Commutators of smooth and nonsmooth vector fields

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Abstract

Commutators or Lie brackets of vector fields play an important role in many contexts. In the first part of the paper I recall some classical results involving vector fields and their commutators such as the asymptotics of the commutator, the commutativity theorem, the simultaneous rectification theorem, the Frobenius theorem and the Chow-Rashevski controllability theorem. A natural question is how much can the smoothness conditions on vector fields be reduced so that these results remain still valid? In the second part of the paper I address the issue of reducing smoothness assumptions on vector fields. I focus primarily on an extension of the notion of iterated commutator and of Chow-Rashevski theorem for nonsmooth vector fields outlining recent results obtained jointly with Franco Rampazzo.

Index Terms

vector fields, iterated Lie brackets or commutators, nonsmooth vector fields, set-valued commutators

1 Introduction

This article aims at being a small survey of both some classical facts involving commutators of vector fields and some results based on notions of generalized iterated commutators for nonsmooth vector fields obtained recently by H. Sussmann, F. Rampazzo and myself.

Many classical results in differential geometry, partial differential equations, control theory, etc, are related to the study of sets of vector fields. In the study of sets of vector fields the notion of Lie bracket or commutator is essential. So it turns out that commutators of vector fields play a fundamental role in stating and proving several results.

In Sect. 2 I recall some classical results from differential geometry, control theory, and partial differential equations involving sufficiently smooth vector fields and their commutators. More precisely, I recall

- the commutativity theorem,
- the simultaneous rectification theorem,
- the Frobenius theorem,
- the Chow-Rashevski controllability theorem,
- the Bony maximum principle for linear PDEs of Hörmander type.

Some of these results continue to make sense for vector fields with much more reduced smoothness. Thus a natural question arises: how much can the smoothness of vector fields be reduced so that the said results continue to remain valid? Even more, by reducing further the smoothness assumptions on vector fields is it possible to extend or generalize these results in some sense by using, if necessary, appropriate generalizations of the notion of Lie bracket?

Therefore, on the other hand I outline here some results related to the issue of reducing the smoothness assumptions on vector fields and obtaining generalizations of some of the above-mentioned classical resutls. Regarding the Chow-Rashevski theorem, a first reduction of regularity, still remaining in the framework of classical iterated commutators, has been achieved in [7]; this result is described in Subsect. 2.2.

In Subsect. 3.1 I relate some of the results of H. Sussman and F. Rampazzo [10], [11]. Based on a notion of set-valued Lie bracket (or commutator) which they had introduced in [10] for Lipschitz continuous vector fields, they have extended in [11] the commutativity theorem to Lipschitz vector fields: the flows of two Lipschitz vector fields commute if and only if their commutator vanishes everywhere (i.e., equivalently, if their commutator vanishes almost everywhere). In [11] they also extend the asymptotic formula that gives an estimate of the lack of commutativity of two vector fields in terms of their Lie bracket, and prove a simultaneous rectification (or flow) box theorem for commuting families of Lipschitz vector fields.

Finally, I conclude by relating recent results obtained jointly with F. Rampazzo [7], [6] in Subsect 3.2. It turns out that a mere iteration of the construction of commutator proposed in [10] is in a sense not appropriate for producing iterated commutators. The problem is, as observed in [11, Section 7], that the resulting "iterated commutators" would be two small for a natural asymptotic formula to be valid. The integral formulas obtained in [7] suggest another definition of iterated commutator for tuples of appropriately nonsmooth vector fields which does not suffer from that problem. This construction, albeit more complicated, yields also an upper semicontinuous set-valued vector field which reduced to the singleton consisting of the classical iterated commutator for sufficiently smooth vector fields. With this notion of iterated commutator at hand, Rampazzo and I have been able to formulate an extended version of the Lie Algebra Rank Condition (LARC), also known as Hörmander condition, and a nonsmooth version of the classical Chow-Rashevski theorem. I give also some hints on the proofs in Subsect. 3.3. I conclude by stating some controllability problems involving nonsmooth vector fields.

Notation. In this paper a differential manifold M is always assumed finite-dimensional, second countable and Hausdorff as a topological space. The tangent space of M at a point $x \in M$ is denoted by T_xM.

I recall that for a (possibly set-valued) vector field f in a differentiable manifold M, that is, a map $f : M \ni x \mapsto f(x) \subset T_xM$, its **flow** Φ^f is the possibly partially defined and possibly set-valued map $M \times \mathbb{R} \ni (x,t) \mapsto \Phi^f(x,t) \subset M$ such that for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $\Phi(x,t)$ is the (possibly empty) set of those states $y \in M$ such that there exists an absolutely continuous curve $\xi : I_t \to M$ such that $\xi(0) = x$, $x(t) = y$, $\dot{\xi}(s) \in f(\xi(s))$ for a.e. $s \in I_t$, where $I_t = [0 \wedge t, 0 \vee t]$. The curve ξ is called an **integral curve** or **trajectory** of f. If f is of class $C^{-1,1}$, that is, upper semicontinues as a set-valued map with compact convex values, then for any compact $K \subset M$ there exists $T > 0$ such that $\Phi^f(x,t)$ is not empty for all $(x,t) \in K \times [0,T]$, see [3]. Let us call $\mathrm{Dom}(\Phi^f)$, the set of those (x,t) such that $\Phi^f(x,t) \neq \emptyset$. If f is of class $C^{0,1}$, that is, f is a locally Lipschitz single-valued map, $\Phi^f(x,t)$ is a singleton, for each $(x,t) \in \mathrm{Dom}(\Phi^f)$, which we identify with the element that it contains. In other words, we now see the flow of f as a possibly partially defined single-valued map $M \times \mathbb{R} \ni (x,t) \mapsto \Phi^f(x,t) \in M$. For a fixed t, I use also the notation Φ^f_t to denote the map $x \mapsto \Phi^f(x,t)$.

