A Novel Insight to the SBR2 Algorithm for Diagonalising Para-Hermitian Matrices

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***Abstract*—The second order sequential best rotation (SBR2) algorithm was originally developed for achieving the strong decorrelation of convolutively mixed sensor array signals. It was observed that the algorithm always seems to produce spectrally majorized output signals, but this property has not previously been proven. In this work, we have taken a fresh look at the SBR2 algorithm in terms of its potential for optimizing the subband coding gain. It is demonstrated how every iteration of the SBR2 algorithm must lead to an increase in the subband coding gain until it comes arbitrarily close to its maximum possible value. Since the algorithm achieves both strong decorrelation and optimal subband coding, it follows that it must also produce spectral majorisation. A new quantity γ associated with the coding gain optimization is introduced, and its monotonic behaviour brings a new insight to the convergence of the SBR2 algorithm.**

***Index Terms*—SBR2, spectral majorization, subband coding, coding gain optimization.**

I. Introduction

In broadband multiple-input multiple-output (MIMO) systems or sensor array processing, given a zero mean data vector $\mathbf{x}[t] \in \mathbb{C}^{p\times 1}$ measured from p sensors, the space-time covariance matrix

$$\mathbf{R}[\tau] = E\left\{\mathbf{x}[t]\mathbf{x}^{\mathrm{H}}[t-\tau]\right\},\ t,\tau \in \mathbb{Z} \quad (1)$$

represents the correlation between pairs of signals sampled over a time delay τ, where $E\{\cdot\}$ denotes the expectation operator, and the superscript $\{\cdot\}^{\mathrm{H}}$ represents Hermitian transpose. The corresponding cross spectral density (CSD) matrix $\underline{\mathbf{R}}(z)$ is a polynomial matrix [1] and can be obtained by taking the z-transform of (1), *i.e.*

$$\underline{\mathbf{R}}(z) = \sum_{\tau=-\infty}^{\infty} \mathbf{R}[\tau]z^{-\tau}. \quad (2)$$

$\underline{\mathbf{R}}(z)$ is para-Hermitian, satisfying $\underline{\mathbf{R}}(z) = \tilde{\underline{\mathbf{R}}}(z)$, where the paraconjugate operator $\tilde{\{\cdot\}}$ denotes Hermitian transpose and time-reversal of the polynomials, *i.e.* $\tilde{\underline{\mathbf{R}}}(z) = \underline{\mathbf{R}}^{\mathrm{H}}(z^{-1})$. To strongly decorrelate the convolutively mixed signals, in other words, to eliminate the cross-correlation terms between the different entries of $\mathbf{x}[t]$ over all time delays, the polynomial eigenvalue decomposition (PEVD) has been proposed [2]. This takes the form

$$\underline{\mathbf{H}}(z)\underline{\mathbf{R}}(z)\tilde{\underline{\mathbf{H}}}(z) \approx \underline{\mathbf{D}}(z) \quad (3)$$

where $\underline{\mathbf{D}}(z)$ is (ideally) a diagonal matrix obtained by diagonalizing $\underline{\mathbf{R}}(z)$ using the similarity transformation $\underline{\mathbf{H}}(z)$ for which $\underline{\mathbf{H}}(z)$ is a paraunitary matrix, *i.e.* $\underline{\mathbf{H}}(z)\tilde{\underline{\mathbf{H}}}(z) = \mathbf{I}$. The approximation sign in (3) indicates that it is not possible in general to compute the PEVD exactly since the paraunitary matrix $\underline{\mathbf{H}}(z)$ is restricted to polynomial (as opposed to rational) form. However it has been shown that a very close approximation can be achieved by letting the polynomial order of $\underline{\mathbf{H}}(z)$ grow arbitrarily large [3]. The PEVD can be seen as an extension of the conventional eigenvalue decomposition (EVD) for para-Hermitian matrices, and several algorithms have been developed for computing it. These include the original SBR2 algorithm [2], its improved version multiple shift version (MS-SBR2) [4] and the familly of sequential matrix diagonalization (SMD) algorithms [5], [6].

