SIAM J. MATH. ANAL.
Vol. 48, No. 4, pp. 2701–2729

HOMOGENIZATION OF A MEAN FIELD GAME SYSTEM IN THE SMALL NOISE LIMIT*

ANNALISA CESARONI[†], NICOLAS DIRR[‡], AND CLAUDIO MARCHI[§]

Abstract. This paper concerns the simultaneous effect of homogenization and of the small noise limit for a second order mean field game (MFG) system with local coupling and quadratic Hamiltonian. We show under some additional assumptions that the solutions of our system converge to a solution of an effective first order system whose effective operators are defined through a cell problem which is a second order system of ergodic MFG type. We provide several properties of the effective operators, and we show that in general the effective system loses the MFG structure.

Key words. mean field games, periodic homogenization, small noise limit, ergodic problem, weak convergence

AMS subject classifications. 35B27, 35B40, 35K40, 35K59, 91A13

DOI. 10.1137/16M1063459

1. Introduction.

We investigate evolutive mean field game (MFG) systems in the small noise limit when the Hamilton–Jacobi equation has a rapidly varying dependence on the state variable x, namely,

$$
\begin{cases} -u^\epsilon_t - \epsilon\Delta u^\epsilon + \frac{1}{2}|\nabla u^\epsilon|^2 = V\left(\frac{x}{\epsilon}, m^\epsilon\right), & x \in \mathbb{R}^n,\ t \in (0,T),\\ m^\epsilon_t - \epsilon\Delta m^\epsilon - \operatorname{div}(m^\epsilon \nabla u^\epsilon) = 0, & x \in \mathbb{R}^n,\ t \in (0,T), \end{cases} \tag{1}
$$

with initial and terminal conditions $u^\epsilon(x,T) = u_0(x)$ and $m^\epsilon(x,0) = m_0(x)$.

MFG systems were introduced by Lasry and Lions in [24, 25, 26] in the study of the overall behavior of a large population of (rational and indistinguishable) individuals in markets, crowd motion, etc. In their approach, in a population of N agents, each single agent is driven by a dynamics perturbed by a random noise and aims to minimize some cost functional which depends only on the empirical distribution of all other players. The Nash equilibria are characterized by a system of $2N$ equations. According to Lasry and Lions, as $N \to +\infty$, this system of PDEs reduces to the system (1) with $\epsilon = 1$, where the first equation gives the value function associated to the "average" player while the second equation describes the evolution of the distribution of players. The rigorous proof of this limit behavior was established by Lasry and Lions in [26] for ergodic differential games, whereas the evolutive case with nonlocal coupling has been addressed in some recent preprints [12, 17, 23].

This approach has been generalized in several directions: long time behavior [13, 14], first order systems [9, 11], and ergodic MFG systems [2, 20]. For a general overview, we refer the reader to [1, 5, 10, 18, 21].

---

*Received by the editors February 29, 2016; accepted for publication (in revised form) June 6, 2016; published electronically August 18, 2016.
http://www.siam.org/journals/sima/48-4/M106345.html
Funding: The research of the first and third authors was partially supported by Indam-Gnampa and by Progetto di Ateneo "Traffic flow on networks: analysis and control." The research of the second author was partially supported by the University of Padova, the Leverhulme Trust RPG-2013-261 and EPSRC EP/M028607/1.
[†]Dip. Scienze Statistiche, Padova University, 35131 Padova, Italy (annalisa.cesaroni@unipd.it).
[‡]School of Mathematics, Cardiff University, Cardiff CF24 4AG, UK (DirrNP@cardiff.ac.uk).
[§]Dip. Ingegneria dell'Informazione, Padova University, 35131 Padova, Italy (claudio.marchi@unipd.it).

Homogenization of a single PDE (and of systems as well) has been investigated exhaustively. A summary of the vast literature on this topic is beyond the aim of this paper. Let us only recall from [6] that, for a single semilinear equation such as the first equation in (1), the expansion

$$u^\epsilon = u^0(x,t) + \epsilon u(x/\epsilon)$$

provides (at least formally) the right "guess" for the cell problem (a single ergodic equation) and for the effective Hamilton–Jacobi equation as $\epsilon \to 0^+$. On the other hand, let us recall (see [29]) that, for the homogenization of a Fokker–Planck equation in the small noise limit such as the second equation in (1), the multiplicative formal expansion

$$m^\epsilon = m^0(t,x)(m(x/\epsilon) + \epsilon m^2(x/\epsilon))$$

takes into account the fact that we expect just weak convergence of the solutions, and provides the right guess for the cell problem and the effective continuity equation.

In this paper we consider periodic homogenization of an MFG system under the simultaneous effect of the small noise limit; in other words, we tackle the limit as $\epsilon \to 0$ to the solutions of (1). This system appears in the limit as $N \to +\infty$ of Nash equilibria for a population of N agents. The dynamics of the ith agent is given by

$$dX_t^i = a_t dt + \sqrt{\epsilon/2}\, dW_t,$$

where W_t is a standard Brownian motion and a is the control chosen in $\mathbb{R}^n$ in order to minimize the cost,

$$L(x,t,a) = \mathbf{E}\int_0^t \left[\frac{|a_s|^2}{2} + V\left(\frac{X_s^i}{\epsilon}, \sum_j \delta_{X_s^j}\right)\right] ds,$$

where we choose for sake of simplicity the standard affine-quadratic dependence on the control. This framework models the case of a differential game which takes place in an environment with heterogeneities of period ϵ, and with dynamics disturbed at a microscopic level by white noise. From a mathematical point of view, this scaling (where the viscosity has the same order of magnitude as the size of the heterogeneities) is the most interesting case. We refer the reader to section 3 for a short discussion of other types of scaling between the viscosity and the period of the running cost. As far as we know, this is the first time that these kinds of problems have been considered in the literature, even though, while completing this work, we became aware of a work in progress on homogenization of an MFG system by Lions and Souganidis [28].

The homogenization limit is interesting as a mathematical question but may also find applications in, e.g., traffic-flow problems. MFGs have widely been used to model traffic flow problems where the cost is the higher the more dense the traffic is. See, for example, [7, 15, 22] and references therein. Changing road conditions—e.g., hills and valleys or a change in the number of lanes—influences the cost functional, resulting in a spatially varying prefactor for the cost depending on local traffic density. If the typical distance between cars is much smaller than the scale on which the road conditions vary, then the behavior on a scale which is in turn much larger than the scale of variation of the road conditions could be described by a homogenized MFG system. We would like to point out that this observation is only a motivation for the problem under consideration here; applications of homogenization to MFG systems modeling traffic flow are beyond the scope of this paper.

Because of the small noise limit considered here, the effective system is expected to be a first order system formed by a Hamilton–Jacobi equation and a transport equation, where the effective Hamiltonian and the effective drift need to be suitably defined. More precisely, in our case, the limit system will be

$$(2)\qquad \begin{cases} -u^0_t + \bar{H}(\nabla u^0, m^0) = 0, & x \in \mathbb{R}^n,\ t \in (0,T), \\ m^0_t - \operatorname{div}(m^0 \bar{b}(\nabla u^0, m^0)) = 0, & x \in \mathbb{R}^n,\ t \in (0,T), \end{cases}$$

with initial/terminal condition $u^0(x,T) = u_0(x)$ and $m^0(x,0) = m_0(x)$. The effective operators $\bar{H}(P,\alpha), \bar{b}(P,\alpha)$ appearing in the limit system are obtained through the following cell problem. For every $P \in \mathbb{R}^n$, $\alpha \geq 0$, find the constant $\bar{H}(P,\alpha)$ for which there exists a solution to the ergodic MFG system

$$(3)\qquad \begin{cases} \text{(i)} - \Delta u + \frac{1}{2}|\nabla u + P|^2 - V(y, \alpha m) = \bar{H}(P,\alpha), & y \in \mathbb{T}^n, \\ \text{(ii)} - \Delta m - \operatorname{div}\left(m\left(\nabla u + P\right)\right) = 0, & y \in \mathbb{T}^n, \\ \text{(iii)} \int_{\mathbb{T}^n} u = 0, \qquad \int_{\mathbb{T}^n} m = 1, \end{cases}$$

while $\bar{b}$ is given by

$$(4)\qquad \bar{b}(P,\alpha) = \int_{\mathbb{T}^n} (\nabla u + P) m\, dy,$$

where (u,m) is the solution to (3). We observe that, with this choice of small noise and local coupling, the cell problem has an MFG structure. On the other hand, in the case of homogenization of MFG systems without small noise or with nonlocal smoothing term V, we expect a cell problem almost decoupled: (3)(i) no longer depends on m, so we can solve it as a standard ergodic equation in $\mathbb{T}^n$. In the case of strong noise, i.e., a second order part of the form Δu, we expect an explicit formula for the effective Hamiltonian $\bar{H}(P,\alpha) = \frac{1}{2}|P|^2 - \int_{\mathbb{T}^n} V(y,\alpha)dy$ (see section 3).

The purpose of this paper is twofold: it contains a study of the effective operators $\bar{H}$ and $\bar{b}$ and a convergence result for problem (1). We provide the following properties of $\bar{H}$ and $\bar{b}$: local Lipschitz continuity, monotonicity in α, coercivity in P of $\bar{H}$, and asymptotic behavior of $\bar{H}$ and $\bar{b}$ with respect to P. As a matter of fact, we establish the following formula:

$$\nabla_P \bar{H}(P,\alpha) = \bar{b}(P,\alpha) - \int_{\mathbb{T}^n} V_m(y, \alpha m)\alpha \tilde{m} m\, dy,$$

where m is the solution to (3) while $\tilde{m}$ is a function with values in $\mathbb{R}^n$ defined as a solution of a suitable system (see problem (22)) and, roughly speaking, coincides with $\nabla_P m$.

The interesting feature is that the limit system (2) may lose the MFG structure because $\nabla_P \bar{H}(P,\alpha)$ may not coincide with $\bar{b}(P,\alpha)$. We provide an explicit example where this phenomenon appears.

We provide the rigorous convergence result of the solution (u^ϵ, m^ϵ) to (1) to a solution (u^0, m^0) of (2) respectively strongly in L^2 and weakly in L^p with $1 \leq p < (n+2)/n$ if $n \geq 3$ and any $p < 2$ if $n = 2$ in the following special case: the initial/terminal data are affine for the Hamilton–Jacobi equation and constant for the Fokker–Planck equation; that is,

$$u_0(x) = P \cdot x, \qquad m_0(x) \equiv 1, \qquad P \in \mathbb{R}^n.$$

The result is obtained by getting careful a priori estimates on the difference, appropriately rescaled, between the exact solution of the perturbed problem (1) and a first order asymptotic expansion. We discuss also a conditional convergence result, under some additional restrictive conditions.

This paper is organized as follows. In section 2 we list the standing assumptions and establish the well-posedness of system (1). In section 3, we heuristically provide the effective limit system, and we also discuss the cases of alternative asymptotic expansion, nonlocal coupling, and strong noise. In section 4, we define the effective operators, show their continuity, and provide a variational characterization, coercivity, and monotonicity of $\bar{H}$. Section 5 is devoted to local Lipschitz continuity of effective operators and to the computation of $\nabla_P \bar{H}$. Section 6 contains the asymptotic behavior with respect to P and an explicit example where the limit system loses the MFG structure. Finally, in section 7, we prove the homogenization result for affine constant initial data, and in section 7.3 we discuss some further perspectives.

2. Standing assumptions.

In this section, we collect the assumptions on system (1) that will hold throughout the paper unless explicitly stated otherwise.

Assumptions on the potential V. We will assume that

$$V(y,m) : \mathbb{T}^n \times \mathbb{R} \to \mathbb{R}$$

- is a C^1 bounded function and without loss of generality (w.l.o.g.) $V \geq 0$;
- is monotone increasing with respect to m; that is, for every compact interval $K \subset \mathbb{R}$, there exists a positive constant $\gamma_K > 0$ such that

$$V(y,m) - V(y,n) \geq \gamma_K (m-n) \quad \forall m,n \in K, y \in \mathbb{T}^n. \tag{5}$$

Assumptions on the initial and terminal data. $u_0 \in C^2(\mathbb{T}^n)$. m_0 is a smooth nonnegative function on $\mathbb{T}^n$, such that $\int_{\mathbb{T}^n} m_0(x)dx = 1$.

We conclude by recalling some well-known results about existence and uniqueness of the solution to the MFG system.

Proposition 1. *For $\epsilon = \frac{1}{k}$ $(k \in \mathbb{N})$, there exists a unique classical solution to* (1).

Proof. For the existence, we follow the arguments in [13, Lemma 4.2]. To this end, it is sufficient to observe that the function $w^\epsilon := \exp\{-\frac{u^\epsilon}{2\epsilon}\}$ satisfies the linear problem

$$-w_t^\epsilon - \epsilon \Delta w^\epsilon + \frac{V w^\epsilon}{2\epsilon} = 0.$$

As for uniqueness, we refer the reader to [26] and [10, Thm. 3.8]. □

3. Formal asymptotic expansions.

In this section we formally derive the cell problem and the effective equation by the method of asymptotic expansions. More precisely, we make the formal ansatz that the solution to the system (1) satisfies the following asymptotic expansion up to first order in the small parameter ϵ:

$$\begin{cases} u^\epsilon(x,t) = u^0(x,t) + \epsilon u\left(\frac{x}{\epsilon}\right), \\ m^\epsilon(x,t) = m^0(x,t)\left(m\left(\frac{x}{\epsilon}\right) + \epsilon m^2\left(\frac{x}{\epsilon}\right)\right), \end{cases} \tag{6}$$

where m and m^2 are assumed to be periodic with average, respectively, 1 and 0.

We insert this asymptotic expansion into (1), denoting $P = \nabla u^0$, $\alpha = m^0$, and $y = \frac{x}{\epsilon}$.

At order ϵ^{-1} in the second equation, we get

$$-\Delta_y m - \mathrm{div}_y(m(P + \nabla_y u)) = 0. \tag{7}$$

Note that a positive solution to (7), for P, and $\nabla_y u$ measurable and bounded, exists according to [3, Thm. 4.3, page 136].

