Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

EVENTUAL SMOOTHNESS AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO A CHEMOTAXIS SYSTEM PERTURBED BY A LOGISTIC GROWTH

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(Communicated by the associate editor name)

Abstract. In this paper we study the chemotaxis-system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + g(u) & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u & x \in \Omega, t > 0, \end{cases}$$

defined in a convex smooth and bounded domain Ω of $\mathbb{R}^n$, $n \geq 1$, with $\chi > 0$ and endowed with homogeneous Neumann boundary conditions. The source g behaves similarly to the logistic function and satisfies $g(s) \leq a - bs^\alpha$, for $s \geq 0$, with $a \geq 0$, $b > 0$ and $\alpha > 1$. Continuing the research initiated in [33], where for appropriate $1 < p < \alpha < 2$ and $(u_0, v_0) \in C^0(\bar{\Omega}) \times C^2(\bar{\Omega})$ the global existence of very weak solutions (u, v) to the system (for any $n \geq 1$) is shown, we principally study boundedness and regularity of these solutions after some time. More precisely, when $n = 3$, we establish that

- for all $\tau > 0$ an upper bound for $\frac{a}{b}, ||u_0||_{L^1(\Omega)}, ||v_0||_{W^{2,\alpha}(\Omega)}$ can be prescribed in a such a way that (u, v) is bounded and Hölder continuous beyond τ;
- for all (u_0, v_0), and sufficiently small ratio $\frac{a}{b}$, there exists a $T > 0$ such that (u, v) is bounded and Hölder continuous beyond T.

Finally, we illustrate the range of dynamics present within the chemotaxis system in one, two and three dimensions by means of numerical simulations.

1. Introduction and motivations.

According to the *logistic model* in population dynamics (Pierre-François Verhulst, 1838), the self-limiting growth of a biological

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2010 *Mathematics Subject Classification.* Primary: 35B40, 35K55, 35B65, 92C17; Secondary: 35Q92.

Key words and phrases. Nonlinear parabolic systems, chemotaxis, logistic source, boundedness, eventual smoothness.

TEW would like to thank St John's College, Oxford and the Mathematical Biosciences Institute (MBI) at Ohio State University, for financially supporting this research through the National Science Foundation grant DMS 1440386 and BBSRC grant BKNXBKOO BK00.16.

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population size at a certain time, $p(t)$, is described by the equation

$$\frac{dp}{dt} = rp(1 - \frac{p}{K}).$$

The constant $r > 0$ defines the growth rate and $K > 0$ is the carrying capacity of the species, which is also associated to the death rate of the same species (see [30]). Once an initial size for the population is given, i.e. $p(0) = p_0$, the aforementioned equation has an explicit solution whose expression is, for any $t \geq 0$,

$$p(t) = \frac{Kp_0e^{rt}}{K + p_0(e^{rt} - 1)}.$$

This shows that the total population increases progressively from p_0 at time $t = 0$ to the limit K, which is reached when $t \to \infty$. In particular, the size of the population remains bounded for all time, and 0 and K are the only stationary points of p; the first represents an unstable situation while the second an asymptotically stable equilibrium.

This formulation does not express how the population distributes in the space it occupies, neither considers the presence of further factors which induce the migration of the population from one zone to another. Indeed, *chemotaxis* is the movement of cells present in an environment in response to a chemical stimulus therein inhomogeneously distributed.

The mathematical model for the description of the chemotaxis proposed by Keller and Segel in 1970 (see [11]) is defined by two parabolic differential equations, one for the distribution of the cells, $u = u(x, t)$, and the other for the concentration of chemical signal, $v = v(x, t)$, where naturally x is the spatial variable and t the temporal one:

$$\begin{cases} u_t = \nabla \cdot (\varrho \nabla u - \chi u \nabla v), \\ v_t = \Delta v - \kappa v + u. \end{cases} \tag{1}$$

In system (1), the parameters ϱ, χ and κ are positive constants. Although we will consider such cross diffusive terms in terms of chemotactic movement, they are able to appear for a variety of reasons, including growth of the underlying solution space [5, 14, 43, 44] and inhomogeneities in the underlying environment [4].

In the present case we prescribe the following description to the equations. Chemoattractant, v, spreads diffusively, decays with rate κ and is also produced by the bacteria with rate 1. The bacteria diffuse with mobility ϱ and drift in the direction of the gradient of concentration of the chemoattractant with velocity $\chi|\nabla v|$; χ is called chemosensitivity. Hence, once the initial cells distribution and chemical concentration (that is $u_0(x) = u(x, 0)$ and $v_0(x) = v(x, 0)$) are given, under zero-flux boundary conditions on both u and v, the previous problem describes the chemotactic dynamics of a cells population in a totally insulated domain.

