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Recent Mathematical Developments in Quantum Field Theory

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Abstract. This workshop has focused on three areas in mathematical quantum field theory and their interrelations: 1) conformal field theory, 2) constructions of interacting models of quantum field theory by various methods, and 3) several approaches studying the interplay of quantum field theory and gravity.

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Introduction by the Organisers

Several decades after its invention, quantum field theory (QFT) remains the basis of the theoretical understanding of elementary particle physics, and an important tool in the study of condensed matter systems, making it a topic of prime interests for many physicists. But in view of the rich mathematical structure of QFT, and the many different formulations it allows, QFT is by now also a field of research in mathematics, acting as a bridge for the interchange of ideas, concepts and methods between mathematics and theoretical physics. On the one hand, it is widely expected that new mathematical insights are needed in order to make further progress on the many open questions in QFT, regarding for example the mathematical status of concrete field theoretic models. On the other hand, the structures found within QFT provide a major stimulus and incentive for pushing forward the current frontiers of mathematics.

The workshop "Recent mathematical developments in quantum field theory" focused on several topics connected to this interplay between physics and mathematics.

A particularly well developed area within QFT is the field of *conformal* QFT, in which the usual Poincaré symmetry group is dramatically enlarged to the conformal group, leading to many special features and simplifications. The simplifications are particularly drastic in two spacetime dimensions, and, as a result, such theories can to some extent be classified, and many models are more or less exactly solvable. As a striking example of this, Teschner gave a talk on the solution of Liouville field theory on a two-dimensional cylinder.

Recently, also the analysis of conformal field theories in more than two dimensions by "bootstrap methods" has again been a very active area of research. Schomerus reported on this program and explained how solutions can be approached by mapping conformal blocks of 4-point functions to quantum mechanical eigenfunctions of a specific Pöschl-Teller Hamiltonian in one dimension.

Complementary to explicit model constructions, there exist also (several) well-developed operator algebraic approaches to conformal field theory. Kawahigashi gave a talk on chiral heterotic theories within the operator-algebraic formulation of conformal field theory, and Carpi reported on recent progress in understanding the relations of this approach with the one based on vertex operator algebras.

In the absence of conformal invariance, the *construction of models* with non-trivial interaction becomes even harder. Depending on the type of model, several different methods have been developed, such as functional integral methods, renormalization group approaches, operator product expansions, inverse scattering, or operator algebraic methods. A number of talks in the workshop presented current developments in these subjects: Fröb gave a talk on the operator product expansion in Yang Mills theory as a short-distance expansion in a perturbative setting, and in particular presented novel functional equations for these quantities which can be used to construct them order-by-order in perturbation theory, and possibly even non-perturbatively. The talk of Imbrie focused on the use of renormalization group methods in the construction of the eigenstates of a many-body Hamiltonian displaying the phenomenon of many-body localization. While his concrete model was non-relativistic, the methods are related to those of constructive QFT which feature a delicate diagrammatic analysis involving large versus small field decompositions. The talk by Knörrer described an ongoing long-term research program (in collaboration with Balaban, Feldman and Trubowitz) on the construction of correlation functions and thermodynamic quantities for an interacting Bose gas, based on a functional integral representation. This model is a non-relativistic, yet very ambitious test case for renormalization group techniques developed originally for QFTs since it aims in particular at a rigorous control of spontaneous symmetry breaking and the Goldstone modes related to Bose-Einstein condensation.

A completely different perspective on the construction problem in QFT was given by Jäkel, who presented a research program aiming at the construction of QFTs on deSitter space with the help of Tomita-Takesaki modular theory, where the interaction is meant to be encoded not in terms of a Lagrangian density, but rather in terms of a suitable vector from the canonical cone of a von Neumann algebra w.r.t. the "vacuum" state. Also the talk by Cadamuro was dedicated to the construction of certain (integrable) two-dimensional QFTs using operator algebraic techniques, this time focusing on inverse scattering theory and bound states. In such integrable models, the (two-particle) S-matrix is simple enough to be usable as a meaningful description of the interaction.

In a general QFT in higher dimensions, this is far from being true, and also the understanding of scattering theory is by no means as complete as in the non-relativistic case of quantum mechanics. Dybalski gave a talk on the current state of the art regarding asymptotic completeness in QFT.

In the last 10 years, interesting progress has been made in the study of the "Grosse-Wulkenhaar model", a four-dimensional QFT which was originally motivated by QFT on a (Euclidean) non-commutative space. This model has seen several variations and refinements, and can now be formulated as a QFT on the usual (commutative) four-dimensional Euclidean space. As Wulkenhaar explained in his talk, this model shows certain aspects of integrability despite living in higher dimensions, and there is partial evidence that it might be possible to translate it to a Minkowski QFT by a Wick rotation.

Another talk related to constructive field theory was given by Chandra, who reported on recent progress in the theory of stochastic PDEs. Such theories have a close relationship with QFTs on Euclidean space, and in fact can be seen as an alternative route to a (non-perturbative) construction of such models. In his talk, he reviewed some of the recent progress in this area due in particular to Hairer, Gubinelli, and others. He then explained how a BPHZ theorem can be formulated within Hairer's theory of regularity structures for certain types of stochastic PDEs corresponding to super-renormalizable QFT models, providing thus a new approach to these types of QFTs at the non-perturbative level.

The third main topic of the workshop was the question about the *combination of QFT with gravity*, a subject that has attracted considerable attention for decades. Several approaches exist, ranging from studying QFT on fixed but curved Lorentzian spacetimes, string theory, loop quantum gravity, to attempts of establishing quantum theories of gravity by quantization of classical general relativity.

Hack reported on a research program on a perturbative quantization of gravity and applications to cosmology, making use of the framework of perturbative algebraic quantum field theory. A very different approach to the problem of quantizing gravity in four dimensions was presented by Rivasseau: "Tensor field theory" draws its inspiration from Regge calculus, random matrix theory and 2D gravity, and views quantum gravity as a theory of random geometries formulated in terms of random tensors, which is then investigated with renormalization group methods.

Bär and Sanders gave talks within the setting of quantum field theories on curved but classical spacetimes. Bär presented a new type of index theorem for Dirac operators on globally hyperbolic spacetime with spin structures and discussed how this result can be applied to understanding anomalies. Sanders' talk was about a generalization of so-called “modular nuclearity conditions” to QFT on curved spacetimes and the use of this concept for characterizing states as well as for obtaining estimates on entanglement entropies.

At the interface of conformal field theory and QFT on curved spacetimes there also lies research on the famous AdS-CFT correspondence. We had three talks on different aspects of this subject. Gottschalk reviewed the correspondence in the Euclidean setting and presented particular results in a model example for this theory based on the Liouville model. Samberg talked about p-adic AdS-CFT which can be seen as a toy model for the more familiar Euclidean AdS-CFT. Following well-known methodology in number theory, the idea is to replace the continuous hyperbolic space given by the Euclidean AdS by a discrete hyperbolic space, namely, an infinite tree whose conformal boundary is most elegantly described in terms of the field $\mathbb{Q}_p$ of p-adic numbers. Also related to the AdS-CFT correspondence, Zahn gave a talk about a holographic relation concerning a massive scalar field on $(d+1)$-dimensional Minkowski space with a d-dimensional boundary.

The topic of QFT on curved spacetimes was also the subject of a special evening lecture by Wald. His talk was devoted to the question of “information loss” in the context of black hole evaporation, and gave a critical overview of the lively debate of this question in the current literature/media.

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Recent Mathematical Developments in Quantum Field Theory

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Abstracts

Lessons from Liouville field theory

Jörg Teschner

Most interacting quantum field theories have been constructed in the sense of formal power series in coupling parameters only. The fact that the perturbative expansions of quantum field theories generically do not converge raises interesting questions. One may, in particular, wonder if information on non-perturbative effects is necessary in order to characterise quantum field theories completely on the non-perturbative level. Exactly solvable examples like the Liouville theory (LFT) defined classically on Euclidean two-dimensional manifolds Σ by the action

$$S = \int_{\Sigma} \frac{d^2 z}{\pi} \left(\partial_z \phi \partial_{\bar{z}} \phi + \pi\mu e^{2b\phi}\right), \tag{1}$$

may allow us to gain insights into such issues.

1. Exact results on LFT

There is a set of known exact results that characterise the LFT completely as a conformal field theory. Two pieces of information are crucial

- **Spectrum:** The Hilbert space is $\mathcal{H} \simeq \int_{\mathbb{R}+}^{\oplus} dP\, \mathcal{V}_P \otimes \bar{\mathcal{V}}_P$, where $\mathcal{V}_P$ (resp. $\bar{\mathcal{V}}_P$) is the irreducible unitary representation of the Virasoro algebra with generators L_n (resp. $\bar{L}_n$), central charge $c = 1 + 6Q^2$, $Q = b + b^{-1}$, and highest weight vectors v_P (resp. $\bar{v}_P$) satisfying $L_n v_P = \delta_{n,0}(Q^2/4 + P^2)v_P$ (resp. $\bar{L}_n \bar{v}_P = \delta_{n,0}(Q^2/4 + P^2)\bar{v}_P$).
- **Observables:** The local observables can all be obtained from the fields $V_\alpha(w, \bar{w})$ on the Euclidean cylinder with coordinates $w = \tau + i\sigma$ which are quantum counterparts of exponential functions $e^{2\alpha\phi(w,\bar{w})}$. The fields $V_\alpha(w, \bar{w})$ are fully characterised by the properties

$$[L_n, V_\alpha(w, \bar{w})] = e^{nw}(\partial_w + \Delta n)V_\alpha(w, \bar{w})\,, \tag{2}$$

$$[\bar{L}_n, V_\alpha(w, \bar{w})] = e^{n\bar{w}}(\bar{\partial}_{\bar{w}} + \Delta n)V_\alpha(w, \bar{w})\,, \tag{3}$$

allowing us to calculate arbitrary matrix elements from the particular matrix elements

$$C\big(\tfrac{Q}{2} + iP, \alpha, \tfrac{Q}{2} + iP'\big) := \langle\, P \,|\, V_\alpha(0,0) \,|\, P' \,\rangle \qquad |\, P \,\rangle = v_P \otimes \bar{v}_P\,. \tag{4}$$

An explicit formula for the function $C(\alpha_3, \alpha_2, \alpha_1)$ was proposed in [1, 2]. It was shown in [3] that the theory fully characterised by the function proposed in [1, 2] satisfies the usual consistency conditions like locality and crossing symmetry. The exact formula for $C(\alpha_3, \alpha_2, \alpha_1)$ displays a remarkable symmetry under $b \to b^{-1}$ which seems difficult to understand using standard quantum field theoretical methods.

2. Standard perturbative approaches

One of the earliest approaches to construct LFT was based on canonical quantisation. Splitting $\int_0^{2\pi} \frac{d\sigma}{2\pi} : e^{2b\phi(\sigma,0)} := e^{2b\phi_0} + \int_0^{2\pi} \frac{d\sigma}{2\pi}\big(: e^{2b\phi(\sigma,0)} : - e^{2b\phi_0}\big)$, where $\phi_0 = \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \phi(\sigma, 0)$, allows one to define a perturbative expansion of the matrix elements $N_b(k, \beta, k') := \langle\, bk \,|\, V_{b\beta}(0, 0) \,|\, bk' \,\rangle$ in powers of b^2. The terms up to order b^{10} have been calculated in [4]. The results have been compared with the first terms in the asymptotic expansion of $C(\frac{Q}{2}+ibk, b\beta, \frac{Q}{2}+ibk')$ in powers of b^2 in [5]. Perfect agreement was found. It was furthermore shown in [5] that the expansion of $C(\frac{Q}{2} + ibk, b\beta, \frac{Q}{2} + ibk')$ in powers of b^2 is *not* convergent.

Expansions around classical solutions have also been studied, see [6, 7] and references therein. Such expansions can be in particular be used to define an expansion of $C(b\eta_1, b\eta_2, b\eta_3)$ in powers of b. It was observed that the resulting expansion is very sensitive to the renormalisation prescriptions. In the case of LFT it turns out to be necessary and sufficient [6, 7] to specify a definition of the propagator at coinciding points. There exists a choice which allows one to reproduce the leading coefficients in the expansion of $C(b\eta_1, b\eta_2, b\eta_3)$ [7].

3. Expansion in powers of μ

Expansions in powers of μ have not much been studied. This was believed to be useless as it was known for quite a while that the function $C(\alpha_3, \alpha_2, \alpha_1)$ has a simple μ-dependence proportional to $\mu^{1+b^{-2}-b^{-1}(\alpha_1+\alpha_2+\alpha_3)}$. A way out is to consider matrix elements such as $\langle\, p \,|\, V_\alpha(w, \bar{w}) \,|\, \psi \,\rangle$, where $|\psi\rangle$ is a linear combination of Virasoro primaries of the form $\int dp\, \psi(p)|p\rangle$. $\psi(p)$ can be taken from a suitable space of test functions which are, in particular, analytic in the upper half-plane. The perturbative expansion of $\langle\, p \,|\, V_\alpha(w, \bar{w}) \,|\, \psi \,\rangle$ in powers of μ can be defined in the usual way. It turns out that the coefficients of this expansion have singularities at rational values of b^2. This indicates that the theory having the matrix elements defined in this way will not be well-defined as a quantum field theory on the non-perturbative level.

Applying the same procedure to the action $S = \int \frac{d^2z}{\pi}\big(\partial_z\phi\partial_{\bar{z}}\phi + \pi\mu e^{2b\phi} + \pi\mu' e^{2b^{-1}\phi}\big)$, with $\tilde{\mu}$ chosen such that the relation $\pi\gamma(b^{-2})\mu' = (\pi\gamma(b^2)\mu)^{b^{-2}}$ is satisfied, it may be shown that the resulting expansion in powers of μ is non-singular for all values of b^2. The singularities caused by the two interaction terms $\pi\mu e^{2b\phi}$ and $\pi\tilde{\mu} e^{2b^{-1}\phi}$ cancel each other completely. One may use this observation to construct a well-defined expansion in powers of μ that coincides with the same type of expansion obtained using the exact results.

4. Concluding remarks

Exponential interactions like those appearing in Liouville theory have been studied by the methods of constructive field theory in the work [8]. Within this framework it was shown that there exists a real number b_* such that exponential interactions like $e^{2b\phi}$ are trivial for $b > b_*$. The exact solution yields a definition for LFT that is well-defined and non-trivial for all values of b. It would be interesting to resolve this apparent discrepancy.

It is not clear at this stage if there exists an unambiguous resummation of the expansion in powers of b. However, our observations concerning the expansion in powers of μ indicate that there exists a basically canonical way to cancel the singularities at rational values of b^2 which occur in the μ-expansion. The necessary modification is non-perturbative in b, and can not be seen in any expansion in powers of b. The cancellation mechanism for the singularities in the μ-expansion offers an explanation for the remarkable self-duality of LFT under $b \to b^{-1}$.

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The integrable conformal bootstrap in $d \geqslant 2$

Volker Schomerus
(joint work with M. Isachenkov)

Conformal quantum field theories capture the universal low energy behavior of important quantum systems (e.g. 3D Ising model). While many 2-dimensional models have been solved, there exist no analytical techniques so far to compute e.g. the exact critical exponents of higher dimensional systems.

The conformal bootstrap programme was designed in the 1970$^{\mathrm{s}}$ to make progress, but the mathematical difficulties of the equations could not be overcome at the time. More recently, numerical studies of these equations were shown to provide results of remarkable precision.

In my talk, I described a new approach to the kinematical input of the analytic bootstrap programme. In particular, I showed that the so-called conformal blocks of scalar 4-point functions, which play a central role in the bootstrap programme, can be mapped to eigenfunctions of a 2-particle hyperbolic Calogero-Sutherland model. The latter describes two coupled particles in a Pöschl-Teller potential. Their interaction, which depends smoothly on the dimension, is integrable. The observation brings in new analytical tools, mosty developed since the 1980$^{\mathrm{s}}$, that could lead to exact solutions of the conformal bootstrap programme.

Relative tensor products of heterotic full conformal field theories

Yasuyuki Kawahigashi

Recall a subfactor of M is described with a Q-system (θ, v, w), a triple of an endomorphism of M and isometries $v \in \mathrm{Hom}(\mathrm{id}, \theta)$, $w \in \mathrm{Hom}(\theta, \theta^2)$ satisfying the following identities:

$$v^* w = \theta(v^*) w \in \mathbb{R}_+,$$
$$\theta(w) w = w^2.$$

A Q-system is also called a C^*-Frobenius algebra of a C^*- category containing θ as an object.