Then we collect the terms of order ϵ^0 in the expansion of the m-equation and get

$$\begin{aligned} &m^0(-\Delta_y m^2 - \mathrm{div}_y(m^2(P + \nabla_y u))) \\ &\qquad = -(mm^0)_t - \mathrm{div}_x(m^0 m(P + \nabla_y u)) + 2\nabla_x m_0 \cdot \nabla_y m. \end{aligned} \tag{8}$$

Observe that if m^2 is a solution, then $m^2 + km$ is still a solution of the same equation for every $k \in \mathbb{R}$, which allows us to satisfy the mean zero constraint for m^2.

The solvability condition for m^2 gives (integrating the fast variable y over $\mathbb{T}^n$, while treating the slow variable x as a parameter)

$$m_t^0 - \mathrm{div}_x\left(m^0\left(\int_{\mathbb{T}^n} m(P + \nabla_y u)dy\right)\right) = 0,$$

which is the expected limit equation for m^0. Here we used that m has average equal to 1 and that the other two terms average to 0 since m, u are assumed to be periodic in y and so the average of ∇m is 0.

Inserting the asymptotic expansion ansatz into the first equation (the equation for u) in (1), we get that the terms of order ϵ^0 are

$$-u_t^0 - \Delta_y u + \frac{1}{2}|\nabla_y u + \nabla_x u^0|^2 - V(y, m^0 m) = 0,$$

which gives the formula for the effective Hamiltonian: for every $P \in \mathbb{R}^n$ and $\alpha \geq 0$, $\bar{H}(P, \alpha)$ is the unique constant for which there exists a periodic solution $u = u(y)$ to

$$-\Delta u + \frac{1}{2}|\nabla u + P|^2 - V(y, \alpha m) = \bar{H}(P, \alpha),$$

and $u_t = \bar{H}(\nabla u_0, \alpha)$ has to hold.

Summing up, the cell problem is the ergodic MFG system

$$\begin{cases} -\Delta u + \frac{1}{2}|\nabla u + P|^2 - V(y, \alpha m) = \bar{H}(P, \alpha), \\ -\Delta m - \mathrm{div}\,(m\,(\nabla u + P)) = 0, \\ \int_{\mathbb{T}^n} u = 0, \quad \int_{\mathbb{T}^n} m = 1. \end{cases}$$

This cell problem permits us to define the effective operators $\bar{H}(P, \alpha)$ and

$$\bar{b}(P, \alpha) = \int_{\mathbb{T}^n} m(\nabla u + P)dy.$$

So the expected limit system is (2).

3.1. Other asymptotic expansions.

3.1.1. Time-dependent asymptotic expansion and ϵ-correction to initial data. We also consider a fast time-scale dependence in the asymptotic expansion. This allows for order ϵ-corrections to the initial/terminal data and leads to a different (time-dependent) cell problem but the same system of effective equations. More precisely, consider the ansatz

$$\begin{cases} u^\epsilon(x,t) = u^0(x,t) + \epsilon u\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right), \\ m^\epsilon(x,t) = m^0(x,t)\left(m\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) + \epsilon m^2\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right)\right). \end{cases}$$

Then, at the highest order (i.e., at order ϵ^0 for the u-equation and at order ϵ^{-1} for the m-equation) we get the new parabolic cell problem

$$\begin{cases} c - u_\tau - \Delta_y u + \frac{1}{2}|\nabla_y u + P|^2 - V(y, \alpha m) = 0, \\ m_\tau - \Delta_y m - \operatorname{div}_y(m(P + \nabla_y u)) = 0 \end{cases} \tag{9}$$

with $c \equiv -u^0_t$, $P \equiv \nabla_x u^0$, and $\alpha \equiv m^0$. The effective equation for u^0 is determined through solvability conditions for (9). Note that u^ϵ will converge to u^0 if

$$\lim_{\epsilon\to 0} \epsilon u\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) = 0, \tag{10}$$

whereas m^ϵ converges weakly to m^0 if, at least, m belongs to some $L^p(\mathbb{T}^n \times (0, +\infty))$ space. Then there exists a unique constant $\bar{H}(P, \alpha)$ such that (9) admits a solution with these properties and $u^0_t = \bar{H}(\nabla_x u^0, m^0)$.

This $\bar{H}$ is expected to be the same as the one in (3) by the analysis of the long time behavior of (9), which is similar to the result in [13]. More precisely, one can expect that $\epsilon u\left(\frac{t}{\epsilon}, y\right) \to (1-t)\lambda$, uniformly in y and $t \in (0,1)$, where λ is the unique constant for which there exists a periodic solution to the system

$$\begin{cases} \lambda - \Delta_y u + \frac{1}{2}|\nabla_y u + P|^2 - \widetilde{V}(y, \alpha m) = 0, \\ -\Delta_y m - \operatorname{div}(m(P + \nabla_y u)) = 0, \end{cases}$$

where $\widetilde{V}(y, \alpha m) := V(y, \alpha m) - c$. Choosing c as the unique constant for which (3) has a solution, by uniqueness of the ergodic constant, there holds $\lambda = 0$ (see [13]). Finally, we get (10) for solutions of (9), and, as $c = u^0_t$, we recover the limiting u-equation in (2). On the other hand, again by [13], we expect that $m\left(y, \frac{t}{\epsilon}\right)$ converges weakly to $m(y)$ in some L^p space.

The limit equation in m was found as a solvability condition for m^2, which now should be bounded in $\mathbb{T}au$, a necessary condition for $m(x/\epsilon, t/\epsilon)$ to converge weakly in some L^p space. As the equation at order ϵ^0 is now of the form

$$m^2_{\mathbb{T}} au = \Delta_y m^2 + R,$$

where R contains all nonlinear terms of order zero, we see that the solution can only be periodic in y and bounded as $\mathbb{T}au \to \infty$ if the mean of R vanishes, which yields the same effective equation as before.

3.1.2. Case of finite noise. We consider the MFG system

$$
\begin{cases} -u_t^\epsilon - \Delta u^\epsilon + \frac{1}{2}|Du^\epsilon|^2 = V\left(\frac{x}{\epsilon}, m^\epsilon\right), & x \in \mathbb{R}^n,\ t \in (0,T), \\ m_t^\epsilon - \Delta m^\epsilon - \operatorname{div}(m^\epsilon \nabla u^\epsilon) = 0, & x \in \mathbb{R}^n,\ t \in (0,T). \end{cases} \tag{11}
$$

In this case, the natural formal ansatz for the solution to the system (11) is

$$
u^\epsilon(x,t) = u^0(x,t) + \epsilon^2 u\left(\frac{x}{\epsilon}\right), \quad m^\epsilon(x,t) = m^0(x,t)\left(m\left(\frac{x}{\epsilon}\right) + \epsilon^2 m^2\left(\frac{x}{\epsilon}\right)\right). \tag{12}
$$

We insert this asymptotic expansion into (11), denoting $X = \Delta_x u^0$, $P = \nabla u^0$, $\alpha = m^0$, and $y = \frac{x}{\epsilon}$.

At order ϵ^{-2} in the second equation we obtain $-\Delta m = 0$, where $m(\cdot)$ is assumed to be periodic and with mean 1. This implies $m \equiv 1$; i.e., $m^\epsilon(x,t) = m^0(x,t)(1 + \epsilon^2 m^2\left(\frac{x}{\epsilon}\right))$, or, in other words, *strong* convergence of m^ϵ to m^0 can be expected. Inserting the asymptotic expansion into the first equation in (11), we get the following cell problem: for every $X \in M_n(\mathbb{R}^n)$, symmetric matrix, $P \in \mathbb{R}^n$, and $\alpha \geq 0$, $\bar{H}(X,P,\alpha)$ is the unique constant for which there exists a periodic solution $u(\cdot)$ to

$$
-\Delta u - \operatorname{tr} X + \frac{1}{2}|P|^2 - V(y,\alpha) = \bar{H}(X,P,\alpha).
$$

So $\bar{H}(X,P,\alpha) = -\operatorname{tr} X + \frac{1}{2}|P|^2 - \int_{\mathbb{T}^n} V(y,\alpha)dy$ (see [6, Chap. 2]).

On the other hand, the solvability condition for m^2 gives the limit equation for m^0. In conclusion, the expected effective system is

$$
\begin{cases} -u_t^0 - \Delta u^0 + \frac{1}{2}|\nabla u^0|^2 - \int_{\mathbb{T}^n} V(y, m^0)dy = 0, \\ m_t^0 - \Delta m^0 - \operatorname{div}\left(m^0 \nabla u^0\right) = 0. \end{cases}
$$

In this case the cell problem decouples, while the limit problem still has an MFG structure. Note that the limit coupling $\bar{V}(m) = \int_{\mathbb{T}^n} V(y,m)dy$ satisfies (5). Hence, the effective system fulfills uniqueness of solutions by a well-known argument (see [26]).

3.1.3. Case of nonlocal coupling. In the case of nonlocal coupling we also expect that the cell problem decouples. We consider the following example to illustrate this issue (see [10, Rem. 2.10]). We define $\mathcal{L}(m) = w$ as the periodic solution to $-\Delta w = m - \int_{\mathbb{T}^n} m(x)dx$ with $w(0) = 0$, and we consider

$$
\begin{cases} -u_t^\epsilon - \epsilon\Delta u^\epsilon + \frac{1}{2}|Du^\epsilon|^2 = V\left(\frac{x}{\epsilon}, \mathcal{L}(m^\epsilon(\cdot,t))\right), & x \in \mathbb{T}^n,\ t \in (0,T), \\ m_t^\epsilon - \epsilon\Delta m^\epsilon - \operatorname{div}(m^\epsilon \nabla u^\epsilon) = 0, & x \in \mathbb{T}^n,\ t \in (0,T). \end{cases} \tag{13}
$$

We consider the formal asymptotic expansion (6). We add the ansatz that also $w^\epsilon = \mathcal{L}(m^\epsilon)$ satisfies the asymptotic expansion which is the standard expansion for homogenization of second order equations

$$
w^\epsilon(x,t) = w^0(x,t) + \epsilon^2 w\left(\frac{x}{\epsilon}\right),
$$

where $-\Delta w^0 = m^0 - 1$, and w is the periodic function with zero average such that $-\Delta w(y) = m^0(x)(m(y) - 1)$. This implies that in the cell problem m no longer appears in the first equation. So, the cell system is

$$
\begin{cases} -\Delta u + \frac{1}{2}|\nabla u + P|^2 - V(y,\alpha) = \bar{H}(P,\alpha), \\ -\Delta m - \operatorname{div}\left(m\left(\nabla u + P\right)\right) = 0, \\ \int_{\mathbb{T}^n} u = 0, \qquad \int_{\mathbb{T}^n} m = 1, \end{cases}
$$

where $P = \nabla_x u^0$ and $\alpha = w^0$.

In this case the expected limit system is given by

$$\begin{cases} -u_t^0 + \bar{H}(\nabla u^0, \mathcal{L}(m^0(\cdot,t))) = 0, & x \in \mathbb{R}^n,\ t \in (0,T), \\ m_t^0 - \operatorname{div}(m^0 \bar{b}(\nabla u^0, m^0)) = 0, & x \in \mathbb{R}^n,\ t \in (0,T). \end{cases}$$

4. Cell problem, effective Hamiltonian, and effective vector field. This section is devoted to the definition and the properties of the effective operators. In the first proposition, we tackle the solvability of the cell problem obtained by the formal asymptotic expansion.

PROPOSITION 2. *Under the standing assumptions, for every $P \in \mathbb{R}^n$ and $\alpha \geq 0$, consider the system*

$$\begin{cases} \text{(i)} & -\Delta u + \frac{1}{2}|\nabla u + P|^2 - V(y, \alpha m) = \bar{H}(P,\alpha), \qquad x \in \mathbb{T}^n, \\ \text{(ii)} & -\Delta m - \operatorname{div}\left(m\left(\nabla u + P\right)\right) = 0, \qquad x \in \mathbb{T}^n, \\ \text{(iii)} & \fint_{\mathbb{T}^n} u = 0, \qquad \fint_{\mathbb{T}^n} m = 1. \end{cases} \tag{14}$$

Then there exists a unique constant $\bar{H}(P,\alpha)$ such that the system admits a solution (u, m). Moreover, this solution is unique, and $u \in C^{2,\gamma}$, $m \in W^{1,p}$ for all $\gamma \in (0,1)$ and all $p > 1$.

Finally, there exists a constant $c > 0$ such that $m \geq c > 0$.

Proof. Observe that if $\alpha = 0$, in the first equation there is no m-dependence. So, we solve the ergodic problem (14)(i) (see [4]). Then we plug the solution u into (14)(ii), getting a solution m (see [3]), with the desired properties.

So, let us assume that $\alpha > 0$. We use an argument similar to that in [8]. We consider $K := \{m \in C^{0,\gamma}(\mathbb{T}^n) \mid m$ Lebesgue-density of a measure$\}$; observe that K is a closed subset of $C^{0,\gamma}(\mathbb{T}^n)$. We introduce an operator $T : K \to K$, $m' \mapsto m$ as follows: given $m' \in K$, we solve the ergodic equation (14)(i) with m replaced by m' coupled with the first condition in (14)(iii). Then, we get m solving (14)(ii) coupled with the second condition in (14)(iii).

Let us prove that T is well-posed. By [4], there exists a unique pair $(u, \bar{H}) \in W^{2,p}(\mathbb{T}^n) \times \mathbb{R}$ for any $p \geq 1$ (u unique up to an additive constant) solving (14)(i). In particular, ∇u is Hölder continuous. By [3, Thm. 4.2], problem (14)(ii) admits a solution $m \in W^{1,p}(\mathbb{T}^n)$ which is unique up to multiplicative constants. Hence, the mapping T is well defined.

Let us now prove the T is continuous and compact. To this end, we recall from [4] that there hold

$$|\bar{H}| \leq K + |P|^2, \qquad \|u\|_{W^{2,p}(\mathbb{T}^n)} \leq C,$$

where K depends on $\|V\|_\infty$ and C is a constant depending only on $\|V\|_\infty$, P, and p. On the other hand, we also recall from [3, Thm. 4.2] and [2, Lemma 2.3] that there holds

$$\|m\|_{W^{1,p}(\mathbb{T}^n)} \leq C',$$

where C' is a constant depending only on $\|\nabla u\|_\infty$, and so in particular only from $\|V\|_\infty$ and P.

