Supplementary Material for
CofiFab: Coarse-to-Fine Fabrication of Large 3D Objects

1 Volumes of convex polyhedrons

To compute the volume $V(\mathbf{P})$ for a convex polyhedron P with vertices $\mathbf{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$, we first introduce a new vertex

$$\mathbf{p}(f_j) = \frac{1}{\sigma(j)} \sum_{k=1}^{\sigma(j)} \mathbf{p}_{j_k}$$

for every non-triangular face f_j with vertices $\mathbf{p}_{j_1}, \mathbf{p}_{j_2}, \ldots, \mathbf{p}_{j_{\sigma(j)}}$. Connecting $\mathbf{p}(f_j)$ with all vertices of f_j results in a triangulation of the polyhedron. Then the volume of the polyhedron can be computed as [Allgower and Schmidt 1986]

$$V(\mathbf{P}) = \frac{1}{6} \sum_{t_i \in \mathcal{T}} \det\left(\mathbf{p}^1(t_i), \mathbf{p}^2(t_i), \mathbf{p}^3(t_i)\right), \qquad (1)$$

where $\mathcal{T}$ is the set of faces for the triangulated polyhedron, and $\mathbf{p}^1(t_i), \mathbf{p}^2(t_i), \mathbf{p}^3(t_i)$ are the vertices of triangle t_i in positive orientation. In our optimization, the positive orientation is determined from the initial polyhedron shape, by choosing a consistent ordering of triangle vertices such that Equation (1) produces a positive value.

2 Surface sampling for convex polyhedrons

Our optimization requires sample points $\{\mathbf{q}_i\}$ on the surface of a polyhedron P, represented as $\mathbf{q}_i = \mathbf{P}\mathbf{b}_i$, where $\mathbf{b}_i \in \mathbb{R}^n$ are pre-computed convex combination coefficients with respect to the polyhedron vertex positions. To generate the samples and compute the coefficient vectors $\{\mathbf{b}_i\}$, we first triangulate the polyhedron by introducing new vertices on non-triangular faces (see Section 1). We then compute three types of sample points from the triangulated polyhedron T:

1. *Vertices of T*: such a sample point $\mathbf{q}_i$ is either a vertex of the original polyhedron P, or an interior point on a face of P. In the former case, vector $\mathbf{b}_i$ has exactly one non-zero element of value 1. In the latter case, there are $\sigma(j)$ non-zero elements in $\mathbf{b}_i$, each with value $1/\sigma(j)$, where $\sigma(j)$ is the number of vertices of the original polyhedron face that contains $\mathbf{q}_i$ (see Equation (1)).

2. *Interior points on an edge e_i of T*: such a point can be represented as a convex combination of the two vertex sample points that belongs to e_i. In our implementation, we generate K internal sample points for each edge. Let $\mathbf{q}_{i_1}, \mathbf{q}_{i_2}$ be the coefficient vectors for the two end vertex samples for e_i, then the K interior samples on e_i are computed as:

$$\mathbf{q}_j(e_i) = \frac{j}{K+1}\mathbf{q}_{i_1} + \frac{K-j+1}{K+1}\mathbf{q}_{i_2}, \quad j = 1, \ldots, K.$$

3. *Interior points on a triangle t_i of T*: such a point can be represented as a convex combination of the three vertex sample points that belongs to t_i. Let $\mathbf{q}_{i_1}, \mathbf{q}_{i_2}, \mathbf{q}_{i_3}$ be the coefficient vectors for the vertex samples, then according to the parameter K the sample points are computed as:

$$\mathbf{q}_{a,b,c}(t_i) = \frac{a}{K+1}\mathbf{q}_{i_1} + \frac{b}{K+1}\mathbf{q}_{i_2} + \frac{c}{K+1}\mathbf{q}_{i_3},$$

where $a, b, c \in \mathbb{N}$ and $a + b + c = K + 1$.

We determine the value of K from a user-specified parameter N_s for the preferred number of samples. K is chosen as the smallest number such that the total number of sample points is at least N_s.

3 Computation of centroids

To compute the centroid $\mathbf{C}$ of the final model, we consider the final model as the combination of a hollow polyhedron made from uniform thin-sheet materials, and a 3D volume shell with uniform density. Then

$$\mathbf{C} = \frac{(\mathbf{C}_1 V_1 - \mathbf{C}_3 V_3)\rho_1 + \mathbf{C}_2 A_2 \rho_2}{(V_1 - V_3)\rho_1 + A_2\rho_2},$$

where $\mathbf{C}_1, \mathbf{C}_3$ are the solid centroids of the target shape and the polyhedron, respectively; $\mathbf{C}_2$ is the surface centroid of the polyhedron; V_1, V_3 are the internal volumes of the target surface and the polyhedron, respectively; A_2 is the polyhedron surface area; ρ_1 and ρ_2 are parameters for the volume density of the 3D printed part and the area density of the laser-cut material, respectively. Here $\mathbf{V}_1, \mathbf{V}_3$ can be computed using Equation (1). Using the same notation as Equation (1), the solid centroid of a polyhedron shape can be computed as

