Student-like models for risky asset with dependence

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Abstract

We present a new construction of the Student and Student-like fractal activity time model for risky asset. The construction uses the diffusion processes and their superpositions and allows for specified exact Student or Student-like marginal distributions of the returns and for flexible and tractable dependence structure. The fractal activity time is asymptotically self-similar, which is a desired feature seen in practice.

Key Words: Risky asset model; Student distribution; geometric Brownian motion; Fractal activity time; reciprocal gamma diffusion; option pricing formula.

1 Introduction

The fit of the classical Black-Scholes-Merton (BSM) geometric Brownian motion (GBM) has been questioned in the last decade based on real financial data. Empirically derived 'stylized features' of returns (logarithmic asset returns) summarized by Granger [13] includes: good approximation of the returns by uncorrelated identically distributed random variables, dependence of squared returns; and distributions that are heavier-tailed and higher-peaked than Gaussian distributions. A number of models that incorporate non-Gaussian distributions and/or dependence in returns have been proposed in [1; 8; 9; 14; 15; 16; 18; 20; 21], and many others. The stochastic volatility model of Barndorff-Nielsen and Shephard [1] generalized GBM by considering a continuous time stochastic volatility process constructed via superpositions of the Ornstein-Uhlenbeck (OU) processes.

In the present paper, we use a fractal activity time geometric Brownian motion (FATGBM) model, proposed by Heyde [14] and elaborated on for Variance Gamma and Student distributions [8; 9; 19; 20; 21]. Models with Variance Gamma and normal inverse Gaussian distributions of the returns were considered in [8; 9; 15; 19; 20; 21] using OU type and diffusion-type processes and their superpositions. In this paper, we consider two constructions of activity time. First construction is based on the reciprocal gamma diffusion type processes [15], [18]. This construction leads to stationary returns with exact Student marginal distribution. Under short-range dependence of the returns, we obtain explicit pricing formula based on the asymptotic self-similarity of the activity time. The second construction uses a superposition of two reciprocal gamma diffusion type processes. The important advantages of the second construction are the ability to incorporate more complicated dependence structure.

Some model for a risky assets with the heavy-tailed returns has been proposed by Borland [6], where the random variable affecting returns had been modeled according to an anomalous Wiener process characterized by a Tsallin distribution of index q, the so called q-Gaussian distribution. The q-Gaussian distribution is in fact the Student distribution with different parametrization.

2 The Fractal Activity Time Model

Let T_t, $t \geq 0$, be a random time change or fractal activity time, that is, positive nondecreasing process such that $T_0 = 0$, and let W_t, $t \geq 0$, be a standard Brownian motion independent of the process T_t.

We consider the model for the stock price

$$P_t = P_0 e^{\mu t + \theta T_t + \sigma W_{T_t}}, \ t \geq 0, \tag{1}$$

where parameters $\mu \in R$ and $\sigma > 0$ reflect drift and diffusion, and $\theta \in R$. It is known that under some set of conditions, P_t, $t \geq 0$, is the strong solution of the following stochastic differential equation (SDE) (see Kobayashi [17])

$$dP_t = \mu P_t dt + (\theta + \sigma^2/2) P_t dT_t + \sigma P_t dW_{T_t}, \ t \geq 0.$$

The increments over unit time are $\tau_t = T_t - T_{t-1}$, $t = 1, 2, \ldots$ and the returns are given by

$$X_t = \log\left(\frac{P_t}{P_{t-1}}\right) \stackrel{d}{=} \mu + \theta\tau_t + \sigma\tau_t^{\frac{1}{2}} W_1, \tag{2}$$

where $\stackrel{d}{=}$ denotes equality in distribution.

For our construction, the fractal activity time T_t is defined as follows (see Kerss et al [16]):

$$T_0 = 0, \quad T_t = \sum_{i=1}^{[t]} \tau_i + \tau_{[t]+1}(t - [t]), \tag{3}$$

where $\{\tau_s, \ s \geq 0\}$ is a stationary process with heavy-tailed marginal distribution.

For modeling the increments over time τ_t with a stationary OU processes with Inverse Gamma, Inverse Gaussian or Tampered Stable distribution, see Leonenko et al [16; 19]. These constructions modeled the returns by stochastic processes with Student, Normal Inverse Gaussian or Normal Tempered Stable distribution correspondingly.

In this paper we use two different models for the stationary process $\{\tau_s,\ s \geq 0\}$:

Model 1: Define $\tau_i = \tau_i^{(1)} = Y_i,\ i = 0, 1, 2, \ldots$, where $\{Y_t,\ t \geq 0\}$ is an ergodic Markov process such that

$$dY_t = -\omega \left(Y_t - \frac{\alpha}{\beta - 1} \right) dt + \sqrt{\frac{2\omega}{\beta - 1} Y_t^2}\, dW_t, \quad t \geq 0, \tag{4}$$

where $\omega > 0$, $\alpha > 0$, $\beta > 1$ and $W = \{W_t, t \geq 0\}$ is a standard Brownian motion.

The stochastic differential equation given by (4) has a unique Markovian weak solution and the diffusion process Y that solves it is ergodic with reciprocal gamma $R\Gamma(\beta, \alpha)$ as invariant density (see [18] and formula (5) below).

