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# High contrast homogenisation in nonlinear elasticity under small loads 

Mikhail Cherdantsev ${ }^{1}$, Kirill Cherednichenko ${ }^{2}$, and Stefan Neukamm ${ }^{3}$<br>${ }^{1}$ School of Mathematics, Cardiff University, Senghennydd Road, Cardiff, CF24 4AG, UK<br>${ }^{2}$ Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK<br>${ }^{3}$ TU Dresden, Fachrichtung Mathematik, Institut für Wissenschaftliches, Rechnen, D-01062 Dresden, Germany

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#### Abstract

We study the homogenisation of geometrically nonlinear elastic composites with high contrast. The composites we analyse consist of a perforated matrix material, which we call the "stiff" material, and a "soft" material that fills the remaining pores. We assume that the pores are of size $0<\varepsilon \ll 1$ and are periodically distributed with period $\varepsilon$. We also assume that the stiffness of the soft material degenerates with rate $\varepsilon^{2 \gamma}, \gamma>0$, so that the contrast between the two materials becomes infinite as $\varepsilon \downarrow 0$. We study the homogenisation limit $\varepsilon \downarrow 0$ in a low energy regime, where the displacement of the stiff component is infinitesimally small. We derive an effective two-scale model, which, depending on the scaling of the energy, is either a quadratic functional or a partially quadratic functional that still allows for large strains in the soft inclusions. In the latter case, averaging out the small scale-term justifies a single-scale model for high-contrast materials, which features a non-linear and non-monotone effect describing a coupling between microscopic and the effective macroscopic displacements.


Keywords: High-contrast homogenisation; Nonlinear elasticity; Two-scale $\Gamma$-convergence.

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## 1 Introduction

We consider a geometrically nonlinear elastic composite material that consists of a "stiff" matrix material and periodically distributed pores filled by a "soft" material: for $\varepsilon>0$ and a fixed scaling parameter $\gamma>0$ we consider the energy functional of non-linear elasticity

$$
\begin{equation*}
\mathcal{I}_{\varepsilon}(u):=\int_{\Omega}\left(\varepsilon^{2 \gamma} W^{0}(\nabla u) \chi_{\varepsilon}+W^{1}(\nabla u)\left(1-\chi_{\varepsilon}\right)\right) d x-\int_{\Omega} f_{\varepsilon} \cdot u d x, \quad u \in H^{1}(\Omega) . \tag{1}
\end{equation*}
$$

Here $\Omega$ denotes a Lipschitz domain in $\mathbb{R}^{d}$ (the reference domain of the elastic body) and $u: \Omega \rightarrow \mathbb{R}^{d}$ is a deformation satisfying clamped boundary conditions: $u(x)=x$ on $\partial \Omega$. We denote by $f_{\varepsilon}: \Omega \rightarrow \mathbb{R}^{d}$ the density of the applied body forces, $W^{0}$ and $W^{1}$ are frameindifferent, non-degenerate energy densities (see Section 2 below for the precise assumptions), and $\chi_{\varepsilon}$ denotes the indicator function of the pores, i.e. of the domain occupied by the "soft" material component. As will be made precise in Section 2, we assume that the pores are of size $\varepsilon$ and are periodically distributed in the interior of $\Omega$ with period $\varepsilon$. As can be seen from (1), in the homogenisation limit $\varepsilon \downarrow 0$ the stiffness of the "soft" material degenerates with rate $\varepsilon^{2 \gamma}(\gamma>0)$, while the stiffness of the "stiff" material remains unchanged. Hence, the contrast between the soft material (occupying the pores) and the stiff material (occupying the perforated matrix) becomes infinite in the limit $\varepsilon \downarrow 0$. We therefore refer to the corresponding limit procedure as high-contrast homogenisation. Our goal is to identify the effective behaviour of the minimisation problem associated with $\mathcal{I}_{\varepsilon}$ by studying its limit under a proper rescaling.

Summary and discussion of our result. To illustrate our result, here in the introduction we restrict ourselves to the special case $\gamma=1$. If we assume that the density of the body forces is small in magnitude, in the sense that $f_{\varepsilon}=\varepsilon^{\alpha} f$ for some $\alpha \geq 1$, and has vanishing first moment, i.e. $\int_{\Omega} f(x) \cdot x d x=0$, then (1) can be expressed as

$$
\begin{align*}
I_{\varepsilon}^{\alpha}(\varphi) & :=\frac{1}{\varepsilon^{2 \alpha}} \mathcal{I}_{\varepsilon}(u)  \tag{2}\\
& =\int_{\Omega}\left(\frac{1}{\varepsilon^{2(\alpha-1)}} W^{0}\left(I+\varepsilon^{\alpha} \nabla \varphi\right) \chi_{\varepsilon}+\frac{1}{\varepsilon^{2 \alpha}} W^{1}\left(I+\varepsilon^{\alpha} \nabla \varphi\right)\left(1-\chi_{\varepsilon}\right)\right) d x-\int_{\Omega} f \cdot \varphi d x
\end{align*}
$$

where

$$
\begin{equation*}
\varphi(x)=\frac{u(x)-x}{\varepsilon^{\alpha}}, \quad x \in \Omega, \quad \varphi \in H_{0}^{1}\left(\Omega ; \mathbb{R}^{d}\right) \tag{3}
\end{equation*}
$$

denotes the scaled displacement, and $I$ stands for the identity matrix in $\mathbb{R}^{d \times d}$. In this paper we analyse the asymptotics of the minimisation problem associated with $I_{\varepsilon}^{\alpha}$ in the limit $\varepsilon \downarrow 0$ by appealing to the concept of $\Gamma$-convergence. The latter goes back to De Giorgi (e.g. see [19] for a standard reference). In a metric setting it is defined as follows:

Definition 1 ( $\Gamma$-convergence). Let $(X, d)$ denote a metric space. A sequence of functionals $I_{\varepsilon}: X \rightarrow[-\infty, \infty] \Gamma$-converges to a functional $I: X \rightarrow[-\infty, \infty]$, if
(a) (lower bound). For every $x_{0} \in X$ and every $x_{\varepsilon} \rightarrow x_{0}$ in $X$ we have $\liminf _{\varepsilon \rightarrow 0} I_{\varepsilon}\left(x_{\varepsilon}\right) \geq I_{0}\left(x_{0}\right)$.
(b) (recovery sequence). For every $x_{0} \in X$ there exists a sequence $x_{\varepsilon} \rightarrow x_{0}$ in $X$ such that $\lim _{\varepsilon \rightarrow 0} I_{\varepsilon}\left(x_{\varepsilon}\right)=I_{0}\left(x_{0}\right)$.

In that case we call $I_{0}$ the $\Gamma$-limit of the sequence $I_{\varepsilon}$.
A fundamental property of $\Gamma$-convergence is the following fact: If a sequence of functionals $\Gamma$-converges and the functionals are equicoercive, then the associated sequence of minima (resp. minimisers) converge (up to a subsequence) to the minimum (resp. a minimiser) of the $\Gamma$-limit, and any minimiser of the $\Gamma$-limit can be obtained as a limit of a minimizing sequence of the original functionals. Thanks to this property, $\Gamma$-convergence is especially useful for the study of the asymptotics of parametrised minimisation problems. The $\Gamma$-limit, if it exists, is unique; yet, the question weather a sequence of functionals $\Gamma$-converges or not, and the form of the $\Gamma$-limit depend on the topology of $X$. In particular, a sequence of functionals is more likely to $\Gamma$-converge in a stronger topology, while it is more likely to be equicoercive in a weaker topology. Therefore, it is natural to consider the strongest notion of convergence on $X$ for which the functionals remain equicoercive. In the situation we are interested in, namely the asymptotics of the functionals $I_{\varepsilon}^{\alpha}$ defined on $X=H_{0}^{1}(\Omega)$, it turns out that due to the presence of high-contrast, a natural and appropriate notion of convergence on $H_{0}^{1}(\Omega)$ is a variant of two-scale convergence that we discuss next. Let us first recall the standard notion of two-scale convergence from [32] and [1]:

Definition 2. We say that a sequence $f_{\varepsilon} \in L^{2}(\Omega)$ weakly two-scale converges to $f \in L^{2}(\Omega \times Y)$ if the sequence $f_{\varepsilon}$ is bounded and

$$
\lim _{\varepsilon \downarrow 0} \int_{\Omega} f_{\varepsilon}(x) \varphi(x, x / \varepsilon) d x=\iint_{\Omega \times Y} f(x, y) \varphi(x, y) d x d y
$$

for all $\varphi \in C_{c}^{\infty}\left(\Omega, C_{\#}^{\infty}(Y)\right)$, where $C_{\#}^{\infty}(Y)$ denotes the space of $Y:=(0,1)^{d}$-periodic functions in $C^{\infty}\left(\mathbb{R}^{d}\right)$. We say that a sequence $f_{\varepsilon} \in L^{2}(\Omega)$, strongly two-scale converges to $f \in L^{2}(\Omega \times Y)$, if the sequence $f_{\varepsilon}$ weakly two-scale converges to $f$ and one has $\left\|f_{\varepsilon}\right\|_{L^{2}(\Omega)} \rightarrow\|f\|_{L^{2}(\Omega \times Y)}$ as $\varepsilon \downarrow 0$. For vector-valued functions two-scale convergence is defined component-wise.

In the study of the asumptotics of $I_{\varepsilon}^{\alpha}$ we work with a variant of this definition that is tailor made to capture the effects of high-contrast. For $\gamma=1$ it can be summarised as follows (for details and the general case see Section 3.1): Given a sequence of displacements $\varphi_{\varepsilon}$ in $H_{0}^{1}(\Omega)$ we consider the unique decomposition $\varphi_{\varepsilon}=g_{\varepsilon}^{0}+g_{\varepsilon}^{1}$ into a harmonic function $g_{\varepsilon}^{0} \in H_{0}^{1}\left(\Omega_{\varepsilon}^{0}\right)$ and a remainder $g_{\varepsilon}^{1} \in H_{0}^{1}(\Omega)$. We then write $\varphi_{\varepsilon} \xrightarrow{2}\left(g^{0}, g^{1}\right)$ and say that $\varphi_{\varepsilon}$ converges to a pair $\left(g^{0}, g^{1}\right)$ with $g^{0} \in L^{2}\left(\Omega ; H_{0}^{1}\left(Y_{0}\right)\right)$ and $g^{1} \in H_{0}^{1}(\Omega)$, if $g_{\varepsilon}^{1} \rightharpoonup g^{1}$ weakly in $H^{1}(\Omega)$, and $g_{\varepsilon}^{0} \stackrel{2}{\rightharpoonup} g^{0}, \varepsilon \nabla g_{\varepsilon}^{0} \stackrel{2}{\rightharpoonup} \nabla_{y} g^{0}$ weakly two-scale. The component $g^{1}$ of a limit pair $\left(g^{0}, g^{1}\right)$ describes the (scaled) macroscopic displacement of the body, and $g^{0}$ is a two-scale function describing the (scaled) microscopic displacement on the pores relative to the deformed matrix material. As a main result (see Theorem 1 and Theorem 3) we prove (in fact in a slightly more general situation) that $I_{\varepsilon}^{\alpha} \Gamma$-converges w.r.t. the type of convergence introduced above. It turns out that two different regimes emerge for $\alpha>1$ and $\alpha=1$.

In the small strain regime $(\alpha>1)$, the strain $\varepsilon^{\alpha} \nabla \varphi$ becomes infinitesimally small in the entire domain $\Omega$, and the limit behaviour is expressed by a linearised, two-scale energy

$$
\begin{align*}
I_{\text {small }}: g^{0} \in L^{2}\left(\Omega ; H_{0}^{1}\left(Y_{0}\right)\right) \times & H_{0}^{1}\left(\Omega ; \mathbb{R}^{d}\right) \rightarrow \mathbb{R} \\
I_{\text {small }}\left(g^{0}, g^{1}\right):= & \int_{\Omega \times Y} Q^{0}\left(\nabla_{y} g^{0}(x, y)\right) d y+Q_{\mathrm{hom}}^{1}\left(\nabla g^{1}(x)\right) d x  \tag{4}\\
& -\int_{\Omega}\left(\int_{Y_{0}} g^{0}(x, y) d y+g^{1}(x)\right) \cdot f(x) d x \tag{5}
\end{align*}
$$

Here $Q^{0}$ and $Q^{1}$ are the quadratic forms of the quadratic expansions of $W^{0}$ and $W^{1}$ at the identity, and $Q_{\text {hom }}^{1}$ denotes the homogenised energy density obtained from $Q^{1}$, see (12) and (23) for details. The functional $I_{\text {small }}$ is the two-scale $\Gamma$-limit of the sequence $I_{\varepsilon}^{\alpha}$ in the sense that (cf. Theorem 1):
(a) (lower bound). For every $\left(g^{0}, g^{1}\right)$ and every $\varphi_{\varepsilon} \stackrel{2}{\longrightarrow}\left(g^{0}, g^{1}\right)$ we have $\liminf _{\varepsilon \downarrow 0} I_{\varepsilon}^{\alpha}\left(\varphi_{\varepsilon}\right) \geq$ $I_{\text {small }}\left(g^{0}, g^{1}\right)$.
(b) (recovery sequence). For every $\left(g^{0}, g^{1}\right)$ there exists a sequence $\varphi_{\varepsilon} \stackrel{2}{\longrightarrow}\left(g^{0}, g^{1}\right)$ such that $\lim _{\varepsilon \downarrow 0} I_{\varepsilon}^{\alpha}\left(\varphi_{\varepsilon}\right) \geq I_{\text {small }}\left(g^{0}, g^{1}\right)$.
In addition, we prove that the functional $\left(I_{\varepsilon}^{\alpha}\right)_{\varepsilon>0}$ are equicoercive and deduce convergence of the associated minimisation problems (see Theorem 1 and Proposition 1). In Theorem 2 we establish a two-scale expansion showing that if $\varphi^{\varepsilon}$ is an (almost) minimiser of $I_{\varepsilon}^{\alpha}$, then

$$
\begin{equation*}
\varphi_{\varepsilon}(x) \approx g_{*}^{1}(x)+\varepsilon \psi(x, x / \varepsilon)+g_{*}^{0}(x, x / \varepsilon) \tag{6}
\end{equation*}
$$

where $\left(g_{*}^{0}, g_{*}^{1}\right)$ is a minimiser of $I_{\text {small }}$, and $\psi$ denotes a corrector function that only depends on $g_{*}^{1}$. Finally, we illustrate that by averaging out the fast variable $y$, the limit $I_{\text {small }}$ can be further simplified. In fact, Proposition 2 shows that $I_{\varepsilon}^{\alpha} \Gamma$-converges (w.r.t. weak convergence in $\left.L^{2}(\Omega)\right)$ to the functional $\bar{I}_{\text {small }}: L^{2}(\Omega) \rightarrow \mathbb{R} \cup\{+\infty\}$ given by

$$
\begin{align*}
\bar{I}_{\mathrm{small}}(\varphi):= & \min \left\{\int_{\Omega} Q_{\mathrm{hom}}^{1}\left(\nabla g^{1}\right)+\bar{Q}(\bar{G}): g^{1} \in H_{0}^{1}(\Omega), \bar{G} \in L^{2}(\Omega) \text { with } g^{1}+\bar{G}=\varphi\right\} \\
& -\int_{\Omega} f \cdot \varphi d x \tag{7}
\end{align*}
$$

with a (positive definite) quadratic form $\bar{Q}: \mathbb{R}^{d} \rightarrow[0, \infty)$ defined by

$$
\bar{Q}(\bar{G}):=\min \left\{\int_{Y^{0}} Q^{0}\left(\nabla g^{0}(y)\right) d y: g^{0} \in H_{0}^{1}\left(Y^{0}\right), \int_{Y^{0}} g^{0} d y=\bar{G}\right\}
$$

$\bar{Q}$ captures the influence of the pores (and their geometry) on the effective behavior. The minimiser $\varphi_{*} \in L^{2}(\Omega)$ to $\bar{I}_{\text {small }}$ takes the form $\varphi_{*}=g_{*}^{1}+\bar{G}_{*}$ with $\bar{G}_{*}:=\int_{Y_{0}} g_{*}^{0}(\cdot, y) d y$. In view of (6) the field $\bar{G}_{*}$ can be interpreted as the gap between the macroscopic displacement and the microscopic displacements in the pores.

In the finite strain regime, which corresponds to $\alpha=1$, the displacement gradient $\varepsilon \nabla \varphi_{\varepsilon}$ becomes infinitesimally small only in the stiff component, while large strains still may occur in the soft pores. Therefore, the $\Gamma$-limit is a non-convex (partially linearised) functional of the form

$$
\begin{aligned}
I_{\text {finite }}\left(g^{0}, g^{1}\right):= & \int_{\Omega \times Y} \mathcal{Q} W^{0}\left(I+\nabla_{y} g^{0}(x, y)\right) d y+Q_{\mathrm{hom}}^{1}\left(\nabla g^{1}(x)\right) d x \\
& -\int_{\Omega}\left(\int_{Y_{0}} g^{0}(x, y) d y+g^{1}(x)\right) \cdot f(x) d x
\end{aligned}
$$

where $\mathcal{Q} W^{0}$ denotes the quasiconvex envelope of $W^{0}$. Similarly to the small strain regime, one can average out the fast scale $y$ and obtain $\Gamma$-convergence of $I_{\varepsilon}^{0}$ to the functional $\bar{I}_{\text {finite }}$ : $L^{2}(\Omega) \rightarrow \mathbb{R} \cup\{+\infty\}$ given by

$$
\begin{align*}
\bar{I}_{\text {finite }}(\varphi):= & \min \left\{\int_{\Omega} Q_{\mathrm{hom}}^{1}\left(\nabla g^{1}\right)+\bar{V}(\bar{G}): g^{1} \in H_{0}^{1}(\Omega), \bar{G} \in L^{2}(\Omega) \text { with } g^{1}+\bar{G}=\varphi\right\}  \tag{8}\\
& -\int_{\Omega} f \cdot \varphi d x,
\end{align*}
$$

with non-convex potential $\bar{V}: \mathbb{R}^{d} \rightarrow[0, \infty)$ defined by

$$
\bar{V}(\bar{G}):=\min \left\{\int_{Y^{0}} \mathcal{Q} W^{0}\left(I+\nabla g^{0}(y)\right) d y: g^{0} \in H_{0}^{1}\left(Y^{0}\right), \int_{Y^{0}} g^{0} d y=\bar{G}\right\},
$$

see Remark 2. In contrast to the small strain regime, where $\bar{Q}$ is quadratic, the potential $\bar{V}$ is non-convex and expresses a nonlinear (and non-monotone) coupling between the macroscopic and microscopic displacement.