2 COMMUTATORS OF SMOOTH VECTOR FIELDS

Here I give a small selection of classical theorems from differential geometry, control theory and partial differential equations where commutators of vector fields appear. This is a good occasion to recall to the nonspecialist reader some of these results and convince him/her of the utility of this concept.

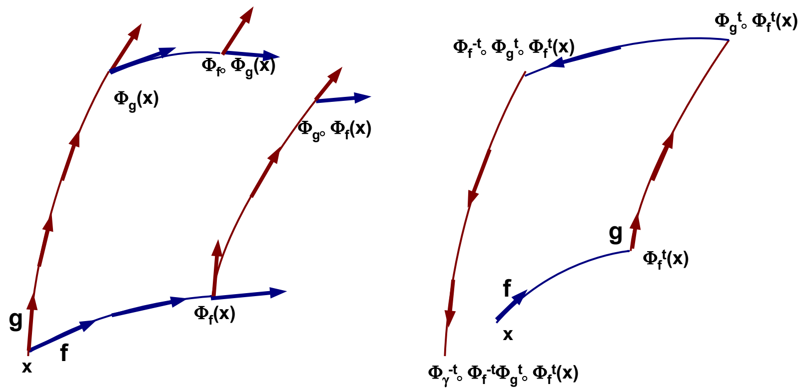

Fig. 1: Asymptotic formulas

2.1 Theorems from differential geometry

Definition 2.1. *The* **Lie bracket** *of* C^1 *vector fields* f, g *is*

$$[\mathbf{f}, \mathbf{g}] := \mathbf{Dg} \cdot \mathbf{f} - \mathbf{Df} \cdot \mathbf{g}$$

Basic properties:

1) $[f, g]$ is a vector field (i.e. it is an intrinsic object)
2) $[f, g] = -[g, f]$ (antisymmetry) ($\Longrightarrow$ $[f, f] = 0$)
3) $[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$ (Jacobi identy)

Much of the commutators utility stems from the following asymptotic formulas which show that the commutator of two vector fields measures the lack of commutativity between their flows.

$$\Phi^t_f \circ \Phi^t_g(x) - \Phi^t_g \circ \Phi^t_f(x) = t^2[f, g](x) + o(t^2) \tag{2.1}$$

$$\Phi^{-s}_g(x) \circ \Phi^{-t}_f \circ \Phi^t_g \circ \Phi^t_f(x) = x + t^2[f, g](x) + o(t^2) \tag{2.2}$$

as $t \to 0$.

Roughly speaking the rectification theorem states that if a vector field f does not vanish at a point then it is possible to find a coordinate chart around that point such that the coordinate representation of f in the chart is a constant vector field. Given two or more vector fields $f_1, f_2, \dots$ one may ifc it is possible to find a coordinate chart such the coordinate representations of $f_1, f_2, \dots$ are constant vector fields. The answer to this question is given by the following theorem.

Theorem 2.2. *Let* $f_1, \dots, f_m$ *be linearly independent vector fields of class* C^1*. Then*

1) $[f_i, f_j] \equiv 0 \quad \forall i, j = 1, \dots, m$

if and only if

2) simultaneous rectification holds true for $f_1, \dots, f_m$

if and only if

3) the flows commute:

$$\Phi^t_{f_i} \circ \Phi^s_{f_j}(x) = \Phi^s_{f_j} \circ \Phi^t_{f_i}(x), \quad \forall x \in M, \quad \forall i, j = 1, \dots, m\,.$$

The result (1) $\Longleftrightarrow$ (3) is known as the **commutativity theorem** while. (1) $\Longleftrightarrow$ (2) as the **simultaneous rectification theorem**.

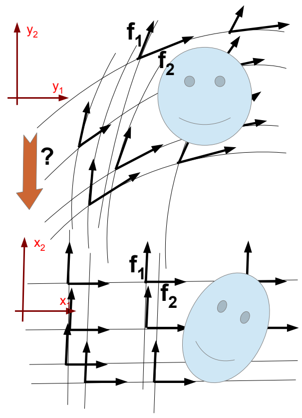

Fig. 2: Simultaneous rectification

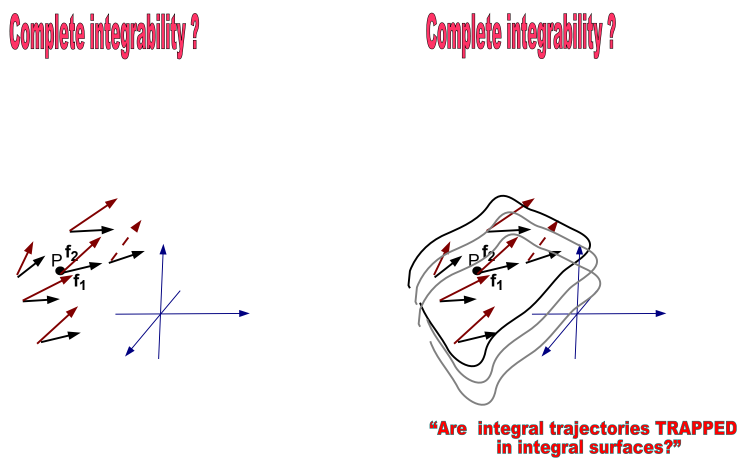

Fig. 3: Integrability of sets of vector fields

It is known that for any Lipschitz vector field and any point in a manifold there exists a (unique curve) passing from the given point whose tangent at each of its points has the same direction as the given vector field at those points. This is Picard-Lindeloff (or Cauchy-Lipschitz) theorem on the existence and uniqueness of solutions for initial value problems for ODEs. In other words, an integral curve may be assigned to each vector field. Suppose, now we are given two vector fields in the three-dimensional space. Does there exist a surface whose tangent plane at each point si spanned by the values of the given vector fields at that point. This question is answered by the following result known as **Frobenius theorem**.

