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# STOCHASTIC REPRESENTATION OF FRACTIONAL BESSEL-RIESZ MOTION

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ABSTRACT. This paper derives the stochastic solution of a Cauchy problem for the distribution of a fractional diffusion process. The governing equation involves the Bessel-Riesz derivative (in space) to model heavy tails of the distribution, and the Caputo-Djrbashian derivative (in time) to depicts the memory of the diffusion process. The solution is obtained as Brownian motion with time change in terms of the Bessel-Riesz subordinator on the inverse stable subordinator. This stochastic solution, named fractional Bessel-Riesz motion, provides a method to simulate a large class of stochastic motions with memory and heavy tails.

## 1. INTRODUCTION

Bochner (1949) and Feller (1952) demonstrated the connection between the stable distribution and fractional calculus. Specifically, Bochner (1949) proposed the Cauchy problem

$$(1.1) \quad \frac{\partial}{\partial t} p(t, x) = -(-\Delta)^{\alpha/2} p(t, x), \quad p(0, x) = \delta(x),$$

where  $\alpha \in (0, 2]$ ,  $\delta(x)$  is the Dirac delta function. The solution is the density of the symmetric  $\alpha$ -stable distribution. Feller (1952) extended (1.1) to a more general situation by replacing the fractional Laplacian  $-(-\Delta)^{\alpha/2}$  by a pseudodifferential operator with symbol  $-|\lambda|^\alpha \exp(i \operatorname{sign}(\lambda) \theta \pi/2)$ ,  $\lambda \in \mathbb{R}$ ,  $\alpha$  being the index of stability,  $\theta$  the index of skewness (asymmetry). The corresponding solutions generate all stable distributions.

Extending this class, Anh and McVinish (2004) proposed the Bessel-Riesz distribution, which is the solution to the Cauchy problem

$$(1.2) \quad \frac{\partial}{\partial t} p(t, x) = -\kappa (-\Delta)^{\alpha/2} (I - \Delta)^{\gamma/2} p(t, x), \quad p(0, x) = \delta(x), \quad \kappa > 0,$$

where  $(-\Delta)^{\alpha/2}$  and  $(I - \Delta)^{\gamma/2}$  are the inverses of the Riesz potential and the Bessel potential respectively. They showed that the solution of (1.2) defines a strongly continuous bounded holomorphic semigroup of angle  $\pi/2$  on  $L^p(\mathbb{R}^n)$  for  $\alpha > 0, \alpha + \gamma \geq 0$  and any  $p \geq 1$ . This solution is the characteristic function of a type  $G$  distribution for all  $t \geq 0$  if and only if  $\alpha \in (0, 1], \alpha + \gamma \in [0, 1]$ . This type of distribution, together with its relevance and significance, is elaborated on in Section 3. The resulting stochastic solution of (1.2), named the Bessel-Riesz motion, is then represented as  $W(L_{\alpha, \gamma}(t))$ , where  $W(t)$  is Brownian motion and  $L_{\alpha, \gamma}(t)$  is the Bessel-Riesz Lévy subordinator. The exponent  $\alpha$  indicates how often

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large jumps occur, while the combined effect of  $\alpha$  and  $\gamma$  describes the small-scale behavior of the process. Depending on the sum  $\alpha + \gamma$ , the Bessel-Riesz motion will be either a compound Poisson process, a pure jump process with jumping times dense in  $[0, \infty)$  or the sum of a compound Poisson process and an independent Brownian motion. Thus the two-parameter model (1.2) is able to generate a range of behaviors for the stochastic solution.

In this paper we extend the above setting to incorporate memory into the Bessel-Riesz motion. Specifically, we follow Anh and Leonenko (2001) and propose the Cauchy problem

$$(1.3) \quad \frac{\partial^\beta}{\partial t^\beta} p(t, x) = -\kappa (-\Delta)^{\alpha/2} (I - \Delta)^{\gamma/2} p(t, x), \quad p(0, x) = \delta(x), \quad \kappa > 0,$$