Connection to acoustic wave propagation in high-contrast materials. The sequence of functionals $\varepsilon^{-2 \alpha} \mathcal{I}_{\varepsilon}$, in either of the two regimes described above, occupies an intermediate position between a fully nonlinearly elastic composite and fully linearised models, as $\varepsilon \rightarrow 0$. Notably, linear models with high contrast, which are suitable for the description of small displacement fields (that often occur, say, in acoustic wave propagation) already exhibit a coupling between the macroscopic part $g^{0}$ and microscopic part $g^{1}$ of the minimiser of $I_{\text {small }}$, which in our case is obtained as a limit in the small-strain regime $\alpha>1$. This can be seen by considering the time-harmonic solitons to the equations of elastodynamics with the elastic part of the energy given by (4), away from the sources of the elastic motion. In this case the function $f$ in (5) (which in our analysis we assume to be independent of the fast variable $x / \varepsilon$ for simplicity, an assumption that can be relaxed with no changes in the proofs needed) has to be replaced by the sum $g^{0}+g^{1}$, with the integration in (5) carried over $Y_{0}$ and $Q$ at the same time, i.e. the work of the external forces (5) is replaced by the expression for the work of "self-forces"

$$
-\omega^{2} \int_{\Omega} \int_{Y_{0}}\left(g^{0}(x, y)+g^{1}(x)\right) \cdot\left(g^{0}(x, y)+g^{1}(x)\right) d y d x
$$

where $\omega$ is the frequency. The solution to the Euler-Lagrange equation for the resulting functional is a coupled system of equations for $g^{0}, g^{1}$, so that when the equation for $g^{0}$ is solved in terms of $g^{1}$ and substituted into the second equation, it takes the form (away from the sources):

$$
\begin{equation*}
\mathcal{A}^{\mathrm{hom}} g^{1}=\omega^{2} \beta\left(\omega^{2}\right) g^{1} \tag{9}
\end{equation*}
$$

for some non-negative self-adjoint differential operator $\mathcal{A}^{\text {hom }}$ and a special nonlinear function $\beta$, which takes positive and negative values on alternating intervals of the real axis (leading to "lacunae", or "band gaps" in the spectrum of the corresponding operator) and is obtained from the spectral decomposition of $g^{0}$ and the subsequent averaging over $Y_{0}$, see [38]. From this point of view, the non-quadratic finite-strain functional $I_{\text {finite }}$ is a "matching", "partially quadratic", homogenised model corresponding, e.g., to finite-amplitude, rather than smallamplitude, wave motions that can no longer be treated using a quadratic model such as $I_{\text {small }}$ but can still be used in place of models of nonlinear elasticity where the elastic energy terms on both components of the composite (stiff and soft) are non-quadratic.

Methods and previous results. In this paper we appeal to analytic methods that have been developed in the last two decades in the areas of nonlinear elasticity and homogenisation. Among these are the notion of two-scale convergence introduced in [32], [1]) and periodic unfolding (see [18] and references therein). The convergence statements of our main results are expressed in the language of $\Gamma$-convergence (see [19] and references therein). In order to treat the geometric nonlinearity of the considered functional, we make use of the geometric rigidity estimate (see [23]). Since we consider a low energy regime, linearisation and homogenisation take place at the same time. The simultaneous treatment of both effects is inspired by recent works [28], [29], [30], [31], [25] of the third author, where various problems involving simultaneous homogenisation, linearisation and dimension reduction are studied. The homogenisation of the kind of high-contrast composites that we study is related to the homogenisation for periodically perforated domains (e.g. see [33], [10]). For instance, we make use of extensions across the pores. As a side result we prove a version of the geometric rigidity estimate for perforated domains (see Lemma 4 below). We would like to remark that while the present work is one of the few papers, along with [17], [8], that treat the fully nonlinear high-contrast case, during the last decade there has been a significant amount of literature devoted to the mathematical analysis of phenomena associated with, or modelled by, a high degree of contrast between the properties of the materials constituting a composite, in the linearised setting. The first contributions in this direction are due to Zhikov [38], and Bouchitté and Felbacq [6], following an earlier paper by Allaire [1] and the collection of papers by Hornung et al. [24] (see also the references therein), where the special role of high-contrast elliptic PDE was pointed out albeit not studied in detail. These works demonstrated that the behaviour of the field variable in such models is of a two-scale type in the homogenisation limit, i.e. the limit model cannot be reduced to a one-scale formulation and fields that depend on the fast variable remain in the effective model. They also noticed that the spectrum of such materials has a band-gap structure, as in (9), and indicated how this fact could be exploited for high-resolution imaging and cloaking. It has since been an adopted approach to the theoretical construction of "negative refraction" media, or more generally "metamaterials", which is now a hugely popular area of research in physics (see e.g. [34] and references therein). On the analytical side, a number of further works followed, in particular [3], [4], [11], [13], [14], [16], [26], [15], [35], [7], [9], where various consequences of high contrast (or, mathematically speaking, the property of non-uniform ellipticity) in the underlying equations have been explored. Among these are the "non-locality" and "micro-torsion" effects in materials with high-contrast inclusions in the shape of fibres extending in one or more directions, the "partial band-gap" wave propagation due to the high degree of anisotropy of one of the constituent media, and the localisation of energy in high-contrast media with a defect ("photonic crystal fibres"), all of which can be thought of as examples of "non-standard", or "non-classical", behaviour in composites, which is not available in the usual moderate-contrast materials. In the present paper we aim to develop further a rigorous high-contrast theory in the context of finite elasticity, where the underlying model is nonlinear.

With this paper we continue the multiscale theme initiated in [17], where the regime of large deformation gradients in the soft component of the composite was considered. Let us emphasise two points that contrast our contribution to some earlier work within the related field. First, we note that, apart from [17], [8], a number of other articles (e.g. [5], [12], [7]) have treated high-contrast periodic composites in the nonlinear context. However, the related results are of limited relevance to nonlinear elasticity, due to the convexity or monotonicity assumptions made in these works. In the present paper we study a class of functionals subject to the requirement of material fame indifference (see assumption (W1) in Section 2), which makes
our analysis fit the fully nonlinear elasticity framework, as opposed to the works mentioned. Second, as was discussed above, the analysis of composites with "soft" inclusions within a "stiff" matrix cannot be reduced to a "decoupled" model where the perforated medium obtained by removing the inclusions is considered first and the displacement within the inclusions is found independently, which from the physics perspective can be viewed as a kind of resonance phenomenon; cf. (9) in the linearisation regime, for which an inherent energy coupling, in the limit as $\varepsilon \rightarrow 0$, between the soft and stiff components of the composite is essential. On a related note, the proof of the key compactness statement (Lemma 1) involves the simultaneous analysis of the displacements on the two components. We would also like to highlight the fact that in [17] the order of the relative scaling of the displacements on the soft and stiff components of the composite are assumed from the outset, while in the present work it is the result of the above compactness argument itself.

Organisation of the paper. In Section 2 we state the assumptions on the geometry of the composite and the material law. In Section 3 we present the main results, starting with results regarding two-scale compactness, convergence results in the small strain regime and finally the convergence result in the finite strain regime. All proofs are presented in Section 4.

### 1.1 Notation

Here we list some notation that we use throughout the text. Additional items will be introduced whenever they are first used in the text.

- $d \geq 2$ is the (integer) dimension of the space occupied by the material.
- $p \geq 1$ is the exponent in the notation $L^{p}$ for a Lebesgue space.
- $Y:=(0,1)^{d}$ the reference period cell; $Y^{0}$ is an open Lipschitz set whose closure is contained in $Y$, and $Y^{1}:=Y \backslash \overline{Y^{0}}$.
- $\Omega, \Omega_{\varepsilon}^{0}$ and $\Omega_{\varepsilon}^{1}$ denote the reference domains of the composite, the set occupied by the pore material, and the domain occupied by the matrix material, respectively, see Section 2 for precise definition.
- Unless stated otherwise, all function spaces $L^{2}(\Omega), H^{1}(\Omega), H_{0}^{1}(\Omega)$, etc. consist of functions taking values in $\mathbb{R}^{d}$.
- Function spaces whose notation contains subscript "c" consist of functions that vanish outside a compact set.
- The function spaces $H_{\#}^{1}, H_{0}^{1}\left(Y^{0}\right)$, and $\mathcal{A}\left(Y^{0}\right)$ are introduced in Section 3.1.
- We write • and : for the canonical inner products in $\mathbb{R}^{d}$ and $\mathbb{R}^{d \times d}$, respectively.
- $\mathrm{SO}(d)$ denotes the set of rotations in $\mathbb{R}^{d \times d}$.
- $\lesssim$ stands for $\leq$ up to a multiplicative constant that only depends on $d, Y^{1}, \Omega$, and on $p$ if applicable.


## 2 Geometric and constitutive setup

The pore geometry. The set $Y^{0}$ defined above describes the "pores" contained within the cell $Y$. Note that $Y^{1}$ is an open, bounded, connected set with Lipschitz boundary. Therefore, to each $\varphi \in H^{1}\left(Y^{1}\right)$ we can associate (see e.g. [33]) a unique harmonic extension $g^{1} \in H^{1}(Y)$ characterised by

$$
\begin{equation*}
g^{1}=\varphi \text { in } Y^{1}, \quad \int_{Y^{0}} \nabla g^{1}: \nabla \zeta d y=0 \quad \forall \zeta \in H_{0}^{1}\left(Y^{0}\right) \tag{10}
\end{equation*}
$$

For this extension the inequality

$$
\begin{equation*}
\left\|\nabla g^{1}\right\|_{L^{2}\left(Y^{0}\right)} \leq C\|\nabla \varphi\|_{L^{2}\left(Y^{1}\right)} \tag{11}
\end{equation*}
$$

holds with a constant $C$ that only depends on $Y^{1}$.
For a given domain $\Omega \subset \mathbb{R}^{d}$ and $\varepsilon>0$, we define the sets $\Omega_{\varepsilon}^{0}$ and $\Omega_{\varepsilon}^{1}$ as follows:

$$
\Omega_{\varepsilon}^{0}:=\bigcup\left\{\varepsilon\left(\xi+Y^{0}\right) \mid \xi \in \mathbb{Z}^{d}, \varepsilon(\xi+Y) \subset \Omega\right\}, \quad \Omega_{\varepsilon}^{1}:=\Omega \backslash \overline{\Omega_{\varepsilon}^{0}} .
$$

Note that by construction $\Omega_{\varepsilon}^{1}$ is a Lipschitz domain. In particular, it is connected and $\partial \Omega \subset$ $\partial \Omega_{\varepsilon}^{1}$. We denote by $\chi_{\varepsilon}$ the indicator function of the set of pores:

$$
\chi_{\varepsilon}(x):= \begin{cases}1, & x \in \Omega_{\varepsilon}^{0} \\ 0, & x \in \mathbb{R}^{d} \backslash \Omega_{\varepsilon}^{0}\end{cases}
$$

The composite. The two materials are described by energy densities $W^{i}: \mathbb{R}^{d \times d} \rightarrow[0,+\infty]$, $i=0,1$. Unless stated otherwise, we assume that for $i=0,1$ :
(W1) $W^{i}$ is frame-indifferent, i.e. $W^{i}(R F)=W^{i}(F)$ for all $R \in \mathrm{SO}(d)$ and all $F \in \mathbb{R}^{d \times d}$;
(W2) The identity matrix $I \in \mathbb{R}^{d \times d}$ is a "natural state", i.e. $W^{i}(I)=0$, and $W^{i}$ is nondegenerate, i.e. for all

$$
W^{i}(F) \geq c_{0} \operatorname{dist}^{2}(F, \mathrm{SO}(d)), \quad \forall F \in \mathbb{R}^{d \times d}, \quad c_{0}>0 .
$$

(W3) $W^{i}$ has a quadratic expansion at $I$, i.e. there exists a non-negative quadratic form $Q^{i}$ on $\mathbb{R}^{d \times d}$ and an increasing function $r^{i}:[0, \infty) \rightarrow[0, \infty]$ with $\lim _{s \downarrow 0} r^{i}(s)=0$, such that

$$
\begin{equation*}
\left|W^{i}(I+G)-Q^{i}(G)\right| \leq|G|^{2} r^{i}(|G|) \quad \forall G \in \mathbb{R}^{d \times d} \tag{12}
\end{equation*}
$$

As shown in [30, Lemma 2.7] the quadratic form $Q^{i}$ associated with $W^{i}$ via (W3) satisfies

$$
\begin{equation*}
c_{1}|\operatorname{sym} G|^{2} \leq Q^{i}(G)=Q^{i}(\operatorname{sym} G) \leq c_{1}^{-1}|\operatorname{sym} G|^{2} \quad \forall G \in \mathbb{R}^{d \times d}, \quad c_{1}>0 . \tag{13}
\end{equation*}
$$

In the finite strain regime we consider a different set of assumptions for $W^{0}$, which are listed in Section 3.3.
The scaling parameter $\gamma$. Throughout the paper $\gamma>0$ denotes a fixed scaling parameter. It is a quantitative measure of the relative contrast between the two components of the composite.

Energy functional. We define the elastic energy as a functional of the displacement, as follows:

$$
\begin{equation*}
\mathcal{E}_{\varepsilon}(u)=\int_{\Omega}\left(\varepsilon^{2 \gamma} W^{0}(I+\nabla u) \chi_{\varepsilon}+W^{1}(I+\nabla u)\left(1-\chi_{\varepsilon}\right)\right) d x, \quad u \in H_{0}^{1}\left(\Omega, \mathbb{R}^{d}\right) \tag{14}
\end{equation*}
$$

## 3 Main results

### 3.1 Compactness and two-scale convergence

We first present an a priori estimate and a two-scale compactness statement for sequences $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ whose energy is equi-bounded in the sense that

$$
\begin{equation*}
\limsup _{\varepsilon \downarrow 0} \Phi_{\varepsilon}^{\gamma}\left(v_{\varepsilon}\right)<\infty, \tag{15}
\end{equation*}
$$

where

$$
\Phi_{\varepsilon}^{\gamma}(v):=\int_{\Omega} \operatorname{dist}^{2}(I+\nabla v(x), \mathrm{SO}(d))\left(\varepsilon^{2 \gamma} \chi_{\varepsilon}+\left(1-\chi_{\varepsilon}\right)\right) d x .
$$

Note that, by virtue of the non-degeneracy assumption (W2) the functional $\Phi_{\varepsilon}^{\gamma}(\cdot)$ bounds below $\mathcal{E}_{\varepsilon}(\operatorname{id}+\cdot)$, where $\operatorname{id}(x)=x, x \in \Omega$. As we shall see in the upcoming Lemma 1 , the inequality (15) implies that the sequence $\varphi_{\varepsilon}$ is bounded in $H^{1}(\Omega)$, and thus weakly converges (up to extracting a subsequence) to a limit displacement $\varphi \in H_{0}^{1}(\Omega)$. For our purpose we require a precise understanding of the oscillations that emerge along that limit. We achieve this by combining two concepts:

- We write a representation for $\varphi_{\varepsilon}$ in the spirit of an asymptotic decomposition as $\varepsilon \downarrow 0$.
- We study the convergence properties of the terms in this decomposition by appealing to two-scale convergence.

In the following lemma we address the first item above.
Lemma 1. Let $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ and $0<\varepsilon \leq 1$.
(a) There exists a unique pair of functions $g_{\varepsilon}^{0} \in H_{0}^{1}\left(\Omega_{\varepsilon}^{0}\right)$ and $g_{\varepsilon}^{1} \in H_{0}^{1}(\Omega)$ such that

$$
\begin{align*}
& \text { (i) } \varphi_{\varepsilon}=g_{\varepsilon}^{1}+\varepsilon^{1-\gamma} g_{\varepsilon}^{0}, \\
& \text { (ii) } \int_{\Omega_{\varepsilon}^{0}} \nabla g_{\varepsilon}^{1}: \nabla \zeta=0 \quad \forall \zeta \in H_{0}^{1}\left(\Omega_{\varepsilon}^{0}\right) . \tag{16}
\end{align*}
$$

(b) There exists a positive constant $C$ that only depends on $\Omega, Y^{0}$ such that

$$
\begin{equation*}
\left\|g_{\varepsilon}^{0}\right\|_{L^{2}(\Omega)}^{2}+\left\|\varepsilon \nabla g_{\varepsilon}^{0}\right\|_{L^{2}(\Omega)}^{2}+\left\|g_{\varepsilon}^{1}\right\|_{H^{1}(\Omega)}^{2} \leq C \Phi_{\varepsilon}^{\gamma}\left(\varphi_{\varepsilon}\right) \tag{17}
\end{equation*}
$$

where $g_{\varepsilon}^{0}, g_{\varepsilon}^{1}$ and $\varphi_{\varepsilon}$ are related to each other as in (a).
As already explained in the introduction, for our purpose it is convenient to appeal to two-scale convergence, see Definition 2. We use the following shorthand notation:

$$
\begin{array}{rll}
f_{\varepsilon} \rightarrow f_{0} & : \Leftrightarrow & f_{\varepsilon} \text { strongly converges to } f_{0} \text { in } L^{2}(\Omega), \\
f_{\varepsilon} \rightharpoonup f_{0} & : \Leftrightarrow & f_{\varepsilon} \text { weakly converges to } f_{0} \text { in } L^{2}(\Omega), \\
f_{\varepsilon} \stackrel{2}{\rightharpoonup} f & : \Leftrightarrow & f_{\varepsilon} \text { weakly two-scale converges to } f \text { in } L^{2}(\Omega \times Y), \\
f_{\varepsilon} \xrightarrow{2} f & : \Leftrightarrow & f_{\varepsilon} \text { strongly two-scale converges to } f \text { in } L^{2}(\Omega \times Y) .
\end{array}
$$

The upcoming lemma states a two-scale compactness result for the displacements $g_{\varepsilon}^{0}$ and $g_{\varepsilon}^{1}$ that appear in the representation (16). Due to the differential constraint satisfied by $g_{\varepsilon}^{0}$, the corresponding two-scale limits automatically satisfy certain structural properties, which can be captured with the help of the following function spaces:

- $H_{\#}^{1}$ is the space of $[0,1)^{d}$-periodic functions in $H_{\text {loc }}^{1}\left(\mathbb{R}^{d}\right)$.
- $H_{0}^{1}\left(Y^{0}\right)$ is the closed subspace of $H_{\#}^{1}$ consisting of functions $\psi \in H_{\#}^{1}$ with $\psi=0$ on $Y^{1}$.
- $\mathcal{A}\left(Y^{0}\right)$ is the closed subspace of $H_{\#}^{1}$ consisting of functions $\psi \in H_{\#}^{1}$ that satisfy the identity

$$
\int_{Y^{0}} \nabla_{y} \psi: \nabla_{y} \zeta d y=0 \quad \forall \zeta \in H_{0}^{1}\left(Y^{0}\right)
$$

Lemma 2. Consider a sequence $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ and let $\left(g_{\varepsilon}^{0}, g_{\varepsilon}^{1}\right)$ be associated with $\varphi_{\varepsilon}$ via (16). Suppose that there exists a sequence of positive numbers $m_{\varepsilon}$ such that

$$
\underset{\varepsilon \downarrow 0}{\limsup } m_{\varepsilon}^{-2} \Phi_{\varepsilon}^{\gamma}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)<\infty \quad \text { and } \quad m_{\varepsilon}=O\left(\varepsilon^{\gamma}\right)
$$

Then there exist

$$
\begin{equation*}
g^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right), \quad g^{1} \in H_{0}^{1}(\Omega), \quad \psi \in L^{2}\left(\Omega, \mathcal{A}\left(Y^{0}\right)\right) \tag{18}
\end{equation*}
$$

such that, up to selecting a subsequence, one has

$$
\begin{align*}
& g_{\varepsilon}^{0} \stackrel{2}{\rightharpoonup} g^{0}, \quad \varepsilon \nabla g_{\varepsilon}^{0} \stackrel{2}{\rightharpoonup} \nabla_{y} g^{0},  \tag{19}\\
& g_{\varepsilon}^{1} \rightharpoonup g^{1} \text { weakly in } H^{1}(\Omega) \text { and } \nabla g_{\varepsilon}^{1} \stackrel{2}{\rightharpoonup} \nabla g^{1}+\nabla_{y} \psi .
\end{align*}
$$

The identification obtained in the previous lemma is sharp, in the sense of the following statement.