Most of the work reported since then has focused on improving the performance or reducing the computational cost of the PEVD algorithms. In this paper, we take a fresh look at the SBR2 algorithm in terms of its effect on the subband coding gain. This leads to the much desired proof that the SBR2 algorithm does indeed converge towards a spectrally majorised solution. It also suggests a modified form of SBR2 algorithm, explicitly designed to maximize the coding gain, and gives a new perspective on its convergence.

This paper is organised as follows. A brief review of the SBR2 algorithm is given in Sec. II. In Sec. III, the principle of coding gain optimization is discussed in the context of SBR2, followed by the spectral majorization proof and an alternative test for convergence of the SBR2 algorithm which arises from the proof. Finally, simulation results are presented in Sec. IV and conclusions are drawn in Sec. V.

II. Review of the SBR2 Algorithm

The SBR2 algorithm [2] was designed to eliminate the cross-correlation elements for the space-time covariance matrix $\mathbf{R}[\tau]$ over a suitable range of delay τ. It comprises a number of iterative stages which aim to transfer all the off-diagonal elements in $\underline{\mathbf{R}}(z)$ onto the diagonal. For each iteration, the algorithm starts by finding the dominant off-diagonal coefficient $r_{jk}[\tau]$ of $\underline{\mathbf{R}}(z)$. Note that the search is restricted to the upper triangular region due to the para-Hermitian property. Thus the location of $r_{jk}[\tau]$, $(k > j)$ satisfies

$$\{j,k,\tau\} = \arg\max_{j,k,\tau} \|\mathbf{R}[\tau]\|_{\infty}, \quad (4)$$

where j, k and τ are the corresponding row, column and time indices. An elementary delay matrix $\underline{\mathbf{B}}^{(k,\tau)}(z)$ is applied first to shift the entry $r_{jk}[\tau]$ and its conjugate $r_{kj}[-\tau]$ onto the zero-lag ($\tau = 0$) coefficient matrix $\mathbf{R}[0]$ by means of the transformation

$$\underline{\mathbf{R}}'(z) = \underline{\mathbf{B}}^{(k,\tau)}(z)\underline{\mathbf{R}}(z)\tilde{\underline{\mathbf{B}}}^{(k,\tau)}(z), \tag{5}$$

where $\underline{\mathbf{B}}^{(k,\tau)}(z)$ take the form of

$$\underline{\mathbf{B}}^{(k,\tau)}(z) = \text{diag}\{\underbrace{1,\cdots,1}_{k-1}, z^{-\tau}, \underbrace{1,\cdots,1}_{p-k}\}. \tag{6}$$

An elementary rotation matrix $\mathbf{Q}^{(j,k)}(\theta,\phi)$ is then used to transfer the energy of these off-diagonal elements onto the diagonal by means of the update formula

$$\underline{\mathbf{R}}''(z) = \mathbf{Q}^{(j,k)}(\theta,\phi)\underline{\mathbf{R}}'(z)\mathbf{Q}^{(j,k)\text{H}}(\theta,\phi). \tag{7}$$

$\mathbf{Q}^{(j,k)}(\theta,\phi)$ represents a complex Jacobi rotation which takes the form of a $p \times p$ identity matrix except for the 2×2 submatrix $\hat{\mathbf{Q}}(\theta,\phi)$ defined by the intersection of rows j and k with columns j and k. This is given by

$$\hat{\mathbf{Q}}(\theta,\phi) = \begin{bmatrix} \cos\theta & \sin\theta\, e^{i\phi} \\ -\sin\theta\, e^{-i\phi} & \cos\theta \end{bmatrix}. \tag{8}$$

Here the parameters θ and ϕ are chosen to drive the dominant coefficient to zero. It follows from equations (6)and (7) that

$$\underline{\mathbf{R}}''(z) = \underline{\mathbf{G}}(z)\underline{\mathbf{R}}(z)\tilde{\underline{\mathbf{G}}}(z) \tag{9}$$

where the matrix

$$\underline{\mathbf{G}}(z) = \mathbf{Q}^{(j,k)}(\theta,\phi)\underline{\mathbf{B}}^{(k,\tau)}(z). \tag{10}$$