Theorem 2.3 (Frobenius Theorem)**.** *Let $f_1, \ldots, f_m$ be linearly independent vector fields on a manifold M of class C^1.*

- $\{f_1, \ldots, f_m\}$ *is* **completely integrable***, in the sense that through each point of M there exists a submanifold N of M whose tangent space at each of its points is spanned by the given vector fields,*
  if and only if
- *(***involutivity***)* $[f_i, f_j] \in span\{f_1, \ldots, f_m\} \quad \forall i, j = 1, \ldots, m$

2.2 Chow-Rashevski controllability theorem and a strong maximum principle for linear PDEs of Hörmander type

Here I take the viewpoint of geometric control theory in which a **control system** is primarily a family of vector fields $\mathcal{F}$ on some differentiable manifold M. Here I work with finite-dimensional manifolds.

My first task is to clarify what do I mean by **controllability**. The term is quite self-explanatory, but it admits different formalizations in literature, useful for different purposes.

By an $\mathcal{F}$**-trajectory** I mean an absolutely continuous curve $x(\cdot)$ defined on some interval I which is a finite concatenation of integral curves of vector fields in $\mathcal{F}$, that is, the interval I admits a partition $\{t_0, \ldots, t_p\}$ for some $p \in \mathbb{N}$, and there exist vector fields $f_1, \ldots, f_p \in \mathcal{F}$ such that $\dot{x}(t) = f_i(x(t))$ for a.e. $t_{i-1} < t < t_i$ for $i = 1, \ldots, p$ (or $\dot{x}(t) \in f_i(x(t))$ if one allows for differential inclusions (i.e., set-valued vector fields) as at some point it will be done).

We say that $\mathcal{F}$ is **controllable** (in M) if any pair of points $x, y \in M$ can be connected by an $\mathcal{F}$-trajectory (that is, a concatenation of a finite number of integral curves of vector fields in $\mathcal{F}$).

Let

$$x_* \in M. \qquad T \geq 0,$$

we define the **reachable set** of $\mathcal{F}$ from x_* at time T by setting

$$\mathcal{R}_{\mathcal{F}}(x_*, T) := \left\{ x \in M \ : \ \exists x(\cdot) \ \mathcal{F} - \text{trajectory such that } x(0) = x_*, \ x(T) = x \right\}.$$

We say that $\mathcal{F}$ is **small time locally controllable** (STLC) from x_* if x_* is an interior point of $\mathcal{R}_{\mathcal{F}}(x_*, t)$ for any time $t > 0$ (no matter how small).

Let the **reachable set** of $\mathcal{F}$ from x_* be defined by

$$\mathcal{R}_{\mathcal{F}}(x_*) := \bigcup_{T>0} \mathcal{R}_{\mathcal{F}}(x_*, T).$$

Why is the notion of controllability useful, apart from its obvious geometrical and physical interest? I limit myself to two results.

I need to recall that vector fields are identifiable with first-order partial differential operators. Given a vector field f which in a coordinate chart has components

$$(f_1(x), \ldots, f_n(x))$$

where $n := \dim M$, then it is identified to the first-order differential operator

$$\varphi \mapsto f\varphi = \sum_{j=1}^{n} f_j(x) \partial_{x\ j} \varphi.$$

Theorem 2.4 (Bony's theorem)**.** *Let*

$$L = \sum_{i=1}^{q} f_i^2 + f_0$$

be a second-order differential operator, where $q \geq 1$, f_i, $i = 0, \ldots, q$, are first-order linear differential operators (i.e, vector fields). If u is a classical solution of

$$Lu = 0 \qquad \textit{in } M,$$

attaining a maximum at some point $x \in M$, that is,

$$\sup_{y \in M} u(y) = u(x),$$

then, for the **maximum propagation set**

$$\mathrm{Prop}(x) := \{ y \in M \ : \ u(y) = \bar{M} \}$$

contains the reachable set of $\mathcal{F}$ from x: that is,

$$\mathcal{R}_{\mathcal{F}}(x) \subset \mathrm{Prop}(x)\,,$$

where

$$\mathcal{F} := \{ \pm f_i \ : \ i = 1, \dots, q \} \cup \{ f_0 \}\,.$$

Corollary 2.5 (Strong maximum principle). *If above $\mathcal{F}$ is controllable then L satisfies the strong maximum principle: any solution u of L attaining a maximum value is constant.*

The second fact that I recall is the following.

Theorem 2.6 (Optimality necessary condition). *Any time-optimal trajectory $x(\cdot)$ at each time lies in the boundary of the reachable set.*

$$x(\cdot)\ \textit{time-optimal}\ \ \mathcal{F} - \textit{tracetory} \implies x(t) \in \partial \mathcal{R}_{\mathcal{F}}(x,t)\ \forall t > 0\,.$$

So we have the following interesting conclusion: sufficient conditions for small time local controllability are equivalent to necessary conditions for optimality. Thus, for instance, the Pontryagin maximum principle–a first order necessary optimality condition in optimal control theory–can be used to derive meaningful sufficient conditions for small time local controllability.

The set of all C^∞ vector fields on a C^∞ manifold M, denoted by $\mathcal{F}(M)$ forms a Lie algebra when endowed with the Lie bracket operation. The Lie algebra generated by a set of vector fields $\mathcal{F}$ is by definition the smallest vector subspace L of $\mathcal{F}(M)$ such that

- $\mathcal{F} \subset L$,
- $[f, g] \in L$ whenever $f, g \in V$;

we denote such a subalgebra of $\mathcal{F}(M)$ by $L(\mathcal{F})$.

If L is an algebra of vector fields on a differential manifold M and $x \in M$, let

$$L_x := \{ f(x) \ : \ f \in L \};$$

clearly it is a linear subspace of T_xM. In other words $L(\mathcal{F})$ is the linear subspace of $\mathcal{F}(M)$ generated by vectors in $\mathcal{F}$ and the **iterated Lie brackets**

$$\begin{gathered} , [[f_3, f_4], f_5],\ [[f_6, f_7], [f_8, f_9]],\ [[[f_{10}, f_{11}], f_{12}], \\ f_{14}],\ [[[f_{15}, f_{16}], [f_{17}, f_{18}]], f_{19}], \dots \end{gathered} \tag{2.3}$$

as $f_1, f_2, \dots \in \mathcal{F}$.