Model 2: Define

$$\tau_i = \tau_i^{(2)} = Y_i^{(1)} + Y_i^{(2)},\ i = 0, 1, 2, ...,$$

where $\{Y_t^{(1)},\ t \geq 0\}$ and $\{Y_t^{(2)},\ t \geq 0\}$ are independent processes such that

$$dY_t^{(j)} = -\omega^{(j)} \left(Y_t^{(j)} - \frac{\alpha^{(j)}}{\beta^{(j)} - 1} \right) dt + \sqrt{\frac{2\omega^{(j)}}{\beta^{(j)} - 1} (Y_t^{(j)})^2}\, dW_t^{(j)},\ j = 1, 2, \quad t \geq 0,$$

where $W_t^{(j)},\ j = 1, 2$ are the independent copies of the standard Brownian motion.

In this paper we develop the distribution theory for the returns (2) in which the dynamic of τ_t is given by one of these two models respectively and prove the asymptotic self-similarity of the fractal activity time T_t. Concerning the associated stock price processes, we determine the relationship between some parameters in order to achieve the martingatily of the discounted prices and recover a close options pricing formula that improves in respect to the classical BSM formula the reproduction of the real option quotes. Note the idea of superposition belongs to Barndorff-Nielsen and Sheppard (2001) [1] (see also [2] and [4]).

3 Distribution Theory

3.1 Results for model 1

For our first construction the increments of activity time process T_t defined in (3) have a reciprocal gamma $R\Gamma(\beta, \alpha)$ stationary distribution with probability density function (pdf)

$$\mathfrak{rg}(x) = \frac{\alpha^\beta}{\Gamma(\beta)}\, x^{-\beta-1} e^{-\frac{\alpha}{x}} I_{x>0}, \tag{5}$$

where $\alpha > 0$, $\beta > 0$, and $I_.$ is the indicator.

We need to find for our first model the probability distribution function for the returns and show that this construction leads to stationary returns with exact Student marginal distribution. Also we need to find moments for the increments of activity time and for returns. It will be used for options pricing formula. We also examine the auto-correlation structure of returns and make sure that returns are dependent.

Remark 1 *If the distribution of τ_t is $R\Gamma\left(\frac{\nu}{2}, \frac{\delta^2}{2}\right)$, where $\delta > 0$, $\nu > 0$, then moment of the k-th order is given by*

$$\mathrm{E}[\tau_t^k] = \left(\frac{\delta^2}{2}\right)^k \frac{\Gamma(\frac{\nu}{2} - k)}{\Gamma(\frac{\nu}{2})}, \quad \frac{\nu}{2} > k. \tag{6}$$

Thus, we have the explicit expressions for the mean and variance of τ_t:

$$\mathrm{E}[\tau_t] = \frac{\delta^2}{\nu - 2}, \quad \nu > 2, \qquad \mathrm{Var}[\tau_t] = \frac{2\delta^4}{(\nu-2)^2(\nu-4)}, \quad \nu > 4. \tag{7}$$

In the Model 1 assuming $\theta = 0$ and $\sigma = 1$, the returns X_t given by (2) is stationary process with marginal Student $T(\mu, \delta, \nu)$ distribution

$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\delta\sqrt{\pi}\Gamma(\frac{\nu}{2})} \frac{1}{[1 + (\frac{x-\mu}{\delta})^2]^{\frac{\nu+1}{2}}}\ , \quad x \in \mathbb{R}, \tag{8}$$

where $\mu \in \mathbb{R}$ is a location parameter, $\delta > 0$ is a scaling parameter, $\nu > 0$ is a tail index.
The moments of the returns X_t are given by

$$\mathrm{E}\{X_t - \mathrm{E}[X_t]\}^n = \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{\nu-n}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi}} \delta^n I_{\nu > n = 2k,\ k \in \mathbb{N}} = S_n(\nu) I_{\nu > n = 2k,\ k \in \mathbb{N}},$$

where

$$S_n(\nu) = \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{\nu-n}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi}} \delta^n. \tag{9}$$

If $Y_0 \sim R\Gamma\left(\frac{\nu}{2}, \frac{\delta^2}{2}\right)$, then Y_t is a stationary process with marginal density $R\Gamma\left(\frac{\nu}{2}, \frac{\delta^2}{2}\right)$ and the autocorrelation function for Y_t is given by

$$\rho(k) = \mathrm{Corr}(Y_{s+k}, Y_s) = e^{-\omega k},\ k \geq 0,\ s \geq 0.$$

Proposition 1 *If X_t is a stochastic process given by (2), when τ_t as in model 1, then, for any integer $k \geq 0$:*

1. $$\mathrm{Cov}(X_t, X_{t+k}) = \frac{2\theta^2\delta^4}{(\nu-2)^2(\nu-4)} e^{-\omega k}; \tag{10}$$

   in particular $\mathrm{Cov}_{\theta=0}(X_t, X_{t+k}) = 0$.