where  $t > 0$ ,  $x \in \mathbb{R}^n$ ,  $\beta \in (0, 1]$ ,  $\alpha \in (0, 2]$ ,  $\gamma \in [0, \infty)$ , and  $\partial^\beta / \partial t^\beta$  is the Caputo-Djrbashian derivative defined in (4.2) below. See also Saichev and Zaslavsky (1997) for  $\gamma = 0$ . Nigmatullin (1986) was among the earlier papers on diffusion in a porous medium with fractal geometry, specifically, Koch-tree type fractional structure. The aim of this paper is to show that the stochastic solution of (1.3) can be represented as  $W(L_{\alpha, \gamma}(Y_\beta(t)))$ , where  $W$  is the  $n$ -dimensional Brownian motion,  $L_{\alpha, \gamma}(t)$  is the Bessel-Riesz Lévy subordinator, and  $Y_\beta$  is the inverse stable subordinator. All three processes  $W$ ,  $L_{\alpha, \gamma}$ ,  $Y_\beta$  are assumed to be independent. This stochastic solution, named fractional Bessel-Riesz motion, provides a method to simulate a large class of (modified) Lévy motions with memory governed by Eq. (1.3). We will follow the approach of Baeumer and Meerschaert (2001), which derives the stochastic solution of the fractional diffusion equation. For convenience, an outline of this approach is given in the Appendix.

## 2. SYMMETRIC $\alpha$ -STABLE LÉVY PROCESS

In this section we recall some basic definitions and properties of symmetric  $\alpha$ -stable Lévy process, the subordinator, and inverse subordinator. These will be used to construct the Bessel-Riesz motion and its fractional version in the next two sections. Let  $X(t)$ ,  $t \geq 0$  be an  $\mathbb{R}^n$ -valued Lévy process with characteristic function

$$(2.1) \quad \mathbb{E} e^{i\langle z, X(t) \rangle} = e^{-t\psi(z)}, \quad z \in \mathbb{R}^n, t > 0,$$

and Lévy triple  $(a, Q, \nu)$ , where  $a \in \mathbb{R}^n$ ,  $Q$  is a non-negative definite  $(n \times n)$ -matrix, and  $\nu$  is a measure on  $\mathbb{R}^n - \{0\}$  such that  $\int_{\mathbb{R}^n} \min(1, \|z\|^2) \nu(dz) < \infty$ . The Lévy exponent  $\psi(z)$  is given by the Lévy-Khintchin representation

$$(2.2) \quad \psi(z) = i\langle a, z \rangle + \frac{1}{2} \langle z, Qz \rangle + \int_{\mathbb{R}^n} \left( 1 - e^{i\langle z, \lambda \rangle} + i\langle \lambda, \mathbf{1}_{\{\|x\| < 1\}} x \rangle \right) \nu(d\lambda).$$

The infinitesimal generator of the above Lévy process is given by

$$\begin{aligned} \mathcal{G}f(x) &= -\langle a, \nabla f(x) \rangle + \frac{1}{2} \sum_{i,j} Q_{ij} f_{ij}(x) \\ &+ \int_{\mathbb{R}^n} (f(x+y) - f(x) - \langle \mathbf{1}_{\{\|y\| < 1\}} y, \nabla f(x) \rangle) \nu(dy). \end{aligned}$$

For  $a = 0$ ,  $Q = 0$ ,  $\nu(d\lambda) = c/\|\lambda\|^{n+\alpha}$  for some constant  $c$  and some  $\alpha \in (0, 2]$ , we have  $\psi(z) = c_1 \|z\|^\alpha$  from (2.2) for some constant  $c_1$ . The corresponding Lévy process is called the symmetric  $\alpha$ -stable Lévy process in  $\mathbb{R}^n$ . Its infinitesimal generator

$\mathcal{G}$  is the fractional Laplacian  $-(-\Delta)^{\alpha/2}$ ,  $\alpha \in (0, 2]$ . This is a Markov process, and its transition density  $p_\alpha(t, x, y) = p_\alpha(t, x - y)$ ,  $t > 0, x, y \in \mathbb{R}^n$ , is determined by its Fourier transform

$$(2.3) \quad e^{-t\|z\|^\alpha} = \int_{\mathbb{R}^n} e^{i\langle z, y \rangle} p_\alpha(t, y) dy, t > 0, z \in \mathbb{R}^n,$$

that is,  $p_\alpha(t, x, y)$  satisfies the fractional Laplace equation

$$(2.4) \quad \frac{\partial}{\partial t} p_\alpha(t, x, y) = -(-\Delta)^{\alpha/2} p(t, x, y), \quad p_\alpha(0, x - y) = \delta(x - y).$$

The process has right-continuous sample paths, and the transition density satisfies the scaling property

$$p_\alpha(t, x, y) = \frac{1}{t^{n/\alpha}} p_\alpha\left(1, \frac{x}{t^{1/\alpha}}, \frac{y}{t^{1/\alpha}}\right).$$