Definition 2.7 (LARC). *We say that $\mathcal{F}$ of C^∞ satisfies the* **Lie algebra rank condition** *at a point $x_* \in M$ if*

$$L(\mathcal{F})_{x_*} = T_{x_*}M\,. \tag{LARC}$$

Theorem 2.8 (Chow-Rashevski). *Assume that $\mathcal{F}$ is symmetric, that is, $-\mathcal{F} \subset \mathcal{F}$, the vector fields in $\mathcal{F}$ are of class C^∞ and M is of class C^∞. Then*

$$\mathcal{F}\ \textit{satisfies LARC at}\ x_*\ \ \textit{implies}\ F\ \textit{is}\,\textit{STLC at}\ x_*.$$

If in addition M is connected and $\mathcal{F}$ satisfies LARC at any $x \in M$, then $\mathcal{F}$ is controllable.

(In passing, actually, for analytic symmetric systems $\mathcal{F}$, controllability is equivalent to the fulfillment of the LARC condition.)

However, in order to have controllability, usually, much less smoothness is needed. A more precise formulation of LARC is needed which would allow us to reduce the smoothness assumptions as much as possible. Observe that since $\dim M < \infty$ then the linear space in (2.3) is generated by a finite number of iterated Lie brackets evaluated at x_*. In order to compute these iterated brackets it is not needed that the elements of $\mathcal{F}$ be of class C^∞; it just suffices that they be of class C^1.

So I have to introduce some smoothness classes of vector fields that make possible the computation of iterated Lie brackets with the least smoothness requirements (in an appropriate sense).

In general, we may denote an **iterated Lie bracket** by

$$B(\mathbf{f}),$$

where

$$\mathbf{f} = (f_1, \dots, f_m)$$

is the collection of vector fields which occur in the definition of $B(\mathbf{f})$. From now on I will often speak simply of a **bracket** instead of an iterated Lie bracket when no ambiguity arises.

A notion of formal iterated Lie bracket B can be introduced–as a suitable word in a suitable alphabet–and we shall say that the degree of B is m–the number of the vector fields of $\mathbf{f}$ to which B applies in order to obtain the "true" iterated Lie bracket $B(\mathbf{f})$.

With this notation the vector fields themselves are brackets of degree $m = 1$. $[f_1, f_2]$ is a bracket of degree 2. Some brackets of degree 3 are

$$\big[[f_1,\, f_2],\, f_3\big], \qquad \big[f_1,\, [f_2,\, f_3]\big];$$

actually there is "only one" bracket of degree 3 for each of them is just the negative of the other.Some brackets of degree 4 are

$$\Big[\big[[f_1,\, f_2],\, f_3\big], f_4\Big], \quad \big[[f_1,\, f_2]\,[f_3,\, f_4]\big], \dots;$$

and so on...

Any iterated Lie bracket can be regarded as constructed in a recursive way by successive bracketings. In particular, we have that

$$B(\mathbf{f}) = [B_1(\mathbf{f}_{(1)}), B_2(\mathbf{f}_{(2)})]\,,$$

where $\mathbf{f}_{(1)}$, $\mathbf{f}_{(2)}$ constitute a partition of $\mathbf{f}$.

$$\mathbf{f}_{(1)} = (f_1, \dots, f_{m_1}), \qquad \mathbf{f}_{(2)} = (f_{m_1+1}, \dots, f_m)$$

and B_1, B_2 are subbrackets–thus they are called–of B with

$$deg(B_1) = m_1, \qquad deg(B_2) = m_2\,.$$

.

Now an important definition: smoothness classes of vector fields.

Recall that for $k \geq 0$, a vector field f is said to be *of class* C^k if the derivatives $D^j f$ exist and are continuous for every $j = 0, \dots, k$. One says that $\mathbf{f}$ *is of class* C^B, and writes $\mathbf{f} \in C^B$, if all the components of $\mathbf{f}$ are as many times continuously differentiable as it is necessary in order to compute $B(\mathbf{f})$, so that $B(\mathbf{f})$ turns out to be a continuous vector field.

Examples 2.9.

$$\begin{aligned}
&\textit{If } B=[\cdot,\cdot] \textit{ and } \mathbf{f}=(f_1,f_2), \textit{ then } \mathbf{f}\in C^B \textit{ iff } f_1\in C^1, f_2\in C^1\,;\\
&\textit{If } B=\big[[\cdot,\cdot],\ \big] \textit{ and } \mathbf{f}=(f_1,f_2,f_3), \textit{ then } \mathbf{f}\in C^B \textit{ iff } f_1\in C^2, f_2\in C^2, f_3\in C^1\,;\\
&\textit{If } B=\Big[\big[[\cdot,\cdot],\ \big],\ \Big] \textit{ and } \mathbf{f}=(f_1,f_2,f_3,f_4), \textit{ then } \mathbf{f}\in C^B \textit{ iff } f_1\in C^3, f_2\in C^3, f_3\in C^2, f_4\in C^1\,.
\end{aligned}$$

and so on. So, in each of the cases,

$$[f_1,\, f_2],\ \ [[f_1,\, f_2],\, f_3],\ \ \Big[[[f_1,\, f_2],\, f_3],\, f_4\Big],$$

can be computed, yielding a continuous vector field.