Lemma 3. Let $g^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$, $g^{1} \in H_{0}^{1}(\Omega)$ and $\psi \in L^{2}\left(\Omega, \mathcal{A}\left(Y^{0}\right)\right)$. Let $c_{\varepsilon}$ be an arbitrary sequence of positive numbers converging to zero. Then there exist function sequences $g_{\varepsilon}^{0} \in H_{0}^{1}\left(\Omega_{\varepsilon}^{0}\right), g_{\varepsilon}^{1} \in H_{0}^{1}(\Omega)$ such that $\left(g_{\varepsilon}^{0}, g_{\varepsilon}^{1}\right)$ is related to $\varphi_{\varepsilon}:=g_{\varepsilon}^{1}+\varepsilon^{1-\gamma} g_{\varepsilon}^{0}$ as in (16), and

$$
\begin{align*}
& g_{\varepsilon}^{0} \xrightarrow{2} g^{0}, \quad \varepsilon \nabla g_{\varepsilon}^{0} \xrightarrow{2} \nabla_{y} g^{0}, \\
& g_{\varepsilon}^{1} \rightharpoonup g^{1} \text { weakly in } H^{1}(\Omega) \text { and } \nabla g_{\varepsilon}^{1} \xrightarrow{2} \nabla g^{1}+\nabla_{y} \psi,  \tag{20}\\
& \underset{\varepsilon \downarrow 0}{\limsup } c_{\varepsilon}\left\|\nabla \varphi_{\varepsilon}\right\|_{L^{\infty}(\Omega)}=0 .
\end{align*}
$$

Our main result is formulated in terms of the notion of convergence described in the above lemmas. For convenience we use the following notation:

- Given $\varphi_{\varepsilon} \in H^{1}(\Omega)$ we write $g_{\varepsilon}^{1}+\varepsilon^{1-\gamma} g_{\varepsilon}^{0} \stackrel{(16)}{=} \varphi_{\varepsilon}$, if $g_{\varepsilon}^{1} \in H^{1}(\Omega), g_{\varepsilon}^{0} \in H_{0}^{1}\left(\Omega_{\varepsilon}^{0}\right)$, and both functions are related to $\varphi_{\varepsilon}$ as in (16).
- We write $\varphi_{\varepsilon} \stackrel{2}{\longrightarrow}\left(g^{0}, g^{1}\right)$, if $g_{\varepsilon}^{1}+\varepsilon^{1-\gamma} g_{\varepsilon}^{0} \stackrel{(16)}{=} \varphi_{\varepsilon}$ and

$$
\begin{equation*}
g_{\varepsilon}^{0} \stackrel{2}{\rightharpoonup} g^{0}, \quad \varepsilon \nabla g_{\varepsilon}^{0} \stackrel{2}{\rightharpoonup} \nabla_{y} g^{0}, \quad g_{\varepsilon}^{1} \rightharpoonup g^{1} \text { weakly in } H^{1}(\Omega) . \tag{21}
\end{equation*}
$$

- We write $\varphi_{\varepsilon} \xrightarrow{2}\left(g^{0}, g^{1}\right)$, if $g_{\varepsilon}^{1}+\varepsilon^{1-\gamma} g_{\varepsilon}^{0} \stackrel{(16)}{=} \varphi_{\varepsilon}$ and

$$
\begin{equation*}
g_{\varepsilon}^{0} \xrightarrow{2} g^{0}, \quad \varepsilon \nabla g_{\varepsilon}^{0} \xrightarrow{2} \nabla_{y} g^{0}, \quad g_{\varepsilon}^{1} \rightharpoonup g^{1} \text { weakly in } H^{1}(\Omega) . \tag{22}
\end{equation*}
$$

### 3.2 Convergence in the small strain regime $m_{\varepsilon}=o\left(\varepsilon^{\gamma}\right)$

Throughout this section we assume that the densities $W^{0}$ and $W^{1}$ satisfy the conditions (W1)(W3). We show that in the small strain regime the limit functional

$$
\mathcal{E}_{\text {small }}: L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right) \times H_{0}^{1}(\Omega) \rightarrow[0, \infty),
$$

is given by

$$
\mathcal{E}_{\text {small }}\left(g^{0}, g^{1}\right):=\int_{\Omega \times Y}\left(Q^{0}\left(\nabla_{y} g^{0}(x, y)\right)+Q_{\mathrm{hom}}^{1}\left(\nabla g^{1}(x)\right)\right) d x
$$

where

$$
\begin{equation*}
Q_{\mathrm{hom}}^{1}(F):=\min _{\psi \in \mathcal{A}\left(Y^{0}\right)} \int_{Y^{1}} Q^{1}\left(F+\nabla_{y} \psi(y)\right) d y \tag{23}
\end{equation*}
$$

More precisely, the following theorem holds.
Theorem 1. Let $m_{\varepsilon}$ be a sequence of positive numbers and assume that $m_{\varepsilon}=o\left(\varepsilon^{\gamma}\right)$ as $\varepsilon \downarrow 0$.
(a) (Compactness). Suppose that $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ satisfy

$$
\underset{\varepsilon \downarrow 0}{\limsup } m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)<\infty
$$

Then, up to a subsequence, one has $\varphi_{\varepsilon} \xrightarrow{2}\left(g^{0}, g^{1}\right)$ for some $g^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$ and $g^{1} \in H_{0}^{1}(\Omega)$.
(b) (Lower bound). Consider $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ and suppose that $\varphi_{\varepsilon} \xrightarrow{2}\left(g^{0}, g^{1}\right)$ for some $g^{0} \in$ $L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$ and $g^{1} \in H_{0}^{1}(\Omega)$. Then the estimate

$$
\liminf _{\varepsilon \downarrow 0} m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right) \geq \mathcal{E}_{\text {small }}\left(g^{0}, g^{1}\right)
$$

holds.
(c) (Recovery sequence). For all $g^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$ and $g^{1} \in H_{0}^{1}(\Omega)$ there exists a sequence $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ such that $\varphi_{\varepsilon} \xrightarrow{2}\left(g^{0}, g^{1}\right)$ and

$$
\lim _{\varepsilon \downarrow 0} m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)=\mathcal{E}_{\text {small }}\left(g^{0}, g^{1}\right)
$$

In the next result we consider a minimisation problem that involves the density of the "body forces" $\ell_{\varepsilon} \in L^{2}(\Omega)$. We study the variational limit of the (scaled) total energy

$$
\begin{equation*}
I_{\varepsilon}(\varphi):=\frac{1}{m_{\varepsilon}^{2}}\left(\mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi\right)-\int_{\Omega} \ell_{\varepsilon} \cdot\left(m_{\varepsilon} \varphi\right) d x\right) \tag{24}
\end{equation*}
$$

where the scaling factor $m_{\varepsilon}$ is determined by the body forces via

$$
\begin{equation*}
m_{\varepsilon}:=\varepsilon^{1-\gamma}\left\|\ell_{\varepsilon}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{0}\right)}+\left\|\ell_{\varepsilon}\right\|_{L^{2}(\Omega)} . \tag{25a}
\end{equation*}
$$

In the small strain regime we assume that the body forces are small in the sense that

$$
\begin{equation*}
m_{\varepsilon}=o\left(\varepsilon^{\gamma}\right), \quad \varepsilon \downarrow 0 \tag{25b}
\end{equation*}
$$

Moreover, we assume that the (scaled) body-force densities converge, as $\varepsilon \downarrow 0$, in the following way:

$$
\begin{equation*}
m_{\varepsilon}^{-1} \varepsilon^{1-\gamma} \chi_{\varepsilon} \ell_{\varepsilon} \xrightarrow{2} \ell^{0}, \quad m_{\varepsilon}^{-1} \ell_{\varepsilon} \rightharpoonup \ell^{1} . \tag{25c}
\end{equation*}
$$

It follows from Theorem 1 that the variational limit of the total energy (24) is given by the functional

$$
\begin{equation*}
I_{\text {small }}\left(g^{0}, g^{1}\right):=\mathcal{E}_{\text {small }}\left(g^{0}, g^{1}\right)-\int_{\Omega}\left(\int_{Y^{0}} \ell^{0} \cdot g^{0} d y+\ell^{1} \cdot g^{1}\right) d x \tag{26}
\end{equation*}
$$

Proposition 1. Assume that (25a)-(25c) hold.
(b) (Convergence of infima). One has

$$
\lim _{\varepsilon \downarrow 0} \inf _{\varphi \in H_{0}^{1}(\Omega)} I_{\varepsilon}(\varphi)=\min I_{\text {small }}\left(g^{0}, g^{1}\right),
$$

where the minimum on the right-hand side is taken over all $g^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$ and $g^{1} \in H_{0}^{1}(\Omega)$. Moreover, the minimum is attained for a unique pair $\left(g_{*}^{0}, g_{*}^{1}\right)$.
(b) (Convergence of minimisers). Let $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ be a sequence of almost minimisers, i.e.

$$
\begin{equation*}
I_{\varepsilon}\left(\varphi_{\varepsilon}\right) \leq \inf _{\varphi \in H_{0}^{1}(\Omega)} I_{\varepsilon}(\varphi)+o(1), \quad \varepsilon \downarrow 0 \tag{27}
\end{equation*}
$$

Then

$$
\varphi_{\varepsilon} \stackrel{2}{\rightharpoonup}\left(g_{*}^{0}, g_{*}^{1}\right) \quad \text { and } \quad \nabla g_{\varepsilon}^{1} \stackrel{2}{\rightharpoonup} \nabla g_{*}^{1}+\nabla_{y} \psi_{*}
$$

where $\psi_{*} \in L^{2}\left(\Omega, \mathcal{A}\left(Y^{0}\right)\right)$ denotes the unique "corrector" characterised by

$$
\begin{equation*}
Q_{\mathrm{hom}}^{1}\left(\nabla g_{*}^{1}(x)\right)=\int_{Y^{1}} Q^{1}\left(\nabla g_{*}^{1}(x)+\nabla_{y} \psi_{*}(x, y)\right) d y, \quad \int_{Y} \psi_{*}(x, y) d y=0 \tag{28}
\end{equation*}
$$

for almost every $x \in \Omega$.
Next, we prove that almost minimisers $\varphi_{\varepsilon}$ satisfy the asymptotic relation

$$
\begin{equation*}
\varphi_{\varepsilon}=g_{*, \varepsilon}^{1}(x)+\varepsilon^{1-\gamma} g_{*, \varepsilon}^{0}(x)+o(1), \varepsilon \downarrow 0, \quad \text { in } W^{1, p}(\Omega) \quad(p<2), \tag{29}
\end{equation*}
$$

where $g_{*, \varepsilon}^{1}$ and $g_{*, \varepsilon}^{0}$ formally obey the "ansatz"

$$
\begin{equation*}
g_{*, \varepsilon}^{0}(x) \stackrel{\text { formally }}{=} g_{*}^{0}(x, x / \varepsilon), \quad g_{*, \varepsilon}^{1}(x) \stackrel{\text { formally }}{=} g_{*}^{1}(x)+\varepsilon \psi_{*}(x, x / \varepsilon) . \tag{30}
\end{equation*}
$$

Here $\left(g_{*}^{0}, g_{*}^{1}\right)$ and $\psi_{*}$ denote the minimising pair and corrector from Proposition 1. Since the functions on the right-hand sides in (30) are in general not smooth enough to define $g_{*, \varepsilon}^{0}$ and $g_{*, \varepsilon}^{1}$ by (30) directly, we use instead the approximation associated with $\left(g_{*}^{0}, g_{*}^{1}, \psi_{*}\right)$ via Lemma 3.

In addition to the properties of $Y^{0}$ assumed in Section 2, we require the following assumption on the regularity of $Y^{0}$ :

Assumption 1. There exist an exponent $p<2$ and a constant $C$ such that for all $\varphi \in H^{1}\left(Y^{1}\right)$ and $g^{1} \in H^{1}(Y)$ related via (10) we have $\left\|\nabla g^{1}\right\|_{L^{p}\left(Y^{0}\right)} \leq C\|\nabla \varphi\|_{L^{p}\left(Y^{1}\right)}$.

Note that Assumption 1 is satisfied if $Y^{0}$ can be written as the disjoint union of a finite number of Lipschitz domains $Y_{1}^{0}, \ldots Y_{N}^{0}$ with $\partial Y_{i}^{0} \cap \partial Y_{j}^{0}=\emptyset$ for $i \neq j$.

Theorem 2. Assume that (25a)-(25c) hold, and let Assumption 1 be satisfied. Let $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ be a sequence of almost minimisers, i.e.

$$
I_{\varepsilon}\left(\varphi_{\varepsilon}\right) \leq \inf _{\varphi \in H_{0}^{1}(\Omega)} I_{\varepsilon}(\varphi)+o(1), \quad \varepsilon \downarrow 0
$$

Let $\left(g_{*}^{0}, g_{*}^{1}\right)$ be the minimiser of $I_{\text {small }}$ and let $\psi_{*}$ be defined through (28). Let $\left(g_{*, \varepsilon}^{0}, g_{*, \varepsilon}^{1}\right)$ and $\varphi_{*, \varepsilon}=g_{*, \varepsilon}^{1}+\varepsilon^{1-\gamma} g_{*, \varepsilon}^{0}$ be associated with $\left(g_{*}^{0}, g_{*}^{1}, \psi_{*}\right)$ as in Lemma 3, i.e. $\varphi_{*, \varepsilon} \xrightarrow{2}\left(g_{*}^{0}, g_{*}^{1}\right)$ and $\nabla g_{*, \varepsilon}^{1} \xrightarrow{2} \nabla g_{*}^{1}+\nabla_{y} \psi_{*}$. Then for $g_{\varepsilon}^{1}+\varepsilon^{1-\gamma} g_{\varepsilon}^{0} \stackrel{(16)}{=} \varphi_{\varepsilon}$ one has

$$
\begin{equation*}
\left\|g_{\varepsilon}^{0}-g_{*, \varepsilon}^{0}\right\|_{L^{p}\left(\Omega_{\varepsilon}^{0}\right)}+\left\|\varepsilon \nabla g_{\varepsilon}^{0}-\varepsilon \nabla g_{*, \varepsilon}^{0}\right\|_{L^{p}\left(\Omega_{\varepsilon}^{0}\right)}+\left\|g_{\varepsilon}^{1}-g_{*, \varepsilon}^{1}\right\|_{W^{1, p}(\Omega)} \rightarrow 0 \quad \text { as } \varepsilon \downarrow 0 . \tag{31}
\end{equation*}
$$

Remark 1. To illustrate the result of Theorem 2, consider the case $\gamma=1$ with $\ell_{\varepsilon}:=m_{\varepsilon} \ell\left(x, \frac{x}{\varepsilon}\right)$, where $\ell(x, y)$ is smooth both in $x$ and $y$ and is periodic in $y$. If the domain $\Omega$ and the pore set $Y^{0}$ are sufficiently regular, the minimisers $\left(g_{*}^{0}, g_{*}^{1}\right)$ and $\psi_{*}$ are smooth, by the classical elliptic regularity theory, see e.g. [21]. In that case we may set

$$
g_{*, \varepsilon}^{0}(x):=g_{*}^{0}(x, x / \varepsilon) \text { and } g_{*, \varepsilon}^{1}:=g_{*}^{1}(x)+\varepsilon \psi_{*}(x, x / \varepsilon) \text {, }
$$

and the asymptotic formula for $\varphi_{\varepsilon}$ reads

$$
\varphi_{\varepsilon}=g_{*}^{0}(x, x / \varepsilon)+g_{*}^{1}(x)+\varepsilon \psi_{*}(x, x / \varepsilon)+R_{\varepsilon}(x),
$$

where $\left\|R_{\varepsilon}\right\|_{W^{1, p}(\Omega)} \rightarrow 0$ as $\varepsilon \downarrow 0$.
In the remainder of this section, we restrict to the special case of the introduction (in the small strain regime), i.e. we assume that $\alpha>1, \gamma=1, m_{\varepsilon}=\varepsilon^{\alpha}$ and $\ell_{\varepsilon}=\varepsilon^{\alpha} f$ for some $f \in L^{2}(\Omega)$, so that the functionals $I_{\varepsilon}^{\alpha}$ and $I_{\varepsilon}$ defined in (2) and (24) are identical. Hence, Theorem 1 and Proposition 1 prove two-scale $\Gamma$-convergence of $I_{\varepsilon}^{\alpha}$ and the convergence of the associated minimisation problems (as claimed in the introduction), and Theorem 2 yields the two-scale expansion (6). We argue that the functionals $I_{\varepsilon}^{\alpha} \Gamma$-converge to the (single scale) limit $\bar{I}_{\text {small }}$ :

Proposition 2. For $\alpha>1$ and $f \in L^{2}(\Omega)$ consider $I_{\varepsilon}^{\alpha}$ and $\bar{I}_{\text {small }}$ defined in (2) and (7). We extend $I_{\varepsilon}^{\alpha}$ to a functional on $L^{2}(\Omega)$ by setting $I_{\varepsilon}^{\alpha}:=+\infty$ on $L^{2}(\Omega) \backslash H_{0}^{1}(\Omega)$. Then:
(a) (Compactness). Suppose that $\varphi_{\varepsilon} \in L^{2}(\Omega)$ satisfy

$$
\limsup _{\varepsilon \downarrow 0} I_{\varepsilon}^{\alpha}\left(\varphi_{\varepsilon}\right)<\infty
$$

Then, up to a subsequence, we have $\varphi_{\varepsilon} \rightharpoonup \varphi$ weakly in $L^{2}(\Omega)$.
(b) (Lower bound). For every $\varphi_{\varepsilon} \in L^{2}(\Omega)$ with $\varphi_{\varepsilon} \rightharpoonup \varphi$ weakly in $L^{2}(\Omega)$ we have

$$
\liminf _{\varepsilon \downarrow 0} I_{\varepsilon}^{\alpha}\left(\varphi_{\varepsilon}\right) \geq \bar{I}_{\text {small }}(\varphi)
$$

(c) (Upper bound). For every $\varphi \in L^{2}(\Omega)$ we can find $\varphi_{\varepsilon} \rightharpoonup \varphi$ weakly in $L^{2}(\Omega)$ such that

$$
\lim _{\varepsilon \downarrow 0} I_{\varepsilon}^{\alpha}\left(\varphi_{\varepsilon}\right)=\bar{I}_{\text {small }}(\varphi)
$$

(d) (Convergence of the minimisation problem). Let $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ denote an infimizing sequence of $I_{\varepsilon}^{\alpha}$, i.e.

$$
I_{\varepsilon}^{\alpha}\left(\varphi_{\varepsilon}\right) \leq \inf _{\varphi \in L^{2}(\Omega)} I_{\varepsilon}^{\alpha}(\varphi)+o(1), \quad \varepsilon \downarrow 0
$$

Let $\left(g_{*}^{0}, g_{*}^{1}\right)$ denote the unique minimiser of $I_{\text {small }}$ and $\varphi_{*}$ denote the unique minimiser of $\bar{I}_{\text {small }}$. Then $\inf _{L^{2}(\Omega)} I_{\varepsilon}^{\alpha} \rightarrow \min _{L^{2}(\Omega)} \bar{I}, \varphi_{\varepsilon} \rightharpoonup \varphi_{*}$ weakly in $L^{2}(\Omega)$, and

$$
\varphi_{*}=g_{*}^{1}+\int_{Y_{0}} g^{0} d y, \quad \bar{I}_{\text {small }}\left(\varphi_{*}\right)=I_{\text {small }}\left(g_{*}^{0}, g_{*}^{1}\right) .
$$

