CONNECTIVE C^*-ALGEBRAS

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ABSTRACT. Connectivity is a homotopy invariant property of separable C^*-algebras which has three notable consequences: absence of nontrivial projections, quasidiagonality and a more geometric realization of KK-theory for nuclear C^*-algebras using asymptotic morphisms. The purpose of this paper is to further explore the class of connective C^*-algebras. We give new characterizations of connectivity for exact and for nuclear separable C^*-algebras and show that an extension of connective separable nuclear C^*-algebras is connective. We establish connectivity or lack of connectivity for C^*-algebras associated to certain classes of groups: virtually abelian groups, linear connected nilpotent Lie groups and linear connected semisimple Lie groups.

1. INTRODUCTION

Connectivity of separable C^*-algebras was introduced in our earlier paper [10] under different terminology, see Definition 2.1 below. The initial motivation for studying it stemmed from our search for homotopy-symmetric C^*-algebras. By a result of Loring and the first author [9], these are precisely the separable C^*-algebras for which one can unsuspend in the E-theory of Connes and Higson [6]. Using a result of Thomsen [34], we proved in [10] that connectivity is equivalent to homotopy-symmetry for all separable nuclear C^*-algebras. Moreover, we showed that connectivity has a number of important permanence properties. These facts allowed us to exhibit new classes of homotopy-symmetric C^*-algebras.

The purpose of this paper is to further explore the class of connective C^*-algebras. We are motivated by the following three properties they share:

(i) If A is a separable nuclear C^*-algebra, then $KK(A, B)$ is isomorphic to the homotopy classes of completely positive and contractive (cpc) asymptotic morphisms from A to $B \otimes \mathcal{K}$ for any separable C^*-algebra B.

(ii) Connective C^*-algebras are quasidiagonal. In fact, if A is connective, then $A \otimes B$ is quasidiagonal for any C^*-algebra B. ($A \otimes B$ denotes the minimal tensor product.)

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(iii) Connective C^*-algebras do not have nonzero projections. In fact, if A is connective, then $A \otimes B$ does not have nonzero projections for any C^*-algebra B.

Connectivity is of particular interest in the case of group C^*-algebras. A countable discrete group G is called connective if the augmentation ideal $I(G)$ defined as the kernel of the trivial representation $\iota\colon C^*(G) \to \mathbb{C}$ is a connective C^*-algebra. In view of properties (ii) and (iii) connectivity of G may be viewed as a stringent topological property that accounts simultaneously for the quasidiagonality of $C^*(G)$ and the verification of the Kadison-Kaplansky conjecture for certain classes of groups. Examples of nonabelian connective groups were exhibited in [10] and [11].

In this paper we give new characterizations of connectivity for exact and nuclear separable C^*-algebras, see Prop. 2.2, 2.3. We prove that connectivity of separable nuclear C^*-algebras is preserved under extensions, see Thm. 2.4. This is a key permanence property which does not hold for quasidiagonal C^*-algebras.

There is a close connection between the topology of the spectrum and connectivity, which we employ to reveal an obstruction to connectivity by using work of Blackadar and Cuntz [1] and Pasnicu and Rørdam [30]. In particular, we show that a countable discrete group G is not connective if the trivial representation ι is a shielded point of the unitary dual of G in the sense of Def. 2.9.

Motivated by this, we give a complete description of the neighborhood of ι in the spectrum of the Hantzsche-Wendt group G, which is a torsion free crystallographic group with holonomy $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, in Sec. 3.1. This allows us to prove that G is not connective in this case (Cor. 3.2). Moreover, we show that this group provides a counterexample to a conjecture from [8]. Specifically, we prove that the natural map $[[I(G), \mathcal{K}]] \to K^0(I(G))$ is not an isomorphism (Lem. 3.3). In contrast, we show that all torsion free crystallographic groups with *cyclic holonomy* are connective (Thm. 3.8).

Next, we investigate connectivity for C^*-algebras associated to Lie groups. We show that all noncompact linear connected nilpotent Lie groups have connective C^*-algebras (Thm. 4.3). Using classic results from representation theory in conjunction with permanence properties of connectivity, we show that if G is a linear connected complex semisimple Lie group, then $C^*_r(G)$ is connective if and only if G is not compact (Thm. 4.5). Moreover, if G is a linear connected real reductive Lie group, then $C^*_r(G)$ is connective if and only if G does not have a compact Cartan subgroup (Thm. 4.6).

A common denominator of our results concerning group C^*-algebras is that in all the cases we analyzed, $C^*(G)$ contains a large connective ideal.

2. Connective C^*-algebras

2.1. **Definitions and background.** For a C^*-algebra A, the *cone over* A is defined as $CA = C_0[0,1)\otimes A$ the *suspension of* A as $SA = C_0(0,1)\otimes A$.

The first of the following two notions was introduced in [10, Def. 2.6 (i)], under a different terminology which we have abandoned. We use the abbreviation *cpc map* for a completely positive and contractive map.

Definition 2.1. Let A be a C^*-algebra.
(a) A is *connective* if there is a $*$-monomorphism

$$\Phi\colon A \to \prod_n CL(\mathcal{H})/\bigoplus_n CL(\mathcal{H})$$

which is liftable to a cpc map $\varphi\colon A \to \prod_n CL(\mathcal{H})$.
(b) A is *almost connective*, if there is a (not necessarily liftable) $*$-monomorphism $\Phi\colon A \to \prod_n CL(\mathcal{H})/\bigoplus_n CL(\mathcal{H})$.

For a discrete group G, we define $I(G)$ to be the augmentation ideal, i.e. the kernel of the trivial representation $C^*(G)\to\mathbb{C}$. We will sometimes say that a discrete amenable group G is connective if the C^*-algebra $I(G)$ is connective. Note that (almost) connective C^*-algebras do not have nonzero projections. Thus any connective C^*-algebra is nonunital. Our definition allows that the zero C^*-algebra $\{0\}$ is connective.

Let A and B be C^*-algebras. An *asymptotic morphism* is a family of maps $\{\varphi_t\colon A\to B\}_{t\in[0,\infty)}$ such that
a) for each $a\in A$ the map $t\mapsto \varphi_t(a)$ is norm-continuous and bounded,
b) for all $a,b\in A$ and $\lambda\in\mathbb{C}$, we have

$$\begin{aligned}\lim_{t\to\infty}\|\varphi_t(a+\lambda\, b)-(\varphi_t(a)+\lambda\,\varphi_t(b))\| &= 0\\ \lim_{t\to\infty}\|\varphi_t(ab)-\varphi_t(a)\,\varphi_t(b)\| &= 0\\ \lim_{t\to\infty}\|\varphi_t(a^*)-\varphi_t(a)^*\| &= 0\ .\end{aligned}$$

A *discrete asymptotic morphism* $(\varphi_n)_{n\in\mathbb{N}}$ between A and B is a family of maps $\varphi_n\colon A\to B$ that satisfies the analogous conditions as a) and b) above with the index set replaced by $\mathbb{N}$. A homotopy between two (discrete) asymptotic morphisms $(\varphi_t^0)_{t\in I}$ and $(\varphi_t^1)_{t\in I}$ is a (discrete) asymptotic morphism $H_t\colon A\to C[0,1]\otimes B$, such that $\mathrm{ev}_i\circ H_t=\varphi_t^i$ for all $t\in I$, where I either denotes $[0,\infty)$ or $\mathbb{N}$. We will say that a (discrete) asymptotic morphism $(\varphi_t)_{t\in I}$ is completely positive and contractive (cpc) if each of the maps φ_t is cpc. The corresponding homotopy classes will be denoted as follows:

- $[[A,B]]$ – homotopy classes of asymptotic morphisms,
- $[[A,B]]_{\mathbb{N}}$ – homotopy classes of discrete asymptotic morphisms,
- $[[A,B]]_{\mathbb{N}}^{cp}$ – homotopy classes of discrete cpc asymp. morphisms

2.2. **Characterizations of connectivity.** In the following we give two more characterizations of connectivity for exact and respectively nuclear C^*-algebras.

Proposition 2.2. *Let A be a separable exact C^*-algebra. Then A is connective if and only if there is an injective $*$-homomorphism $\pi\colon A \to \mathcal{O}_2$ which is null-homotopic as a discrete cpc asymptotic morphism. This means that $[[\pi]] = 0$ in the set $[[A, \mathcal{O}_2]]^{cp}_{\mathbb{N}}$.*

Proof. ($\Rightarrow$) By assumption, there is a cpc discrete asymptotic morphism $\{\varphi_n : A \to C[0,1] \otimes \mathcal{O}_2\}_n$ such that $\varphi_n^{(0)} = \pi$ is an injective $*$-homomorphism and $\varphi_n^{(1)} = 0$. Thus, we can view $\{\varphi_n\}_n$ as an injective discrete asymptotic morphism $\{\varphi_n : A \to C_0[0,1) \otimes \mathcal{O}_2 \subset CL(\mathcal{H})\}_n$ and hence A is connective.