2. $$\mathrm{Cov}(X_t^2, X_{t+k}^2) = \frac{\delta^4}{(\nu-2)(\nu-4)} \left[\frac{2(\sigma^2 + 2\theta\mu)^2}{\nu - 2} e^{-\omega k} + \right.$$
   $$\left. \theta^4\delta^4 \left[\frac{8}{(\nu-4)(\nu-6)^2(\nu-8)} e^{-\omega k \frac{\nu-4}{\nu-2}} + \frac{8}{(\nu-2)(\nu-6)^2} e^{-\omega k}\right] + (\theta^2\sigma^2 + 2\theta^3\mu) \frac{8\delta^2}{(\nu-2)(\nu-6)} e^{-\omega k}\right]; \tag{11}$$

in particular

$$\mathrm{Cov}_{\theta=0}(X_t^2, X_{t+k}^2) = \frac{2\delta^4\sigma^4}{(\nu-2)^2(\nu-4)}e^{-\omega k}.$$

Proof. For the covariance function, we obtain:

$$\begin{gathered}\mathrm{Cov}(X_t, X_{t+k}) = \mathrm{Cov}(\mu + \theta\tau_t + \sigma\sqrt{\tau_t}W_1^{(1)}, \mu + \theta\tau_{t+k} + \sigma\sqrt{\tau_{t+k}}W_1^{(2)}) = \\ \mathrm{E}[(\theta\tau_t + \sigma\sqrt{\tau_t}W_1^{(1)} - \theta\mathrm{E}[\tau_t])(\theta\tau_{t+k} + \sigma\sqrt{\tau_{t+k}}W_1^{(2)} - \theta\mathrm{E}[\tau_{t+k}])] = \\ = \theta^2(\mathrm{E}[\tau_t\tau_{t+k}] - \mathrm{E}[\tau_t]\mathrm{E}[\tau_{t+k}]) = \theta^2\mathrm{Cov}(\tau_t, \tau_{t+k}),\end{gathered} \tag{12}$$

where the second equation follows from the fact that $\mathrm{E}[X_t] = \mu + \theta\mathrm{E}[\tau_t]$, and the third from the independence between the two stochastic processes.

We have

$$\begin{gathered}\mathrm{Cov}(X_t^2, X_{t+k}^2) = (\sigma^4 + 4\theta^2\mu^2 + 4\theta\mu\sigma^2)\mathrm{Cov}(\tau_t, \tau_{t+k}) + \\ \theta^4\mathrm{Cov}(\tau_t^2, \tau_{t+k}^2) + (\theta^2\sigma^2 + 2\theta^3\mu)(\mathrm{Cov}(\tau_t^2, \tau_{t+k}) + \mathrm{Cov}(\tau_t, \tau_{t+k}^2)) = \\ (\sigma^2 + 2\theta\mu)^2\mathrm{Cov}(\tau_t, \tau_{t+k}) + \theta^2\mathrm{Cov}(\tau_t^2, \tau_{t+k}^2) + 2(\theta^2\sigma^2 + 2\theta^3\mu)\mathrm{Cov}(\tau_t^2, \tau_{t+k}).\end{gathered} \tag{13}$$

Moreover, τ_t is a reciprocal Gamma diffusion, then $\mathrm{Corr}(\tau_t, \tau_{t+k}) = e^{-\omega k}$ with variance given by (6). Using an exact form of transition density of a reciprocal gamma diffusion obtained in [18], we have

$$\begin{aligned}E[\tau_t\tau_{t+k}] &= \frac{\delta^4}{(\nu-2)^2}\left(\frac{2}{(\nu-4)}e^{-\omega k} + 1\right), \\ E[\tau_t^2\tau_{t+k}^2] &= \frac{\delta^8}{(\nu-2)(\nu-4)}\left[\frac{8}{(\nu-4)(\nu-6)^2(\nu-8)}e^{-\omega k\frac{\nu-4}{\nu-2}} + \frac{8e^{-\omega k}}{(\nu-2)(\nu-6)^2} + \frac{1}{(\nu-2)(\nu-4)}\right], \\ E[\tau_t\tau_{t+k}^2] &= E[\tau_t^2, \tau_{t+k}] = \frac{\delta^6}{(\nu-2)^2(\nu-4)}\left(\frac{4}{\nu-6}e^{-\omega k} + 1\right).\end{aligned} \tag{14}$$

Elaborating these expressions, we recover

$$\begin{aligned}Cov(\tau_t, \tau_{t+k}) &= \frac{2\delta^4}{(\nu-2)^2(\nu-4)}e^{-\omega k}, \\ Cov(\tau_t^2, \tau_{t+k}^2) &= \frac{\delta^8}{(\nu-2)(\nu-4)}\left[\frac{8}{(\nu-4)(\nu-6)^2(\nu-8)}e^{-\omega k\frac{\nu-4}{\nu-2}} + \frac{8}{(\nu-2)(\nu-6)^2}e^{-\omega k}\right], \\ Cov(\tau_t^2, \tau_{t+k}) &= \frac{\delta^6}{(\nu-2)^2(\nu-4)}\left(\frac{4}{\nu-6}e^{-\omega k}\right),\end{aligned} \tag{15}$$

and thus, substituting (14), (15) back into (12), (13), we obtain (10), (11).

■

3.2 Results for model 2

In order to increase the correlation of returns, we introduce the second model, in which τ_t is the sum of two independent stochastic processes with reciprocal gamma marginal distributions. Indeed, let the activity time process T_t be given by (3), in which $\tau_i = \tau_i^{(2)} = Y_i^{(1)} + Y_i^{(2)},\ i = 0, 1, 2, ...,$ where $Y_t^{(1)}$ and $Y_t^{(2)}$ are independent stationary processes such that $Y_t^{(j)} \sim R\Gamma(\beta_j, \alpha_j)$, $j = 1, 2$.
The stochastic process $Y_t^{(1)} + Y_t^{(2)},\ t \geq 0$ is stationary and its marginal density is a convolution of two reciprocal gamma density.
Similarly to Kerss et al [16], our approach for option pricing formulas depends upon the knowledge of the density function of the sum of two independent reciprocal gamma distribution.
An important result in this direction is given by Giron and Castillo [12], who had showed that under some restrictions on the shape parameters the convolution of reciprocal gamma distributions is distributed as a finite mixture of reciprocal gamma distributions all having the same scale parameter.
If $Y_t^{(1)} \sim R\Gamma\left(n + \frac{1}{2}, \alpha_1\right)$ and $Y_t^{(2)} \sim R\Gamma\left(m + \frac{1}{2}, \alpha_2\right)$ then the convolution of $Y_t^{(1)}$ and $Y_t^{(2)}$ is distributed as the following mixture (see [12]):