$$\mathbf{C}(\mathbf{P}) = \frac{\sum_{t_i \in \mathcal{T}} \det\left(\mathbf{p}^1(t_i), \mathbf{p}^2(t_i), \mathbf{p}^3(t_i)\right)\left(\mathbf{p}^1(t_i) + \mathbf{p}^2(t_i) + \mathbf{p}^3(t_i)\right)}{4 \cdot \sum_{t_i \in \mathcal{T}} \det\left(\mathbf{p}^1(t_i), \mathbf{p}^2(t_i), \mathbf{p}^3(t_i)\right)}, \qquad (2)$$

while the surface area of a polyhedron is

$$A_{\mathbf{P}} = \frac{1}{2} \sum_{t_i \in \mathcal{T}} \|[\mathbf{p}^2(t_i) - \mathbf{p}^1(t_i)] \times [\mathbf{p}^3(t_i) - \mathbf{p}^1(t_i)]\|, \qquad (3)$$

and its surface centroid is

$$\mathbf{C}_A(\mathbf{P}) = \frac{\sum_{t_i \in \mathcal{T}} \|[\mathbf{p}^2(t_i) - \mathbf{p}^1(t_i)] \times [\mathbf{p}^3(t_i) - \mathbf{p}^1(t_i)]\| \sum_{k=1}^{3} \mathbf{p}^k(t_i)}{\sum_{t_i \in \mathcal{T}} \|[\mathbf{p}^2(t_i) - \mathbf{p}^1(t_i)] \times [\mathbf{p}^3(t_i) - \mathbf{p}^1(t_i)]\|}, \qquad (4)$$

$\mathbf{C}_1, \mathbf{C}_3$ are computed using formula (2), while A_s and $\mathbf{C}_2$ are computed using formulas (3) and (4), respectively.

4 Constraints for optimizing multiple polyhedrons

The two faces (f_k^i, f_l^j) chosen for the connection between two polyhedrons must satisfy the following conditions:

1. f_k^i, f_l^j are parallel, with their outward normals pointing towards each other;

2. there exists a cylinder with radius r and with its axis parallel to the normals of f_k^i, f_l^j, such that its two ends touch the two faces (f_k^i, f_l^j) and lie within the interior of each face, and the whole cylinder lie inside the target shape.

For the first condition, we require

$$\mathbf{n}_k^i + \mathbf{n}_l^j = \mathbf{0},$$

where $\mathbf{n}_k^i$ and $\mathbf{n}_l^j$ are the outward normal variables for the two faces. For the second condition, we introduce auxiliary variables $\mathbf{c}_k^i, \mathbf{c}_l^j \in \mathbb{R}^3$ for the centers of the circles, where the cylinder touches the two faces. $\mathbf{c}_k^i$ and $\mathbf{c}_l^j$ are required to lie on the two faces, respectively. The line segment between these two points must be orthogonal to the two faces, thus requiring

$$\mathbf{c}_k^i + t_k^i \mathbf{n}_k^i = \mathbf{c}_l^j,$$

with auxiliary variable $t_k^i > 0$. Moreover, each face must be kept inside a disc with radius r and center $\mathbf{c}_k^i$ (or $\mathbf{c}_l^j$, respectively). Taking face f_k^i as an example, we require

$$(\mathbf{c}_k^i - \mathbf{p}_{j_1}) \cdot \frac{\mathbf{n}_k^i \times (\mathbf{p}_{j_1} - \mathbf{p}_{j_2})}{\|\mathbf{n}_k^i \times (\mathbf{p}_{j_1} - \mathbf{p}_{j_2})\|} \geq r,$$

where $\mathbf{p}_{j_1}, \mathbf{p}_{j_2}$ are two adjacent vertices in f_k^i in an appropriate order. A similar constraint is defined for face f_l^j. Finally, we compute a set of sample points $\{\mathbf{q}\}$ on the cylinder, and enforce a constraint

$$D(\mathbf{q}) \geq d_{\min},$$

where D is the signed distance function from the surface of the whole object. Each sample $\mathbf{q}$ is computed as

$$\mathbf{q} = a\mathbf{c}_k^i + (1-a)\mathbf{n}_l^j + r(\mathbf{e}_1^{k,i} \cos b + \mathbf{e}_2^{k,i} \cos b),$$

where parameters $a \in [0,1]$ and $b \in [0, 2\pi]$ are pre-determined, $\mathbf{e}_1^{k,i}, \mathbf{e}_2^{k,i}$ are auxiliary variables that form an orthonormal frame with $\mathbf{n}_k^i$, previously used for enforcing the bounding rectangle constraints.

References

ALLGOWER, E. L., AND SCHMIDT, P. H. 1986. Computing volumes of polyhedra. *Math. Comput. 46*, 173, 171–174.