Consider a sequence $\{m'_n\}$ with $m'_n \in K$ and $m'_n \to m'$ in the $C^{0,\gamma}$-topology. Therefore, the corresponding solutions $(u'_n, \bar{H}_n)$ to (14)(i) with m replaced by m'_n coupled with the first condition in (14)(iii) are uniformly bounded in $W^{2,p}(\mathbb{T}^n) \times \mathbb{R}$. Consequently, the corresponding solutions m_n to (14)(ii) coupled with the second condition in (14)(iii) are uniformly bounded in $W^{1,p}(\mathbb{T}^n)$. So, by possibly passing to a subsequence, we get $u_n' \to u^*$ and $\bar{H}_n \to \bar{H}^*$. By stability and by uniqueness of problem (14)(i), we get $u^* = u$ and $\bar{H}^* = \bar{H}$. In a similar manner, we get $m_n \to m$. Hence, we have accomplished the proof of the continuity of T.

The map T is also compact because $m \in W^{1,p}(\mathbb{T}^n)$ for any $p \geq 1$; in particular, $m \in C^{0,\alpha'}(\mathbb{T}^n)$ for any $\alpha' \in (0,1)$. By the compactness of the embedding $C^{0,\gamma'}(\mathbb{T}^n) \to C^{0,\gamma}(\mathbb{T}^n)$ with $\gamma' > \gamma$, we infer the compactness of T.

We conclude, by Schauder's fixed point theorem, the proof of the existence of a solution $(u,m) \in W^{2,p}(\mathbb{T}^n) \times W^{1,p}(\mathbb{T}^n)$ to (14) for any $p > 1$. By a standard bootstrap argument, we obtain the claimed regularity, i.e., $u \in C^{2,\gamma}$.

The last statement is proved in [3, Thm. 4.3, page 136]. ◻

We now prove some properties of the effective Hamiltonian $\bar{H}$ and the effective vector field $\bar{b}$, defined, respectively, in (14) and (4).

Proposition 3. *$\bar{H}$ is coercive in P, that is,*

$$\frac{|P|^2}{2} - \|V\|_\infty \leq \bar{H}(P,\alpha) \leq \frac{|P|^2}{2},$$

and is decreasing in α.

Moreover, for any $\gamma \in (0,1)$, $p \in (1,+\infty)$, the maps

$$(P,\alpha) \to \bar{H}(P,\alpha) \in \mathbb{R}, \qquad (P,\alpha) \to \bar{b}(P,\alpha) \in \mathbb{R}, \qquad (P,\alpha) \to (u,m) \in C^{1,\gamma} \times W^{1,p}$$

are all continuous, where (u,m) is the solution to (14).

Proof. Coercivity. Let $y \in \mathbb{T}^n$ be a maximum point of u, which is a solution to (14)(i). Then

$$\bar{H}(P,\alpha) \geq \frac{|P|^2}{2} - \|V\|_\infty,$$

which implies the desired coercivity. Moreover, if we compute (14)(i) at a minimum point of u, recalling that $V \geq 0$, we get

$$\bar{H}(P,\alpha) \leq \frac{|P|^2}{2}.$$

Monotonicity. Denote $m^{\alpha_1} = \alpha_1 m^1$, where (u^1, m^1) solves (14)(ii) with (P, α_1) and $m^{\alpha_2} = \alpha_2 m^2$, where (u^2, m^2) solves (14)(ii) with (P, α_2). Let $\bar{u} = u^1 - u^2$ and $\bar{m} = m^{\alpha_1} - m^{\alpha_2}$. Then $(\bar{u}, \bar{m})$ solves

$$\begin{cases} -\Delta\bar{u} + \frac{1}{2}|\nabla u^1 + P|^2 - \frac{1}{2}|\nabla u^2 + P|^2 - V(y, m^{\alpha_1}) + V(y, m^{\alpha_2}) \\ \qquad = \bar{H}(P,\alpha_1) - \bar{H}(P,\alpha_2), \\ -\Delta\bar{m} - \operatorname{div}(m^{\alpha_1}(\nabla u^1 + P) - m^{\alpha_2}(\nabla u^2 + P)) = 0, \\ \int_{\mathbb{T}^n} \bar{u} = 0, \quad \int_{\mathbb{T}^n} \bar{m} = \alpha_1 - \alpha_2. \end{cases}$$

We multiply the first equation by $\bar{m}$ and the second by $\bar{u}$, we integrate, and we subtract one equation from the other to obtain

$$(\bar{H}(P,\alpha_1) - \bar{H}(P,\alpha_2))(\alpha_1 - \alpha_2) = -\int (V(y, m^{\alpha_1}) - V(y, m^{\alpha_2}))(m^{\alpha_1} - m^{\alpha_2})$$

$$+ \int \bar{m}\left(\frac{1}{2}|\nabla u^1 + P|^2 - \frac{1}{2}|\nabla u^2 + P|^2\right) - \nabla\bar{u} \cdot (m^{\alpha_1}(\nabla u^1 + P) - m^{\alpha_2}(\nabla u^2 + P)).$$

It is easy to check that if $q_1, q_2 \in \mathbb{R}^n$ and $n_1, n_2 \geq 0$, then

$$(n_1 - n_2)\left(\frac{|q_1|^2}{2} - \frac{|q_2|^2}{2}\right) - (q_1 - q_2)\cdot(n_1 q_1 - n_2 q_2) = -\frac{n_1 + n_2}{2}|q_1 - q_2|^2 \leq 0.$$

Then, applying this equality to $q_1 = \nabla u^1 + P$, $q_2 = \nabla u^2 + P$, $n_1 = m^{\alpha_1}$, $n_2 = m^{\alpha_2}$, we get that

$$(\bar{H}(P,\alpha_1) - \bar{H}(P,\alpha_2))(\alpha_1 - \alpha_2) \leq -\int (V(y, m^{\alpha_1}) - V(y, m^{\alpha_2}))(m^{\alpha_1} - m^{\alpha_2}) \leq 0.$$

Continuity. Consider a sequence $(P_n, \alpha_n) \to (P, \alpha)$. Observe that $\bar{H}(P_n, \alpha_n)$ are uniformly bounded in n due to coerciveness. For every n, let the pair (u_n, m_n) be the solution to the cell problem (14) with (P, α) replaced by (P_n, α_n). By the same argument as in [4], we get a priori bounds on ∇u_n independent of n; i.e., there exists $C > 0$ such that $|\nabla u_n| \leq C$ for every n. So, up to a subsequence, $\nabla u_n \to v$ weakly* in L^∞. Moreover, $\|m_n\|_{W^{1,p}} \leq K$ for every $p \in (1, \infty)$ and every n (see [2, Lemma 2.3]). So, by Morrey's inequality, for p sufficiently large, there exists a subsequence m_n converging in $C^{0,\gamma}$ to m. So, up to extracting a converging sequence, we get, by stability of viscosity solutions, that $u_n \to u$, where u is a viscosity solution to

$$-\Delta u + \frac{1}{2}|\nabla u + P|^2 - V(y, \alpha m) = \lim_n \bar{H}(P_n, \alpha_n).$$

Moreover, u is Lipschitz, and, by a standard argument, therefore even smooth. This implies that $v = \nabla u$, and then by uniqueness, along any converging subsequences, $\lim_n \bar{H}(P_n, \alpha_n) = \bar{H}(P, \alpha)$, $\lim_n u_n = u$ uniformly, and $\lim_n m_n = m$ uniformly and in $W^{1,p}$ for $p \in (1, +\infty)$, where (u, m) is the solution to (14). By standard elliptic regularity theory and a priori bounds on the solution to (14)(i) with uniformly Holder continuous source term (see also Proposition 2), the convergence of u_n is also in $C^{1,\gamma}$. ∎

We conclude by recalling a variational characterization of the effective Hamiltonian.

Following [26], we introduce the following energy functional in $H^1(\mathbb{T}^n) \times H^1(\mathbb{T}^n)$ for every fixed $P \in \mathbb{R}^n$ and $\alpha \geq 0$:

$$E_{P,\alpha}(v, n) = \int_{\mathbb{T}^n} n \frac{|\nabla v + P|^2}{2} + \nabla n \cdot (\nabla v + P) - \Phi_\alpha(y, n) dy, \tag{15}$$

where $(\Phi_\alpha)_n \equiv \frac{\partial \Phi_\alpha}{\partial n}(y, n) = V(y, \alpha n)$.

LEMMA 4.

$$\frac{\partial E_{P,\alpha}}{\partial m}(u, m) = 0 \quad \textit{iff } (u, m) \textit{ solves } (14)(\mathrm{i}),$$

$$\frac{\partial E_{P,\alpha}}{\partial u}(u, m) = 0 \quad \textit{iff } (u, m) \textit{ solves } (14)(\mathrm{ii}).$$

Moreover, for the (u, m) *solution to* (14), *there holds true*

$$\bar{H}(P, \alpha) = E_{P,\alpha}(u, m) + \int_{\mathbb{T}^n} (\Phi_\alpha(y, m) - V(y, \alpha m) m)\, dy. \tag{16}$$

Proof. The first statement is a straightforward computation. We give a quick sketch of it. Fix m, and let u be the solution to (14)(i) with this fixed m. Then, for every $\epsilon > 0$ small and smooth ϕ with $\int_{\mathbb{T}^n} \phi = 0$, we get

$$\begin{aligned}\frac{E_{P,\alpha}(u, m+\epsilon\phi) - E_{P,\alpha}(u,m)}{\epsilon} &= \int_{\mathbb{T}^n} \phi \frac{|\nabla u + P|^2}{2} + \nabla\phi \cdot (\nabla u + P) \\ &\quad - \int_{\mathbb{T}^n} \frac{\Phi_\alpha(y, m+\epsilon\phi) - \Phi_\alpha(y,m)}{\epsilon}.\end{aligned}$$

So, letting $\epsilon \to 0$, and using the fact that u solves (14)(i), we get

$$\lim_{\epsilon\to 0} \frac{E_{P,\alpha}(u, m+\epsilon\phi) - E_{P,\alpha}(u,m)}{\epsilon} = \bar{H}(P,\alpha) \int_{\mathbb{T}^n} \phi = 0.$$

The other implication is obtained by reverting this argument. Analogous arguments work for the second statement.

Finally, in order to prove (16), we first observe that multiplying (14)(ii) by u and integrating over $\mathbb{T}^n$ leads to

$$\int \Delta u \; m = \int m\,(\nabla u + P)\,\nabla u.$$

Now, multiplying (14)(i) by m and integrating over $\mathbb{T}^n$, we infer (16). ∎

5. Regularity properties of the effective operators. In this section we study the relation between the effective Hamiltonian and the effective vector field. The main result is the computation of $\nabla_P \bar{H}(P,\alpha)$ established in Theorem 7. To do this, we first derive local Lipschitz estimates of $\bar{H}$. Finally, we will provide some related regularity results also for the vector field $\bar{b}$.

5.1. Variation of $\bar{H}$ with respect to P. Now fix δ small, and consider for every $i = 1, \dots, n$ the cell problem (14) associated to $(P + \delta e_i, \alpha)$, where $\{e_i\}_{i=1,\dots,n}$ is an orthonormal basis of $\mathbb{R}^n$. We denote with (u_i^δ, m_i^δ) the solution. First, by Proposition 3,

$$\lim_{\delta\to 0} u_i^\delta = u \;\text{ in } C^{1,\gamma}, \qquad \lim_{\delta\to 0} m_i^\delta = m \quad \text{in } C^0, \tag{17}$$

where (u,m) is the solution to (14) associated to (P,α).

Our aim is to characterize the following functions and compute their limits as $\delta \to 0^+$:

$$w_i^\delta(x) := \frac{u_i^\delta - u}{\delta}, \qquad n_i^\delta = \frac{m_i^\delta - m}{\delta}. \tag{18}$$

Note that (w_i^δ, n_i^δ) is a solution of the following system:

$$\begin{cases} \text{(i)} & -\Delta w_i^\delta + (\nabla w_i^\delta + e_i) \cdot \frac{2P + \delta e_i + \nabla u + \nabla u_i^\delta}{2} - V_m(y, \alpha\widetilde{n}^\delta)\alpha n_i^\delta \\ & = \frac{\bar{H}(P+\delta e_i,\alpha) - \bar{H}(P,\alpha)}{\delta}, \\ \text{(ii)} & -\Delta n_i^\delta - \operatorname{div}\big((P + \delta e_i + \nabla u_i^\delta) n_i^\delta\big) = \operatorname{div}(m(\nabla w_i^\delta + e_i)), \\ \text{(iii)} & \int_{\mathbb{T}^n} n_i^\delta = \int_{\mathbb{T}^n} w_i^\delta = 0, \end{cases} \tag{19}$$

where we used the mean value theorem to write

$$\frac{1}{\delta}(V(y, \alpha m_i^\delta) - V(y, \alpha m)) = V_m(y, \alpha\widetilde{n}^\delta)\alpha n_i^\delta \tag{20}$$

for some $\widetilde{n}^\delta$ such that $n^\delta(y) \in (m(y), m_i^\delta(y))$.

First we prove a priori estimates on w_i^δ, n_i^δ and prove that, as $\delta \to 0$, the right-hand side of (19)(i) does not explode.

PROPOSITION 5. *Let $\alpha > 0$. There is a constant C depending on (P, α), such that, for any δ sufficiently small,*

$$\|n_i^\delta\|_2^2 + \|\nabla w_i^\delta\|_2^2 \le C$$

and

$$\frac{\bar{H}(P+\delta e_i, \alpha) - \bar{H}(P, \alpha)}{\delta} \le C. \tag{21}$$

This implies in particular that $\bar{H}$ is locally Lipschitz in P.