$$Y_t^{(1)} + Y_t^{(2)} \sim \sum_{i=1}^{m+1} p_i R\Gamma\left(n - \frac{1}{2} + i, (\sqrt{\alpha_1} + \sqrt{\alpha_2})^2\right), \tag{16}$$

where the weights $p_i \geq 0$, $\sum_{i=1}^{m+1} p_i = 1$ can be obtained from the following formula, derived in [12]:

$$p_{m+1} = \frac{\sqrt{\pi}\Gamma(n + m + \frac{1}{2})}{\Gamma(n + \frac{1}{2})\Gamma(m + \frac{1}{2}))} \frac{(\sqrt{\alpha_1})^n (\sqrt{\alpha_2})^m}{(\sqrt{\alpha_1} + \sqrt{\alpha_2})^{n+m}},$$

$$p_{j+1} = 2^{2j+2}\Gamma\left(n + \frac{1}{2} + j\right)\left(\frac{c\gamma_{n+j}}{2^{2j+2}\sqrt{\pi}(\sqrt{\alpha_1} + \sqrt{\alpha_2})^{n+j}} - \sum_{i=j+2}^{m+1} \frac{p_i}{2^{2j}\Gamma(n - \frac{1}{2} + i)} \frac{(n + 2j - 2 - j)!}{(n + j)!(i - 1 - j)!}\right),$$
$$j = 0, 1, 2, ..., m - 1,$$

where

$$c = \frac{\pi}{2^{2(n+m)}\Gamma(n + \frac{1}{2})\Gamma(m + \frac{1}{2})},$$

$$\gamma_k = 2^{2k} \sum_{i=0}^{k} \frac{(2n - i)!}{(i)!(n - i)!} \frac{(2m - k + i)!}{(k - i)!(m - k + i)!} (\sqrt{\alpha_1})^i (\sqrt{\alpha_2})^{k-i},\ k = 0, 1, ..., n,$$

$$\gamma_{n+k} = 2^{2(n+k)} \sum_{i=0}^{n} \frac{(2n - i)!}{(i)!(n - i)!} \frac{(2m - n - k + i)!}{(n + k - i)!(m - n - k + i)!} (\sqrt{\alpha_1})^i (\sqrt{\alpha_2})^{n+k-i},\ k = 0, 1, ..., m - n,$$

$$\gamma_{m+k} = 2^{2(m+k)} \sum_{i=k}^{n} \frac{(2n - i)!}{(i)!(n - i)!} \frac{(m - k + i)!}{(m + k - i)!(i - k)!} (\sqrt{\alpha_1})^i (\sqrt{\alpha_2})^{m+k-i},\ k = 0, 1, ..., n.$$

Remark 1 *We obtain the density of the increments of activity time process T_t as follows: if $\tau_i = Y_i^{(1)} + Y_i^{(2)},\ i = 0, 1, 2, ...,$ then it is distributed as the mixture (16) with density*

$$f_{Y_t^{(1)}+Y_t^{(2)}}(x) = \sum_{i=1}^{m+1} p_i f_{R\Gamma}\left(n - \frac{1}{2} + i, (\sqrt{\alpha_1} + \sqrt{\alpha_2})^2, x\right),\ x > 0. \tag{17}$$

Remark 2 *Denoting*

$$(\sqrt{\alpha_1} + \sqrt{\alpha_2})^2 = \frac{\delta^2}{2}, \quad n - \frac{1}{2} + i = \frac{\nu_i}{2},\ i = 1, 2, ..., m+1, \tag{18}$$

then moments of all orders are given by

$$\mathrm{E}[\tau_t^k] = \sum_{i=1}^{m+1} p_i \left(\frac{\delta^2}{2}\right)^k \frac{\Gamma\left(\frac{\nu_i}{2} - k\right)}{4\Gamma\left(\frac{\nu_i}{2}\right)}, \quad \frac{\nu_i}{2} > k.$$

In particular,

$$\mathrm{E}[\tau_t] = \sum_{i=1}^{m+1} p_i \frac{\delta^2}{\nu_i - 2}, \min_{i=1,...m+1} \nu_i > 2; \qquad \mathrm{Var}[\tau_t] = \sum_{i=1}^{m+1} p_i \frac{2\delta^4}{(\nu_i - 2)^2(\nu_i - 4)}, \quad \min_{i=1,...m+1} \nu_i > 4. \tag{19}$$

The characteristic function of the returns can be expressed in form of the mixture of the modified Bessel function of the third kind, $K_\lambda(x)$, as shown in the next statement.

