\nabla^{S \boxtimes E}$ and the metric and the second is Clifford multiplication will be called the *projective Dirac operator* twisted by E .

The bundle gerbe L_{spin} provides a canonical trivialization Q of $L_{\text{spin}} \boxtimes L_{\text{spin}}$ with the property that $Q(S^* \boxtimes S) \cong \mathbb{C}\ell(M)$ as a bundle of $\mathbb{C}\ell(M)$ - $\mathbb{C}\ell(M)$ -bimodules. This identifies the operator $D^S: C^\infty(M, \mathbb{C}\ell(M)) \rightarrow C^\infty(M, \mathbb{C}\ell(M))$ with the right Clifford linear Dirac operator. As described in [3, Section 24.5], D^S yields a class in the K -homology group $KK(C(M, \mathbb{C}\ell(M)), \mathbb{C})$.

Since the connection is even and Clifford multiplication is an odd operation, D^E has the decomposition $D_{\pm}^E: C^\infty(M, Q(S_+ \boxtimes E)) \rightarrow C^\infty(M, Q(S_- \boxtimes E))$.

Theorem 3.7. *Let S, E and Q be as in the last paragraph and denote by D_+^E the positive part of the projective Dirac operator twisted by E , then*

$$\text{ind}(D_+^E) = \int_M \widehat{A}(M) \text{ch}(E) .$$

In particular, the index of D_+^E does not depend on the choice of Q .

Proof. Let $\pi_M: T^*M \rightarrow M$ be the bundle projection and denote by $D(T^*M)$ and $S(T^*M)$ the disc and sphere bundle of the cotangent bundle. The symbol $\sigma: \pi_M^*Q(S_+ \boxtimes E) \rightarrow \pi_M^*Q(S_- \boxtimes E)$ of D_+^E yields an element $[Q(S_+ \boxtimes E), Q(S_- \boxtimes E), \sigma] \in K_A^0(D(T^*M), S(T^*M)) \cong K_A^0(T^*M)$. Denote by

$$\phi: H^*(M, \mathbb{R}) \rightarrow H^{*+\dim(M)}(D(T^*M), S(T^*M); \mathbb{R})$$

the Thom isomorphism. From the Mishchenko-Fomenko index theorem [18] together with Lemma 3.2 we obtain

$$\begin{aligned} \text{ind}(D_+^E) &= \int_M \text{Td}(M) \phi^{-1}(\text{ch}([Q(S_+ \boxtimes E), Q(S_- \boxtimes E), \sigma])) \\ &= \int_M \text{Td}(M) \phi^{-1}(\text{ch}([S_+, S_-, \sigma_D])) \text{ch}(E) = \int_M \widehat{A}(M) \text{ch}(E) \end{aligned}$$

where the identification of $\text{Td}(M) \phi^{-1}(\text{ch}([S_+, S_-, \sigma_D]))$ with the \widehat{A} -genus described in [16] works in the twisted case as well. \square

Theorem 3.8. *Let D be the projective Dirac operator. Let $E \rightarrow Y$ be a finitely generated projective twisted Hilbert A -module bundle for a bundle gerbe L with $dd(L) = dd(L_{\text{spin}})$. Let Q be a trivialization of $L^* \boxtimes L_{\text{spin}}$ and $[E^S] := \rho_Q([E]) = [C(M, Q(S^* \boxtimes E))] \in KK(\mathbb{C}, C(M, \mathbb{C}\ell(M) \otimes A))$. Then*

$[E^S] \otimes_{C(M, \mathcal{C}\ell(M))} [D^S] \in KK(\mathbb{C}, A) \cong K_0(A)$ is the class representing the Mishchenko-Fomenko index of D^E .

Proof. It suffices to show that the intersection product coincides with the Fredholm module:

$$\left[L^2(Q(S^* \boxtimes E)), D^E \left(1 + (D^E)^2 \right)^{-\frac{1}{2}} \right] \in KK(\mathbb{C}, A) .$$

The proof of this is just a slight modification of [25, Theorem 5.22], therefore we omit it. \square

From this decomposition of the index in twisted K -theory, we obtain the following naturality result in analogy to the untwisted case [11, Lemma 3.1].