When  $\alpha = 2$ ,  $X(t)$ ,  $t \geq 0$  is the usual  $n$ -dimensional Brownian motion, but runs at twice the speed. That is, if  $\alpha = 2$ , then  $X(t) = W(2t)$ ,  $t \geq 0$  and

$$p_2(t, x, y) = (4\pi t)^{-\frac{n}{2}} \exp\left\{-\frac{\|x - y\|^2}{4t}\right\}, t \geq 0, x, y \in \mathbb{R}^n.$$

When  $\alpha = 1$ ,  $X(t)$ ,  $t \geq 0$  is the Cauchy process in  $\mathbb{R}^n$ , whose transition density is given by the Cauchy distribution or Poisson kernel:

$$p_1(t, x, y) = \frac{c_n t}{\left(t^2 + \|x - y\|^2\right)^{\frac{n+1}{2}}}, t > 0, x \in \mathbb{R}^n, c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}.$$

Since we concentrate on the symmetric case in this paper, we will write the transition density as  $p(t, x)$  for  $p(t, x - y)$  from now on.

Let us next obtain a stochastic representation of the process defined by the Cauchy problem (2.4) using the concept of time change by a subordinator. A subordinator is a non-negative non-decreasing Lévy process starting from 0. A subordinator  $L(t)$ ,  $t \geq 0$  is characterized by its Laplace exponent

$$(2.5) \quad \phi(s) = as + \int_0^\infty (1 - e^{-s\lambda}) \nu(d\lambda),$$

where  $a \geq 0$ , and  $\nu$  is its Lévy measure, that is,  $\mathbb{E} \exp\{-sL(t)\} = \exp\{-t\phi(s)\}$ ,  $t \geq 0, s > 0$ , and

$$\int_0^\infty (1 \wedge \lambda) \nu(d\lambda) < \infty.$$

A function  $\phi : (0, \infty) \rightarrow (0, \infty)$  is the Laplace exponent of some subordinator if and only if it is the Bernstein function, which is defined by  $\phi(0+) = 0$ , and  $(-1)^m \phi^{(m)}(s) \leq 0, m = 1, 2, \dots$ . Thus, if  $\nu$  has a Lévy density  $\mu$ , then  $(-1)^m \mu^{(m)}(s) \geq 0, m = 1, 2, \dots$ , or  $\mu(s)$  is a completely monotone function, i.e., for some finite measure  $\rho$  on  $(0, \infty)$

$$(2.6) \quad \mu(s) = \int_0^\infty e^{-su} d\rho(u).$$

If  $\phi(s) = s^{\alpha/2}, \alpha \in (0, 2)$ , we call  $L(t)$  the  $\alpha$ -stable subordinator and denote it as  $L_\alpha(t)$ .

Let  $W(t)$ ,  $t \geq 0$  be the  $n$ -dimensional standard Brownian motion, and  $L(t)$ ,  $t \geq 0$  be a subordinator independent of  $W$ . Then the process  $X(t) = W(L(t))$ ,  $t \geq 0$

has Lévy exponent  $\psi(z) = \phi(\|z\|^2)$ , and its infinitesimal generator is  $-\phi(-\Delta)$ . In the case that  $L(t)$  is the  $\alpha$ -stable subordinator  $L_\alpha(t)$  with Laplace exponent  $\phi(s) = s^{\alpha/2}$ ,  $\alpha \in (0, 2)$ , independent of  $W$ , then the infinitesimal generator of the process  $X(t) = W(L_\alpha(t))$ ,  $t \geq 0$  is  $(-\Delta)^{\alpha/2}$ ,  $0 < \alpha < 2$ . It is therefore a symmetric  $\alpha$ -stable Lévy process.

### 3. THE BESSEL-RIESZ MOTION

In this section we derive a stochastic representation for the Bessel-Riesz motion defined by the Cauchy problem

$$(3.1) \quad \frac{\partial p(t, x)}{\partial t} = -\kappa(-\Delta)^{\alpha/2} (I - \Delta)^{\gamma/2} p(t, x), \quad p(0, x) = \delta(x),$$

$t > 0$ ,  $x \in \mathbb{R}^n$ ,  $\kappa > 0$ ,  $\alpha \in (0, 2]$ ,  $\gamma \in [0, \infty)$ . The spatial Fourier transform of the Green function of (3.1) is

$$(3.2) \quad \widehat{G}(t, z) = \exp \left\{ -\kappa t \|z\|^\alpha \left( 1 + \|z\|^2 \right)^{\gamma/2} \right\}, \quad z \in \mathbb{R}^n,$$

see Angulo *et al.* (2000).