Proposition 2 *Let $\tau_t = Y_t^{(1)} + Y_t^{(2)}$ be a sum of two independent reciprocal gamma diffusions: $Y_t^{(1)} \sim R\Gamma\left(\frac{\nu_1}{2}, \frac{\delta_1^2}{2}\right)$, $Y_t^{(2)} \sim R\Gamma\left(\frac{\nu_2}{2}, \frac{\delta_2^2}{2}\right)$ Then, the characteristic function of the returns 2 for $\theta = 0$, $\sigma = 1$, is given by*

$$\phi_X(\zeta) = \mathrm{E}e^{i\zeta X_t} = e^{i\zeta\mu} \sum_{i=1}^{m+1} p_i \phi_T(\zeta, \delta, \nu_i), \tag{20}$$

where $\phi_T(\zeta, \delta, \nu_i) = K_{\nu_i/2}(\delta|\zeta|)(\delta|\zeta|)^{\nu_i/2} 2^{1-(\nu_i/2)}/\Gamma\left(\frac{\nu_i}{2}\right)$, $i \in \{1, 2, ..., m+1\}$ is the characteristic function of the symmetric Student distribution.
The moments of returns X_t are of the form:

$$\mathrm{E}\{X_t - \mathrm{E}[X_t]\}^n = \sum_{i=1}^{m+1} p_i S_n(\nu_i) I_{\nu_i > n = 2k,\ k \in \mathbb{N}},\ t = 1, 2, ..., \tag{21}$$

denoting $S_n(\nu_i)$ as in (9).

Proof. The characteristic function of the returns $X_t \stackrel{d}{=} \mu + \sqrt{\vartheta_t} Z$, where the random variable Z has the standard normal distribution $N(0,1)$ and ϑ_t is distributed as the mixture (16), is of the form:

$$\phi_X(\zeta) = e^{i\zeta\mu} \int_0^\infty e^{-(\frac{\zeta^2}{2})x} f_\vartheta(x) dx =$$

$$= e^{i\zeta\mu} \int_0^\infty e^{-(\frac{\zeta^2}{2})x} \sum_{i=1}^{m+1} p_i f_{R\Gamma}\left(n - \frac{1}{2} + i, (\sqrt{\alpha_1} + \sqrt{\alpha_2})^2\right) dx.$$

Using the Remark 2, we have

$$\phi_X(\zeta) = e^{i\zeta\mu}\sum_{i=1}^{m+1} p_i \int_0^\infty e^{-(\zeta^2/2)x} f_{R\Gamma}(\frac{\nu_i}{2},\frac{\delta^2}{2})dx = e^{i\zeta\mu}\sum_{i=1}^{m+1} p_i K_{\nu_i/2}(\delta\,|\zeta|)(\delta\,|\zeta|)^{\nu_i/2} 2^{1-(\nu_i/2)}/\Gamma\left(\frac{\nu_i}{2}\right).$$

The moments of the returns X_t can be then obtained easily. ∎

Remark 3 *We obtained an explicit expressions for the first four central moments of the returns:*

$$\mu_2 = \mathrm{E}[(X_t-\mu)^2] = \sum_{i=1}^{m+1} p_i \frac{\delta^2}{\nu_i - 2}, \quad \min_{i=1,\dots m+1}\{\nu_i\} > 2 \quad \mu_3 = \mathrm{E}[(X_t-\mu)^3] = 0,$$

$$\mu_4 = \mathrm{E}[(X_t-\mu)^4] = \sum_{i=1}^{m+1} p_i \frac{3\delta^4}{(\nu_i-4)(\nu_i-2)}, \min_{i=1,\dots m+1}\{\nu_i\} > 4.$$

Let $\tau_t = Y_t^{(1)} + Y_t^{(2)}$ be a sum of two independent reciprocal Gamma diffusions, $Y_t^{(j)} \sim R\Gamma\left(\frac{\nu_j}{2},\frac{\delta_j^2}{2}\right)$ with correlation coefficients ω_j, $j=1,2$. Let us find $\mathrm{Cov}(\tau_t,\tau_{t+k})$, $\mathrm{Cov}(\tau_t^2,\tau_{t+k})$ and $\mathrm{Cov}(\tau_t^2,\tau_{t+k}^2)$. We have

$$\mathrm{Cov}(\tau_t,\tau_{t+k}) = \mathrm{Cov}(Y_t^{(1)}+Y_t^{(2)}, Y_{t+k}^{(1)}+Y_{t+k}^{(2)}) = \mathrm{Cov}(Y_t^{(1)},Y_{t+k}^{(1)}) + Cov(Y_t^{(2)},Y_{t+k}^{(2)}),$$

$$\mathrm{Cov}(\tau_t^2,\tau_{t+k}^2) = \mathrm{Cov}\left((Y_t^{(1)}+Y_t^{(2)})^2, (Y_{t+k}^{(1)}+Y_{t+k}^{(2)})^2\right) =$$

$$\mathrm{Cov}\left((Y_t^{(1)})^2+(Y_t^{(2)})^2+2Y_t^{(1)}Y_t^{(2)}, (Y_{t+k}^{(1)})^2+(Y_{t+k}^{(2)})^2+2Y_{t+k}^{(1)}Y_{t+k}^{(2)}\right) =$$

$$\mathrm{Cov}\left((Y_t^{(1)})^2,(Y_{t+k}^{(1)})^2\right) + \mathrm{Cov}\left((Y_t^{(2)})^2,(Y_{t+k}^{(2)})^2\right) + 4\mathrm{E}[Y_t^{(2)}][\mathrm{E}[(Y_t^{(1)})^2 Y_{t+k}^{(1)}] - \mathrm{E}[Y_t^{(1)}]\mathrm{E}[(Y_t^{(1)})^2]]$$

$$+4\mathrm{E}[Y_t^{(1)}][\mathrm{E}[(Y_t^{(2)})^2 Y_{t+k}^{(2)}] - \mathrm{E}[Y_t^{(2)}]\mathrm{E}[(Y_t^{(2)})^2]] + 4\mathrm{E}[Y_t^{(1)}Y_{t+k}^{(1)}]\mathrm{E}[Y_t^{(2)}Y_{t+k}^{(2)})] - 4\mathrm{E}[Y_t^{(1)}]^2\mathrm{E}[Y_t^{(2)}]^2.$$