Real observations show that this movement may eventually lead to aggregation processes, in which the density of the cells spatially coalesces and grows without bound (*chemotactic collapse*). Mathematically, this collapse implies that (possibly) u becomes unbounded at one or more points of its domain at a certain instant (blow-up time). It is known that, in a one-dimensional domain, all the solutions of (1) are global and uniformly bounded in time (see [21]), while that in the n-dimensional setting, with $n \geq 2$, unbounded solutions to the same problem have been detected (see, for instance, [8] and [39]).

In line with the chemotactic scenario, in [10] for radial solutions and in [20] for non-radial, the authors prove that under suitable assumptions the bacteria concentration blows up in finite time, for certain domains of $\mathbb{R}^2$ and in the cases in which the second differential equation of (1) is replaced by $0 = \Delta v - v + u$ (parabolic-elliptic case). Moreover, for the classical parabolic-parabolic (or fully parabolic) case, estimates from below and numerical computations for the blow-up time of unbounded solutions to (1) are derived in [24] and [7], respectively (see also [17] for a more general analysis).

Furthermore, a number of interesting results concerning properties of solutions to chemotaxis-systems have been also attained for a broader class of problems, in which the first equation of (1) reads $u_t = \nabla \cdot (S(u)\nabla u) - \nabla \cdot (T(u)\nabla v)$. Precisely, bounded or unbounded solutions of the corresponding problem is determined by the asymptotic behaviour of the ratio $T(u)/S(u)$, especially in terms of the space dimension; we refer, for instance, to [6] and [40] for the parabolic-elliptic case and to [9, 18, 27, 28, 37] for the parabolic-parabolic case.

As an approach towards the model of self-organizing behaviour of cells populations, it seems coherent to adapt the original Keller-Segel formulation to the case in which the temporal evolution of a cells distribution may be perturbed by the proliferation and the death of the cells themselves. Mathematically it is possible by adding a linear combination of power functions depending on u and, possibly, on $|\nabla u|$ to the first equation of system (1) (see details in [1, 16, 32] for pure chemotaxis-systems, but also in [31] for weakly coupled systems).

Conforming to the previous paragraph, this investigation focuses on fully parabolic chemotaxis-systems which are complemented by *logistic-type* effects. To the best of our knowledge, the following are the most recent and partial results in this regard; under Neumann boundary conditions and in a convex smooth and bounded domain Ω of $\mathbb{R}^n$, $n \geq 1$:

i) For the problem
$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v) + au - bu^2 & x \in \Omega, \ t > 0, \\ v_t = \Delta v - v + u & x \in \Omega, \ t > 0, \end{cases} \tag{2}$$
the existence of global weak solutions is proven for any nonnegative and sufficient regular initial data (u_0, v_0) and arbitrarily small values of $b > 0$. Moreover, if $n = 3$ and a is not too large, these solutions become classical after some time (see [13]).

ii) For the problem
$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (u\nabla v) + g(u) & x \in \Omega, \ t > 0, \\ \tau v_t = \Delta v - v + u & x \in \Omega, \ t > 0, \end{cases} \tag{3}$$
where g generalizes the logistic source in (2), and satisfies $g(0) \geq 0$ and $g(s) \leq a - bs^2$, for $s \geq 0$, and with $a \geq 0$, b, χ, τ positive constants, it is proved in [36] that if b is big enough, for all sufficiently smooth and nonnegative initial data, (u_0, v_0), the problem possesses a unique bounded and global-in-time classical solution. Furthermore, even though [22] yields global classical solutions to (3) for any (not necessarily large) $b > 0$, which remain bounded in a (convex smooth and bounded) domain of $\mathbb{R}^2$, the same conclusion is not clear to occur for $n \geq 3$.

iii) For the same problem (3), but with source term g controlled, respectively from below and above, by $-c_0(s + s^\alpha)$ and $a - bs^\alpha$, for $s \geq 0$, and with some

$\alpha > 1$, $a \geq 0$ and $b, c_0 > 0$, global existence of very weak solutions is attained in [33]. Moreover, for $n = 3$, sufficient conditions on the initial data and the coefficients of the source g which ensure the boundedness of such solutions are discussed in [34].