### 3.3 Convergence in the finite strain regime $m_{\varepsilon}=\varepsilon^{\gamma}$

Throughout this section we assume that

- $W^{1}$ satisfies the conditions (W1)-(W3).
- $W^{0}: \mathbb{R}^{d \times d} \rightarrow[0, \infty)$ is continuous and satisfies the growth condition

$$
\begin{equation*}
c_{0} \operatorname{dist}^{2}(F, \mathrm{SO}(d)) \leq W^{0}(F) \leq c_{0}^{-1}\left(1+|F|^{2}\right) \quad \forall F \in \mathbb{R}^{d \times d} \tag{32a}
\end{equation*}
$$

and the local Lipschitz condition

$$
\begin{equation*}
\left|W^{0}(F+G)-W^{0}(F)\right| \leq c_{0}^{-1}(1+|F|+|G|)|G| \quad \forall F, G \in \mathbb{R}^{d \times d} \tag{32b}
\end{equation*}
$$

We prove that in the finite strain regime the limit functional

$$
\mathcal{E}_{\text {finite }}: L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right) \times H_{0}^{1}(\Omega) \rightarrow[0, \infty)
$$

is given by

$$
\mathcal{E}_{\text {finite }}\left(g^{0}, g^{1}\right):=\iint_{\Omega \times Y^{0}} \mathcal{Q} W^{0}\left(I+\nabla_{y} g^{0}(x, y)\right) d y d x+\int_{\Omega} Q_{\mathrm{hom}}^{1}\left(\nabla g^{1}(x)\right) d x
$$

where $\mathcal{Q} W^{0}$ denotes the quasiconvex envelope of $W^{0}$ (see e.g. [20]). The associated limit of the total energy $I^{\varepsilon}$, see (24), is given by (cf. (26))

$$
I_{\text {finite }}\left(g^{0}, g^{1}\right):=\mathcal{E}_{\text {finite }}\left(g^{0}, g^{1}\right)-\int_{\Omega}\left(\int_{Y^{0}} \ell^{0} \cdot g^{0} d y+\ell^{1} \cdot g^{1}\right) d x
$$

where $\ell^{0}, \ell^{1}$ are defined in the same way as in (25c).
Theorem 3. (a) (Lower bound). Consider a sequence $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ and the associated decomposition $\varepsilon^{1-\gamma} g_{\varepsilon}^{0}+g_{\varepsilon}^{1} \stackrel{(16)}{=} \varphi_{\varepsilon}$. If $\varphi_{\varepsilon} \stackrel{2}{\longrightarrow}\left(g^{0}, g^{1}\right)$ and $g_{\varepsilon}^{0} \xrightarrow{2} g^{0}$, then

$$
\liminf _{\varepsilon \downarrow 0} \varepsilon^{-2 \gamma} \mathcal{E}_{\varepsilon}\left(\varepsilon^{\gamma} \varphi_{\varepsilon}\right) \geq \mathcal{E}_{\text {finite }}\left(g^{0}, g^{1}\right)
$$

(b) (Recovery sequence). For any $g^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$ and $g^{1} \in H_{0}^{1}(\Omega)$ there exists a sequence $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ such that $\varphi_{\varepsilon} \stackrel{2}{\rightharpoonup}\left(g^{0}, g^{1}\right)$ and

$$
\lim _{\varepsilon \downarrow 0} \varepsilon^{-2 \gamma} \mathcal{E}_{\varepsilon}\left(\varepsilon^{\gamma} \varphi_{\varepsilon}\right)=\mathcal{E}_{\text {finite }}\left(g^{0}, g^{1}\right)
$$

(c) Suppose that the force densities $\ell_{\varepsilon} \in L^{2}(\Omega)$ satisfy (25a) and (25c) with $m_{\varepsilon}=\varepsilon^{\gamma}$. Then the infima converge, i.e.

$$
\begin{equation*}
\lim _{\varepsilon \downarrow 0} \inf _{\varphi \in H_{0}^{1}(\Omega)} I_{\varepsilon}(\varphi)=\inf I_{\text {finite }}\left(g^{0}, g^{1}\right) \tag{33}
\end{equation*}
$$

where the infimum on the right-hand side is taken over all functions $g^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$ and $g^{1} \in H_{0}^{1}(\Omega)$. Moreover, there exist a minimising pair $\left(g_{*}^{0}, g_{*}^{1}\right)$ and a recovery sequence $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ with $\varphi_{\varepsilon} \xrightarrow{2}\left(g_{*}^{0}, g_{*}^{1}\right)$ such that

$$
\begin{equation*}
I_{\varepsilon}\left(\varphi_{\varepsilon}\right) \rightarrow I_{\text {finite }}\left(g_{*}^{0}, g_{*}^{1}\right)=\min I_{\text {finite }}\left(g^{0}, g^{1}\right) \quad \text { as } \varepsilon \downarrow 0 \tag{34}
\end{equation*}
$$

Remark 2 (Example in Section 1). If we consider Theorem 3 and Proposition 1 in the case $\gamma=1, m_{\varepsilon}=\varepsilon$ and $\ell_{\varepsilon}=\varepsilon f$ for some $f \in L^{2}(\Omega)$, then we recover the special case (in the finite strain regime) presented in the introduction. In particular, we deduce that the functionals $I_{\varepsilon}^{\alpha}$ two-scale $\Gamma$-converge (in the sense of Theorem 3) to $I_{\text {finite }}$. Arguing as in the small-strain regime, cf. Proposition 2, we deduce that $I_{\varepsilon}^{\alpha} \Gamma$-converges (with respect to the weak topology in $\left.L^{2}(\Omega)\right)$ to $\bar{I}_{\text {finite }}$, cf. (8). We leave the details to the readers.

## 4 Proofs

We start by proving the auxiliary results discussed in Section 3.1. Sections 4.2 and 4.3 contain the proofs of the main statements in the small strain and finite strain cases, respectively.

### 4.1 Proofs of Lemma 1, Lemma 2, and Lemma 3: a priori estimate, compactness and approximation

A key ingredient in the proof of Lemma 1 is the geometric rigidity estimate by Friesecke et al. [23]:
Theorem 4 (Geometric rigidity estimate, see [23]). Let $U$ be an open, bounded Lipschitz domain in $\mathbb{R}^{d}, d \geq 2$. There exists a constant $C(U)$ with the following property: for each $v \in H^{1}(U)$ there is a rotation $R \in \mathrm{SO}(d)$ such that

$$
\int_{U}|\nabla v(x)-R|^{2} d x \leq C(U) \int_{U} \operatorname{dist}^{2}(\nabla v(x), \mathrm{SO}(d)) d x .
$$

Moreover, the constant $C(U)$ is invariant under uniform scaling of $U$.
In fact, we need the following modified version, which is adapted to perforated domains.
Lemma 4. There exists a constant $C>0$ that only depends on $\Omega$ and $Y^{1}$ such that for all $\varepsilon>0$ and $v \in H^{1}(\Omega)$ satisfying

$$
\begin{equation*}
\int_{\Omega_{\varepsilon}^{0}} \nabla v: \nabla \zeta d x=0 \quad \forall \zeta \in H_{0}^{1}\left(\Omega_{\varepsilon}^{0}\right) \tag{35}
\end{equation*}
$$

the estimates

$$
\begin{align*}
\|\operatorname{dist}(\nabla v, \mathrm{SO}(d))\|_{L^{2}(\Omega)} & \leq C\|\operatorname{dist}(\nabla v, \mathrm{SO}(d))\|_{L^{2}\left(\Omega_{\varepsilon}^{1}\right)}  \tag{36}\\
\|\nabla v-R\|_{L^{2}\left(\Omega_{\varepsilon}^{1}\right)} & \leq C\|\operatorname{dist}(\nabla v, \mathrm{SO}(d))\|_{L^{2}\left(\Omega_{\varepsilon}^{1}\right)} \tag{37}
\end{align*}
$$

hold for some $R \in \mathrm{SO}(d)$, which may depend on $v$. In addition, if $v(x)=x+c$ on $\partial \Omega$ for some constant $c$, then we may set $R=I$.

Proof of Lemma 4. Step 1. The proof of the inequality (36).
Let $\widehat{\Omega}_{\varepsilon}:=\bigcup\left\{\varepsilon(\xi+Y) \mid \xi \in \mathbb{Z}^{d}, \varepsilon(\xi+Y) \subset \Omega\right\}$ denote the union of $\varepsilon$-cells that are completely contained in $\Omega$. Since $\Omega \backslash \widehat{\Omega}_{\varepsilon} \subset \Omega_{\varepsilon}^{1}$, it suffices to prove (36) for $\Omega$ replaced by $\widehat{\Omega}_{\varepsilon}$, respectively. In fact we shall prove the following stronger estimate: for all $\xi \in \mathbb{Z}^{d}$ with $\varepsilon(\xi+Y) \subset \Omega$ we have

$$
\begin{equation*}
\int_{\varepsilon(\xi+Y)} \operatorname{dist}^{2}(\nabla v, \mathrm{SO}(d)) d x \lesssim \int_{\varepsilon\left(\xi+Y^{1}\right)} \operatorname{dist}^{2}(\nabla v, \mathrm{SO}(d)) d x \tag{38}
\end{equation*}
$$

For the argument fix an admissible $\xi \in \mathbb{Z}^{d}$. Application of Theorem 4 with $U=\varepsilon\left(\xi+Y^{1}\right)$ yields a rotation $R \in \mathrm{SO}(d)$ such that

$$
\begin{equation*}
\int_{\varepsilon\left(\xi+Y^{1}\right)}|\nabla v-R|^{2} d x \lesssim \int_{\varepsilon\left(\xi+Y^{1}\right)} \operatorname{dist}^{2}(\nabla v, \mathrm{SO}(d)) d x \tag{39}
\end{equation*}
$$

Note that the multiplicative constant in the estimate above only depends on $Y^{1}$, since $\varepsilon\left(\xi+Y^{1}\right)$ is a dilation and translation of $Y^{1}$. On the other hand, since $\varepsilon\left(\xi+Y^{0}\right) \ni x \mapsto(v(x)-R x)$ is harmonic, we have (cf. (11)):

$$
\int_{\varepsilon(\xi+Y)} \operatorname{dist}^{2}(\nabla v, \mathrm{SO}(d)) d x \leq \int_{\varepsilon(\xi+Y)}|\nabla v-R|^{2} \lesssim \int_{\varepsilon\left(\xi+Y^{1}\right)}|\nabla v-R|^{2}
$$

Combined with (39), inequality (38) follows.
Step 2 . The proof of the rigidity estimate (37).
From (36) and Theorem 4 (applied with $U=\Omega$ ) we deduce that for some $R \in \mathrm{SO}(d)$ :

$$
\begin{equation*}
\|\nabla v-R\|_{L^{2}(\Omega)} \lesssim\|\operatorname{dist}(\nabla v, \mathrm{SO}(d))\|_{L^{2}\left(\Omega_{\varepsilon}^{1}\right)} \tag{40}
\end{equation*}
$$

which in particular implies (37). Finally we argue that one can set $R=I$, if $v=x+c$ on $\partial \Omega$. In view of (40), it suffices to show that $\int_{\Omega}|\nabla v-I|^{2} d x \leq \int_{\Omega}|\nabla v-R|^{2} d x$ for all $R \in \mathrm{SO}(d)$. This inequality can be seen as follows: Consider $\varphi(x):=v(x)-x-c$ and note that $\varphi$ vanishes on $\partial \Omega$, so that
$\int_{\Omega}|\nabla v-I|^{2} d x=\int_{\Omega}|\nabla \varphi|^{2} d x \leq \int_{\Omega}|\nabla \varphi|^{2}+|I-R|^{2} d x=\int_{\Omega}|\nabla \varphi+(I-R)|^{2} d x=\int_{\Omega}|\nabla v-R|^{2} d x$,
which in fact holds for an arbitrary matrix $R$.
We are now in position to present the proofs of Lemmas 1 and 2.
Proof of Lemma 1. In the following, the symbol $\lesssim$ stands for $\leq$ up to a multiplicative constant that only depends on $Y^{1}$ and $\Omega$.
Step 1. Existence of the decomposition (16) and derivation of the estimate for $g_{\varepsilon}^{1}$.
Let $g_{\varepsilon}^{1}$ denote the unique function in $H^{1}(\Omega)$ characterised by $g_{\varepsilon}^{1}=\varphi_{\varepsilon}$ in $\Omega_{\varepsilon}^{1}$ and (16) (ii). Since $\partial \Omega_{\varepsilon}^{0}$ is Lipschitz, we deduce that $g_{\varepsilon}^{0}:=\varepsilon^{\gamma-1}\left(\varphi_{\varepsilon}-g_{\varepsilon}^{1}\right) \in H_{0}^{1}\left(\Omega_{\varepsilon}^{0}\right)$. This proves the existence of the decomposition. We claim that

$$
\begin{equation*}
\int_{\Omega_{\varepsilon}^{0}}\left|\nabla g_{\varepsilon}^{1}\right|^{2} d x \lesssim \int_{\Omega_{\varepsilon}^{1}}\left|\nabla \varphi_{\varepsilon}\right|^{2} d x=\int_{\Omega_{\varepsilon}^{1}}\left|\nabla g_{\varepsilon}^{1}\right|^{2} d x . \tag{41}
\end{equation*}
$$

Since $\Omega_{\varepsilon}^{0}$ is defined as the union of the sets $\varepsilon\left(\xi+Y^{0}\right)$ with $\xi \in Z_{\varepsilon}:=\left\{\xi \in \mathbb{Z}^{d}: \varepsilon(\xi+Y) \subset \Omega\right\}$, it suffices to prove $\int_{\varepsilon\left(\xi+Y^{0}\right)}\left|\nabla g_{\varepsilon}^{1}\right|^{2} d x \lesssim \int_{\varepsilon\left(\xi+Y^{1}\right)}\left|\nabla \varphi_{\varepsilon}\right|^{2} d x$. The latter follows from (11) by a
scaling argument, since the rescaled functions $y \mapsto \varphi_{\varepsilon}(\varepsilon(\xi+y))$ and $y \mapsto g_{\varepsilon}^{1}(\varepsilon(\xi+y))$ satisfy (10).

Next, we prove (17). Consider $v_{\varepsilon}(x):=x+g_{\varepsilon}^{1}(x)$ and note that $v_{\varepsilon}(x)$ satisfies (35). Hence, (41) and Lemma 4 yield

$$
\begin{equation*}
\int_{\Omega}\left|\nabla g_{\varepsilon}^{1}\right|^{2} d x \lesssim \int_{\Omega_{\varepsilon}^{1}}\left|\nabla g_{\varepsilon}^{1}\right|^{2} d x \leq \int_{\Omega}\left|\nabla v_{\varepsilon}-I\right|^{2} \stackrel{(37)}{\leq} C \int_{\Omega_{\varepsilon}^{1}} \operatorname{dist}^{2}\left(I+\nabla \varphi_{\varepsilon}(x), \mathrm{SO}(d)\right) d x \leq \Phi_{\varepsilon}^{\gamma}\left(\varphi_{\varepsilon}\right) . \tag{42}
\end{equation*}
$$

Since $g_{\varepsilon}^{1}$ vanishes on the boundary of $\Omega$, the estimate upgrades (by Poincaré's inequality) to $\left\|g_{\varepsilon}^{1}\right\|_{H^{1}(\Omega)} \leq C \Phi_{\varepsilon}^{\gamma}\left(\varphi_{\varepsilon}\right)$.
Step 2. Derivation of the estimate for $g_{\varepsilon}^{0}$.
Since we have an improved Poincaré inequality (see e.g. [24, Lemma 1.6]):

$$
\begin{equation*}
\forall g \in H_{0}^{1}\left(\Omega_{\varepsilon}^{0}\right): \quad\|g\|_{L^{2}\left(\Omega_{\varepsilon}^{0}\right)} \lesssim \varepsilon\|\nabla g\|_{L^{2}\left(\Omega_{\varepsilon}^{0}\right)} \tag{43}
\end{equation*}
$$

it suffices to prove

$$
\left\|\varepsilon \nabla g_{\varepsilon}^{0}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{0}\right)}^{2} \lesssim \Phi_{\varepsilon}^{\gamma}\left(\varphi_{\varepsilon}\right)
$$

To this end, notice that since $\varphi_{\varepsilon}$ vanishes on the boundary of $\Omega$, we have

$$
\begin{align*}
\left\|\nabla \varphi_{\varepsilon}\right\|_{L^{2}(\Omega)}^{2} & =\min _{R \in \operatorname{SO}(d)}\left\|I+\nabla \varphi_{\varepsilon}-R\right\|_{L^{2}(\Omega)}^{2} \stackrel{\text { Theorem } 4}{\lesssim} \int_{\Omega} \operatorname{dist}^{2}\left(I+\nabla \varphi_{\varepsilon}(x), \mathrm{SO}(d)\right) d x  \tag{44}\\
& \leq \varepsilon^{-2 \gamma} \Phi_{\varepsilon}^{\gamma}\left(\varphi_{\varepsilon}\right) .
\end{align*}
$$

Thanks to the first identity in (16), we get by triangle inequality:

$$
\left\|\varepsilon^{1-\gamma} \nabla g_{\varepsilon}^{0}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{0}\right)} \leq\left\|\nabla \varphi_{\varepsilon}\right\|_{L^{2}(\Omega)}+\left\|\nabla g_{\varepsilon}^{1}\right\|_{L^{2}(\Omega)}
$$

Combined with (42) and (44) we finally get

$$
\left\|\varepsilon \nabla g_{\varepsilon}^{0}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{0}\right)}^{2}=\varepsilon^{2 \gamma}\left\|\varepsilon^{1-\gamma} \nabla g_{\varepsilon}^{0}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{0}\right)}^{2} \lesssim \Phi_{\varepsilon}^{\gamma}\left(\varphi_{\varepsilon}\right) .
$$

Proof of Lemma 2. Step 1. A priori estimate and basic compactness.
From Lemma 1 we deduce that

$$
\begin{equation*}
\underset{\varepsilon \downarrow 0}{\limsup }\left(\left\|g_{\varepsilon}^{0}\right\|_{L^{2}(\Omega)}^{2}+\left\|\varepsilon \nabla g_{\varepsilon}^{0}\right\|_{L^{2}(\Omega)}^{2}+\left\|g_{\varepsilon}^{1}\right\|_{H^{1}(\Omega)}^{2}\right) \leq C \underset{\varepsilon \downarrow 0}{\lim \sup ^{2}} m_{\varepsilon}^{-2} \Phi_{\varepsilon}^{\gamma}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)<\infty . \tag{45}
\end{equation*}
$$

Hence, by standard results concerning two-scale convergence (cf. [1, Proposition 1.14] and [36, Proposition 4.2]), there exist $g^{1} \in H_{0}^{1}(\Omega), \psi \in L^{2}\left(\Omega, H_{\#}^{1}\right)$ and $g^{0} \in L^{2}\left(\Omega, H_{\#}^{1}\right)$ such that, up to a subsequence, one has

$$
\begin{aligned}
& g_{\varepsilon}^{1} \rightharpoonup g^{1} \text { weakly in } H^{1}(\Omega), \quad \nabla g_{\varepsilon}^{1} \stackrel{2}{\rightharpoonup} \nabla g^{1}+\nabla_{y} \psi, \\
& g_{\varepsilon}^{0} \stackrel{2}{\rightharpoonup} g^{0}, \quad \varepsilon \nabla g_{\varepsilon}^{0} \stackrel{2}{\rightharpoonup} \nabla_{y} g^{0} .
\end{aligned}
$$