Corollary 3.9. *Let D be the projective Dirac operator and let $E \rightarrow Y$ be a twisted Hilbert A -module bundle for a bundle gerbe L with $dd(L) = dd(L_{spin})$. Given a C^* -algebra homomorphism $\varphi : A \rightarrow B$ define the twisted Hilbert B -module bundle F via $F = E \otimes_{\varphi} B$. Then: $\varphi_*([D^E]) = [D^F]$, where $\varphi_* : K_0(A) \rightarrow K_0(B)$ denotes the induced map on K -theory.*

Proof. Let $[E^S] := [C(M, Q(S^* \boxtimes E))] \in KK(\mathbb{C}, C(M, \mathcal{C}\ell(M) \otimes A))$ and choose a trivialization Q of $L^* \boxtimes L_{spin}$. The naturality of the Kasparov product yields

$$\begin{aligned} \varphi_*([D^E]) &= \varphi_*([E^S] \otimes_{C(M, \mathcal{C}\ell(M))} [D^S]) \\ &= ((\text{id}_{C(M, \mathcal{C}\ell(M))} \otimes \varphi_*)[C(M, Q(S^* \boxtimes E))]) \otimes_{C(M, \mathcal{C}\ell(M))} [D^S] \\ &= [C(M, Q(S^* \boxtimes F))] \otimes_{C(M, \mathcal{C}\ell(M))} [D^S] = [D^F] . \quad \square \end{aligned}$$

4. ENLARGEABLE MANIFOLDS WITH SPIN UNIVERSAL COVER

In this section we will extend the result of [11, 12] about the Rosenberg index obstruction for closed enlargeable spin manifolds to closed enlargeable orientable manifolds with spin structure on the universal cover. Since \widetilde{M} may be non-compact, we have to work in a twistedly equivariant setting.

Definition 4.1. A connected closed oriented manifold M with fixed metric g is called *enlargeable* if the following holds: For every $\varepsilon > 0$, there is a connected cover $\bar{M} \rightarrow M$ carrying a spin structure and an ε -contracting map

$$\bar{\psi}_{\varepsilon} : (\bar{M}, \bar{g}) \rightarrow (S^n, g_0)$$

which is constant outside a compact subset of \bar{M} and of nonzero degree. Here, \bar{g} denotes the induced metric on \bar{M} and g_0 is the standard metric on S^n .

Definition 4.2. Let M be a closed even-dimensional oriented Riemannian manifold and let \mathcal{V}_{\max} be the maximal twisted Mishchenko-Fomenko bundle (Def. 2.16). The (*maximal*) *twisted Stolz-Rosenberg obstruction* is given by $\theta^{\max}(M) = \text{ind}(D^{\mathcal{V}_{\max}}) \in K_0(C_{\max}^*(\widehat{\pi} \rightarrow \pi))$.

4.1. Almost flat twisted bundles. In our proof of the non-vanishing of $\theta^{\max}(M)$ for closed enlargeable M with spin universal cover we will extend the argument given in [11, 12] to the twisted setting. Thus, it will rely on almost flat *twisted* bundles.

Definition 4.3. Let $L_{\widehat{\pi}}$ be the Mishchenko-Fomenko bundle gerbe. A sequence $E_i \rightarrow \widetilde{M}$, $i \in \mathbb{N}$ of smooth twisted vector bundles for $L_{\widehat{\pi}}$ equipped with connections ∇^i will be called a *sequence of almost flat twisted bundles*, if $\lim_{i \rightarrow \infty} \|\Omega_i\| = 0$, where Ω_i is the curvature of the connection ∇^i . The norm is induced by the pointwise norm on $\text{end}(E_i) \rightarrow M$ and the maximum norm on the unit sphere bundle in $\Lambda^2(M)$. Moreover, the twistings $\gamma_i^g: E_i \rightarrow g^*E_i$ considered as sections $C(\widetilde{M}, \text{Hom}(E_i, g^*E_i))$ should be locally Lipschitz continuous with a global Lipschitz constant C independent of i , i.e. each point $\widetilde{m} \in \widetilde{M}$ has an open neighborhood U of \widetilde{m} , such that E_i and g^*E_i are trivial over U and such that $\gamma_i^g|_U$ viewed as an element in $C(U, U(V))$ (V being the typical fiber of E_i) satisfies

$$\|\gamma_i^g(\widetilde{m}_1) - \gamma_i^g(\widetilde{m}_2)\| \leq C d(\widetilde{m}_1, \widetilde{m}_2)$$

where the metric is induced by the Riemannian structure pulled back to \widetilde{M} .