Additionally, analogous conclusions dealing with parabolic-elliptic versions of models related to (2) or (3) are also available. For instance, in [29] it is proven that weak solutions exist for arbitrary $b > 0$; moreover they are smooth and globally classical if $b > (n-2)/n$. Finally, with source term g as in the above item iii), global existence of very weak solutions and their boundedness and eventual smoothness properties are established in [35].

2. **Objectives and main results.** In agreement with all of the above, this present research is dedicated to the following problem

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + g(u) & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0 \text{ and } v(x,0) = v_0(x) \geq 0 & x \in \Omega, \end{cases} \tag{4}$$

defined in a convex smooth and bounded domain Ω of $\mathbb{R}^n$, $n \geq 1$, $\chi > 0$, and where (u_0, v_0) is a pair of nonnegative functions from $C^0(\bar{\Omega}) \times C^2(\bar{\Omega})$, with $\partial v_0/\partial\nu = 0$ on $\partial\Omega$, $\partial/\partial\nu$ indicating the outward normal derivative. Moreover, the function g belongs to $C^1([0,\infty))$, satisfies $g(0) \geq 0$ and for some $\alpha > 1$ it is such that

$(H1_\alpha)$ $g(s) \leq a - bs^\alpha$, for all $s \geq 0$, with $a \geq 0$ and $b > 0$,
$(H2_\alpha)$ $g(s) \geq -c_0(s + s^\alpha)$, for all $s \geq 0$, with $c_0 > 0$.

Our starting point are the contributions [33] and [34], where these partials results are, respectively, presented:

1) *existence of global very weak solutions*: for any $n \geq 1$ and $\alpha \in (1,2)$ satisfying $\alpha > 2 - \frac{1}{n}$, the global existence of very weak solutions (u, v) to the system is shown for any nonnegative initial data, $(u_0, v_0) \in C^0(\bar{\Omega}) \times C^2(\bar{\Omega})$, and under zero-flux boundary condition on v_0;
2) *boundedness of very weak solutions*: in the most realistic three dimensional setting, these very weak solutions derived in *1)* are bounded. More precisely, if the ratio a/b does not exceed a certain value and the initial data are such that $||u_0||_{L^p(\Omega)}$ and $\|\nabla v_0\|_{L^4(\Omega)}$ are small enough then, for appropriate $9/5 < p < \alpha < 2$, (u, v) is uniformly-in-time bounded in $(0, \infty)$.

A natural and complementary question connected to the above results *1)* and *2)* is to show that singularities of solutions to (4), possibly arising after some finite time, disappear. In other words, our main objective is to analyse whether solutions to (4), with "unclear" behaviour over a certain period, become eventually bounded and smooth beyond some time; additionally, it is also interesting to investigate and characterize their long time behaviour. Specifically, we address the following issues, which are directly related to each other.

- *For any fixed time $\tau > 0$, is it possible to claim that the very weak solutions of* (4) *improve their regularity and become bounded beyond such τ?* According to Theorem 2.1 below, by imposing suitable smallness conditions on u_0 and v_0, measured in proper norms, the very weak solutions are eventually bounded and Hölder continuous provided the ratio a/b does not exceed a certain value.

- *Is it possible to claim that any of the very weak solutions of* (4) *improve their regularity and become bounded beyond some time, regardless the initial sizes of* u_0 *and* v_0*?* Again under smallness assumption on the ratio a/b, this question is positively shown in the forthcoming Theorem 2.2. It is proved that it is always possible to find a $T > 0$ such that the very weak solution are bounded and Hölder continuous beyond T, independently by some norm of the initial data u_0 and v_0; clearly, the time beyond which it occurs depends on such initial norm.
- *Is it possible to characterize the asymptotics of bounded solutions, i.e. their behaviour for* $t \to \infty$*?* If on the one hand *all* the solutions of the Pierre-François Verhulst model converge to the constant steady state K (as shown in §1), then the chemotaxis-diffusion-growth models may lead to a spatially uniform steady state, or to a spatially heterogeneous steady state, as well as irregular spatiotemporal solutions, possibly defined by time-periodic or time-irregular pattern formations (see [2, 23]). The theoretical analysis dealing with the behaviour of the solutions to (4) when the time increases is currently unclear and goes beyond the scope of this paper; here, we present an important number of numerical simulations (in one, two and three dimensions) concerning the long time behaviour of bounded solutions and also the blow-up scenario of unbounded ones (see §6).