Let $\{A_1(I)\}$, $\{A_2(I)\}$ be completely rational local conformal nets in the sense of [17]. Each describes a chiral conformal field theory on S^1. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be the finite dimensional representation categories of $\{A_1(I)\}$, $\{A_2(I)\}$, respectively. They are modular tensor categories by [17]. A full conformal field theory $\{B_1(I \times J)\}$ is described with a local Q-system (θ, v, w) with $\theta \in \mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}}$ as in [16], where "rev" means the braiding structure is reversed. Such θ has a decomposition $\bigoplus_{\lambda \in \mathcal{C}_1, \mu \in \mathcal{C}_2} Z^1_{\lambda\mu} \lambda \boxtimes \bar{\mu}$. The full conformal field theory $\{B_1(I \times J)\}$ has a trivial representation theory if and only if Z^1 has the modular invariance property, $S_1 Z^1 = Z^1 S_2$, $T_1 Z^1 = Z^1 T_2$, where S_1, S_2 are the S-matrices of $\mathcal{C}_1, \mathcal{C}_2$, respectively and T_1, T_2 are the T-matrices of $\mathcal{C}_1, \mathcal{C}_2$, respectively. This fact was conjectured by Rehren [23] and proved by Müger [21] and Kawahigashi-Longo independently. We say that the Q-system is Lagrangian when this modular invariance property holds.

If $\mathcal{C}_1 = \mathcal{C}_2$, then Z^1 is a usual modular invariant matrix. It is easy to see that the modular invariance property is preserved under multiplication of two modular invariants except for the normalization property $Z_{00} = 1$. Then it has been observed a product of two modular invariants often decomposes into a sum of modular invariants. For example, for $SU(2)_{17}$, we have three modular invariants, labeled as A_{17}, D_{10}, E_7 and here A_{17} is the identity matrix. We have the following "fusion rules" among these.

$$D_{10} \otimes D_{10} = 2D_{10},$$
$$D_{10} \otimes E_7 = E_7 \otimes D_{10} = 2E_7,$$
$$E_7 \otimes E_7 = D_{10} \oplus E_7.$$

The fusion rules of this type were studied in [11] and understood in [13] and [14] through braided products of Q-systems and Rehren's construction [22] of a Q-system based on α-induction [20], [24], [4], [5], [6], [7], [8], [9], [10]. He we would like to make a more direct construction of this type of "relative products" without assuming $\mathcal{C}_1 = \mathcal{C}_2$.

Suppose we have local Lagrangian Q-systems (θ_1, v_1, w_1) on $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{rev}}$ with $\theta_1 = \bigoplus_{\lambda \in \mathcal{C}_1, \mu \in \mathcal{C}_2} Z^1_{\lambda\mu} \lambda \boxtimes \bar{\mu}$ and (θ_2, v_2, w_2) on $\mathcal{C}_2 \boxtimes \mathcal{C}_3^{\mathrm{rev}}$. with $\theta_2 = \bigoplus_{\mu \in \mathcal{C}_2, \nu \in \mathcal{C}_3} Z^2_{\mu\nu} \mu \boxtimes \bar{\nu}$. Then we have the following result.

We have a decomposition $Z^1Z^2 = \sum_k Z^{3,k}$ of a matrix product Z^1Z^2, where each $\bigoplus_{\lambda\in\mathcal{C}_1,\nu\in\mathcal{C}_3} Z^{3,k}_{\lambda\nu}\lambda\boxtimes\bar{\nu}$ is the endomorphism part of a local Lagrangian Q-system.

This gives an irreducible decomposition of a relative tensor product of two Q-systems over a modular tensor category $\mathcal{C}_2$. There is a formal similarity between this construction and the product considered in [3] in connection to [2]. This construction also has some similarity of the fusion of defects considered in [1]. We hope to clarify the meaning of the similarity.

The above construction is done on the level of Q-systems and modular tensor categories. The meaning in the context of full conformal field theory is also not clear.

In [15], we connected a local Lagrangian Q-system to a gapped domain wall between topological phases [18]. From this viewpoint, our construction is regarded as a mathematical realization of the notion of "composition" of two gapped domain walls in [18].

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Representations of conformal nets and vertex operator algebra modules

Sebastiano Carpi
(joint work with Mihály Weiner and Feng Xu)

Vertex operator algebras (VOAs) and conformal nets give two different axiomatizations for chiral two-dimensional conformal field theory (chiral CFT). VOAs are mainly of algebraic nature [4, 5, 8, 10]. A VOA (over $\mathbb{C}$) is a complex vector space V together with a linear map $V \ni a \mapsto Y(a,z)$ satisfying certain assumptions related to the underlying CFT interpretation. The *vertex operators* $Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ are formal power series with coefficients $a_{(n)} \in \mathrm{End}(V)$ or, equivalently, operator-valued formal distributions on $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. They should be interpreted as the quantum fields of the theory. The map $a \mapsto Y(a,z)$ is called the *state field correspondence* and among its properties we mention here the so called *Borcherds identity* which is deeply related with the locality property of quantum fields [8].

On the other hand conformal nets are defined in terms of operator algebras on Hilbert spaces and hence they are mainly functional analytic objects [6, 9]. They are the chiral CFT version of algebraic quantum field theory (AQFT) [7]. A conformal net $\mathcal{A}$ on S^1 is a map $I \mapsto \mathcal{A}(I)$ from the set $\mathcal{I}$ of open, non-dense, non-empty intervals of the unit circle S^1 into the family of von Neumann algebras acting on a fixed Hilbert space (the *vacuum Hilbert space*) satisfying certain assumptions which, also in this case, are related to the underlyng CFT interpretation. Among these assumptions we mention here *locality* which means that the von Neumann algebras associated to any pair $I_1, I_2 \in \mathcal{I}$ of disjoint intervals commute.

Despite their significant mathematical differences, these two formulations show their common CFT origin through many structural similarities. Moreover, many interesting chiral CFT unitary models can be considered from both point of view with similar outputs. However, a direct general connection between unitary VOAs and conformal nets has been studied for the first time only recently by Y. Kawahigashi, R. Longo, M. Weiner and me in [1]. For every sufficiently nice unitary VOA V we have shown how to define a corresponding conformal net $\mathcal{A}_V$ and how to recover V with its VOA structure from $\mathcal{A}_V$.

We shortly outline the construction. Let V be a simple unitary VOA [1, 3] whose vertex operators satisfy certain *energy bounds* and let $f \in C^\infty(S^1)$ be a smooth complex valued function on the circle. Then, every operator valued distribution $Y(a, z)$, $z \in S^1$, gives rise to a closed (unbounded) operator $Y(a, f)$ acting on the Hilbert space completion $\mathcal{H}_V$ of V (the *smeared vertex operator*). Moreover, for every open interval $I \in \mathcal{I}$, the family of operators $\{Y(a, f) : a \in V, f \in C_c^\infty(I)\}$ generates a von Neumann algebra $\mathcal{A}_V(I)$ on $\mathcal{H}_V$, i.e. the smallest von Neumann algebra with which every operator in the family is *affiliated*. Then, V is said to be *strongly local* if the map $I \mapsto \mathcal{A}_V(I)$ satisfies locality. It turns out that if V is strongly local then $\mathcal{A}_V$ satisfies also remaining properties and hence it is a conformal net. Many known examples of unitary VOAs such as the unitary Virasoro VOAs, the unitary affine Lie algebras VOAs, the known $c = 1$ unitary VOAs, the moonshine VOA $V^\natural$, together with their coset and orbifold subVOAs, have been shown to be strongly local in [1] and it has been conjectured that every simple unitary VOA is strongly local.

VOAs and conformal nets have very interesting representation theories (theory of superselection sectors) but these play only a marginal role in [1]. A first important step towards the analysis of the representation theory aspects of the map $V \mapsto \mathcal{A}_V$ has been recently made by M. Weiner, F. Xu and me [2]. Let V be a strongly local VOA and let $\mathcal{A}_V$ be the corresponding conformal net. A VOA module for V is a complex vector space M together with linear map $a \mapsto Y_M(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)}^M z^{-n-1}$, $a_{(n)}^M \in \mathrm{End}(M)$, from V into the set of operator valued formal distributions on M, which is compatible, in an appropriate sense, with the vertex operator algebra structure of V. In particular the *represented vertex operators* $Y_M(a, z)$ satisfy the Borcherds identity on M.

On the other hand, a representation π of $\mathcal{A}_V$ is a family $\{\pi_I : I \in \mathcal{I}\}$ where each π_I is a representation of the von Neumann algebra $\mathcal{A}_V(I)$ on a fixed Hilbert space $\mathcal{H}_\pi$. If π is locally normal, i.e. if every π_I is continuous with respect to the σ-weak topology (the natural topology of von Neumann algebras), then, each representation π_I naturally extends to the unbounded closed operators affiliated with $\mathcal{A}(I)$. In particular, if $f \in C_c^\infty(I)$ then $\pi_I(Y(a, f))$ is a well defined closed operator on $\mathcal{H}_\pi$. This is the starting point for the notion of *strongly integrable module* introduced in [2].

Let M be a VOA module for the strongly local VOA V. We assume that M is unitary, i.e. that it has a scalar product $(\cdot|\cdot)_M$ which is compatible with the unitary structure of V, see [3]. Furthermore, we assume that the represented vertex operators $Y_M(a, z)$ satisfy energy bounds similar to those of the vertex operators $Y(a, z)$. Then we can consider the *represented smeared vertex operators*

$Y_M(a, f)$, $f \in C^\infty(S^1)$. In [2] M is defined to be strongly integrable if there exists a (necessarily unique) locally normal representation π^M of the conformal net $\mathcal{A}_V$ such that $\pi^M_I(Y(a, f)) = Y_M(a, f)$ for all $a \in V$, all $I \in \mathcal{I}$ and all $f \in C^\infty_c(I)$.

The corresponding map $M \mapsto \pi^M$ and its inverse preserve unitary equivalence, direct sums and irreducibility. Moreover, using free Femi field constructions, one can give many interesting examples of strongly integrable modules. In particular all the VOA modules of the type A unitary affine VOAs and the related coset VOA modules are strongly integrable. As a consequence, various results previously obtained by Feng Xu for the representation theory of type A diagonal coset conformal nets by means of subfactor theory methods [11, 12] can be transported to the VOA setting giving a solution of various long standing open problems in VOA representation theory. In this way we obtain e.g. a solution to certain purely VOA irreducibility problems for diagonal type A coset VOA modules thanks to the power of subfactor theory.

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AdS/CFT correspondence in the Euclidean setting - triviality for the exponential interaction

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(joint work with Horst Thaler)

The AdS/CFT correspondence [1] relates expected values from type II string theory on the AdS space with the expected value of Yang Mills theory on its conformal boundary. As already motivated by Witten [2], scalar quantum fields are frequently used as a toy model for this conjecture. However, in contrast to the usual definition of effective actions via the Laplace transform of a functional measure, here source terms are related to the prescription of boundary values at the conformal boundary

$$Z(h)/Z(0) = \frac{1}{Z(0)} \int e^{-S_0(\phi) - V_\Lambda(\phi)} \delta(\partial\phi - h)\, \mathcal{D}\phi, \tag{1}$$

where $S_0(\phi) = \frac{1}{2}\int_{\mathbb{H}_d} |\nabla\phi|^2 + m^2\phi^2\, d_g x$ is the Euclidean action of the free, scalar massive field and $V_\Lambda(\phi) = \int_\Lambda : \exp(\alpha\phi) : \, d_g x$ is an exponential interaction term with bounded infra red cut off region Λ. Also, $d_g x$ stands for integration wrt the metric on $\mathbb{H}_d$ and $\partial\phi$ is the restriction of ϕ to the conformal boundary $\partial_c\mathbb{H}_d$ of $\mathbb{H}_d$, the $d-$dimensional hyperbolic space, which is the Euclidean analogue of the Lorentzian AdS$_d$ space. For a rigorous theory of Wick rotation from AdS$_d$ to $\mathbb{H}_d$ confer [3, 4].

In [5], Dütsch and Rehren conjectured that (1) can be equivalently expressed in terms of an ordinary QFT-like generating functional

$$\tilde{Z}(h)/\tilde{Z}(0) = \frac{1}{\tilde{Z}(0)} \int e^{-S_0(\phi) - V_\Lambda(\phi)} e^{\int_{\partial_c\mathbb{H}_d} \partial\phi\, h dx}\, \mathcal{D}\phi. \tag{2}$$

such that $\tilde{Z}(ch)/\tilde{Z}(0) = Z(h)/Z(0)$ for some constant $c > 0$. Here it is important that both prescriptions of the AdS/CFT correspondence use different boundary conditions for the Laplace operator at the conformal boundary.

In subsequent mathematical work [7], mathematical definitions for both expressions (1) and (2) have been given in terms of rigorously defined Gaussian functional integrals perturbed by an exponential interaction with cut-off Λ in dimension $d = 2$. Also the equivalence of both theories, as proposed by Rehren and Dütsch, is established. The conformal invariance of the resulting boundary theory and further structural properties like reflection positivity have been proven as well, under the hypothesis of the existence and uniqueness of the thermodynamic limit $\Lambda \nearrow \mathbb{H}_d$.

As already observed in [8], the thermodynamic limit in the AdS/CFT case is quite different from the thermodynamic limit in standard constructive QFT [6]. For the case of polynomial interactions with ultra-violet cut-off we proved the triviality of the boundary theory using hypercontractivity estimates.

In a recent work [9], we investigated the thermodynamic limit for the Euclidean AdS/CFT correspondence for the case of exponential interaction. By standard techniques one can show that

$$\tilde{Z}(ch)/\tilde{Z}(0) = e^{\frac{1}{2}\alpha_+(h,h)} \int e^{-V_\Lambda(\phi + cH_+h)} d\mu_+(\phi) \tag{3}$$

where μ_+ is a Gaussian measure of the distributions over $\mathbb{H}_d$ prescribed by the covariance function of bulk -to-bulk propagator G_+, whereas H_+ is the so called boundary-to-bulk propagator and α_- a certain boundary-to-boundary propagator.

Using the structure of the exponential interaction, we see that

$$V_\Lambda(\phi + cH_+h) = \int_\Lambda :\exp(\alpha\phi(x)): e^{\alpha c H_+ h(x)} d_g x. \tag{4}$$

For massive theories with test function $h(\xi) > 0$, $\xi \in \partial_c \mathbb{H}_d$, we realize that the space-dependent coupling constant $e^{\alpha c H_+ h(x)}$ diverges, as $H_+h((z,\xi)) \to \infty$ as $z \to 0$ in the half-space model of $\mathbb{H}_2$ with conformal boundary at $z = 0$, $x = (z, \xi)$.

Since the divergence of coupling constants only takes place in the denominator of (2), one expects a very different scaling behaviour of $\tilde{Z}_\Lambda(ch)$ as compared with the sum over states $\tilde{Z}_\Lambda(0)$ as $\Lambda \nearrow \mathbb{H}_d$. Using decoupling inequalities [6] and hyperbolic triangulations of the disk model of $\mathbb{H}_2$, we can in fact rigorously show [9] that this leads to

$$\tilde{Z}_\Lambda(ch)/\tilde{Z}_\Lambda(0) \to 0 \tag{5}$$

for some sequences $\Lambda \nearrow \mathbb{H}_d$. We conclude that either the thermodynamic limit, if it exists, is trivial.

Whether this observation of triviality can be cured by procedures like holographic renormalization [10] is left for future investigation.

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p-adic AdS/CFT

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(joint work with S. S. Gubser, J. Knaute, S. Parikh, P. Witaszczyk)

A p-adic analog of the AdS/CFT correspondence [1, 2, 3] is constructed. Motivations for investigating the possibility of a discrete version of AdS/CFT stem, among others, from segmented strings [4, 5] and tensor networks associated with holography [6]. Within string theory, p-adic numbers and related trees had already been considered in lieu of real worldsheets almost 30 years ago [7, 8, 9]. For details on the results reported on here see [10].

Let p be a prime number. The field $\mathbb{Q}_p$ of p-adic numbers is the completion of the rational numbers by inclusion of the limits of all Cauchy sequences with respect to the ultrametric induced by the p-adic absolute value $|x|_p := p^{-v_p(x)}$. Here, the valuation $v_p(x) \in \mathbb{Z}$ for non-zero $x \in \mathbb{Q}$ is uniquely defined by writing $x = p^{v_p(x)} \frac{a}{b}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ and $p \nmid a, b$, and additionally $|0|_p := 0$. Every p-adic number $x \in \mathbb{Q}_p$ now has a unique series expansion, $x = p^{v_p(x)} \sum_{n=0}^{\infty} a_n p^n$ with 'digits' $a_n \in \{0, 1, \ldots, p-1\}$ and $a_0 \neq 0$, and still $v_p(x) \in \mathbb{Z}$. The norm defined via $|\cdot|_p$ is extended in the obvious way to $\mathbb{Q}_p$. The ring of p-adic integers is defined as $\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid v_p(x) \geq 0\}$, and $\mathbb{U}_p := \{x \in \mathbb{Z}_p \mid v_p(x) = 0\} = \mathbb{Z}_p^\times$ denotes the set of units in $\mathbb{Z}_p$.