is termed an elementary paraunitary matrix and equation (9) constitutes an elementary similarity transformation. The algorithm continues by making the substitution $\underline{\mathbf{R}}(z) \leftarrow \underline{\mathbf{R}}''(z)$ and repeating the process mentioned above until all the off-diagonal elements are smaller than a given threshold ϵ which can be set to a very small value to achieve sufficient accuracy. Assuming that the algorithm has converged by the N^{th} iteration, the diagonalized para-Hermitian matrix in equation (3) takes the form

$$\underline{\mathbf{D}}(z) = \text{diag}\{\underline{d}_{11}(z), \underline{d}_{22}(z), \cdots, \underline{d}_{pp}(z)\}, \tag{11}$$

and the paraunitary matrix generated in the process is given by

$$\underline{\mathbf{H}}(z) = \underline{\mathbf{G}}_N(z)\cdots\underline{\mathbf{G}}_2(z)\underline{\mathbf{G}}_1(z). \tag{12}$$

For further details of the SBR2 algorithm, see [2].

III. Filter Bank Based Subband Coding

A. Optimal Coding Gain

The subband coder is a generalization of the transform coder and has been used for several applications including data compression [7]. A subband coder aims to maximize the coding gain, *i.e.* to minimize the mean square reconstruction error due to subband quantization. Kirac and Vaidyanathan [7], [8] derived the necessary and sufficient conditions for maximizing the coding gain:

(1) *strong (or total) decorrelation* – this means that the CSD matrix has been diagonalized as in (11), or equivalently the subband signals $\mathbf{v}[t] = \sum_{\tau=0}^{T}\mathbf{H}[\tau]\mathbf{x}[t-\tau]$ are totally uncorrelated, *i.e.* $E\{v_k[t]v_l^*[t-\tau]\} = 0, k \neq l, \forall\, t,\tau$, Here $\{\cdot\}^*$ denotes the complex conjugate operator;

(2) *spectral majorization* – the power spectral densities (PSDs) $\underline{d}_{ll}(e^{j\Omega}) = \underline{d}_{ll}(z)|_{z=e^{j\Omega}}, l = 1,2,\cdots,p$ satisfy $\underline{d}_{11}(e^{j\Omega}) \geq \underline{d}_{22}(e^{j\Omega}) \geq \cdots \geq \underline{d}_{pp}(e^{j\Omega})$, $\forall\,\Omega$. In other words, the PSD matrix $\underline{\mathbf{R}}_{xx}(e^{j\Omega})$ of $\mathbf{x}[t]$ is diagonalized at every angular frequency Ω such that the eigenvalues of $\underline{\mathbf{R}}_{xx}(e^{j\Omega})$ are arranged in descending order.

Denoting the CSD matrix of $\mathbf{x}[t]$ by $\underline{\mathbf{R}}_{xx}(z)$, the coding gain, whose maximization requires diagonalization and spectral majorization, is measured as the ratio of the arithmetic and geometric means of the channel variances. For the i^{th} iteration and the l^{th} channel, this variance is given by $r_{ll}^{(i)}[0]$, so the coding gain is defined as [5]

$$G^{(i)} = \frac{\frac{1}{p}\sum_{l=1}^{p} r_{ll}^{(i)}[0]}{\left(\prod_{l=1}^{p} r_{ll}^{(i)}[0]\right)^{\frac{1}{p}}}. \tag{13}$$

Note that the trace

$$\text{tr}\Big\{\mathbf{R}^{(i)}[0]\Big\} = \sum_{l=1}^{p} r_{ll}^{(i)}[0] = \text{tr}\{\mathbf{R}[0]\} = \text{tr}\{\mathbf{D}[0]\} \tag{14}$$

is invariant under paraunitary transformations and so maximizing the coding gain is equivalent to minimizing the product of variances in the denominator of equation (13).

B. Spectral Majorization

The SBR2 algorithm has been adopted successfully in the design of the paraunitary (orthonormal) filter banks for subband coding [9], [10]. In effect it has demonstrated the capability of a principle component filter bank (PCFB) by achieving the optimal coding gain. However, there is no proof in the existing literature that the SBR2 algorithm will always produce the necessary spectral majorization. In the rest of this section a proof of this important property will be derived.

Theorem (Spectral Majorization of the SBR2 Algorithm): If strong decorrelation is achieved using the SBR2 algorithm, the resulting PSDs must also be spectrally majorised.