Linked to the previous first two questions, these represent exactly our main theoretical assertions:

Theorem 2.1. *Let Ω be a convex smooth and bounded domain of $\mathbb{R}^3$, $\chi > 0$ and $g \in C^1([0,\infty))$, with $g(0) \geq 0$, such that for some $9/5 < \alpha < 2$ both assumptions* $(H1_\alpha)$ *and* $(H2_\alpha)$ *are verified. Then, for any $\tau > 0$ there exists a positive real $\delta(\tau) > 0$ such that if*

$$\max\left\{\left(\frac{a}{b}\right)^{\frac{1}{\alpha}}|\Omega|, ||u_0||_{L^1(\Omega)}, ||v_0||^{\alpha}_{W^{2,\alpha}(\Omega)}\right\} < \delta(\tau), \tag{5}$$

problem (4) *admits a very weak solution, (u, v), which is bounded in $\Omega \times (\tau, \infty)$. Moreover, (u, v) is such that for all $t > \tau$*

$$\|u\|_{C^{2,1}\left(\bar{\Omega}\times[t+3,t+4]\right)} + \|v\|_{C^{2,1}\left(\bar{\Omega}\times[t+3,t+4]\right)} \leq C_\tau, \tag{6}$$

for some $C_\tau > 0$.

Theorem 2.2. *Let Ω be a convex smooth and bounded domain of $\mathbb{R}^3$, $\chi > 0$ and $g \in C^1([0,\infty))$, with $g(0) \geq 0$, such that for some $9/5 < \alpha < 2$ both assumptions* $(H1_\alpha)$ *and* $(H2_\alpha)$ *are verified. Then, there exist positive real numbers $\tilde{\delta}$ and T such that if $a/b < \tilde{\delta}$ problem* (4) *admits a very weak solution, (u, v), which is bounded in $\Omega \times (T, \infty)$. Moreover, (u, v) is such that for all $t > T$*

$$\|u\|_{C^{2,1}\left(\bar{\Omega}\times[t+3,t+4]\right)} + \|v\|_{C^{2,1}\left(\bar{\Omega}\times[t+3,t+4]\right)} \leq C_T, \tag{7}$$

for some $C_T > 0$.

3. **Preliminaries and definition of suitable solution.** The following results are herein formulated according to our purposes, and they are used through the paper to prove the claimed theorems. For the sake of clarity, we also close the section with the definition of very weak solutions to (4).

Lemma 3.1. *(The Jensen inequality) Let f be a nonnegative function belonging to $L^1([t_1,t_2])$, with $t_1 < t_2$. Then, for any concave function $\varphi:\mathbb{R}\to\mathbb{R}$ this inequality holds*

$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\varphi(f(t))dt \le \varphi\left(\frac{1}{t_2-t_1}\int_{t_1}^{t_2} f(t)dt\right). \tag{8}$$

Proof. See Theorem 3.4 of [25]. □

Now we collect some properties regarding the Neumann heat semigroup $\left(e^{t\Delta}\right)_{t\ge 0}$ in $\Omega\subset\mathbb{R}^n$, $n\ge 1$.

Lemma 3.2. *For $p\in(1,\infty)$, let us consider the operator $-\Delta$ defined in the domain $D(-\Delta):=\left\{f\in W^{2,p}(\Omega)\;\middle|\;\frac{\partial f}{\partial\nu}|_{\partial\Omega}=0\right\}$. Then the operator $(-\Delta+1)$ is sectorial in $L^p(\Omega)$ and for any $\rho\ge 0$ possesses fractional powers $(-\Delta+1)^\rho$, with dense domain $D((-\Delta+1)^\rho)$. Moreover, there exist positive constants C_S and μ_1 such that*

- *for all $t>0$ and $p\le q<\infty$ the following L^p-L^q estimates hold*

$$\|(-\Delta+1)^\rho e^{t(\Delta-1)}f\|_{L^q(\Omega)}\le C_S t^{-\rho-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}e^{-\mu_1 t}\|f\|_{L^p(\Omega)} \quad \forall\, f\in L^p(\Omega), \tag{9}$$

- *for all $t>0$ and $1<p\le q<\infty$ the operator $e^{t\Delta}\nabla\cdot$ possesses a uniquely determined extension to an operator from $L^p(\Omega)$ to $L^q(\Omega)$ obeying this L^p-L^q estimate*

$$\|e^{t\Delta}\nabla\cdot\boldsymbol{f}\|_{L^q(\Omega)}\le C_S\left(1+t^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\right)e^{-\mu_1 t}\|\boldsymbol{f}\|_{L^p(\Omega)} \quad \forall\,\boldsymbol{f}\in L^p(\Omega;\mathbb{R}^n), \tag{10}$$

- *for all $t>0$ and $1\le p\le q\le\infty$ this L^p-L^q estimate holds*

$$\|\nabla e^{t\Delta}f\|_{L^q(\Omega)}\le C_S\left(1+t^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\right)e^{-\mu_1 t}\|f\|_{L^p(\Omega)} \quad \forall\, f\in L^p(\Omega). \tag{11}$$

Proof. See Section 2. of [9] and Lemma 1.3 of [38]. □

We also make use of these elementary results.