A tree of coordination number $p+1$, the Bruhat–Tits tree [11], is naturally associated with the p-adics $\mathbb{Q}_p$, as illustrated in Fig. 1. A ray through the tree corresponds, via choice of the valuation (which means choosing a bush on the main trunk, drawn in red) and choices of digits, to a p-adic number. The Bruhat–Tits tree takes the role of the bulk in ordinary AdS/CFT, and its boundary is seen to be $\mathbb{Q}_p \cup \{\infty\} = \mathbb{P}^1(\mathbb{Q}_p)$. We introduce a depth coordinate z_0 on the tree, taking the discrete values $z_0 = p^\omega$ for $\omega \in \mathbb{Z}$. At fixed z_0, the equivalence relation $x \sim y$ iff $x - y \in z_0\mathbb{Z}_p$ partitions $\mathbb{Q}_p$ into countably many equivalence classes of the form $z + z_0\mathbb{Z}_p$ (arranged on a horizontal slice in Fig. 1). Every such equivalence class is uniquely associated with a node on the tree. We thus use tuples (z_0, z) with $z \in \mathbb{Q}_p$ as our bulk coordinates.

So far, we have a tentative setup for an analog to $\mathrm{AdS}_2/\mathrm{CFT}_1$. Aiming for higher-dimensional versions $\mathrm{AdS}_{n+1}/\mathrm{CFT}_n$, we would like to have trees with a boundary that is an n-dimensional vector space over $\mathbb{Q}_p$. Let us denote the unique unramified degree-n field extension of $\mathbb{Q}_p$ by $\mathbb{Q}_q$ where $q = p^n$. As is well known, there is a unique extension of the valuation and norm on $\mathbb{Q}_p$ to $\mathbb{Q}_q$ which we denote by $v_q(\cdot)$ and $|\cdot|_q$, respectively. $\mathbb{Q}_q$ can be constructed by adjoining to $\mathbb{Q}_p$ a primitive $(p^n - 1)$-th root of unity (see for instance [12]). In fact, $\mathbb{Q}_q$ is an n-dimensional vector space over $\mathbb{Q}_p$, and its projective completion $\mathbb{Q}_q \cup \{\infty\} = \mathbb{P}^1(\mathbb{Q}_q)$ is naturally realized as the boundary of a Bruhat–Tits tree with coordination number $q+1$.

There is a natural action of $\mathrm{PGL}(2, \mathbb{Q}_q) = \mathrm{GL}(2, \mathbb{Q}_q)/\mathbb{Q}_q^\times$ on $\mathbb{P}^1(\mathbb{Q}_q)$ via fractional linear transformations $x \mapsto (\alpha x + \beta)/(\gamma x + \delta)$ where $\alpha, \beta, \gamma, \delta \in \mathbb{Q}_q$. Correlators that we compute holographically below have transformation behavior under

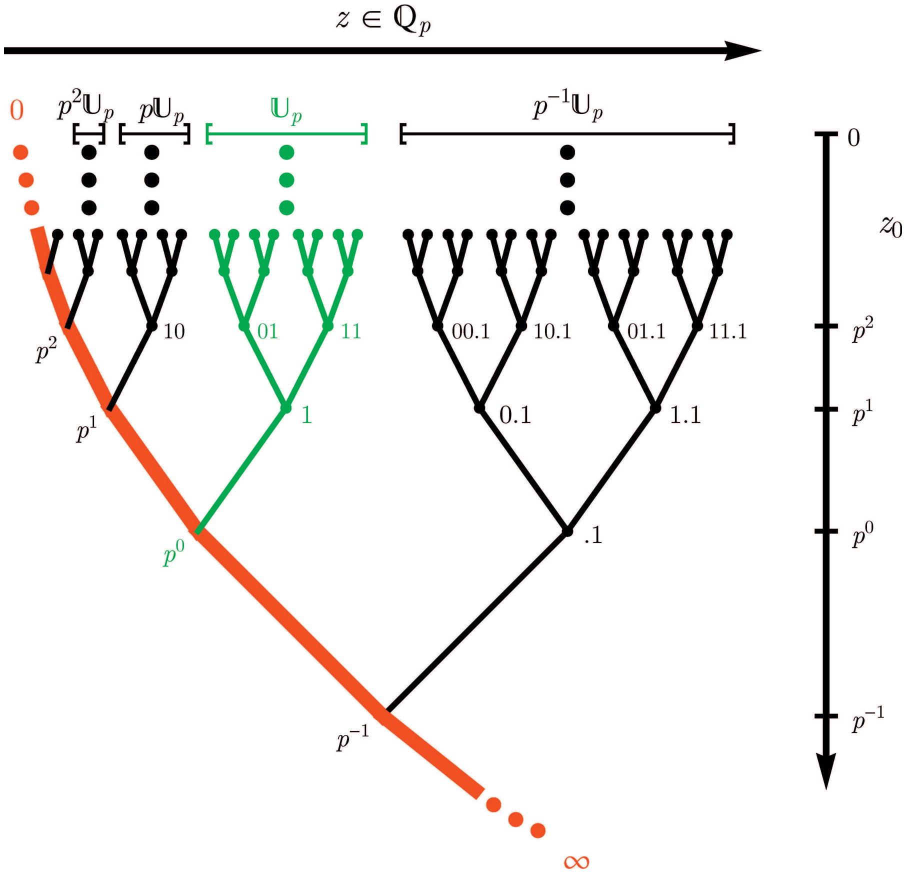

FIGURE 1. The Bruhat–Tits tree for $\mathbb{Q}_p$ with $p = 2$. Coordinates (z_0, z) with $z_0 = p^\omega, \omega \in \mathbb{Z}$, and $z \in \mathbb{Q}_p$ are used to label the nodes on the tree.

this group analogous to the behavior of correlators in real Euclidean AdS/CFT under conformal transformations.

We study an interacting massive-scalar model on the tree, defined by the action

$$S[\phi] = \eta_p \sum_{\langle ab \rangle} \frac{1}{2}(\phi_a - \phi_b)^2 + \eta_p \sum_a \left(\frac{1}{2} m_p^2 \phi_a^2 + \frac{g_3}{3!}\phi_a^3 + \frac{g_4}{4!}\phi_a^4 \right), \tag{1}$$

where a and b label nodes on the tree and $\langle ab \rangle$ indicates that the sum is over nearest-neighbor sites, i. e., over the edges. η_p, m_p^2, g_3, and g_4 are real constants. Employing the usual AdS/CFT prescription we compute the boundary-theory 2- and 3-point correlators as well as the 4-point correlator arising from bulk contact diagrams. To this end, the bulk-to-bulk and bulk-to-boundary propagators in the bulk theory need to be determined first. The bulk-to-bulk propagator $G(a, b)$ for two bulk points a, b is the Green's function of the free part of the equation of motion associated with Eq. (1), while the bulk-to-boundary propagator $K(a, x)$, where a is a node on the tree and $x \in \mathbb{Q}_q$, is a limit of $G(a, b)$ subject to a certain normalization condition. We derive expressions for G and K that are analogous in structure to the well-known results in ordinary AdS/CFT but are considerably simpler in details. Knowledge of K allows to construct bulk solutions $\phi(z_0, z)$ from boundary data $\phi_0(z)$. The on-shell bulk action $S_{\text{on-shell}}$ can then be expressed in terms of $\phi_0(z)$. The conjectured fundamental relation of AdS/CFT posits that $\exp(-S_{\text{on-shell}}[\phi_0(z)])$ is the generating functional for correlation functions of the operator $\mathcal{O}(z)$ dual to the bulk field ϕ; the latter's boundary data $\phi_0(z)$ is the source conjugate to $\mathcal{O}(z)$. Details and subtleties in applying this in the context of our construction can be found in [10].

The results for the 2- and 3-point correlators are

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_p = \eta_p \, \frac{p^\Delta}{\zeta_p(2\Delta - n)} \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta - n)} \frac{1}{|x_{12}|_q^{2\Delta}} \,, \tag{2}$$

$$\langle \mathcal{O}(\vec{x}_1)\mathcal{O}(\vec{x}_2)\rangle_\infty = \eta_\infty (2\Delta - n) \frac{\zeta_\infty(2\Delta)}{\zeta_\infty(2\Delta - n)} \frac{1}{|\vec{x}_{12}|^{2\Delta}} \,, \tag{3}$$

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle_p = -\eta_p g_3 \frac{\zeta_p(\Delta)^3 \, \zeta_p(3\Delta - n)}{\zeta_p(2\Delta - n)^3} \frac{1}{|x_{12}x_{23}x_{13}|_q^\Delta} \,, \tag{4}$$

$$\langle \mathcal{O}(\vec{x}_1)\mathcal{O}(\vec{x}_2)\mathcal{O}(\vec{x}_3)\rangle_\infty = -\eta_\infty g_3 \frac{\zeta_\infty(\Delta)^3 \, \zeta_\infty(3\Delta - n)}{2\, \zeta_\infty(2\Delta - n)^3} \frac{1}{(|\vec{x}_{12}||\vec{x}_{23}||\vec{x}_{13}|)^\Delta} \,. \tag{5}$$

Besides our p-padic results we have also given the results from ordinary $\mathrm{AdS}_{n+1}/\mathrm{CFT}_n$, indicated by a formal subscript '∞'. We have written $x_{ij} := x_i - x_j$, and analogously for $\vec{x}_{ij}$. Using the local zeta functions $\zeta_p(s) := 1/(1 - p^{-s})$ and $\zeta_\infty(s) := \pi^{-s/2}\,\Gamma(s/2)$ in the expressions above, we observe striking similarities between the p-adic correlators and the corresponding correlation functions in ordinary AdS/CFT. Furthermore, we observe a match between the p-adic 4-point correlator and the leading logarithmic term in the 4-point correlator in ordinary AdS/CFT. These close analogies might point to some interesting underlying adelic structure that appears certainly worth exploring further.

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Interacting Quantum Theories on de Sitter Space

Christian Jäkel
(joint work with João C.A. Barata and Jens Mund)

In 1975 Figari, Høegh-Krohn and Nappi [4] constructed the $\mathcal{P}(\varphi)_2$ model on the two-dimensional de Sitter space

$$dS \doteq \left\{ x \in \mathbb{R}^{1+2} \mid x_0^2 - x_1^2 - x_2^2 = -r^2 \right\} , \quad r > 0 .$$

In this talk, I will present a novel construction of this model [1], which emphasizes group theoretic and operator algebraic aspects.

I will start with a brief discussion of the causality structure of de Sitter space. In the sequel, I will describe the free classical dynamical system in both its covariant and its canonical form, and present the associated quantum one-particle KMS structures. The latter are related to representations of $O(1,2)$. While a covariant representation of $O(1,2)$ was provided by Bros and Moschella in [3], the *unitary irreducible representations* of $SO_0(1,2)$ for both the principal and the complementary series on the Hilbert space spanned by wave functions with support on a (closed) space-like geodesic are due to the authors. Second quantisation provides a description of free bosons on the de Sitter space in terms of canonical fields φ and canonical momenta π associated to the Cauchy surface. This formulation is unitarily equivalent to the one provided in [3] by Bros and Moschella. It, however, remains to justify that the Fock zero-particle vector $\Omega_\circ$ represents the physically relevant de Sitter vacuum state. This is not completely obvious: on de Sitter space, there is no global time evolution (in terms of a one-parameter group of automorphisms) and hence no natural notion of energy. Consequently, one can not require that the *vacuum state* is a state of minimal energy. One may still require that a *de Sitter vacuum state* is invariant under the action of the Lorentz group. But in itself, this requirement does *not* guarantee the necessary stability. The so-called the *geodesic KMS condition* proposed by Borchers and Buchholz [2] ensures stability: it requires that the restriction of the vacuum state to the wedge

$$W_1 \doteq \left\{ x \in dS \mid x_2 > |x_0| \right\} ,$$

is a thermal state with respect to the dynamics provided by the one-parameter group $t \mapsto \exp(itL_\circ)$ of boosts which leaves the wedge W_1 invariant. The *unique* rotation invariant state satisfying this condition is the one induced by the Fock vacuum vector $\Omega_\circ$. The temperature of this thermal state is thereby fixed, and it is the Hawking temperature $T = (2\pi r)^{-1}$.

Exploring Euclidean methods, I will then describe how one can add an interaction. Instead of introducing stochastic processes, which are frequently used in the literature to describe interacting Euclidean quantum fields, I prefer to work directly on the Fock space associated to the free Euclidean field on the Euclidean sphere. As I will show, the latter contains a rotation invariant vector which represents the interaction Euclidean vacuum state for the $\mathcal{P}(\varphi)_2$ model. This new Euclidean vacuum state satisfies reflection positivity[1] with respect to reflections leaving invariant a great circle on the Euclidean sphere and thereby gives rise to a new representation of $SO(1,2)$ on the two-dimensional de Sitter space.

As the free Euclidean field on the sphere satisfies the Markov property, the Osterwalder-Schrader reconstruction is given by a projection from the Euclidean Fock space for the sphere to the canonical Hilbert space for the free massive field on the de Sitter space, introduced in the first part of my talk. Hence, the interacting vacuum can be represented by a vector in the Fock space associated to the canonical free field on the de Sitter space. This is not surprising, as the ultraviolet problems are tame in $1+1$ space-time dimensions and the spatial volume is compact. Physical *infrared* problems are absent on de Sitter space despite the fact that Lorentzian perturbation theory is plagued by infrared problems. (We note that artificial infrared problems were introduced in [4] by an unfortunate choice of coordinates. However, these infrared problems are absent in our formulation.) Hence, the $\mathcal{P}(\varphi)_2$ model on the de Sitter space is in some sense the simplest model which satisfies all the basic expectations of an interacting relativistic quantum (field) theory, such as finite speed of light, particle production, causality, and so on.

Looking at the more technical aspects, we note that local symmetric semi-groups techniques [5, 7] are used to justify the existence of the operator sum

$$L := \overline{L_\circ + V} \; , \qquad V = \int_{S^1} r \cos\psi \, \mathrm{d}\psi \;\; {:}\mathcal{P}(\varphi(0,\psi)){:} \; , \tag{1}$$

with $\mathcal{P}$ a real valued polynomial, bounded from below. The theory of virtual representations [6] is used to prove that the newly defined one-parameter unitary groups actually give rise to a representation of $SO(1,2)$. In fact, the boosts $t \mapsto \exp(itL)$ together with the free rotations, which preserve the Cauchy surface, generate a new reducible, unitary *representation* of the *Lorentz group* on Fock space. The resulting *interacting* quantum field, defined by setting

$$\Phi(x) \doteq U(\Lambda)\varphi(0)U^{-1}(\Lambda) \; , \quad x = \Lambda \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix} \right) \; ,$$

coincides with the free field on the Cauchy surface $x_0 = 0$ and elsewhere on dS it satisfies the following non-linear *equation of motion*:

$$(\Box_{dS} + m^2)\Phi = -\mathcal{P}'(\Phi) \; , \qquad m > 0 \; .$$

The sum given in (1) provides the crucial link between the free and the interacting quantum field theory, as L is the generator of the modular group which leaves the von Neumann algebra $\mathcal{A}_\circ(W_1)$ for the free field associated to the wedge W_1

[1]This version of reflection positivity thus differs from the pioneering work by Figari, Høegh-Krohn and Nappi [4], where two antipodal points were taken out of the sphere and reflection positivity was formulated with respect to a half-circle connecting these two points.

invariant. The vector representing the interacting vacuum is given by Araki's perturbation theory of modular automorphisms:

$$\Omega = \frac{\mathrm{e}^{-\pi H}\Omega_\circ}{\|\mathrm{e}^{-\pi H}\Omega_\circ\|}\ , \qquad H := L_\circ + \int_0^\pi r\cos\psi\,\mathrm{d}\psi\ :\!\mathcal{P}(\varphi(0,\psi))\!:\ .$$

The de Sitter vacuum state induced by Ω, characterised by the geodesic KMS condition [2], has some surprising properties. Due to thermalisation effects introduced by the curvature of space-time, it is unique even for large coupling constants, despite the fact that different phases occur in the limit of curvature to zero (i.e., the Minkowskian limit).