Let M be a closed orientable smooth manifold and let \bar{M} be a cover of M equipped with a spin structure, let $\pi = \pi_1(M)$ and $\bar{\pi} = \pi_1(\bar{M}) \subset \pi$. The bundle projection $P_{\text{Spin}(\bar{M})} \rightarrow \bar{M}$ of the principal $\text{Spin}(n)$ -bundle induces an isomorphism $\pi_1(P_{\text{Spin}(\bar{M})}) \rightarrow \bar{\pi}$, which in turn yields an injective group homomorphism $\sigma: \bar{\pi} \rightarrow \pi_1(P_{\text{Spin}(\bar{M})}) \rightarrow \pi_1(P_{\text{SO}(\bar{M})}) \rightarrow \widehat{\pi}$, where the first map is the inverse of the above isomorphism and the other two homomorphism arise from the respective coverings. We call a (set-theoretic) split $s: \pi \rightarrow \widehat{\pi}$ *compatible*, if $s(gh) = s(g)\sigma(h)$ for $g \in \pi$ and $h \in \bar{\pi}$.

Lemma 4.4. *Let M , \bar{M} , π and $\bar{\pi}$ be as in the last paragraph. The choice of a compatible set-theoretic split $s: \pi \rightarrow \widehat{\pi}$ turns the Hilbert space $H = \ell^2(\pi/\bar{\pi})$ into a projective representation of π , which corresponds to an honest representation of $\widehat{\pi}$.*

Proof. Let $e \in \widehat{\pi}$ be the image of the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ and let $q = \frac{1}{2}(1 - e) \in \mathbb{C}[\widehat{\pi}]$. Since q is central in the group ring, it yields a well-defined projection $q: \ell^2(\widehat{\pi}/\sigma(\bar{\pi})) \rightarrow \ell^2(\widehat{\pi}/\sigma(\bar{\pi}))$. The space $q\ell^2(\widehat{\pi}/\sigma(\bar{\pi}))$ is a representation of $\widehat{\pi}$, on which e acts by multiplication with -1 , i.e. a projective representation of π .

A compatible split now yields an injective map $\pi/\bar{\pi} \rightarrow \widehat{\pi}/\sigma(\bar{\pi})$ via $[g] \mapsto [s(g)]$. It induces an isometric isomorphism $\ell^2(\pi/\bar{\pi}) \rightarrow q\ell^2(\widehat{\pi}/\sigma(\bar{\pi}))$ \square

Theorem 4.5. *Let M be an even-dimensional closed orientable manifold that is enlargeable in the sense of Definition 4.1. Let $i \in \mathbb{N}$ be a positive natural number. Then there is a C^* -algebra C_i (which will be constructed in the proof) and a twisted Hilbert C_i -module bundle $E_i \rightarrow \widetilde{M}$ for the Mishchenko-Fomenko bundle gerbe $L_{\widehat{\pi}}$ together with a twisted connection ∇_i that has the following properties: The curvature Ω_i of E_i satisfies*

$$\|\Omega_i\| \leq \frac{1}{i} C$$

where C is a constant depending only on $\dim(M)$. Moreover, there is a split extension $0 \rightarrow \mathbb{K} \rightarrow C_i \rightarrow X_i \rightarrow 0$ with a certain C^* -algebra X_i . In particular, each $K_0(C_i)$ splits off a $\mathbb{Z} = K_0(\mathbb{K})$ summand and the $K_0(\mathbb{K})$ -part of the index of the projective Dirac operator $D_+^{E_i}$ is different from 0.