Proof: As expressed in (9), the polynomial matrices $\underline{\mathbf{R}}(z)$ and $\underline{\mathbf{R}}''(z)$ are related by a generalized similarity transformation. Let us now introduce the parameter

$$\gamma \triangleq \prod_{l=1}^{p} r_{ll}[0] \tag{15}$$

where $r_{ll}[0], l = 1,2,\cdots,p$ represent the diagonal element of $\mathbf{R}[0]$. Following the elementary delay step, the SBR2 algorithm employs a Jacobi rotation as shown in (8) to transfer the energy of the off-diagonal element $r'_{jk}[0] = r_{jk}[\tau]$ and its conjugate $r'_{kj}[0] = r_{kj}[-\tau]$ onto the diagonal of $\mathbf{R}[0]$ by choosing the rotation parameters such that

$$\begin{bmatrix} c & se^{i\phi} \\ -se^{-i\phi} & c \end{bmatrix}\begin{bmatrix} r'_{jj}[0] & r'_{jk}[0] \\ r'_{kj}[0] & r'_{kk}[0] \end{bmatrix}\begin{bmatrix} c & -se^{i\phi} \\ se^{-i\phi} & c \end{bmatrix}$$

$$= \begin{bmatrix} r''_{jj}[0] & 0 \\ 0 & r''_{kk}[0] \end{bmatrix}, \tag{16}$$

where c and s denote $\cos\theta$ and $\sin\theta$ respectively. Since the transformations are unitary it follows that

$$\det\left\{\begin{bmatrix} r'_{jj}[0] & r'_{jk}[0] \\ r'_{kj}[0] & r'_{kk}[0] \end{bmatrix}\right\} = \det\left\{\begin{bmatrix} r''_{jj}[0] & 0 \\ 0 & r''_{kk}[0] \end{bmatrix}\right\}, \tag{17}$$

i.e. $$r''_{jj}[0]r''_{kk}[0] = r'_{jj}[0]r'_{kk}[0] - |r'_{jk}[0]|^2$$

$$= r_{jj}[0]r_{kk}[0] - |r_{jk}[\tau]|^2 \tag{18}$$

where we have taken account of the fact that $r'_{jj}[0] = r_{jj}[0]$, $r'_{kk}[0] = r_{kk}[0]$ and $r'_{jk}[0] = r_{jk}[\tau]$. Since $r_{jk}[\tau] \neq 0$, it follows that

$$r''_{jj}[0]r''_{kk}[0] < r_{jj}[0]r_{kk}[0]. \tag{19}$$

and, since only the j^{th} and k^{th} diagonal elements are altered during the iteration, we have

$$\gamma'' \triangleq \prod_{l=1}^{p} r''_{ll}[0] < \gamma \quad . \tag{20}$$

Clearly the denominator in (13) which is directly related to $\gamma^{(i)}$, is monotonically reduced at each iteration in SBR2, *i.e.* $\gamma^{(i)} < \gamma^{(i-1)}$, until no further reduction is possible ($|r_{jk}[\tau]| < \epsilon$). It follows that the coding gain $G^{(i)}$ increases monotonically to attain its maximum value $G^{(N)}$.

It was clearly demonstrated by Vaidyanathan [7] that the optimum coding gain (which requires a PCFB) can be achieved if and only if strong decorrelation and spectral majorization have been obtained. Thus it follows that the SBR2 algorithm, which was explicitly designed to achieve strong decorrelation, must not only achieve that objective, but also produce spectral majorization. ■

C. Modified SBR2 Algorithm

Instead of looking for the dominant off-diagonal element $|r_{jk}[\tau]|$, it is now possible to consider the coding gain $G^{(i)}$ as a convergence indicator for the SBR2 algorithm. This gives us a useful new insight whereby the SBR2 algorithm converges uniformly due to the monotonic behaviour of $\gamma^{(i)}$ by contrast with the original convergence factor $|r_{jk}[\tau]|$ whose value does not reduced monotonically. An alternative approach therefore, is to monitor the gradient of the coding gain $\rho^{(i)} = G^{(i)} - G^{(i-1)}$. As the value of $\rho^{(i)}$ is not guaranteed to reduce monotonically, the average value $\hat{\rho}$ of the gradients over a suitable range $W \in \mathbb{Z}$ is calculated, *i.e.* $\hat{\rho} = \frac{1}{W}\sum_{k=i-W+1}^{i} \rho^{(k)}$. Then the iterative process stops when the value of $\hat{\rho}$ is sufficiently small. The modified SBR2 algorithm is summarized in Tab. I.