($\Leftarrow$) Suppose that A is a separable exact connective C^*-algebra. By [10, Prop. 2.11] it follows that $[[A, \mathcal{O}_2 \otimes \mathcal{K}]]^{cp}_{\mathbb{N}}$ is an abelian group. By Kirchberg's embedding theorem, there is an injective $*$-homomorphism $\pi\colon A \to \mathcal{O}_2 \otimes \mathcal{K}$. Moreover $\pi \oplus \pi\colon A \to \mathcal{O}_2 \otimes \mathcal{K}$ is unitarily homotopy equivalent to π. It follows that $[[\pi]] \oplus [[\pi]] = [[\pi]]$ in the group $[[A, \mathcal{O}_2 \otimes \mathcal{K}]]^{cp}_{\mathbb{N}}$ and hence $[[\pi]] = 0$. After embedding $\mathcal{O}_2 \otimes \mathcal{K}$ into $\mathcal{O}_2$ we obtain the desired conclusion. □

Proposition 2.3. *Let A be a separable nuclear C^*-algebra. The following properties are equivalent.*

(i) *A is connective.*
(ii) *$A \otimes B$ is connective for some C^*-algebra B that contains a nonzero projection.*
(iii) *$A \otimes B$ is connective for all C^*-algebras B*
(iv) *$[[A, \mathcal{O}_2 \otimes \mathcal{K}]] = 0$.*
(v) *$[[A, L(\mathcal{H}) \otimes \mathcal{K}]] = 0$.*

Proof. The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ were established in [10]. (For $(ii) \Rightarrow (i)$ observe that A is a subalgebra of $A \otimes B$ if B contains a nonzero projection.)

$(i) \Rightarrow (iv)$ and $(i) \Rightarrow (v)$. Let B be a σ-unital C^*-algebra such that $KK(A, B) = 0$, for instance $B = \mathcal{O}_2$ or $B = L(\mathcal{H})$. If A is connective, then A is homotopy symmetric and hence $[[A, B \otimes \mathcal{K}]] \cong KK(A, B) = 0$ by [10, Thm. 3.1]. Note that even though [10, Thm. 3.1] was stated for separable C^*-algebras B it is routine to extend the result to general C^*-algebras using the separability of A.

$(iv) \Rightarrow (i)$ and $(v) \Rightarrow (i)$. Fix an embedding $\pi\colon A \to \mathcal{O}_2 \subset L(\mathcal{H}) \otimes \mathcal{K}$ and regard it as a constant asymptotic morphism $\{\pi_t\colon A \to L(\mathcal{H}) \otimes \mathcal{K}\}_t$. By assumption, $[[\pi_t]] = 0$ in $[[A, L(\mathcal{H}) \otimes \mathcal{K}]]$ and hence by restriction, $[[\pi_n]] = 0$ in $[[A, L(\mathcal{H}) \otimes \mathcal{K}]]_{\mathbb{N}}$. We shall view $L(\mathcal{H}) \otimes \mathcal{K}$ as a subalgebra of $L(\mathcal{H})$. The corresponding homotopy from the constant discrete morphism $\{\pi_n\}_n$ to zero will induce an embedding $\Phi\colon A \to \prod_n CL(\mathcal{H})/\bigoplus_n CL(\mathcal{H})$ which is liftable to a cpc map $\varphi\colon A \to \prod_n CL(\mathcal{H})$ by the nuclearity of A. □

2.3. **Extensions of connective C^*-algebras.** Connectivity of C^*-algebras has a plethora of permanence properties as proven in [10, Thm. 3.3]. In particular, it is inherited by split extensions [10, Thm. 3.3 (d)]. In the following theorem this result is extended to non-split extensions as well.

Theorem 2.4. *Let $0 \to J \to A \to B \to 0$ be an exact sequence of separable nuclear C^*-algebras. If J and B are connective, then A is connective.*

Proof. Since connectivity passes to nuclear subalgebras we may replace the given extension by

$$0 \to J \otimes \mathcal{O}_2 \otimes \mathcal{K} \to A \otimes \mathcal{O}_2 \otimes \mathcal{K} \to B \otimes \mathcal{O}_2 \otimes \mathcal{K} \to 0.$$

Adding to this extension a trivial absorbing extension, using the addition in Ext-theory, we obtain an absorbing extension

$$0 \to J \otimes \mathcal{O}_2 \otimes \mathcal{K} \to E \to B \otimes \mathcal{O}_2 \otimes \mathcal{K} \to 0, \tag{1}$$

which by construction has the property that $A \subset A \otimes \mathcal{O}_2 \otimes \mathcal{K} \subset E$, see for instance [4, Lemma 2.2]. Since $Ext(B \otimes \mathcal{O}_2, J \otimes \mathcal{O}_2) = 0$ as $\mathcal{O}_2$ is KK-contractible and we are dealing with an absorbing extension, it follows that the extension (1) splits by [24, Sec. 7] and so E is connective by [10, Thm. 3.3]. We conclude that $A \subset E$ is connective. □

In the sequel we will need to use the following result from [10], which is based on [2].

Theorem 2.5 ([10], Cor. 3.4.)**.** *Let A be a separable continuous field of nuclear C^*-algebras over a compact connected metrizable space X. If one of the fibers of A is connective, then A is connective.*

Corollary 2.6. *Let A be a separable continuous field of nuclear C^*-algebras over a locally compact metrizable space X that has no compact open subsets. Then A is connective.*

Proof. Let $Y = X \cup \{y_0\}$ be the one-point compactification of X. Then Y is a compact metrizable space which must be connected. Indeed, arguing by contradiction, say that $Y = U \cup V$ with U, V open and nonempty with $y_0 \in U$ and $U \cap V = \emptyset$. Then $V = V \cap X$ is both an open and compact subset of X.

We can view A as a continuous field over Y (see the remark on page 145 of [2]) and the fiber over y_0 satisfies $A(y_0) = \{0\}$. It follows that A is connective by Thm. 2.5. □

2.4. **Obstructions to connectivity.** If A is a C^*-algebra, we denote by $\widehat{A}$ the spectrum of A, which consists of all unitary equivalence classes of irreducible representations and by $\mathrm{Prim}(A)$ the primitive spectrum of A consisting of kernels of irreducible representations. The unitary dual $\widehat{G}$ of a group G identifies with $\widehat{C^*(G)}$. Recall that $\widehat{A}$ is topologized by pulling-back the Jacobson topology of $\mathrm{Prim}(A)$ under the natural map $\widehat{A} \to \mathrm{Prim}(A)$, $\pi \mapsto \ker \pi$, [14]. Let π and $(\pi_n)_n$ be irreducible representations of A acting on the same separable Hilbert space H. Suppose that $\|\pi_n(a)\xi - \pi(a)\xi\| \to 0$ for all $a \in A$ and $\xi \in H$. Then the sequence $(\pi_n)_n$ converges to π in the topology of $\widehat{A}$, see [14, Sec. 3.5].

Proposition 2.7. *Let A be a separable C^*-algebra.*

(i) If $\mathrm{Prim}(A)$ *contains a non-empty compact open subset, then A is not connective.*
(ii) If A is nuclear and $\mathrm{Prim}(A)$ *is Hausdorff, then A is connective if and only if* $\mathrm{Prim}(A)$ *does not contain a non-empty compact open subset.*

Proof. (i) Set $X = \mathrm{Prim}(A)$. If X has a non-empty compact open subset, then $A \otimes \mathcal{O}_2$ contains a nonzero projection by [30, Prop. 2.7] and hence A cannot be connective.

(ii) One implication follows from (i). For the other implication suppose that $X = \mathrm{Prim}(A)$ does not contain a non-empty compact open subset. Since X is Hausdorff by assumption, A is a nuclear separable continuous field over the locally compact space X, [16]. This is explained in detail in [3, Sec. 2.2.2]. Now we apply Cor. 2.6. $\square$

We would like thank Gabor Szabo for pointing out the following invariance property of connectivity.

Proposition 2.8. *Let A and B be separable nuclear C^*-algebra with homeomorphic primitive spectra. Then A is connective if and only if B is connective.*

Proof. Kirchberg's classification theorem [26] implies that if A and B are as in the statement, then $A \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong B \otimes \mathcal{O}_2 \otimes \mathcal{K}$. The desired conclusion follows now from Proposition 2.3. $\square$

Definition 2.9. Let A be a separable C^*-algebra. A point $\pi \in \widehat{A}$ is called *shielded*, if $\widehat{A} \setminus \{\pi\} \neq \emptyset$ and any sequence $(\pi_n)_n$ in $\widehat{A} \setminus \{\pi\}$ which converges to π also converges to another point $\eta \in \widehat{A} \setminus \{\pi\}$.