We recall that a random vector  $X \in \mathbb{R}^n$  is said to be of type  $G$  if  $X \stackrel{d}{=} \sqrt{V}Z$ , where  $V \geq 0$  is an infinitely divisible random variable, and  $Z$  is the normal random vector  $\mathcal{N}_n(0, \Sigma)$  independent of  $V$ . It is known that  $X$  is infinitely divisible and  $\mathbb{E} \exp \{i \langle z, X \rangle\} = \exp \left\{ -g \left( \frac{1}{2} z' \Sigma^{-1} z \right) \right\}$ ,  $z \in \mathbb{R}^n$ , where  $g(u) = -\log \mathbb{E} \exp \{-Vu\}$ . Since  $V \geq 0$  and  $V$  is infinitely divisible, its Laplace exponent has the spectral representation (2.5) with Lévy measure  $\nu$ , and  $\mu$  is the Lévy density of  $\nu$ , represented by (2.6).

We need a result from Kelker (1971), which is formulated in Theorem 1 below.

**Theorem 1.** *The function  $\phi : (0, \infty) \rightarrow (0, \infty)$  is the Laplace exponent of some infinitely divisible distribution on  $\mathbb{R}_+$  if and only if  $\phi(s)$  is a function with a completely monotone derivative and  $\phi(0) = 0$ .*

**Theorem 2.** *Assume that*

$$(3.3) \quad \alpha \in (0, 2], \alpha + \gamma \in [0, 2].$$

*Then,*

(1) *there exists a Lévy subordinator  $L_{\alpha, \gamma}(t)$ ,  $t \geq 0$  with Laplace exponent*

$$(3.4) \quad \phi(s) = \kappa s^{\alpha/2} (1 + s)^{\gamma/2}, \quad s > 0;$$

(2) *the distribution with characteristic function (3.2) for  $t = 1$  is of type  $G$ , hence is infinitely divisible.*

*Proof.* Without loss of generality, we assume that  $\kappa = 1$ . According to Theorem 1, we have to show that the function

$$h(u) = \frac{d}{du} \left[ u^{\frac{\alpha}{2}} (1 + u)^{\frac{\gamma}{2}} \right] = u^{\frac{\alpha}{2}-1} (1 + u)^{\frac{\gamma}{2}-1} \left( \frac{\alpha}{2} + \frac{1}{2}(\alpha + \gamma)u \right)$$

is completely monotone for all  $u > 0$ . For  $m \geq 1$

$$\begin{aligned} h^{(m)}(u) &= u^{\frac{\alpha}{2}-m-1} (1 + u)^{\frac{\gamma}{2}-m-1} \\ &\times \left[ P_m(x) \left( \left( \frac{\alpha}{2} - m \right) + \left( \frac{\alpha + \gamma}{2} - 2m \right) u \right) + P'_m(u) u (1 + u) \right], \end{aligned}$$

where  $P_m(u) = \sum_{r=0}^m C_m^r u^r$  is a polynomial of degree  $m$ . By mathematical induction, one can prove that the coefficients  $C_m^r$  satisfy the following recurrence relations:

$$(3.5) \quad C_{m+1}^0 = \left(\frac{\alpha}{2} - m\right) C_m^0;$$

$$(3.6) \quad C_{m+1}^r = \left(\frac{\alpha}{2} + r - m\right) C_m^r + \left(\frac{\alpha + \gamma}{2} + r - 2m - 1\right) C_m^{r-1};$$

$$(3.7) \quad C_{m+1}^{r+1} = \left(\frac{\alpha + \gamma}{2} - m\right) C_m^m,$$

with

$$C_1^0 = \frac{\alpha}{2}, \quad C_1^1 = \frac{\alpha + \gamma}{2}.$$

Also

$$(3.8) \quad C_m^{m+1} = C_m^{m-1} \left(\frac{\alpha + \gamma}{2} - m\right).$$

From (3.5) to (3.8), it follows that all non-zero coefficients of the polynomial  $P_m(u)$  have the sign  $(-1)^{m+1}$  if (3.3) hold. Hence  $h(u)$  has a completely monotone derivative. Then both (1) and (2) follow.  $\square$

Consider the standard  $n$ -dimensional Brownian motion  $W(t)$ ,  $t \geq 0$  and the Lévy subordinator with Laplace exponent (3.4) independent of  $W$ ; then the process

$$(3.9) \quad X(t) = W(L_{\alpha,\gamma}(t)), \quad t \geq 0$$

is a Lévy process with Lévy exponent (3.2) if (3.3) holds. The infinitely divisible  $n$ -dimensional distribution with characteristic function (3.2) for  $t = 1$  is called the Bessel-Riesz distribution.