Proof. We set $f := \nabla u + P + \delta e_i$. Note that $\|f\|_2^2$ is bounded uniformly in δ by Proposition 2. Hence, we can write (19)(i)–(ii) as

$$\begin{cases} \text{(i)} \quad -\Delta w_i^\delta + (\nabla w_i^\delta + e_i) \cdot f + \frac{\delta}{2}(|\nabla w_i^\delta|^2 - 1) - V_m(y, \alpha\widetilde{n}^\delta)\alpha n_i^\delta \\ \qquad = \frac{\bar{H}(P+\delta e_i, \alpha) - \bar{H}(P, \alpha)}{\delta}, \\ \text{(ii)} \quad -\Delta n_i^\delta - \operatorname{div}\big((f + \delta\nabla w_i^\delta) n_i^\delta\big) = \operatorname{div}\big(m(\nabla w_i^\delta + e_i)\big). \end{cases}$$

Now we test (i) with n_i^δ and (ii) with w_i^δ. Note that n_i^δ has mean zero, so the constant term on the right-hand side of (i) drops out. We subtract and get

$$\int \left((e_i \cdot f) n_i^\delta + \frac{\delta}{2}(|\nabla w_i^\delta|^2 - 1) n_i^\delta - V_m(y, \alpha\widetilde{n}^\delta)\alpha (n_i^\delta)^2 - \delta |\nabla w_i^\delta|^2 n_i^\delta \right)$$
$$= \int \left(m |\nabla w_i^\delta|^2 + (\nabla w_i^\delta \cdot e_i) m \right);$$

i.e., recalling that $\int n_i^\delta = 0$,

$$\int \left((e_i \cdot f) n_i^\delta - (\nabla w_i^\delta \cdot e_i) m \right) = \int \left(V_m(y, \alpha\widetilde{n}^\delta)\alpha (n_i^\delta)^2 + \left(\frac{\delta}{2} n_i^\delta + m\right) |\nabla w_i^\delta|^2 \right).$$

Since m and m_i^δ are bounded in L^∞ uniformly in δ (i.e., no concentrations phenomena appear), our assumptions on V (monotone with derivative bounded away from zero) allow us to estimate $V_m(\cdot) > \gamma_K > 0$ for some constant γ_K depending on m (by (5), (17), and (20)), and moreover $\frac{\delta}{2} n_i^\delta + m = \frac{1}{2}\left(m_i^\delta + m\right) > c > 0$ by Proposition 2 and (17). So the right-hand side is $\ge c(\|n_i^\delta\|_2^2 + \|\nabla w_i^\delta\|_2^2)$, and we conclude with Young's inequality that there exists a constant C which depends only on a priori estimates of the correctors, on α and P, such that

$$\|n_i^\delta\|_2^2 + \|\nabla w_i^\delta\|_2^2 \le C(\nabla u, m).$$

This together with testing (i) with the constant 1 gives (21). ◻

In order to characterize the limit as $\delta \to 0$ of the w_i^δ, n_i^δ solution to (19), we introduce an auxiliary system, and we study existence and uniqueness of its solutions.

LEMMA 6. *Let (u, m) be the solution to (14), with $\alpha > 0$. Then for every $i = 1, \ldots, n$ there exists a unique $c_i(P, \alpha) \in \mathbb{R}$ such that there exists a solution $(\tilde{u}_i, \tilde{m}_i)$ to* *the system*

$$
(22)\quad \begin{cases} \text{(i)} & -\Delta\tilde{u}_i + \nabla\tilde{u}_i\cdot(\nabla u + P) + (\nabla u + P)\cdot e_i - V_m(y,\alpha m)\alpha\tilde{m}_i = c_i(P,\alpha),\\ \text{(ii)} & -\Delta\tilde{m}_i - \operatorname{div}\big((P+\nabla u)\tilde{m}_i\big) = \operatorname{div}(m(\nabla\tilde{u}_i + e_i)),\\ \text{(iii)} & \int_{\mathbb{T}^n}\tilde{m}_i = \int_{\mathbb{T}^n}\tilde{u}_i = 0. \end{cases}
$$

Moreover, the solution is unique and smooth; i.e., $(\tilde{u}_i,\tilde{m}_i)\in C^{2,\gamma}\times W^{1,p}$ *for all* $\gamma\in(0,1)$ *and all* $p>1$.

Proof. Let us observe that the adjoint operator to the one in (i) coincides with the operator in (14)(ii) which has a 1-dimensional kernel (see [3, Thm. 4.3]), so the compatibility condition for the existence of a solution to (i) reads as follows:

$$
(23)\qquad c_i(P,\alpha) = \int_{\mathbb{T}^n}((\nabla u + P)_i m - V_m(y,\alpha m)\alpha\tilde{m}_i m)\,dy.
$$

For the uniqueness of the solution of the system, the argument is quite standard. Let $(\tilde{u}_i,\tilde{m}_i,c_i(P,\alpha))$ and $(\tilde{v}_i,\tilde{n}_i,k_i(P,\alpha))$ be two solutions to (22). Let $\tilde{u}=\tilde{u}_i-\tilde{v}_i$ and $\tilde{m}=\tilde{m}_i-\tilde{n}_i$. Then $\tilde{u},\tilde{m}$ solve the following system:

$$
\begin{cases} \text{(i)} & -\Delta\tilde{u} + \nabla\tilde{u}\cdot\nabla u - V_m(y,\alpha m)\alpha\tilde{m} = c_i(P,\alpha) - k_i(P,\alpha),\\ \text{(ii)} & -\Delta\tilde{m} - \operatorname{div}\big(\nabla u\tilde{m}\big) = \operatorname{div}(m\nabla\tilde{u}),\\ \text{(iii)} & \int_{\mathbb{T}^n}\tilde{m} = \int_{\mathbb{T}^n}\tilde{u} = 0. \end{cases}
$$

We multiply (i) by $\tilde{m}$ and (ii) by $\tilde{u}$, subtract (ii) from (i), and integrate on the torus $\mathbb{T}^n$: so we obtain, after some easy integrations by parts,

$$
-\int_{\mathbb{T}^n}\big(V_m(y,\alpha m)\alpha\tilde{m}^2 + |\nabla u|^2 m\big) = 0.
$$

Due to assumption (5) on V and on the fact that $m>0$, both terms in the previous integral are positive. This implies that $\tilde{m}=0$, so $\tilde{m}_i=\tilde{n}_i$, and moreover $c_i(P,\alpha)=k_i(P,\alpha)$ and $u_i=v_i$.

For the existence of a solution, we argue by standard fixed point argument (see, e.g., [2, 10]). We briefly sketch the argument. First note that both equations in (22) are linear, with coefficients in $C^{0,\gamma}$ (due to our assumptions on V and to Proposition 2).

Fix now $\bar{n}\in C^{0,\gamma}(\mathbb{T}^n)$, with $\int_{\mathbb{T}^n}\bar{n}=0$, and solve (i) with this fixed $\bar{n}$ in place of $\tilde{m}_i$. We obtain that there exists a unique constant $c_i^{\bar{n}}(P,\alpha)$, given by (23), for which the equation admits a solution v. This solution is unique, by the constraint on the average, and smooth, say in $C^{2,\gamma}$ (since it is Lipschitz, and then we apply standard elliptic regularity theory). Now, we replace m with v in (22)(ii) and solve it. We get that there exists a unique solution $l\in W^{1,p}$ for every $p>1$, with the constraint that $\int_{\mathbb{T}^n}l=0$. Indeed, the existence of a one parameter family of solutions to (ii) in $W^{1,2}$ is obtained by the Fredholm alternative (see, e.g., [3]), and uniqueness is obtained by adding the constraint on the average. The enhanced regularity can be obtained as in [2, Lemma 2.3].

So, we constructed a map

$$
(24)\qquad T:\mathcal{B} := \left\{\bar{n}\in C^{0,\gamma}\ \Big|\ \int_{\mathbb{T}^n}\bar{n}=0\right\}\longrightarrow\mathcal{B}
$$

such that $T:\bar{n}\to(v,c_i^n(P,\alpha))\to l$. The continuity of such a map can be obtained as in [10, Thm. 3.1]. Let m_n be a sequence in $\mathcal{B}$ converging uniformly to $\bar{n}$; let u_n be the solution to (i) with m_n and v the solution to (i) with n. Then $c_i^{m_n}(P,\alpha)\to c_i^{\bar{n}}(P,\alpha)$ and $V_m(y,\alpha m)\alpha m_n\to V_m(y,\alpha m)\alpha\bar{n}$ uniformly. By stability of viscosity solutions, we get that $u_n\to v$ uniformly. Moreover, ∇u_n are uniformly bounded in $C^{0,\gamma}$ (due to standard elliptic regularity theory and uniform convergence of the coefficients of equation (i); see also [2, Lemma 2.2]), so we can extract a subsequence $\nabla u_n\to\nabla v$ uniformly. Let μ_n and ν be the solutions to (ii) with ∇u replaced by, respectively, ∇u_n and ∇v, so $\mu_m=T(m_n)$ and $\nu=T(\bar{n})$. By the L^∞ uniform bound on ∇u_n, we get that μ_n are uniformly bounded in $W^{1,p}$ for every $p>1$ (see [2, Lemma 2.3]), and then by Sobolev embedding, they are uniformly bounded in $C^{0,\gamma}$. Passing to a converging subsequence, we get that $\mu_n\to l$ uniformly, and moreover l is a weak solution to (ii), with ∇v. By uniqueness we conclude that $l=\nu$; moreover, again by uniqueness of the limits, we have convergence for the full sequence. This gives continuity of the operator T. Compactness can be obtained as in [2, Thm. 2.1]. This allows us to conclude by Schauder's fixed point theorem. ◻

THEOREM 7. *Let $\alpha>0$. For every $i=1,\dots,n$,*

$$\lim_{\delta\to0}\frac{\bar{H}(P+\delta e_i,\alpha)-\bar{H}(P,\alpha)}{\delta}=\bar{b}_i(P,\alpha)-\int_{\mathbb{T}^n}V_m(y,\alpha m)\alpha\tilde{m}_i m dy,$$

where m is the solution to (14)(ii) *and $\tilde{m}_i$ is the solution to* (22)(ii).

Proof. Note that the coefficients of the system (19), due to (17), are converging to the coefficients of the system (22). Moreover, $\frac{\bar{H}(P+\delta e_i,\alpha)-\bar{H}(P,\alpha)}{\delta}$ is bounded by (21); then up to a subsequence, we can assume it is converging to some constant. Moreover, due to the a priori bounds in Proposition 5, we can extract subsequences $\nabla w_i^\delta, n_i^\delta$ converging weakly in L^2. By stability and by uniqueness of the solution to (22), we get the convergence of w_i^δ, n_i^δ to $\tilde{u}_i,\tilde{m}_i$, and by uniqueness of the constant c_i for which the system (22) admits a solution we get that

$$c_i(P,\alpha)=\lim_{\delta\to0}\frac{\bar{H}(P+\delta e_i,\alpha)-\bar{H}(P,\alpha)}{\delta},$$

which gives the desired conclusion, recalling formula (23). ◻

5.2. Variation of $\bar{H}$ with respect to α. We shall proceed as in the previous section to compute the variation of $\bar{H}$ with respect to α.

LEMMA 8. *Let $\alpha>0$. Then $\bar{H}$ is locally Lipschitz in α, i.e., there is a constant C depending on P,α, such that, for any δ sufficiently small,*

$$\frac{\bar{H}(P,\alpha+\delta)-\bar{H}(P,\alpha)}{\delta}\le C.$$

Proof. The proof is similar to that of Proposition 5.

For δ small we consider the solution (u^δ,m^δ) to the cell problem (14) associated to $(P,\alpha+\delta)$. By Proposition 3, we get that as $\delta\to0$, $u^\delta\to u$ in $C^{1,\gamma}$, $m^\delta\to m$ in C^0. The functions

$$w^\delta=\frac{u^\delta-u}{\delta},\quad n^\delta=\frac{m^\delta-m}{\delta}$$

fulfill

$$
(25) \qquad \begin{cases} \text{(i)} & -\Delta w^\delta + \delta \frac{|\nabla w^\delta|^2}{2} + \nabla w^\delta \cdot (\nabla u + P) - V_m(y, \alpha \tilde{n}^\delta)(\alpha n^\delta + m^\delta) \\ & = \frac{\bar{H}(P,\alpha+\delta) - \bar{H}(P,\alpha)}{\delta}, \\ \text{(ii)} & -\Delta n^\delta - \operatorname{div}\big((P + \nabla u^\delta) n^\delta\big) = \operatorname{div}(m(\nabla w^\delta)), \\ \text{(iii)} & \int_{\mathbb{T}^n} n^\delta = \int_{\mathbb{T}^n} w^\delta = 0 \end{cases}
$$

for some $\tilde{n}^\delta(y) \in (m(y), (1 + \frac{\delta}{\alpha}) m^\delta(y))$. We multiply (i) by n^δ and (ii) by w^δ, we subtract (ii) from (i), and, recalling that w^δ, n^δ have mean zero, we get

$$
\int \frac{(m^\delta + m)}{2} |\nabla w^\delta|^2 + V_m(y, \alpha \tilde{n}^\delta) \alpha (n^\delta)^2 = - \int V_m(y, \alpha \tilde{n}^\delta) n^\delta m^\delta.
$$

By the Young inequality, this implies

$$
\int \frac{(m^\delta + m)}{2} |\nabla w^\delta|^2 + \frac{1}{2} V_m(y, \alpha \tilde{n}^\delta) \alpha (n^\delta)^2 \le \frac{1}{2\alpha} \int V_m(y, \alpha \tilde{n}^\delta)(m^\delta)^2,
$$

which, in particular, recalling that $V_m \ge \gamma_K$ and $m^\delta + m > 0$, implies $\|n^\delta\|_2^2 + \|\nabla w^\delta\|_2^2 \le C$, with a constant depending on m, u, P, α.