Proof. Let $2n = \dim(M)$ and $\pi = \pi_1(M)$. Since the Chern character is rationally an isomorphism, there is a vector bundle $F \rightarrow S^{2n}$ with non-vanishing top Chern class $c_n(F) \neq 0$. Choose a connection η_F on F and fix $i \in \mathbb{N}$. Since M is enlargeable, there exists a spin covering $\bar{M} \rightarrow M$ together with a $\frac{1}{i}$ -contracting map $\psi: \bar{M} \rightarrow S^{2n}$, which is constant outside a compact subset K of \bar{M} . Let P_F be the principal $U(n)$ -bundle of frames in F . Since ψ is constant on $\bar{M} \setminus K$ we can choose a trivialization for the principal $U(n)$ -bundle ψ^*P_F over this set:

$$(\psi^*P_F)|_{\bar{M} \setminus K} \cong (\bar{M} \setminus K) \times U(n)$$

such that the pullback connection $\psi^*\eta_F|_{\bar{M} \setminus K}$ is flat. Let $\rho: \bar{M} \rightarrow M$ be the covering map, $\tilde{\rho}: \tilde{M} \rightarrow M$ the universal cover and $\tilde{\pi} = \pi_1(\tilde{M})$. As described in the proof of [12, Proposition 1.5] we can cover M by open sets $U_j, j \in I$, such that each component $V_{\lambda,j} \subset \bar{M}$ of $\rho^{-1}(U_j)$ maps diffeomorphically onto U_j , intersects only one component $V_{\mu,k}$ of $\rho^{-1}(U_k)$ for any k and such that $\psi^*P_F|_{V_{\lambda,j}}$ trivializes. Let $J_j = \pi_0(\rho^{-1}(U_j))$ be the index set labeling the components, likewise set $\tilde{J}_j = \pi_0(\tilde{\rho}^{-1}(U_j))$. Let $\tilde{\varphi}_{\alpha,j}: \tilde{J}_j \rightarrow \pi/\tilde{\pi}$ be the map that sends αg to $[g] \in \pi/\tilde{\pi}$ for $g \in \pi$, where π acts on \tilde{J}_j by deck transformations. Since $\bar{M} = \tilde{M}/\tilde{\pi}$, this induces bijections $\varphi_{\lambda,j}: J_j \rightarrow \pi/\tilde{\pi}$ for each $\lambda \in J_j$. Note that

$$(7) \quad \varphi_{[\alpha g],j} = g^{-1} \cdot \varphi_{[\alpha],j} .$$

Moreover, if $\lambda \in J_j$ and $\mu \in J_k$ belong to components with non-empty intersection, then $\varphi_{\lambda,j}(\kappa) = \varphi_{\mu,k}(\tau)$ if τ and κ intersect.

Consider the Hilbert space $H = \ell^2(\pi/\tilde{\pi}) \otimes \mathbb{C}^n$. Let $C_S \subset \mathcal{B}(H)$ be the C^* -algebra generated by the group of all permutations of $\pi/\tilde{\pi}$ and all multiplications by functions $f: \pi/\tilde{\pi} \rightarrow S^1$. So we have permutation operators with S^1 -entries as a generating set of C_S . Let $C_T \subset \mathcal{B}(H)$ be the C^* -algebra generated by linear transformations, which are of the form

$$T: H \rightarrow H \quad ; \quad T([g] \otimes v) = [h] \otimes T'v \quad \text{and} \quad T|_{([g] \otimes \mathbb{C}^n)^\perp} = 0 .$$

for some matrix $T' \in M_n(\mathbb{C})$ and $[g], [h] \in \pi/\tilde{\pi}$. Let $C_{S,T}$ be the C^* -algebra generated by C_S and C_T inside of $\mathcal{B}(H)$ and note that C_T is a 2-sided ideal in $C_{S,T}$. Moreover, C_T is isomorphic either to the compact operators or to a matrix algebra. Applying the stabilization trick of [12] we can without loss of generality assume that the former is the case. Let $C_i = \{(c_1, c_2) \in C_{S,T} \times C_{S,T} \mid c_1 - c_2 \in C_T\}$. This algebra fits into a split exact sequence $0 \rightarrow C_T \rightarrow C_i \rightarrow C_{S,T} \rightarrow 0$ with the splitting induced by the diagonal map, $C_T \rightarrow C_i$ via $a \mapsto (a, 0)$ and $C_i \rightarrow C_{S,T}$ via $(a, b) \mapsto b$.

