Space periodic Jacobi elliptic solution for triad modified Schrödinger equations

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Abstract

We present an analytical solution for triad nonlinear evolution equations with modified Schrödinger terms. An example for application in compressible water waves is presented.

1 Introduction

We consider a two-dimensional problem of an interacting wave triad of the form

$$\Psi_{j,t}(x,t) = i\alpha_j \left[\Psi_{j,xx}(x,t) + \delta_j^2 \Psi_j(x,t)\right] + \gamma_j V_j(t)\Psi_j(x,t) \qquad j = 1,2,3 \tag{1}$$

with α_j, δ_j, and γ are parameters of the physical problem. We also assume that the relations

$$V_1\Psi_1 = \Psi_2\Psi_3, \quad V_2\Psi_2 = \Psi_1\Psi_3^*, \quad V_3\Psi_3 = \Psi_1\Psi_2^*, \tag{2}$$

are satisfied, $\delta_2 = \delta_3 = \delta_1/2 \equiv \delta/2$, and asterisks denote complex conjugates. The objective is to derive a periodic analytical solution, for application in long and short wave triad interactions.

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2 Solution

Redefine $\Psi_j(x,t) = g_j(x) f_j(t)$, so that equations (1) can be written as

$$g_1 f_{1,t} = i\alpha_1 \left[g_{1,xx} + \delta_1^2 g_1\right] f_1 + \gamma_1 g_2 g_3 f_2 f_3 \tag{3}$$

$$g_2 f_{2,t} = i\alpha_2 \left[g_{2,xx} + \delta_2^2 g_2\right] f_2 + \gamma_2 g_1 g_3^* f_1 f_3^* \tag{4}$$

$$g_3 f_{3,t} = i\alpha_3 \left[g_{3,xx} + \delta_3^2 g_3\right] f_3 + \gamma_3 g_1 g_2^* f_1 f_2^* \tag{5}$$

Based on (2), $g_1 = g_2 g_3$, $g_2 = g_1 g_3^*$, and $g_3 = g_1 g_2^*$, so that equations (3), (4), and (5), can be rewritten as

$$f_{1,t} = \gamma_1 f_2 f_3 + i\alpha_1 \left[g_{1,xx} + \delta_1^2 g_1\right] f_1 g_1^{-1} \tag{6}$$

$$f_{2,t} = \gamma_2 f_1 f_3^* + i\alpha_2 \left[g_{2,xx} + \delta_2^2 g_2\right] f_2 g_2^{-1} \tag{7}$$

$$f_{3,t} = \gamma_3 f_1 f_2^* + i\alpha_3 \left[g_{3,xx} + \delta_3^2 g_3\right] f_3 g_3^{-1} \tag{8}$$

Now we can seek a solution in two parts. The first part requires that

$$g_{j,xx} + \delta_j^2 g_j = 0, \qquad j = 1, 2, 3. \tag{9}$$

A general solution of (9) is given by

$$g_j = a_j e^{i\delta_j x} + b_j e^{-i\delta_j x}. \tag{10}$$

For the second part of the solution we need to solve the following simplified system of three ordinary differential equations that amplitudes satisfy

$$f_{1,t} = \gamma_1 f_2 f_3, \qquad f_{2,t} = \gamma_2 f_1 f_3^*, \qquad f_{3,t} = \gamma_3 f_1 f_2^* \tag{11}$$

Multiplying (11) by f_j^* and adding its conjugate multiplied by f_j, for j=1,2,3 respectively, we obtain the following set of equations

$$f_1^* f_{1,t} + f_1 f_{1,t}^* = \gamma_1 \left(f_1^* f_2 f_3 + f_1 f_2^* f_3^*\right) \tag{12}$$

$$f_2^* f_{2,t} + f_2 f_{2,t}^* = \gamma_2 \left(f_1^* f_2 f_3 + f_1 f_2^* f_3^*\right) \tag{13}$$

$$f_3^* f_{3,t} + f_3 f_{3,t}^* = \gamma_3 \left(f_1^* f_2 f_3 + f_1 f_2^* f_3^*\right) \tag{14}$$

