Integrality Gaps of Integer Knapsack Problems

Iskander Aliev[1], Martin Henk[2], and Timm Oertel[3]

[1] Mathematics Institute, Cardiff University, UK
alievi@cardiff.ac.uk
[2] Department of Mathematics, TU Berlin, Germany
henk@math.tu-berlin.de
[3] Mathematics Institute, Cardiff University, UK
oertelt@cardiff.ac.uk

Abstract. We obtain optimal lower and upper bounds for the (additive) integrality gaps of integer knapsack problems. In a randomised setting, we show that the integrality gap of a "typical" knapsack problem is drastically smaller than the integrality gap that occurs in a worst case scenario.

1 Introduction

Given an integer $m \times n$ matrix A, integer vector $\boldsymbol{b} \in \mathbb{Z}^m$ and a cost vector $\boldsymbol{c} \in \mathbb{Q}^n$, consider the linear integer programming problem

$$\min\{\boldsymbol{c} \cdot \boldsymbol{x} : A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \in \mathbb{Z}^n_{\geq 0}\}\,. \tag{1}$$

The linear programming relaxation to (1) is obtained by dropping the integrality constraint

$$\min\{\boldsymbol{c} \cdot \boldsymbol{x} : A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \in \mathbb{R}^n_{\geq 0}\}\,. \tag{2}$$

We will denote by $IP_{\boldsymbol{c}}(A, \boldsymbol{b})$ and $LP_{\boldsymbol{c}}(A, \boldsymbol{b})$ the optimal values of (1) and (2), respectively.

While the problem (2) is polynomial time solvable [20], it is well known that (1) is NP-hard [14]. There are many examples, where relaxation on the integrality constraints are used to approximate, or even to solve, integer programming problems. Prominent examples can be found in the areas of cutting plane algorithms, such us Gomory cuts [15], and approximation algorithms for combinatorial problems. For further details see [3], [8] and [28]. Therefore, a natural question is to compare the optimal values $IP_{\boldsymbol{c}}$ and $LP_{\boldsymbol{c}}$ with each other.

Suppose that (1) is feasible and bounded. The *(additive) integrality gap* $IG_{\boldsymbol{c}}(A, \boldsymbol{b})$ is a fundamental characteristic of the problem (1), defined as

$$IG_{\boldsymbol{c}}(A, \boldsymbol{b}) = IP_{\boldsymbol{c}}(A, \boldsymbol{b}) - LP_{\boldsymbol{c}}(A, \boldsymbol{b})\,.$$

The problem of computing bounds for the additive integrality gaps has been studied by Hoşten and Sturmfels [18], Sullivant [27], Eisenbrand and Shmonin [12] and, more recently, by Eisenbrand et al [11]. Specifically, given a tuple $(A, \boldsymbol{c})$ one asks for the upper bounds on $IG_{\boldsymbol{c}}(A, \boldsymbol{b})$ as $\boldsymbol{b}$ varies. In this setting, the optimal bound is given by the *integer programming gap* $\mathrm{Gap}_{\boldsymbol{c}}(A)$, defined by Hoşten and Sturmfels [18] as

$$\mathrm{Gap}_{\boldsymbol{c}}(A) = \max_{\boldsymbol{b}} IG_{\boldsymbol{c}}(A, \boldsymbol{b})\,,$$

where $\boldsymbol{b}$ ranges over integer vectors such that (1) is feasible and bounded. Note that, $\mathrm{Gap}_c(A) = 0$ for all $c \in \mathbb{Z}^n$, if and only if A is totally unimodular [25, Theorem 19.2].

Hoşten and Sturmfels [18] showed that for fixed n the value of $\mathrm{Gap}_{\boldsymbol{c}}(A)$ can be computed in polynomial time. Eisenbrand and Shmonin [12] extended this result to integer programs in the canonical form.

Eisenbrand et al [11] studied a closely related problem of testing upper bounds for $IG_{\boldsymbol{c}}(A,\boldsymbol{b})$ in context of a generalised *integer rounding property*. Following [11], the tuple $(A,\boldsymbol{c})$ with $\boldsymbol{c}\in\mathbb{Z}^n$ has the *additive integrality gap of at most* γ if

$$IP_{\boldsymbol{c}}(A,\boldsymbol{b}) \le \lceil LP_{\boldsymbol{c}}(A,\boldsymbol{b})\rceil + \gamma$$

for each $\boldsymbol{b}$ for which the linear programming relaxation (2) is feasible.

The classical case $\gamma = 0$ corresponds to the integer rounding property and can be tested in polynomial time [25, Section 22.10]. The integer rounding property, in its turn, implies solvability of (1) in polynomial time [7]. The computational complexity of the problem drastically changes already for $\gamma = 1$. Eisenbrand et al [11] showed that it is NP-hard to test whether $(A,\boldsymbol{c})$ has additive gap of at most γ even if $m = \gamma = 1$.

A bound for the additive integrality gap in terms of A and $\boldsymbol{c}$ can be derived from the results of Cook et al [9] on distances between optimal solutions to integer programs in canonical form and their linear programming relaxations. Let $\hat{A}$ be an integer $d\times n$ matrix and let $\hat{\boldsymbol{b}}$ and $\boldsymbol{c}$ be rational vectors such that $\hat{A}\boldsymbol{x}\le\hat{\boldsymbol{b}}$ has an integer solution and $\min\{\boldsymbol{c}\cdot\boldsymbol{x} : \hat{A}\boldsymbol{x}\le\hat{\boldsymbol{b}}, \boldsymbol{x}\in\mathbb{R}^n\}$ exists. Note that, in this setting $\hat{\boldsymbol{b}}$ is not required to be integer. Then Corollary 2 in [9], applied in the minimisation setting, gives the bound