Step 2. The proof of the inclusion $\psi \in L^{2}\left(\Omega, \mathcal{A}\left(Y^{0}\right)\right)$.
By a density argument, it suffices to show that

$$
\begin{equation*}
\iint_{\Omega \times Y^{0}} \nabla_{y} \psi(x, y): \nabla_{y}\left(\zeta_{1}(x) \zeta_{2}(y)\right) d x d y=0 \tag{46}
\end{equation*}
$$

for all scalar functions $\zeta_{1} \in C_{\mathrm{c}}^{\infty}(\Omega)$, and all $\zeta_{2} \in C_{\mathrm{c}}^{\infty}\left(Y^{0}\right)$. To this end, we identify $\zeta_{2}$ with its unique $Y$-periodic extension to $\mathbb{R}^{d}$ that vanishes on $Y^{1}$, and set

$$
\zeta_{\varepsilon}(x):=\varepsilon \zeta_{1}(x) \zeta_{2}(x / \varepsilon), \quad x \in \Omega .
$$

Thanks to (16) we have

$$
\int_{\Omega} \nabla g_{\varepsilon}^{1}: \nabla \zeta_{\varepsilon} d x=0
$$

As can be easily checked, we have $\nabla \zeta_{\varepsilon} \xrightarrow{2} \zeta_{1}(x) \nabla_{y} \zeta_{2}(y)$, so that

$$
\begin{aligned}
0 & =\lim _{\varepsilon \downarrow 0} \int_{\Omega} \nabla g_{\varepsilon}^{1}: \nabla \zeta_{\varepsilon} d x \\
& =\iint_{\Omega \times Y^{0}}\left(\nabla g^{1}(x)+\nabla_{y} \psi(x, y)\right):\left(\zeta_{1}(x) \nabla_{y} \zeta_{2}(y)\right) d x d y \\
& =\iint_{\Omega \times Y^{0}} \nabla_{y} \psi(x, y):\left(\zeta_{1}(x) \nabla_{y} \zeta_{2}(y)\right) d x d y,
\end{aligned}
$$

where the last identity holds thanks to the periodicity of $\zeta_{2}$. This proves (46).
Step 3. The proof of the inclusion $g^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$.
By a density argument, it suffices to show that

$$
\begin{equation*}
\iint_{\Omega \times Y} g^{0}(x, y) \cdot\left(\zeta_{1}(x) \zeta_{2}(y)\right) d x d y=0 \tag{47}
\end{equation*}
$$

for all scalar functions $\zeta_{1} \in C_{\mathrm{c}}^{\infty}(\Omega)$ and all $\zeta_{2} \in H_{\#}^{1}$ with $\zeta_{2}=0$ on $Y^{0}$. We argue by considering the function $\zeta_{\varepsilon}(x):=\zeta_{1}(x) \zeta_{2}\left(\frac{x}{\varepsilon}\right), x \in \Omega$, the support of which is contained in $\Omega_{\varepsilon}^{1}$ for $\varepsilon \ll 1$. Since $\zeta_{\varepsilon} \xrightarrow{2} \zeta_{1}(x) \zeta(y)$, and since $g_{\varepsilon}^{0}$ is supported in $\Omega_{\varepsilon}^{0}$, we deduce that

$$
0=\lim _{\varepsilon \downarrow 0} \int_{\Omega} g_{\varepsilon}^{0}(x) \cdot \zeta_{\varepsilon}(x) d x=\iint_{\Omega \times Y} g^{0}(x, y) \cdot\left(\zeta_{1}(x) \zeta_{2}(y)\right) d x d y
$$

This completes the argument.
In the proof of Lemma 3 we appeal to the construction of a diagonal sequence that is due to Attouch, see [2]:

Lemma 5. For any $h:[0, \infty)^{2} \rightarrow[0,+\infty]$, there exists a mapping $(0,1) \ni \varepsilon \mapsto \delta(\varepsilon) \in(0,1)$ such that

$$
\lim _{\varepsilon \downarrow 0} \delta(\varepsilon)=0 \quad \text { and } \quad \limsup _{\varepsilon \downarrow 0} h(\varepsilon, \delta(\varepsilon)) \leq \underset{\delta \downarrow 0}{\limsup } \limsup _{\varepsilon \downarrow 0} h(\varepsilon, \delta) \text {. }
$$

Proof of Lemma 3. Step 1. Characterisation of strong two-scale convergence via unfolding.
For $f_{\varepsilon}: \Omega \rightarrow \mathbb{R}$ and $f: \Omega \times Y \rightarrow \mathbb{R}$ define

$$
d_{\varepsilon}\left(f_{\varepsilon}, f\right):=\int_{\mathbb{R}^{d}} \int_{Y}\left|\tilde{f}_{\varepsilon}(\varepsilon\lfloor x / \varepsilon\rfloor+\varepsilon y)-\tilde{f}(x, y)\right|^{2} d y d x
$$

where $\tilde{f}_{\varepsilon}$ denotes the extension by zero of $f_{\varepsilon}$ to $\mathbb{R}^{d}, \tilde{f}$ denotes the extension by zero of $f$ to $\mathbb{R}^{d} \times Y$, and $\lfloor z\rfloor$ denotes the unique element in $\mathbb{Z}^{d}$ with $z-\lfloor z\rfloor \in[0,1)^{d}$. We recall from [36] that

$$
\begin{equation*}
f_{\varepsilon} \xrightarrow{2} f \quad \Leftrightarrow \quad d_{\varepsilon}\left(f_{\varepsilon}, f\right) \rightarrow 0 \tag{48}
\end{equation*}
$$

The characterisation extends in the obvious way to vector-valued functions.
Step 2. Construction of $g_{\varepsilon}^{1}$.
We claim that there exists a sequence $g_{\varepsilon}^{1}$ in $H_{0}^{1}(\Omega)$ whose elements satisfy (16)(ii) and

$$
\begin{equation*}
g_{\varepsilon}^{1} \rightharpoonup g^{1} \text { weakly in } H^{1}(\Omega), \quad \nabla g_{\varepsilon}^{1} \xrightarrow{2} g^{1}+\nabla_{y} \psi, \quad \limsup _{\varepsilon \downarrow 0} c_{\varepsilon}\left\|\nabla g_{\varepsilon}^{1}\right\|_{L^{\infty}(\Omega)}=0 \tag{49}
\end{equation*}
$$

Indeed, by a density argument there exist $g^{1, \delta} \in C_{\mathrm{c}}^{\infty}(\Omega), \delta \in(0,1)$, and $\psi \in C_{\mathrm{c}}^{\infty}\left(\Omega, C_{\#}^{\infty}(Y)\right)$ such that

$$
\left\|g^{1, \delta}-g^{1}\right\|_{H^{1}(\Omega)}+\left\|\nabla_{y} \psi^{\delta}-\nabla_{y} \psi\right\|_{L^{2}(\Omega \times Y)} \leq \delta \quad \forall \delta
$$

For $\varepsilon>0, \delta \in(0,1)$, define

$$
g_{\varepsilon}^{1, \delta}(x):=g^{1, \delta}(x)+\varepsilon \psi^{\delta}(x, x / \varepsilon)
$$

and set

$$
d_{\varepsilon}^{\delta}:=d_{\varepsilon}\left(\nabla g_{\varepsilon}^{1, \delta}, \nabla g^{1}+\nabla_{y} \psi\right)+\left\|g_{\varepsilon}^{1, \delta}-g^{1}\right\|_{L^{2}(\Omega)}+c_{\varepsilon}\left\|\nabla g_{\varepsilon}^{1, \delta}\right\|_{L^{\infty}(\Omega)}
$$

By construction, we have $\lim _{\delta \downarrow 0} \lim _{\sup }^{\varepsilon \downarrow 0} d_{\varepsilon}^{\delta}=0$, and Lemma 5 yields a function $\varepsilon \mapsto \delta(\varepsilon)$ with $\lim _{\varepsilon \downarrow 0} d_{\varepsilon}^{\delta(\varepsilon)}=0$. In view of Step 1, this implies that the diagonal sequence $\tilde{g}_{\varepsilon}^{1}:=g_{\varepsilon}^{1, \delta(\varepsilon)}$ satisfies (49). Now, for each $\varepsilon>0$, let $g_{\varepsilon}^{1}$ denote the function satisfying (16)(ii) and such that $g_{\varepsilon}^{1}=\tilde{g}_{\varepsilon}^{1}$ on $\Omega_{\varepsilon}^{1}$. To conclude the argument, we only need to show that $g_{\varepsilon}^{1}$ satisfies (49). Consider the difference $\eta_{\varepsilon}:=\tilde{g}_{\varepsilon}^{1}-g_{\varepsilon}^{1}$. Since $\eta_{\varepsilon}$ is bounded in $H^{1}(\Omega)$ and $\eta_{\varepsilon}=0$ in $\Omega_{\varepsilon}^{1}$, we have $\eta_{\varepsilon} \rightharpoonup 0$ in $H^{1}(\Omega)$, and, up to a subsequence, $\nabla \eta_{\varepsilon} \stackrel{2}{\rightharpoonup} \nabla_{y} \varphi$ for some $\varphi \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$. On the other hand, since $g_{\varepsilon}^{1}$ satisfies (16) (ii) and $\tilde{g}_{\varepsilon}^{1}$ satisfies (49), we deduce that

$$
\left\|\nabla \eta_{\varepsilon}\right\|_{L^{2}(\Omega)}^{2}=\int_{\Omega_{\varepsilon}^{0}} \nabla \eta_{\varepsilon}: \nabla \eta_{\varepsilon} d x=\int_{\Omega_{\varepsilon}^{0}} \nabla \tilde{g}_{\varepsilon}^{1}: \nabla \eta_{\varepsilon} d x \rightarrow \iint_{\Omega \times Y^{0}}\left(\nabla g^{1}+\nabla_{y} \psi\right): \nabla_{y} \varphi d x d y
$$

Since $\nabla g^{1}$ is independent of $y$, and because $\psi \in L^{2}\left(\Omega, \mathcal{A}\left(Y^{0}\right)\right)$, the integral on the right-hand side vanishes. Hence, $\left\|\nabla \eta_{\varepsilon}\right\|_{L^{2}(\Omega)}^{2} \rightarrow 0$, and thus $g_{\varepsilon}^{1}$ satisfies (49).
Step 3. Conclusion.
As can be shown by appealing to a combination of a density argument and a diagonalsequence argument, similar to Step 1 , there exists a sequence $g_{\varepsilon}^{0} \in H_{0}^{1}\left(\Omega_{\varepsilon}^{0}\right)$ such that

$$
g_{\varepsilon}^{0} \xrightarrow{2} g^{0}, \quad \varepsilon \nabla g_{\varepsilon}^{0} \xrightarrow{2} \nabla_{y} g^{0}, \quad \limsup _{\varepsilon \downarrow 0} c_{\varepsilon}\left\|\varepsilon^{1-\gamma} \nabla g_{\varepsilon}^{0}\right\|_{L^{\infty}(\Omega)}=0 .
$$

Now define $\varphi_{\varepsilon}(x):=\varepsilon^{1-\gamma} g_{\varepsilon}^{0}+g_{\varepsilon}^{1}$, and note that $\left(g_{\varepsilon}^{0}, g_{\varepsilon}^{1}\right)$ satisfy (16). In view of the convergence of $g_{\varepsilon}^{0}$ and $g_{\varepsilon}^{1}$, the sequence $\varphi_{\varepsilon}$ has the required properties.

### 4.2 Proof of Theorems 1, 2 and Propositions 1, 2: small strain regime

As a preliminary remark, we note that two different effects play a role when passing to the limit $\varepsilon \downarrow 0$ in the small strain regime:

- The non-convex energy functional is linearised at identity map (which is a stress-free state for $\left.\mathcal{E}_{\varepsilon}\right)$ - this corresponds to the passage from nonlinear to linearised elasticity.
- The obtained linearised, still oscillating, convex-quadratic energy is homogenised.

The following lemma is used to treat both effects simultaneously. Its proof combines convex homogenisation methods (e.g. [37, Proposition 1.3]) with a "careful Taylor expansion" in the spirit of [23, Proof of Theorem 6.2]. For notational convenience, we introduce two "linearised" functionals:

- For $G_{\varepsilon}=\left(G_{\varepsilon}^{0}, G_{\varepsilon}^{1}\right) \in L^{2}\left(\Omega, \mathbb{R}^{d \times d}\right) \times L^{2}\left(\Omega, \mathbb{R}^{d \times d}\right)$ set

$$
\mathcal{Q}_{\varepsilon}\left(G_{\varepsilon}\right)=\mathcal{Q}_{\varepsilon}\left(G_{\varepsilon}^{0}, G_{\varepsilon}^{1}\right):=\int_{\Omega_{\varepsilon}^{0}} Q^{0}\left(G_{\varepsilon}^{0}(x)\right) d x+\int_{\Omega_{\varepsilon}^{1}} Q^{1}\left(G_{\varepsilon}^{1}(x)\right) d x
$$

- For $G=\left(G^{0}, G^{1}\right) \in L^{2}\left(\Omega \times Y, \mathbb{R}^{d \times d}\right) \times L^{2}\left(\Omega \times Y, \mathbb{R}^{d \times d}\right)$ set

$$
\mathcal{Q}(G)=\mathcal{Q}\left(G^{0}, G^{1}\right):=\int_{\Omega \times Y^{0}} Q^{0}\left(G^{0}(x, y)\right) d x d y+\int_{\Omega \times Y^{1}} Q^{1}\left(G^{1}(x, y)\right) d x d y
$$

Lemma 6. Consider sequences $g_{\varepsilon}^{0}, g_{\varepsilon}^{1} \in H^{1}(\Omega)$ that satisfy

$$
\limsup _{\varepsilon \downarrow 0}\left(\left\|\varepsilon \nabla g_{\varepsilon}^{0}\right\|_{L^{2}(\Omega)}+\left\|\nabla g_{\varepsilon}^{1}\right\|_{L^{2}(\Omega)}\right)<\infty
$$

Set $\varphi_{\varepsilon}:=\varepsilon^{1-\gamma} g_{\varepsilon}^{0}+g_{\varepsilon}^{1}$ and $G_{\varepsilon}=\left(G_{\varepsilon}^{0}, G_{\varepsilon}^{1}\right):=\left(\varepsilon \nabla g_{\varepsilon}^{0}, \nabla g_{\varepsilon}^{1}\right)$.
(a) If $G_{\varepsilon} \stackrel{2}{\rightharpoonup} G$ and $m_{\varepsilon}=o\left(\varepsilon^{\gamma}\right)$ as $\varepsilon \downarrow 0$, then

$$
\begin{equation*}
\liminf _{\varepsilon \downarrow 0} m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right) \geq \liminf _{\varepsilon \downarrow 0} \mathcal{Q}_{\varepsilon}\left(\theta_{\varepsilon} G_{\varepsilon}\right) \geq \mathcal{Q}(G) \tag{50}
\end{equation*}
$$

where $\theta_{\varepsilon}: \Omega \rightarrow\{0,1\}$ is defined by

$$
\theta_{\varepsilon}(x):= \begin{cases}1, & \text { if }\left|\nabla \varphi_{\varepsilon}\right| \leq\left(m_{\varepsilon} \varepsilon^{\gamma}\right)^{-1 / 2} \\ 0, & \text { otherwise }\end{cases}
$$

(b) If $G_{\varepsilon} \xrightarrow{2} G, m_{\varepsilon}=o\left(\varepsilon^{\gamma}\right)$ as $\varepsilon \downarrow 0$, and

$$
\begin{equation*}
\limsup _{\varepsilon \downarrow 0}\left\|m_{\varepsilon} \nabla \varphi_{\varepsilon}\right\|_{L^{\infty}(\Omega)}=0 \tag{51}
\end{equation*}
$$

then

$$
\lim _{\varepsilon \downarrow 0} m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)=\mathcal{Q}(G)
$$

Remark 3. In Lemma 6, following [23], the function $\theta_{\varepsilon}$ is introduced, in order to truncate the peaks of $G_{\varepsilon}$. This is needed for exploiting the quadratic expansion (W3). Since both $\varepsilon \nabla g_{\varepsilon}^{0}$ and $\nabla g_{\varepsilon}^{1}$ are assumed to be bounded sequences in $L^{2}(\Omega)$, we deduce, from the definition of $\theta_{\varepsilon}$, the fact that $m_{\varepsilon} \varepsilon^{-\gamma}=o(1)$ and the Chebyshev inequality, that

$$
\begin{equation*}
\forall r<\infty \quad\left\|\theta_{\varepsilon}-1\right\|_{L^{r}(\Omega)} \rightarrow 0, \quad \text { and } \quad\left\|\theta_{\varepsilon}\right\|_{L^{\infty}(\Omega)} \leq 1 \tag{52}
\end{equation*}
$$

In the proof of Lemma 6 we need to pass to the limit in products of the form $f_{\varepsilon} \theta_{\varepsilon}$, where $\theta_{\varepsilon}$ satisfies (52), or $f_{\varepsilon} \chi_{\varepsilon}$, where $\chi_{\varepsilon}$ denotes the indicator of $\Omega_{\varepsilon}^{0}$. This is done by appealing to the next two lemmas, the proofs of which are elementary and left to the reader.