Lemma 3.3. *Let y be a positive real number satisfying $y\le c(y^l+1)$ for some $c>0$ and $0<l<1$. Then $y\le\max\{1,(2c)^{\frac{1}{1-l}}\}$.*

Proof. Let us suppose $y\ge 1$ (if $y\le 1$ there is nothing left to show): $y^l\ge 1$ so that $y\le c(y^l+1)\le 2cy^l$ and hence $y\le(2c)^{\frac{1}{1-l}}$. □

Lemma 3.4. *For any $A,B\ge 0$ and $\mathfrak{p}>1$ we have*

$$A^{\frac{1}{\mathfrak{p}}}+B^{\frac{1}{\mathfrak{p}}}\ge(A+B)^{\frac{1}{\mathfrak{p}}}. \tag{12}$$

Proof. Let us prove relation (12) for both A and B strictly positive; otherwise the inequality is obvious. If we set $t=A/B$, it is sufficient to show that the function $h(t)=(t^{\frac{1}{\mathfrak{p}}}+1)-(t+1)^{\frac{1}{\mathfrak{p}}}$ is increasing for $t>0$ and $\mathfrak{p}>1$. We have $h'(t)=\frac{1}{\mathfrak{p}}t^{\frac{1}{\mathfrak{p}}-1}-\frac{1}{\mathfrak{p}}(t+1)^{\frac{1}{\mathfrak{p}}-1}$; since $\mathfrak{p}>1$ implies $(t+1)^{1-\frac{1}{\mathfrak{p}}}>t^{1-\frac{1}{\mathfrak{p}}}$, we conclude that $h'(t)>0$ for all $t>0$. □

Finally, we give these definitions.

Definition 3.5. Let $T>0$. A pair (u,v) of nonnegative functions

$$u\in L^1(\Omega\times(0,T)),\ v\in L^1(0,T;W^{1,1}(\Omega)),$$

is said to be a very weak subsolution of (4) in $\Omega\times(0,T)$ if

$$g(u) \textit{ and } u\nabla v \textit{ belong to } L^1(\Omega\times(0,T)),$$

if

$$-\int_0^T\int_\Omega u\varphi_t-\int_\Omega u_0\varphi(\cdot,0)\le\int_0^T\int_\Omega u\Delta\varphi+\chi\int_0^T\int_\Omega u\nabla v\cdot\nabla\varphi+\int_0^T\int_\Omega g(u)\varphi,$$

for all nonnegative $\varphi\in C_0^\infty(\bar\Omega\times[0,T))$, such that $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega\times(0,T)$, and if

$$-\int_0^T\int_\Omega v\psi_t-\int_\Omega v_0\psi(\cdot,0)+\int_0^T\int_\Omega\nabla v\cdot\nabla\psi+\int_0^T\int_\Omega v\psi=\int_0^T\int_\Omega u\psi,$$

for all nonnegative $\psi\in C_0^\infty\left(\bar\Omega\times[0,T)\right)$.

Definition 3.6. Let $T>0$ and $\gamma\in(0,1)$. A pair (u,v) of nonnegative functions

$$u\in L^1(0,T;L^\gamma(\Omega)),\ v\in L^2(0,T;W^{1,2}(\Omega)),$$

is said to be a weak γ-entropy supersolution of (4) in $\Omega\times(0,T)$ if

$$\begin{cases}u^{\gamma-2}|\nabla u|^2,u^{\gamma-1}g(u)\ \textit{and}\ u^\gamma\nabla v\ \textit{belong to}\ L^1(\Omega\times(0,T)),\\ u^{\gamma-1}|\nabla u|\ \textit{belongs to}\ L^2(\Omega\times(0,T)),\end{cases}$$