$$\begin{aligned}\min\{\boldsymbol{c}\cdot\boldsymbol{x} : \hat{A}\boldsymbol{x}\le\hat{\boldsymbol{b}}, \boldsymbol{x}\in\mathbb{Z}^n\} - \min\{\boldsymbol{c}\cdot\boldsymbol{x} : \hat{A}\boldsymbol{x}\le\hat{\boldsymbol{b}}, \boldsymbol{x}\in\mathbb{R}^n\}\\ \le n\Delta(A)\|\boldsymbol{c}\|_1\,,\end{aligned} \tag{3}$$

where $\Delta(A)$ stands for the maximum sub-determinant of A and $\|\boldsymbol{c}\|_1 = \sum_{i=1}^n |c_i|$ denotes the l_1-*norm* of $\boldsymbol{c}$. The estimate (3) strengthened previous results of Blair and Jeroslow [4], [5]. Given that $\hat{\boldsymbol{b}}$ does not have to be integer, one can show that the bound (3) is essentially tight (see Remark 1). However, considering that we study linear integer programming, it is natural to assume that also $\hat{\boldsymbol{b}}$ is integer, but then it is not clear whether (3) remains optimal. By studying linear integer programming problems in standard form we naturally require $\boldsymbol{b}$ and respectively $\hat{\boldsymbol{b}}$ to be integer.

This paper will focus on the problem (1) with $m = 1$, to which we refer to as the *integer knapsack problem*. Note that, usually the integer knapsack problem is defined in the literature as $\min\{\bar{\boldsymbol{c}}\cdot\boldsymbol{x} : \bar{A}\boldsymbol{x}\le\boldsymbol{b}, \boldsymbol{x}\in\mathbb{Z}^n_{\ge 0}\}$. However, this problem can be brought into standard form (1), by lifting the polytope by one dimension and defining $A = (\bar{A}\ 1)$ and $\boldsymbol{c} = \binom{\bar{c}}{0}$. We will assume that the entries of A are positive. For the integer knapsack problem the positivity assumption guarantees that the feasible region of its linear programming relaxation (2) is bounded (or empty) for all $\boldsymbol{b}$. Conversely, for $m = 1$ any linear problem (2) with bounded feasible region can be written with A satisfying the positivity assumption. Without loss of generality, we also assume that $n\ge 2$ and the entries of A are coprime. That is the following conditions are assumed to hold:

$$\begin{aligned}&(i)\ \ A = (a_1,\ldots,a_n)\,, n\ge 2\,, a_i\in\mathbb{Z}_{>0}\,, i = 1,\ldots,n\,,\\ &(ii)\ \gcd(a_1,\ldots,a_n) = 1\,.\end{aligned} \tag{4}$$

For $A\in\mathbb{Z}^{1\times n}$ we denote by $\|A\|_\infty$ its *maximum norm*, i.e., $\|A\|_\infty = \max_{i=1,\ldots,n}|a_i|$. Applying (3) with

$$\hat{A} = \begin{pmatrix} A\\ -A\\ -I_n\end{pmatrix}, \hat{\boldsymbol{b}} = \begin{pmatrix} b\\ -b\\ \boldsymbol{0}\end{pmatrix},$$

where I_n is the $n \times n$ identity matrix and $\mathbf{0}$ is the n dimensional zero vector, we obtain the bound

$$\mathrm{Gap}_{\boldsymbol{c}}(A) \le n\|A\|_\infty \|\boldsymbol{c}\|_1 \,. \tag{5}$$

How far is the bound (5) from being optimal? Does $\mathrm{Gap}_{\boldsymbol{c}}(A)$ admit a natural lower bound? To answer these questions we will establish a link between the integer programming gaps, covering radii of simplices and Frobenius numbers. Our first result gives an upper bound on the integer programming gap that improves (5) with factor $1/n$. We also show that the obtained bound is optimal.

Theorem 1

(i) Let A satisfy (4) and let $\boldsymbol{c} \in \mathbb{Q}^n$. Then

$$\mathrm{Gap}_{\boldsymbol{c}}(A) \le (\|A\|_\infty - 1)\, \|\boldsymbol{c}\|_1 \,. \tag{6}$$

(ii) For any positive integer k there exist A with $\|A\|_\infty = k$ satisfying (4) and $\boldsymbol{c} \in \mathbb{Q}^n$ such that

$$\mathrm{Gap}_{\boldsymbol{c}}(A) = (\|A\|_\infty - 1)\, \|\boldsymbol{c}\|_1 \,. \tag{7}$$

We will say that the tuple $(A, \boldsymbol{c})$ is *generic* if for any positive $b \in \mathbb{Z}$ the linear programming relaxation (2) has a unique optimal solution. An optimal lower bound for $\mathrm{Gap}_{\boldsymbol{c}}(A)$ with generic $(A, \boldsymbol{c})$ can be obtained using recent results [1] on the *lattice programming gaps* associated with the group relaxations to (1).

A subset τ of $\{1, \ldots, n\}$ partitions $\boldsymbol{x} \in \mathbb{R}^n$ as $\boldsymbol{x}_\tau$ and $\boldsymbol{x}_{\bar\tau}$, where $\boldsymbol{x}_\tau$ consists of the entries indexed by τ and $\boldsymbol{x}_{\bar\tau}$ the entries indexed by the complimentary set $\bar\tau = \{1, \ldots, n\} \setminus \tau$. Similarly, the matrix A is partitioned as A_τ and $A_{\bar\tau}$. Assume that (A, c) is generic and (4) holds. Then, let $\tau = \tau(A, \boldsymbol{c})$ denote the unique index of the basic variable for the optimal solution to the linear relaxation (2) with a positive $b \in \mathbb{Z}$. The index τ is well-defined. We also define $\boldsymbol{l}(A, \boldsymbol{c}) = \boldsymbol{c}_{\bar\tau} - \boldsymbol{c}_\tau A_\tau^{-1} A_{\bar\tau}$. Note that the vector $\boldsymbol{l} = \boldsymbol{l}(A, \boldsymbol{c})$ is positive for generic tuples $(A, \boldsymbol{c})$.