The Lévy process (3.9) is Markovian, and its transition density satisfies (3.1). The Lévy measure  $\nu$  of the subordinator  $L_{\alpha,\gamma}(t)$ ,  $t \geq 0$  with Laplace exponent (3.4) can be computed by inverting the Laplace transform in terms of Kummer's confluent hypergeometric function

$${}_1F_1(a; b; z) = \sum_{m=0}^{\infty} \frac{(a)_m z^m}{(b)_m m!}, \quad z \in \mathbb{C},$$

where  $(a)_m = a(a+1) \dots (a+m-1)$ . It has the form

$$(3.10) \quad \nu(d\lambda) = \kappa \left[ \frac{\alpha + \gamma}{2\Gamma\left(1 - \frac{\alpha + \gamma}{2}\right)} \frac{1}{\lambda^{\frac{\alpha + \gamma}{2}}} {}_1F_1\left(1 - \frac{\gamma}{2}; 2 - \frac{\alpha + \gamma}{2}; -\lambda\right) \right. \\ \left. + \frac{\alpha + \gamma}{2\Gamma\left(1 - \frac{\alpha + \gamma}{2}\right)} \frac{1}{\lambda^{\frac{\alpha + \gamma}{2} - 1}} {}_1F_1\left(1 - \frac{\gamma}{2}; 1 - \frac{\alpha + \gamma}{2}; -\lambda\right) \right] d\lambda.$$

For  $\alpha = 2$  it can be simplified to

$$\nu(d\lambda) = \frac{\kappa}{\Gamma\left(-\frac{\gamma}{2}\right)} \left( \frac{e^{-\lambda}}{\lambda^{1 + \frac{\gamma}{2}}} + \frac{e^{-\lambda}}{\lambda^{2 + \frac{\gamma}{2}} \left(1 + \frac{\gamma}{2}\right)} \right) d\lambda.$$

It follows that, with  $\alpha = 2$ , the subordinator  $L_{\alpha,\gamma}(t)$ ,  $t \geq 0$  is the sum of a compound Poisson process with intensity  $\lambda$  and gamma distributed jumps, and an exponentially tempered stable subordinator.

For  $\alpha + \gamma = 0$

$$\nu(d\lambda) = \frac{\alpha}{2} \kappa \cdot {}_1F_1\left(1 + \frac{\alpha}{2}; 2; -\lambda\right) d\lambda;$$

thus  $L_{\alpha, \gamma}(t), t \geq 0$  is a compound Poisson subordinator and also a generalized gamma convolution.

For  $\alpha + \gamma = 2$ , one can show that

$$\nu(d\lambda) = \kappa t \left( \sum_{k=0}^{\infty} \frac{1}{\lambda^{1-k}} \frac{1}{k!} \left( c\left(k+1, -\frac{\alpha}{2}\right) \right) + \frac{\alpha}{2} c\left(k, -\frac{\alpha}{2}\right) \right) d\lambda,$$

where

$$c\left(k, -\frac{\alpha}{2}\right) = (-1)^k \frac{\left(\frac{\alpha}{2}\right)_k}{(1)_k},$$

or

$$\nu(d\lambda) = \kappa t \left( \frac{\alpha}{2\lambda} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} k - \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \frac{\left(\frac{\alpha}{2}\right)_{k+1}}{(1)_{k+1}} \right) d\lambda.$$

Thus  $L_{\alpha, \gamma}(t), t \geq 0$  is again a compound Poisson process with drift  $\kappa t$ , since this Lévy measure is finite for  $\alpha + \gamma = 2$ .

A subordinator for  $L$  is self-decomposable if

$$\mathbb{E} \exp\{-sL(t)\} = \exp\left\{-cts + t \int_0^{\infty} (e^{-\lambda s} - 1) \frac{\varkappa(\lambda)}{\lambda} d\lambda\right\},$$

where  $c \geq 0$ , and  $\varkappa(\lambda)$  is a non-negative function, decreasing and right-continuous on  $(0, \infty)$ , and such that

$$(3.11) \quad \int_0^{\infty} (1 \wedge \lambda) \frac{\varkappa(\lambda)}{\lambda} d\lambda < \infty.$$