if

$$-\int_0^T\int_\Omega u^\gamma\varphi_t-\int_\Omega u_0^\gamma\varphi(\cdot,0)\ge\gamma(1-\gamma)\int_0^T\int_\Omega u^{\gamma-2}|\nabla u|^2\varphi+\int_0^T\int_\Omega u^\gamma\Delta\varphi$$
$$+\chi\gamma\int_0^T\int_\Omega u^\gamma\nabla v\cdot\nabla\varphi+\gamma\int_0^T\int_\Omega u^{\gamma-1}g(u)\varphi+\chi(\gamma-1)\int_0^T\int_\Omega\varphi u^{\gamma-1}\nabla u\cdot\nabla v,$$

for all nonnegative $\varphi\in C_0^\infty(\bar\Omega\times[0,T))$, such that $\partial\varphi/\partial\nu=0$ on $\partial\Omega\times(0,T)$, and if

$$-\int_0^T\int_\Omega v\psi_t-\int_\Omega v_0\psi(\cdot,0)+\int_0^T\int_\Omega\nabla v\cdot\nabla\psi+\int_0^T\int_\Omega v\psi=\int_0^T\int_\Omega u\psi,$$

for all nonnegative $\psi\in C_0^\infty(\bar\Omega\times[0,T))$.

Definition 3.7. Let $T>0$. A pair (u,v) of functions is called *very weak solution* for problem (4) in $\Omega\times(0,T)$ if it is both a very weak subsolution and a weak γ-entropy supersolution of (4) in $\Omega\times(0,T)$, in the sense of Definitions 3.5 and 3.6.

A global very weak solution of (4) is a pair (u,v) of functions defined in $\Omega\times(0,\infty)$ which is a very weak solution of (4) in $\Omega\times(0,T)$ for all $T>0$.

4. **Approximate problem and existence of very weak solutions.** In preparation for the main estimates, by means of a parameter $\varepsilon\in(0,1)$, we define a perturbed chemotaxis-system, properly constructed to approximate and hence solve and analyze the original problem (4). Precisely, we rely on these specific results.

Proposition 1. *Let Ω be a convex smooth and bounded domain of $\mathbb{R}^n$, with $n\ge1$, and $\beta>n+2$. Moreover, for some $\alpha>1$, let us assume $g\in C^1([0,\infty))$, such that $g(0)\ge0$, and satisfies $(H1_\alpha)$. Then, for any $\varepsilon\in(0,1)$ and nonnegative functions $u_0\in C^0(\bar\Omega)$ and $v_0\in C^2(\bar\Omega)$, with $\partial v_0/\partial\nu=0$ on $\partial\Omega$, the following problem*

$$\begin{cases}u_{\varepsilon t}=\Delta u_\varepsilon-\chi\nabla\cdot(u_\varepsilon\nabla v_\varepsilon)+g(u_\varepsilon)-\varepsilon u_\varepsilon^\beta & x\in\Omega,t>0,\\ v_{\varepsilon t}=\Delta v_\varepsilon-v_\varepsilon+u_\varepsilon & x\in\Omega,t>0,\\ \frac{\partial u_\varepsilon}{\partial\nu}=\frac{\partial v_\varepsilon}{\partial\nu}=0 & x\in\partial\Omega,t>0,\\ u_\varepsilon(x,0)=u_0(x)\ge0\ \textit{and}\ v_\varepsilon(x,0)=v_0(x)\ge0 & x\in\Omega,\end{cases}\tag{13}$$

admits a unique classical and global solution $(u_\varepsilon,v_\varepsilon)$, for which both u_ε and v_ε are nonnegative and bounded in $\Omega\times(0,\infty)$.

Proof. See Proposition 1 of [33]. □

We also present this result concerning both the existence and boundedness properties of global very weak solutions to system (4).

Proposition 2. *Let Ω be a convex smooth and bounded domain of $\mathbb{R}^n$, with $n \geq 1$. For some $\alpha \in (1,2)$, with $\alpha > 2 - \frac{1}{n}$, let us assume $g \in C^1([0,\infty))$, such that $g(0) \geq 0$, and verifying both assumptions ($H1_\alpha$) and ($H2_\alpha$). Then for all nonnegative functions $(u_0(x), v_0(x)) \in C^0(\bar{\Omega}) \times C^2(\bar{\Omega})$, with $\frac{\partial v_0}{\partial \nu} = 0$ on $\partial\Omega$, problem (4) admits at least one global very weak solution (u,v), according to Definition 3.7. More precisely, this solution is the limit of a sequence of globally bounded couples of functions $(u_\varepsilon, v_\varepsilon)$ which classically solve the approximate problem (13), in the sense that, for any $t > 0$ and $1 < p < \alpha < 2$, as $\varepsilon \to 0$ we have*

$$\begin{cases} u_\varepsilon \to u & \text{in } L^p(\Omega \times (0,t)), \\ u_\varepsilon^\gamma \rightharpoonup u^\gamma & \text{in } L^2(0,t; W^{1,2}(\Omega)), \\ u_\varepsilon \rightharpoonup u & \text{in } L^\alpha(\Omega \times (0,t)), \\ v_\varepsilon \to v & \text{in } L^{\frac{\alpha}{\alpha-1}}(0,t; W^{1,\frac{\alpha}{\alpha-1}}(\Omega)). \end{cases}$$