Let ρ_d denote the *covering constant* of the standard d-dimensional simplex, defined in Section 2.

Theorem 2

(i) Let A satisfy (4) and let $\boldsymbol{c} \in \mathbb{Q}^n$. Suppose that $(A, \boldsymbol{c})$ is generic. Then for $\tau = \tau(A, \boldsymbol{c})$ and $\boldsymbol{l} = \boldsymbol{l}(A, \boldsymbol{c})$ we have

$$\mathrm{Gap}_{\boldsymbol{c}}(A) \ge \rho_{n-1}(|A_\tau| l_1 \cdots l_{n-1})^{1/(n-1)} - \|\boldsymbol{l}\|_1 \,. \tag{8}$$

(ii) For any $\epsilon > 0$, there exists a matrix A, satisfying (4) and $\boldsymbol{c} \in \mathbb{Q}^n$ such that $(A, \boldsymbol{c})$ is generic and, in the notation of part (i), we have

$$\mathrm{Gap}_{\boldsymbol{c}}(A) < (\rho_{n-1} + \epsilon)(|A_\tau| l_1 \cdots l_{n-1})^{1/(n-1)} - \|\boldsymbol{l}\|_1 \,. \tag{9}$$

The only known values of ρ_d are $\rho_1 = 1$ and $\rho_2 = \sqrt{3}$ (see [13]). It was proved in [2], that $\rho_d > (d!)^{1/d} > d/\mathrm{e}$. For sufficiently large d this bound is not far from being optimal. Indeed, $\rho_d \le (d!)^{1/d}(1 + O(d^{-1} \log d))$ (see [10] and [21]).

How large is the integer programming gap of a "typical" knapsack problem? To tackle this question we will utilize the recent strong results of Strömbergsson [26] (see also Schmidt

[24] and references therein) on the asymptotic distribution of Frobenius numbers. The main result of this paper will show that for any $\epsilon > 2/n$ the ratio

$$\frac{\mathrm{Gap}_{\boldsymbol{c}}(A)}{\|A\|_\infty^\epsilon \|\boldsymbol{c}\|_1}$$

is bounded, on average, by a constant that depends only on dimension n. Hence, for fixed $n > 2$ and a "typical" integer knapsack problem with large $\|A\|_\infty$, its linear programming relaxation provides a drastically better approximation to the solution than in the worst case scenario, determined by the optimal upper bound (6).

For $T \geq 1$, let $Q(T)$ be the set of $A \in \mathbb{Z}^{1\times n}$ that satisfy (4) and

$$\|A\|_\infty \leq T\,.$$

Let $N(T)$ be the cardinality of $Q(T)$. For $\epsilon \in (0,1)$ let

$$N_\epsilon(t,T) = \#\left\{ A \in Q(T) : \max_{\boldsymbol{c}\in\mathbb{Q}^n} \frac{\mathrm{Gap}_{\boldsymbol{c}}(A)}{\|A\|_\infty^\epsilon \|\boldsymbol{c}\|_1} > t \right\}. \tag{10}$$

In what follows, $\ll_n$ will denote the Vinogradov symbol with the constant depending on n. That is $f \ll_n g$ if and only if $|f| \leq c|g|$, for some positive constant $c = c(n)$. The notation $f \asymp_n g$ means that both $f \ll_n g$ and $g \ll_n f$ hold.

Theorem 3 *For $n \geq 3$*

$$\frac{N_\epsilon(t,T)}{N(T)} \ll_n t^{-\alpha(\epsilon,n)} \tag{11}$$

uniformly over all $t > 0$ and $T \geq 1$. Here

$$\alpha(\epsilon,n) = \frac{n-2}{(1-\epsilon)n}\,.$$

From (11) one can derive an upper bound on the average value of the (normalised) integer programming gap.

Corollary 4 *Let $n \geq 3$. For $\epsilon > 2/n$*

$$\frac{1}{N(T)} \sum_{A\in Q(T)} \max_{\boldsymbol{c}\in\mathbb{Q}^n} \frac{\mathrm{Gap}_{\boldsymbol{c}}(A)}{\|A\|_\infty^\epsilon \|\boldsymbol{c}\|_1} \ll_n 1\,. \tag{12}$$

The last theorem of this paper shows that the bound in Corollary 4 is not far from being optimal. We include its proof in the Appendix.

Theorem 5 *For T large*

$$\frac{1}{N(T)} \sum_{A\in Q(T)} \max_{\boldsymbol{c}\in\mathbb{Q}^n} \frac{\mathrm{Gap}_{\boldsymbol{c}}(A)}{\|A\|_\infty^{1/(n-1)} \|\boldsymbol{c}\|_1} \gg_n 1\,. \tag{13}$$

Hence, the optimal value of ϵ in (12) cannot be smaller than $1/(n-1)$.

Remark 1.

(i) An example due to L. Lovász [25, Section 17.2], with $\varDelta(A) = 1$, shows that the bound (3) is best possible in this particular case. We would like to point out that by a small adaptation of Lovász's example one can show that this bound is, in all its generality, best possible up to a constant factor, i.e., the upper bound for the additive integrality gap is in $\varTheta(\varDelta(A)n)$. Let $\delta \in \mathbb{Z}_{>0}$ and $0 < \beta < 1$. We define

$$A = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & & \ddots & & \\ & & -1 & 1 & \\ & & & -\delta & 1 \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} \beta \\ \vdots \\ \beta \end{pmatrix} \text{ and } \boldsymbol{c} = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix}.$$

By construction $\varDelta(A) = \delta$. The unique solution of the linear relaxation is $\boldsymbol{x}^T = (\beta, 2\beta, \ldots, (n-1)\beta, (\delta(n-1)+1)\beta)$ and the unique optimal integer solution is $\boldsymbol{z}^T = (0, \ldots, 0)$. Thus $\|\boldsymbol{x} - \boldsymbol{z}\|_\infty = (\delta(n-1)+1)\beta \approx n\varDelta(A)$.