It follows that for  $\alpha + \gamma = 0$  or  $\alpha + \gamma = 2$ , the subordinator  $L_{\alpha, \gamma}(t), t \geq 0$  is not self-decomposable, while for  $\alpha = 2$ , the function

$$\varkappa(\lambda) = \frac{\kappa}{\Gamma\left(-\frac{\gamma}{2}\right)} \left( \frac{e^{-\lambda}}{\lambda^{\gamma/2}} + \frac{e^{-\lambda}}{\lambda^{1+\frac{\gamma}{2}}} \frac{1}{\left(1 + \frac{\gamma}{2}\right)} \right)$$

has derivative

$$\varkappa'(\lambda) = -\kappa \frac{1}{\Gamma\left(-\frac{\gamma}{2}\right)} \frac{e^{-\lambda}}{\lambda^{2+\frac{\gamma}{2}}} \left( \lambda^2 + (1+\gamma)\lambda + \left(1 + \frac{\gamma}{2}\right)^2 \right).$$

Thus  $\varkappa(\lambda)$  is decreasing if and only if

$$4\left(1 + \frac{\gamma}{2}\right)^2 - (1+\gamma)^2 > 0.$$

Hence  $L_{\alpha, \gamma}$  is self-decomposable if  $\gamma > -\frac{3}{2}$  for the case  $\alpha = 2$ , see Anh and McVinish (2004) for more details.

In general, the function  $\varkappa(\lambda)$  satisfies (3.11), and it follows that if  $\varkappa(\lambda) = h(\lambda)$ , then

$$\int_0^{\infty} e^{-\lambda x} (\lambda \varkappa'(\lambda)) d\lambda = -(h(x) + xh'(x)).$$

From Bernstein's Theorem (see Feller 1971, Theorem 1a of Chapter XIII-5),  $L_{\alpha, \gamma}$  is self-decomposable if and only if the function

$$x^{\frac{\alpha}{2}-1} (1+\gamma)^{\frac{\alpha}{2}-2} \left[ \left(\frac{\alpha}{2}\right)^2 + \left(\alpha \left(\frac{\alpha+\gamma}{2}\right) + \frac{\gamma}{2}\right) x + \left(\frac{\alpha+\gamma}{2}\right)^2 x^2 \right]$$

is completely monotone. The latter function is non-negative for  $\alpha > 0$  if  $\gamma < 0$  and

$$\frac{\alpha}{4} \left( (\alpha + \gamma)^2 + \gamma \right)^2 - \frac{\alpha^2}{4} (\alpha + \gamma) \geq 0.$$

Since the Bessel-Riesz process (3.9) is a subordinated Brownian motion, its Lévy measure is of the form

$$(3.12) \quad \tilde{\nu}(d\lambda) = \int_0^\infty \exp\left(-\frac{\|\lambda\|^2}{4s}\right) \frac{1}{(4\pi s)^{n/2}} \nu(ds) d\lambda, \quad \lambda \in \mathbb{R}^n,$$

where the Lévy measure  $\nu$  of the subordinator  $L_{\alpha,\gamma}$  is given by (3.10). It is self-decomposable if  $\alpha = 2$ ,  $\gamma > -\frac{3}{2}$ . As with the Lévy subordinator  $L_{\alpha,\gamma}(t)$ ,  $t \geq 0$ , the qualitative behavior of the paths of the Bessel-Riesz process  $W(L_{\alpha,\gamma}(t))$ ,  $t \geq 0$  changes with the value of  $\alpha + \gamma$ . For  $\alpha + \gamma < 2$ ,  $L_{\alpha,\gamma}(t)$ ,  $t \geq 0$  and  $W(L_{\alpha,\gamma}(t))$ ,  $t \geq 0$  display similar behavior. For  $\alpha + \gamma = 2$ ,  $W(L_{\alpha,\gamma}(t))$ ,  $t \geq 0$  is the sum of a compound Poisson process and an independent Brownian motion.

#### 4. THE FRACTIONAL BESSEL-RIESZ MOTION

We now consider the Cauchy problem

$$(4.1) \quad \frac{\partial^\beta}{\partial t^\beta} p(t, x) = -\kappa (-\Delta)^{\alpha/2} (I - \Delta)^{\gamma/2} p(t, x), \quad p(0, x) = \delta(x),$$

where  $t > 0$ ,  $x \in \mathbb{R}^n$ ,  $\kappa > 0$ ,  $0 < \beta \leq 1$ ,  $\alpha \in (0, 2]$ ,  $\gamma \in [0, \infty)$ , and  $\partial^\beta / \partial t^\beta$  is the Caputo-Djrbashian fractional derivative:

$$(4.2) \quad \frac{\partial^\beta}{\partial t^\beta} p(t, x) = \begin{cases} \frac{\partial}{\partial t} p(t, x), & \text{if } \beta = 1 \\ \frac{1}{\Gamma(1-\beta)} \left[ \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\beta} u(\tau, x) d\tau - \frac{p(0,x)}{t^\beta} \right], & \text{if } 0 < \beta < 1 \end{cases}$$

(see, e.g., Anh and Leonenko (2001), Mainardi (2010), Meerschaert and Sikorskii (2012)).