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Wedge-local fields in interacting quantum field theories with bound states

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The construction of interacting quantum field theories is a hard task due to the complicated structure of local observables in the presence of interaction. In the class of *quantum integrable models* in $1+1$-dimensional Minkowski spacetime the construction becomes more tractable due to the particular type of interaction. Integrable models describe systems of relativistic particles subject to elastic scattering, where the momenta of the particles and the particle number are conserved, yielding infinitely many conserved quantities. As a consequence, the S-matrix is of "factorizing" type in the sense that the scattering of n particles is the product of two particle scattering processes.

These models have been treated with various methods. Using the perturbative approach, one computes the S-matrix elements, and therefore the Green's functions, of the theory. However, Lagrangians are usually complicated enough that the construction remains at perturbative level. An exception is the sine-Gordon model where Fröhlich and Seiler [1] compute the Euclidean Green's functions and show that the S-matrix is non-trivial. However, it has not been proven that the

S-matrix is factorizing. Alternatively, the *Form Factor Programme* [2, 3] bases the construction on an inverse scattering problem. One postulates the form of the S-matrix and, via a list of axioms deduced from physical requirements and consistent with the properties of the S-matrix, constructs the n-point functions of the theory by expanding them in a series of *form factors* (i.e., certain matrix elements of the interacting fields). These yield infinite series expansions whose convergence is hard to control.

A more recent idea due to Schroer [4] avoids these infinite series by considering, instead of strictly local operators, observables localized in unbounded wedge-shaped regions. This weaker localization property allows to construct observables with a simpler expression in momentum space. Strictly local observables can then be recovered by taking intersection of the algebras generated by observables in *right* and *left* wedges. Using an abstract argument based on a certain phase space property called *modular nuclearity*, one would show that this intersection is non-trivial. Finally, one would solve the inverse scattering problem by computing the S-matrix of the input by methods of Haag-Ruelle scattering theory. This *operator-algebraic approach* has proved to be successful for the construction of a large class of integrable models. These are models of bosons with an S-matrix analytic in the *physical strip* (a certain region of the momentum complex plane), including the Ising and sinh-Gordon models with one particle species [5], Federbush-type models [10] with several particle species; there are also partial results in the $O(N)$ non-linear sigma models [6, 7].

However, models where S-matrix components have singularites in the physical strip have not been treated in this framework before. In [8, 9], together with Y. Tanimoto, I obtained first results in this direction. The models we consider have S-matrices whose components have a certain pole structure in the physical strip. Examples are the Bullough-Dodd model, the $Z(N)$-Ising model, the affine-Toda field theories, the sine-Gordon and Thirring models. These models are of interest since a pole in the physical strip is believed to correspond to a bound state. The idea is that two particles of type α, β can scatter with the exchange of a unitary factor, the S-matrix component $S^{\alpha\beta}_{\beta\alpha}(\theta_1 - \theta_2)$, but the type of particles stays the same in scattering. In this case, the S-matrix is said to be "diagonal". We denote with θ_1, θ_2 the *rapidities* of the particles α, β, which parametrize their momenta. The two particles can also fuse into a third particle of type γ in the following sense:

$$p_{m_\alpha}(\theta + i\theta_{(\alpha\beta)}) + p_{m_\beta}(\theta - i\theta_{(\beta\alpha)}) = p_{m_\gamma}(\theta), \tag{1}$$

i.e., the momenta of two "virtual" particles add to the momentum of a third "real" particle (the bound particle) which lies on the mass shell. The numbers $\theta_{(\alpha\beta)}, \theta_{(\beta\alpha)}$ are solutions of this equation, once the masses of the particles are fixed. The bound state formed would correspond to a pair of simple poles of the component $S^{\alpha\beta}_{\beta\alpha}(\zeta)$ in the physical strip $0 < \operatorname{Im}\zeta < \pi$. These are the *s-channel* pole, which is located at $\zeta = i\theta_{\alpha\beta} = i\theta_{(\alpha\beta)} + i\theta_{(\beta\alpha)}$ and the *t-channel* pole, which arises due symmetry properties of the S-matrix. The possible fusion processes, that we denote by the symbol $(\alpha\beta) \to \gamma$, are characteristic of each model and they form their fusion tables. Here, we will focus on the $Z(N)$-Ising model, where the particles are of type $1, \dots, N-1$, the possible fusions are of the form $(\alpha\beta) \to \alpha + \beta \bmod N$, and the anti-particle $\bar{\alpha}$ of a particle of type α is $N - \alpha$.

For each pair α, β, the component $S^{\alpha\beta}_{\beta\alpha}(\zeta)$ of the S-matrix is a meromorphic function on $\mathbb{C}$, fulfilling a number of axioms [9, Sec. 2.1], including unitarity, crossing symmetry and the Bootstrap equation. As an example, the crossing symmetry relation reads $S^{\alpha\beta}_{\beta\alpha}(i\pi - \zeta) = S^{\bar{\beta}\alpha}_{\alpha\bar{\beta}}(\zeta)$, for $\zeta \in \mathbb{C}$. In the $Z(N)$-Ising model, the component $S^{11}_{11}(\zeta)$, fulfilling the above properties, is given by

$$S^{11}_{11}(\zeta) = \frac{\sinh \frac{1}{2}\left(\zeta + \frac{2\pi i}{N}\right)}{\sinh \frac{1}{2}\left(\zeta - \frac{2\pi i}{N}\right)}, \tag{2}$$

and one can construct all the other S-matrix components by using the Bootstrap equation. Specfically, the components $S^{\alpha\beta}_{\beta\alpha}$ with only indices of type 1 and $N-1$ have at most two simple poles in the physical strip and no other poles. These components play a crucial role in the proof of *weak wedge-commutativity*, as we will explain below.

For an S-matrix $S^{\alpha\beta}_{\beta\alpha}$ analytic in the physical strip, Lechner constructed quantum fields ϕ, ϕ' with the property that they commute when smeared with test functions supported in the left and right wedge, respectively. This computation relies on a shift of an integral contour where the integrand contains $S^{\alpha\beta}_{\beta\alpha}$. In the case where the S-matrix has poles in the physical strip, shifting the integral contour yields the residues of $S^{\alpha\beta}_{\beta\alpha}$, and ϕ, ϕ' are no longer wedge-local. To overcome this problem, we introduce a new field $\tilde{\phi} = \phi + \chi$ by adding the so called *bound state operator* χ to the field ϕ, so that the commutator of χ with its reflected operator (by the action of the CPT operator) χ' cancels the above residues. χ acts on a one-particle vector ξ^β as follows, if $(\alpha\beta)$ fuse into some γ,

$$\left(\chi_{1,\alpha}(f)\xi\right)^\gamma(\theta) := -i\eta^\gamma_{\alpha\beta} f^+_\alpha(\theta + i\theta^\gamma_{(\alpha\beta)})\xi^\beta(\theta - i\theta_{(\beta\alpha)}), \tag{3}$$

where the matrix elements $\eta^\gamma_{\alpha\beta}$ are related to the residues of the S-matrix. One can show that this operator is densely defined and symmetric on a suitable domain of vectors, but it is not self-adjoint on a naive domain. To prove (weak) wedge-commutativity, one shows that the commutator of the reflected field $\tilde{\phi}'$ with $\tilde{\phi}$ vanishes for test functions supported in right and left wedges, respectively, in the weak sense, i.e. in matrix elements between suitable vectors. In this computation in the $Z(N)$-Ising model, the components of the test functions, and therefore the S-matrix components, are restricted to particles of type 1 and $N-1$, limiting the number of poles in the physical strip. Strong commutativity has not been proven yet. For this, one would need to show the existence of self-adjoint extensions of $\tilde{\phi}(f)$ and $\tilde{\phi}'(g)$, and select the ones that strongly commute. This is a non-trivial task, but some progress has been made in the Bullough-Dodd model. $\tilde{\phi}(f)$ is also a polarization-free generator but not temperate. In the case of many particle species, we can prove the Reeh-Schlieder property only for models with two species of particles.

In the operator-algebraic approach, we are interested in the following question. Let us suppose that strong commutativity hold for a certain extension $\tilde{\phi}^-$. We consider the von Neumann algebras

$$\mathcal{A}(W_L + x) = \{e^{i\tilde{\phi}(f)^-} : \operatorname{supp} f \subset W_L + x\}'' , \tag{4}$$

and similarly for the right wedge. Observables localized in bounded regions are obtained as intersections of von Neumann algebras

$$\mathcal{A}(\mathcal{O}) := \mathcal{A}(W_L + x) \cap \mathcal{A}(W_R - y), \quad \mathcal{O} = W_L + x \cap W_R - y. \tag{5}$$

The problem would then be to show that this intersection is non-trivial, i.e., technically, that the vacuum of the theory is cyclic and separating for the local algebra. Finally, *Haag-Ruelle scattering theory* would be applied to compute the S-matrix of the input and solve the inverse scattering problem. This would yield a complete construction of the theory, and is part of our future work.

Concluding, the construction of integrable models with bound states is a new promising direction in constructive quantum field theory. The Bullough-Dodd model, the $Z(N)$-Ising model and affine-Toda field theories are among those models which we hope to fully construct using operator-algebraic techniques. An interesting problem would be to extend our construction to the sine-Gordon and Thirring models, comparing our construction to the Euclidean methods in [1]. It would also be interesting to investigate whether such models can be seen as deformation of a free field theory in the spirit of Lechner's work, as well as to study the quantum group symmetry of the affine-Toda field theories.

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Generalized Wentzell boundary conditions and holography

Jochen Zahn

Holography has been a main theme in theoretical high energy physics and quantum gravity in the last two decades. Inspired by the gauge/gravity duality, studies of holographic aspects were often considering $d+1$ dimensional Anti deSitter space (AdS) and its (conformal) boundary, d dimensional Minkowski space. However, holography is a generic aspect of quantum field theory on space-times with time-like boundaries, raising the question which of the properties of holography on AdS are generic, and which ones are special to AdS. Let us list some of the properties of holography on AdS:

- The boundary theory is a conformal field theory. This is a generic feature on AdS [1, 2], independent of the concrete choice of the bulk fields.
- The correspondence maps bulk observables localized in bounded regions to boundary observables localized in bounded regions.
- For a bulk theory with local observables, the boundary theory will not fulfill the time-slice axiom [1].
- The boundary conformal field in general has a positive anomalous dimension. The basic example is the massive scalar field [3].

In the following, we study the holographic relation between a massive scalar field on $d+1$ dimensional Minkowski space with d dimensional time-like boundaries. Not surprisingly, we find that the boundary field theory is not conformal, and that the bulk observables localized in bounded space-time regions are mapped to boundary observables that are delocalized. The first two properties in the above list thus seems to be specific to AdS. Also in our setting, the time-slice axiom does not hold for the boundary theory.

Regarding the last point in the above list, it seems obvious that the boundary field, being the boundary limit of the bulk field, inherits its short-distance behavior. One would this expect that the boundary field generically has an anomalous dimension. However, it turns out that there are boundary conditions for which this is not the case. These are so-called generalized Wentzell boundary conditions, which we discuss in this contribution, which is based on [4]. Concretely, we consider a free scalar field on the $(d+1)$-dimensional space-time $M = \mathbb{R} \times \Sigma$, with the spatial slices Σ having a boundary $\partial\Sigma$. The main example will be the strip $\Sigma = \mathbb{R}^{d-1} \times [-S, S]$. We do not impose boundary conditions by hand, but supplement the bulk action with an action for the boundary, which is of the same form. Concretely,

$$\mathcal{S} = \mathcal{S}_{\mathrm{bk}} + \mathcal{S}_{\mathrm{bd}} = -\frac{1}{2}\int_M g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mu^2\phi^2 - \frac{c}{2}\int_{\partial M} h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mu^2\phi^2 \qquad (1)$$

where g is the Minkowski metric on the bulk M and h is the induced metric on the boundary ∂M. c is a positive constant with the dimension of a length. Similar actions are used for the description of strings with masses at the ends [5] and, in the context of the AdS/CFT correspondence, for *holographic renormalization* [6, 7]. However, it is interesting to note that in the latter context, the analog of c is usually negative, whereas we require a positive c in order to prove the well-posedness of the Cauchy problem.

Variation of the action (1) yields, upon integration by parts, the equations of motion

$$-\Box_g \phi + \mu^2 \phi = 0 \qquad \text{in } M, \tag{2}$$

$$-\Box_h \phi + \mu^2 \phi = c^{-1} \partial_\perp \phi \qquad \text{in } \partial M. \tag{3}$$

Here $\partial_\perp$ denotes the inward pointing normal derivative. Boundary conditions involving second order derivatives are known as *generalized Wentzell*, *Feller-Wentzell type*, *kinetic*, or *dynamical*, [8, 9, 10, 11, 12].

From the action (1), one derives the symplectic form

$$\sigma((\phi, \dot{\phi}), (\psi, \dot{\psi})) = \int_\Sigma \phi\dot{\psi} - \dot{\phi}\psi + c \int_{\partial\Sigma} \phi\dot{\psi} - \dot{\phi}\psi.$$

It is thus natural to introduce the Hilbert space $H = L^2(\Sigma) \oplus L^2(\partial\Sigma)$. An element of H is typically written as $\Phi = (\phi_{\mathrm{bk}}, \phi_{\mathrm{bd}})$, with the scalar product

$$\langle \Phi, \Psi \rangle = \langle \phi_{\mathrm{bk}}, \psi_{\mathrm{bk}} \rangle_{L^2(\Sigma)} + c \langle \phi_{\mathrm{bd}}, \psi_{\mathrm{bd}} \rangle_{L^2(\partial\Sigma)}.$$

Note that for negative c, one would have to work with a Krein space. On H, the equation of motion (2), (3) can be written as

$$-\partial_t^2 \Phi = \Delta \Phi,$$

with

$$\Delta = \begin{pmatrix} -\Delta_\Sigma + \mu^2 & 0 \\ -c^{-1}\partial_\perp \cdot | & -\Delta_{\partial\Sigma} + \mu^2 \end{pmatrix},$$

where we encode the boundary condition in the domain

$$D = \left\{ (\phi_{\mathrm{bk}}, \phi_{\mathrm{bd}}) \in H \mid \phi_{\mathrm{bk}} \in H^2(\Sigma), \phi_{\mathrm{bd}} \in H^2(\partial\Sigma), \phi_{\mathrm{bk}}|_{\partial\Sigma} = \phi_{\mathrm{bd}} \right\}.$$

One can show that Δ is self-adjoint.

Given the self-adjoint operator Δ, one may construct energy Hilbert spaces for Cauchy data in complete analogy to the case of Dirichlet boundary conditions. Taking the intersection of these, one establishes the existence of a unique smooth solution for smooth initial data that fulfill certain boundary conditions and are of finite energy. The restriction to finite energy can be lifted by establishing causal propagation. This proceeds by identifying bulk and boundary stress energy tensors fulfilling the dominant energy condition. One shows that no energy flows through the boundary and thus establishes local energy estimates. Also at this step, the positivity of c is essential.

Using Δ, it is also straightforward to canonically quantize the system. One establishes that

- The time-zero fields can be restricted to the boundary.
- The space-time fields are as Wightman-like as possible.
- One may restrict the space-time fields to the boundary, obtaining a generalized free field ϕ_{bd}.

- The short-distance singularity of $\phi_{\rm bd}$ is that of a scalar field in d space-time dimensions, i.e., it does not have an anomalous dimension.
- The boundary field $\phi_{\rm bd}$ satisfies the Reeh-Schlieder property but does not fulfill the time-slice axiom.
- There is a holographic relation between bulk and boundary fields, i.e., $\phi_{\rm bk}(f) = \phi_{\rm bd}(F(f))$, where $F : \mathcal{S}(M) \to \mathcal{S}(\partial M)$. An analogous map also exists for Wick powers. It is in general not possible to choose $F(f) \in \mathfrak{D}(M)$ here, i.e., locality is lost. For the half-space $\Sigma = \mathbb{R}^{d-1} \times [0, \infty)$, locality is lost even further, as $F(f)$ will in general only be a smooth L^2 function.