(ii) In the proof of Theorem 1 (and, subsequently, Theorem 3) we estimate the integrality gap using a covering argument that guarantees existence of a solution to (1) in an $(n-1)$-dimensional simplex of sufficiently small diameter, translated by a solution to (2). Here the diameter of the simplex is independent of $\boldsymbol{c}$. The argument allows us, in particular, to restate Theorem 1 (i) in terms of the infinity norm:

$$\mathrm{Gap}_{\boldsymbol{c}}(A) \leq 2\left(\|A\|_\infty - 1\right)\|\boldsymbol{c}\|_\infty .$$

Depending on $\boldsymbol{c}$ this gives a stronger bound.

2 Coverings and Frobenius numbers

In what follows, $\mathcal{K}^d$ will denote the space of all d-dimensional *convex bodies*, i.e., closed bounded convex sets with non-empty interior in the d-dimensional Euclidean space $\mathbb{R}^d$.

By $\mathcal{L}^d$ we denote the set of all d-dimensional lattices in $\mathbb{R}^d$. Given a matrix $B \in \mathbb{R}^{d\times d}$ with $\det B \neq 0$ and a set $S \subset \mathbb{R}^d$ let $BS = \{B\boldsymbol{x} : \boldsymbol{x} \in S\}$ be the image of S under linear map defined by B. Then we can write $\mathcal{L}^d = \{B\,\mathbb{Z}^d : B \in \mathbb{R}^{d\times d}, \det B \neq 0\}$. For $\varLambda = B\,\mathbb{Z}^d \in \mathcal{L}^d$, $\det(\varLambda) = |\det B|$ is called the *determinant* of the lattice $\varLambda$.

Recall that the *Minkowski sum* $X + Y$ of the sets $X, Y \subset \mathbb{R}^d$ consists of all points $\boldsymbol{x} + \boldsymbol{y}$ with $\boldsymbol{x} \in X$ and $\boldsymbol{y} \in Y$. For $K \in \mathcal{K}^d$ and $\varLambda \in \mathcal{L}^d$ the *covering radius* of K with respect to $\varLambda$ is the smallest positive number μ such that any point $\boldsymbol{x} \in \mathbb{R}^d$ is covered by $\mu K + \varLambda$, that is

$$\mu(K, \varLambda) = \min\{\mu > 0 : \mathbb{R}^d = \mu K + \varLambda\}.$$

For further information on covering radii in the context of the geometry of numbers see e.g. Gruber [16] and Gruber and Lekkerkerker [17].

Let $\varDelta = \{\boldsymbol{x} \in \mathbb{R}^d_{\geq 0} : x_1 + \cdots + x_d \leq 1\}$ be the standard d-dimensional simplex. The optimal lower bound in Theorem 2 is expressed using the covering constant $\rho_d = \rho_d(\varDelta)$ defined as

$$\rho_d = \inf\{\mu(\varDelta, \varLambda) : \det(\varLambda) = 1\}.$$

We will be also interested in coverings of $\mathbb{Z}^d$ by lattice translates of convex bodies. For this purpose we define

$$\mu(K, \varLambda; \mathbb{Z}^d) = \min\{\mu > 0 : \mathbb{Z}^d \subset \mu K + \varLambda\}.$$

Given $A = (a_1, \dots, a_n)$ satisfying (4) the *Frobenius number* $g(A)$ is least so that every integer $b > g(A)$ can be represented as $b = a_1 x_1 + \cdots + a_n x_n$ with nonnegative integers $x_1, \dots, x_n$.

Kannan [19] found a nice and very useful connection between $g(A)$ and geometry of numbers. Let us consider the $(n-1)$-dimensional simplex

$$S_A = \left\{ \boldsymbol{x} \in \mathbb{R}^{n-1}_{\geq 0} : a_1\, x_1 + \cdots + a_{n-1}\, x_{n-1} \leq 1 \right\}$$

and the $(n-1)$-dimensional lattice

$$\Lambda_A = \left\{ \boldsymbol{x} \in \mathbb{Z}^{n-1} : a_1\, x_1 + \cdots + a_{n-1}\, x_{n-1} \equiv 0 \bmod a_n \right\}.$$

Kannan [19] established the identities

$$\mu(S_A, \Lambda_A) = g(A) + a_1 + \cdots + a_n$$

and

$$\mu(S_A, \Lambda_A; \mathbb{Z}^{n-1}) = g(A) + a_n. \tag{14}$$

3 Proof of Theorem 1

The proof of the upper bound in part (i) will be based on two auxiliary lemmas. First we will need the following property of $\mu(K, \Lambda; \mathbb{Z}^{n-1})$.

Lemma 1. *For any $\boldsymbol{y} \in \mathbb{Z}^{n-1}$ the set $\mu(K, \Lambda; \mathbb{Z}^{n-1})K$ contains a point of the translated lattice $\boldsymbol{y} + \Lambda$.*

Proof. By the definition of $\mu(K, \Lambda; \mathbb{Z}^{n-1})$ we have $\mathbb{Z}^{n-1} \subset \mu(K, \Lambda; \mathbb{Z}^{n-1})K + \Lambda$. Therefore for any integer vector $\boldsymbol{y}$ we have $(\boldsymbol{y} + \Lambda) \cap \mu(K, \Lambda; \mathbb{Z}^{n-1})K \neq \emptyset$. □

The next lemma gives an upper bound for the integer programming gap in terms of the Frobenius number associated with vector A.