Definition 2.10 (LARC). *Let $x_* \in M$. $k\in\mathbb{N}$. We say that $\mathcal{F}$ satisfies the* Lie algebra rank condition (RANC) of step k at x_* *if there exist formal iterated Lie brackets $B_1,\dots,B_\ell$ and finite collections of vector fields $\mathbf{f}_1,\dots,\mathbf{f}_\ell$ in $\mathcal{F}$ such that:*

- *i) k is k is the maximum of the degrees of the iterated Lie brackets B_j;*
- *ii) M is of class C^k;*
- *iii) for every $j=1,\dots,\ell$, $\mathbf{f}_j\in C^{B_j}$,*
- *iv) span$\Big\{B_1(\mathbf{f}_1)(x_*),\dots,B_\ell(\mathbf{f}_\ell)(x_*)\Big\}=T_{x_*}M$.*

Thus the Chow-Rashevski’s theorem can be formulated as follows:

Theorem 2.11 (Chow-Rashevski)**.** *Let $\mathcal{F}$ be a symmetric ($-\mathcal{F}\subset\mathcal{F}$) family of ($C^1$) vector fields on a differentiable manifold M, and $x_*\in M$. If $\mathcal{F}$ satisfies the LARC of step k at x_* for some $k\in\mathbb{N}$, then the control system $\mathcal{F}$ is STLC from x_*. More precisely, if d is a Riemannian distance defined on an open set A containing the point x_*, then there exist a neighborhood $U\subset A$ of x_* and a positive constant C such thatfor every $x\in U$ one has*

$$T(x)\le C(d(x,x_*))^{1/k}.$$

If M is connected and LARC holds at any $x_*\in M$ (not necessarily with the same iterated Lie brackets at each point x_*), then $\mathcal{F}$ is controllable in M, (that is, any pair of points in M can be connected by a finite concatenation of integral curves of vector fields in $\mathcal{F}$).

These are in some sense the minimal smoothness classical assumptions under which the Chow theorem can be stated and proved.

I claim that this result is new for a couple of reasons: First, some of the involved vector fields–and precisely those which are applied first degree brackets–are only continuous and hence their flows are set-valued maps in general, thus classical analysis is not applicable, at least not straightforwardly. But even if we assume all vector field of class C^1 the result is probably still new for the usual proof relies on asymptotic formulas for iterated brackets $B(\mathbf{f})$ which are known to be valid only for vector fields of class C^ℓ, where ℓ is at least the lowest order of differentiation of vector fields in f needed to in order to make sense of $B(\mathbf{f})$. However, $B(\mathbf{f})$ makes sense for f of class C^B and indeed in [7] these asymptotic formulas have been extended to such vector fields, which allow extending Chow-Rashevski’s theorem as Theorem 2.11.

I resume this discourse in Subsect. 3.2 where I outline my current work with F. Rampazzo on obtaining a generalization of Chow-Rashevski’s theorem for nonsmooth vector fields which is based on a notion of a generalized (set-valued) iterated commutator.

3 Commutators of nonsmooth vector fields

How much can the smoothness assumptions be reduced? What about nonsmooth counterparts of the notion of commutator or Lie bracket (similarly to notions of generalized differential in nonsmooth analysis)? Do there exist nonsmooth generalizations of the "smooth results" stated in the previous section.

Starting in the $90's$ there have been quite a few papers on these issues, especially, on commutativity and Frobenius type theorems under more and more weak hypotheses on vector fields, in the Control, Dynamical Systems, and PDE literature, with a.e. notions of Lie bracket (Simic [13], Calcaterra [5], Cardin-Viterbo, Montanari-Morbidelli [8], Luzzatto-Tureli-War, ...)

3.1 The Rampazzo-Sussmann commutator and applications to the geometry of nonsmooth vector fields

I limit myself to describing some results of Rampazzo and Sussmann [10], [11]. Then I give an outline of my own results with Rampazzo in defining a generalized iterated commutator and proving a nonsmooth version of Chow-Rashevski's controllability theorem.

In [10] Rampazzo and Sussmann have introduced a natural notion of commutator for Lipschiz continues vector fields (much in the spirit of Clarke's generalized differential).

Definition 3.1. *If f_1, f_2 are Lipschitz continuous, one sets*

$$[\mathbf{f_1}, \mathbf{f_2}]_{\mathbf{set}}(\mathbf{x}) := \overline{\mathbf{co}} \left\{ \mathbf{v} = \lim_{\mathbf{j}\to\infty} [\mathbf{f_1}, \mathbf{f_2}](\mathbf{x_j}), \right\}$$

where

1. $x_j \in \mathcal{D}iff(f_1) \cap \mathcal{D}iff(f_2)$ *for all* j,
2. $\lim_{j\to\infty} x_j = x$;

here $\mathcal{D}iff(f_k)$ $(k = 1, 2)$ is the set of points where f_k is differentiable, a full measure set in M by Rademacher's theorem.

Here are some elementary properties of $[f_1, f_2]_{set}$.

Proposition 3.2. *1) $x \mapsto [f_1, f_2]_{set}(x)$ is an upper semicontinous set-valued vector field with compact convex nonempty values.*

2) $[f_1, f_2]_{set} = -[f_2, f_1]_{set}$ (in particular, $[f_1, f_1]_{set} = 0$).

3) $[f_1, f_2]_{set}(x) = \{[f_1, f_2](x)\}$ are of class C^1 around x.

4) $[f_1, f_2]_{set} \equiv \{0\} \iff [f_1, f_2] = 0$ almost everywhere.

With this notion of commutator they have generalized the asimptotic formulas, the commutativity theorem, the rectification theorem, and in [10] a nonsmooth version of Chow-Rashevski's theorem for iterated brackets of degree 2. Moreover, Rampazzo [9] has aslo generalized Frobenius theorem. Thus the following results have been proved.