Lemma 2.10. *Let A be a unital separable C^*-algebra. If a point $\pi \in \widehat{A}$ is closed and shielded, then $I = \ker \pi$ is not connective.*

Proof. Observe that $I \neq \{0\}$, since $\widehat{A} \setminus \{\pi\} \neq \emptyset$ and $\{\pi\}$ is closed. By Proposition 2.7 it suffices to show that $\mathrm{Prim}(I) = \mathrm{Prim}(A) \setminus \{I\}$ is a nonempty compact-open subset of $\mathrm{Prim}(A)$. Since $\{\pi\}$ is closed, it follows that $q^{-1}(I) = \{\pi\}$ and hence $q(\widehat{A} \setminus \{\pi\}) = \mathrm{Prim}(A) \setminus \{I\}$. The quotient map $q\colon \widehat{A} \to \mathrm{Prim}(A)$ is continuous and open, since the topology of $\widehat{A}$ is defined as the preimage of the topology of $\mathrm{Prim}(A)$. Therefore, the lemma follows if we show that $\widehat{A} \setminus \{\pi\}$ is compact and open. $\widehat{A} \setminus \{\pi\}$ is open because $\{\pi\}$ is closed. Since $\widehat{A}$ is compact and satisfies the second axiom of countability [14], it suffices to show that $\widehat{A} \setminus \{\pi\}$ is sequentially compact, [25, p. 138]. Let $(\pi_n)_n$ be a sequence in $\widehat{A} \setminus \{\pi\}$. By compactness of $\widehat{A}$ it contains a subsequence $(\pi_{n_k})_k$ converging in $\widehat{A}$. If it converges to $\pi \in \widehat{A}$, then it also converges to some other point $\eta \in \widehat{A} \setminus \{\pi\}$, because π is shielded. Hence $\widehat{A} \setminus \{\pi\}$ is also compact. □

Corollary 2.11. *Let G be a countable discrete group. If the trivial representation $\iota \in \widehat{G}$ is shielded, then $I(G)$ is not connective.*

Proof. Since ι is a one-dimensional representation, it follows that $\{\iota\}$ is closed in $\widehat{G}$. Thus, the statement follows from Lemma 2.10. □

3. Connectivity of crystallographic groups

It is known that there are precisely 10 closed flat 3-dimensional manifolds. Conway and Rossetti [7] call these manifolds platycosms ("flat universes"). The Hantzsche-Wendt manifold [17], or the didicosm in the terminology of [7], is the only platycosm with finite homology. Its fundamental group G, known as the Hantzsche-Wendt group, is generated by two elements x and y subject to two relations:

$$x^2yx^2 = y, \quad y^2xy^2 = x.$$

The group G is one of the classic torsion free 3-dimensional crystallographic groups, [17, 7]. It is useful to introduce the notation $z = (xy)^{-1}$.

A concrete realization of G as rigid motions of $\mathbb{R}^3$ is given by the following transformations X, Y, Z that correspond to the group elements x, y and z.

$$X(\xi) = A\xi + a, \quad Y(\xi) = B\xi + b, \quad Z(\xi) = C\xi + c, \quad \xi \in \mathbb{R}^3,$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$a = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}, \quad c = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix}.$$

The transformations X^2, Y^2 and Z^2 are just translations by unit vectors in the positive directions of the coordinate axes.

One shows (independently of the previous concrete realization) that the elements x^2, y^2 and z^2 commute. Moreover one has the following relations in G:

$$\begin{aligned} xx^2x^{-1} &= x^2, & xy^2x^{-1} &= y^{-2}, & xz^2x^{-1} &= z^{-2} \\ yx^2y^{-1} &= x^{-2}, & yy^2y^{-1} &= y^2, & yz^2y^{-1} &= z^{-2} \\ zx^2z^{-1} &= x^{-2}, & zy^2z^{-1} &= y^{-2}, & zz^2z^{-1} &= z^2 \end{aligned}$$

The subgroup N of G generated by x^2, y^2 and z^2 is normal in G and it is isomorphic to $\mathbb{Z}^3 \cong \mathbb{Z}x^2 \oplus \mathbb{Z}y^2 \oplus \mathbb{Z}z^2$. Let $q : G \to H = G/N$ denote the quotient map.

$$1 \longrightarrow N \longrightarrow G \xrightarrow{q} H \longrightarrow 1$$

H is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ with generators are $q(x)$ and $q(y)$.

For later use, we will need the following identities that hold in G.

$$\begin{gathered} x^{-1}y = yxy^2z^{-2} = zx^2z^{-2}, \quad x^{-1}z = yz^2, \\ y^{-1}x = zx^2, \quad y^{-1}z = x(x^{-2}y^2). \end{gathered} \tag{2}$$

3.1. **Induced representations and the unitary dual of G.** Based on Corollary 2.11 we will show that $I(G)$ for the Hantzsche-Wendt group G is not connective. This requires a thorough analysis of the spectrum $\widehat{G}$.

Our basic reference for this section is the book of Kaniuth and Taylor [23]. The unitary dual of G consists of unitary equivalence classes of irreducible unitary representations of G and is denoted by $\widehat{G}$. G acts on $\widehat{N} \cong \mathbb{T}^3$ by $g \cdot \chi = \chi(g \cdot g^{-1})$. If we identify the character $\chi \in \widehat{N}$ with the point $(\chi(x^2), \chi(y^2), \chi(z^2)) = (u, v, w) \in \mathbb{T}^3$, then the action of G is described as follows:

$$x \cdot (u, v, w) = (u, \bar{v}, \bar{w}), \quad y \cdot (u, v, w) = (\bar{u}, v, \bar{w}), \quad z \cdot (u, v, w) = (\bar{u}, \bar{v}, w).$$

The stabilizer of a character χ is the subgroup G_χ of G defined by $G_\chi = \{g \in G \colon \chi(g \cdot g^{-1}) = \chi(\cdot)\}$. It is clear that $N \subset G_\chi$ and that there is a bijection from G/G_χ onto the orbit of χ. In particular, the orbits of the action of G on $\widehat{N}$ can only have length 1, 2 or 4. Mackey has shown that each irreducible representation $\pi \in \widehat{G}$ is supported by the orbit of some character $\chi \in \widehat{N}$, in the sense that the restriction of π to N is unitarily equivalent to some multiple m_π of the direct sum of the characters in the orbit of χ.

$$\pi_{|N} \sim m_\pi \bigoplus_{g\in G/G_\chi} \chi(g \cdot g^{-1}).$$

In the sum above g runs through a set of coset representatives.

Mackey's theory has a particularly nice form for virtually abelian discrete groups. Let $\Omega \subset \widehat{N}$ be a subset which intersects each orbit of G exactly once. For each $\chi \in \widehat{N}$, let $\widehat{G_\chi}$ be the unitary dual of the stabilizer group G_χ and denote by $\widehat{G_\chi}^{(\chi)}$ the subset of $\widehat{G_\chi}$ consisting of classes of irreducible representations σ of G_χ such the restriction of σ to N is unitarily equivalent to a multiple of χ. Then, according to [23, Thm. 4.28]

Theorem 3.1. $\widehat{G} = \left\{ \mathrm{ind}_{G_\chi}^G(\sigma) \colon \sigma \in \widehat{G_\chi}^{(\chi)},\ \chi \in \Omega \right\}$.

Let ι be the trivial representation of G. We will prove that $\iota \in \widehat{G}$ is shielded by showing that any sequence $(\pi_n)_n$ of points in $\widehat{G} \setminus \{\iota\}$ that converges to ι has a subsequence which is convergent to a point $\eta \neq \iota$.

Let $R_\ell \subset \widehat{G}$ consist of those classes of irreducible representations which lie over ℓ-orbits, i.e. the orbits of length ℓ. Write $\widehat{G}$ as the disjoint union $\widehat{G} = R_1 \cup R_2 \cup R_4$. It suffices to assume that all the elements π_n belong to the same subset R_ℓ. We distinguish the three possible cases for ℓ:

1-orbits. Consider the characters of N of the form $\chi = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}$. These are precisely the points in $\widehat{N}$ which are fixed under the action of G. In other words $G_\chi = G$. Let $(\pi_n)_n$ be a sequence of elements in $R_1 \subset \widehat{G}$ and such that $(\pi_n)_n$ is convergent to ι. Since the restriction of π_n to N is a multiple of a character $\chi_n = (\varepsilon_1(n), \varepsilon_2(n), \varepsilon_3(n))$, it follows that $\varepsilon_1(n) = \varepsilon_2(n) = \varepsilon_3(n) = 1$ for all sufficiently large n and hence since π_n is irreducible, there is m such that $\pi_n = \iota$ for $n \geq m$. Hence there is no sequence in $R_1 \setminus \{\iota\}$ which converges to ι.