Additionally, for $n = 3$, there exists a positive real $\delta > 0$ such that if a and b are such that $a/b < \delta$ then for all $9/5 < p < \alpha < 2$ it is possible to find a $\lambda > 0$ with the following property: if $(u_0(x), v_0(x)) \in C^0(\bar{\Omega}) \times C^2(\bar{\Omega})$ are also such that $\|u_0\|_{L^p(\Omega)} < \left(\frac{p}{2}\right)^{\frac{1}{p}} \lambda$ and $\|\nabla v_0\|_{L^4(\Omega)} < \lambda^{\frac{p}{4}}$, the global very weak solution (u,v) is uniformly-in-time bounded in $(0,\infty)$.

Proof. For the existence question, see the proof of Theorem 2.1 in [33]. For the boundedness, Theorem 2.1 in [34]. □

Remark 1. By taking into account the regularity properties that the limit (u,v) provided by Proposition 2 inherits from the sequence $(u_\varepsilon, v_\varepsilon)$, we have that for $n = 1$ the Sobolev inequalities imply that both $W^{1,2}(\Omega)$ and $W^{1,\frac{\alpha}{\alpha-1}}(\Omega)$ are compactly embedded in $C(\bar{\Omega})$. As a consequence, in the one dimensional setting, for any $t > 0$ the solution $(u(\cdot,t), v(\cdot,t))$ to problem (4) belongs to $L^\infty(\Omega)$ for any choice of the initial distribution (u_0, v_0) and the parameters a and b which determine the behaviour of function g.

5. A priori estimates and proof of the theorems.

In this section our principal objective is to infer some ε-independent and uniform-in-time estimates for both u_ε and v_ε components of the solution $(u_\varepsilon, v_\varepsilon)$ to (13). In this sense, the following lemma includes some inequalities which are strongly employed with this aim.

Lemma 5.1. *Let Ω be a convex smooth and bounded domain of $\mathbb{R}^3$ and $(u_\varepsilon, v_\varepsilon)$ the solution of problem (13) provided by Proposition 1. Then for any $9/5 < p < \alpha < 2$ and $t_2 > t_1 \geq 0$, $(u_\varepsilon, v_\varepsilon)$ verifies*

$$\begin{cases} \displaystyle\int_\Omega u_\varepsilon(\cdot,t) \leq m + e^{-\alpha a^{\frac{\alpha-1}{\alpha}} b^{\frac{1}{\alpha}}(t-t_1)} \left(\int_\Omega u_\varepsilon(\cdot,t_1) - m \right) & \forall\, t \geq t_1, \\ \displaystyle\int_\Omega u_\varepsilon(\cdot,t) \leq M_\varepsilon(t_1) & \forall\, t \geq t_1, \\ \displaystyle\int_{t_1}^{t_2} \int_\Omega u_\varepsilon^\alpha \leq \frac{a|\Omega|(t_2-t_1) + M_\varepsilon(t_1)}{b} & \forall\, t_2 > t_1, \end{cases}$$

as well,

$$\begin{aligned}\int_{t_1}^{t_2}\left[\frac{1}{p}\int_\Omega u_\varepsilon^p+\left(\frac{1}{2}\int_\Omega|\nabla v_\varepsilon|^4\right)^{\frac{\alpha}{4}}\right]\leq&\frac{t_2-t_1}{p}|\Omega|\left(\frac{a}{b}\right)^{\frac{p}{\alpha}}\\&+\frac{((t_2-t_1)|\Omega|)^{\frac{\alpha-p}{\alpha}}}{pb^{\frac{p}{\alpha}}}M_\varepsilon(t_1)^{\frac{p}{\alpha}}\\&+C_\Omega M_\varepsilon(t_1)\frac{b+1}{b}+C_\Omega\frac{a|\Omega|(t_2-t_1)}{b},\end{aligned}\tag{15}$$

where $M_\varepsilon(t_1)=\max\Big\{m,||u_\varepsilon(\cdot,t_1)||_{L^1(\Omega)},||v_\varepsilon(\cdot,t_1)||^\alpha_{W^{2,\alpha}(\Omega)}\Big\}$, *with* $m=(a/b)^{\frac{1}{\alpha}}|\Omega|$, C_Ω *being a positive constant not depending on* ε.