TABLE I
THE MODIFIED SBR2 ALGORITHM

Input $p \times p$ para-Hermitian matrix $\underline{\mathbf{R}}(z)$.
Specify maximum number of iterations, $maxiter$, convergence parameter, ϵ and trim factor μ.
Initialization: $iter \leftarrow 0$, $\hat{\rho} \leftarrow 1 + \epsilon$ and $\underline{\mathbf{H}}(z) \leftarrow \mathbf{I}_p$.
while $iter < maxiter$ && $\hat{\rho} > \epsilon$
  locate the dominant off-diagonal element $r_{jk}[\tau]$.
  set $g = |r_{jk}[\tau]|$.
  if $iter = 0$ && $g = 0$
    break;
  else
    set $\underline{\mathbf{R}}'(z) = \underline{\mathbf{B}}^{(k,\tau)}(z)\underline{\mathbf{R}}(z)\tilde{\underline{\mathbf{B}}}^{(k,\tau)}(z)$;
    set $\underline{\mathbf{H}}'(z) = \underline{\mathbf{B}}^{(k,\tau)}(z)\underline{\mathbf{H}}(z)$;
    compute rotation parameters (θ, ϕ);
    update $\underline{\mathbf{R}}(z) = \mathbf{Q}^{(j,k)}(\theta,\phi)\underline{\mathbf{R}}'(z)\mathbf{Q}^{(j,k)\mathrm{H}}(\theta,\phi)$;
    update $\underline{\mathbf{H}}(z) = \mathbf{Q}^{(j,k)}(\theta,\phi)\underline{\mathbf{H}}'(z)$;
    $iter \leftarrow iter + 1$;
    trim $\underline{\mathbf{R}}(z)$ and $\underline{\mathbf{H}}(z)$ based on trim factor μ;
    compute $G^{(iter)}$ and $\rho^{(iter)}$ according to (13);
    assign value to W;
    if $iter \geq W$
      set $\hat{\rho} = \frac{1}{W}\sum_{k=iter-W+1}^{iter} \rho^{(k)}$;
    end
  end
end

IV. SIMULATIONS AND RESULTS

In order to investigate the SBR2 algorithm in terms of the coding gain optimization, we have chosen one of the examples which was used to test the algorithm in the original SBR2 paper [2] , *i.e.* a convolutively mixed signal $\mathbf{x}[t]$ was generated from a 2×3 MIMO channel model with the mixing process represented by a 3×2 polynomial matrix $\underline{\mathbf{A}}(z)$ whose entries comprised order-5 FIR filters, and Gaussian random noise was added to each sensor output with a signal-to-noise ratio (SNR) of 5.3 dB.

The CSD matrix $\underline{\mathbf{R}}(z)$ of the received signals $\mathbf{x}[t]$ is plotted in Fig. 1. After applying the modified SBR2 algorithm to diagonalize this matrix with $\epsilon = 10^{-5}$ representing the threshold of the average gradient $\hat{\rho}$, $W = 10$, and the trim factor $\mu = 10^{-4}$, the algorithm converged in 110 iterations to a point where the average gradient $\hat{\rho} = 0.96 \times 10^{-5}$. Fig. 2 shows that the product of the subband variances $\gamma^{(i)}$ is monotonically reduced as the iteration goes. On the contrary, the coding gain $G^{(i)}$ is monotonically increasing as shown in Fig. 3. As opposed to the original SBR2 algorithm, Fig. 4 shows the behaviour of the convergence factor $g = |r_{jk}[\tau]|$ for which it converged at $g = 0.0224$. Finally the diagonalized CSD matrix is plotted in Fig. 5.

V. CONCLUSION

In this paper, we have investigated the SBR2 algorithm in terms of optimizing the subband coding gain, leading to a first proof that it must also achieve spectral majorization. In addition, the monotonically increasing behaviour of the coding gain has been exploited to obtain a more reliable test of convergence for the algorithm.