2-orbits. The characters $\chi \in \widehat{N}$ with orbits of length two are those $\chi = (u, v, w)$ where precisely only one of the coordinates is not equal to ± 1. Let us argue first that if $(\pi_n)_n$ is a sequence of elements in $R_2 \subset \widehat{G}$ such that π_n lies over the orbit of $\chi_n = (u_n, v_n, w_n)$ and $(\pi_n)_n$ is convergent to ι, then two of the coordinates u_n, v_n, w_n must be equal to 1 for all sufficiently large n.

Suppose that each π_n lies over the orbit of a character χ_n of the form $\chi_n = (u_n, \varepsilon_2(n), \varepsilon_3(n))$ where $u_n \neq \pm 1$ and $\varepsilon_2(n), \varepsilon_3(n) \in \{\pm 1\}$. Then G_{χ_n} is generated by x, y^2, z^2 and $\{e, y\}$ are coset representatives for G/G_{χ_n}.

Since $\pi_n|_N \sim m_n(\chi_n(\cdot) \oplus \chi_n(y \cdot y^{-1}))$, it follows that

$$\pi_n(y^2) \sim m_n \begin{pmatrix} \varepsilon_2(n) & 0 \\ 0 & \varepsilon_2(n) \end{pmatrix}, \quad \pi_n(z^2) \sim m_n \begin{pmatrix} \varepsilon_3(n) & 0 \\ 0 & \varepsilon_3(n) \end{pmatrix}$$

and hence if $(\pi_n)_n$ converges to ι, then we must have $\varepsilon_2(n) = \varepsilon_3(n) = 1$ for all sufficiently large n. The cases $\chi_n = (\varepsilon_1, v_n, \varepsilon_3)$ and $\chi_n = (\varepsilon_1, \varepsilon_2, w_n)$ are treated similarly.

In view of the discussion above, it suffices to focus on characters of N the form $\chi = (u, 1, 1)$. The orbit of χ consists of two points, $(u, 1, 1)$ and $(\bar{u}, 1, 1)$. The corresponding stabilizer G_χ is generated by x, y^2 and z^2. In particular $G_\chi = N \cup xN$ and $G = G_\chi \cup y\, G_\chi$. The exact sequence

$$1 \longrightarrow N \longrightarrow G_\chi \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

does not split since G is torsion free. The quotient G/G_χ is generated by the coset of y. Let $\sigma \in \widehat{G}_\chi$ be an irreducible representation of G_χ whose restriction to N is a multiple of χ. Since $\chi(y^2) = \chi(z^2) = 1$, it follows that σ factors through $G_\chi/\mathbb{Z}^2$. Moreover we have a nontrivial central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow G_\chi/\mathbb{Z}^2 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

where the normal subgroup is generated by the image of x^2 under the map $N \to N/\langle y^2, z^2\rangle$ and the quotient group is generated by the image of $q(x)$ under the map $H \to H/\langle q(y)\rangle$. Since $G_\chi/\mathbb{Z}^2$ is an abelian group, σ must be a character such that $\sigma(x)^2 = \sigma(x^2) = \chi(x^2) = u$. Thus $\sigma(x) = a \in \mathbb{T}$ with $a^2 = u$. Let us compute the representation $\pi = \mathrm{ind}_{G_\chi}^G(\sigma)$ of G induced by σ. It acts on the Hilbert space

$$H_\pi = \{\xi : G \to \mathbb{C} \colon \xi(gh) = \sigma(h^{-1})\xi(g), \quad g \in G, h \in G_\chi\}.$$

Since $G = G_\chi \cup y\, G_\chi$, we can identify H_π with $\mathbb{C}^2$ via the isometry $\xi \mapsto (\xi(e), \xi(y))$. Then $\pi(g)\xi = \xi(g^{-1}\cdot)$ can be described using (2) as follows:

$$\begin{aligned}
\pi(x)\xi(e) &= \xi(x^{-1}) = \sigma(x)\xi(e) = a\xi(e) \\
\pi(x)\xi(y) &= \xi(x^{-1}y) = \xi(yxy^2z^{-2}) = \sigma(z^2y^{-2}x^{-1})\xi(y) = \bar{a}\xi(y) \\
\pi(y)\xi(e) &= \xi(y^{-1}) = \xi(y \cdot y^{-2}) = \sigma(y^2)\xi(y) = \xi(y) \\
\pi(y)\xi(y) &= \xi(y^{-1} \cdot y) = \xi(e)
\end{aligned}$$

which produces the following matrices with respect to the basis given above:

$$\pi(x) = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi(z) = \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix} \tag{3}$$

Corresponding to the characters $(1, v, 1)$ and $(1, 1, w)$ we obtain the irreducible representations, where we use the isometries $\xi \mapsto (\xi(e), \xi(x))$ and $\xi \mapsto (\xi(e), \xi(y))$ respectively:

$$\pi(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix}, \quad \pi(z) = \begin{pmatrix} 0 & \bar{b} \\ b & 0 \end{pmatrix}, \quad b^2 = v, \tag{4}$$

and

$$\pi(x)=\begin{pmatrix}0&\bar c\\ c&0\end{pmatrix},\quad \pi(y)=\begin{pmatrix}0&1\\ 1&0\end{pmatrix},\quad \pi(z)=\begin{pmatrix}c&0\\ 0&\bar c\end{pmatrix},\quad c^2=w. \tag{5}$$

Let $(\pi_n)_n$ be a sequence in R_2 that converges to ι in $\widehat{G}$. Arguing by symmetry, we may assume that each π_n is given by the formulas (3) corresponding to a sequence of points $u_n\in\mathbb{T}$ with $u_n\notin\{\pm1\}$. Since $\pi_n\to\iota$ it follows from the equation (3) that $u_n\to 1$. Again from (3) we can compute the limits of the sequences $\pi_n(x)$ and $\pi_n(y)$ in $U(2)$. This gives the representation $\pi\colon G\to U(2)$:

$$\pi(x)=\begin{pmatrix}1&0\\ 0&1\end{pmatrix},\quad \pi(y)=\begin{pmatrix}0&1\\ 1&0\end{pmatrix}$$

It is clear that π is a representation of G that factors through the left regular representation of $\mathbb{Z}/2$. Decompose π into a direct sum of characters $\pi\sim\iota\oplus\eta$. Then η is not equivalent to ι and $\pi_n\to\eta$ in $\widehat{G}$.

4-orbits. Let $\chi=(u,v,w)\in\mathbb{T}^3$ be a character of N with $u,v,w\notin\{\pm1\}$. Its orbit under the action of G consists of four points and $G_\chi=N$. Let us compute the representation $\pi=\mathrm{ind}_N^G(\chi)$ of G induced by σ. It acts on the Hilbert space $H_\pi=\{\xi : G\to\mathbb{C}\colon \xi(gh)=\chi(h^{-1})\xi(g),\quad g\in G, h\in N\}$. Thus one can identify H_π with $\mathbb{C}^4$ via the isometry $\xi\mapsto(\xi(e),\xi(x),\xi(y),\xi(z))$. Using the identities (2), we verify that $\pi(g)\xi=\xi(g^{-1}\cdot)$ is described as follows:

$$\begin{aligned}
\pi(x)\xi(e)&=\xi(x^{-1})=\xi(x\cdot x^{-2})=\chi(x^2)\xi(x)=u\xi(x)\\
\pi(x)\xi(x)&=\xi(e)\\
\pi(x)\xi(y)&=\xi(x^{-1}y)=\xi(zx^2z^{-2})=\chi(z^2x^{-2})\xi(z)=w\bar u\xi(z)\\
\pi(x)\xi(z)&=\xi(x^{-1}z)=\xi(yz^2)=\chi(z^{-2})\xi(y)=\bar w\xi(y)
\end{aligned}$$

$$\begin{aligned}
\pi(y)\xi(e)&=\xi(y^{-1})=\xi(y\cdot y^{-2})=\chi(y^2)\xi(y)=v\xi(y)\\
\pi(y)\xi(x)&=\xi(y^{-1}x)=\xi(zx^2)=\chi(x^{-2})\xi(z)=\bar u\xi(z)\\
\pi(y)\xi(y)&=\xi(e)\\
\pi(y)\xi(z)&=\xi(y^{-1}z)=\xi(x\cdot x^{-2}y^2)=\chi(y^{-2}x^2)\xi(x)=\bar v u\xi(x)
\end{aligned}$$

producing the matrices:

$$\pi(x)=\begin{pmatrix}0&u&0&0\\ 1&0&0&0\\ 0&0&0&\bar u w\\ 0&0&\bar w&0\end{pmatrix},\quad \pi(y)=\begin{pmatrix}0&0&v&0\\ 0&0&0&\bar u\\ 1&0&0&0\\ 0&u\bar v&0&0\end{pmatrix} \tag{6}$$