The mixed covariance is calculated in the same way and it is given by:

$$\mathrm{Cov}(\tau_t^2,\tau_{t+k}) = \mathrm{Cov}(\tau_t,\tau_{t+k}^2) =$$

$$\mathrm{Cov}\left((Y_t^{(1)})^2,Y_{t+k}^{(1)}\right) + \mathrm{Cov}\left((Y_t^{(2)})^2,Y_{t+k}^{(2)}\right) + 2\mathrm{E}[Y_t^{(2)}]\mathrm{Cov}\left(Y_t^{(1)},Y_{t+k}^{(1)}\right) + 2\mathrm{E}[Y_t^{(1)}]\mathrm{Cov}\left(Y_t^{(2)},Y_{t+k}^{(2)}\right).$$

Using (14), (15) and (7) for diffusions $Y_t^{(1)}$ and $Y_t^{(2)}$, we can get

$$\mathrm{Cov}(\tau_t,\tau_{t+k}) = e^{-\omega_1 k}\frac{2\delta_1^4}{(\nu_1-2)^2(\nu_1-4)} + e^{-\omega_2 k}\frac{2\delta_2^4}{(\nu_2-2)^2(\nu_2-4)}, \tag{22}$$

$$\begin{aligned} &\mathrm{Cov}(\tau_t^2,\tau_{t+k}^2) = \\ &= \frac{\delta_1^8}{(\nu_1-2)(\nu_1-4)}\left[\frac{8}{(\nu_1-4)(\nu_1-6)^2(\nu_1-8)}e^{-\omega_1 k\frac{\nu_1-4}{\nu_1-2}} + \frac{8e^{-\omega_1 k}}{(\nu_1-2)(\nu_1-6)^2}\right] + \\ &+ \frac{\delta_2^8}{(\nu_2-2)(\nu_2-4)}\left[\frac{8}{(\nu_2-4)(\nu_2-6)^2(\nu_2-8)}e^{-\omega_2 k\frac{\nu_2-4}{\nu_2-2}} + \frac{8e^{-\omega_2 k}}{(\nu_2-2)(\nu_2-6)^2}\right] + \\ &+ \frac{4\delta_2^2}{\nu_2-2}\left[\frac{\delta_1^6}{(\nu_1-2)^2(\nu_1-4)}\left(\frac{4}{\nu_1-6}e^{-\omega_1 k}\right)\right] + \frac{4\delta_1^2}{\nu_1-2}\left[\frac{\delta_2^6}{(\nu_2-2)^2(\nu_2-4)}\left(\frac{4}{\nu_2-6}e^{-\omega_2 k}\right)\right] + \\ &+ \frac{4\delta_1^4\delta_2^4}{(\nu_1-2)^2(\nu_2-2)^2}\left(\frac{4}{(\nu_1-4)^2(\nu_2-2)^2}e^{-(\omega_1+\omega_2)k} + \frac{2}{(\nu_1-4)^2}e^{-\omega_1 k} + \frac{2}{(\nu_2-4)^2}e^{-\omega_2 k}\right); \end{aligned} \tag{23}$$

$$\mathrm{Cov}(\tau_t^2,\tau_{t+k}) =$$
$$= \frac{\delta_1^6}{(\nu_1-2)^2(\nu_1-4)}\left(\frac{4}{\nu_1-6}e^{-\omega_1 k}\right) + \frac{\delta_2^6}{(\nu_2-2)^2(\nu_2-4)}\left(\frac{4}{\nu_2-6}e^{-\omega_2 k}\right) +$$
$$\frac{4\delta_2^2\delta_1^4 e^{-\omega_1 k}}{(\nu_2-2)(\nu_1-2)^2(\nu_1-4)} + \frac{4\delta_1^2\delta_2^4 e^{-\omega_2 k}}{(\nu_1-2)(\nu_2-2)^2(\nu_2-4)}. \tag{24}$$

Using (12) and (13) we can easy get the correlation structure of the stochastic process X_t for model 2, as shown in the next proposition.

Proposition 3 *If X_t is the stochastic process given by (2), when $\tau_t = Y_t^{(1)} + Y_t^{(2)}$ is a sum of two independent reciprocal Gamma diffusions as in model 2, then, for any integer $k \geq 0$ we have:*

1.
$$\mathrm{Cov}(X_t, X_{t+k}) = \theta^2 \mathrm{Cov}(\tau_t, \tau_{t+k}) = \theta^2 \left(e^{-\omega_1 k}\frac{2\delta_1^4}{(\nu_1-2)^2(\nu_1-4)} + e^{-\omega_2 k}\frac{2\delta_2^4}{(\nu_2-2)^2(\nu_2-4)}\right),$$
in particular $\mathrm{Cov}_{\theta=0}(X_t, X_{t+k}) = 0$;