REFERENCES

[1] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall, 1993.

[2] J.G. McWhirter, P.D. Baxter, T. Cooper, S. Redif, J. Foster, "An EVD Algorithm for Para-Hermitian Polynomial Matrices," *IEEE Trans. SP*, 55(5):2158–2169, May 2007.

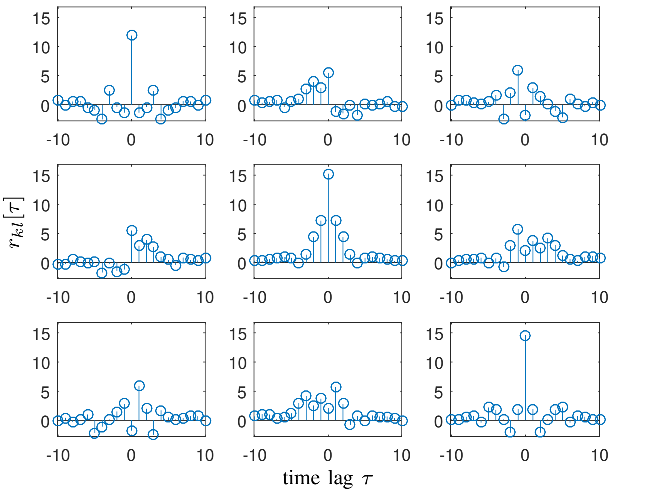

Fig. 1. Space-time covariance matrix $\mathbf{R}[\tau]$ for three mixed signals generated from two i.i.d. source signals at 5.3 dB SNR.

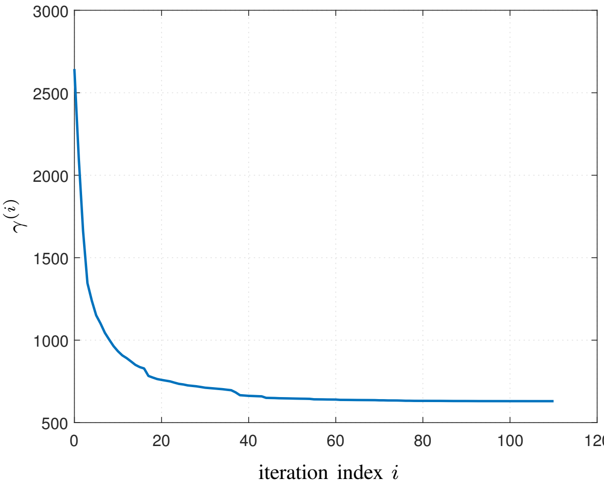

Fig. 2. Convergence of the SBR2 algorithm for diagonalizing the space-time covariance matrix in Fig. 1, showing the behaviour of $\gamma^{(i)}$.

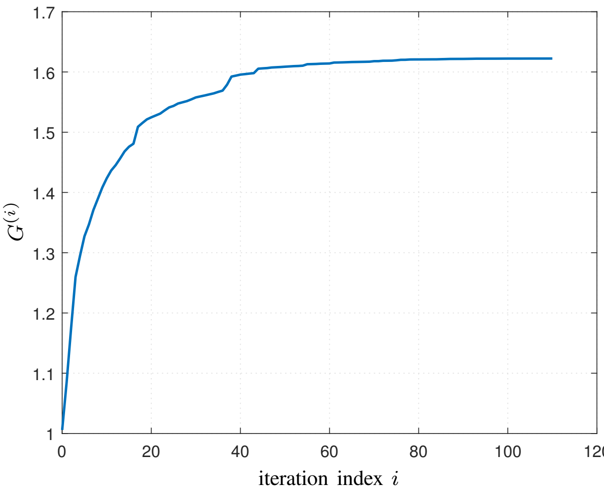

Fig. 3. Convergence of the SBR2 algorithm for diagonalizing the space-time covariance matrix in Fig. 1, showing the behaviour of the coding gain $G^{(i)}$.