It follows that

$$\pi(x^2)=\begin{pmatrix} u&0&0&0\\ 0&u&0&0\\ 0&0&\bar{u}&0\\ 0&0&0&\bar{u}\end{pmatrix},\quad \pi(y^2)=\begin{pmatrix} v&0&0&0\\ 0&\bar{v}&0&0\\ 0&0&v&0\\ 0&0&0&\bar{v}\end{pmatrix},\quad \pi(z^2)=\begin{pmatrix} w&0&0&0\\ 0&\bar{w}&0&0\\ 0&0&\bar{w}&0\\ 0&0&0&w\end{pmatrix} \tag{7}$$

Let $(\pi_n)_n$ be a sequence in R_4 converging to ι in $\widehat{G}$. Each π_n is given by the formulas (6) corresponding to a sequence of points $(u_k, v_k, w_k) \in \mathbb{T}^3$ with $u_k, v_k, w_k \notin \{\pm 1\}$. Since $\pi_n \to \iota$ it follows from equation (7) that $u_k, v_k, w_k \to 1$. Again from (6) we can compute the limits of the sequences $\pi_n(x)$ and $\pi_n(y)$ in the space of unitary operators $U(4)$. This gives the representation $\pi\colon G \to U(4)$:

$$\pi(x)=\begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0\end{pmatrix},\quad \pi(y)=\begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\end{pmatrix}$$

It is clear that π is a representation of G that factors through $H = \mathbb{Z}/2\times\mathbb{Z}/2$. Decompose π into a direct sum of characters η_i. Since π is not equivalent to a multiple of the trivial representation, it follows that at least one of these characters is not equivalent to ι. On the other hand $\pi_n \to \eta_i$ in $\widehat{G}$ for all i. Combining the above analysis with Corollary 2.11 we obtain immediately

Corollary 3.2. *If G is the Hantzsche-Wendt group, then $I(G)$ is not connective.*

It was conjectured in [8] that if G is a torsion free discrete amenable group, then $[[I(G),\mathcal{K}]] \cong KK(I(G),\mathbb{C})$. We argue now that this conjecture fails for the Hantzsche-Wendt group. Indeed this follows from the previous corollary in conjunction with the following lemma.

Lemma 3.3. *Let G be a residually finite torsion free discrete amenable group which admits a classifying space with finitely generated K-homology group $K_1(BG)$. Then $[[I(G),\mathcal{K}]] \cong KK(I(G),\mathbb{C})$ if and only if $I(G)$ is connective.*

Proof. Suppose first that $[[I(G),\mathcal{K}]] \cong KK(I(G),\mathbb{C})$. Since G is amenable and residually finite it follows that $C^*(G)$ is residually finite dimensional. Since G is amenable, G satisfies the Baum-Connes conjecture and $C^*(G)$ satisfies the UCT by results of Higson and Kasparov [22] and Tu [35]. In particular we have a short exact sequence

$$0 \to Ext^1(K_1(C^*(G)),\mathbb{Z}) \to KK(C^*(G),\mathbb{C}) \to \mathrm{Hom}(K_0((C^*(G)),\mathbb{Z}) \to 0$$

Let $\pi_n : C^*(G) \to M_{d(n)}(\mathbb{C})$ be a separating sequence of finite dimensional representations. The restriction of π_n to $I(G)$ will be denoted by σ_n. By [8, Prop. 3.2] $(\pi_n)_* = d(n)\iota_* : K_0(C^*(G)) \to \mathbb{Z}$ and hence $[\sigma_n] \in Ext^1(K_1(I(G)),\mathbb{Z}) \subset KK(I(G),\mathbb{C})$ is a torsion element since $K_1(I(G)) \cong$

$K_1(BG)$ is finitely generated. After replacing π_n by a suitable multiple of itself we have arranged that $[\sigma_n] = 0$ in $KK(I(G), \mathbb{C})$ and hence $[[\sigma_n]] = 0$ in $[[I(G), \mathcal{K}]]$. Since the sequence (σ_n) separates the elements of $I(G)$ it follows that $I(G)$ is connective.

The converse is contained in the main result of [10] which shows that if A is a separable nuclear connective C^*-algebra, then $[[A, \mathcal{K}]] \cong KK(A, \mathbb{C})$. □

3.2. **Crystallographic groups with cyclic holonomy.** In this section we are going to show that torsion free crystallographic groups with cyclic holonomy are connective. Apart from this we isolate a lemma, which proves that $I(G)$ for a group G which is a finite extension of a connective group always contains a "big" connective ideal. In particular, the lemma also holds for the Hantzsche-Wendt group.

The proof of both results uses some tools from the index theory of C^*-subalgebras. A reference is [39]. Let Γ and G be discrete groups and let H be a finite group. Suppose that they fit into an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow G \xrightarrow{q} H \longrightarrow 1 \ .$$

Let $E\colon C^*(G) \to C^*(\Gamma)$ be the faithful conditional expectation [39, Ex. 1.2.3] given on group elements by

$$E(g) = \begin{cases} g & \text{if } g \in \Gamma \\ 0 & \text{else} \end{cases} \ .$$

Choose a lift $g_h \in G$ for each $h \in H$. The pairs (g_h^{-1}, g_h) form a quasi-basis in the sense of [39, Def. 1.2.2]. Let $\mathcal{E} = C^*(G)$ considered as a right Hilbert $C^*(\Gamma)$-module, where the right action is induced by the inclusion $C^*(\Gamma) \to C^*(G)$ and the inner product is given by $\langle a, b\rangle = E(a^*b)$ [39, Sec. 2.1]. Note that $\mathcal{E}$ is complete [39, Prop. 2.1.5]. The quasi-basis induces an isometric isomorphism of right Hilbert $C^*(\Gamma)$-modules $u\colon \mathcal{E} \to \ell^2(H) \otimes C^*(\Gamma)$ with

$$u(a) = \sum_h \delta_h \otimes E(g_h a)$$

and inverse $u^*\colon \ell^2(H) \otimes C^*(\Gamma) \to \mathcal{E}$ with $u^*(\delta_h \otimes b) = g_h^{-1} b$. Let $\mathcal{L}_{C^*(\Gamma)}(\mathcal{E})$ be the bounded adjointable operators on $\mathcal{E}$ and denote by $\mathcal{K}_{C^*(\Gamma)}(\mathcal{E})$ the compact ones. Then we have $\mathcal{L}_{C^*(\Gamma)}(\mathcal{E}) \cong \mathcal{K}_{C^*(\Gamma)}(\mathcal{E}) \cong \mathcal{K}(\ell^2(H)) \otimes C^*(\Gamma)$. The left multiplication of $C^*(G)$ on $\mathcal{E}$ induces a $*$-homomorphism

$$\psi\colon C^*(G) \to \mathcal{K}(\ell^2(H)) \otimes C^*(\Gamma)$$

with matrix entries $\psi_{h',h}(a) = E(g_{h'}\, a\, g_h^{-1})$. Suppose we have $a \in C^*(G)$ with $\psi(a) = 0$. Then

$$a = \frac{1}{|H|}\sum_{h,h'} g_{h'}^{-1} E(g_{h'}\, a\, g_h^{-1}) g_h = 0 \ .$$

Hence, ψ is injective.

Lemma 3.4. *Let Γ be a connective group and let H be a finite group. Suppose that the group G fits into a short exact sequence of the form*

$$1 \longrightarrow \Gamma \longrightarrow G \xrightarrow{q} H \longrightarrow 1 \ .$$

Then $I(G,H) = \ker(I(q)\colon I(G) \to I(H))$ is connective as well.

Proof. Let $\iota\colon C^*(\Gamma) \to \mathbb{C}$ be the trivial representation and let ψ be the injective $*$-homomorphism constructed above. For all $b \in C^*(\Gamma) \subset C^*(G)$ we have $\psi_{h',h}(b) = \delta_{h',h}\, g_h\, b\, g_h^{-1}$. In particular, ψ embeds the ideal J generated by $I(\Gamma)$ into $\ker(\mathrm{id} \otimes \iota) = \mathcal{K}(\ell^2(H)) \otimes I(\Gamma)$, which is connective. Hence J is connective as well. It is clear that $J \subseteq I(G,H)$. Let $x \in I(G,H)$. By the property of the quasi-basis

$$0 = q(x) = \sum_{h \in H} q(E(x\, g_h^{-1}))\, h \quad \Rightarrow \quad q(E(x\, g_h^{-1})) = 0 \quad \forall h \in H \ .$$

Since $I(G,H) \cap C^*(\Gamma) = I(\Gamma)$ we obtain that $E(x\, g_h^{-1}) \in I(\Gamma)$ for all $h \in H$ and therefore $x = \sum_h E(x\, g_h^{-1}))\, g_h \in J$, hence $J = I(G,H)$. □

The proof of the second result uses an induction over the rank of the free abelian subgroup based on the following observation.