From Anh and Leonenko (2001), the Green function of the initial-value problem (4.1) is of the form

$$(4.3) \quad G(t, x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}} e^{i\langle \lambda, x \rangle} E_\beta \left( -\kappa t^\beta \|\lambda\|^\alpha \left( 1 + \|\lambda\|^2 \right)^{\gamma/2} \right) d\lambda,$$

where

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)}, \quad z \in \mathbb{C}, 0 < \beta \leq 1,$$

is the one-parameter Mittag-Leffler function.

We now consider the Lévy subordinator  $L_\beta(t)$ ,  $t \geq 0$  with Laplace exponent given by

$$\mathbb{E} e^{-sL_\beta(1)} = e^{-s^\beta}, \quad s > 0, 0 < \beta < 1,$$

and the inverse stable subordinator

$$(4.4) \quad Y_\beta(t) = \inf \{u > 0, L_\beta(u) > t\}, \quad t > 0.$$

It is known (see, e.g., Meerschaert and Sikorskii (2012), Leonenko, Meerschaert and Sikorskii (2013a)) that

$$\mathbb{E} e^{-sY_\beta(t)} = E_\beta(-st^\beta), \quad s > 0,$$



and the process  $Y_\beta(t)$ ,  $t > 0$  has density

$$(4.5) \quad h(t, x) = \frac{d}{dx} \mathbb{P} \{Y_\beta(t) \leq x\} = \frac{t}{\beta} \frac{1}{x^{1+\frac{1}{\beta}}} g_\beta \left( \frac{t}{x^{\frac{1}{\beta}}} \right),$$

where

$$(4.6) \quad g_\beta(x) = \frac{d}{dx} \mathbb{P} (L_\beta(1) \leq x) = \frac{1}{\pi} \sum_{\kappa=0}^{\infty} (-1)^{\kappa+1} \frac{\Gamma(\beta\kappa+1)}{\kappa!} \frac{1}{x^{\beta\kappa+1}} \sin \pi\kappa\beta.$$

Note that

$$g_\beta(x) = \frac{\beta}{\Gamma(1-\beta)} \frac{1}{x^{1+\beta}} (1 + o(1)), \quad x \rightarrow \infty.$$

From (4.3) and (3.9), we obtain the following result on the stochastic solution of (4.1)

**Theorem 3.** *Assume that*

$$\alpha \in (0, 2], \quad \beta \in (0, 1), \quad \gamma \in [0, \infty), \quad \alpha + \gamma \in [0, 2],$$

$W$  is the  $n$ -dimensional Brownian motion,  $L_{\alpha,\gamma}$  is the Lévy subordinator described in Theorem 2,  $Y_\beta$  is the inverse stable subordinator (4.4), and all three processes  $W$ ,  $L_{\alpha,\gamma}$ , and  $Y_\beta$  are jointly independent.

*Then the stochastic process*

$$(4.7) \quad X_{BR}(t) = W(L_{\alpha,\gamma}(Y_\beta(t))), \quad t > 0$$

*is the stochastic solution of (4.1), that is, its density function (or propagator)  $p(t, x)$ ,  $t > 0$ ,  $x \in \mathbb{R}$  satisfies the fractional Bessel-Riesz differential equation (4.1) with the point-source initial condition.*

The process  $Y_\beta$  is not Markovian with non-stationary and non-independent increments. The process  $W(L_{\alpha,\gamma}(Y_\beta(t)))$ , which provides the stochastic solution to the fractional differential equation (4.1), is not Markovian and not a Lévy process unless  $\beta = 1$ , in which case it reduces to the Lévy process (3.9).

If  $n = 1$ ,

$$(4.8) \quad p(t, x) = \frac{d}{dx} \mathbb{P} \{X_{BL}(t) \leq x\} = \int_0^\infty h(t, \theta) w(\theta, x) d\theta,$$

where

$$w(\theta, x) = \frac{d}{dx} \mathbb{P} \{W(L_{\alpha,\gamma}(\theta)) \leq x\},$$

and the function

$$h(t, \theta) = \frac{d}{d\theta} \mathbb{P} \{Y_\beta(t) \leq \theta\}$$

is given by (4.5). By inverting the characteristic function (3.2) for  $n = 1$ , one has

$$(4.9) \quad w(t, x) = \int_{\mathbb{R}} e^{ixz} e^{-\kappa t |z|^\alpha (1+|z|^2)^{\gamma/2}} dz.$$

Thus, the formula (4.8) with (4.9) and (4.5) gives an explicit solution of the fractional equation (4.1) for  $n = 1$ .