We choose trivializations of ψ^*P_F over the sets $V_{\lambda,j} \subset \bar{M}$, where we take the trivialization fixed above if $V_{\lambda,j}$ is a subset of $\bar{M} \setminus K$. This way we get a cocycle on the double intersections $V_{(\lambda,\mu),(j,k)} = V_{\lambda,j} \cap V_{\mu,k}$:

$$T'_{(\lambda,\mu),(j,k)}: V_{(\lambda,\mu),(j,k)} \rightarrow U(n) .$$

We can extend $T'_{(\lambda,\mu),(j,k)}$ to a cocycle with values in the unitary group $U(C_{S,T})$ as follows:

$$T_{(\lambda,\mu),(j,k)}^1(x)(\varphi_{\lambda,j}(\kappa) \otimes v) = \varphi_{\mu,k}(\tau) \otimes T'_{(\kappa,\tau),(j,k)}(\bar{x})(v) ,$$

where $\tau \in \pi/\bar{\pi}$ is the index of the component of $\rho^{-1}(U_k)$ that intersects $V_{\kappa,j}$ and \bar{x} denotes the lift of $\rho(x)$ to the component $V_{\kappa,j}$. This map actually does nothing to the first tensor factor by our previous considerations. Let $T_{(\lambda,\mu),(j,k)}^2$ be the constant map with value $1 \in U(C_{S,T})$. $T'_{(\kappa,\tau),(j,k)}$ is different from the identity only for finitely many pairs (κ, τ) . Thus,

$$T_{(\lambda,\mu),(j,k)}: V_{(\lambda,\mu),(j,k)} \rightarrow U(C_i) \quad ; \quad T_{(\lambda,\mu),(j,k)} = (T_{(\lambda,\mu),(j,k)}^1, T_{(\lambda,\mu),(j,k)}^2) .$$

is a well-defined cocycle with values in $U(C_i)$. We therefore get a smooth Hilbert C_i -module bundle $\bar{E}_i \rightarrow \bar{M}$, whose pullback to \widetilde{M} will be $E_i = \widetilde{M} \times_M \bar{E}_i$. By Lemma 4.4, the space $\ell^2(\pi/\bar{\pi})$ carries a projective unitary representation of π , which induces a projective representation $r: \pi \rightarrow U(C_i)$. For $\alpha, \beta \in \widetilde{J}_j$ denote the corresponding components of $\widetilde{\rho}^{-1}(U_j)$ by $W_{\alpha,j}$ and $W_{\beta,j}$ respectively. Let $\lambda = [\alpha]$, $\mu = [\beta]$, $\lambda' = [\alpha g^{-1}]$ and $\mu' = [\beta g^{-1}] \in J_j$. We define

$$\gamma^g: W_{\alpha,j} \times C_i \rightarrow W_{\alpha g^{-1},j} \times C_i$$

by left multiplication with $r(g)$. Due to equation (7) and with $\varphi_{\lambda,j}(\kappa) = \varphi_{\mu,k}(\tau) = [h] \in \pi/\bar{\pi}$ we have

$$\begin{aligned} (T_{(\lambda',\mu'),(j,k)}(x) \cdot r(g)) (\varphi_{\lambda,j}(\kappa) \otimes v) &= c_{\bar{\pi}}(g, h) \varphi_{\mu',j}(\tau) \otimes T'_{(\kappa,\tau),(j,k)}(v) \\ &= (r(g) \cdot T_{(\lambda,\mu),(j,k)}(x)) (\varphi_{\lambda,j}(\kappa) \otimes v) . \end{aligned}$$

Thus, γ^g intertwines the transition functions of E_i and $g^*(E_i)$. Therefore it yields a well-defined twisting map $\gamma^g: E_i \rightarrow g^*(E_i)$. This clearly satisfies the Lipschitz condition, since it even is locally constant.

Let $\eta_{\kappa,j} \in \Omega^1(V_{\kappa,j}, \mathfrak{u}(n))$ be the pullback of η_F via the trivialization. These induce forms in $\Omega^1(V_{\lambda,j}, C_{S,T}^a)$, where $C_{S,T}^a$ denotes the anti-selfadjoint operators in $C_{S,T}$, via

$$\left(\eta_{\lambda,j}^{E_i} \right)_x (\xi) (\varphi_{\lambda,j}(\kappa) \otimes v) = \varphi_{\lambda,j}(\kappa) \otimes (\eta_{\kappa,j})_x (\xi) \cdot v .$$