By testing (25)(i) by 1 and integrating over $\mathbb{T}^n$, these bounds imply the desired local Lipschitz continuity. ∎

Lemma 9. *Let (u, m) be the solution to* (14)*. Then there exists a unique $k(P, \alpha)$ such that there exists a solution $(\bar{u}, \bar{m})$ to the system*

$$
(26) \qquad \begin{cases} \text{(i)} & -\Delta \bar{u} + \nabla \bar{u} \cdot (\nabla u + P) - V_m(y, \alpha m) \alpha \bar{m} - V_m(y, \alpha m) m = k(P, \alpha), \\ \text{(ii)} & -\Delta \bar{m} - \operatorname{div}\big((P + \nabla u) \bar{m}\big) = \operatorname{div}(m \nabla \bar{u}), \\ \text{(iii)} & \int_{\mathbb{T}^n} \bar{m} = \int_{\mathbb{T}^n} \bar{u} = 0. \end{cases}
$$

Moreover, the solution is unique and smooth; i.e., $(\bar{u}, \bar{m}) \in C^{2,\gamma} \times W^{1,p}$ for all $\gamma \in (0, 1)$ and all $p > 1$.

Moreover,

$$
k(P, \alpha) = - \int_{\mathbb{T}^n} \left[V_m(y, \alpha m)(m + \alpha \bar{m})^2 + \alpha m |\nabla \bar{u}|^2 \right] dy.
$$

Proof. The proof is completely analogous (with some minor modifications) to the proof of Lemma 6, so we omit it.

To obtain the representation formula for $k(P, \alpha)$, we multiply (26)(i) by $(m + \alpha \bar{m})$, where m solves (14)(ii), and integrate on $\mathbb{T}^n$. We obtain, after some integration by parts,

$$
k(P, \alpha) = \int_{\mathbb{T}^n} \alpha \operatorname{div}(m \nabla \bar{u}) \bar{u} - V_m(y, \alpha m)(m + \alpha \bar{m})^2 dy,
$$

which gives the desired conclusion. ∎

Theorem 10. *Let $\alpha > 0$.*

$$
\lim_{\delta \to 0} \frac{\bar{H}(P, \alpha + \delta) - \bar{H}(P, \alpha)}{\delta} = - \int_{\mathbb{T}^n} \left[V_m(y, \alpha m)(m + \alpha \bar{m})^2 + \alpha m |\nabla \bar{u}|^2 \right] dy < 0,
$$

where m is the solution to (14)(ii) *and $(\bar{u}, \bar{m})$ is the unique solution to* (26)*. In particular, $\bar{H}$ is strictly decreasing in α.*

Proof. The proof is completely analogous (with some minor modifications) to the proof of Theorem 7, so we omit it. ∎

5.3. Regularity properties of $\bar{b}$.

THEOREM 11. *Let $P \in \mathbb{R}^n$ and $\alpha > 0$. Then $\bar{b}$ is locally Lipschitz continuous with respect to P and α. Moreover,*

$$\lim_{\delta \to 0} \frac{\bar{b}(P+\delta e_i, \alpha) - \bar{b}(P,\alpha)}{\delta} = e_i + \int_{\mathbb{T}^n} [\tilde{m}_i \nabla u + m \nabla \tilde{u}_i] \, dy,$$

where $(\tilde{u}_i, \tilde{m}_i)$ is the solution to (22) and

$$\lim_{\delta \to 0} \frac{\bar{b}(P, \alpha+\delta) - \bar{b}(P,\alpha)}{\delta} = \int_{\mathbb{T}^n} [\bar{m} \nabla u + m \nabla \bar{u}] \, dy,$$

where $(\bar{u}, \bar{m})$ is the solution to (26).

Proof. Note that

$$\frac{\bar{b}(P+\delta e_i, \alpha) - \bar{b}(P,\alpha)}{\delta} = \int_{\mathbb{T}^n} \left[n_i^\delta (\nabla u_i^\delta + P + \delta e_i) + m(\nabla w_i^\delta + e_i) \right] dy,$$

where n_i^δ and w_i^δ are as defined in (18). Note that the right-hand side in the previous equality is bounded by a constant depending on P, α, u, m due to Proposition 5 and to (17). This gives locally Lipschitz continuity of $\bar{b}$ with respect to P.

So, by the proof of Theorem 7 and by (17),

$$\lim_{\delta \to 0} \frac{\bar{b}(P+\delta e_i, \alpha) - \bar{b}(P,\alpha)}{\delta} = \int_{\mathbb{T}^n} [\tilde{m}_i (\nabla u + P) + m(\nabla \tilde{u}_i + e_i)] \, dy.$$

An analogous argument gives the statement for the variation with respect to α. ∎

6. Qualitative properties of the effective operators. In this section we provide some qualitative properties of the effective operators $\bar{H}, \bar{b}$: their asymptotic limit as $|P| \to +\infty$, and an explicit example where the effective system (2) loses the MFG structure, i.e., $\nabla_P \bar{H} \neq \bar{b}$.

PROPOSITION 12. *Let $\bar{H}$ and $\bar{b}$ be the effective operators defined in (14) and (4). Then*

$$\lim_{|P| \to +\infty} \frac{\bar{H}(P,\alpha)}{|P|^2} = \frac{1}{2} \qquad \text{and} \qquad \lim_{|P| \to +\infty} \frac{|\bar{b}(P,\alpha) - P|}{|P|} = 0,$$

uniformly for $\alpha \in [0, +\infty)$.

Proof. We multiply (14)(i) by $\frac{m-1}{|P|^2}$. We integrate and get, recalling the periodicity assumptions and that m has mean 1,

$$\begin{aligned} 0 &= \int_{\mathbb{T}^n} -\frac{m \Delta u}{|P|^2} + \frac{1}{2|P|^2} \left| \nabla u + P \right|^2 (m-1) - \frac{V(y, \alpha m)}{|P|^2}(m-1) \\ &= \int_{\mathbb{T}^n} -\frac{m \Delta u}{|P|^2} + \frac{|\nabla u|^2}{2|P|^2}(m-1) + \nabla u \cdot \frac{P}{|P|^2} m - \frac{V(y,\alpha m)}{|P|^2}(m-1). \end{aligned} \tag{27}$$

We multiply (14)(ii) by $\frac{u}{|P|^2}$, integrate, and get, recalling the periodicity assumptions,

$$0 = \int_{\mathbb{T}^n} -\frac{u \Delta m}{|P|^2} + m \frac{|\nabla u|^2}{|P|^2} + m \nabla u \cdot \frac{P}{|P|^2}. \tag{28}$$

We subtract (27) from (28) and get

$$\int_{\mathbb{T}^n} \frac{|\nabla u|^2}{2|P|^2}(m+1) + \frac{V(y,\alpha m)}{|P|^2}(m-1) = 0. \tag{29}$$

We observe that, by monotonicity of V,

$$V(y,\alpha m)(m-1) \geq V(y,\alpha)(m-1).$$

So (29) gives

$$\int_{\mathbb{T}^n} \frac{|\nabla u|^2}{2|P|^2}(m+1) + \frac{V(y,\alpha)}{|P|^2} m dy \leq \int_{\mathbb{T}^n} \frac{V(y,\alpha)}{|P|^2} dy.$$

Recalling that $V \geq 0$ and $m \geq 0$, we obtain

$$\int_{\mathbb{T}^n} \frac{|\nabla u|^2}{2|P|^2}(m+1) dy \leq \frac{\int_{\mathbb{T}^n} V(y,\alpha) dy}{|P|^2}. \tag{30}$$

Since $m \geq 0$, by Proposition 2, this implies that $\frac{\nabla u}{|P|} \to 0$ in L^2 as $|P| \to +\infty$ (and then also $\frac{u}{|P|} \to 0$ in L^2 as $|P| \to +\infty$ by the Poincaré inequality).

We multiply (14)(i) by $\frac{1}{|P|^2}$ and integrate to get

$$\int_{\mathbb{T}^n} \frac{|\nabla u + P|^2}{2|P|^2} - \frac{V(y,\alpha m)}{|P|^2} dy = \frac{\bar{H}(P,\alpha)}{|P|^2}. \tag{31}$$

We recall that V is bounded and, moreover, by the triangle inequality

$$\frac{1}{\sqrt{2}} - \left(\int_{\mathbb{T}^n} \frac{|\nabla u|^2}{2|P|^2}\right)^{1/2} \leq \left(\int_{\mathbb{T}^n} \frac{|\nabla u + P|^2}{2|P|^2} dy\right)^{1/2} \leq \frac{1}{\sqrt{2}} + \left(\int_{\mathbb{T}^n} \frac{|\nabla u|^2}{2|P|^2}\right)^{1/2}.$$

Then

$$\lim_{|P| \to +\infty} \left(\int_{\mathbb{T}^n} \frac{|\nabla u + P|^2}{2|P|^2} dy\right)^{1/2} = \frac{1}{\sqrt{2}}.$$

So, letting $|P| \to +\infty$ in (31), we get the desired result.

By definition (4), we get

$$\bar{b}(P,\alpha) - P = \int_{\mathbb{T}^n} m(y) \nabla u(y) dy.$$

So,

$$\frac{\left|\bar{b}(P,\alpha) - P\right|}{|P|} \leq \frac{1}{|P|} \int_{\mathbb{T}^n} m(y) |\nabla u(y)| dy.$$

By Hölder's inequality and (30) we get

$$\frac{\left|\bar{b}(P,\alpha) - P\right|}{|P|} \leq \left(\int_{\mathbb{T}^n} \frac{|\nabla u|^2}{|P|^2} m dy\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^n} m dy\right)^{\frac{1}{2}} \leq \frac{\sqrt{2 \int_{\mathbb{T}^n} V(y,\alpha) dy}}{|P|}.$$

So, letting $|P| \to +\infty$, we conclude that

$$\frac{\left|\bar{b}(P,\alpha) - P\right|}{|P|} \to 0.$$

$\square$

6.1. A case in which the limit system is not MFG. We show that in general the effective system (2) is not an MFG system by providing an example in dimension $n = 1$ where

$$\nabla_P \bar{H}(P,\alpha) \neq \bar{b}(P,\alpha). \tag{32}$$

We assume that the potential has the following form:

$$V(y,m) = v(y) + m, \quad \text{with } v \geq 0.$$

So the cell problem reads

$$\begin{cases} -\Delta u + \frac{1}{2}|\nabla u + P|^2 - \alpha m - v(y) = \bar{H}(P,\alpha), & y \in \mathbb{T}^n, \\ -\Delta m - \operatorname{div}\left(m\left(\nabla u + P\right)\right) = 0, & y \in \mathbb{T}^n, \\ \int_{\mathbb{T}^n} u(y)dy = 0, \qquad \int_{\mathbb{T}^n} m(y)dy = 1. & \end{cases} \tag{33}$$

Note that in this case the potential does not satisfy the standing assumptions, since it is not bounded. Moreover, we will see that the previous results still apply.

LEMMA 13. *Let $n \leq 3$.*

(i) *For every $P \in \mathbb{R}^n$, $\alpha \geq 0$ there exists a unique constant $\bar{H}(P,\alpha)$ such that (33) admits a solution (u,m). Moreover, this solution is unique, $m > 0$, and $u \in C^{2,\gamma}$, $m \in W^{1,p}$ for every $\gamma \in (0,1)$ and $p > 1$.*

(ii) *There hold $\lim_{|P|\to+\infty} \frac{\bar{H}(P,\alpha)}{|P|^2} = \frac{1}{2}$ and $\lim_{|P|\to+\infty} \frac{|\bar{b}(P,\alpha)-P|}{|P|} = 0$, locally uniformly for $\alpha \in [0,+\infty)$.*

(iii) *The maps $(P,\alpha) \to \bar{H}(P,\alpha), \bar{b}(P,\alpha)$ are continuous.*

(iv) *There holds*

$$\nabla_P \bar{H}(P,\alpha) = \bar{b}(P,\alpha) - \frac{\alpha}{2}\nabla_P(\|m\|_{L^2}^2).$$

Proof. (i) This existence result can be found in [16, Thm. 1.4]; see also [30] and [19].

(ii) Arguing as in the proof of Proposition 12, we obtain (29), which in this case reads

$$\int_{\mathbb{T}^n} \frac{|\nabla u|^2}{2|P|^2}(m+1) + \frac{v(y) + \alpha m}{|P|^2}(m-1)dy = 0. \tag{34}$$

This implies, recalling that $v \geq 0$,

$$\int_{\mathbb{T}^n} (m+1)\frac{|\nabla u|^2}{2|P|^2} + \frac{\alpha m^2}{|P|^2} \leq \frac{\int_{\mathbb{T}^n} v(y)dy + \alpha}{|P|^2}, \tag{35}$$

which in turns gives, since $m \geq 0$, that $\nabla u/|P|, u/|P| \to 0$ in L^2 as $|P| \to +\infty$, and that, locally uniformly in $\alpha \geq 0$,

$$\frac{\bar{H}(P,\alpha)}{|P|^2} = \int_{\mathbb{T}^n} \frac{|\nabla u + P|^2}{2|P|^2} - \frac{v(y) + \alpha m}{|P|^2}dy \to \frac{1}{2} \qquad \text{as } |P| \to +\infty.$$

Moreover, arguing as in the proof of Proposition 12, we get also that

$$\frac{|\bar{b}(P,\alpha) - P|}{|P|} \to 0 \qquad \text{as } |P| \to +\infty, \quad \text{locally uniformly in } \alpha \geq 0.$$

(iii) Integrating the first equation in (33), we get

$$\bar{H}(P,\alpha) \geq \frac{|P|^2}{2} - \alpha - \int v(y)dy.$$

Moreover, if we multiply the first equation in (33) by m, the second by u, and integrate, we get that

$$\bar{H}(P,\alpha) \leq \frac{|P|^2}{2}.$$

Arguing again as in the proof of Proposition 3, we get the statement.

(iv) By (iii), the same arguments as in the proof of Theorem 7 apply. So, using the explicit formula of V, we have that

$$\frac{d}{dP_i}\bar{H}(P,\alpha) = \bar{b}_i(P,\alpha) - \int_{\mathbb{T}^n} \alpha\tilde{m}_i m dy = \bar{b}_i(P,\alpha) - \frac{\alpha}{2}\frac{d}{dP_i}(\|m\|^2_{L^2}),$$

where $\tilde{m}_i$ is the solution to (22). ∎

We observe that, for every P, m cannot be a constant; actually, it would imply that u is also a constant, which would contradict the first equation in (33). Jensen's inequality implies

$$\|m\|_{L^2} > 1 = \|m\|^2_{L^1}. \tag{36}$$

Assume for the moment that up to a subsequence

$$\|m\|_{L^2} \to 1 \qquad \text{as } |P| \to +\infty; \tag{37}$$

then necessarily $\frac{d}{dP_i}(\|m\|^2_{L^2}) \neq 0$ at some i, P, α, and so (32) holds true. Then we are left with the proof of (37).