We now address the existence and asymptotic behavior of the correlation function of  $X_{BR}(t)$  for  $n = 1$ . The characteristic function of  $X_{BR}(t)$  is

$$\mathbb{E} e^{i\lambda X_{BR}(t)} = E_\beta \left( -\kappa t^\beta \|\lambda\|^\alpha \left( 1 + \|\lambda\|^2 \right)^{\gamma/2} \right),$$

and it is twice continuously differentiable at  $\lambda = 0$  for  $\alpha \geq 2$ . The application of the parameter restrictions from (4.1) yields  $\alpha = 2$  and  $\gamma = 0$  for the case with finite second moment, so that the correlation function is well-defined. In this case the stochastic solution of (4.1) is  $W(2\kappa Y_\beta(t))$ , the Brownian motion time-changed by the inverse stable subordinator. The correlation function of this process has been considered in Janczura and Wilomanska (2009) and Leonenko *et al.* (2014), and for  $0 < s \leq t$ ,

$$\text{corr}[Z(t), Z(s)] = \left(\frac{s}{t}\right)^{\beta/2}.$$

With this power-law decay, the correlation function is not integrable at infinity for  $0 < \beta < 1$ , which can be viewed as the long-range dependence of the process  $W(2\kappa Y_\beta(t))$ .

## 5. APPENDIX: CAUCHY PROBLEM FOR FRACTIONAL DIFFUSION

This appendix presents a key result of Baeumer and Meerschaert (2001) on stochastic solution of the fractional diffusion equation (see also Baeumer, Meerschaert and Nane (2001), Leonenko, Meerschaert and Sikorskii (2013a, b) for related results). We formulate their result in the context most relevant to this paper. An  $\mathbb{R}^n$ -valued Lévy process  $X(t), t \geq 0$  is said to be a stochastic solution of the Cauchy problem

$$\frac{\partial}{\partial t} u(t, x) = \mathcal{G}f(x), \quad u(0, x) = f(x), \quad t > 0, x \in \mathbb{R}^n,$$

when the Cauchy problem is solved by

$$u(t, x) = T(t)f(x) = \int p(t, x - y)f(y)dy,$$

where  $p(x, t)$  is the density of  $X(t)$ , and operator  $\mathcal{G}$  is the generator of the bounded strongly continuous semigroup  $\{T(t), t \geq 0\}$  in the space  $L^1(\mathbb{R}^n)$ :

$$\mathcal{G}f(x) = \lim_{h \downarrow 0} \frac{T(h)f(x) - f(x)}{h}.$$

This operator is defined on a dense subset of this space (see, for example, Pazy (1983), Arendt *et al.* (2001)), and  $f$  belongs to its domain.

In this setting, Theorem 3.1 of Baeumer and Meerschaert (2001) gives the unique strong solution (in the space of continuity of the semigroup) for the fractional Cauchy problem

$$\frac{\partial^\beta}{\partial t^\beta} u(t, x) = \mathcal{G}u(t, x), \quad u(0, x) = f(x).$$

The solution has the following form:

$$u(t, x) = \int \int_0^\infty p((t/s)^\beta, x - y) g_\beta(s) f(y) ds dy.$$

Here  $g_\beta(t)$  is the density (4.6) of  $L(1)$ , the standard stable subordinator  $\{L_\beta(t), t \geq 0\}$  evaluated at  $t = 1$ . The solution of the fractional Cauchy problem can also be written as

$$u(t, x) = \int \int_0^\infty p(z, x - y) g_\beta\left(\frac{t}{z^\beta}\right) \frac{1}{z^{1+\frac{1}{\beta}}} f(y) dz dy,$$

or

$$u(t, x) = \int \int_0^\infty p(z, x - y) h(t, z) f(y) dz dy,$$

where  $h(t, \cdot)$  is the density of the inverse stable subordinator  $Y_\beta(t)$ . This shows that the density of the non-Markovian process  $Z(t) = X(Y_\beta(t))$ ,  $t > 0$ , where  $X$  and  $Y_\beta$  are independent,

$$q(t, x) = \int_0^\infty p(z, x) h(t, z) dz,$$

is the fundamental solution of this fractional Cauchy problem.

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