2. $\mathrm{Cov}(X_t^2, X_{t+k}^2)$ *can be expressed as (13), where* $\mathrm{Cov}(\tau_t,\tau_{t+k})$, $\mathrm{Cov}(\tau_t^2,\tau_{t+k})$, $\mathrm{Cov}(\tau_t^2,\tau_{t+k}^2)$ *are given by (22), (23), (24) correspondingly; in particular*
$$\mathrm{Cov}_{\theta=0}(X_t^2, X_{t+k}^2) = \sigma^4 \mathrm{Cov}(\tau_t, \tau_{t+k}) = \sigma^4 \left(e^{-\omega_1 k}\frac{2\delta_1^4}{(\nu_1-2)^2(\nu_1-4)} + e^{-\omega_2 k}\frac{2\delta_2^4}{(\nu_2-2)^2(\nu_2-4)}\right).$$

3.3 Asymptotic self-similarity of the activity time process

Let $D[0,1]$ be the Skorokhod space (see [5]). We will show that the activity time process T_t is asymptotically self-similar in both models. We remain that the standard Brownian motion is self-similar with Hurst parameter $H = 1/2$, that is $W_{at} \stackrel{d}{=} \sqrt{a} W_t$, for any $a > 0,\ t \geq 0$.

Proposition 4 *For $M = 1, 2$ and $t \in [0,1]$:*

$$\frac{1}{c_M\sqrt{N}}(T_{[Nt]} - \mathrm{E}[T_{[Nt]}]) \Rightarrow W_t, \ \text{as } N \to \infty \quad t \in [0,1] \tag{25}$$

in the sense of weak convergence "$\Rightarrow$" in the space $D[0,1]$ with Skorokhod topology. The normalizing constant is given by

$$c_1^2 = \frac{2\delta^4}{(\nu-2)^2(\nu-4)}\frac{e^{\omega_1}+1}{e^{\omega_1}-1}, \qquad M = 1, \tag{26}$$

for the first model and by

$$c_2^2 = \sum_{i=1}^{m+1} p_i \frac{2\delta^4}{(\nu_i-2)^2(\nu_i-4)}\frac{e^{\omega_2}+1}{e^{\omega_2}-1}, \qquad M = 2,$$

for the second model.

Proof. If $M = 1$, then $\tau_t = \tau_t^{(1)} = Y_t, t = 0, 1, 2, ...,$ is a strong mixing process. We note that the strong mixing condition with exponential mixing rate for a class of mean-reversing diffusions are proven in [11]. Moreover in [18] these conditions are checked for inverted gamma mean-reversing diffusion processes. Thus, the stochastic process $Y_t, t = 0, 1, 2, ...,$ satisfies the strong mixing condition with exponential mixing rate and according to Theorem 20.1 (see [5]) we obtain (25) with normalizing constant (26).

Really in this case $(\tau_1 = \tau)$:

$$\begin{aligned} \mathrm{Var}[\sum_{i=1}^{N} \tau_i] &= \mathrm{Var}(\tau) \sum_{i=1}^{N} \sum_{j=1}^{N} \mathrm{Corr}(\tau_i, \tau_j) = \\ &= \mathrm{Var}(\tau) N \sum_{r=-(N-1)}^{N-1} (1 - \frac{|r|}{N}) \mathrm{Corr}(\tau_0, \tau_r), \end{aligned}$$

and since

$$\sum_{r=-\infty}^{\infty} |\mathrm{Corr}(\tau_0, \tau_r)| < \infty, \sum_{r=-\infty}^{\infty} \mathrm{Corr}(\tau_0, \tau_r) \neq 0,$$

we have as $N \to \infty$

$$\lim \frac{1}{N} \mathrm{Var}[\sum_{i=1}^{N} \tau_i] = \mathrm{Var}(\tau) \sum_{r=-\infty}^{\infty} \mathrm{Corr}(\tau_0, \tau_r) =$$

$$= \mathrm{Var}(\tau)[1 + 2 \sum_{r=1}^{\infty} \mathrm{Corr}(\tau_1, \tau_{r+1})] =$$

$$= \mathrm{Var}(\tau) + 2 \sum_{i=1}^{\infty} \mathrm{Cov}(\tau_1, \tau_{i+1}) = c_1^2,$$

where

$$c_1^2 = \frac{2\delta^4}{(\nu-2)^2(\nu-4)} + 2 \frac{2\delta^4}{(\nu-2)^2(\nu-4)} \sum_{i=1}^{\infty} e^{-\omega_1 i} = \frac{2\delta^4}{(\nu-2)^2(\nu-4)} \frac{e^{\omega_1} + 1}{e^{\omega_1} - 1}.$$

For the model 1 and model 2, $\tau_t^{(1)}$ and $\tau_t^{(2)}$, $t = 0, 1, 2, ...,$ are strong mixing processes with exponential mixing rates. Sum of two strong mixing processes is also strong mixing processes (see [10]), and weak convergence in $D[0, 1]$ holds by standard arguments. The constant can be computed using the distribution theory for model 2. Namely,

$$c_2^2 = \sum_{i=1}^{m+1} p_i \frac{2\delta^4}{(\nu_i-2)^2(\nu_i-4)} + 2 \sum_{i=1}^{m+1} p_i \frac{2\delta^4}{(\nu_i-2)^2(\nu_i-4)} \sum_{j=1}^{\infty} e^{-\omega_2 j} =$$

$$= \sum_{i=1}^{m+1} p_i \frac{2\delta^4}{(\nu_i-2)^2(\nu_i-4)} \frac{e^{\omega_2} + 1}{e^{\omega_2} - 1}.$$

Therefore, the asymptotic self-similarity with Hurst parameter $H = \frac{1}{2}$ holds for both models. ■

The asymptotic self-similarity established in the previous section can be used to compute prices of the European call options.