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The operator product expansion for Yang-Mills theories

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(joint work with Jan Holland and Stefan Hollands)

In a quantum field theory, the *operator product expansion* (OPE) is the statement that correlation functions of composite operators $\mathcal{O}_A$ in any physically reasonable quantum state $|\psi\rangle$ (say, of finite total energy) can be approximated by a sum over state-independent coefficients $\mathcal{C}^B_{A_1 \cdots A_s}$ (the *OPE coefficients*) and expectation values of a single composite operator,

$$\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_s}(x_s) \rangle_\psi \approx \sum_B \mathcal{C}^B_{A_1 \cdots A_s}(\boldsymbol{x}) \, \langle \mathcal{O}_B(x_s) \rangle_\psi \ ,$$

where $\boldsymbol{x} \equiv (x_1, \ldots, x_s)$. The quality of the approximation depends on the state $|\psi\rangle$ and on how many composite operators $\mathcal{O}_B$ are included on the right-hand side. All the following theorems apply to arbitrary power-counting renormalisable (Euclidean) quantum field theories to all orders in formal perturbation theory in $\hbar$, and in Ref. [2] we have shown

Theorem 1. *The OPE holds as an asymptotic expansion:*

$$\lim_{\tau\to 0} \tau^{\Delta} \left[\langle \mathcal{O}_{A_1}(\tau x_1) \cdots \mathcal{O}_{A_s}(\tau x_s) \rangle_\psi - \sum_{B\colon [\mathcal{O}_B] < D} \mathcal{C}^{B}_{A_1 \cdots A_s}(\tau \boldsymbol{x}) \, \langle \mathcal{O}_B(\tau x_s) \rangle_\psi \right] = 0 \,,$$

where $\Delta \equiv [\mathcal{O}_{A_1}] + \cdots + [\mathcal{O}_{A_s}] - D + \delta$ with an arbitrary $\delta > 0$, and $[\mathcal{O}]$ denotes the engineering dimension of the composite operator $\mathcal{O}$. It holds in all states $|\psi\rangle$ which are obtained by acting with smeared field operators on the vacuum, for massless theories as long as the smearing does not involve exceptional momenta.

We have furthermore proven in Ref. [2] that the OPE coefficients are well-defined distributions, and that they are associative,

Theorem 2. *For all insertion points x_i satisfying*

$$\max_{1 \le i < k} |x_i - x_k| < \min_{k < j \le s} |x_j - x_k| \,,$$

the OPE coefficients fulfill the associative or factorisation property

$$\mathcal{C}^{B}_{A_1 \cdots A_s}(\boldsymbol{x}) = \sum_C \mathcal{C}^{C}_{A_1 \cdots A_k}(x_1, \ldots, x_k) \mathcal{C}^{B}_{C A_{k+1} \cdots A_s}(x_k, \ldots, x_s) \,.$$

In particular, this implies that one can perform the OPE for three or more operators in two or more steps and obtains the same result, as long as one first performs the OPE for the operators which lie closest together.

In Yang-Mills theories, the basic dynamical field is a Lie-algebra valued one-form A and the action is given by

$$S = \frac{1}{2} \int \mathrm{tr}\, F \wedge \star F \,,$$

where $F \equiv \mathrm{d}A + \mathrm{i}g[A, A]$ is the field strength, g is a coupling constant and tr denotes contraction with the Cartan-Killing metric. It is well known that because of gauge invariance – the invariance of S under $A \to A - \mathrm{i}\,\mathrm{d}f + g[A, f]$ for any smooth function f – a direct perturbative quantisation is impossible. The modern way to deal with this is the Batalin-Vilkovisky formalism, where one introduces additional fields (e.g., the ghost c and antighost $\bar{c}$, which are Grassmann and Lie-algebra valued functions) and antifields $A^\ddagger$, $c^\ddagger$, $\bar{c}^\ddagger$, ..., for each field, which have opposite Grassmann parity and form degree to the field. One then defines an *antibracket* $(\cdot, \cdot)$ by linearity and a graded Leibniz rule and by declaring field and antifield to be conjugate. The action S is also augmented in such a way that $(S, S) = 0$, and one checks that the *Slavnov-Taylor differential* $\hat{\mathsf{s}}$ defined by $\hat{\mathsf{s}}F \equiv (S, F)$ satisfies $\hat{\mathsf{s}}^2 = 0$ and $\hat{\mathsf{s}}\,\mathrm{d} + \mathrm{d}\,\hat{\mathsf{s}} = 0$. We can thus define the cohomology classes $H^{g,p}(\hat{\mathsf{s}})$ of local functionals of form degree p and ghost number g [where ghost number is additive, 1 for c and -1 for $\bar{c}$ and 0 for all other fields, and $g(\phi^\ddagger) = -g(\phi) - 1$], and the observables of the original Yang-Mills theory are recovered as elements of $H^{0,p}(\hat{\mathsf{s}})$. The theory defined by the augmented action can be perturbatively quantised, and the quantum observables are obtained as elements of $H^{0,p}(\hat{\mathsf{q}})$ where $\hat{\mathsf{q}} = \hat{\mathsf{s}} + \mathcal{O}(\hbar)$ is the *quantum Slavnov-Taylor differential.* To have a consistent theory, the correlation functions must be well defined on the cohomology classes, which are the *Ward identities* in the quantum theory. We have proven in Ref. [1] that the proper Ward identities hold if the relative cohomology class (invariant under the global E(4) symmetry) $H^{1,4}_{\mathrm{E}(4)}(\hat{\mathsf{s}}|\,\mathrm{d})$ is empty, and thus in particular for pure Yang-Mills theory based on a semi-simple Lie algebra.

The Ward identities for the correlation functions translate into Ward identities for the OPE coefficients, and in Ref. [2] we have proven

Theorem 3. *The OPE coefficients fulfil the Ward identity*

$$\begin{aligned} 0 = \sum_{k=1}^{s} \sum_{C} \mathcal{Q}_{A_k}{}^{C} \mathcal{C}^{B}_{A_1 \cdots A_{k-1} C A_{k+1} \cdots A_s}(\boldsymbol{x}) - \sum_{C} \mathcal{Q}_{C}{}^{B} \mathcal{C}^{C}_{A_1 \cdots A_s}(\boldsymbol{x}) \\ - \hbar \sum_{1 \leq k < l \leq s} \sum_{E,w} \mathcal{C}^{B}_{A_1 \cdots A_{k-1} E A_{k+1} \cdots A_{l-1} A_{l+1} \cdots A_s}(\boldsymbol{x}) \mathcal{B}^{E,w}_{A_k A_l} \partial^w_{x_k} \delta^4(x_k - x_l) \end{aligned}$$

if $H^{1,4}_{\mathrm{E}(4)}(\hat{\mathsf{s}}|\,\mathrm{d}) = 0$. *Here, the coefficients* $\mathcal{Q}_A{}^B$ *are defined by*

$$\hat{\mathsf{q}} \mathcal{O}_A \equiv \sum_{B\colon [\mathcal{O}_B] \leq [\mathcal{O}_A]+1} \mathcal{Q}_A{}^B \mathcal{O}_B \,,$$

and the coefficients $\mathcal{B}^{C,w}_{AB}$ *are defined by*

$$(\mathcal{O}_A(x), \mathcal{O}_B(y))_\hbar \equiv \sum_{C\colon [\mathcal{O}_C] \leq [\mathcal{O}_A]+[\mathcal{O}_B]-3} \mathcal{O}_C(x) \sum_w \mathcal{B}^{C,w}_{AB} \partial^w_x \delta^4(x-y) \,,$$

where $(\cdot,\cdot)_\hbar = (\cdot,\cdot) + \mathcal{O}(\hbar)$ *is the* quantum antibracket*, and* w *is a multiindex.*

Especially, this identity implies that the OPE of gauge-invariant operators only involves gauge-invariant operators on the right-hand side, where a gauge-invariant operator in the quantum theory is an element of $H^{0,p}(\hat{\mathsf{q}})$.

Lastly, we have also proven in Ref. [2] that one can recursively construct the OPE coefficients, the quantum Slavnov-Taylor differential and the quantum antibracket as a formal power series in the coupling constant g, given as only input the corresponding quantities in the free theory and a choice of *interaction operator* $\mathcal{O}_\mathcal{I}$ of ghost number 0 and dimension $1 \leq [\mathcal{O}_\mathcal{I}] \leq 4$

$$\mathcal{O}_\mathcal{I} = \sum_{A\colon 1 \leq [\mathcal{O}_A] \leq 4} \mathcal{I}^A \mathcal{O}_A \,,$$

which is defined up to a total derivative and fulfills

$$\mathcal{O}_\mathcal{I} = \partial_g S|_{g=0} + \mathcal{O}(g) + \mathcal{O}(\hbar) \,, \qquad \hat{\mathsf{q}} \mathcal{O}_\mathcal{I} = \mathrm{d}\mathcal{O}' \,.$$

for some other operator $\mathcal{O}'$. Namely, we have shown that

Theorem 4. *The derivative of the OPE coefficients and the quantum Slavnov-Taylor differential and antibracket with respect to the coupling constant g reads*

$$\hbar \partial_g \mathcal{C}^B_{A_1 \cdots A_s}(\boldsymbol{x}) = \int \sum_E \mathcal{I}^E \Bigg[- \mathcal{C}^B_{EA_1 \cdots A_s}(y, \boldsymbol{x}) + \sum_{C\colon [\mathcal{O}_C] < [\mathcal{O}_B]} \mathcal{C}^C_{A_1 \cdots A_s}(\boldsymbol{x}) \mathcal{C}^B_{EC}(y, x_s)$$
$$+ \sum_{k=1}^{s} \sum_{C\colon [\mathcal{O}_C] \leq [\mathcal{O}_{A_k}]} \mathcal{C}^C_{EA_k}(y, x_k) \mathcal{C}^B_{A_1 \cdots A_{k-1} C A_{k+1} \cdots A_s}(\boldsymbol{x}) \Bigg] \mathrm{d}^4 y \,,$$

$$\hbar \partial_g \mathcal{Q}_A{}^B = \int \sum_E \mathcal{I}^E \Bigg[\sum_{C\colon [\mathcal{O}_C] \leq [\mathcal{O}_A]} \mathcal{C}^C_{EA}(y, 0) \mathcal{Q}_C{}^B - \sum_{C\colon [\mathcal{O}_B] \leq [\mathcal{O}_C]} \mathcal{Q}_A{}^C \mathcal{C}^B_{EC}(y, 0) \Bigg] \mathrm{d}^4 y$$
$$+ \hbar \sum_E \mathcal{I}^E \widetilde{\mathcal{B}}^B_{EA} \,,$$

and

$$\hbar \partial_g \widetilde{\mathcal{B}}^B_{A_1 A_2} = \int \sum_E \mathcal{I}^E \Bigg[\sum_{C\colon [\mathcal{O}_C] \leq [\mathcal{O}_{A_1}]} \widetilde{\mathcal{B}}^B_{CA_2} \mathcal{C}^C_{EA_1}(y, 0) + \sum_{C\colon [\mathcal{O}_C] \leq [\mathcal{O}_{A_2}]} \widetilde{\mathcal{B}}^B_{A_1 C} \mathcal{C}^C_{EA_2}(y, 0)$$
$$- \sum_{C\colon [\mathcal{O}_C] = [\mathcal{O}_A] + [\mathcal{O}_B] - 3} \widetilde{\mathcal{B}}^C_{A_1 A_2} \mathcal{C}^B_{EC}(y, 0) \Bigg] \mathrm{d}^4 y \,.$$

Here, the coefficients $\widetilde{\mathcal{B}}^C_{A_1 A_2}$ are in a one-to-one (but somewhat complicated) relation with the coefficients $\mathcal{B}^{C,w}_{AB}$ of theorem 3, and are concretely given by

$$\int \left(\mathcal{O}_A(x), \mathcal{O}_B(y) \right)_\hbar \mathrm{d}^4 x = \sum_{C\colon [\mathcal{O}_C] = [\mathcal{O}_A] + [\mathcal{O}_B] - 3} \widetilde{\mathcal{B}}^C_{AB} \mathcal{O}_C(y) \,.$$

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Symmetry Breaking in a Gas of Bosons

Horst Knörrer

(joint work with T.Balaban, J.Feldman and E.Trubowitz)

A report on the status of a program to prove symmetry breaking in the thermodynamic limit of a weakly interacting system of bosons on a three dimensional lattice. This is joint work with T.Balaban, J.Feldman and E.Trubowitz.

Having made the standard coherent state functional integral approach rigorous [1], and having treated the temporal ultraviolet problem [2], we are faced with a functional integral

$$\int e^{\mathcal{A}_0(\psi^*,\psi)} \prod_{x\in\mathbb{Z}/L^t\mathbb{Z}\times\mathbb{Z}^3/L^t\mathbb{Z}} \tfrac{d\psi^*(x)\wedge d\psi(x)}{2\pi i}$$

whose limit $s\to\infty,\ t\to\infty$ describes the partition function of the system up to "nonperturbatively small" terms (that is, terms that are smaller than any power of the coupling constant), and similar expressions for the correlations. Here, ψ is a complex valued field on $\mathcal{X}_0=\mathbb{Z}/L^t\mathbb{Z}\times\mathbb{Z}^3/L^s\mathbb{Z}$, the integral is over a set of ψ that satisfy a "small field bound" (proportional to a positive power of the coupling constant). The action is of the form

$$\mathcal{A}_0(\psi^*,\psi) \;=\; -\langle\psi^*,\,(\partial_0+\Delta)\psi\rangle \;-\; \lambda\mathcal{V}(\psi^*,\psi) \;+\; \mu\langle\psi^*,\psi\rangle \;+p(\psi^*,\psi)$$

where ∂_0 and Δ are the lattice "time derivative" and spatial Laplacian, respectively. $\mathcal{V}$ is a quartic expression for the interaction, λ is the coupling constant, μ the chemical potential, and the "perturbative correction" $p(\psi_*,\psi)$ is a power series in the independent fields ψ_*,ψ that converges in the small field region.

In the talk, I described how iterated block spin transformations transform this functional integral representation for the partition function into one where the corresponding action has a deep circular potential well (Mexican hat). This can be seen as the onset of the formation of a Goldstone boson and of symmetry breaking.

Block spin transformation associates to a function $F_0(\psi_*,\psi)$ of fields on $\mathcal{X}_0$ a function $B_1(\theta_*,\,\theta)$ of fields on the subset $\mathcal{X}_{-1}=L^2\mathbb{Z}/L^t\mathbb{Z}\times L\mathbb{Z}^3/L^s\mathbb{Z}^3$ of $\mathcal{X}_0$ as follows:

$$B_1(\theta_*,\,\theta)=\tfrac{1}{N}\int\prod_{x\in\mathcal{X}_0}\tfrac{d\psi^*(x)\wedge d\psi(x)}{2\pi i}\;e^{-\frac{1}{L^2}\langle\theta_*-Q\psi^*\,,\,\theta-Q\psi\rangle}\;F_0(\psi^*,\,\psi)$$

where $(Q\psi)(y)$ is the average of ψ over a block of sides $L^2\times L\times L\times L$ centered at $y\in\mathcal{X}_{-1}$, and N is the normalization constant (that depends only on L and the precise form of Q) that makes the integral of B_1 equal to the integral of F_0. After the block spin transformation, we rescale B_1 to a function F_1 on the unit lattice $\mathbb{Z}/L^{t-2}\mathbb{Z}\times\mathbb{Z}^3/L^{s-1}\mathbb{Z}^3$.

We apply this transformation with $F_0=e^{\mathcal{A}_0}$ and iterate the construction to obtain functions $F_1,F_2,\cdots$. The main result is the following representation of these functions:
For $n\le O(\log_L\frac{1}{\lambda})$ and a positive chemical potential of order λ,

$$F_n(\psi^*,\psi)=\tfrac{1}{\mathcal{Z}_n}\,e^{A_n(\psi^*,\psi)+p_n(\psi^*,\psi)}\;+\;\text{(nonperturbatively small error)}$$

with a normalization constant $\mathcal{Z}_n$ and the action

$$A_n=-\langle(\psi^*-Q_n\,\phi_{*n}),\,(\psi-Q_n\,\phi_n)\rangle+\langle\phi_{*n},\;(\partial_0+\Delta+\mu_n)\phi_n\rangle-\mathcal{V}_n(\phi_{*n},\phi_n)$$

where

- Q_n is an averaging operator on $\mathcal{X}_0$ over blocks of size $L^{2n} \times L^n \times L^n \times L^n$, followed by a rescaling to $\mathcal{X}^{(n)} = \mathbb{Z}/L^{t-2n}\mathbb{Z} \times \mathbb{Z}^3/L^{s-n}\mathbb{Z}^3$.
- $\mu_n \approx L^{2n}\mu$ is the renormalized chemical potential.
- $\mathcal{V}_n$ is approximately $\frac{\lambda}{L^n}$ times a rescaled $\mathcal{V}$.
- For each pair of fields (ψ_*, ψ) on $\mathcal{X}^{(n)}$ satisfying small field conditions, the "background fields" $\phi_{*n}(\psi_*, \psi)$, $\phi_n(\psi_*, \psi)$ on $\mathcal{X}_n$ are critical points of the functional

  $$(f_*, f) \mapsto \langle(\psi_* - Q_n f_*), (\psi - Q_n f)\rangle - \langle f_*, (\partial_0 + \Delta + \mu_n) f)\rangle + \mathcal{V}_n(f_*, f)$$

  They are are holomorphic on the set of (ψ_*, ψ) on $\mathcal{X}^{(n)}$ that satisfy the small field conditions.
- The "perturbative correction $p_n(\psi^*, \psi)$ is an analytic function on the small field domain that does not contain any quadratic terms.