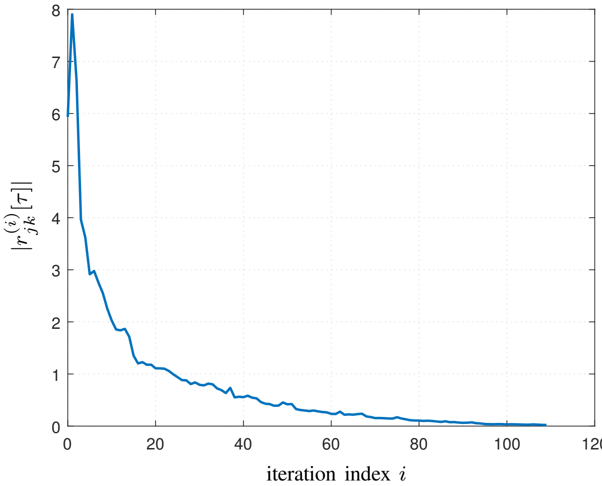

Fig. 4. Convergence of the SBR2 algorithm for diagonalizing the space-time covariance matrix in Fig. 1, showing the behaviour of the off-diagonal element $|r_{jk}^{(i)}[\tau]|$.

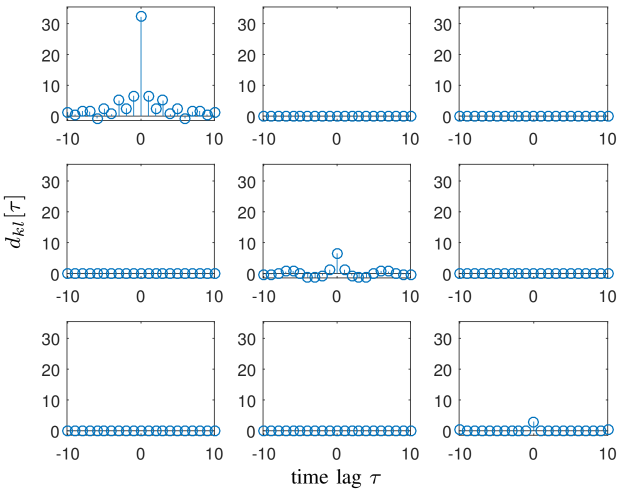

Fig. 5. Strongly decorrelated space-time covariance matrix $\mathbf{D}[\tau]$ produced by the SBR2 algorithm using paraunitary transformation $\underline{\mathbf{H}}(z)$.

[3] S. Icart, P. Comon, "Some properties of Laurent polynomial matrices," in *IMA Conference on Mathematics in Signal Processing*, Birmingham, UK, Dec. 2012.

[4] Z. Wang, J.G. McWhirter, J. Corr, S. Weiss, "Multiple Shift Second Order Sequential Best Rotation Algorithm for Polynomial Matrix EVD," in *23rd EUSIPCO*, pp. 844–848, Nice, France, Aug. 2015.

[5] S. Redif, S. Weiss, J.G. McWhirter, "Sequential Matrix Diagonalization Algorithms for Polynomial EVD of Parahermitian Matrices," *IEEE Trans. SP*, 63(1):81–89, Jan. 2015.

[6] J. Corr, K. Thompson, S. Weiss, J.G. McWhirter, S. Redif, I.K. Proudler, "Multiple Shift Maximum Element Sequential Matrix Diagonalisation for Parahermitian Matrices," in *IEEE SSP Workshop*, pp. 312–315, Gold Coast, Australia, Jun. 2014.

[7] P.P. Vaidyanathan, "Theory of optimal orthonormal subband coders," *IEEE Trans. SP*, 46(6):1528-1543, Jun. 1998.

[8] A. Kirac, P.P. Vaidyanathan, "On existence of FIR principal component filter banks," in *IEEE ICASSP*, pp. 1329-1332, May 1998.

[9] S. Weiss, S. Redif, T. Copper, C. Liu, P.D. Baxter, J.G. McWhirter, "Paraunitary Oversampled Filter Bank Design for Channel Coding," *EURASIP J. Adv. Signal Process.*, vol. 2006:1–10, Mar. 2006.

[10] S. Redif, J.G. McWhirter, and S. Weiss, "Design of FIR Paraunitary Filter Banks for Subband Coding Using a Polynomial Eigenvalue Decomposition," *IEEE Trans. SP*, 59(11):5253-5264, Nov. 2011.