4 Option Pricing

Let $C(Y,K)$ be the price of a European call option with expiry Y (time to mature) and strike price K. Let r be the interest rate. Using approach known as skew-correcting martingale (see [15]), one can prove the following statement.

Proposition 5 *Let $C(Y,K)$ be the price of an European call option with strike price K and time to mature Y, let the market evolves with the following dynamics for the risky and non-risky asset price with interest rate r:*

$$P_t = P_0 e^{\mu t + \theta T_t + \sigma W_{T_t}},\ t \geq 0, \qquad B_t = B_0 e^{rt},$$

where T_t is positive, nondecreasing stochastic process.
Then the price $C(Y,K)$ is given by:

$$C(Y,K) = \int_0^\infty (P_0 \Phi(d_1) - K e^{-rY} \Phi(d_2)) f_{T_Y}(t)\ dt, \tag{27}$$

where

$$d_1 = \frac{\log \frac{P_0}{K} + rY + \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}},\ d_2 = \frac{\log \frac{P_0}{K} + rY - \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}} \tag{28}$$

are both functions of t, and $\Phi(\cdot)$ is a standard normal cumulative distribution function. The density f_{T_Y} for model 1 can be taken approximately as the density of the random variable:

$$\frac{1}{\sqrt{Y}} f_{R\Gamma}\left(\frac{u - \mathrm{E}[\tau_1^{(1)}](\sqrt{Y} - Y)}{\sqrt{Y}}, \frac{\nu}{2}, \frac{\delta^2}{2}\right), \tag{29}$$

where $\mathrm{E}[\tau_1^{(1)}]$ is given by (7). The density f_{T_Y} for model 2 can be taken approximately as the density of the random variable:

$$\frac{1}{\sqrt{Y}} \sum_{i=1}^{m+1} p_i f_{R\Gamma}\left(\frac{u - \mathrm{E}[\tau_1^{(2)}](\sqrt{Y} - Y)}{\sqrt{Y}}, \frac{\nu_i}{2}, \frac{\delta^2}{2}\right), \tag{30}$$

where $\mathrm{E}[\tau_1^{(2)}]$ is given by (19).

Proof. The proof is based on the ideas of the paper [15]. We need to show that $\{e^{-rt}P_t\}$ is a martingale, where P_t is the price of a risky asset and e^{-rt} is the discounting factor. Consider the σ-algebra $\mathcal{F}_s$ of the information available up until time s: $\mathcal{F}_s = \sigma\{\{B(u), u \leq T_s\}, \{T_u, u \leq s\}\}$. Then we have

$$E(e^{-rt}P_t|\mathcal{F}_s) = e^{-rs}P_s e^{(\mu - r)(t-s)} E(e^{(\theta + \frac{1}{2}\sigma^2)(T_t - T_s)}|\mathcal{F}_s),$$

as shown in Finlay and Seneta (2006). Start with a real-world model with no θ parameter ($\theta = 0$), and introduce θ in the form $\theta = -\frac{1}{2}\sigma^2$ to price options. Then $e^{-rt}P_t$ is a martingale as desired. This is a "skew-correcting martingale" because θ determines skewness. Under this risk-neutral model, the European call option price is determined by (27). Given P_0, K, Y, r, σ, and f_{T_Y}, expectation (27) can be numerically evaluated. Note that we have made no assumptions about the distribution of T_Y, so as long as our model has the subordinator structure and all expectations are finite, then this pricing formula is valid. The approximation of density f_{T_Y} is based on the asymptotic self-similarity of T_Y. As showed in [15] using self-similarity (25), the density f_{T_Y} can be taken as the density of $Y\mathrm{E}[\tau_1] + \sqrt{Y}(T_1 - \mathrm{E}[\tau_1])$ approximately. Thus, for model 1 the distribution of $T_1 \overset{d}{=} \tau_1^{(1)}$ is $R\Gamma(x, \frac{\nu}{2}, \frac{\delta^2}{2})$ and therefore we get (29). In the same spirit one can prove that for the model 2 the density f_{T_Y} is approximated by (30). ∎

Remark 2 *If $T_Y = t$, then (27) reduces to the classical BSM formula.*

Remark 3 *The put option price comes from call-put parity relation:*

$$put\ price \ = \ call\ price - stock\ price + present\ value\ of\ exercise\ price.$$

5 Conclusion

The paper developed a FATGBM model with the following features of stock returns, which are quite well documented in the financial and econometric literature: i) the stochastic processes have continuous paths; ii) the returns processes is uncorrelated; iii) dependence is presented in squared returns; iv) the returns have Student or Student like empirical distribution; v) the fractal activity time has a self-similar limit.

The proposed models with properties i)-v) are different from the models in existing literature ([1], [6],[8], [9], [10], [14], [15], [16], [19], [20], [21]). For instance, the models based on the Ornstein-Uhlenbeck processes and their superpositions ([19],[21]) do not satisfy the property i), while the model proposed in [6] uses the Student distribution with time-dependent parameters. The model in [8], [15] uses the Student distribution with only two parameters. Moreover, the number of degrees of freedom is an integer.

How the models are actually fitted the real financial data is future work beyond the scope of the present paper.

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