Theorem 3.3 (Asymptotic Formula [11])**.** *Given Lipschitz continuous f_1, f_2 vector fields on a manifold M of class C^2 and $x* \in M$, then*

$$\lim_{(t_1,t_2,x)\to(0,0,x_*)} \operatorname{dist}\left(\frac{\Phi^{f_2}_{-t_2} \circ \Phi^{f_1}_{-t_1} \circ \Phi^{f_2}_{t_2} \circ \Phi^{f_1}_{t_1}(x) - x_*}{t_1 t_2}, [f_1, f_2]_{set}(x_*) \right) = 0\,.$$

Theorem 3.4 (Commutativity [11])**.** *Let f_1, f_2 be vector fields of a manifold of class C^2. Then the flows of f_1, f_2 commute if and only if $[f_1, f_2]_{set}(x) = \{0\}$ for every $x \in M$ if and only if $[f_1, f_2](x)$ for almost every $x \in M$.*

Theorem 3.5 (Simultaneous rectification [11]). *Let $f_1, \ldots, f_m$ be locally Lipschitz vector fields on a manifold of class C^2, $x_* \in M$, and assume that $f_1(x_*), \ldots, f_m(x_*)$ are linearly independent. Assume that $[f_i, f_j] = 0$ almost everywhere in a neighborhood of x_* (In view of Proposition 3.2, (4), this is equivalent to $[f_1, f_2]_{set} = \{0\}$ in a neighborhood of x_*.) Then there exists a coordinate chart near x_* such that the coordinate representations of $f_1, \ldots, f_m$ in that chart are constant vector fields.*

Theorem 3.6 (Frobenius theorem [9]). *Let $\Delta = (\Delta_x)_{x \in M}$ be a* **distribution** *spanned by a set of locally Lipschitz and linearly independent vector fields $f_1, f_2, \ldots f_m$ on a manifold M of class C^2, that is, for all $x \in M$, $\Delta_x = \mathrm{span}\,\{f_1, f_2, \ldots f_m\}$.*

The following statements are equivalent.

1) *Δ is completely integrable, that is, for each $x_* \in M$ there exists a submanifold N of class $C^{1,1}$ passing through x_* (that is, $x_* \in N$) such that $T_xN = \Delta(x) = \mathrm{span}\,\{f_(x), \ldots, f_m(x)\}$ for every $x \in N$.*
2) *Δ is set-involutive, that is, $[f_i, f_j]_{set}(x) \subset \Delta_x$ for every $i, j = 1, \ldots, m$.*
3) *Δ is a.e. involutive, that is, $[f_i, f_j]_(x) \subset \Delta_x$ for every $i, j = 1, \ldots, m$ and for almost every $x \in M$.*

3.2 Nonsmooth iterated commutators and a nonsmooth Chow-Rashevski theorem

Resuming the discourse of Subsect. 2.2 we can reduce further the smoothness assumptions, requiring that that the vector fields $\mathbf{f} = (f_1, \ldots, f_m)$ participating in the definition of an iterated Lie bracket be as many times differentiable as it is necessary that for the said iterated Lie bracket $B(\mathbf{f})$ to be computed only almost everywhere.

Recall we have already defined what does it mean for $\mathbf{f}$ to be of class C^B. Now we shall give a meaning to the phrase "$\mathbf{f}$ is of class $C^{B-1;L}$".

Definition 3.7. *Let B a formal bracket of degree m and $\mathbf{f} = (f_1, \ldots, f_m)$ a m-tuple of vector fields. We shall say that* $\mathbf{f}$ is of class $C^{B-1;L}$ *(and write $\mathbf{f} \in C^{B-1;L}$) if:*

- *i) the components of $\mathbf{f}$ possess differentials up to one order less that would be the case if we required that $\mathbf{f}$ be of class C^B;*
- *ii) the highest order differentials are locally Lipschitz continuous.*

Recall that a vector field f is said to be *of class* $C^{k-1,1}$ if it is of class C^{k-1} and Df^{k-1} is locally Lipschitz continuous.

Examples 3.8.

$$\begin{aligned}
&\textit{If } B = [\cdot,\, \cdot] \textit{ and } \mathbf{f} = (f_1, f_2), \textit{ then } \mathbf{f} \in C^{B-1;L} \textit{ iff } f_1 \in C^{0,1}, f_2 \in C^{0,1}\,;\\
&\textit{If } B = \big[[\cdot,\, \cdot],\ \big] \textit{ and } \mathbf{f} = (f_1, f_2, f_3), \textit{ then } \mathbf{f} \in C^{B-1;L} \textit{ iff } f_1, f_2 \in C^{1,1}, f_3 \in C^{0,1}\,;\\
&\textit{If } B = \Big[\big[[\cdot,\, \cdot],\ \big],\ \Big] \textit{ and } \mathbf{f} = (f_1, f_2, f_3, f_4), \textit{ then } \mathbf{f} \in C^{B} \textit{ iff } f_1, f_2 \in C^{2,1}, f_3 \in C^{1,1}, f_4 \in C^{0,1}\,,
\end{aligned}$$

and so on.

Now let $M = \mathbb{R}^m$. If f is a vector field on M and $h \in \mathbb{R}^n$ is a translation, we denote by f^h the result of the action of this translation h on the vector field f: that is, we denote by f^h the vector field defined by setting $f^h(x) = f(x + h)$ for all $x \in M$.

Let B be formal bracket of degree m and $\mathbf{f} = (f_1, \ldots, f_m)$ an m-tuple of vector fields which is of class $C^{B-1;L}$ (so that $B(\mathbf{f})(x)$ is well-defined for almost every $x \in M$). We define for all $x \in M$

$$B_{set}(\mathbf{f})(x)$$

to be the convex hull all the limits

$$v = \lim_{\mathbf{h} \to 0} B(\mathbf{f^h})(x)$$

for $\mathbf{h} = (h_1, \dots, h_m) \in \mathbb{R}^m$, where $\mathbf{f}^{\mathbf{h}} := (f_1^{h_1}, \dots, f_m^{h_m})$ (observe that $B(\mathbf{f}^{\mathbf{h}})(x)$ is well-defined for almost every $x \in \mathbb{R}^m$).

Fact This notion of set-valued bracket can be defined in any differentiable manifold M by working on coordinate charts. This notion is chart invariant. The result is a set-valued "vector field", that is, a map

$$M \ni x \mapsto B(\mathbf{f})(x) \subset T_x M$$

with nonempty compact, convex values.