Since $\eta_{\kappa,j}$ is non-zero only for finitely many κ , we can extend $\eta_{\lambda,j}^{E_i}$ to a well-defined 1-form with values in the anti-selfadjoint operators of C_i by setting it to zero in the second component. These 1-forms inherit their transformation behaviour from the forms $\eta_{\kappa,j}$. Thus, they yield a C_i -linear connection ∇^i on sections of E_i . Just like above it follows from (7) that ∇^i is a twisted connection. Since the norm of the curvature Ω_i of ∇^i coincides with that of $\psi^*\Omega_F$, we have

$$\|\Omega_i\| = \|\psi^*\Omega_F\| \leq \frac{1}{i} C .$$

It remains to be shown that the $K_0(\mathbb{K})$ -part of $\text{ind}(D_+^{E_i})$ does not vanish. Here we proceed exactly as in [12]: Let $\mathcal{T} \subset C_T \cong \mathbb{K}$ be the trace class ideal and let D_i be the algebra given by

$$D_i = \{(c_1, c_2) \in C_{S,T} \times C_{S,T} \mid c_1 - c_2 \in \mathcal{T}\} .$$

Since the proof of Lemma [12, lemma 2.4] applies to D_i with the changed $C_{S,T}$ as well, D_i is a unital local C^* -algebra with a trace $\tau(c_1, c_2) = \text{tr}(c_1 - c_2)$, which coincides with the trace of the element after projecting it from C_i to \mathcal{T} . Its C^* -completion is C_i . Since $K_0(C_i) \cong K_0(D_i)$, we can extend \dim_τ from Definition 3.4 to a functional on $K_0(C_i)$ and it suffices to prove that $\dim_\tau(\text{ind}(D_+^{E_i})) \neq 0$. The transition functions in the definition of E_i actually take values in $U(D_i)$ and thus lead to a twisted D_i -module bundle \mathcal{E}_i in the sense of Section 3.2. By Theorem 3.7 we have

$$\begin{aligned} \dim_\tau(\text{ind}(D_+^{E_i})) &= \int_M \widehat{A}(M) \dim_\tau(\text{ch}(E_i)) \\ &= \int_M \widehat{A}(M) \dim_\tau(\text{ch}(\mathcal{E}_i)) = \int_M \widehat{A}(M) \text{ch}_\tau(\mathcal{E}_i) . \end{aligned}$$

We can identify $\Omega_{\mathcal{E}_i} \in \Omega^2(M, \text{end}(\mathcal{E}_i))$ with an equivariant form in $\Omega^2(\widetilde{M}, \text{End}(\mathcal{E}_i))$. If we carry out the integration over a single subset $U_j \subset M$, we could integrate instead over the subset $W_{\alpha,j} \subset \widetilde{M}$ for some $\alpha \in \widetilde{J}_j$. This is independent of the choice of α by equivariance. But over $W_{\alpha,j}$ the form $\tau(\Omega_{\mathcal{E}_i} \wedge \cdots \wedge \Omega_{\mathcal{E}_i})|_{W_{\alpha,j}} \in \Omega^{\text{even}}(W_{\alpha,j}, \mathbb{R})$ coincides with the sum of all $\Omega_{\psi^*F} \wedge \cdots \wedge \Omega_{\psi^*F}|_{V_{\kappa,j}} \in \Omega^{\text{even}}(V_{\kappa,j}, \mathbb{R})$ over $\kappa \in J_j$ by the definition of the trace. Using a partition of unity on M we see that

$$\int_M \widehat{A}(M) \text{ch}_\tau(\mathcal{E}_i) = \int_{\widetilde{M}} \widehat{A}(\widetilde{M}) \text{ch}(\psi^*F - \underline{\mathbb{C}}^n) ,$$

Since the class of $\text{ch}(\psi^*F - \underline{\mathbb{C}}^n)$ is concentrated in degree n we see that the above term is non-vanishing. \square

Remark 4.6. Due to the stabilization trick mentioned in the proof the fibers of E_i are isomorphic to $t_i C_i$ for some projection $t_i \in C_i$, where $t_i = 1$ if \widetilde{M} is non-compact.