Lemma 3.5. *Let $m > 1$ and let Γ and G be countable discrete groups that fit into the following short exact sequence*

$$1 \longrightarrow \Gamma \longrightarrow G \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 1 \ .$$

Suppose that Γ is connective and there are group homomorphisms $\varphi\colon G \to \mathbb{Z}$ and $q\colon \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, such that $\pi = q \circ \varphi$. Then G is connective as well.

Proof. Let $\psi\colon C^*(G) \to \mathcal{K}(\ell^2(H)) \otimes C^*(\Gamma)$ be the injective $*$-homomorphism constructed above and let $\iota\colon C^*(\Gamma) \to \mathbb{C}$ be the trivial representation. Observe that $\rho = (\mathrm{id} \otimes \iota) \circ \psi$ satisfies

$$\rho(g\,\gamma)_{h',h} = \iota(E(g_{h'} g\,\gamma\, g_h^{-1})) = \iota(E(g_{h'} g\, g_h^{-1})\, g_h\,\gamma\, g_h^{-1}) = \iota(E(g_{h'} g\, g_h^{-1}))$$

for all $g \in G$ and $\gamma \in \Gamma$. In particular, ρ factors through the $*$-homomorphism $C^*(G) \to C^*(\mathbb{Z}/m\mathbb{Z})$ induced by π. By assumption this decomposes as

$$C^*(G) \xrightarrow{\varphi} C^*(\mathbb{Z}) \xrightarrow{q} C^*(\mathbb{Z}/m\mathbb{Z}) \ .$$

Altogether we obtain that ρ decomposes into a direct sum of one-dimensional representations, each of which is homotopic through representations to the trivial one. Hence, to show that G is connective, it suffices to construct a path through discrete asymptotic morphisms connecting a faithful morphism with a direct sum of copies of ρ.

Choose a path witnessing the connectivity of Γ, i.e. a discrete asymptotic morphism

$$H_n \colon C^*(\Gamma) \to C([0,1]) \otimes M_n(\mathbb{C})$$

such that for $H_n^{(t)} = \mathrm{ev}_t \circ H_n \colon C^*(\Gamma) \to M_n(\mathbb{C})$ we have that $H_n^{(0)}$ is faithful and $H_n^{(1)}$ is a multiple of ι. Then $(\mathrm{id}_{M_m(\mathbb{C})} \otimes H_n) \circ \psi$ has the desired properties. Hence, G is connective. □

We need the following elementary fact:

Lemma 3.6. *Let $a, b > 1$ be integers and consider the exact sequence*

$$0 \longrightarrow \mathbb{Z}/a\mathbb{Z} \xrightarrow{\cdot b} \mathbb{Z}/ab\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/b\mathbb{Z} \longrightarrow 0$$

with $\pi(x) = x \mod b$. Any generator of $\mathbb{Z}/b\mathbb{Z}$ lifts to a generator of $\mathbb{Z}/ab\mathbb{Z}$.

Proof. Let $\bar{y} \in \mathbb{Z}/b\mathbb{Z}$ be a generator and let $y \in \{0, \dots, b-1\}$ be a representative. Let $p_1, \dots, p_s$ be the disctinct prime factors of ab, such that $p_1, \dots, p_r$ for $r \leq s$ are the ones not dividing y and $p_{r+1}, \dots, p_s$ divide y. Since $\gcd(y, b) = 1$, the primes $p_{r+1}, \dots, p_s$ do not divide b. Let $x = y + p_1 \dots p_r\, b$. We have for $i \in \{1, \dots, r\}$ and $j \in \{r+1, \dots, s\}$

$$x \equiv y \not\equiv 0 \mod p_i\ ,$$
$$x \equiv p_1 \dots p_r\, b \not\equiv 0 \mod p_j\ .$$

Hence $\gcd(x, ab) = 1$ and $x \in \mathbb{Z}/ab\mathbb{Z}$ is a generator with $\pi(x) = \bar{y}$. □

To start the induction we need the following lemma.

Lemma 3.7. *Let G be a countable torsion free discrete group, which fits into an exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

Then G is isomorphic to $\mathbb{Z}$, hence in particular connective.

Proof. This can be proven by calculating $H^2(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$ for all $\mathbb{Z}/m\mathbb{Z}$-module structures on $\mathbb{Z}$, but we give a direct argument here.

Let $x \in G$ be a lift of $1 \in \mathbb{Z}/m\mathbb{Z}$. Then G is generated by x and $\mathbb{Z}$. Moreover, $x^m \in \mathbb{Z} \cap Z(G)$, where $Z(G)$ denotes the center of G. We have $\mathrm{Aut}(\mathbb{Z}) \cong GL_1(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. If $t \in \mathbb{Z}$ denotes a generator, we therefore can only have $xtx^{-1} = t^{-1}$ or $xt = tx$. Suppose the first is true, then

$$x^m = x\, x^m\, x^{-1} = x^{-m} \quad \Rightarrow \quad x^{2m} = e$$

contradicting that G is torsion free. Thus, t and x commute and $x^m = t^n$ for some $n \in \mathbb{Z}$. Without loss of generality we can assume $\gcd(m, n) = 1$. Indeed, if $m = m'\ell$ and $n = n'\ell$ with $\ell > 1$, then $(x^{m'}t^{-n'})^\ell = e$ and therefore $x^{m'} = t^{n'}$ also holds in G. Consider

$$\alpha\colon G \to \mathbb{Z} \quad ; \quad x^k t^\ell \mapsto k\,n + \ell\,m\ .$$

This is a well-defined group homomorphism, which is easily seen to be bijective as a consequence of $\gcd(m, n) = 1$. $\square$

Theorem 3.8. *Let G be a countable, torsion free, discrete group, which fits into an exact sequence of the form*

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow G \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

for some $n, m \in \mathbb{N}$. Then G is connective.

Proof. This will be proven by induction over the rank of the free abelian subgroup. The case $n = 1$ follows from Lemma 3.7.

Observe that $Z(G) \neq \{e\}$. Indeed, let $x \in G$ be a lift of the generator of $\mathbb{Z}/m\mathbb{Z}$. Then G is generated by $\mathbb{Z}^n$ and x. Moreover, $x^m \neq e$ since G is torsion free and $\pi(x^m)$ is trivial, hence $x^m \in \mathbb{Z}^n$. Thus, x^m commutes with $\mathbb{Z}^n$ and x, hence with all elements of G, i.e. $x^m \in Z(G)$.

This implies that the transfer homomorphism $T\colon G \to \mathbb{Z}^n$ associated to the finite index subgroup $\mathbb{Z}^n$ is non-trivial. Therefore there exists a surjective group homomorphism $\varphi\colon G \to \mathbb{Z}$. Let $q\colon \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be the canonical quotient homomorphism and let $\bar{\varphi} = q \circ \varphi$. Let $H = \ker(\varphi)$. We have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & H \arrow[r] \arrow[d] & \mathbb{Z}/a\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[d] & G \arrow[r, "\pi"] \arrow[d, "\varphi"] & \mathbb{Z}/m\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & b\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r, "\pi'"] \arrow[d] & \mathbb{Z}/b\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

The value of a is chosen in such a way that $\mathrm{Im}(\pi|_H) \cong \mathbb{Z}/a\mathbb{Z}$ and b satisfies $m = ab$. The homomorphism π' is surjective since π and $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/b\mathbb{Z}$ are. The vertical arrow on the left hand side is induced by $\varphi|_{\mathbb{Z}^n}$.

Suppose $H \subset \ker(\pi) = \mathbb{Z}^n$. Then $a = 1$, $b = m$ and $\pi = \pi' \circ \varphi$. By Lemma 3.5, G is then connective. So we may assume $a > 1$.

We claim that there is an element $g \in G$ such that $\varphi(g) = 1$ and $\pi(g)$ is a generator of $\mathbb{Z}/m\mathbb{Z}$. This is constructed as follows: If $b = 1$, we can choose $g \in G$, such that $\varphi(g) = 1$ and modify it by an element in H to achieve that $\pi(g)$ becomes a generator. Otherwise, choose $g' \in G$ such that $\varphi(g') = 1$ and note that $\pi'(\varphi(g'))$ is a generator of $\mathbb{Z}/b\mathbb{Z}$ by surjectivity. We can lift $\pi'(\varphi(g'))$ to a generator $x \in \mathbb{Z}/m\mathbb{Z}$ by Lemma 3.6. Note that $\pi(g') - x \in \mathbb{Z}/a\mathbb{Z}$ and lift this difference to an element $h \in H$. Let $g = g'\,h^{-1}$. Then $\varphi(g) = \varphi(g') = 1$ and $\pi(g) = \pi(g') - \pi(h) = \pi(g') - \pi(g') + x = x$.