Evaluating the action A_n at constant fields $\psi_* = z_*$, $\psi = z$ one gets a function proportional to $\big(|z|^2 - L^{3n}\frac{\mu}{\lambda}\big)^2 - \big(L^{3n}\frac{\mu}{\lambda}\big)^2$, which has a circle of degenerate minima (Mexican hat). This structure is generally seen as an indication for symmetry breaking. The construction as described cannot be continued over $n \approx \log_L \frac{1}{\lambda}$, because the radius of analyticity for the background fields and of the the perturbative correction p_n gets smaller than the radius of the potential well. We presently continue our program by introducing radial and tangential (Goldstone) fields near a point of the potential well and following a flow in terms of such fields with techniques similar to those which have led to the result above.

An induction step in our proof of the result above mainly concerns the block spin integral

$$\int \prod_{x \in \mathcal{X}^{(n)}} \frac{d\psi^*(x) \wedge d\psi(x)}{2\pi i}\; e^{-\frac{1}{L^2}\langle \theta_* - Q\psi^*, \theta - Q\psi\rangle}\; e^{A_n(\psi^*, \psi) + p_n(\psi^*, \psi)}$$

There are the following main issues

- "Large field generates small factors": If the fields were real (which they are not) and and θ were large, one would see a nonpertubatively small factor by the term $\langle \theta_* - Q\psi^*, \theta - Q\psi\rangle$ if ψ is small, or by the quartic term in A_n if ψ is large. One has to show that a similar mechanism still works in the framework of complex fields. Then one can restrict the integral to (θ_*, θ) that fulfil a small field conditions,
- "Holomorphic form representation" is just the observation that the block spin integral can be viewed as the integral of the holomorphic differential form

  $$\omega = e^{\mathcal{A}_{\rm eff}(\theta_*, \theta; \psi_*, \psi) + p_n(\psi_*, \psi)} \bigwedge_{x \in \mathcal{X}_0^{(n)}} \frac{d\psi_*(x) \wedge d\psi(x)}{2\pi \imath}$$

  over the "real" subspace $\psi_* = \psi^*$. Here,

  $$\mathcal{A}_{\rm eff}(\theta_*, \theta; \psi_*, \psi) = -\tfrac{1}{L^2}\langle \theta_* - Q\psi_*, \theta - Q\psi\rangle + A_n(\psi_*, \psi)$$

- To perform stationary phase for this integral, we determine, for each pair (θ_*, θ) that fulfils small field conditions, the critical points $\psi_{*\mathrm{cr}}(\theta_*,\, \theta)$, $\psi_{\mathrm{cr}}(\theta_*,\, \theta)$ of the map
$$(\psi_*, \psi) \longrightarrow \mathcal{A}_{\mathrm{eff}}(\theta_*, \theta;\, \psi_*, \psi)$$
It turns out that they are analytic functions of (θ_*, θ), but that they are in general not real (i.e. $\psi_{*\mathrm{cr}}(\theta_*,\, \theta) \neq \psi^*_{\mathrm{cr}}(\theta_*,\, \theta)$, even when $\theta_* = \theta^*$. This is due to the first time derivative in the action.
- Stationary phase amounts to writing the integral as a product
$$\text{(value of integrand at the critical point)} \times \text{(fluctuation integral)}$$
The value at the integrand looks like $e^{A_{n+1}}$, but with non renormalized chemical potential and with incorrect “perturbative correction”.
- The fluctuation variables for the fluctuation integral are fields ζ_*, ζ such that $\psi_{(*)} = \psi_{(*)\mathrm{cr}} + \zeta_{(*)}$. Since $\psi_{*\mathrm{cr}} \neq \psi^*_{\mathrm{cr}}$, the integral of the holomorphic form ω over the real subspace leads to an integral in these variables over a subspace where, in general, $\zeta_* \neq \zeta^*$. The holomorphic form is closed; we find a “generalized contour” to apply Stokes theorem to replace the integral by one where $\zeta_* = \zeta^*$. We check that the boundary terms in this argument are nonperturbatively small.
- The methods of [3] now give a representation for the resulting integral. Its logarithm contains quadratic parts, which leads to the renormalization of the chemical potential. Rescaling then completes the induction step.

Progress in this program is documented on http://www.math.ubc.ca/~feldman/bec/.

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Constructing the eigenstates of a many-body Hamiltonian: an RG/KAM approach

John Z. Imbrie

Abstract. For a weakly interacting quantum spin chain with random local interactions, we prove that many-body localization follows from a physically reasonable assumption that limits the extent of level attraction in the statistics of eigenvalues. In a KAM-style construction, a sequence of local unitary transformations is used to diagonalize the Hamiltonian by deforming the initial tensor-product basis into a complete set of exact many-body eigenfunctions.

1. What is Localization?

Consider the Anderson model:

$$H_{xy} = v_x \delta_{xy} + J_{xy}, \text{ with } x, y \in \Lambda \subset \mathbb{Z}^D, \text{ and } J_{xy} = \begin{cases} \gamma & \text{if } |x-y| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Here v_x are iid random variables; each represents the potential energy of a particle if it sits at site x. With weak hopping or strong disorder, the diagonal matrix entries v_x are usually much larger than γ, so the Hamiltonian is predominantly diagonally dominant. As a result, the time evolution $\exp(itH)(x,y)$ has a random walk expansion, which leads to rapid extinction $|\exp(itH)(x,y)| \leq \gamma^{|x-y|}$ for γ small. This is called dynamical localization, and it implies an absence of particle transport. Also, we obtain exponential decay of the eigenfunctions of H.

But resonances can spoil this picture. For example, the matrix $\begin{pmatrix} v_1 & \gamma \\ \gamma & v_2 \end{pmatrix}$ has eigenvectors that are very close to $\binom{1}{0}$ and $\binom{0}{1}$ only when $v_1 - v_2$ is not close to zero. In this case, the eigenvectors resemble the basis vectors. When $v_1 - v_2$ is small, the eigenvectors are a nontrivial mixture of the basis vectors. A similar situation prevails for the Hamiltonian on Λ, except for a dilute collection of sites that are resonant, which leads to tunneling and spreading of eigenfunctions. But as long as resonances do not percolate, localization prevails.

So let us consider a new characterization of localization: There is a way to "deform" the eigenstates into the basis vectors. Then the eigenstates will "resemble" the basis vectors. This characterization is one that generalizes to many-body quantum systems.

A Result on Localization. The following result demonstrates exponential decay of the eigenfunctions of H with integrated bounds uniform in Λ for small γ.

Theorem. *There is a $p > 0$ such that if γ is sufficiently small (depending only on D), the eigenfunction correlator obeys the bound*

$$\int d\lambda(v) \sum_\alpha |\psi_\alpha(x)\psi_\alpha(y)| \ \leq \gamma^{p|x-y|}.$$

This result was first proven by Aizenman and Molchanov using the fractional moment method. My recent work [1] gives a proof via multi-scale analysis, and this forms the basis for a method that works for many-body localization. One defines a sequence of rotation operators that diagonalize the Hamiltonian. Away from resonant regions, these rotations are generated by matrices that decay exponentially in the distance between the affected sites. We call such operators *quasilocal.*

2. Localization in Many-Body Systems

Does the phenomenon of localization persist in a more realistic model with interacting particles? My goal here is to discuss a proof of MBL, modulo an assumption on level statistics. Consider a random field, random transverse field, random exchange Ising model on $\Lambda = [-K, K] \cap \mathbb{Z}$:

$$H = \sum_{i=-K}^{K} h_i S_i^z + \sum_{i=-K}^{K} \gamma_i S_i^x + \sum_{i=-K-1}^{K} J_i S_i^z S_{i+1}^z.$$

This operates on the Hilbert space $\mathcal{H} = \bigotimes_{i \in \Lambda} \mathbb{C}^2$, with

$$S_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, S_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

operating on the i^{th} variable. Assume $\gamma_i = \gamma \Gamma_i$ with γ small. Random variables h_i, Γ_i, J_i are independent and bounded, with bounded probability densities.

Assumption LLA(ν, C). *Consider the Hamiltonian H in boxes of size n. Its eigenvalues satisfy*

$$P\left(\min_{\alpha \neq \beta} |E_\alpha - E_\beta| < \delta\right) \leq \delta^\nu C^n,$$

for all $\delta > 0$ and all n.

Remark. One expects Poisson statistics ($\nu = 1$) or repulsive statistics ($\nu > 1$, like GOE), but not level attraction ($\nu < 1$). Although proving this result seems to be out of reach at the moment for many-body systems, a promising approach was presented in [2] in the single-body context.

Theorem. (See [3, 4].) *Let ν, C be fixed. There exists a $\kappa > 0$ such that for γ sufficiently small,* ***LLA(ν,C)*** *implies the following estimate:*

$$\mathbb{E}\, \mathrm{Av}_\alpha \, |\langle S_0^z \rangle_\alpha| = 1 - O(\gamma^\kappa),$$

where $\mathbb{E}$ denotes the disorder average, Av_α denotes an average over α, and $\langle \cdot \rangle_\alpha$ denotes the expectation in the eigenstate α. All bounds are uniform in Λ.

Implication. The eigenstates do indeed resemble the basis vectors from whence they came (*i.e.* eigenvectors of $\{S_i^z\}_{i \in \Lambda}$), because $\langle S_0^z \rangle_\alpha$ is close to ± 1.

3. KAM Method

We obtain a complete diagonalization of the H by successively eliminating low-order off-diagonal terms. This is accomplished with rotations that can be written as a convergent power series, provided nonresonant conditions are satisfied; they are quasilocal. Resonant regions are diagonalized as blocks in quasidegenerate perturbation theory.

First Step. Initially, the only off-diagonal term is $\gamma_i S_i^x$, which is local. Let the spin configuration $\sigma^{(i)}$ be equal to σ with the spin at i flipped. The associated change in energy is

$$\Delta E_i \equiv E(\sigma) - E(\sigma^{(i)}) = 2\sigma_i(h_i + J_i \sigma_{i+1} + J_{i-1}\sigma_{i-1}).$$

We say that the site i is resonant if $|\Delta E_i| < \varepsilon \equiv \gamma^{1/20}$ for at least once choice of $\sigma_{i-1}, \sigma_{i+1}$. Then for nonresonant sites the ratio $\gamma_i/\Delta E_i$ is $\leq \gamma^{19/20}$. A site is resonant with probability $\sim 4\varepsilon$. Hence resonant sites form a dilute set ($\sim$ large field region) where perturbation theory breaks down.

Perturbation Theory. Let $H = H_0 + \mathcal{J}$ with H_0 diagonal and $\mathcal{J}$ off-diagonal. Put $\mathcal{J} = J^{\text{res}} + J^{\text{per}}$, where J^{res} contains terms $J(i) \equiv \gamma_i S_i^x$ with i resonant, and J^{per} contains the rest. Then define an antisymmetric matrix

$$A \equiv \sum_{\text{nonresonant } i} A(i) \text{ with } A(i)_{\sigma\sigma^{(i)}} = \frac{J(i)_{\sigma\sigma^{(i)}}}{E_\sigma - E_{\sigma^{(i)}}}.$$

Using e^{-A} for a basis change, we obtain a renormalized Hamiltonian:

$$H^{(1)} = e^A H e^{-A} = H + [A, H] + \frac{[A, [A, H]]}{2!} + \ldots = H_0 + J^{\text{res}} + J^{(1)}.$$

After the change of basis, J^{per} is gone, while J^{res} remains. The new interaction $J^{(1)}$ is quadratic and higher order in γ. Note that $A(i)$ commutes with $A(j)$ or $J(j)$ if $|i - j| > 1$. Thus we preserve quasi-locality of $J^{(1)}$; it can be written as $\sum_g J^{(1)}(g)$, where g is a sum of connected graphs involving spin flips $J(i)$ and associated energy denominators.

We also perform exact rotations in small, isolated resonant blocks to diagonalize the Hamiltonian there. This paves the way for reintegrating such regions into the perturbative framework in subsequent steps ($\sim$ recycling of large field regions). The process continues on a sequence of length scales $L_k = (15/8)^k$, with off-diagonal elements of order γ^m, $m \in [L_k, L_{k+1})$ eliminated in the k^{th} step. Letting $k \to \infty$, we eliminate all off-diagonal terms in H.

Dressing Transformation. The small, quasilocal rotations allow us to label each interacting eigenfunction with the basis vector it came from. Then one can apply the inverse similarity transformation to the spin operators S_i^z and obtain a complete set of local integrals of motion (LIOMs) that commute with H. In this way we demonstrate that H is integrable; it is non-ergodic. The existence of a complete set of LIOMs is one definition of a fully MBL system.

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Integrability in a 4D QFT model

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(joint work with Harald Grosse)

We start from a regularisation of the $\lambda\phi^4_4$-model on noncommutative Moyal space in finite volume [1],

(1)
$$S[\phi] = \frac{1}{64\pi^2}\int d^4x \Big(\frac{Z}{2}\phi\star\big(-\Delta+\Omega^2\|2\Theta^{-1}x\|^2 + \mu^2_{bare}\big)\phi + \frac{\lambda_{bare}Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)\;,$$

where $Z, \lambda_{bare}, \mu_{bare}$ are functions of renormalised values λ, μ and of the regulators $\Omega, \Theta, \mathcal{N}$ encoded in the oscillator potential and the $\star$-product. We expand $\phi(x) = \sum_{\underline{m},\underline{n}\in\mathbb{N}^2} \Phi_{\underline{m}\underline{n}} f_{m_1n_1}(x^0,x^1) f_{m_2n_2}(x^3,x^4)$ in the matrix basis of the Moyal product

$$f_{mn}(y^0,y^1) = 2(-1)^m\sqrt{\tfrac{m!}{n!}}\Big(\sqrt{\tfrac{2}{\theta}}y\Big)^{n-m} L_m^{n-m}\Big(\tfrac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}, \tag{2}$$

which satisies $f_{\underline{m}\underline{n}} \star f_{\underline{k}\underline{l}} = \delta_{\underline{n}\underline{k}} f_{\underline{m}\underline{l}}$ and $\int \frac{dx}{64\pi^2} f_{\underline{m}\underline{n}}(x) = V\delta_{\underline{m}\underline{n}}$ with $V := (\frac{\theta}{4})^2$. At the special point $\Omega = 1$ one then obtains a matrix model $S[\Phi] = V\mathrm{Tr}(ZE\Phi^2 + \frac{Z^2\lambda}{4}\Phi^4)$ with $E = (E_{\underline{m}}\delta_{\underline{m}\underline{n}}) = \frac{\mu^2_{\mathrm{bare}}}{2} + \frac{1}{\sqrt{V}}\mathrm{diag}(0,1,1,2,2,2,\dots)$ which admits a natural cut-off $\mathcal{N}$. The resulting partition function $\mathcal{Z}[J] = \int \mathcal{D}\Phi \exp(-S[\Phi] + \mathrm{tr}(J\Phi))$ is merely considered as a device to extract the equations of motions, i.e. Schwinger-Dyson equations. The matrix model structure induces a refinement of N-point functions into partitions $N = N_1 + \cdots + N_B$ and a corresponding expansion

(3)
$$V^{-2}\log\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \sum_{B=1}^{\infty}\sum_{1\le N_1\le\cdots\le N_B}^{\infty}\sum_{\underline{p}_1^1,\dots,\underline{p}^B_{N_B}} \frac{G_{|\underline{p}_1^1\dots\underline{p}^1_{N_1}|\dots|\underline{p}_1^B\dots\underline{p}^B_{N_B}|}}{S_{N_1\dots N_B}}\prod_{\beta=1}^{B}\frac{J_{\underline{p}_1^\beta\underline{p}_2^\beta}\cdots J_{\underline{p}^\beta_{N_\beta}\underline{p}_1^\beta}}{VN_\beta}.$$

The Ward identity for the $U(\mathcal{N})$ group action [2] is used to collapse — in a coupled limit $\sqrt{V},\mathcal{N}\to\infty$ with their ratio fixed — the tower of Schwinger-Dyson equations into a self-consistent formula for the 2-point function alone,

$$G_{|\underline{a}\underline{b}|} = \frac{1}{Z(E_{\underline{a}}+E_{\underline{b}})} - \frac{Z\lambda}{(E_{\underline{a}}+E_{\underline{b}})}\frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}}\Big(G_{|\underline{a}\underline{b}|}G_{|\underline{a}\underline{p}|} - \frac{G_{|\underline{p}\underline{b}|}-G_{|\underline{a}\underline{b}|}}{Z(E_{\underline{p}}-E_{\underline{a}})}\Big), \tag{4}$$

and a hierarchy of linear equations for all higher correlation functions [3]. These equations are algebraic if one $N_i > 2$, e.g. $G_{|\underline{a}\underline{b}\underline{c}\underline{d}|} = (-\lambda)\frac{G_{|\underline{a}\underline{b}|}G_{|\underline{c}\underline{d}|}-G_{|\underline{a}\underline{d}|}G_{|\underline{c}\underline{b}|}}{(E_{\underline{a}}-E_{\underline{c}})(E_{\underline{b}}-E_{\underline{d}})}$ which proves that the β-function is zero, otherwise (e.g. for $G_{|\underline{a}\underline{b}|\underline{c}\underline{d}|}$) complicated but linear.