Proposition 14. *Let $n = 1$ and (u, m) be the solution to* (33). *Then* (37) *holds true.*

Proof. By (35) we get that $\|m\|_{L^2}$ and $\|u\|_{W^{1,2}}$ are uniformly bounded with respect to $|P|$.

Now we prove that $\sqrt{m}$ is uniformly bounded with respect to P in $W^{1,2}$ using the same argument of [13, Lemma 2.5]. We multiply the second equation in (33) by $\log m$ and integrate. We get, using periodicity,

$$\int_{\mathbb{T}^1} \frac{|m'|^2}{m} + m'u'dy = 0.$$

So, by the Cauchy–Schwarz inequality,

$$\int_{\mathbb{T}^1} \frac{|m'|^2}{m}dy = -\int_{\mathbb{T}^1} m'u'dy \leq \frac{1}{2}\int_{\mathbb{T}^1} \frac{|m'|^2}{m}dy + \frac{1}{2}\int_{\mathbb{T}^1} m|u'|^2dy.$$

Therefore, by (35), we conclude

$$\int_{\mathbb{T}^1} |(\sqrt{m})'|^2dy = \frac{1}{4}\int_{\mathbb{T}^1} \frac{|m'|^2}{m}dy \leq \frac{1}{4}\int_{\mathbb{T}^1} m|u'|^2dy \leq \frac{\alpha + \int_{\mathbb{T}^1} v(y)dy}{4}.$$

Since $n = 1$, the embedding of $W^{1,2}(0,1)$ in $C(0,1)$ is compact, so we conclude that, possibly passing to a subsequence, $\sqrt{m} \to \sqrt{m_\infty}$ uniformly as $|P| \to +\infty$. This implies that $m \to m_\infty$ in $C(0,1)$ as $P \to +\infty$, and then also strongly in L^2.

We are left to prove that $m_\infty = 1$.

It is sufficient to prove that $m \to 1$ weakly in L^2 as $|P| \to +\infty$. We apply to the second equation in (33) a smooth periodic test function ϕ, and we divide by $|P|$. By periodicity, we get

$$\int_{\mathbb{T}^1} m\phi' dy = \int_{\mathbb{T}^1} \frac{m\phi''}{|P|} - \frac{m\phi' \cdot u'}{|P|} dy.$$

Letting $|P| \to +\infty$ and recalling that $\nabla u/|P| \to 0$ in L^2 and that $\|m\|_{L^2}$ is uniformly bounded with respect to $|P|$, we get

$$\lim_{|P|\to+\infty} \int_{\mathbb{T}^1} m\phi' dy = 0 \tag{38}$$

for every smooth periodic test function ϕ. We observe that every smooth periodic function ψ can be written as $c + \phi'$, where $c = \int_0^1 \psi$ and ϕ is still periodic. So, since $\int_0^1 m = 1$, we get

$$\lim_{|P|\to+\infty} \int_{\mathbb{T}^1} (m-1)\psi dy = \lim_{|P|\to+\infty} \int_{\mathbb{T}^1} [(m-1)c + m\phi' - \phi'] dy = \lim_{|P|\to+\infty} \int_{\mathbb{T}^1} m\phi' dy = 0. \qquad \square$$

7. Convergence for affine-constant initial data. We prove the homogenization result in the special case where the initial and terminal data of system (1) are affine and constant, respectively. Our arguments are based on some a priori estimates which are inspired by estimates used in [13] for investigating the long time behavior for MFG systems.

Fix $P \in \mathbb{R}^n$, and consider the MFG system

$$\begin{cases} -u_t^\epsilon - \epsilon\Delta u^\epsilon + \frac{1}{2}|\nabla u^\epsilon|^2 = V\left(\frac{x}{\epsilon}, m^\epsilon\right), & x \in \mathbb{R}^n,\ t \in (0,T), \\ m_t^\epsilon - \epsilon\Delta m^\epsilon - \operatorname{div}(m^\epsilon \nabla u^\epsilon) = 0, & x \in \mathbb{R}^n,\ t \in (0,T), \\ u^\epsilon(x,T) = P \cdot x, \qquad m^\epsilon(x,0) \equiv 1, & x \in \mathbb{R}^n. \end{cases} \tag{39}$$

Actually the initial data $P \cdot x$ is not periodic, so Proposition 1 does not apply directly. In order to preserve the periodicity, from now on $\epsilon = \frac{1}{k}$, with $k \in \mathbb{N}$.

We construct a solution as follows.

LEMMA 15. *There exists a smooth solution (u^ϵ, m^ϵ) to the MFG system (39). Moreover, the maps $x \mapsto u^\epsilon(x,t) - P \cdot x$, $x \mapsto \nabla u^\epsilon(x,t)$, $x \mapsto m^\epsilon(x,t)$ are $\epsilon\mathbb{Z}^n$-periodic for every $t \in [0,T]$. Finally, the solution is unique among all solutions such that $u^\epsilon(x,t) - P \cdot x$ and m^ϵ are $\epsilon\mathbb{Z}^n$-periodic.*

Proof. Let (w^ϵ, m^ϵ) be the unique $\epsilon\mathbb{Z}^n$-periodic smooth solution to the MFG system

$$\begin{cases} -w_t^\epsilon - \epsilon\Delta w^\epsilon + H^P(\nabla w^\epsilon) = V\left(\frac{x}{\epsilon}, m^\epsilon(x,t)\right), & x \in \mathbb{R}^n,\ t \in (0,T), \\ m_t^\epsilon - \epsilon\Delta m^\epsilon - \operatorname{div}(m^\epsilon \nabla_P H^P(\nabla w^\epsilon)) = 0, & x \in \mathbb{R}^n,\ t \in (0,T), \\ w^\epsilon(x,T) = 0, \qquad m^\epsilon(x,0) = 1, \end{cases} \tag{40}$$

where

$$H^P(q) = \frac{|q+P|^2}{2}.$$

Note that H^P is strictly convex and has superlinear growth; moreover, V is monotone in the second argument, so (40) admits a unique $\epsilon\mathbb{Z}^n$-periodic solution (see [13, Lemma 4.2]).

Define $u^\epsilon(x,t) = w^\epsilon(x,t) + P\cdot x$. Then (u^ϵ, m^ϵ) is a smooth solution to (39). Finally, $u^\epsilon(x,t) - P\cdot x$, $m^\epsilon(x,t)$ and ∇u^ϵ are $\epsilon\mathbb{Z}^n$-periodic with respect to x. ◻

We recall the definition of $\bar{H}(P,\alpha)$ and $\bar{b}(P,\alpha)$ given, respectively, in Proposition 2 and in (4). In particular, $\bar{H}(P,1)$ is the unique constant such that there exists a solution (u^1, m^1) to the cell system

$$\begin{cases} -\Delta u^1 + \frac{1}{2}|\nabla u^1 + P|^2 - V(y, m^1) = \bar{H}(P,1), \\ -\Delta m^1 - \operatorname{div}\left(m^1\left(\nabla u^1 + P\right)\right) = 0, \\ \int_{\mathbb{T}^n} u^1 = 0, \qquad \int_{\mathbb{T}^n} m^1 = 1, \end{cases} \tag{41}$$

and

$$\bar{b}(P,1) = \int_{\mathbb{T}^n} (\nabla u^1 + P) m^1(y)dy.$$

We recall that, by [3, Thm. 4.3, page 136], $m^1 > 0$; in particular, there exist constants $\delta > 0$ and $K > 0$ such that $0 < \delta \le m^1(y) \le K$.

It is easy to check that (u_0, m_0) with

$$u^0(t,x) = P\cdot x + (t-T)\bar{H}(P,1), \qquad m^0(t,x) \equiv 1 \tag{42}$$

is a solution to the limit system

$$\begin{cases} -u_t + \bar{H}(\nabla u, m) = 0, & x\in\mathbb{R}^n,\ t\in(0,T), \\ m_t - \operatorname{div}(\bar{b}(\nabla u, m)m) = 0, & x\in\mathbb{R}^n,\ t\in(0,T), \\ u(x,T) = P\cdot x, \quad m(x,0) = 1, & x\in\mathbb{R}^n. \end{cases} \tag{43}$$

Let us now state our convergence result; its proof is postponed to section 7.2.

THEOREM 16. *Let (u^ϵ, m^ϵ) be the solution of the MFG system (39) defined in Lemma 15. Then for every compact Q in $\mathbb{R}^n$,*

(i) *$u^\epsilon \to u^0$ in $L^2(Q\times[0,T])$,*

(ii) *$m^\epsilon \to m^0$ weakly in $L^p(Q\times[0,T])$ for $1 \le p < (n+2)/n$ if $n \ge 3$ and for $p < 2$ if $n = 2$,*

where (u^0, m^0) is the solution to the effective problem (43) defined in (42).

7.1. A priori estimates. Let (u^ϵ, m^ϵ) be the solution to (39) given by Lemma 15. Consider the functions

$$\begin{cases} v^\epsilon(y,t) = \frac{1}{\epsilon}u^\epsilon(\epsilon y, t) - P\cdot y - \frac{1}{\epsilon}(t-T)\bar{H}(P,1) - u^1(y), \\ n^\epsilon(y,t) = m^\epsilon(\epsilon y, t) - m^1(y). \end{cases}$$

By Lemma 15, we get that v^ϵ, n^ϵ are both $\mathbb{Z}^n$-periodic and smooth functions. It is a straightforward computation to check that they solve

$$\begin{cases} -\epsilon v_t^\epsilon - \Delta v^\epsilon + H_P(y, \nabla v^\epsilon) = V\left(y, m^1 + n^\epsilon\right) - V\left(y, m^1\right), & y\in\mathbb{T}^n,\ t\in(0,T), \\ \epsilon n_t^\epsilon - \Delta n^\epsilon - \operatorname{div}(n^\epsilon \nabla_q H_P(y, \nabla v^\epsilon)) = \operatorname{div}(m^1 \nabla v^\epsilon), & y\in\mathbb{T}^n,\ t\in(0,T), \\ v^\epsilon(y,T) = -u^1(y) \quad n^\epsilon(y,0) = 1 - m^1(y), & y\in\mathbb{T}^n, \end{cases} \tag{44}$$

where

$$H_P(y,q) = \frac{|q|^2}{2} + q\cdot(P+\nabla u^1(y)). \tag{45}$$

Note that system (44) is not an MFG system because of the presence of the term $\operatorname{div}(m^1\nabla v^\epsilon)$. Nevertheless, due to the divergence structure, since $\int_{\mathbb{T}^n} n^\epsilon(y,0)dy = 0$ and $n^\epsilon(\cdot,t)$, $\nabla v^\epsilon(\cdot,t)$, $\nabla u^1(\cdot)$ are $\mathbb{Z}^n$-periodic, we get that for every $t\in[0,T]$,

$$\int_{\mathbb{T}^n} n^\epsilon(y,t)dy = 0. \tag{46}$$

LEMMA 17. *For every $t_1,t_2\in[0,T]$ with $t_1\le t_2$, we get*

$$-\epsilon\left[\int_{\mathbb{T}^n} v^\epsilon n^\epsilon\right]_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\mathbb{T}^n} \frac{2m^1+n^\epsilon}{2}|\nabla v^\epsilon|^2 + \left[V(y,m^1+n^\epsilon)-V(y,m^1)\right]n^\epsilon dy. \tag{47}$$

Proof. We multiply the first equation in (44) by n^ϵ and the second by v^ϵ, subtract the first equation from the second, and integrate in $\mathbb{T}^n\times[t_1,t_2]$:

$$\begin{aligned} 0 &= \int_{t_1}^{t_2}\int_{\mathbb{T}^n} -\epsilon(v^\epsilon n^\epsilon)_t - \Delta v^\epsilon n^\epsilon + \Delta n^\epsilon v^\epsilon + \operatorname{div}(n^\epsilon(\nabla v^\epsilon + P + \nabla u^1))v^\epsilon \\ &\quad + \int_{t_1}^{t_2}\int_{\mathbb{T}^n} n^\epsilon\left[\frac{|\nabla v^\epsilon|^2}{2} - \big(V(y,m^1+n^\epsilon)-V(y,m^1)\big) + \nabla v^\epsilon\cdot(P+\nabla u^1)\right] \\ &\quad + \int_{t_1}^{t_2}\int_{\mathbb{T}^n} \operatorname{div}(m^1\nabla v^\epsilon)v^\epsilon. \end{aligned}$$

Recalling the periodicity of $n^\epsilon, v^\epsilon, m^1$, we get (47). ∎

We define the following functional on $H^1(\mathbb{T}^n)\times H^1(\mathbb{T}^n)$:

(48)
$$E(v,n) = \int_{\mathbb{T}^n}(n(y)+m^1(y))H_P(y,\nabla v(y)) + \nabla v(y)\cdot(\nabla n(y)+\nabla m^1(y)) - \Phi^1(y,n)dy,$$

where (u^1,m^1) is the solution to the cell problem (41), H_P is as defined in (45), and

$$\Phi^1(y,n) = \int_0^n V(y,s+m^1)-V(y,m^1)ds. \tag{49}$$

Note that due to the fact that V is increasing in the second variable, $\Phi^1\ge 0$.