Let $G' = \ker(\bar{\varphi}) = \{g \in G \mid \varphi(g) = \ell \cdot m \text{ for } \ell \in \mathbb{Z}\} \supset H$. Hence, the following diagram has exact rows.

$$\begin{tikzcd} 1 \arrow[r] & G' \arrow[r] \arrow[d, "\varphi|_{G'}"'] & G \arrow[r, "\bar{\varphi}"] \arrow[d, "\varphi"] & \mathbb{Z}/m\mathbb{Z} \arrow[r] \arrow[d, "="'] & 0 \\ 1 \arrow[r] & m\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r, "q"'] & \mathbb{Z}/m\mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{8}$$

The group G is generated by $\mathbb{Z}^n$ and the element g constructed above. We have $g^m \in \mathbb{Z}^n \cap Z(G)$ and $\varphi(g^m) = m$. In particular, $g^m \in Z(G')$. Let

$$\psi \colon H \times \mathbb{Z} \to G' \quad ; \quad (h, k) \mapsto h \cdot g^{mk} .$$

This is a group homomorphism, since g^m is central and it fits into the commutative diagram with exact upper and lower part

$$\begin{tikzcd} & & G' \arrow[dr, "\varphi|_{G'}/m"] & & \\ 1 \arrow[r] & H \arrow[ur] \arrow[dr, "i"'] & & \mathbb{Z} \arrow[r] & 0 \\ & & H \times \mathbb{Z} \arrow[uu, "\psi"] \arrow[ur, "\mathrm{pr}"'] & & \end{tikzcd}$$

proving that ψ is in fact an isomorphism. By the upper row in diagram (8) and Lemma 3.5, the connectivity of G follows if $H \times \mathbb{Z}$, hence H, is connective [10, Thm. 4.1]. But H fits into a short exact sequence of the form

$$0 \to A \to H \to \mathbb{Z}/a\mathbb{Z} \to 0$$

where A is the free abelian kernel of the nonzero homomorphism $\mathbb{Z}^n \to b\mathbb{Z}$ from above, which has rank $(n-1)$. This completes the induction step. $\square$

4. Connectivity of Lie group C^*-algebras

In this section we determine which linear connected nilpotent Lie groups and which linear connected reductive Lie groups have connective reduced C^*-algebras. Let us recall that nilpotent connected Lie groups are liminary as shown by Dixmier [13] and Kirillov [27] and semisimple connected Lie groups are liminary as shown by Harish-Chandra [18].

4.1. **Solvable and nilpotent Lie groups.** A locally compact group N is compactly generated if $N = \bigcup_n V^n$ for some compact subset V of N. Every connected locally compact group is automatically compactly generated. The structure of abelian compactly generated locally compact groups is known. If N is such a group, then $N \cong \mathbb{R}^n \times \mathbb{Z}^m \times K$ for integers $n, m \geq 0$ and K a compact group, [12, Thm. 4.4.2].

Proposition 4.1. *If G is a second countable locally compact amenable group (for example a solvable Lie group) whose center contains a noncompact closed connected subgroup, then $C^*(G)$ is connective.*

Proof. Let N be a noncompact closed connected subgroup of $Z(G)$. Then, by the structure theorem quoted above, N must have a closed subgroup isomorphic to $\mathbb{R}$. Consider the central extension:

$$0 \to \mathbb{R} \to G \to H \to 0.$$

Since G is amenable, by [29, Thm. 1.2] (as explained in [15, Lemma 6.3]), $C^*(G)$ has the structure of a continuous field of C^*-algebras over $\widehat{\mathbb{R}} \cong \mathbb{R}$. The desired conclusion follows from Cor. 2.6 since $\mathbb{R}$ has no compact open subsets. $\square$

Example 4.2. We give here two examples that complement Proposition 4.1.

(i) Simply connected solvable Lie groups can have discrete noncompact centers. This is the case for $G = \mathbb{C} \rtimes_\alpha \mathbb{R}$ where $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathbb{C})$ is defined by $\alpha(t)(z) = e^{it}z$ for $t \in \mathbb{R}$ and $z \in \mathbb{C}$. In this case $Z(G) = \{0\} \times 2\pi\mathbb{Z}$.

Nevertheless in this case $C^*(G)$ is connective. Consider the extension

$$0 \to Z(G) \to G \to G/Z(G) \cong \mathbb{C} \times \mathbb{T} \to 0.$$

Then $C^*(G)$ is a continuous $C(\mathbb{T})$-algebra whose fiber at 1 is the algebra $C^*(\mathbb{C} \times \mathbb{T})$. Since $C^*(\mathbb{C} \times \mathbb{T}) \cong C_0(\mathbb{R}^2) \otimes c_0(\mathbb{Z})$ is connective, so is $C^*(G)$.

(ii) Both the real and the complex "$ax + b$" groups

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in F^\times, b \in F \right\}$$

where $F = \mathbb{R}$ or $F = \mathbb{C}$ are solvable with trivial center and their C^*-algebras contain a copy of the compacts $\mathcal{K}$, see [33], and so they are not connective.

Theorem 4.3. *Let G be a (real or complex) linear connected nilpotent Lie group. Then $C^*(G)$ is connective if and only if G is not compact.*

Proof. We view G as a real Lie group. By [37, Chap. 2, Thm. 7.3], if G is a linear connected nilpotent Lie group, then G decomposes as a direct product $G = T \times N$ of a torus T and a simply connected nilpotent group N. If G is compact, then $G = T$ and $C^*(G)$ is isomorphic to a direct sum of $\mathbb{C}$ so that it is not connective. If G is noncompact, then N is nontrivial and so the center of G is given by $Z(G) = T \times Z(N)$, where the center $Z(N)$ of N is isomorphic to $\mathbb{R}^n$ for some $n \geq 1$. We conclude the proof by applying Proposition 4.1. □

Remark 4.4. It is not true that a liminary (CCR) C^*-algebra is connective if and only if does not have nonzero projections. Indeed

$$A = \left\{ f \in C([0,1], M_2(\mathbb{C}))\colon f(0) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \, f(1) = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \, \lambda \in \mathbb{C} \right\}$$

does not contain nonzero projections but is not connective since $\mathrm{Prim}(A)$ is homeomorphic to a circle S^1 and hence it is compact (and open in itself).

4.2. **Reductive Lie groups.** A linear connected reductive group G is a closed group of real or complex matrices that is closed under conjugate transpose. In other words G is a closed and selfadjoint subgroup of the general linear group over either $\mathbb{R}$ or $\mathbb{C}$. A linear connected semisimple group is a linear connected reductive group with finite center [28].

Say $G \subset GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$. Define $K = G \cap O(n)$, or $K = G \cap U(n)$ in the complex case. If G is linear connected reductive, then K is compact, connected and is a maximal compact subgroup of G [28, Prop.1.2].

Let $G = KAN$ be the Iwasawa decomposition of the linear connected semisimple Lie group G. A is abelian and N is nilpotent and both are closed simply connected subgroups of G, [28, Thm. 5.12].

First we consider the case of complex Lie groups.

Theorem 4.5. *If G is a linear connected complex semisimple Lie group, then $C^*_r(G)$ is connective if and only if G is not compact.*

Proof. If G is compact, $C^*_r(G)$ is isomorphic to a direct sum of matrix algebras and hence it is not connective as it contains nonzero projections.

Conversely, suppose now that G is a non-compact linear connected semisimple complex Lie group. Note that from the Cartan decomposition $G =$

KAK [28, Thm. 5.20] it follows that since G is non-compact, so is A and therefore $A \cong \mathbb{R}^n$, for some $n \geq 1$.

Let M be the centralizer of A in K. By Lemma 3.3 and Proposition 4.1 of [31], it follows that

$$C_r^*(G) \subset C_0(\widehat{M} \times \widehat{A}, \mathcal{K}).$$

Since $\widehat{M} \times \widehat{A} \cong \widehat{M} \times \mathbb{R}^n$ does not have nonempty compact open subsets, it follows from Proposition 2.7(ii) that $C_0(\widehat{M} \times \widehat{A}, \mathcal{K})$ is connective. This completes the proof since connectivity passes to C^*-subalgebras. □

Next we consider the case of linear connected real reductive Lie groups. An element $g \in G$ is semisimple if it can be diagonalized over $\mathbb{C}$ when viewed as a matrix $g \in M_n(\mathbb{C})$.