Lemma 7. Let $f_{\varepsilon}, \theta_{\varepsilon}$ be sequences in $L^{2}(\Omega)$ and assume that $\theta_{\varepsilon}$ satisfies (52), then the following implications are valid:

$$
\begin{aligned}
& \underset{\varepsilon \downarrow 0}{\limsup }\left\|f_{\varepsilon}\right\|_{L^{2}(\Omega)}<\infty \quad \Rightarrow \quad\left\|\theta_{\varepsilon} f_{\varepsilon}-f_{\varepsilon}\right\|_{L^{p}(\Omega)} \rightarrow 0 \quad \forall p<2, \\
& f_{\varepsilon} \rightharpoonup f \quad \Rightarrow \quad \theta_{\varepsilon} f_{\varepsilon} \rightharpoonup f, \\
& f_{\varepsilon} \stackrel{2}{\rightharpoonup} f \quad \Rightarrow \quad \theta_{\varepsilon} f_{\varepsilon} \stackrel{2}{\longrightarrow} f, \\
& \left\{\begin{array}{l}
f_{\varepsilon} \rightharpoonup f \\
\left(\left|f_{\varepsilon}\right|^{2}\right) \text { equi-integrable }
\end{array}\right\} \quad \Rightarrow \quad\left\|\theta_{\varepsilon} f_{\varepsilon}-f_{\varepsilon}\right\|_{L^{2}(\Omega)} \rightarrow 0 .
\end{aligned}
$$

Lemma 8. Suppose that $f_{\varepsilon}$ be a sequence in $L^{2}(\Omega)$ and, as above, let $\chi_{\varepsilon}$ denote the set indicator function of $\Omega_{\varepsilon}^{0}$. Then the following implications hold:

$$
\begin{array}{rll}
f_{\varepsilon} \stackrel{2}{\rightarrow} f & \Rightarrow & \chi_{\varepsilon} f_{\varepsilon} \stackrel{2}{\longrightarrow} \chi(y) f(x, y), \\
f_{\varepsilon} \xrightarrow{2} f & \Rightarrow & \chi_{\varepsilon} f_{\varepsilon} \xrightarrow{2} \chi(y) f(x, y),
\end{array}
$$

where $\chi$ denotes the indicator function of $Y^{0}$.
Proof of Lemma 6. Step 1. Linearisation.
We claim that the following statement holds for $i=1,2$ : Let $F_{\varepsilon}$ denote a sequence in $L^{2}\left(\Omega, \mathbb{R}^{d \times d}\right)$, and let $c_{\varepsilon}$ be a sequence of positive numbers converging to zero, such that

$$
\begin{equation*}
\limsup _{\varepsilon \rightarrow 0} c_{\varepsilon}\left\|F_{\varepsilon}\right\|_{L^{\infty}(\Omega)}=0 \tag{53}
\end{equation*}
$$

Then the convergence

$$
\begin{equation*}
\lim _{\varepsilon \downarrow 0}\left|c_{\varepsilon}^{-2} \int_{\Omega} W^{i}\left(I+c_{\varepsilon} F_{\varepsilon}\right) d x-\int_{\Omega} Q^{i}\left(F_{\varepsilon}(x)\right) d x\right|=0 \tag{54}
\end{equation*}
$$

holds. Indeed, thanks to (W3) we have

$$
\left|c_{\varepsilon}^{-2} W^{i}\left(I+c_{\varepsilon} F_{\varepsilon}\right)-Q^{i}\left(F_{\varepsilon}\right)\right| \leq\left|F_{\varepsilon}\right|^{2} r^{i}\left(c_{\varepsilon}\left|F_{\varepsilon}\right|\right) \leq\left|F_{\varepsilon}\right|^{2} r^{i}\left(c_{\varepsilon}\left\|F_{\varepsilon}\right\|_{L^{\infty}(\Omega)}\right) \quad \text { a.e. }
$$

Thanks to (53), and since $F_{\varepsilon}$ is bounded in $L^{2}(\Omega)$, the right-hand side converges to zero in $L^{1}(\Omega)$, and (54) follows.
Step 2. Proof of part (a).
Since the energy densities $W^{0}, W^{1}$ are minimised at the identity, $c f$. (W2), we have

$$
\begin{equation*}
m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(\varphi_{\varepsilon}\right) \geq\left(m_{\varepsilon} \varepsilon^{-\gamma}\right)^{-2} \int_{\Omega} W^{0}\left(I+m_{\varepsilon} \varepsilon^{-\gamma} \theta_{\varepsilon} F_{\varepsilon}^{0}(x)\right) d x+m_{\varepsilon}^{-2} \int_{\Omega} W^{1}\left(I+m_{\varepsilon} \theta_{\varepsilon} F_{\varepsilon}^{1}(x)\right) d x \tag{55}
\end{equation*}
$$

where

$$
\begin{equation*}
F_{\varepsilon}^{0}:=\chi_{\varepsilon}\left(G_{\varepsilon}^{0}+\varepsilon^{\gamma} G_{\varepsilon}^{1}\right), \quad F_{\varepsilon}^{1}(x):=\left(1-\chi_{\varepsilon}\right) G_{\varepsilon}^{1} . \tag{56}
\end{equation*}
$$

Thanks to the definition of $\theta_{\varepsilon}$ we have $\left\|m_{\varepsilon} \varepsilon^{-\gamma} \theta_{\varepsilon} F_{\varepsilon}^{0}\right\|_{L^{\infty}(\Omega)}+\left\|m_{\varepsilon} \theta_{\varepsilon} F_{\varepsilon}^{1}\right\|_{L^{\infty}(\Omega)} \rightarrow 0$, so that we may apply (54) to the right-hand side in (55). We get

$$
\liminf _{\varepsilon \downarrow 0} m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(\varphi_{\varepsilon}\right) \geq \liminf _{\varepsilon \downarrow 0} \mathcal{Q}_{\varepsilon}\left(\theta_{\varepsilon} F_{\varepsilon}^{0}, \theta_{\varepsilon} F_{\varepsilon}^{1}\right)=\liminf _{\varepsilon \downarrow 0} \mathcal{Q}_{\varepsilon}\left(\theta_{\varepsilon} G_{\varepsilon}^{1}, \theta_{\varepsilon} G_{\varepsilon}^{0}\right),
$$

where for the last identity we used the facts that $F_{\varepsilon}^{1}=G_{\varepsilon}^{1}$ on $\Omega_{\varepsilon}^{1}$ and

$$
\left\|F_{\varepsilon}^{0}-G_{\varepsilon}^{0}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{0}\right)}=\left\|\varepsilon^{\gamma} \nabla g_{\varepsilon}^{1}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{0}\right)} \rightarrow 0 .
$$

It remains to argue that

$$
\liminf _{\varepsilon \downarrow 0} \mathcal{Q}_{\varepsilon}\left(\theta_{\varepsilon} G_{\varepsilon}\right) \geq \mathcal{Q}(G) .
$$

In order to show this, notice that

$$
\begin{equation*}
Q_{\varepsilon}\left(\theta_{\varepsilon} G_{\varepsilon}\right)=\int_{\Omega} Q^{0}\left(\theta_{\varepsilon} \chi_{\varepsilon} G_{\varepsilon}^{0}\right) d x+\int_{\Omega} Q^{1}\left(\theta_{\varepsilon}\left(1-\chi_{\varepsilon}\right) G_{\varepsilon}^{1}\right) d x \tag{57}
\end{equation*}
$$

From $G_{\varepsilon} \stackrel{2}{\rightharpoonup} G$ we deduce, using Lemma 7, Remark 3 and Lemma 8, that

$$
\theta_{\varepsilon} \chi_{\varepsilon} G_{\varepsilon}^{0} \stackrel{2}{\rightharpoonup} \chi(y) G^{0}(x, y), \quad \theta_{\varepsilon}\left(1-\chi_{\varepsilon}\right) G_{\varepsilon}^{1} \stackrel{2}{\rightharpoonup}(1-\chi(y)) G^{1}(x, y) .
$$

By appealing to the lower semicontinuity of convex integral functionals with respect to weak two-scale convergence (cf. [37, Proposition 1.3]), we deduce that the liminf of the right-hand side in (57) is bounded below by $\mathcal{Q}(G)$. This completes the argument.
Step 3. Proof of part (b).
We claim that

$$
\begin{equation*}
\lim _{\varepsilon \downarrow 0} \left\lvert\, \frac{1}{m_{\varepsilon}^{2}} \int_{\Omega_{\varepsilon}^{0}} W\left(\mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)-\mathcal{Q}_{\varepsilon}\left(\varepsilon \nabla g_{\varepsilon}^{0}, \nabla g_{\varepsilon}^{1}\right) \mid=0 .\right.\right. \tag{58}
\end{equation*}
$$

Note that

$$
m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(\varphi_{\varepsilon}\right)=\left(m_{\varepsilon} \varepsilon^{-\gamma}\right)^{-2} \int_{\Omega} W^{0}\left(I+m_{\varepsilon} \varepsilon^{-\gamma} F_{\varepsilon}^{0}(x)\right) d x+m_{\varepsilon}^{-2} \int_{\Omega} W^{1}\left(I+m_{\varepsilon} F_{\varepsilon}^{1}(x)\right) d x
$$

where $F_{\varepsilon}^{0}$ and $F_{\varepsilon}^{1}$ are defined in (56). By (51) we have $\left\|m_{\varepsilon} \varepsilon^{-\gamma} F_{\varepsilon}^{0}\right\|_{L^{\infty}(\Omega)}+\left\|m_{\varepsilon} F_{\varepsilon}^{1}\right\|_{L^{\infty}(\Omega)} \rightarrow 0$ and (54) yields

$$
\left|m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)-\mathcal{Q}_{\varepsilon}\left(F_{\varepsilon}^{0}, F_{\varepsilon}^{1}\right)\right| \rightarrow 0 \text { as } \varepsilon \downarrow 0 .
$$

Since $G_{\varepsilon} \xrightarrow{2} G$ we have, thanks to Lemma 8:

$$
\chi_{\varepsilon} F_{\varepsilon}^{0} \xrightarrow{2} \chi(y) G^{0}(x, y), \quad\left(1-\chi_{\varepsilon}\right) F_{\varepsilon}^{1} \xrightarrow{2}(1-\chi(y)) G^{1}(x, y) .
$$

Hence, the continuity of convex integral functionals with respect to strong two-scale convergence (cf. [36]) yields

$$
\mathcal{Q}_{\varepsilon}\left(F_{\varepsilon}^{0}, F_{\varepsilon}^{1}\right) \rightarrow \mathcal{Q}(G),
$$

which completes the argument.
We are now in a position to prove the $\Gamma$-convergence statement for the energies $\mathcal{E}_{\varepsilon}$.
Proof of Theorem 1. Step 1. Part (a) (Compactness).
Thanks to (W2) we have

$$
m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right) \geq c_{0} m_{\varepsilon}^{-2} \Phi_{\varepsilon}^{\gamma}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)
$$

Hence, the claim of Theorem 1(a) directly follows from Lemma 2.
Step 2. Part (b) (Lower bound).

Without loss of generality we assume that

$$
\liminf _{\varepsilon \downarrow 0} m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)=\limsup _{\varepsilon \downarrow 0} m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)<\infty .
$$

Furthermore, thanks to Lemma 2, we can assume in addition that $\nabla g_{\varepsilon}^{1} \stackrel{2}{\rightharpoonup} \nabla g^{1}+\nabla_{y} \psi$ for some $\psi \in L^{2}\left(\Omega, \mathcal{A}\left(Y^{0}\right)\right)$, so that

$$
G_{\varepsilon}:=\left(\varepsilon \nabla g_{\varepsilon}^{0}, \nabla g_{\varepsilon}^{1}\right) \xrightarrow{2}\left(\nabla_{y} g^{0}, \nabla g^{1}+\nabla_{y} \psi\right)=: G .
$$

Applying Lemma 6(a) yields

$$
\liminf _{\varepsilon \downarrow 0} m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right) \geq \int_{\Omega \times Y^{0}} Q^{0}\left(\nabla_{y} g^{0}(x, y)\right) d x d y+\int_{\Omega \times Y^{1}} Q^{1}\left(\nabla g^{1}(x)+\nabla_{y} \psi(x, y)\right) d x d y
$$

This completes the argument, since the right-hand side is bounded from below by $\mathcal{E}_{\text {small }}\left(g^{0}, g^{1}\right)$.
Step 3. Part (c) (Upper bound).
Choose $\psi \in L^{2}\left(\Omega, \mathcal{A}\left(Y^{0}\right)\right)$ such that

$$
\begin{equation*}
\int_{\Omega \times Y^{1}} Q\left(\nabla g^{1}(x)+\nabla_{y} \psi(x, y)\right) d x d y=\int_{\Omega} Q_{\mathrm{hom}}^{1}\left(\nabla g^{1}(x)\right) d x \tag{59}
\end{equation*}
$$

Let $\varphi_{\varepsilon}$ denote the sequence associated with $g^{0}, g^{1}$ and $\psi$ via Lemma 3 with $c_{\varepsilon}:=m_{\varepsilon}$. In view of (20), applying Lemma 6 (b) yields

$$
\lim _{\varepsilon \downarrow 0} m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)=\int_{\Omega \times Y} Q^{0}\left(\nabla_{y} g^{0}(x, y)\right) d x d y+\int_{\Omega \times Y^{1}} Q^{1}\left(\nabla g^{1}(x)+\nabla_{y} \psi(x, y)\right) d x d y
$$

It follows from (59) that the right-hand side equals $\mathcal{E}_{\text {small }}\left(g^{0}, g^{1}\right)$.
Proof of Proposition 1. Step 1. A priori estimate.
We claim that for every sequence $\varphi_{\varepsilon}$ in $H_{0}^{1}(\Omega)$ the following implication holds:

$$
\begin{equation*}
\limsup _{\varepsilon \downarrow 0} I_{\varepsilon}\left(\varphi_{\varepsilon}\right)<\infty \quad \Rightarrow \quad \limsup _{\varepsilon \downarrow 0} m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)<\infty \tag{60}
\end{equation*}
$$

Indeed, we have

$$
\begin{aligned}
\left|\int_{\Omega} \ell_{\varepsilon} \cdot m_{\varepsilon} \varphi_{\varepsilon} d x\right| & \leq m_{\varepsilon}\left(\left\|\ell_{\varepsilon}\right\|_{L^{2}(\Omega)}\left\|g_{\varepsilon}^{1}\right\|_{L^{2}(\Omega)}+\varepsilon^{1-\gamma}\left\|\ell_{\varepsilon}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{0}\right)}\left\|g_{\varepsilon}^{0}\right\|_{L^{2}\left(\Omega_{\varepsilon}^{0}\right)}\right) \\
& \stackrel{(25 \mathrm{a})}{\leq} m_{\varepsilon}\left(\left\|m_{\varepsilon} g_{\varepsilon}^{1}\right\|_{L^{2}(\Omega)}+\left\|m_{\varepsilon} g_{\varepsilon}^{0}\right\|_{L^{2}(\Omega)}\right) \stackrel{(17)}{\lesssim} m_{\varepsilon}^{2} \sqrt{m_{\varepsilon}^{-2} \phi_{\varepsilon}^{\gamma}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)} \\
& \stackrel{(\mathrm{W} 2)}{\lesssim} m_{\varepsilon}^{2} \sqrt{m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right)} .
\end{aligned}
$$

Combining this with the definition of $I_{\varepsilon}$ we get

$$
m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right) \lesssim I_{\varepsilon}\left(\varphi_{\varepsilon}\right)+\sqrt{m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(\varphi_{\varepsilon}\right)}
$$

which implies (60).
Step 2. The proof of parts (a) and (b).

The existence of a minimiser to $I_{\text {small }}$ follows by the direct method. The minimiser $\left(g_{*}^{0}, g_{*}^{1}\right)$ is unique, since the implication

$$
\int_{\Omega \times Y} Q^{0}\left(\nabla_{y} \tilde{g}^{0}(x, y)\right) d x d y+\int_{\Omega} Q_{\mathrm{hom}}^{1}\left(\nabla \tilde{g}^{1}(x)\right) d x=0 \quad \Rightarrow \quad \tilde{g}^{0}=\tilde{g}^{1}=0
$$

holds for all $\tilde{g}^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$ and $\tilde{g}^{1} \in H_{0}^{1}(\Omega)$.
The remaining claims of Proposition 1 follow from the standard $\Gamma$-convergence arguments (cf. [19, Corollary 7.20]), provided the functionals $I_{\varepsilon}, \varepsilon>0$, are equi-coercive and $\Gamma$-converge to $I_{\text {small }}$. Indeed, thanks to (25c), it is easy to check that $\varphi_{\varepsilon} \stackrel{2}{\longrightarrow}\left(g^{0}, g^{1}\right)$ implies

$$
\begin{equation*}
\lim _{\varepsilon \downarrow 0} \frac{1}{m_{\varepsilon}^{2}} \int_{\Omega} \ell_{\varepsilon} \cdot m_{\varepsilon} \varphi_{\varepsilon} d x=\int_{\Omega}\left(\int_{Y^{0}} \ell^{0} \cdot g^{0} d y+\ell^{1} \cdot g^{1}\right) d x, \tag{61}
\end{equation*}
$$

since the integral on the left-hand side only involves products of weakly and strongly two-scale convergent factors, $c f$. [36, Proposition 2.8]). In combination with Theorem 1, this implies that $I_{\varepsilon} \Gamma$-converges to $I_{\text {small }}$. In addition, the trivial inequality

$$
\inf _{\varphi \in H_{0}^{1}(\Omega)} I_{\varepsilon}(\varphi) \leq I_{\varepsilon}(0)=0,
$$

combined with (60) and Lemma 2, proves that the functionals $I_{\varepsilon}$ are equi-coercive.
For the proof of Theorem 2 we make use of the following lemma:
Lemma 9 (Decomposition Lemma, see [22], [27]). Let $v \in H_{0}^{1}(\Omega)$ and let $v_{\varepsilon} \in H^{1}(\Omega)$ be a sequence with $v_{\varepsilon} \rightharpoonup v$ in $H^{1}(\Omega)$. Then there exists a sequence $\bar{V}_{\varepsilon} \in H^{1}(\Omega)$ such that the following properties hold for a subsequence of $v_{\varepsilon}$ (not relabelled):
(a) $\bar{V}_{\varepsilon} \rightharpoonup v$ in $H^{1}(\Omega)$;
(b) $\bar{V}_{\varepsilon}=v_{\varepsilon}$ in a neighbourhood of $\partial \Omega$;
(c) $\left(\left|\nabla \bar{V}_{\varepsilon}\right|^{2}\right)$ is equi-integrable;
(d) $\left|\left\{x \in \Omega: v_{\varepsilon}(x) \neq \bar{V}_{\varepsilon}(x)\right\}\right| \rightarrow 0$ as $\varepsilon \downarrow 0$.