Lemma 2. *For A satisfying (4) and $\boldsymbol{c} \in \mathbb{Q}^n$*

$$\mathrm{Gap}_{\boldsymbol{c}}(A) \leq \frac{(g(A) + \|A\|_\infty)\|\boldsymbol{c}\|_1}{\min_i a_i}. \tag{15}$$

Proof. Let b be a nonnegative integer. Consider the *knapsack polytope*

$$P(A, b) = \{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : A\boldsymbol{x} = b\}.$$

Clearly, $P(A, b)$ is a simplex with vertices

$$(b/a_1, 0, \dots, 0), (0, b/a_2, \dots, 0), \dots, (0, \dots, 0, b/a_n)$$

and

$$P(A, b) \subset \left[0, \frac{b}{\min_i a_i}\right]^n. \tag{16}$$

Notice also that

$$bS_A = \pi_n(P(A, b)), \tag{17}$$

where $\pi_n(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection that forgets the last coordinate.

Rearranging the entries of A, if necessary, we may assume that the optimal value $LP_{\boldsymbol{c}}(A, b)$ is attained at the vertex $\boldsymbol{v} = (0, \ldots, 0, b/a_n)$ of $P(A, b)$.

If $b \leq \mu(S_A, \Lambda_A; \mathbb{Z}^{n-1})$ then (14) and (16) imply that the integrality gap is bounded by the right hand side of (15).

Suppose now that $b > \mu(S_A, \Lambda_A; \mathbb{Z}^{n-1})$. Then, in view of (17),

$$\mu(S_A, \Lambda_A; \mathbb{Z}^{n-1})S_A \subset \pi_n(P(A, b)) . \tag{18}$$

Let $\Lambda(A, b) = \{\boldsymbol{x} \in \mathbb{Z}^n : A\boldsymbol{x} = b\}$ be the set of integer points in the affine hyperplane $A\boldsymbol{x} = b$. There exists $\boldsymbol{y} \in \mathbb{Z}^{n-1}$ such that

$$\pi_n(\Lambda(A, b)) = \boldsymbol{y} + \Lambda_A . \tag{19}$$

By Lemma 1, there is a point $(z_1, \ldots, z_{n-1}) \in \pi_n(\Lambda(A, b)) \cap \mu(S_A, \Lambda_A; \mathbb{Z}^{n-1})S_A$. Hence

$$\boldsymbol{z} = \left(z_1, \ldots, z_{n-1}, \frac{b}{a_n} - \frac{a_1 z_1 + \cdots + a_{n-1} z_{n-1}}{a_n}\right) \in \Lambda(A, b) \cap P(A, b) \tag{20}$$

is a feasible integer point for the knapsack problem (1).

Since $(z_1, \ldots, z_{n-1}) \in \mu(S_A, \Lambda_A; \mathbb{Z}^{n-1})S_A$, we have

$$||\boldsymbol{v} - \boldsymbol{z}||_\infty \leq \frac{\mu(S_A, \Lambda_A; \mathbb{Z}^{n-1})}{\min_i a_i} \leq \frac{g(A) + \|A\|_\infty}{\min_i a_i} , \tag{21}$$

where the last inequality follows from (14). Therefore, the integrality gap is bounded by the right hand side of (15). □

To complete the proof of part (i) we need the classical upper bound for the Frobenius number due to Schur (see Brauer [6]):

$$g(A) \leq (\min_i a_i)\|A\|_\infty - (\min_i a_i) - \|A\|_\infty . \tag{22}$$

Combining (15) and (22) we obtain (6).

To prove part (ii), we set $A = (k, \ldots, k, 1)$, $b = k - 1$ and $\boldsymbol{c} = \boldsymbol{e}_n$, where $\boldsymbol{e}_i$ denotes the i-th unit-vector. Note that A fulfils the conditions (4). The integer programming problem (1) has precisely one feasible, and therefore optimal, integer point, namely $(k-1) \cdot \boldsymbol{e}_n$. Thus $IP_{\boldsymbol{c}}(A, b) = k - 1$. The corresponding linear relaxation (2) has the, in general not unique, optimal solution $\frac{k-1}{k} \cdot \boldsymbol{e}_1$ with $LP_{\boldsymbol{c}}(A, b) = 0$. Hence, $\mathrm{Gap}_{\boldsymbol{c}}(A) \geq IG_{\boldsymbol{c}}(A, b) = k - 1 = (\|A\|_\infty - 1)\|\boldsymbol{c}\|_1$.

4 Proof of Theorem 2

We will first establish a connection between $\mathrm{Gap}_{\boldsymbol{c}}(A)$ and the lattice programming gap associated with a certain lattice program.

For a vector $\boldsymbol{w} \in \mathbb{Q}^{n-1}_{>0}$, a $(n-1)$-dimensional lattice $\Lambda \subset \mathbb{Z}^{n-1}$ and $\boldsymbol{r} \in \mathbb{Z}^{n-1}$ consider the lattice program (also referred to as the *group problem*)

$$\min\{\boldsymbol{w} \cdot \boldsymbol{x} : \boldsymbol{x} \equiv \boldsymbol{r}(\bmod \Lambda), \boldsymbol{x} \in \mathbb{R}^{n-1}_{\geq 0}\} . \tag{23}$$

Here $\boldsymbol{x} \equiv \boldsymbol{r}(\bmod \Lambda)$ if and only if $\boldsymbol{x} - \boldsymbol{r}$ is a point of Λ.

Let $m(\Lambda, \boldsymbol{w}, \boldsymbol{r})$ denote the value of the minimum in (23). The *lattice programming gap* $\mathrm{Gap}(\Lambda, \boldsymbol{w})$ of (23) is defined as

$$\mathrm{Gap}(\Lambda, \boldsymbol{w}) = \max_{\boldsymbol{r} \in \mathbb{Z}^{n-1}} m(\Lambda, \boldsymbol{w}, \boldsymbol{r}) \,. \tag{24}$$

The lattice programming gaps were introduced and studied for sublattices of all dimensions in $\mathbb{Z}^{n-1}$ by Hoşten and Sturmfels [18].