More compactly we can write $|f_j|^2_{,t} = 2\gamma_j \Im\{f_1^* f_2 f_3\}$, where the Hamiltonian $\Im\{f_1^* f_2 f_3\}$ is a constant of the motion (*Holm and Lynch* (2002)). Now define $\mathbb{Z}_{,t} = \Im\{f_1^* f_2 f_3\}$ gives $|f_j|^2 = 2\gamma_j \mathbb{Z} + \psi_{0j}^2$. In order to carry on with the solution the signs of γ_j have to be determined. Note that for a resonating triad, $\gamma_1 + \gamma_2 + \gamma_3 = 0$, (*Lynch et al.* (2003)), thus one has a different sign than the others. Assume, with no loss of generality, that γ_1 is negative, that $\Psi_1(x, t = 0) = \psi_{01} = 0$, and that $|\psi_{03}| < |\psi_{02}|$ we obtain

$$\mathbb{Z}_{,t} = \sqrt{-8|\gamma_1|\gamma_2\gamma_3\mathbb{Z}\left(\mathbb{Z} + \frac{\psi_{02}^2}{\gamma_2}\right)\left(\mathbb{Z} + \frac{\psi_{03}^2}{\gamma_3}\right)} \tag{15}$$

This is an elliptic function with a solution given by (see *Byrd and Friedman* (1971), equation 236.00, p.79)

$$\mathbb{Z} = -\frac{\psi_{03}^2}{\gamma_3}\mathrm{sn}^2(u, k) \tag{16}$$

where $\mathrm{sn}(u, k)$ is the sine amplitude Jacobian elliptic function of argument u, and modulus k given by

$$u = \sqrt{2|\gamma_1|\gamma_3}|\psi_{02}|t, \qquad k = \frac{|\psi_{03}|}{|\psi_{02}|}\sqrt{\frac{\gamma_2}{\gamma_3}} \tag{17}$$

and the expression for $|f_j|$ are

$$|f_j|^2 = |\psi_{0j}|^2 - 2\gamma_j\frac{|\psi_{03}|^2}{\gamma_3}\mathrm{sn}^2(u, k) \tag{18}$$

Finally, the analytical solution is given by

$$|\Psi_1(x, t)|^2 = -2\gamma_1\frac{|\psi_{03}|^2}{\gamma_3}\mathrm{sn}^2(u, k)\left[\exp(2i\delta x) + \exp(-2i\delta x)\right] \tag{19}$$

$$|\Psi_2(x, t)|^2 = \left[|\psi_{02}|^2 - 2\gamma_2\frac{|\psi_{03}|^2}{\gamma_3}\mathrm{sn}^2(u, k)\right]\left[\exp(i\delta x) + \exp(-i\delta x)\right] \tag{20}$$

$$|\Psi_3(x, t)|^2 = |\psi_{03}|^2\left[1 - 2\mathrm{sn}^2(u, k)\right]\left[\exp(i\delta x) + \exp(-i\delta x)\right] \tag{21}$$

3 Application

The solution presented here can be applied in various long-short wave triad interactions, such as Rossby-type waves (see *Pedlosky* (1987); *Charney* (1948)), or wave motion in an inhomogeneous plasma (*Hasegawa and Mima* (1977)). Nevertheless, the following example considers the interaction of two surface gravity waves with an acoustic wave in a mechanism similar to that proposed by *Longuet–Higgins* (1950), and more recently by *Kadri and Stiassnie* (2013) and *Kadri* (2015).

Given an acoustic wave and two gravity waves with potential amplitudes ϕ_a, ϕ_{g1}, and ϕ_{g2}, satisfying the following evolution equations

$$\phi_{a,t} = -\frac{ic^2\delta}{2\omega h}\left(\phi_{a,xx} + \frac{4\omega^2}{c^2}\phi_a\right) - \frac{2\omega}{hc}\phi_{g1}\phi_{g2} \tag{22}$$

$$\phi_{g1,t} = \frac{2\omega^3}{gc}\phi_a\phi_{g2}^*; \quad \phi_{g2,t} = \frac{2\omega^3}{gc}\phi_a\phi_{g1}^* \tag{23}$$

where $c = 1500$ m/s, is the speed of sound in water, ω is the frequency of the gravity waves, h is the water depth. The solution of evolution equations is then given by

$$|\Phi_a(x,t)|^2 = \frac{2g|\phi_{0(g2)}|^2}{\omega h}\mathrm{sn}^2\left(u,k\right)\left(e^{4i\omega x/c} + e^{-4i\omega x/c}\right) \tag{24}$$

$$|\Phi_{g1}(x,t)|^2 = \left[|\phi_{0(g1)}|^2 - 2|\phi_{0(g2)}|^2\mathrm{sn}^2(u,k)\right]\left(e^{2i\omega x/c} + e^{-2i\omega x/c}\right) \tag{25}$$

$$|\Phi_{g2}(x,t)|^2 = |\phi_{0(g2)}|^2\left[1 - 2\mathrm{sn}^2(u,k)\right]\left(e^{2i\omega x/c} + e^{-2i\omega x/c}\right) \tag{26}$$

with $u = 2\sqrt{2/gh}\omega^2/c$, and $k = |\phi_{0(g2)}|/|\phi_{0(g1)}|$.

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