Proof of Theorem 2. It suffices to prove the theorem for a subsequence. Throughout the proof we write

$$
G_{\varepsilon}:=\left(\varepsilon \nabla g_{\varepsilon}^{0}, \nabla g_{\varepsilon}^{1}\right), \quad G:=\left(\nabla_{y} g_{*}^{0}, \nabla g_{*}^{1}+\nabla_{y} \psi_{*}\right) .
$$

Furthermore, we make use of the functionals $\mathcal{Q}_{\varepsilon}$ and $\mathcal{Q}$ introduced at the beginning of Section 4.2. Recall that

$$
\mathcal{E}_{\text {small }}\left(g_{*}^{0}, g_{*}^{1}\right)=Q(G) .
$$

Step 1. Convergence of $\varphi_{\varepsilon}$ and of the corresponding energy values.
We claim that, as $\varepsilon \downarrow 0$, one has

$$
\begin{align*}
& G_{\varepsilon} \stackrel{2}{\rightharpoonup} G  \tag{62}\\
& I_{\varepsilon}\left(\varphi_{\varepsilon}\right) \rightarrow I_{\text {small }}\left(g_{*}^{0}, g_{*}^{1}\right)  \tag{63}\\
& m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right) \rightarrow \mathcal{E}_{\text {small }}\left(g_{*}^{0}, g_{*}^{1}\right) . \tag{64}
\end{align*}
$$

Indeed, from Proposition 1 we immediately deduce that $\varphi_{\varepsilon} \xrightarrow{2}\left(g_{*}^{0}, g_{*}^{1}\right)$ and (63). Furthermore, in view of the continuity of the loading term, $c f$. (61), this implies (64). For (62), it remains to argue that $\nabla g_{\varepsilon}^{1} \stackrel{2}{\rightharpoonup} \nabla g_{*}^{1}+\nabla_{y} \psi_{*}$. Thanks to $\varphi_{\varepsilon} \stackrel{2}{\rightharpoonup}\left(g_{*}^{0}, g_{*}^{1}\right)$ and Lemma 2 we have, up to a subsequence, $\nabla g_{\varepsilon}^{1} \stackrel{2}{\rightharpoonup} \nabla g_{*}^{1}+\nabla_{y} \psi$ for some $\psi \in L^{2}\left(\Omega, \mathcal{A}\left(Y^{0}\right)\right)$. Furthermore, from (64) and Lemma 6 (a) we infer that

$$
E_{\text {small }}\left(g_{0}^{*}, g_{1}^{*}\right)=\liminf _{\varepsilon \downarrow 0} m_{\varepsilon}^{-2} \mathcal{E}_{\varepsilon}\left(m_{\varepsilon} \varphi_{\varepsilon}\right) \geq \int_{\Omega \times Y^{0}} Q^{0}\left(\nabla_{y} g_{*}^{0}\right) d x d y+\int_{\Omega \times Y^{1}} Q^{1}\left(\nabla g_{*}^{1}+\nabla_{y} \psi\right) d x d y .
$$

This, in particular, implies

$$
\int_{\Omega \times Y^{1}} Q^{1}\left(\nabla g_{*}^{1}+\nabla_{y} \psi\right) d x d y=\int_{\Omega} Q_{\mathrm{hom}}^{1}\left(\nabla g_{*}^{1}\right) d x .
$$

In view of (28) we conclude that $\psi=\psi_{*}$ and (62) follows.
Step 2. Equi-integrable decomposition.
We claim that for a subsequence (not relabelled) there exist sequences $\bar{g}_{\varepsilon}^{0}, \bar{G}_{\varepsilon}^{1} \in H_{0}^{1}(\Omega)$ such that $\bar{G}_{\varepsilon}:=\left(\varepsilon \nabla \bar{G}_{\varepsilon}^{0}, \nabla \bar{G}_{\varepsilon}^{1}\right)$ satisfies

$$
\begin{equation*}
\bar{G}_{\varepsilon}-G_{\varepsilon} \stackrel{2}{\longrightarrow} 0, \quad\left\|\bar{G}_{\varepsilon}-G_{\varepsilon}\right\|_{L^{p}(\Omega)} \rightarrow 0, \tag{65}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{Q}_{\varepsilon}\left(\bar{G}_{\varepsilon}\right) \rightarrow \mathcal{E}_{\text {small }}\left(g_{*}^{0}, g_{*}^{1}\right) . \tag{66}
\end{equation*}
$$

To show the above, notice that thanks to Lemma 9 there exist sequences $\bar{G}_{\varepsilon}^{0}, \bar{G}_{\varepsilon}^{1} \in H_{0}^{1}(\Omega)$ such that

- $\left(\varepsilon^{2}\left|\nabla \bar{G}_{\varepsilon}^{0}\right|^{2}\right)$ and $\left(\left|\nabla \bar{g}_{\varepsilon}^{1}\right|^{2}\right)$ are equi-integrable,
- the indicator function $\bar{\theta}_{\varepsilon}$ defined by

$$
\bar{\theta}_{\varepsilon}(x):= \begin{cases}1, & \text { if } \bar{G}_{\varepsilon}^{0}(x)=g_{\varepsilon}^{0}(x) \text { and } \bar{G}_{\varepsilon}^{1}(x)=g_{\varepsilon}^{1}(x),  \tag{67}\\ 0, & \text { otherwise },\end{cases}
$$

satisfies (52).
Since $\bar{G}_{\varepsilon}-G_{\varepsilon}=\left(1-\bar{\theta}_{\varepsilon}\right)\left(\bar{G}_{\varepsilon}-G_{\varepsilon}\right)$ (and $p<2$ ), the convergence (65) follows from the boundedness of the sequence $\left(\bar{G}_{\varepsilon}-G_{\varepsilon}\right)$ in $L^{2}(\Omega)$, Lemma 7 , and Hölder's inequality.

We prove (66). Thanks to (62) we have $\bar{G}_{\varepsilon} \stackrel{2}{\sim} G$, so that (due to the lower semicontinuity of convex integral functionals with respect to weak two-scale convergence, cf. [37, Proposition 1.3]):

$$
\liminf _{\varepsilon \downarrow 0} \mathcal{Q}_{\varepsilon}\left(\bar{G}_{\varepsilon}\right) \geq \mathcal{Q}(G)=\mathcal{E}_{\text {small }}\left(g_{*}^{0}, g_{*}^{1}\right)
$$

Hence, for (66) it suffices to prove the opposite estimate, i.e. $\lim \sup _{\varepsilon \downarrow 0} \mathcal{Q}_{\varepsilon}\left(\bar{G}_{\varepsilon}\right) \leq \mathcal{E}_{\text {small }}(G)$, which, thanks to (50) and (64), follows from

$$
\begin{equation*}
\limsup _{\varepsilon \downarrow 0}\left(\mathcal{Q}_{\varepsilon}\left(\bar{G}_{\varepsilon}\right)-\mathcal{Q}_{\varepsilon}\left(\theta_{\varepsilon} G_{\varepsilon}\right)\right) \leq 0 . \tag{68}
\end{equation*}
$$

In order to show (68) notice that since the supports of $\bar{\theta}_{\varepsilon}$ and $\left(1-\bar{\theta}_{\varepsilon}\right)$ are disjoint, and because $\bar{\theta}_{\varepsilon} \theta_{\varepsilon} G_{\varepsilon}=\bar{\theta}_{\varepsilon} \theta_{\varepsilon} \bar{G}_{\varepsilon}$ (cf. (67)), an expansion of the squares yields

$$
\begin{aligned}
\mathcal{Q}_{\varepsilon}\left(\bar{G}_{\varepsilon}\right)-\mathcal{Q}_{\varepsilon}\left(\theta_{\varepsilon} G_{\varepsilon}\right) & =\mathcal{Q}_{\varepsilon}\left(\bar{\theta}_{\varepsilon} \bar{G}_{\varepsilon}\right)+\mathcal{Q}_{\varepsilon}\left(\left(1-\bar{\theta}_{\varepsilon}\right) \bar{G}_{\varepsilon}\right)-\mathcal{Q}_{\varepsilon}\left(\bar{\theta}_{\varepsilon} \theta_{\varepsilon} G_{\varepsilon}\right)-\mathcal{Q}_{\varepsilon}\left(\left(1-\bar{\theta}_{\varepsilon}\right) \theta_{\varepsilon} G_{\varepsilon}\right) \\
& =\mathcal{Q}_{\varepsilon}\left(\bar{\theta}_{\varepsilon}\left(1-\theta_{\varepsilon} \varepsilon \bar{G}_{\varepsilon}\right)+\mathcal{Q}_{\varepsilon}\left(\left(1-\bar{\theta}_{\varepsilon}\right) \bar{G}_{\varepsilon}\right)-\mathcal{Q}_{\varepsilon}\left(\left(1-\bar{\theta}_{\varepsilon}\right) \theta_{\varepsilon} G_{\varepsilon}\right)\right. \\
& \leq \mathcal{Q}_{\varepsilon}\left(\bar{\theta}_{\varepsilon}\left(1-\theta_{\varepsilon}\right) \bar{G}_{\varepsilon}\right)+\mathcal{Q}_{\varepsilon}\left(\left(1-\bar{\theta}_{\varepsilon}\right) \bar{G}_{\varepsilon}\right) .
\end{aligned}
$$

It is easy to check that $\bar{\theta}_{\varepsilon}\left(1-\theta_{\varepsilon}\right)$ and $1-\bar{\theta}_{\varepsilon}$ converge to zero in $L^{r}(\Omega)$ for all $r<\infty$. Hence, since $\left|\bar{G}_{\varepsilon}\right|^{2}$ is equi-integrable, Lemma 7 implies that the right-hand side of the previous estimate converges to zero, and (68) follows.
Step 3. Error estimate.
We claim that

$$
\begin{equation*}
\int_{\Omega_{\varepsilon}^{0}} \varepsilon\left|\operatorname{sym} \nabla\left(\bar{g}_{\varepsilon}^{0}-g_{*, \varepsilon}^{0}\right)\right|^{2} d x+\int_{\Omega_{\varepsilon}^{1}}\left|\operatorname{sym} \nabla\left(\bar{g}_{\varepsilon}^{1}-g_{*, \varepsilon}^{1}\right)\right|^{2} d x \rightarrow 0 \quad \text { as } \varepsilon \downarrow 0 . \tag{69}
\end{equation*}
$$

For the argument set $G_{*, \varepsilon}:=\left(\varepsilon \nabla g_{*, \varepsilon}^{0}, \nabla g_{*, \varepsilon}^{1}\right)$. In view of (13) it suffices to argue that

$$
\mathcal{Q}_{\varepsilon}\left(\bar{G}_{\varepsilon}-G_{*, \varepsilon}\right) \rightarrow 0 \quad \text { as } \varepsilon \downarrow 0 .
$$

The latter can be seen as follows: We have

$$
\mathcal{Q}_{\varepsilon}\left(\bar{G}_{\varepsilon}-G_{*, \varepsilon}\right)=\mathcal{Q}_{\varepsilon}\left(\bar{G}_{\varepsilon}\right)-\mathcal{Q}_{\varepsilon}\left(G_{*, \varepsilon}\right)+2 \mathcal{B}_{\varepsilon}\left(G_{*, \varepsilon} ; G_{*, \varepsilon}-\bar{G}_{\varepsilon}\right),
$$

where $\mathcal{B}_{\varepsilon}$ denotes the bilinear form associated with $\mathcal{Q}_{\mathcal{E}}$.
The difference of the two quadratic terms on the right-hand side converges to zero, since $G_{*, \varepsilon}$ is associated with a recovery sequence, and thanks to (66). On the other hand, since $G_{*, \varepsilon}$ strongly two-scale converges, and $G_{*, \varepsilon}-\bar{G}_{\varepsilon} \stackrel{2}{\rightleftharpoons} 0$ by (65), we deduce that

$$
\lim _{\varepsilon \downarrow 0} \mathcal{B}_{\varepsilon}\left(G_{*, \varepsilon} ; G_{*, \varepsilon}-\bar{G}_{\varepsilon}\right)=0,
$$

as $\mathcal{B}_{\varepsilon}\left(G_{*, \varepsilon} ; G_{*, \varepsilon}-\bar{G}_{\varepsilon}\right)$ only involves products between a weakly and a strongly two-scale convergent factor (cf. [36, Proposition 2.8]).
Step 4. Conclusion (Proof of (31)).
We split the estimate into

$$
\begin{align*}
& \int_{\Omega}\left(\left|g_{\varepsilon}^{0}-g_{*, \varepsilon}^{0}\right|^{p}+\left|\varepsilon \nabla g_{\varepsilon}^{0}-\varepsilon \nabla g_{*, \varepsilon}^{0}\right|^{p}\right) d x \rightarrow 0 \quad \text { as } \varepsilon \downarrow 0,  \tag{70}\\
& \int_{\Omega}\left(\left|g_{\varepsilon}^{1}-g_{*, \varepsilon}^{1}\right|^{p}+\left|\nabla g_{\varepsilon}^{1}-\nabla g_{*, \varepsilon}^{1}\right|^{p}\right) d x \rightarrow 0 \quad \text { as } \varepsilon \downarrow 0 . \tag{71}
\end{align*}
$$

Thanks to (65) and Step 3 we have

$$
\begin{equation*}
\int_{\Omega_{\varepsilon}^{0}} \varepsilon\left|\operatorname{sym} \nabla\left(g_{\varepsilon}^{0}-g_{*, \varepsilon}^{0}\right)\right|^{p} d x+\int_{\Omega_{\varepsilon}^{1}}\left|\operatorname{sym} \nabla\left(g_{\varepsilon}^{1}-g_{*, \varepsilon}^{1}\right)\right|^{p} d x \rightarrow 0 \quad \text { as } \varepsilon \downarrow 0 . \tag{72}
\end{equation*}
$$

Argument for (70): Set $\eta_{\varepsilon}^{0}:=g_{\varepsilon}^{0}-g_{*, \varepsilon}^{0}$. Since $\eta_{\varepsilon}^{0} \in H_{0}^{1}\left(\Omega_{\varepsilon}^{0}\right) \subset H_{0}^{1}(\Omega)$, Korn's inequality yields

$$
\int_{\Omega}\left|\nabla \eta_{\varepsilon}^{0}\right|^{p} d x \leq C \int_{\Omega}\left|\operatorname{sym} \nabla \eta_{\varepsilon}^{0}\right|^{p} d x=C \int_{\Omega_{\varepsilon}^{0}}\left|\operatorname{sym} \nabla \eta_{\varepsilon}^{0}\right|^{p} d x
$$

where $C>0$ only depends on $\Omega$, $p$, and $d$. Combined with the improved Poincaré inequality (43) and (72), (70) follows.

Argument for (71): We claim that (71) follows from

$$
\begin{equation*}
\left\|\operatorname{sym} \nabla \eta_{\varepsilon}^{1}\right\|_{L^{p}\left(\Omega_{\varepsilon}^{0}\right)} \rightarrow 0 \tag{73}
\end{equation*}
$$

where $\eta_{\varepsilon}^{1}:=g_{\varepsilon}^{1}-g_{*, \varepsilon}^{1}$. Indeed, since $\eta_{\varepsilon}^{1}$ vanishes on $\partial \Omega$, (73), (72) and Korn's first inequality yield (71).

Thanks to the definition of $\Omega_{\varepsilon}^{0}$, the argument for (73) can be reduced to the following statement: For all $\xi \in Z_{\varepsilon}:=\left\{\xi \in \mathbb{Z}^{d}: \varepsilon(\xi+Y) \subset \Omega\right\}$ we have

$$
\begin{equation*}
\int_{\varepsilon\left(\xi+Y^{0}\right)}\left|\operatorname{sym} \nabla \eta_{\varepsilon}\right|^{p} d x \lesssim \int_{\varepsilon\left(\xi+Y^{1}\right)}\left|\operatorname{sym} \nabla \eta_{\varepsilon}\right|^{p} d x \tag{74}
\end{equation*}
$$

For the argument consider the rescaled function

$$
\hat{\eta}_{\varepsilon}: Y \mapsto \mathbb{R}^{d}, \quad \hat{\eta}_{\varepsilon}(y):=\eta_{\varepsilon}(\varepsilon(\xi+y))-S y+c
$$

where $S \in \mathbb{R}_{\text {skew }}^{d \times d}$ and $c \in \mathbb{R}^{d}$ are chosen such that the Poincaré and Korn inequalities yield

$$
\begin{equation*}
\int_{Y^{1}}\left(\left|\hat{\eta}_{\varepsilon}\right|^{p}+\left|\nabla \hat{\eta}_{\varepsilon}\right|^{p}\right) d y \lesssim \int_{Y^{1}}\left|\operatorname{sym} \nabla \hat{\eta}_{\varepsilon}\right|^{p} d y . \tag{75}
\end{equation*}
$$

Since both $g_{\varepsilon}^{1}$ and $g_{*, \varepsilon}^{1}$ satisfy (16)(ii), we have $-\triangle \hat{\eta}_{\varepsilon}=0$ in $Y^{0}$ in the distributional sense. Hence, thanks to Assumption 1 and (75), we have

$$
\int_{Y^{0}}\left|\nabla \hat{\eta}_{\varepsilon}\right|^{p} d y \lesssim \int_{Y^{1}}\left|\operatorname{sym} \nabla \hat{\eta}_{\varepsilon}\right|^{p} d y
$$

and thus

$$
\begin{aligned}
\int_{\varepsilon\left(\xi+Y^{0}\right)}\left|\operatorname{sym} \nabla \eta_{\varepsilon}\right|^{p} d x & =\varepsilon^{d-p} \int_{Y^{0}}\left|\operatorname{sym} \nabla \hat{\eta}_{\varepsilon}\right|^{p} d y \lesssim \varepsilon^{d-p} \int_{Y^{1}}\left|\operatorname{sym} \nabla \hat{\eta}_{\varepsilon}\right|^{p} d y \\
& =\int_{\varepsilon\left(\xi+Y^{1}\right)}\left|\operatorname{sym} \nabla \eta_{\varepsilon}\right|^{p} d x
\end{aligned}
$$

Proof of Proposition 2. Step 1. Proof of (a).
 and thus the compactness part of Theorem 1 (and the fact that two-scale convergence implies weak convergence) yields (for a subsequence that we do not relabel):

$$
\begin{equation*}
\varphi_{\varepsilon} \stackrel{2}{\rightharpoonup}\left(g^{0}, g^{1}\right) \quad \text { and } \quad \varphi_{\varepsilon} \rightharpoonup \varphi:=g^{1}+\int_{Y^{0}} g^{0} d y . \tag{76}
\end{equation*}
$$

Step 2. Proof of (b).
We may restrict to the case

$$
\liminf _{\varepsilon \downarrow 0} I_{\varepsilon}^{\alpha}\left(\varphi_{\varepsilon}\right)=\underset{\varepsilon \downarrow 0}{\limsup } I_{\varepsilon}^{\alpha}\left(\varphi_{\varepsilon}\right)<\infty
$$

Thanks to Step 1 we may assume w.l.o.g. that (76) holds. From the lower bound part of Theorem 1 (and the fact that $\int_{\Omega} f \cdot \varphi_{\varepsilon} d x \rightarrow \int_{\Omega} f \cdot \varphi d x$ ) we thus deduce that

$$
\liminf _{\varepsilon \downarrow 0} I_{\varepsilon}^{\alpha}\left(\varphi_{\varepsilon}\right) \geq I_{\text {small }}\left(g^{0}, g^{1}\right)
$$

With the definition of $\bar{Q}$ from the introduction we get with $\bar{G}:=\int_{Y_{0}} g^{0} d y$ the inequality:

$$
\begin{aligned}
I_{\text {small }}\left(g^{0}, g^{1}\right) \geq & \int_{\Omega} Q_{\mathrm{hom}}^{1}\left(\nabla g^{1}(x)\right)-\left(g^{1}(x)+\bar{G}(x)\right) \cdot f(x) d x \\
& +\int_{\Omega} \min \left\{\int_{Y^{0}} Q^{0}\left(\nabla_{y} \tilde{g}^{0}(y)\right): \tilde{g}^{0} \in H_{0}^{1}\left(Y^{0}\right), \int_{Y^{0}} \tilde{g}^{0}(y)=\bar{G}(x)\right\} \\
= & \int_{\Omega} Q_{\mathrm{hom}}^{1}\left(\nabla g^{1}(x)\right)+\bar{Q}(\bar{G}(x))-\left(g^{1}(x)+\bar{G}(x)\right) \cdot f(x) d x
\end{aligned}
$$

Since $\varphi=g^{1}+\bar{G}$ by (76), the right-hand side is bounded from below by $\bar{I}_{\text {small }}(\varphi)$, which completes the argument for (b).
Step 3. Proof of (c).
Let $\varphi \in L^{2}(\Omega)$. It suffices to argue that there exists $g^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$ and $g^{1} \in H_{0}^{1}(\Omega)$ with $\varphi=g^{1}+\int_{Y^{0}} g^{0} d y$ and

$$
\begin{equation*}
\bar{I}_{\mathrm{small}}(\varphi)=I_{\mathrm{small}}\left(g^{0}, g^{1}\right) \tag{77}
\end{equation*}
$$

Indeed, in that case, we can find by part (c) of Theorem 1 a sequence $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ such that $\varphi_{\varepsilon} \stackrel{2}{\rightharpoonup}\left(g^{0}, g^{1}\right)$ and $\lim _{\varepsilon \downarrow 0} I_{\varepsilon}^{\alpha}\left(\varphi_{\varepsilon}\right)=I_{\text {small }}\left(g^{0}, g^{1}\right)$. Since $\varphi_{\varepsilon} \stackrel{2}{\rightharpoonup}\left(g^{0}, g^{1}\right)$ implies $\varphi_{\varepsilon} \rightharpoonup \varphi=$ $g^{1}+\int_{Y^{0}} g^{0} d y$ weakly in $L^{2}(\Omega)$, we deduce from (77) that $\varphi_{\varepsilon}$ is the sought for recovery sequence. In order to prove (77) let $\rho_{i}(i=1,2,3)$ denote the unique minimiser in $H_{0}^{1}\left(Y^{0}\right)$ to

$$
H_{0}^{1}\left(Y^{0}\right) \ni \rho \mapsto \int_{Y^{0}} Q^{0}(\nabla \rho(y)) d y \quad \text { subject to } \int_{Y^{0}} \rho d y=e_{i}
$$

and set

$$
\bar{q}_{i}:=\int_{Y^{0}} Q^{0}\left(\nabla \rho_{i}(y)\right) d y
$$

Then it is easy to check that

$$
\bar{Q}(G)=\sum_{i=1}^{3} G_{i}^{2} \bar{q}_{i} \quad \text { for all } G \in \mathbb{R}^{3}
$$

and since $\bar{q}_{1}, \ldots, \bar{q}_{3}>0$, we deduce that $\bar{Q}$ is a positive definite quadratic form. Hence, since $Q_{\text {hom }}^{1}(F) \geq c|\operatorname{sym} F|^{2}$ for some $c>0$, we deduce that we can find a unique function $g^{1} \in H_{0}^{1}(\Omega)$ that minimizes the functional

$$
\begin{equation*}
H_{0}^{1}(\Omega) \ni g \mapsto \int_{\Omega} Q_{\mathrm{hom}}^{1}(\nabla g(x))+\bar{Q}(\varphi(x)-g(x)) d x \tag{78}
\end{equation*}
$$

Setting $g^{0}(x, y):=\sum_{i=1}^{3}\left((\varphi(x)-g(x)) \cdot e_{i}\right) \rho_{i}(y)$ we deduce that $\int_{\Omega} \bar{Q}(\varphi(x)-g(x)) d x=$ $\int_{\Omega \times Y^{0}} Q^{0}\left(\nabla_{y} g^{0}(x, y)\right) d y$, and thus (thanks to the definition of $\left.\bar{I}_{\text {small }}\right)(77)$ follows.