Having the sequence E_i of almost flat twisted bundles at hand, we can form the C^* -algebra

$$A = \prod_{i \in \mathbb{N}} C_i$$

of bounded sequences with i th entry in C_i , in which the norm closure of the sequences with only finitely many non-zero entries

$$A' = \overline{\bigoplus_{i \in \mathbb{N}} C_i}^{\|\cdot\|}$$

is a two-sided ideal and we set $Q = A/A'$. Let A_i be the ideal in A consisting of sequences that are 0 everywhere, but in the i th entry.

Theorem 4.7. *There is a smooth twisted Hilbert A -module bundle $E \rightarrow \widetilde{M}$ together with a twisted connection*

$$\nabla^E : C^\infty(\widetilde{M}, E) \rightarrow C^\infty(\widetilde{M}, T^*\widetilde{M} \otimes E)$$

such that the following holds:

- $E \cdot A_i$ is isomorphic to E_i as a twisted Hilbert C_i -module bundle.
- The connection preserves the subbundles $E \cdot A_i$.

- The sequence of curvatures $\Omega_i \in \Omega^2(M, \text{end}(E \cdot A_i))$ of the connection induced on $E \cdot A_i$ by ∇^E satisfies $\lim_{i \rightarrow \infty} \|\Omega_i\| = 0$.

Proof. The idea is to see that the product bundle $E_L = \Delta_{\widetilde{M}}^* (\prod_{i \in \mathbb{N}} E_i)$, where

$$\Delta_{\widetilde{M}}: \widetilde{M} \rightarrow \prod_{i \in \mathbb{N}} \widetilde{M}$$

is the diagonal map, has locally Lipschitz continuous transition functions. This parallels the construction given in the proof of [11, Lemma 2.1] with the only difference that we have to work equivariantly over \widetilde{M} , so we just sketch the differences and refer to [11] for the details: We cover M by subsets U_j , each of them diffeomorphic to I^n , where $I = [0, 1]$, such that $\widetilde{M} \rightarrow M$ is trivial over U_j via

$$\phi_j: U_j \times \pi \rightarrow \widetilde{M} \Big|_{U_j}.$$

We can find trivializations

$$\psi_{i,j}^1: \phi_j^* E_i \Big|_{U_j \times \{1\}} \rightarrow I^n \times t_i C_i$$

of $\phi_j^* E_i \Big|_{U_j \times \{1\}}$, such that constant sections of $\phi_j^* E_i$ over $I^k \times \{0, \dots, 0\}$ are parallel with respect to ∇_{∂_l} for $1 \leq l \leq k$, where ∇ denotes the connection induced by ∇^{E_i} . Using the twisting we can extend $\psi_{i,j}^1$ to a trivialization $\psi_{i,j}$ of $\phi_j^* E_i \Big|_{U_j \times \pi}$ with components $\psi_{i,j}^g$ with $g \in \pi$. Let $\eta_{i,j}^g \in \Omega^1(I^n, t_i C_i t_i)$ be the pullback of the connection 1-form of ∇^{E_i} . The way the trivializations are constructed is crucial to prove the estimate given in [11, Lemma 2.3], which now still holds and we have $\|\eta_{i,j}^g\| \leq n \cdot \|\Omega_{i,j}^g\|$, where $\Omega_{i,j}^g$ denotes the curvature of $\eta_{i,j}^g$. The right hand side of the above inequality is independent of $g \in \pi$. Thus, our control of the curvatures carries over to an upper bound on the local connection 1-forms. The trivializations $\psi_{i,j}$ induce transition maps

$$\psi_{i,(j,k)}: (U_j \cap U_k) \times \pi \rightarrow U(t_i C_i t_i)$$

and the upper bound on the local connection 1-forms yields an upper bound on the norm of the derivative $D_{(x,g)} \psi_{i,(j,k)}$ just as described in [11, Lemma 2.5, Proposition 2.6] proving Lipschitz continuity of the transition functions. The Lipschitz condition on the twisting maps ensures that the product of the γ_i^g is continuous, when considered as an element in $C(\widetilde{M}, \text{Hom}(E_L, g^* E_L))$. Thus, E_L is a continuous twisted Hilbert A -module bundle. Note that $\pi = \pi_1(M)$ acts via the adjoint action unitarily on C_i and we set $\mathcal{A} = \widetilde{M} \times_{\text{Ad}} C_i$. E_L corresponds to a projection $t_L \in C(M, \mathcal{A})$, which we can approximate by a projection in $C^\infty(M, \mathcal{A})$ to obtain a smooth twisted Hilbert A -module bundle E by the construction given in Theorem 2.10. We have $E \cdot A_i \cong E_L \cdot A_i = E_i$. The isomorphism may only be continuous, but it can be smoothed.