We shall say that a family $\mathcal{F}$ of vector fields on M satisfies the **nonsmooth LARC** at a point $x_* \in M$ if there exist formal brackets $B_1, \dots, B_\ell$ and tuples of vector fields $\mathbf{f}_1, \dots, \mathbf{f}_\ell$ such that

- i) M is of class C^k where k is the maximum of the degrees of the brackets B_j for $j = 1, \dots, \ell$;
- ii) $\mathbf{f}_j$ is of class $C^{B_j - 1;L}$ for $j = 1, \dots, \ell$;
- iii) for all $v_j \in B_{set}()(x_*)$, $j = 1, \dots, \ell$,

$$\operatorname{span}\left\{v_1, \dots, v_\ell\right\} = T_{x_*} M\,.$$

So the nonsmooth version of Chow-Rashevski theorem can be stated as follows:

Theorem 3.9 (A nonsmooth Chow-Rashevski theorem [6]). *Let $\mathcal{F}$ by a symmetric family of vector fields on M and $x_* \in M$.*

$$\mathcal{F} \text{ satisfies nonsmooth LARC at } x_* \Longrightarrow \mathcal{F} \text{ is STLC from } x_*\,.$$

If $\mathcal{F}$ satisfies the nonsmooth LARC at any $x \in M$ and M is connected, than $\mathcal{F}$ is controllable.

3.3 Sketch of the proofs

The commutators play an important role in controllability because they reveal "hidden" directions of motion apart from the obvious ones, that is, the given vector fields. For the usual commutator this can be seen by the asymptotic formula (2.2). A useful tool is the so-called Agrachev-Gamkrelidze formalism (see [1], [2], [11]) which greatly simplifies computations. Here is the same formula (2.2) in this formalism:

$$x e^{t_1 f_1} e^{t_2 f_2} e^{-t_1 f_1} e^{-t_2 f_2} = x + t_1 t_2 [f_1, f_2](x) + o(t_1 t_2) \tag{SAF}$$

as $(t_1, t_2) \to (0, 0)$.

In fact, formulas can for any iterated commutator $B(\mathbf{f})$; indeed associating recursively to each commutator a multiflow (that is a product of flows) in the following way: If

$$B(\mathbf{f}) = [B_1(\mathbf{f}_{(1)}), B_2(\mathbf{f}_{(2)})],$$

where $\mathbf{f} = (f_1, \dots, f_m)$, $m = deg(B)$, $B = [B_1, B_2]$ with $deg(B_1) = m_1$, $deg(B_2) = m_2$, $\mathbf{f}_{(1)} = (f_1, \dots, f_{m_1})$, $\mathbf{f}_{(2)} = (f_{m_1+1}, \dots, f_m)$, then

$$\begin{aligned}\Phi_B^{\mathbf{f}}(t_1, \dots, t_m) := \Phi_{B_1}^{\mathbf{f}_{(1)}}(t_1, \dots, t_{m_1}) \Phi_{B_2}^{\mathbf{f}_{(2)}}(t_{m_1+1}, \dots, t_m) \\ \left(\Phi_{B_1}^{\mathbf{f}_{(1)}}(t_1, \dots, t_{m_1})\right)^{-1} \left(\Phi_{B_2}^{\mathbf{f}_{(2)}}(t_{m_1+1}, \dots, t_m)\right)^{-1}\end{aligned}$$

Of course for any bracket of degree 1, that is, for any formal bracket B of degree one and any vector field f we sat $\Phi_B^f(t) := e^{tf}$ for all $t \in \mathbb{R}$ for which it makes sense.

Then the asymptotic formula says

Theorem 3.10 (Asymptotic formulas [7]). *If $\mathbf{f} \in C^B$, then*

$$x \Phi_B^{\mathbf{f}}(t_1, \dots, t_m) = x + t_1 \cdots t_m B(\mathbf{f})(x) + o(t_1 \cdots t_m)$$

as $(t_1, \ldots, t_m) \to 0_{\mathbb{R}^m}$.

Some examples: If $f_1, f_2, \in C^2$, $f_3 \in C^1$,

$$xe^{t_1f_1}e^{t_2f_2}e^{-t_1f_1}e^{-t_2f_2}e^{t_3f_3}e^{t_2f_2}e^{t_1f_1}e^{-t_2f_2}e^{-t_1f_1}e^{-t_3f_3} = x + t_1t_2t_3\big[[f_1,\ f_2],\ f_3\big](x) + o(t_1t_2t_3)$$

as $(t_1, t_2, t_3) \to (0, 0, 0)$.

If $f_1 \in C^3$, $f_2 \in C^3$, $f_3 \in C^2$, $f_4 \in C^1$, then

$$\begin{aligned} &xe^{t_1f_1}e^{t_2f_2}e^{-t_1f_1}e^{-t_2f_2}e^{t_3f_3}e^{t_2f_2}e^{t_1f_1}e^{-t_2f_2}e^{-t_1f_1}e^{-t_3f_3}e^{t_4f_4} \\ &\qquad e^{t_3f_3}e^{t_1f_1}e^{t_2f_2}e^{-t_1f_1}e^{-t_2f_2}e^{-t_3f_3}e^{t_2f_2}e^{t_1f_1}e^{-t_2f_2}e^{-t_1f_1}e^{-t_4f_4} \\ &\qquad\qquad = x + t_1t_2t_3t_4\Big[\big[[f_1,\ f_2],\ f_3\big],\ f_4\Big](x) + o(t_1t_2t_3t_4) \end{aligned}$$

as as $(t_1, t_2, t_3, t_4) \to (0, 0, 0, 0)$.