LEMMA 18. *Let (v^ϵ,n^ϵ) be the solution to (44). Then there exists a constant C_P depending on P (independent of ϵ) such that*

$$E(v^\epsilon(\cdot,t),n^\epsilon(\cdot,t))\le C_P \quad \forall t\in[0,T].$$

Moreover,

$$\|\nabla v^\epsilon(\cdot,0)\|_{L^2(\mathbb{T}^n)}\le C_P. \tag{50}$$

Proof. First we compute

$$\begin{aligned} \frac{d}{dt}E(v^\epsilon(\cdot,t),n^\epsilon(\cdot,t)) &= \int_{\mathbb{T}^n} n_t^\epsilon H_P(y,\nabla v^\epsilon) + (n^\epsilon+m^1)\nabla_q H_P(y,\nabla v^\epsilon)\nabla v_t^\epsilon dy \\ &\quad + \int_{\mathbb{T}^n}\nabla v_t^\epsilon\cdot(\nabla n^\epsilon+\nabla m^1) + \nabla v^\epsilon\cdot\nabla n_t^\epsilon - \Phi_n^1(y,n^\epsilon)n_t^\epsilon dy; \end{aligned}$$

integrating by parts and recalling (44) and (41), we get

$$\frac{d}{dt}E(v^\epsilon(\cdot,t),n^\epsilon(\cdot,t)) = \int_{\mathbb{T}^n} n_t^\epsilon \left[H_P(y,\nabla v^\epsilon(y,t)) - \Delta v^\epsilon - V(y,n^\epsilon+m^1) + V(y,m^1)\right]$$
$$+ \int_{\mathbb{T}^n} v_t^\epsilon \left[-\mathrm{div}(n^\epsilon \nabla_q H_P(y,\nabla v^\epsilon)) - \mathrm{div}(m^1\nabla v^\epsilon) - \Delta n^\epsilon\right]$$
$$+ \int_{\mathbb{T}^n} v_t^\epsilon \left[-\Delta m^1 - \mathrm{div}(m^1(P+\nabla u^1))\right] = \int_{\mathbb{T}^n} \epsilon v_t^\epsilon n_t^\epsilon - \epsilon v_t^\epsilon n_t^\epsilon$$
$$= 0.$$

Therefore, for every $t \in [0,T]$

$$E(v^\epsilon(\cdot,t),n^\epsilon(\cdot,t)) \equiv M_T \quad \forall t \in [0,T].$$

Then, recalling that $\Phi^1 \geq 0$ and $v^\epsilon(y,T) = -u^1(y)$,

$$M_T = E(v^\epsilon(\cdot,T),n^\epsilon(\cdot,T)) \leq \int_{\mathbb{T}^n} (\Delta u^1 + H_P(y,-\nabla u^1))(n^\epsilon(y,T)+m^1(y)).$$

By definition of H_P in (45) and by the fact that $n^\epsilon + m^1 \geq 0$ with $\int_{\mathbb{T}^n} n^\epsilon(y,T) + m^1(y) = 1$ (by (46)),

$$M_T \leq \int_{\mathbb{T}^n} \left(\Delta u^1 - \nabla u^1\cdot P - \frac{|\nabla u^1|^2}{2}\right)(n^\epsilon(y,T)+m^1(y))$$
$$\leq \int_{\mathbb{T}^n} \left(\Delta u^1 + \frac{|P|^2}{2}\right)(n^\epsilon(y,T)+m^1(y))dy \leq \|u^1\|_{C^2} + \frac{|P|^2}{2} := C_P,$$

which amounts to the first part of the statement. Let us now prove (50). Recalling that $n^\epsilon(0,y) + m^1(y) \equiv 1$,

$$C_P \geq E(v^\epsilon(0,\cdot),n^\epsilon(0,\cdot)) = \int_{\mathbb{T}^n} H_P(y,\nabla v^\epsilon(0,y)) - \Phi^1(y,1-m^1(y))$$
$$\geq \int_{\mathbb{T}^n} \frac{1}{4}|\nabla v^\epsilon(0,y)|^2 - |P+\nabla u^1|^2 - \Phi^1(y,1-m^1(y))dy,$$

where the last inequality is a consequence of the Cauchy–Schwarz inequality. Note that by definition (49) we have that

$$\Phi^1(y,1-m^1(y)) = \int_0^{1-m^1(y)} V(y,s+m^1) - V(y,m^1)ds \leq (V(y,1)-V(y,m^1))(1-m^1).$$

Then,

$$\int_{\mathbb{T}^n} |\nabla v^\epsilon(0,y)|^2 \leq 4C_P + 4\int_{T^n} \left[|P+\nabla u^1|^2 + (V(y,1)-V(y,m^1))(1-m^1)\right]dy,$$

which implies that $|\nabla v^\epsilon(0,\cdot)|$ is bounded in $L^2(\mathbb{T}^n)$. ∎

We are ready to prove the main lemma on a priori bounds.

LEMMA 19. *There exists a constant C_P depending on P such that*

$$\int_0^T \int_{\mathbb{T}^n} \frac{2m^1+n^\epsilon}{2}|\nabla v^\epsilon|^2 + \left[V(y,m^1+n^\epsilon) - V(y,m^1)\right] n^\epsilon dy \leq C_P\epsilon.$$

Moreover,

$$\lim_{\epsilon\to 0}\int_0^T\int_{\mathbb{T}^n}|\nabla v^\epsilon(y,t)|^2dydt=0 \qquad \text{and} \qquad \lim_{\epsilon\to 0}\int_0^T\int_{\mathbb{T}^n}|n^\epsilon(y,t)|dydt=0.$$

Proof. By the initial/terminal data of system (44) and since $\int_{\mathbb{T}^n} m^1 = 1$, we obtain that

$$\begin{aligned}\left[\int_{\mathbb{T}^n} v^\epsilon n^\epsilon\right]_0^T &= \int_{\mathbb{T}^n} v^\epsilon(0,y)(1-m^1(y))dy+\int_{\mathbb{T}^n} n^\epsilon(T,y)u^1(y)\\ &=\int_{\mathbb{T}^n}(v^\epsilon(0,y)-\int_{\mathbb{T}^n} v^\epsilon(0,z)dz)(1-m^1(y))dy\\ &\quad+\int_{\mathbb{T}^n}(n^\epsilon(T,y)+m^1(y))u^1(y)-\int_{\mathbb{T}^n} m^1(y)u^1(y).\end{aligned}$$

So using the Cauchy–Schwarz and Poincaré inequalities in the first term, we get

$$\left[\int_{\mathbb{T}^n} v^\epsilon n^\epsilon\right]_0^T\le C\|\nabla v^\epsilon(0,\cdot)\|_{L^2(\mathbb{T}^n)}+K_P,$$

where $K_P=\|u^1\|_\infty(1+\|m^1\|_\infty)$. So, by Lemma 17 and by (50), we get that

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^n}\frac{2m^1+n^\epsilon}{2}|\nabla v^\epsilon|^2+\left[V(y,m^1+n^\epsilon)-V(y,m^1)\right]n^\epsilon dy&\\ =-\epsilon\left[\int_{\mathbb{T}^n} v^\epsilon n^\epsilon\right]_0^T\le C_P\epsilon,&\end{aligned}\tag{51}$$

and in particular, by the monotonicity of V,

$$\int_0^T\int_{\mathbb{T}^n}\frac{2m^1+n^\epsilon}{2}|\nabla v^\epsilon|^2\le C_P\epsilon.$$

Now we recall that $m^\epsilon(t,\epsilon y)=m^1(y)+n^\epsilon(t,y)\ge 0$ for every $t\in[0,T]$ and $y\in\mathbb{T}^n$; moreover, $m^1\ge\delta>0$. So we get that

$$\int_0^T\int_{\mathbb{T}^n}|\nabla v^\epsilon|^2\le\frac{C_P}{\delta}\epsilon.$$

Again by (51) we get

$$\lim_{\epsilon\to 0}\int_0^T\int_{\mathbb{T}^n}(V(y,m^1+n^\epsilon)-V(y,m^1))n^\epsilon=0.$$

For every $K>0$, we write

$$\begin{aligned}O(\epsilon)&=\int_0^T\int_{\mathbb{T}^n}(V(y,m^1+n^\epsilon)-V(y,m^1))n^\epsilon\\ &=\int_0^T\int_{\mathbb{T}^n}1_{\{|n^\epsilon|\le K\}}(V(y,m^1+n^\epsilon)-V(y,m^1))n^\epsilon\\ &\quad+\int_0^T\int_{\mathbb{T}^n}1_{\{|n^\epsilon|\ge K\}}(V(y,m^1+n^\epsilon)-V(y,m^1))n^\epsilon.\end{aligned}$$

Taking advantage of the strict monotonicity (5), we have that $(V(y,m^1+n^\epsilon) - V(y,m^1))n^\epsilon \geq 0$ and also that there exists C depending on K and m^1 such that

$$O(\epsilon) = \int_0^T \int_{\mathbb{T}^n} 1_{\{|n^\epsilon|\leq K\}}(V(y,m^1+n^\epsilon) - V(y,m^1))n^\epsilon$$
$$\geq C \int_0^T \int_{\mathbb{T}^n} 1_{\{|n^\epsilon|\leq K\}}|n^\epsilon|^2 \geq \frac{C}{T}\left(\int_0^T \int_{\mathbb{T}^n} 1_{\{|n^\epsilon|\leq K\}}|n^\epsilon|\right)^2. \tag{52}$$

Again by (5), possibly increasing C, there holds

$$|V(y,n^\epsilon+m^1) - V(y,m^1)| > C \qquad \text{on } \{y\in\mathbb{T}^n:\ |n^\epsilon|\geq K\}.$$

Hence

$$O(\epsilon) = \int_0^T \int_{\mathbb{T}^n} 1_{\{|n^\epsilon|\geq K\}}(V(y,m^1+n^\epsilon) - V(y,m^1))n^\epsilon$$
$$\geq C \int_0^T \int_{\mathbb{T}^n} 1_{\{|n^\epsilon|\geq K\}}|n^\epsilon|. \tag{53}$$

In conclusion, by (52) and (53), we get $n^\epsilon \to 0$ in $L^1(\mathbb{T}^n\times[0,T])$. ◻

Proposition 20. *We have that*

$$n^\epsilon(x,t) = m^\epsilon(\epsilon x,t) - m^1(x) \to 0 \quad \text{ in } L^p(\mathbb{T}^n\times[0,T])$$

for any $p < (n+2)/n$ if $n\geq 3$ and for any $p<2$ for $n=2$.

Proof. Following the arguments in [13, Lemma 2.5], we get that there exists a constant C independent of ϵ such that

$$\int_0^T \int_{\mathbb{T}^n} |\nabla\sqrt{m^\epsilon(\epsilon x,t)}|^2 dxdt \leq C.$$

From this, by the same arguments as in [13, Corollary 2], we get that $m^\epsilon(\epsilon x,t)$ is bounded in $L^{p'}(\mathbb{T}^n\times[0,T])$ for any $p'\leq (n+2)/n$ if $n\geq 3$ and for any $p'<2$ for $n=2$. Hence, possibly passing to a subsequence, $m^\epsilon(\epsilon x,t)$ weakly converges to some function $\mu(x,t)$ in the same space. Now, we have that $\mu\geq 0$ a.e., $\int_{\mathbb{T}^n}\mu = 1$, and, moreover, for every smooth test function $\phi=\phi(x)$, by the weak convergence and Lemma 19, we have

$$\int_{t_1}^{t_2} \int_{\mathbb{T}^n} [-\mu\Delta\phi + \mu(\nabla u^1 + P)\cdot\nabla\phi] dxdt = 0.$$

This implies, by [13, Lemma 2.6], that μ is independent of t and it coincides with m^1. Hence, we have proved that $m^\epsilon(\epsilon x,t) - m^1(x) \to 0$ weakly in $L^{p'}(\mathbb{T}^n\times[0,T])$. Moreover, since $n^\epsilon\to 0$ in $L^1(\mathbb{T}^n\times[0,T])$ by Lemma 19, we get the strong convergence for $1\leq p<p'$. ◻

7.2. Proof of Theorem 16. We start by proving the convergence of m^ϵ stated in point (ii). We observe that, by Proposition 20, we have that

$$\lim_{\epsilon\to 0}\int_0^T \int_{\mathbb{T}^n} |m^\epsilon(\epsilon y,t) - m^1(y)|^p dydt = 0$$

for $p < (n+2)/n$ for $n \geq 3$ and $p < 2$ for $n = 2$. By the change of variables $z = \epsilon y$, we get

$$\lim_{\epsilon\to 0} \epsilon^{-n} \int_0^T \int_{\epsilon\mathbb{T}^n} |m^\epsilon(z,t) - m^1(z/\epsilon)|^p dzdt = 0.$$

Fix a compact Q in $\mathbb{R}^n$; then there exists a constant $K_Q \in \mathbb{N}$ such that $Q \subset K_Q[0,1]^n$. So, using periodicity, we have

$$\int_0^T \int_Q |m^\epsilon(z,t) - m^1(z/\epsilon)|^p dzdt \leq \int_0^T \int_{K_Q[0,1]^n} |m^\epsilon(z,t) - m^1(z/\epsilon)|^p dzdt = o(1).$$

On the other hand, by weak convergence of a periodic function to its mean (see [27]), $m^1(x/\epsilon)$ weakly converge to 1 in $L^p_{\text{loc}}(\mathbb{R}^n)$ for every p. Hence, point (ii) of the statement is completely proved.

Now we pass to the proof of (i). Integrating the first equation in (40) on $(0,T)\times\mathbb{T}^n$, we get

$$\int_{\mathbb{T}^n} w^\epsilon(\epsilon x, t)dx + \int_t^T \int_{\mathbb{T}^n} \frac{1}{2}|\nabla w^\epsilon(\epsilon x, s) + P|^2 dxds - \int_t^T \int_{\mathbb{T}^n} V(x, m^\epsilon(\epsilon x, s))dxds = 0. \tag{54}$$

Since, by Lemma 19, there holds

$$\int_0^T \int_{\mathbb{T}^n} |(\nabla w^\epsilon(\epsilon x,t) + P) - (\nabla u^1(x) + P)|^2 dxdt \leq C\epsilon,$$

the second term in (54) converges to $\int_0^T \int_{\mathbb{T}^n} \frac{1}{2}|\nabla u^1(x) + P|^2$. On the other hand, we claim that

$$V(x, m^\epsilon(\epsilon x,t)) \to V(x, m^1(x)) \quad \text{in } L^1([0,T]\times\mathbb{T}^n).$$

Indeed, following an argument similar to that in [13, Thm. 2.1], we get that, calling L the Lipschitz constant of V in $\mathbb{T}^n \times [0, \|m^1\|_\infty + 1]$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^n} |V(x, m^\epsilon(\epsilon x,t)) - V(x, m^1(x))|dxdt \\
&\leq \int_0^T \int_{\{m^\epsilon(\epsilon x,t) > m^1(x)+1\}} (V(x, m^\epsilon(\epsilon x,t)) - V(x, m^1(x)))(m^\epsilon(\epsilon x,t) - m^1(x))dxdt \\
&\quad + L\int_0^T \int_{\mathbb{T}^n} |m^\epsilon(\epsilon x,t) - m^1(x)|dxdt \\
&\leq C_P\epsilon + L\int_0^T \int_{\mathbb{T}^n} |m^\epsilon(\epsilon x,t) - m^1(x)|dxdt,
\end{aligned}$$

where the last inequality is due to Lemma 19. By Proposition 20, we get that the right-hand side converges to 0 as $\epsilon \to 0$. Hence, our claim is completely proved.