To proceed with the proof of the part (i), we assume without loss of generality that $\tau(A, \boldsymbol{c}) = \{n\}$. Then for $\boldsymbol{l} = \boldsymbol{l}(A, \boldsymbol{c})$ the lattice programs

$$\min\{\boldsymbol{l} \cdot \boldsymbol{x} : \boldsymbol{x} \equiv \boldsymbol{r} \pmod{\Lambda_A}, \boldsymbol{x} \in \mathbb{R}^{n-1}_{\geq 0}\}\,, \; \boldsymbol{r} \in \mathbb{Z}^{n-1} \tag{25}$$

are the *group relaxations* to (1).

Indeed, for any positive $b \in \mathbb{Z}$ and any integer solution $\boldsymbol{z}$ of the equation $A\boldsymbol{x} = b$ the lattice program (25) with $\boldsymbol{r} = \pi_n(\boldsymbol{z})$, is a group relaxation to (1). On the other hand, for any integer vector $\boldsymbol{r}$ the lattice program (25) is a group relaxation to (1) with $b = \pi_n(A)\boldsymbol{u}$ for a nonnegative integer vector $\boldsymbol{u}$ from $\boldsymbol{r} + \Lambda_A$.

In both cases

$$IG_{\boldsymbol{c}}(A, b) \geq m(\Lambda_A, \boldsymbol{l}, \boldsymbol{r})$$

and, consequently,

$$\mathrm{Gap}_{\boldsymbol{c}}(A) \geq \mathrm{Gap}(\Lambda_A, \boldsymbol{l}) \,. \tag{26}$$

Note that for $n = 2$ we have $\mathrm{Gap}(\Lambda_A, \boldsymbol{l}) = l_1(|A_\tau| - 1)$ and thus (26) implies (8). For $n > 2$, the bound (8) immediately follows from (26) and Theorem 1.2(i) in [1].

The proof of the part (ii) will be based on the following lemma.

Lemma 3. *Let A satisfy (4), $\boldsymbol{c} = (a_1, \ldots, a_{n-1}, 0)^t \in \mathbb{Q}^n$ and $\boldsymbol{l} = (a_1, \ldots, a_{n-1})^t \in \mathbb{Q}^{n-1}_{>0}$. Then*

$$\mathrm{Gap}_{\boldsymbol{c}}(A) = \mathrm{Gap}(\Lambda_A, \boldsymbol{l}) \,. \tag{27}$$

Proof. Observe that assumption (i) in (4) implies that the linear programming relaxation (2) is feasible if and only if b is nonnegative. Recall that $\Lambda(A, b) = \{\boldsymbol{x} \in \mathbb{Z}^n : A\boldsymbol{x} = b\}$ denotes the set of integer points in the affine hyperplane $A\boldsymbol{x} = b$ and $P(A, b) = \{\boldsymbol{x} \in \mathbb{R}_{\geq 0} : A\boldsymbol{x} = b\}$ denotes the knapsack polytope. Suppose that for a nonnegative b the knapsack problem (1) is feasible, with solution $\boldsymbol{y} \in \mathbb{Z}^n_{\geq 0}$. Then for $\boldsymbol{r} = \pi_n(\boldsymbol{y}) \in \mathbb{Z}^{n-1}_{\geq 0}$

$$\pi_n(\Lambda(A, b)) = \boldsymbol{r} + \Lambda_A \,.$$

As $c_n = 0$, the optimal value of the linear programming relaxation $LP_{\boldsymbol{c}}(A, b) = 0$. Therefore, noting that $\boldsymbol{c} = (a_1, \ldots, a_{n-1}, 0)^t$ and $\boldsymbol{l} = \pi_n(\boldsymbol{c})$,

$$IG_{\boldsymbol{c}}(A, b) = \min\{\boldsymbol{l} \cdot \boldsymbol{x} : \boldsymbol{x} \in \boldsymbol{r} + \Lambda_A \,, \boldsymbol{x} \in \pi_n(P(A, b))\} \,. \tag{28}$$

Since

$$\pi_n(P(A, b)) = bS_A = \{\boldsymbol{x} \in \mathbb{R}^{n-1}_{\geq 0} : \boldsymbol{l} \cdot \boldsymbol{x} \leq b\}$$

and $\boldsymbol{l} \cdot \boldsymbol{r} \leq A\boldsymbol{y} = b$, the constraint $\boldsymbol{x} \in \pi_n(P(A, b))$ in (28) can be removed. Consequently, we have

$$IG_{\boldsymbol{c}}(A, b) = m(\Lambda_A, \boldsymbol{l}, \boldsymbol{r}) \,.$$

Hence, by (24), we obtain

$$\mathrm{Gap}_{\boldsymbol{c}}(A) \le \mathrm{Gap}(\Lambda_A, \boldsymbol{l})\,. \tag{29}$$

Suppose now that $\mathrm{Gap}(\Lambda_A, \boldsymbol{l}) = m(\Lambda_A, \boldsymbol{l}, \boldsymbol{r}_0)$. Then

$$IG_{\boldsymbol{c}}(A, A\boldsymbol{r}_0) = m(\Lambda, \boldsymbol{l}, \boldsymbol{r}_0)\,.$$

Together with (29), this implies (27). □

As was shown in the proof of Theorem 1.1 in [1], for $\boldsymbol{l} = (a_1, \ldots, a_{n-1})^t$

$$\mathrm{Gap}(\Lambda_A, \boldsymbol{l}) = g(A) + a_n\,.$$

Thus we obtain the following corollary.