The asymptotic formulas follow from the following integral formulas:

Theorem 3.11 (Integral formulas [7]). *Let* $\mathbf{f} = (f_1, \ldots, f_m) \in C^B$. *There are multiflows, that is, products of*

$$\{e^{t_if_i}\}, \quad \{e^{s_jf_j}\},$$

and their inverses, as many as is the number of subbrackets B' *of* B *of* $deg(B') \geq 2$, *say* r, *and call them* $\Phi_i = \Phi_i(t_1, \ldots, t_m, s_1, \ldots, s_m)$, *for* $i = 1, \ldots, r$, *such that*

$$x\Phi_B^{\mathbf{f}}(t_1, \ldots, t_m) = x + \int_0^{t_1} \cdots \int_0^{t_m} x\Phi_B^{\mathbf{f}}\Phi_1^{\#}\left[\Phi_2^{\#}[\cdots, \cdots], \Phi_3^{\#}[\cdots, \cdots]\right] ds_1 \cdots ds_m$$

Here are the explicit formulas for brackets of low degree $(2, 3)$ in the Agrachev-Gamkrelidze formalism.

$$xe^{t_1f_1}e^{t_2f_2}e^{-t_1f_1}e^{-t_2f_2} = x + \int_0^{t_1}\int_0^{t_2} xe^{t_1f_1}e^{s_2f_2}e^{(s_1-t_1)f_1}[f_1, f_2]e^{-s_1f_1}e^{-s_2f_2}ds_1\, ds_2.$$

$$\begin{aligned} &xe^{t_1f_1}e^{t_2f_2}e^{-t_1f_1}e^{-t_2f_2}e^{t_3f_3}e^{t_2f_2}e^{t_1f_1}e^{-t_2f_2}e^{-t_1f_1}e^{-t_3f_3} - x = \\ &\qquad \int_0^{t_1}\int_0^{t_2}\int_0^{t_3} xe^{t_1f_1}e^{t_2f_2}e^{-t_1f_1}e^{-t_2f_2}e^{s_3f_3}e^{t_2f_2}e^{t_1f_1}e^{(s_2-t_2)f_2}e^{-t_1f_1}e^{-s_2f_2} \\ &\qquad\qquad \left[e^{s_2f_2}e^{s_1f_1}[f_1, f_2]e^{-s_1f_1}e^{-s_2f_2},\ f_3\right]e^{s_2f_2}e^{t_1f_1}e^{-s_2f_2}e^{-t_1f_1}e^{-s_3f_3}ds_1\, ds_2\, ds_3. \end{aligned}$$

If f is of class $C^{B-1;L}$ the asymtotic formulas above do not make sense in general, however, suitable generalizations can be stated making use of the notion of a set-valued bracket.

Theorem 3.12. *If* $\mathbf{f}$ *is of class* $C^{B-1;L}$, $deg(B) = m$, $x_* \in M$, *then*

$$\operatorname{dist}\left(\Phi_B^{\mathbf{f}}(t_1, \ldots, t_m)(x) - x,\ t_1 \cdots t_m B(\mathbf{f})(x_*)\right) = |t_1 \cdots t_m| o(1)$$

as as $|(t_1, \ldots, t_m)| + |x - x_*| \to 0$.

The nonsmooth Chow-Rashevski theorem is proved by using a generalized differentiation theory due to H. Sussmann with good open mapping and chain rule properties.

The result under minimum classical smoothness assumptions for C^1 vector fields can be proved via classical analysis.

3.4 Open problems

The first open problem is to prove or disprove the following generalization of a result of Brunovsky. Let $M = \mathbb{R}^n$. We say that a family of vector fields on M is od iff whenever $f \in \mathcal{F}$, then $x \mapsto -f(-x)$ belongs to $\mathcal{F}$ too.

Brunovsky has proved that,

Theorem 3.13 (Brunovsky). *If $\mathcal{F}$ is odd, its elements are sufficiently smooth, and satisfies the LARC at $x_* = 0$, then $\mathcal{F}$ is STLC at $x_* = 0$.*

(Actually, for odd families of analytic vector fields STLC at 0 is equivalent to the LARC at 0.) Determine whether the following statement is true or not.

Statement. *If $\mathcal{F}$ is odd, and satisfies the nonsmooth LARC at $x_* = 0$, then $\mathcal{F}$ is STLC at $x_* = 0$.*

Let us define recursively for sufficiently smooth vector fields f, g the classical iterated Lie brackets:

$$\mathrm{ad}_f^k g := g$$

for $k = 0$, and

$$\mathrm{ad}_f^k g := \left[f,\, \mathrm{ad}_f^{k-1} g\right]$$

for $k \geq 1$.

Examples 3.14. $\mathrm{ad}_f^1 g = [f, g]$, $\mathrm{ad}_f^2 g = \left[f,\, [f,\, g]\right]$, $\mathrm{ad}_f^3 g = \Big[f,\, \left[f,\, [f,\, g]\right]\Big]$.

If f, g are of class $C^{k-1,1}$ let us denote–with a slight abuse of notation–also by

$$\mathrm{ad}_f^k g$$

the set-valued iterated Lie bracket that "corresponds to the classical bracket $\mathrm{ad}_f^k g'$.

Prove or disprove the following statement.

Statement. *Consider a control system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x)\,,$$

where f, g_i are vector fields on a differentiable manifold M, the controls $(u_1, \ldots, u_m)$ take values in a subset U of $\mathbb{R}^m$ which is a neighborhood of the origin. Let $x_* \in M$. Assume that for each $i = 1, \ldots, m$ exist $k_i \in \mathbb{N}$ such that f, g_i are of class $C^{k_i-1;1}$, and

$$\Big\{\mathrm{ad}_f^k g(x_*) \;:\; i = 1, \ldots, m,\; 0 \leq k \leq k_i\Big\} \cup \Big\{[g_i,\, g_j](x_*) \;:\; 1 \leq i, j \leq m,\; k_i, k_j \geq 1\Big\}$$

span all T_{x_*}. Then the given system is STLC at x_*.

If f, g_i are sufficiently smooth the result is proved in Sussmann 1987.

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