So, by the definition of w^ϵ and of $\bar{H}(P,1)$, we get from (54) that for every $t \in [0,T]$ there holds

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_{\mathbb{T}^n} (u^\epsilon(\epsilon x,t) - \epsilon P\cdot x)dx &= \int_t^T \int_{\mathbb{T}^n} \left[-\frac{1}{2}|\nabla u^1(x) + P|^2 + V(x, m^1(x))\right] dxdt \\
&= (t-T)\bar{H}(P,1).
\end{aligned}$$

This implies that the function $\bar{v}^\epsilon(t) := \int_{\mathbb{T}^n} v^\epsilon(x,t)dx$ satisfies

$$\lim_{\epsilon\to 0} \epsilon\bar{v}^\epsilon(t) = \lim_{\epsilon\to 0}\int_{\mathbb{T}^n}[u^\epsilon(\epsilon x,t) - \epsilon P\cdot x - (t-T)\bar{H}(P,1) - \epsilon u^1(x)]dx = 0.$$

Note that since all the previous estimates are independent of t, the convergence is also uniform for $t \in [0,T]$. By the Poincaré inequality, recalling Lemma 19, we get

$$\int_0^T\int_{\mathbb{T}^n}|v^\epsilon(x,t) - \bar{v}^\epsilon|^2 \le C\int_0^T\int_{\mathbb{T}^n}|\nabla v^\epsilon|^2 \le C_P\epsilon.$$

So, we get that $v^\epsilon(x,t) - \bar{v}^\epsilon(t) \to 0$ in $L^2(\mathbb{T}^n\times[0,T])$. In particular, the last two relations imply that $\epsilon v^\epsilon(x,t)\to 0$ in $L^2(\mathbb{T}^n\times[0,T])$; namely,

$$\lim_{\epsilon\to 0}\int_0^T\int_{\mathbb{T}^n}|u^\epsilon(\epsilon x,t) - \epsilon P\cdot x - (t-T)\bar{H}(P,1) - \epsilon u^1(x)|^2dx = 0.$$

Since $\epsilon u^1\to 0$ in L^2, we get that

$$\lim_{\epsilon\to 0}\int_0^T\int_{\mathbb{T}^n}|u^\epsilon(\epsilon x,t) - \epsilon P\cdot x - (t-T)\bar{H}(P,1)|^2dx = 0.$$

Performing the change of variables $z = \epsilon x$ and choosing the constant K_Q as before, we get that

$$\lim_{\epsilon\to 0} K_Q^n\int_0^T\int_Q|u^\epsilon(z,t) - P\cdot z - (t-T)\bar{H}(P,1)|^2dz = 0,$$

which implies the desired result.

7.3. Final remark on the convergence in the general case. Here we briefly sketch some arguments for extending the convergence result beyond affine data, but still under restrictive assumptions; this issue will be addressed in future work. Assume that

- (2) has a classical solution (this could hold in a small time interval for regular data),
- the solution to (14) has a regular dependence on P and α,
- the data are well prepared: they agree with those of the asymptotic expansions up to a sufficiently high order.

Let $(u(\cdot;x,t), m(\cdot;x,t))$ be the solution to (14) with $(P,\alpha) = (\nabla u^0(x,t), m^0(x,t))$, and let $m^2(\cdot;x,t)$ be the solution to (8). We put $\epsilon = \frac{1}{k}$, with $k\in\mathbb{N}$. We define the error

$$\begin{cases} v^\epsilon(x,t) = u^\epsilon(x,t) - u^0(x,t) - \epsilon u\left(\frac{x}{\epsilon};x,t\right) = u^\epsilon(x,t) - \bar{u}^\epsilon(x,t),\\ n^\epsilon(x,t) = m^\epsilon(x,t) - m^0(x,t)\left(m\left(\frac{x}{\epsilon};x,t\right) + \epsilon m^2\left(\frac{x}{\epsilon};x,t\right)\right) = m^\epsilon(x,t) - \bar{m}^\epsilon(x,t).\end{cases}$$

The assumption that the data are well prepared reads as follows: $v^\epsilon(x,T) = 0$ and $n^\epsilon(x,0) = 0$. This may be relaxed slightly using the alternative expansion in section 3.1.1.

For (u^0,m^0) and (u,m), m^2 regular, as in section 3 we get that v^ϵ, n^ϵ satisfy

$$\begin{cases} -v_t^\epsilon - \epsilon\Delta v^\epsilon + \frac{1}{2}|\nabla v^\epsilon|^2 + \nabla\bar{u}^\epsilon\cdot\nabla v^\epsilon - V(\frac{x}{\epsilon}, n^\epsilon + \bar{m}^\epsilon) + V(\frac{x}{\epsilon},\bar{m}^\epsilon) = R_1^\epsilon,\\ n_t^\epsilon - \epsilon\Delta n^\epsilon - \mathrm{div}\,(n^\epsilon(\nabla\bar{u}^\epsilon)) - \mathrm{div}\,(m^\epsilon(\nabla v^\epsilon)) = R_2^\epsilon,\end{cases}$$

where $R_i^\epsilon \to 0$ uniformly as $\epsilon \to 0$, $i \in \{1,2\}$. We multiply the first equation by n^ϵ and the second by v^ϵ, integrate on $\mathbb{T}^n \times [0,T]$, and subtract one equation from the other. Taking into account that the data are well prepared and $\frac{1}{2}n^\epsilon - m^\epsilon = -\frac{1}{2}(m^\epsilon + \bar{m}^\epsilon) < 0$, we get

$$\int_0^T \int_{\mathbb{T}^n} \frac{1}{2}(m^\epsilon + \bar{m}^\epsilon)|\nabla v^\epsilon|^2 + \left[V\left(\frac{x}{\epsilon}, n^\epsilon + \bar{m}^\epsilon\right) - V\left(\frac{x}{\epsilon}, \bar{m}^\epsilon\right)\right] n^\epsilon dxdt = O(\epsilon). \tag{55}$$

Since $\bar{m}^\epsilon$ is bounded away from zero and uniformly bounded by our assumptions, and $m^\epsilon \geq 0$, we can argue as in Lemma 19. Then we get that $|\nabla v^\epsilon| \to 0$ in $L^2(\mathbb{T}^n \times [0,T])$. This (together with the Poincaré inequality) implies that $v^\epsilon \to 0$ in $L^2(\mathbb{T}^n \times [0,T])$, and, moreover, $n^\epsilon \to 0$ in $L^1(\mathbb{T}^n \times [0,T])$. In conclusion, $u^\epsilon \to u^0$ in $L^2(\mathbb{T}^n \times [0,T])$, and $m^\epsilon \to m^0$ weakly in $L^1(\mathbb{T}^n \times [0,T])$.

Let us stress that even in this case, the oscillations at the highest order of $\bar{m}^\epsilon$ prevent any stronger convergence of m^ϵ to m^0.

REFERENCES

[1] Y. Achdou, *Finite difference methods for mean field games*, in Hamilton-Jacobi Equations: Approximations, Numerical Analysis and Applications, P. Loreti and N. A. Tchou, eds., Lecture Notes in Math. 2074, Springer, Heidelberg, 2013, pp. 1–47, doi:10.1007/978-3-642-36433-4.

[2] M. Bardi and E. Feleqi, *Nonlinear Elliptic Systems and Mean Field Games*, preprint, http://cvgmt.sns.it/paper/2655/, 2015.

[3] A. Bensoussan, *Perturbation Methods in Optimal Control*, John Wiley & Sons, Chichester, UK, Gauthier-Villars, Montrouge, 1988.

[4] A. Bensoussan and J. Frehse, *On Bellman equation of ergodic type with quadratic growth Hamiltonians*, in Contributions to Modern Calculus of Variations, L. Cesari, ed., Pitman Res. Notes Math. Ser. 148, Longman, Harlow, UK, 1987, pp. 13–25.

[5] A. Bensoussan, J. Frehse, and P. Yam, *Mean Field Games and Mean Field Type Control Theory*, Springer Briefs in Mathematics, Springer, New York, 2013, doi:10.1007/978-1-4614-8508-7.

[6] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, 2nd ed., AMS Chelsea, Providence, RI, 2011.

[7] M. Burger, M. Di Francesco, P. Markowich, and M.-T. Wolfram, *Mean field games with nonlinear mobilities in pedestrian dynamics*, Discrete Contin. Dyn. Syst. Ser. B, 19 (2014), pp. 1311–1333, doi:10.3934/dcdsb.2014.19.1311.

[8] F. Camilli and C. Marchi, *Stationary mean field games systems defined on networks*, SIAM J. Control Optim., 54 (2016), pp. 1085–1103, doi:10.1137/15M1022082.

[9] P. Cardaliaguet, *Weak solutions for first order mean field games with local coupling*, in Analysis and Geometry in Control Theory and Its Applications, Springer INdAM Ser. 11, Springer, Cham, 2015, pp. 111–158, doi:10.1007/978-3-319-06917-3.

[10] P. Cardaliaguet, *Note on Mean Field Games*, technical report, available online at https://www.ceremade.dauphine.fr/~cardalia/MFG20130420.pdf.

[11] P. Cardaliaguet and P. J. Graber, *Mean field games systems of first order*, ESAIM Control Optim. Calc. Var., 21 (2015), pp. 690–722 doi:10.1051/cocv/2014044.

[12] P. Cardaliaguet, F. Delarue, J.-M. Lasry, and P.-L. Lions, *The Master Equation and the Convergence Problem in Mean Field Games*, preprint, arXiv:1509.02505 [math.AP], 2015.

[13] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, and A. Porretta, *Long time average of mean field games*, Netw. Heterog. Media, 7 (2012), pp. 279–301, doi:10.3934/nhm.2012.7.279.

[14] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, and A. Porretta, *Long time average of mean field games with a nonlocal coupling*, SIAM J. Control Optim., 51 (2013), pp. 3558–3591, doi:10.1137/120904184.

[15] G. Chevalier, J. Le Ny, and R. Malham, *A micro-macro traffic model based on mean-field games*, in 2015 American Control Conference (ACC), Chicago, IL, 2015, pp. 1983–1988, doi:10.1109/ACC.2015.7171024.

[16] M. Cirant, *Stationary focusing mean field games*, Comm. Partial Differential Equations, 41 (2016), pp. 1324–1346, doi:10.1080/03605302.2016.1192647.

[17] M. Fischer, *On the Connection Between Symmetric N-player Games and Mean Field Games*, preprint, arXiv:1405.1345 [math.PR], 2014; Ann. Appl. Probab., to appear.

[18] D. A. Gomes, L. Nurbekyan, and E. Pimentel, *Economic Models and Mean-Field Games Theory*, IMPA Mathematical Publications, Rio de Janeiro, Brazil, 2015.

[19] D. A. Gomes, E. Pimentel, and H. Sánchez-Morgado, *Time-dependent mean-field games in the superquadratic case*, ESAIM Control Optim. Calc. Var., 22 (2016), pp. 562–580, doi:10.1051/cocv/2015029.

[20] D. A. Gomes, G. E. Pires, and H. Sánchez-Morgado, *A-priori estimates for stationary mean-field games*, Netw. Heterog. Media, 7 (2012), pp. 303–314, doi:10.3934/nhm.2012.7.303.

[21] D. A. Gomes and J. Sáude, *Mean field games models—a brief survey*, Dyn. Games Appl., 4 (2014), pp. 110–154, doi:10.1007/s13235-013-0099-2.

[22] A. Lachapelle and M.-T. Wolfram, *On a mean field game approach modeling congestion and aversion in pedestrian crowds*, Transportation Res. Part B Methodological, 45 (2011), pp. 1572–1589, doi:10.1016/j.trb.2011.07.011.

[23] D. Lacker, *A general characterization of the mean field limit for stochastic differential games*, Probab. Theory Related Fields, 165 (2016), pp. 581–648, doi:10.1007/s00440-015-0641-9.

[24] J.-M. Lasry and P.-L. Lions, *Jeux à champ moyen.* I. *Le cas stationnaire*, C. R. Math. Acad. Sci. Paris, 343 (2006), pp. 619–625, doi:10.1016/j.crma.2006.09.019.

[25] J.-M. Lasry and P.-L. Lions, *Jeux à champ moyen.* II. *Horizon fini et contrôle optimal*, C. R. Math. Acad. Sci. Paris, 343 (2006), pp. 679–684, doi:10.1016/j.crma.2006.09.018.

[26] J.-M. Lasry and P.-L. Lions, *Mean field games*, Japan. J. Math. (N.S.), 2 (2007), pp. 229–260, doi:10.1007/s11537-007-0657-8.

[27] D. Lukkassen and P. Wall, *On weak convergence of locally periodic functions*, J. Nonlinear Math. Phys., 9 (2002), pp. 42–57, doi:10.2991/jnmp.2002.9.1.5.

[28] P.-L. Lions and P. E. Souganidis, *Work in progress*, private communication.

[29] G. A. Pavliotis and A. M. Stuart, *Multiscale Methods. Averaging and Homogenization*, Springer, New York, 2008, doi:10.1007/978-0-387-73829-1.

[30] E. Pimentel and V. Voskanyan, *Regularity theory for second order stationary mean-field games*, Indiana Univ. Math. J., to appear.