Step 4. Proof of (d).

By Proposition 1 we have $\varphi_{\varepsilon} \xrightarrow{2}\left(g_{*}^{0}, g_{*}^{1}\right)$ and thus $\varphi_{\varepsilon} \rightharpoonup \tilde{\varphi}:=g_{*}^{1}+\int_{Y^{0}} g_{*}^{0} d y$. By definition of $\bar{I}_{\text {small }}$ we have

$$
\begin{equation*}
I_{\text {small }}\left(g_{*}^{0}, g_{*}^{1}\right) \geq \bar{I}_{\text {small }}(\tilde{\varphi}) \geq \inf _{L^{2}(\Omega)} \bar{I}_{\text {small }} \tag{79}
\end{equation*}
$$

On the other hand, the map $L^{2}(\Omega) \ni \varphi \mapsto g \in H_{0}^{1}(\Omega)$ with $g$ minimizing the functional in (78) is linear and bounded; hence, we deduce that $\bar{I}_{\text {small }}$ is quadratic and strictly convex. It thus admits a unique minimiser $\varphi_{*} \in L^{2}(\Omega)$. By Step 3 (cf. (77)) we may associated with $\varphi_{*}$ a pair $\left(g^{0}, g^{1}\right)$ such that $\bar{I}_{\text {small }}\left(\varphi_{*}\right)=I_{\text {small }}\left(g^{0}, g^{1}\right)$. Combined with (79) we get

$$
\min _{L^{2}(\Omega)} \bar{I}_{\text {small }}=\bar{I}_{\text {small }}\left(\varphi_{*}\right)=I_{\text {small }}\left(g^{0}, g^{1}\right) \geq I_{\text {small }}\left(g_{*}^{0}, g_{*}^{1}\right) \geq \bar{I}_{\text {small }}(\tilde{\varphi}) \geq \min _{L^{2}(\Omega)} \bar{I}_{\text {small }} .
$$

Hence, equality holds everywhere and the claimed identities follow from the strict convexity of $\bar{I}_{\text {small }}$. The convergence of $\inf _{L^{2}(\Omega)} I_{\varepsilon}^{\alpha} \rightarrow \min _{L^{2}(\Omega)} \bar{I}_{\text {small }}$ follows from part (b) and (c) by standard arguments from $\Gamma$-convergence.

### 4.3 Proof of Theorem 3: finite strain regime

We define, for $g_{\varepsilon}^{0} \in H_{0}^{1}\left(\Omega_{\varepsilon}^{0}\right), g^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$, and $g_{\varepsilon}^{1}, g^{1} \in H_{0}^{1}(\Omega)$, the following functionals:

$$
\begin{aligned}
I_{\varepsilon}^{0}\left(g_{\varepsilon}^{0}\right) & :=\int_{\Omega_{\varepsilon}^{0}}\left(W^{0}\left(I+\varepsilon \nabla g_{\varepsilon}^{0}(x)\right)-\varepsilon^{1-2 \gamma} \ell_{\varepsilon} \cdot g_{\varepsilon}^{0}\right) d x, \\
I_{0}^{0}\left(g^{0}\right) & :=\int_{\Omega \times Y^{0}}\left(\mathcal{Q} W^{0}\left(I+\nabla_{y} g^{0}(x, y)\right)-\ell^{0} \cdot g^{0}\right) d x d y, \\
I_{\varepsilon}^{1}\left(g_{\varepsilon}^{1}\right) & :=\varepsilon^{-2 \gamma} \int_{\Omega_{\varepsilon}^{1}}\left(W^{1}\left(I+\varepsilon^{\gamma} \nabla g_{\varepsilon}^{1}(x)\right)-\int_{\Omega} \varepsilon^{-\gamma} \ell_{\varepsilon} \cdot g_{\varepsilon}^{1}\right) d x, \\
I_{0}^{1}\left(g^{1}\right) & :=\int_{\Omega}\left(Q_{\mathrm{hom}}^{1}\left(\nabla g^{1}(x)\right)-\ell^{1} \cdot g^{1}\right) d x .
\end{aligned}
$$

Thanks to the Lipschitz condition (32b), we can decompose $I_{\varepsilon}$ into the sum $I_{\varepsilon}^{0}+I_{\varepsilon}^{1}$ at the expense of a small error. More precisely, the following lemma holds.

Lemma 10. Suppose that $m_{\varepsilon}=\varepsilon^{\gamma}$. There exists a constant $C>0$ such that for all $\varepsilon>0$ and $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega), \varepsilon^{1-\gamma} g_{\varepsilon}^{0}+g_{\varepsilon}^{1} \stackrel{(16)}{=} \varphi_{\varepsilon}$ we have

$$
\left|I_{\varepsilon}\left(\varphi_{\varepsilon}\right)-\left(I_{\varepsilon}^{0}\left(g_{\varepsilon}^{0}\right)+I_{\varepsilon}^{1}\left(g_{\varepsilon}^{1}\right)\right)\right| \leq C \varepsilon^{\gamma}\left(1+\Phi_{\varepsilon}^{\gamma}\left(\varphi_{\varepsilon}\right)\right) .
$$

Proof. Note that

$$
\begin{aligned}
& \left|I_{\varepsilon}\left(\varphi_{\varepsilon}\right)-\left(I_{\varepsilon}^{0}\left(g_{\varepsilon}^{0}\right)+I_{\varepsilon}^{1}\left(g_{\varepsilon}^{1}\right)\right)\right| \\
& \quad=\left|\int_{\Omega_{\varepsilon}^{0}} W^{0}\left(I+\varepsilon^{\gamma}\left(\varepsilon^{1-\gamma} \nabla g_{\varepsilon}^{0}+\nabla g_{\varepsilon}^{1}\right)\right)-W^{0}\left(I+\varepsilon \nabla g_{\varepsilon}^{0}\right) d x\right| .
\end{aligned}
$$

In view of (32b) and (17), the statement follows.
The following lemma is a simple consequence of [17, Lemma 21, Lemma 22] and (25c):
Lemma 11. Assume (25a), (25c) and $m_{\varepsilon}=e^{\gamma}$.
(a) Consider a sequence $g_{\varepsilon}^{0} \in H_{0}^{1}\left(\Omega_{\varepsilon}^{0}\right)$. If $g_{\varepsilon}^{0} \xrightarrow{2} g^{0}$ and $\varepsilon \nabla g_{0}^{1} \xrightarrow{2} \nabla_{y} g^{0}$, then

$$
\liminf _{\varepsilon \downarrow 0} I_{\varepsilon}^{0}\left(g_{\varepsilon}^{0}\right) \geq I_{0}^{0}\left(g^{0}\right)
$$

(b) For all $g^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$ there exists a sequence $g_{\varepsilon}^{0} \in H_{0}^{1}\left(\Omega_{\varepsilon}^{0}\right)$ such that $g_{\varepsilon}^{0} \xrightarrow{2} g^{0}$, $\varepsilon \nabla g_{\varepsilon}^{0} \stackrel{2}{\rightharpoonup} \nabla_{y} g^{0}$, and

$$
\lim _{\varepsilon \downarrow 0} I_{\varepsilon}^{0}\left(g_{\varepsilon}^{0}\right)=I_{0}^{0}\left(g^{0}\right)
$$

For the stiff part one can prove (similar to Lemma 6) the following lemma:
Lemma 12. Assume that (25a)-(25c) hold.
(a) Consider $g_{\varepsilon}^{1} \in H_{0}^{1}(\Omega)$. If $g_{\varepsilon}^{1} \rightharpoonup g^{1}$ weakly in $H^{1}(\Omega)$, then

$$
\liminf _{\varepsilon \downarrow 0} I_{\varepsilon}^{1}\left(g_{\varepsilon}^{1}\right) \geq I_{0}^{1}\left(g^{1}\right) .
$$

(b) For all $g^{1} \in H_{0}^{1}(\Omega)$ there exists a sequence $g_{\varepsilon}^{1} \in H_{0}^{1}(\Omega)$ such that

$$
g_{\varepsilon}^{1} \rightharpoonup g^{1} \text { weakly in } H^{1}(\Omega), \quad \text { and } \quad \lim _{\varepsilon \downarrow 0} I_{\varepsilon}^{1}\left(g_{\varepsilon}^{1}\right)=I_{0}^{1}\left(g^{1}\right) .
$$

We proceed to the proof of Theorem 3.
Proof of Theorem 3. Step 1. The proof of parts (a) and (b)
Statement (a) and (b) directly follow from Lemma 10, Lemma 11 and Lemma 12.
Step 2. Proof of (33): convergence of the minima.
For brevity set

$$
e_{\varepsilon}:=\inf _{\varphi \in H_{0}^{1}(\Omega)} \varepsilon^{-2 \gamma} I_{\varepsilon}\left(\varphi_{\varepsilon}\right), \quad e_{0}:=\inf _{\substack{g^{0} \in L^{2}\left(\Omega, H_{1}^{1}\left(Y^{0}\right)\right) \\ g^{1} \in H_{0}^{1}(\Omega)}} I_{\text {finite }}\left(g^{0}, g^{1}\right) .
$$

We prove (33) in the form of the two inequalities

$$
\begin{equation*}
\limsup _{\varepsilon \downarrow 0} e_{\varepsilon} \leq e_{0}, \quad \liminf _{\varepsilon \downarrow 0} e_{\varepsilon} \geq e_{0} . \tag{80}
\end{equation*}
$$

The argument for the first inequality in (80) is standard: for $\delta>0$ choose $\left(g^{0}, g^{1}\right)$ with $I_{\text {finite }}\left(g^{0}, g^{1}\right) \leq e_{0}+\delta$. By part (b) there exists a recovery sequence $\varphi_{\varepsilon}$, so that $I_{\varepsilon}\left(\varphi_{\varepsilon}\right) \rightarrow$ $I_{\text {finite }}\left(g^{0}, g^{1}\right)$. Hence

$$
\underset{\varepsilon \downarrow 0}{\limsup } e_{\varepsilon} \leq \underset{\varepsilon \downarrow 0}{\limsup } I_{\varepsilon}\left(\varphi_{\varepsilon}\right)=I_{\text {finite }}\left(g^{0}, g^{1}\right) \leq e_{0}+\delta .
$$

Since this is valid for all $\delta>0$, the first inequality in (80) follows.
Next, we prove the second inequality in (80). Let $\varphi_{\varepsilon}$ denote a sequence with the property $\liminf _{\varepsilon \downarrow 0} e_{\varepsilon}=\liminf _{\varepsilon \downarrow 0} I_{\varepsilon}\left(\varphi_{\varepsilon}\right)$; e.g. choose $\varphi_{\varepsilon} \in H_{0}^{1}(\Omega)$ such that $I_{\varepsilon}\left(\varphi_{\varepsilon}\right) \leq e_{\varepsilon}+\varepsilon$. Combining this with Lemma 10 we deduce that

$$
\liminf _{\varepsilon \downarrow 0} e_{\varepsilon}=\liminf _{\varepsilon \downarrow 0} I_{\varepsilon}\left(\varphi_{\varepsilon}\right)=\liminf _{\varepsilon \downarrow 0}\left(I_{\varepsilon}^{0}\left(g_{\varepsilon}^{0}\right)+I_{\varepsilon}^{1}\left(g_{\varepsilon}^{1}\right)\right) \geq \liminf _{\varepsilon \downarrow 0} I_{\varepsilon}^{0}\left(g_{\varepsilon}^{0}\right)+\liminf _{\varepsilon \downarrow 0} I_{\varepsilon}^{1}\left(g_{\varepsilon}^{1}\right),
$$

where $\varepsilon^{1-\gamma} g_{\varepsilon}^{0}+g_{\varepsilon}^{1} \stackrel{(16)}{=} \varphi_{\varepsilon}$. By passing to a subsequence, we assume without loss of generality that $\varphi_{\varepsilon} \stackrel{2}{\longrightarrow}\left(g^{0}, g^{1}\right)$. Since, thanks to Lemma 12, we have

$$
\liminf _{\varepsilon \downarrow 0} I_{\varepsilon}^{1}\left(g_{\varepsilon}^{1}\right) \geq I_{0}^{1}\left(g^{1}\right) \geq \inf _{\tilde{g}^{1} \in H_{0}^{1}(\Omega)} I_{0}^{1}\left(\tilde{g}^{1}\right),
$$

it remains to argue that

$$
\begin{equation*}
\liminf _{\varepsilon \downarrow 0} I_{\varepsilon}^{0}\left(g_{\varepsilon}^{0}\right) \geq \inf _{\tilde{g}^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)} I_{0}^{0}\left(\tilde{g}^{0}\right) . \tag{81}
\end{equation*}
$$

We identify $g_{\varepsilon}^{0}$ with its extension by zero to $\mathbb{R}^{d}$, and consider the periodic unfolding of $g_{\varepsilon}^{0}$ defined as

$$
G_{\varepsilon}^{0}: \Omega \times Y \rightarrow \mathbb{R}^{d}, \quad G_{\varepsilon}^{0}(x, y):=g_{\varepsilon}^{0}(\varepsilon\lfloor x / \varepsilon\rfloor+\varepsilon y),
$$

where $\lfloor z\rfloor$ stands the unique vector in $\mathbb{Z}^{d}$ such that $z-\lfloor z\rfloor \in[0,1)^{d}$. Further, note that $G_{\varepsilon}^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)$, and for $\xi \in Z_{\varepsilon}:=\left\{\xi \in \mathbb{Z}^{d}: \varepsilon(\xi+Y) \subset \Omega\right\}$ and $y \in Y$ one has

$$
g_{\varepsilon}^{0}(\varepsilon(\xi+y))=G_{\varepsilon}^{0}(\varepsilon(\xi+y), y), \quad \varepsilon \nabla g_{\varepsilon}^{0}(\varepsilon(\xi+y))=\nabla_{y} G_{\varepsilon}^{0}(\varepsilon(\xi+y), y) .
$$

Now we consider $I_{\varepsilon}^{0}\left(g_{\varepsilon}^{0}\right)$, which involves an integral over the set $\Omega_{\varepsilon}^{0}$. Since the latter can be written as a union of sets of the form $\varepsilon\left(\xi+Y^{0}\right)$ with $\xi \in Z_{\varepsilon}$, an elementary calculation shows that

$$
\begin{aligned}
I_{\varepsilon}\left(g_{\varepsilon}^{0}\right) & =\sum_{\xi \in Z_{\varepsilon}} \int_{\varepsilon\left(\xi+Y^{0}\right)}\left(W^{0}\left(I+\varepsilon \nabla g_{\varepsilon}^{0}(x)\right)-\varepsilon^{2 \gamma-1} \ell_{\varepsilon}(x) \cdot g_{\varepsilon}^{0}(x)\right) d x \\
& =\sum_{\xi \in Z_{\varepsilon}} \varepsilon^{d} \int_{Y^{0}}\left(W^{0}\left(I+\nabla_{y} G_{\varepsilon}^{0}(\varepsilon(\xi+y), y)\right)-\varepsilon^{2 \gamma-1} \ell_{\varepsilon}(\varepsilon(\xi+y)) \cdot G_{\varepsilon}^{0}(\varepsilon(\xi+y), y)\right) d y \\
& =\int_{\Omega} \int_{Y^{0}}\left(W^{0}\left(I+\nabla_{y} G_{\varepsilon}^{0}(x, y)\right)-\varepsilon^{2 \gamma-1} \ell_{\varepsilon}(\varepsilon(\lfloor x / \varepsilon\rfloor+y)) \cdot G_{\varepsilon}^{0}(x, y)\right) d y d x
\end{aligned}
$$

Thanks to (25c), the characterisation of strong two-scale convergence introduced in Step 1 of the proof of Lemma 3, and the fact that $\lim \sup _{\varepsilon \downarrow 0}\left\|g_{\varepsilon}^{0}\right\|_{L^{2}(\Omega)}<\infty$, we have

$$
\underset{\varepsilon \downarrow 0}{\limsup }\left|\int_{\Omega \times Y^{0}}\left(\varepsilon^{2 \gamma-1} \ell_{\varepsilon}(\varepsilon(\lfloor x / \varepsilon\rfloor+y))-\ell^{0}(x, y)\right) \cdot G_{\varepsilon}^{0}(\varepsilon(\lfloor x / \varepsilon\rfloor+y), y) d x d y\right|=0 .
$$

Hence, one has

$$
\begin{aligned}
& \liminf _{\varepsilon \downarrow 0} I_{\varepsilon}\left(g_{\varepsilon}^{0}\right) \\
& \quad=\liminf _{\varepsilon \downarrow 0} \int_{\Omega} \int_{Y^{0}}\left(W^{0}\left(I+\nabla_{y} G_{\varepsilon}^{0}(x, y)\right)-\ell^{0}(x, y) \cdot G_{\varepsilon}^{0}(x, y)\right) d y d x \\
& \quad \geq \inf _{g^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)} \int_{\Omega} \int_{Y^{0}}\left(W^{0}\left(I+\nabla_{y} g^{0}(x, y)\right)-\ell^{0}(x, y) \cdot g^{0}(x, y)\right) d y d x \\
& \quad \geq \inf _{g^{0} \in L^{2}\left(\Omega, H_{0}^{1}\left(Y^{0}\right)\right)} \int_{\Omega} \int_{Y^{0}}\left(\mathcal{Q} W^{0}\left(I+\nabla_{y} g^{0}(x, y)\right)-\ell^{0}(x, y) \cdot g^{0}(x, y)\right) d y d x,
\end{aligned}
$$

which proves (81).
Step 3. The proof of the convergence (34).
Since $\mathcal{Q} W^{0}$ is quasiconvex and $Q_{\mathrm{hom}}^{1}$ quadratic, there exists a pair $\left(g_{*}^{0}, g_{*}^{1}\right)$ that minimises $I_{\text {finite }}$. Now the sequence associated with $\left(g_{*}^{0}, g_{*}^{1}\right)$ via Theorem 3 (b) satisfies (34).

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