Corollary 6 *Let $A = (a_1, \ldots, a_n)$ satisfy (4) and $\boldsymbol{c} = (a_1, \ldots, a_{n-1}, 0)^t$. Then*

$$\mathrm{Gap}_{\boldsymbol{c}}(A) = g(A) + a_n\,. \tag{30}$$

For $n = 2$, we have

$$g(A) = a_1 a_2 - a_1 - a_2 \tag{31}$$

by a classical result of Sylvester (see e.g. [22]). Hence the part (ii) immediately follows from Corollary 6. For $n > 2$, noting that $|A_\tau| = a_n$, the part (ii) follows from Corollary 6 and Theorem 1.1 (ii) in [2].

5 Proof of Theorem 3

For convenience, we will work with the quantity

$$f(A) = g(A) + a_1 + \cdots + a_n$$

and the set

$$R = \{A \in \mathbb{Z}^{1\times n} : 0 < a_1 \le \cdots \le a_n\}\,.$$

By Lemma 2, we have

$$N_\epsilon(t, T) \le n!\,\#\left\{A \in Q(T) \cap R : \frac{f(A)}{a_1 a_n^\epsilon} > t\right\}\,. \tag{32}$$

We may assume $t \ge 10$ since otherwise (11) follows from $N_\epsilon(t,T)/N(T) \le 1$. We keep $t' \in [1, t]$, to be fixed later. Then, setting $s(A) = a_{n-1} a_n^{1/(n-1)}$ and noting (32), we get

$$\begin{aligned} N_\epsilon(t, T) &\le n!\,\#\left\{A \in Q(T) \cap R : \frac{f(A)}{s(A)} > t' \text{ or } \frac{s(A)}{a_1 a_n^\epsilon} > \frac{t}{t'}\right\} \\ &\le n!\,\#\left\{A \in Q(T) \cap R : \frac{f(A)}{s(A)} > t'\right\} \\ &\quad + n!\,\#\left\{A \in Q(T) \cap R : \frac{a_{n-1}}{a_1 a_n^{\epsilon - 1/(n-1)}} > \frac{t}{t'}\right\}\,. \end{aligned} \tag{33}$$

The first of the last two terms in (33) can be estimated using a special case of Theorem 3 in Strömbergsson [26].

Lemma 4.

$$\#\left\{A \in Q(T) \cap R : \frac{f(A)}{s(A)} > r\right\} \ll_n \frac{1}{r^{n-1}} N(T)\,. \tag{34}$$

Proof. The inequality (34) immediately follows from Theorem 3 in [26] applied with $\mathcal{D} = [0,1]^{n-1}$. □

To estimate the last term, we will need the following lemma.

Lemma 5.

$$\#\left\{A \in Q(T) \cap R : \frac{a_{n-1}}{a_1 a_n^{\epsilon - 1/(n-1)}} > r\right\} \ll_n \frac{1}{rT^{\epsilon - 1/(n-1)}} N(T)\,. \tag{35}$$

Proof. Since $A \in R$, we have $a_{n-1} \leq a_n$. Hence

$$\#\left\{A \in Q(T) \cap R : \frac{a_{n-1}}{a_1 a_n^{\epsilon - 1/(n-1)}} > r\right\} \leq \#\left\{A \in Q(T) \cap R : a_n^{1+1/(n-1)-\epsilon} > ra_1\right\}\,.$$

Furthermore, all $A \in Q(T) \cap R$ with $a_n^{1+1/(n-1)-\epsilon} > ra_1$ are in the set

$$U = \{A \in \mathbb{Z}^{1\times n} : 0 < a_1 < T^{1+1/(n-1)-\epsilon}/r, 0 < a_i \leq T, i = 2, \ldots, n\}\,.$$

Since $\#(U \cap \mathbb{Z}^n) < T^{n+1/(n-1)-\epsilon}/r$ and $N(T) \asymp_n T^n$ (see e.g. Theorem 1 in [23]), the result follows. □

Then by (33), (34) and (35)

$$\frac{N_\epsilon(t,T)}{N(T)} \ll_n \frac{1}{(t')^{n-1}} + \frac{t'}{tT^{\epsilon - 1/(n-1)}}\,. \tag{36}$$

Next, we will bound T from below in terms of t, similar to Theorem 3 in [26]. The upper bound of Schur (22) implies $f(A) < na_1 a_n$. Thus, using (32),

$$\begin{aligned} N_\epsilon(t,T) &\leq \#\left\{A \in Q(T) \cap R : \frac{f(A)}{a_1 a_n^\epsilon} > t\right\} \\ &\leq \#\left\{A \in Q(T) \cap R : a_n^{1-\epsilon} > \frac{t}{n}\right\}\,. \end{aligned}$$

The latter set is empty if $T \leq (t/n)^{\frac{1}{1-\epsilon}}$. Hence we may assume

$$T > \left(\frac{t}{n}\right)^{\frac{1}{1-\epsilon}}\,. \tag{37}$$

Using (36) and (37), we have

$$\frac{N_\epsilon(t,T)}{N(T)} \ll_n \frac{1}{(t')^{n-1}} + \frac{t'}{t^{1+\frac{1}{1-\epsilon}\left(\epsilon - \frac{1}{n-1}\right)}}\,. \tag{38}$$

To minimise the exponent of the right hand side of (38), set $t' = t^\beta$ and choose β with

$$\beta(n-1) = 1 + \frac{1}{1-\epsilon}\left(\epsilon - \frac{1}{n-1}\right) - \beta\,. \tag{39}$$

We get

$$\beta = \frac{n-2}{n(n-1)(1-\epsilon)}$$

and, by (38) and (39),

$$\frac{N_\epsilon(t,T)}{N(T)} \ll_n t^{-\alpha(\epsilon,n)}$$

with $\alpha(\epsilon, n) = \beta(n-1)$. The theorem is proved.