In a scaling limit $\mathcal{N}, V\to\infty$ with $\frac{\mathcal{N}}{\sqrt{V\mu^4}} = \Lambda$ fixed, sums over $\underline{p}\in\mathbb{N}^2_{\mathcal{N}}$ converge to Riemann integrals of continuous variables $a,b\in[0,\Lambda^2]$, and the finite Hilbert transform $\mathcal{H}^\Lambda_a(f) = \frac{1}{\pi}\mathcal{P}\!\int_0^{\Lambda^2}\frac{f(p)\,dp}{p-a}$ arises. The limit $\Lambda\to\infty$ requires renormalisation which, because of the vanishing β-function, can be directly implemented in (4). Noticing that the difference $G(a,b) - G(a,0)$ satisfies a linear equation, the solution theory of Carleman-Tricomi gives the renormalised limiting function $G(a,b)$ in terms of the boundary $G(a,0)$:

Theorem 1 ([3, 4]). *Define* $\tau_b(a) := \arctan_{[0,\pi]}\left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a\mathcal{H}_a^\Lambda[G(\bullet,0)]}{G(a,0)}}\right)$. *Then*

$$G(a,b)=\frac{\sin(\tau_b(a))}{|\lambda|\pi a}\mathrm{e}^{\mathrm{sign}(\lambda)(\mathcal{H}_0[\tau_0(\bullet)]-\mathcal{H}_a[\tau_b(\bullet)])}\begin{cases}1 & \textit{for } \lambda<0,\\ \left(1+\frac{Ca+bF(b)}{\Lambda^2-a}\right) & \textit{for } \lambda>0.\end{cases} \tag{5}$$

Surprisingly, instantons corresponding to solutions of the homogeneous equation, parametrised by a constant C and a function $F(b)$, live at $\lambda > 0$. This reversal is a consequence of renormalisation, to be discussed below. The remaining equation for $G(a,0)$ reduces to symmetry $G(b,0) = G(0,b)$. For $\lambda < 0$ one has

$$G(b,0) = \frac{1}{1+b}\exp\Big(-\lambda\int_0^b dt\int_0^\infty \frac{dp}{(\lambda\pi p)^2 + (t + \frac{1+\lambda\pi\mathcal{H}_p[G(\bullet,0)]}{G(p,0)})^2}\Big). \tag{6}$$

A numerical iteration of (6) converges and shows a phase transition at $\lambda_c \approx -0.39$ [4]. For $\lambda > 0$ the symmetry $G(a,b) = G(b,a)$ is violated if the instantons are ignored. In [5] we have proved by the Schauder fixed point theorem that a C_0^1-solution $\frac{1}{(1+b)^{1-|\lambda|}} \leq G(0,b) \leq \frac{1}{(1+b)^{1-\frac{|\lambda|}{1-2|\lambda|}}}$ exists for $-\frac{1}{6} \leq \lambda < 0$.

Returning to the original formulation (1) in position space, we define connected Schwinger functions on $\mathbb{R}^4$ as

$$\begin{aligned}&\mu^N S_c(\mu x_1,\dots,\mu x_N)\\ &:= \lim_{\mathcal{N},V\to\infty}\sum_{\underline{m}_i,\underline{n}_i\in\mathbb{N}^2_{\mathcal{N}}} f_{\underline{m}_1\underline{n}_1}(x_1)\cdots f_{\underline{m}_N\underline{n}_N}(x_N)\frac{(V\mu^4)^{-2}\mu^{4N}\partial^N\log\mathcal{Z}[J]}{\partial J_{\underline{m}_1\underline{n}_1}\dots\partial J_{\underline{m}_N\underline{n}_N}}\Big|_{J=0}.\end{aligned} \tag{7}$$

Inserting (3) one gets a partition into $f_{\underline{mn}}$-cycles. Expressing the correlation functions as Laplace-Fourier transform produces $\sum_{m_1,\dots,m_N=0}^\infty\prod_{i=1}^N z_i^{m_i}L_{m_i}^{m_{i+1}-m_i}(r_i)$ which we evaluated in [6]. For the choice of z_i, the $V\to\infty$ limit is $\sim V^0$ for N odd but $\sim V^1$ for N even. Together with the V^{-1}-prefactor in (3) for every B one arrives at:

Theorem 2 ([6]). *Defining* $\mathcal{Y} := \lim_{b\to 0}\frac{(1-G(0,b))}{b}$ *and* $s_\beta := N_1+\dots+N_{\beta-1}$, *the connected Schwinger functions are given by*

$$\begin{aligned}&S_c(\mu x_1,\dots,\mu x_N)\\ &=\frac{1}{64\pi^2}\sum_{\substack{N_1+\dots+N_B=N\\ N_\beta \text{ even}}}\sum_{\sigma\in\mathcal{S}_N}\Big(\prod_{\beta=1}^B\frac{4^{N_\beta}}{N_\beta}\int_{\mathbb{R}^4}\frac{d^4p_\beta}{4\pi^2\mu^4}\,e^{\mathrm{i}\langle p_\beta,\sum_{i=1}^{N_\beta}(-1)^{i-1}x_{\sigma(s_\beta+i)}\rangle}\Big)\\ &\quad\times\frac{1}{S_{N_1\dots N_B}}G\big(\underbrace{\tfrac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})},\cdots,\tfrac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1}\big|\dots\big|\underbrace{\tfrac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})},\cdots,\tfrac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}\big).\end{aligned} \tag{8}$$

Only the face-diagonal matrix correlation functions contribute to the Schwinger functions in position space. This can be viewed as confinement of noncommutativity: Whereas interactions involve the complete matrix structure, Schwinger functions depend only on the projection to the diagonal. Euclidean symmetry is manifest. The Schwinger functions show a restricted kinematics where scattering is such that particle momenta are individually conserved, as it is the case in any integrable model.

We have also pointed out in [6] that reflection positivity of the 2-point function amounts to a Stieltjes representation $G(a,a) = \int_0^\infty dt \frac{\rho(t)}{t+a}$ for a positive measure ρ. This is excluded for $\lambda > 0$, whereas we accumulated a lot of evidence that this is the case for $\lambda_c < \lambda \leq 0$. The preferrence of $\lambda < 0$ is a renormalisation effect. The Stieltjes property is related to the anomalous dimension η in $\hat{S}_2(p) \sim \frac{1}{(\|p\|^2+\mu^2)^{1-\eta/2}}$. Naïvely we have $\eta > 0$, in fact $\eta = +\infty$, for $\lambda > 0$. It turns out that the renormalised anomalous dimension is positive for $\lambda < 0$. Consequently, there is no hope to construct a rigorous measure for the partition function, which is why we based our approach on Schwinger-Dyson equations made rigorous.

Our best results so far (not yet published) start from an ansatz $G(0,x) = {}_4F_3\big(\begin{smallmatrix} a,b_1,b_2,b_3 \\ c_1,c_2,c_3 \end{smallmatrix}\big| -x\big)$ with $0 < a < 1$ and $1 < b_i < c_i$, which is a Stieltjes function. Optimising for a, b_i, c_i we came to the conjecture $a = 1 - \frac{1}{\pi}\arcsin(|\lambda|\pi)$ which we were able to prove. Consequently, we expect the critical coupling constant to be exactly $\lambda_c = -\frac{1}{\pi}$. Such a hypergeometric function ansatz solves the fixed point equation (6) up to an error of 10^{-8}. We can plug it into Theorem 1 and notice that $G(\frac{x}{2}, \frac{x}{2})$ is very close, but not exactly equal, to $G(0,x)$. We thus expect that also $G(\frac{x}{2}, \frac{x}{2})$ is Stieltjes, with an intriguing behaviour of the Källén-Lehmann spectral measure ρ: There is a mass gap $[0,\mu^2[$ but *no further gap* $]\mu^2, 4\mu^4[$! Absence of this second gap — remnant of the cured UV/IR-mixing problem — circumvents several triviality theorems.

The exact solution of the model, its restricted kinematics, the vanishing of the β-function and the striking value $\lambda_c = -\frac{1}{\pi}$ of the critical coupling constant all support the conjecture that these results are due to a hidden integrable structure.

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The Tensor Track

Vincent Rivasseau

The tensor track is a program to explore (Euclidean) random fields which are tensors of general rank d. They include as special cases vector (rank 1) and matrix (rank 2) models. Tensor models were introduced as promising candidates for an *ab initio* quantization of gravity. Indeed they are combinatorial objects which do not refer to any background metric, nor even to any background topology. These tensors were initially introduced as symmetric in their indices, a feature which for a long time prevented to investigate rigorously their behavior. In particular in contrast with the famous 't Hooft 1/N expansion for random matrix models, there was until recently no way to probe the large N limit of these symmetric random tensors at rank $d > 2$.

The modern reformulation by R. Gurau and collaborators unlocked the theory by considering un-symmetrized random tensors. Slightly counterintuitively perhaps, these objects have in fact a larger symmetry than symmetric tensors, and this larger symmetry allows to probe their large N limit through 1/N expansions of a new type. Invariants under this symmetry are exactly d-regular edge-colored graphs.

Random tensor models can be further divided into fully invariant models, in which both propagator and interaction are invariant, and tensor field theories in which the interaction is invariant but the propagator is not. This propagator can also incorporate a further gauge invariance to make contact with group field theory, in which case we call the model a tensor group field theory.

The first-half of the talk focused on the motivations for random tensors, which come from random geometry and quantum gravity. The tensor model action is known to be a natural discretization of the Einstein-Hilbert action and can be considered therefore as an equilateral form of Regge calculus. The recent associated field theories have added renormalization, asymptotic freedom and exploration of the infrared flow to this older picture.

The last point is very important. Indeed a main difficulty in bacground independent approaches to quantum gravity is to identify space-time not as “god-given” initially but as emergent, hence as a condensate phase of a more fiundamental quantum theory. This is conceptually similar to the difficult problem of deducing hadronic and nuclear physics from quantum chromodynamics. A fully analytic solution should not be expected soon, since in physics effective behaviors qualitatively different from the bare ones can almost never be computed analytically. Even for the long-time behavior of the three body problem in Newtonian gravity or the phase transition of the Ising model in three dimensions (not to mention the formation of molecules and crystals in the real world), computer simulations are required at some stage. Therefore it is expected that the investigation of renormalization group flows and phase transitions towards an emergent space-time will also require numerical as well as analytic tools. Fortunately in the tensor track approach the models have simple actions and standard field theoretic methods (such as the *functional renormalization group*) are available to compute numerically their renormalization group flows and effective behavior.

Another important bonus of the tensor track approach is that a constructive analysis of the models is often possible. In the relatively simple case of super-renormalizable models with a quartic interaction, Borel summability of the free energy has already been proven on the "stable side" of the coupling constant. The main tool is the so-called multi-scale loop vertex expansion. It is a constructive technique which bypasses the more traditional cluster expansions and does not require any discretization of space-time by regular lattices.

Of course enormous work lies ahead of the tensor track program, which is only one among many competing approaches to quantum gravity. Among the major problems to tackle in this approach we can list finding Euclidean axioms including the right generalization of the Osterwalder-Schrader positivity axiom, to allow in particular emergence of Lorentzian time and causality; constructive treatment of more than quartic interactions; renormalization group evolution from the arborescent to more realistic macroscopic phase; consequences of the theory for cosmology scenarios and for black holes; and addition of the standard model matter fields to the picture. Altogether we nevertheless have the impression that the tensor track has matured enough to be taken seriously and explored further. At the physical level it suggests an emergent space-time scenario with an initial arborescent phase of the universe. This result should not be immediately discarded as non-physical, since the richness of subdominant tensor interactions could lead this arborescent phase to evolve later into geometries closer to our actual universe.

At the mathematical level, the world of tensor models and tensor field theories is incredibly rich, as tensor interactions encode infinitely many triangulations of any piecewise linear manifold with boundaries, and in particular distinguish in four dimensions not only topology, but also smooth structure. They also generalize non-commutative field theories on Moyal space. Their mathematical exploration has in fact just begun and will most probably continue for decades to burgeon in many fascinating directions.

A BPHZ Theorem for Regularity Structures

Ajay Chandra

I introduced Martin Hairer's theory of regularity structures and then discussed recent joint work with Hairer on a systematic way to choose renormalization constants and prove corresponding stochastic estimates for this theory, this scheme is a generalization of the BPHZ renormalization scheme of perturbative Quantum Field Theory. I began by introducing the Φ^4_d stochastic quantization, explaining that once well-posed it defines an infinite dimensional Markov process which has the Euclidean Φ^4_d QFT measure as an invariant measure.

I then explained the Da Prato - Debussche method for the Φ^4_2 stochastic quantization equation and explained how it completely breaks down when treating Φ^4_3. Regardless of where one truncates the naive perturbative for the solution the fixed point problem for the remainder is always ill-posed because the remainder has insufficient regularity to define products that appear in this fixed point problem.

We adopted the point of view that regularity was the wrong property to demand of our remainder since the regularity of a product of a pair of functions/distributions, when it can be defined, is determined by the how singular the worst of the pair is. On the other hand, if we are willing to adopt a more local point of view, the quality of satisfying homogeneity bound at a specific point behaves better under the products.

Using this as motivation, we move away from "global" perturbative expansions and instead and instead pose the solution problem in a jet of local expansions - here one generalizes the classical notion of Taylor series by including indeterminants which represent certain Gaussian polynomials (Wild trees) built out of the linear solution as monomials in addition to classical polynomials. A key idea is that when the right deterministic and quantitative notion of a distribution being locally well-approximated by explicit Gaussian processes is combined with a probablistic algorithm for defining Gaussian polynomials one gets a method of defining products of this solution. One can view this entire procedure as defining a new notion of regularity in which certain singular distributions have positive regularity.

After this I described the concept of a "model" which is what allows one to associate concrete objects to abstract jets. The key content of a model is (a) a map which associates to each Wild tree indeterminant a concrete space-time distribution which is the "homogenous incarnation" of that tree and (b) a family of parallel transport maps which allow one to move this these local expansions from point to point. In order to define the map of item (a) one must mollify the underlying driving white noise at some scale ϵ and then subtract renormalization constants (which diverge as $\epsilon \downarrow 0$) in order to guarantee a limit as the mollification is removed. However convergence is not enough, one must also perform "recenterings" of this process so that these approximations satisfy, uniformly in ϵ, a homogeneity bound. I then discussed how our BPHZ type theorem used multiscale techniques from constructive field theory in order to show that one can define an automatic procedure to perform these renormalizations and check that after recentering one again gets convergence along with the necessary homogeneity bounds.