To construct the connection ∇^E we only need to give local connection 1-forms over the sets $\phi_j(U_j \times \{1\}) \subset \widetilde{M}$ and extend them equivariantly via γ^g to get connection forms over the images of $U_j \times \pi$, which can be patched together with a partition of unity on M . The construction takes

the local forms of the E_i and uses a convolution argument to get a smooth form on the product. This is exactly the same as in [11]. \square

The twisting γ^g maps the subbundle $E \cdot A'$ into itself, therefore the quotient $\mathcal{W} = E/(E \cdot A')$ is a smooth twisted Hilbert Q -module bundle equipped with a *flat* connection ∇^Q and typical fiber tQ for some projection $t \in Q$. If we now fix a basepoint $\tilde{m} \in \tilde{M}$, we get a projective holonomy representation in the sense of Definition 2.18: $(\pi, c_{\hat{\pi}}) \rightarrow \text{End}(\mathcal{W}_{\tilde{m}}) = tQt$. By the universal property of the maximal twisted group C^* -algebra, this extends to a $*$ -homomorphism $\phi: C_{\max}^*(\hat{\pi} \rightarrow \pi) \rightarrow Q$.

As a consequence of Corollary 3.9, the induced map $\phi_*: K_0(C_{\max}^*(\hat{\pi} \rightarrow \pi)) \rightarrow K_0(Q)$ maps $\theta^{\max}(M)$ to $\text{ind}(D_+^{\mathcal{W}'})$, where

$$\mathcal{W}' = \tilde{M} \times tQ = \tilde{M} \times \mathcal{W}_{\tilde{m}} .$$

Using parallel transport, its equivariance as described before Definition 2.18 with respect to γ^g and flatness of Q we see that \mathcal{W}' is isomorphic to \mathcal{W} as a twisted Hilbert Q -module bundle.

Theorem 4.8. *Let M be a closed smooth orientable even-dimensional manifold with $\dim(M) \geq 3$ and \tilde{M} spin that is enlargeable in the sense of Definition 4.1. Then we have*

$$\theta^{\max}(M) \neq 0 \in K_0(C_{\max}^*(\hat{\pi} \rightarrow \pi)) .$$

Proof. As we saw above, we have $\phi_*(\theta^{\max}(M)) = \text{ind}(D_+^{\mathcal{W}'}) \in K_0(Q)$. By [12], the group $K_0(Q)$ splits off a summand

$$\prod_{i \in \mathbb{N}} K_0(\mathbb{K}) / \bigoplus_{i \in \mathbb{N}} K_0(\mathbb{K}) \cong \prod_{i \in \mathbb{N}} \mathbb{Z} / \bigoplus_{i \in \mathbb{N}} \mathbb{Z}$$

and the image of $\text{ind}(D_+^{\mathcal{W}'})$ in the latter group corresponds to the sequence

$$z_i = \left[p_* \left(\text{ind}(D_+^{E \cdot A_i}) \right) \right] = \left[p_* \left(\text{ind}(D_+^{E_i}) \right) \right] ,$$

where $p: C_i \rightarrow \mathbb{K}$ is the projection. By Theorem 4.5 it has only non-vanishing entries. \square

Remark 4.9. Relaxing the condition about the orientability of M requires incorporating orientation twists of K -theory into the setup, which can be seen as a special case of twisted $\mathbb{Z}/2\mathbb{Z}$ -equivariant K -theory as has been observed by Karoubi [14, Remark 6.16], [13]. These can also be described by gerbes (see the Jandl gerbes in [8] and the functor $K_{\pm}(X)$ in [1], which is naturally equivalent with Karoubi's definition), therefore the above argument generalizes to closed non-orientable manifolds as well. Nevertheless, it seems to be impossible to drop the spin condition for the covers \tilde{M} in the definition of enlargeability, since our construction of a projective representation with the right cocycle relies on it.

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