6 Proof of Corollary 4

For the upper bound we observe, that the conditions $n \geq 3$ and $\epsilon > 2/n$ imply that in (11) $\alpha(\epsilon, n) > 1$. Consider vectors $A \in Q(T)$ with

$$e^{s-1} \leq \max_{\boldsymbol{c}\in\mathbb{Q}^n} \frac{\mathrm{Gap}_{\boldsymbol{c}}(A)}{\|A\|_\infty^\epsilon \|\boldsymbol{c}\|_1} < e^s\,. \tag{40}$$

The contribution of vectors satisfying (40) to the sum

$$\sum_{A\in Q(T)} \max_{\boldsymbol{c}\in\mathbb{Q}^n} \frac{\mathrm{Gap}_{\boldsymbol{c}}(A)}{\|A\|_\infty^\epsilon \|\boldsymbol{c}\|_1}$$

on the left hand side of (12) is

$$\leq N_\epsilon(e^{s-1}, T)e^s \ll_n e^{-\alpha(\epsilon,n)s} e^s N(T)\,,$$

where the last inequality holds by (11). Therefore

$$\frac{1}{N(T)} \sum_{A\in Q(T)} \max_{\boldsymbol{c}\in\mathbb{Q}^n} \frac{\mathrm{Gap}_{\boldsymbol{c}}(A)}{\|A\|_\infty^\epsilon \|\boldsymbol{c}\|_1} \ll_n \sum_{s=1}^{\infty} e^{s(1-\alpha(\epsilon,n))}\,.$$

Finally, observe that the series

$$\sum_{s=1}^{\infty} e^{s(1-\alpha(\epsilon,n))}$$

is convergent for $\alpha(\epsilon, n) > 1$.

References

1. I. Aliev, *On the lattice programming gap of the group problems*, Oper. Res. Lett. **43** (2015), 199-202.
2. I. Aliev, and P. M. Gruber, *An optimal lower bound for the Frobenius problem*, J. Number Theory **123** (2007), 71–79.
3. D. Bertsimas, and R. Weismantel, *Optimization over Integers*, Dynamic Ideas, 2005.
4. C. E. Blair, and R. G. Jeroslow, *The value function of a mixed integer program. I*, Discrete Math. **19** (1977), 121-138.

5. C. E. Blair, and R. G. Jeroslow, *The value function of an integer program*, Math. Programming **23** (1982), 237–273.
6. A. Brauer, *On a problem of partitions*, Amer. J. Math. **64** (1942), 299-312.
7. R. Chandrasekaran, *Polynomial algorithms for totally dual integral systems and extensions*, Studies on graphs and discrete programming (Brussels, 1979), pp. 3951, Ann. Discrete Math., 11, North-Holland, Amsterdam-New York, 1981.
8. M. Conforti, G. Cornuéjols and G. Zambelli, *Integer programming*, Graduate Texts in Mathematics, 271, Springer, 2014.
9. W. Cook, A. M. H. Gerards, A. Schrijver, and É. Tardos, *Sensitivity theorems in integer linear programming*, Math. Programming **34** (1986), 251–264.
10. R. Dougherty, and V. Faber, *The degree-diameter problem for several varieties of Cayley graphs. I. The abelian case*, SIAM J. Discrete Math. **17** (2004), 478–519.
11. F. Eisenbrand, N. Hähnle, D. Pálvölgyi, and G. Shmonin, *Testing additive integrality gaps*, Math. Program. A **141** (2013), 257–271.
12. F. Eisenbrand, and G. Shmonin, *Parametric integer programming in fixed dimension*, Math. Oper. Res. **33** (2008), 839–850.
13. I. Fáry, *Sur la densité des réseaux de domaines convexes*, Bull. Soc. Math. France, **78** (1950) 152–161.
14. M.R. Garey, and D.S. Johnson, *Computers and intractability, A guide to the theory of NP-completeness*, A Series of Books in the Mathematical Sciences, W. H. Freeman and Co., San Francisco, Calif., (1979).
15. R.E. Gomory, *Outline of an algorithm for integer solutions to linear programs*, Bull. Amer. Math. Soc., **64**, (1958), 275–278.
16. P.M. Gruber, *Convex and Discrete Geometry*, Springer, Berlin, 2007.
17. P.M. Gruber, and C.G. Lekkerkerker, *Geometry of Numbers*, North–Holland, Amsterdam 1987.
18. S. Hoşten, and B. Sturmfels, *Computing the integer programming gap*, Combinatorica, **27** (2007) , no. 3, 367–382.
19. R. Kannan, *Lattice translates of a polytope and the Frobenius problem*, Combinatorica, **12** (1992), 161–177.
20. L.G. Khachiyan, *Polynomial algorithms in linear programming*, USSR Computational Mathematics and Mathematical Physics, 1, **20**, (1980), 53–72.
21. J. Marklof, and A. Strömbergsson, *Diameters of random circulant graphs*, Combinatorica, **33** (2013), 429–466.
22. J. L. Ramírez Alfonsín, *The Diophantine Frobenius problem*, Oxford Lecture Series in Mathematics and its Applications 30, 2005.
23. W. M. Schmidt, *Asymptotic formulae for point lattices of bounded determinant and subspaces of bounded height*, Duke Math. J., **35** (1968), 327–339.
24. W. M. Schmidt, *Integer matrices, sublattices of* $\mathbb{Z}^m$*, and Frobenius numbers*, Monatsh. Math. **178** (2015), 405–451.
25. A. Schrijver, *Theory of linear and integer programming*, Wiley-Interscience Series in Discrete Mathematics, 1986.
26. A. Strömbergsson, *On the limit distribution of Frobenius numbers*, Acta Arith. **152** (2012), 81–107.
27. S. Sullivant, *Small contingency tables with large gaps*, SIAM J. Discrete Math. **18** (2005), 787–793.
28. V. V. Vazirani, *Approximation Algorithms*, Springer- Verlag, 2001.