A closed subgroup H of G is called a Cartan subgroup if it is a maximal abelian subgroup consisting of semisimple elements, [21, p.67]. If G is either compact or a complex Lie group, then all Cartan subgroups of G are connected and they are conjugated inside G. In the general case G has finitely many Cartan subgroups up to conjugacy and Cartan subgroups can have finitely many connected components.

We denote by $\widehat{G}_d \subset \widehat{G}$ the discrete series representations. It consists of unitary equivalence classes of square-integrable representations

$$\sigma : G \to U(H_\sigma).$$

Harish-Chandra has shown that the discrete series representations of a semisimple Lie group G are parametrized by compact Cartan subgroups and in particular G has discrete series representations if and only if it has a compact Cartan subgroup, [19, 20].

We recall the following facts from [21, p.72] concerning cuspidal parabolic subgroups. Let H be a Cartan subgroup of G. Then H decomposes as a direct product $H = TA = T \times A$, where T is an abelian compact group and A is a vector group isomorphic to $\mathbb{R}^n$ for $n \geq 0$. The case $n = 0$ occurs when H is a compact Cartan subgroup. The centralizer of A in G denoted by

$$L = C_G(A) = \{g \in G \colon ga = ag, \forall a \in A\}$$

is a Levi subgroup of G. This means that there is a parabolic subgroup of G of the form $P = LN$ (not unique) with L as Levi subgroup. Since A is central in L, H is a relatively compact Cartan subgroup of L, i.e. $H/Z(L)$ is compact. This implies that L has discrete series representations. Such a parabolic subgroup $P = LN$ is called cuspidal.

One can further decompose $L = MA$ to obtain a Langlands decomposition

$$P = MAN = MA \ltimes N,$$

with N a unipotent group. If H is a compact Cartan subgroup, then $L = P = G$ by [21, p.72].

We will write $P = M_P A_P N_P$ whenever we want emphasize the components of P.

The description of $C^*_r(G)$ relies on the analysis of the unitary principal series representations of G associated to parabolic cuspidal subgroups P (also called the P-principal series). They are of the form

$$\mathrm{Ind}_P^G(\sigma \otimes \omega \otimes 1_N),$$

where $\sigma \in \widehat{M_d}$ and $\omega \in \widehat{A}$ and 1_N is the trivial representation of N.

Consider two pairs (P_i, σ_i), $i = 1, 2$ consisting of cuspidal parabolic subgroups of G and irreducible square-integrable unitary representations of the subgroups M_i, $i = 1.2$. We say that the pairs are *associated* if there is $g \in G$ such that $gP_1g^{-1} = P_2$ and $\sigma_1(g \cdot g^{-1})$ is unitarily equivalent to σ_2. This is an equivalence relation [5, Def. 5.2]. We denote by $[P, \sigma]$ the equivalence class of the pair (P, σ).

The following statement is based on the calculation of $C^*_r(G)$ by A. Wassermann [38] although we don't really use the full strength of his results. An expanded treatment of the structure of $C^*_r(G)$ appears in [5].

Let $J(G) = \bigcap_{\pi \in \widehat{G}_d} \ker(\pi) \subset C^*_r(G)$ be the common kernel of the discrete series representations. The following theorem shows that the K-homology of $C^*_r(G)$ can be described in terms of homotopy classes asymptotic morphisms $C^*_r(G) \to \mathcal{K}$ which factor through $J(G)$ and discrete series representations.

Theorem 4.6. *Let G be a linear connected real reductive Lie group. Then $C^*_r(G) \cong J(G) \oplus \bigoplus_{\sigma \in \widehat{G}_d} K(H_\sigma)$ and $J(G)$ is a connective liminary C^*-algebra. Moreover, the following assertions are equivalent:*

(i) *$C^*_r(G)$ is connective,*
(ii) *G does not have discrete series representations,*
(iii) *G does not have a compact Cartan subgroup,*
(iv) *There are no nonzero projections in $C^*_r(G)$.*

Proof. As explained in [38, p.560], [5, p.1306] the reduced C^*-algebra of a linear reductive connected Lie group admits an embedding

$$C^*_r(G) \hookrightarrow \bigoplus_{[P,\sigma]} C_0(\widehat{A}_P, \mathcal{K}(H_\sigma)),$$

where the direct sum is over equivalence classes $[P, \sigma]$ as above. It is important to emphasize that if G has a compact Cartan subgroup, then G itself is one of the cuspidal parabolic subgroups and we have:

$$C^*_r(G) \hookrightarrow \bigoplus_{[P,\sigma]} C_0(\widehat{A}_P, \mathcal{K}(H_\sigma)) \oplus \bigoplus_{\sigma \in \widehat{G}_d} \mathcal{K}(H_\sigma),$$

where the first direct sum involves proper cuspidal parabolic subgroups $P = M_P A_P N_P$ and hence $\dim(\widehat{A}_P) > 0$. Moreover by [38], [5]:

$$C_r^*(G) \cong J(G) \oplus \bigoplus_{\sigma \in \widehat{G}_d} \mathcal{K}(H_\sigma) \tag{9}$$

where

$$J(G) \hookrightarrow \bigoplus_{[P,\sigma]} C_0(\widehat{A}_P, \mathcal{K}(H_\sigma)) \ .$$

Hence, $J(G)$ is connective being a subalgebra of a connective C^*-algebra. The first part of the statement follows now from the decomposition (9).

The equivalence $(ii) \Leftrightarrow (iii)$ is Harish-Chandra's result mentioned earlier. In view of the decomposition (9), (ii) implies that $C_r^*(G) = J(G)$ and hence (i) since $J(G)$ is always connective. Connective C^*-algebras do not contain nonzero projections and hence $(i) \Rightarrow (iv)$. Finally by using (9) again, we see that $(iv) \Rightarrow (ii)$ since $\mathcal{K}(H_\sigma)$ contains nonzero projections if $H_\sigma \neq 0$. □

4.3. **A remark on full C^*-algebras of Lie groups.** The full C^*-algebra $C^*(G)$ of a property (T) Lie group G contains nonzero projections and hence it is not connective, see [36]. Nevertheless, inspection of several classes of examples indicates that $C^*(G)$ has interesting connective ideals that arise naturally from the representation theory of G. We postpone a detailed discussion of what is known for another time, but would like to mention two examples.

If G is a connected semisimple Lie group with finite center, then $C^*(G)$ is liminary (or CCR), see [40, p.115].

Proposition 4.7. *(a) $C^*(SL_2(\mathbb{C}))$ is connective.*
(b) $C^*(SL_3(\mathbb{C})) = I(SL_3(\mathbb{C})) \oplus \mathbb{C}$ and $I(SL_3(\mathbb{C}))$ is connective.

Proof. (a) $C^*(SL_2(\mathbb{C}))$ was computed by Fell [16, Thm. 5.4]. We describe now his result. Let Z be the subspace of $\mathbb{R}^2$ defined by $Z = \bigcup_{n=0}^{\infty} \{n\} \times L_n$ where $L_0 = (-1, \infty)$ and $L_n = (-\infty, \infty)$ for all $n \geq 1$. Endow Z with the induced topology from $\mathbb{R}^2$. Let H_0 be a separable infinite dimensional Hilbert space, let $H = H_0 \oplus \mathbb{C}$ and fix a unitary operator $V : H_0 \to H$. Then $C^*(SL_2(\mathbb{C}))$ is isomorphic to

$$\{F \in C_0(Z, \mathcal{K}(H)) \colon F(0,-1) = V^* F(2,0) V \oplus \lambda, \text{ for some } \lambda \in \mathbb{C}\}$$

Since Z has no nonempty open compact subsets it follows that $C_0(Z, \mathcal{K}(H))$ is connective and therefore so is its subalgebra $C^*(SL_2(\mathbb{C}))$.

(b) This will be obtained as a consequence of the following result on the structure of $C^*(SL_3(\mathbb{C}))$ obtained by Pierrot [32]. Let $G = SL_3(\mathbb{C})$ and denote by $\lambda_G : C^*(G) \to C_r^*(G)$ the morphism induced by the left regular representation and by $\iota_G : C^*(G) \to \mathbb{C}$ the trivial representation. Pierrot proved that the kernel J of the morphism $\lambda_G \oplus \iota_G : C^*(G) \to C^*_r(G) \oplus \mathbb{C}$ is a contractible C^*-algebra. The representation ι_G is isolated since G has property (T). Therefore there is an exact sequence

$$0 \to J \to I(G) \to C^*_r(G) \to 0$$

where J is contractible and $C^*_r(G)$ is connective by Theorem 4.5. We conclude that $I(G)$ is connective by applying Theorem 2.4.

□

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