Computing certain invariants of topological spaces of dimension three

Wojciech Dybalski

It is well-known that the conventional property of asymptotic completeness fails in general in quantum field theory due to the possible presence of pairs of oppositely charged particles in the vacuum sector. However, a generalized concept of complete particle interpretation which takes this phenomenon into account, was formulated in [1, 2]: With the help of suitable asymptotic observables (Araki-Haag detectors) we construct a canonical 'charged-particles free' subspace. The generalized property of asymptotic completeness requires that it coincides with the subspace of Haag-Ruelle scattering states. We show that this property holds in any massive quantum field theory satisfying the Haag-Kastler axioms. Our result can be reformulated as a criterion for conventional asymptotic completeness which should be sharp in theories with trivial superselection structure. The crucial technical step is the proof of convergence of the Araki-Haag detectors on all states from a suitable spectral subspace of the energy-momentum operators. For a restricted class of detectors this was accomplished in [1, 2] by applying the quantum mechanical method of propagation estimates in the relativistic setting. In this talk a novel method will be presented, which applies to a larger class of detectors. It uses a compactness argument and the observation that the problem of convergence is non-trivial only on the orthogonal complement of the subspace of scattering states. (Joint project with C. Gérard).

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Index theory on Lorentzian manifolds and the chiral anomaly

Christian Bär

(joint work with Alexander Strohmaier)

Let M be a Lorentzian manifold with boundary; the boundary is assumed to consist of two smooth and spacelike Cauchy hypersurfaces, one lying in the past of the other. We assume that M carries a spin structure so that the spinor bundle $SM \to M$ is defined. Moreover, let the dimension of M be even; then the spinor bundle splits into the two subbundles of left-handed and right-handed spinors, $SM = S_L M \oplus S_R M$. Finally, let $E \to M$ be a Hermitian vector bundle, equipped with a compatible connection. Then we have the bundles of spinors with coefficients in E, $V_{L/R} = S_{L/R}M \otimes E$.

The boundary is a Riemannian manifold and the induced operator on the boundary is a self-adjoint elliptic differential operator. Therefore the Atiyah-Patodi-Singer boundary conditions make sense in this Lorentzian setting; they say

$$P_+(u|_{\partial M}) = 0$$

where P_+ denotes the spectral projector onto the subspace of L^2-spinors over ∂M spanned by the eigenspinors to the non-negative eigenvalues of the boundary Dirac operator.

The twisted Dirac operator $D : C^\infty(M, V_R) \to C^\infty(M, V_L)$ on M is a *hyperbolic* linear differential operator of first order. Usually, index theory is closely tied to ellipticity of the operator and hyperbolic operators are not Fredholm. Moreover, solutions of $Du = 0$ need not be smooth; they can be very irregular.

In this particular setting however, we have a complete analog to the Atiyah-Patodi-Singer index theorem [1]:

Theorem 1 (Bär-Strohmaier [2]). *Under Atiyah-Patodi-Singer boundary conditions, D is a Fredholm operator. The kernel consists of smooth spinor fields and the index is given by*

$$\mathsf{ind}(D_{\mathsf{APS}}) = \int_M \widehat{\mathsf{A}}(M) \wedge \mathsf{ch}(E) + \int_{\partial M} T(\widehat{\mathsf{A}}(M) \wedge \mathsf{ch}(E)) - \frac{h+\eta}{2}.$$

Here $\widehat{\mathsf{A}}(M)$ is the $\widehat{\mathsf{A}}$-form computable in terms of the curvature of M and ch is the Chern character form, an expression in the curvature of E. By $T(\widehat{\mathsf{A}}(M) \wedge \mathsf{ch}(E))$ we denote the corresponding transgression form and h and η denote the dimension of the kernel and the η-invariant of the boundary operator, respectively.

There are also important differences to the Riemannian case. First of all, it is possible to replace the Atiyah-Patodi-Singer boundary conditions by the complementary *anti-Atiyah-Patodi-Singer boundary conditions*

$$P_-(u|_{\partial M}) = 0.$$

In the Riemannian case this would not yield a Fredholm operator. In the Lorentzian setting the operator turns out to be Fredholm and the same index formula as in Theorem 1 holds, except for a global sign. Moreover, the index can be written as

$$\begin{aligned} \mathsf{ind}(D_{\mathsf{APS}}) &= \mathsf{dim}\,\mathsf{ker}[D : C^\infty_{\mathsf{APS}}(M, V_R) \to C^\infty(M, V_L)] \\ &\quad - \mathsf{dim}\,\mathsf{ker}[D : C^\infty_{\mathsf{aAPS}}(M, V_R) \to C^\infty(M, V_L)] \end{aligned}$$

where the subscripts APS and aAPS indicate that (anti-)Atiyah-Patodi-Singer boundary conditions are imposed. In the corresponding Riemannian formula the negative term would have to be replaced by $-\mathsf{dim}\,\mathsf{ker}[D : C^\infty_{\mathsf{APS}}(M, V_L) \to C^\infty(M, V_R)]$ (up to a subtlety if $h \neq 0$).

In the Lorentzian setup the APS-boundary conditions have a natural physical interpretation in terms of a particle-antiparticle splitting. This allows to use Theorem 1 to directly derive a geometric formula for the *chiral anomaly* in quantum field theory on curved spacetimes without the need to resort to mathematically fishy arguments such as a Wick rotation. See [3] for details and computed examples.

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Modular ℓ^p-conditions for QFT in curved spacetimes

Ko Sanders
(joint work with Gandalf Lechner, Stefan Hollands)

In a very general setting, each state of an algebraic quantum (field) theory gives rise to a modular operator. This operator can be used to define maps Ξ, whose ℓ^p-properties yield an estimate of entanglement entropy [1]. After reviewing the notions of spacetime and locally covariant quantum field theories (LCQFTs) [3], I review recent results on modular ℓ^p-conditions for such theories [2].

1. Modular Operators

A general quantum system consists of a C^*-algebra $\mathcal{A}$ with unit 1 and a selection of suitable algebraic states $\omega : \mathcal{A} \to \mathbb{C}$. Each state gives rise to a GNS-representation $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$, with $\Omega_\omega \in \mathcal{H}_\omega$ implementing ω. To define modular operators in this general case, we compress the representation to a subspace in a canonical way [2]. Let $Q_\omega \in \pi_\omega(\mathcal{A})''$ be the orthogonal projection onto $\mathcal{H}'_\omega := \overline{\pi_\omega(\mathcal{A})'\Omega_\omega}$. In $\mathcal{H}'_\omega$, Ω_ω is cyclic and separating for the compressed von Neumann-algebra $Q_\omega \pi_\omega(\mathcal{A})'' Q_\omega$. The modular operator $\Delta_\omega \geq 0$ is now defined as the self-adjoint operator with kernel $(\mathcal{H}'_\omega)^\perp$, with form core $\pi_\omega(\mathcal{A})\Omega_\omega$, and with

$$\|\Delta_\omega^{\frac{1}{2}} a \Omega_\omega\| = \|Q_\omega a^* \Omega_\omega\|, \qquad a \in \pi_\omega(\mathcal{A}).$$

We conclude that modular operators are always available. When ω is pure, Q_ω projects onto the span of Ω_ω and $\Delta_\omega = Q_\omega$ is rather trivial, but for typical quantum states we expect $Q_\omega = 1$ (e.g. due to the Reeh-Schlieder property).

2. ℓ^p-Operators

ℓ^p-operators are bounded operators between Banach spaces, which can be approximated very well by operators of finite rank. This makes the ℓ^p-property very nice, and closely related to a nuclearity condition [2]. To define such operators, we consider a bounded linear map between Banach spaces, $\Xi : B_1 \to B_2$. For $n \in \mathbb{N}$ we introduce the non-increasing sequence of approximation numbers

$$\alpha_n(\Xi) := \inf_{\Xi_n \text{ of rank } \leq n} \|\Xi - \Xi_n\|.$$

For any $p > 0$, Ξ is an ℓ^p-operator iff the α_n decrease fast enough to have

$$\|\Xi\|_p := \left(\sum_{n=0}^{\infty} \alpha_n(\Xi)^p \right)^{\frac{1}{p}} < \infty.$$

ℓ^p-operators form a linear space and $\|.\|_p$ is a quasi-norm:

$$\|\Xi_1 + \Xi_2\|_p \leq \max\{2, 2^{\frac{2}{p}-1}\} \left(\|\Xi_1\|_p + \|\Xi_2\|_p \right).$$

Furthermore, ℓ^p-operators are ℓ^q for $q \geq p$, ℓ^∞ are all compact operators, and there are nice estimates like $\|\Xi B\|_p \leq \|\Xi\|_p \cdot \|B\|$ for bounded operators B.

3. An application to entanglement entropy

Let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ be commuting subalgebras and $\omega : \mathcal{A} \to \mathbb{C}$ a state. We want to quantify the entanglement of ω between the $\mathcal{A}_i$, $i = 1, 2$. First let $\mathcal{H}_i$ be the GNS-representation space for $\omega_i := \omega|_{\mathcal{A}_i}$ and consider the von Neumann algebra

$$\mathcal{M} := \pi_{\omega_1}(\mathcal{A}_1)'' \otimes \pi_{\omega_2}(\mathcal{A}_2)''$$

acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$. A state ω' on $\mathcal{M}$ is called separable iff

$$\omega' = \sum_j \phi_j \otimes \psi_j,$$

where ϕ_j, ψ_j are normal positive functionals and the sum converges in norm.

Araki defined the relative entropy between ω and ω' is defined in terms of a relative modular operator (extending known formulae in terms of density matrices):

$$H(\omega, \omega') = \begin{cases} \langle \Omega_\omega, \log(\Delta_{\omega,\omega'})\Omega_\omega \rangle & \omega \text{ normal on } \mathcal{M} \\ \infty & \text{else.} \end{cases}$$

The entanglement entropy of ω between $\mathcal{A}_1$ and $\mathcal{A}_2$ is then defined as

$$E_\omega(\mathcal{A}_1, \mathcal{A}_2) := \inf_{\omega' \text{ separable}} H(\omega, \omega').$$

We now want to apply modular operators and ℓ^p-properties to the task of estimating the entanglement entropy. For this we introduce the state $\tilde{\omega} := \langle \Omega_\omega, .\Omega_\omega \rangle$ on $\pi_\omega(\mathcal{A}_2)' \supset \pi_\omega(\mathcal{A}_1)$ and the operator

$$\Xi : \mathcal{A}_1 \to \mathcal{H}_{\tilde{\omega}} : a \mapsto \Delta_{\tilde{\omega}}^{\frac{1}{4}} \pi_{\tilde{\omega}}(a)\Omega_{\tilde{\omega}}.$$

When $\|\Xi\|_1 < \infty$, the nuclear index $\nu_1(\Xi) \leq 2^5 \|\Xi\|_1$ is also finite [2] and [1] shows:

$$E_\omega(\mathcal{A}_1, \mathcal{A}_2) \leq \log(2\nu_1(\Xi)) \leq \log(2^6 \|\Xi\|_1).$$

4. Spacetime

A physical theory is a class of systems $\mathtt{PHYS} = \{\mathcal{A}_1, \ldots\}$ together with subsystem relations $\mathcal{A}_1 \to \mathcal{A}_2$, which make PHYS a category. As in algebraic QFT, the physics is in the morphisms of PHYS. There are many ways to divide degrees of freedom into subsystems, e.g. by organising them by particle, by field type or by frequency range. A particularly effective tool is to organise them by localisation region.

LOC is a category of globally hyperbolic Lorentzian manifolds with suitable isometric embeddings. A theory PHYS is a locally covariant QFT (LCQFT) when there exists a functor

$$\mathtt{A} : \mathtt{LOC} \to \mathtt{PHYS}$$

satisfying suitable axioms. Note that manifolds essentially only enter to determine the structure of the category LOC, which encodes locality and general covariance. Taking seriously the idea that spacetime is an organisation principle, which imposes structure on the theory PHYS, we conclude that spacetime is a functor $\mathtt{A} : \mathtt{LOC} \to \ldots$ (satisfying suitable axioms) [3]. (This argument is even stronger in classical GR, using manifolds with a metric, whose points have no physical significance.)

5. A locally covariant modular ℓ^p-condition

Given a LCQFT $\mathtt{A} : \mathtt{LOC} \to \mathtt{PHYS}$ and an object M in $\mathtt{LOC}$, [2] defines the following condition:

A state ω on $\mathcal{A} := \mathtt{A}(M)$ satisfies the modular ℓ^p-condition iff for all $\alpha \in (0, \frac{1}{2})$, $p > 0$ and all compact inclusions $\tilde{O} \to O \to M$ in $\mathtt{LOC}$, the following map is ℓ^p:

$$\Xi : \pi_{\omega|_O}(\mathtt{A}(\tilde{O}))'' \to \mathcal{H}_{\omega|_O} : a \mapsto \Delta^{\alpha}_{\omega|_O} a \Omega_{\omega|_O}.$$

Here, $\omega|_O$ is an abbreviation for the restriction $\omega|_{\mathtt{A}(O)}$. The modular ℓ^p-condition condition is a generally covariant version of a modular nuclearity condition of Buchholz, d'Antoni and Longo, and it has some nice properties [2]: it is stable under taking pull-backs of states and taking mixtures of states, and when the LCQFT has the time-slice property, it also satisfies a stability property under spacetime deformations.

In addition to these general properties, the condition has been verified for an interesting class of states [2]:

Every quasi-free Hadamard state on the Weyl algebra of a real free scalar quantum field on a globally hyperbolic Lorentzian manifold satisfies the modular ℓ^p-condition.

As a consequence of this result one may show that the entanglement entropy of these states between suitable space-like separated regions of spacetime is finite (as long as the regions do not touch and one of them is bounded).

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Cosmological perturbation theory and perturbative quantum gravity

Thomas-Paul Hack

(joint work with Romeo Brunetti, Klaus Fredenhagen, Nicola Pinamonti, Katarzyna Rejzner)

The quest for a satisfactory and complete theory of quantum gravity is complicated by conceptual and technical obstacles. A more conservative approach to a quantum theory of gravity is perturbative quantum gravity. This approach is expected to be only applicable in a regime where quantum gravitational effects are relatively small. In addition, this approach had been facing its own conceptual problems, an area of research in which considerable progress has been made in the last years:

(1) Initially, perturbative quantum gravity had only been formulated on the most trivial background, Minkowski spacetime. However, a generally covariant formulation on general backgrounds has been developed in [2].
(2) It is well-known that local and diffeomorphism-invariant observables, the building blocks of any quantum field theory, do not even exist in classical general relativity. A concept which comes rather close to local observables are the partial or relational observables well-known from classical general relativity and already implemented in the non-perturbative framework of loop quantum gravity [3, 9, 10]. It has been argued in [2, 6] that these observables can also be used as building blocks in perturbative quantum gravity.
(3) Finally, perturbative quantum gravity is a non-renormalisable quantum field theory, i.e. ever new coupling constants appear upon going to higher and higher orders in perturbation theory. In this respect indications of asymptotic safety found in [7, 8] and subsequent works are encouraging: at the so-called UV fixed point of the renormalisation group flow only a finite number of the infinitely many coupling constants are relevant.

A non-trivial application of perturbative quantum gravity is cosmological perturbation theory: a popular model of the early universe is inflation, wherein one assumes that a classical scalar field coupled to the classical gravitational field has triggered an exponential expansion of the universe, after which the universe was essentially void of any structures. The quantum fluctuations of both the scalar field and the gravitational field are thought to be the seeds of the structures we observe in the universe today; this is the topic of cosmological perturbation theory. The first snapshot of these structures, dated approximately 400000 years after the big bang, is the cosmic microwave background radiation. This is well-modelled by linearised perturbative quantum gravity and can thus be considered as the major observational signature of quantum gravitational effects. However, a full understanding of the cosmic microwave background radiation requires the consideration of higher orders in perturbative quantum gravity and so far only rather ad-hoc or conceptually unsatisfactory analyses of this issue seemed to have been discussed in the literature.

In this talk I will describe how perturbative quantum gravity leads to a consistent all-order description of quantized fluctuations around a cosmological background. A central obstacle to overcome is the apparent lack of sufficiently many relational/partial observables: such observables may be equivalently understood as using coordinates which are diffeomorphism-equivariant functionals of the dynamical fields of the model, the scalar field and the gravitation field. In perturbative quantum gravity, these coordinates have to be well-defined and non-degenerate on the background of the theory. However, as cosmological backgrounds have a high degree of symmetry, all local functionals of the background fields have linearly dependent gradients. As I will discuss in detail, this problem can be overcome by constructing suitable non-local and non-degenerate functionals of the background fields. I will discuss the properties of these functionals and I will argue that their non-locality is not problematic for a consistent perturbative quantization. Examples of higher-order gauge-invariant quantities constructed by means of these equivariant coordinates will be presented as well. Finally, I will argue that a consequent all-order implementation of the idea to quantize perturbative gravity directly in terms of diffeomorphism-invariant fundamental fields, rather then in terms of diffeomorphism-equivariant fields and auxiliary fields like in the so-called BV-BRST framework [4, 5], appears to be possible. The talk is largely based on results reported in [1].

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