Proof. Let be $t_1\geq 0$; an integration over Ω of the first equation in (13) yields

$$\frac{d}{dt}\int_\Omega u_\varepsilon(\cdot,t)=\int_\Omega g(u_\varepsilon)-\varepsilon\int_\Omega u_\varepsilon^\beta\leq a|\Omega|-b\int_\Omega u_\varepsilon^\alpha-\varepsilon\int_\Omega u_\varepsilon^\beta,\tag{16}$$

where also $(H1_\alpha)$ and the Neumann boundary condition on u_ε were considered. Another integration over (t_1,t) of this inequality gives

$$b\int_{t_1}^t\int_\Omega u_\varepsilon^\alpha+\varepsilon\int_{t_1}^t\int_\Omega u_\varepsilon^\beta\leq a|\Omega|(t-t_1)-\int_\Omega u_\varepsilon dx+\int_\Omega u_\varepsilon(\cdot,t_1)\quad\forall\, t\geq t_1.\tag{17}$$

Moreover, for any $\alpha>1$ the Hölder inequality yields $\int_\Omega u_\varepsilon\leq|\Omega|^{\frac{\alpha-1}{\alpha}}(\int_\Omega u_\varepsilon^\alpha)^{\frac{1}{\alpha}}$; hence for all $t>0$ relation (16) implies

$$\frac{d}{dt}\int_\Omega u_\varepsilon(\cdot,t)\leq a|\Omega|-b|\Omega|^{1-\alpha}\left(\int_\Omega u_\varepsilon\right)^\alpha.$$

By setting $z(t)=\int_\Omega u_\varepsilon(\cdot,t)-m$, with $m=(a/b)^{\frac{1}{\alpha}}|\Omega|$, we have

$$\begin{aligned}z'&\leq a|\Omega|-b|\Omega|^{1-\alpha}(m+z)^\alpha\leq bm^\alpha|\Omega|^{1-\alpha}-b|\Omega|^{1-\alpha}m^\alpha\left(1+\frac{z}{m}\right)^\alpha\\&\leq-\alpha a^{\frac{\alpha-1}{\alpha}}b^{\frac{1}{\alpha}}z\quad\forall\, t>0,\end{aligned}$$

where we have used the relation $(1+A)^\alpha\geq 1+\alpha A$, with $A\geq 0$.

By complementing this inequality with the initial condition $z(t_1)=\int_\Omega u_\varepsilon(\cdot,t_1)-m$, it is seen that

$$\int_\Omega u_\varepsilon(\cdot,t)\leq m+e^{-\alpha a^{\frac{\alpha-1}{\alpha}}b^{\frac{1}{\alpha}}(t-t_1)}\left(\int_\Omega u_\varepsilon(\cdot,t_1)-m\right)\quad\forall\, t\geq t_1,$$

which implies (14a) and (14b). Hence, (14c) results from (17) and (14a) and the non negativity of $\int_\Omega u_\varepsilon$.

Now, for any $t_2>t_1$, let us independently estimate the terms $\int_{t_1}^{t_2}\frac{1}{p}\int_\Omega u_\varepsilon^p$ and $\int_{t_1}^{t_2}\left(\frac{1}{2}\int_\Omega|\nabla v_\varepsilon|^4\right)^{\frac{\alpha}{4}}$. Since $p/\alpha<1$, then the Hölder inequality allows us to write

$$\int_{t_1}^{t_2}\frac{1}{p}\int_\Omega u_\varepsilon^p\leq\int_{t_1}^{t_2}\frac{1}{p}\left(\int_\Omega u_\varepsilon^\alpha\right)^{\frac{p}{\alpha}}|\Omega|^{\frac{\alpha-p}{\alpha}}.\tag{18}$$

In addition, since $p/\alpha < 1$, the function $t \mapsto \left(\int_\Omega u_\varepsilon^\alpha\right)^{\frac{p}{\alpha}}$ is concave; an application of the Jensen inequality (8) in the interval $[t_1, t_2]$ provides

$$\begin{aligned}\int_{t_1}^{t_2}\left(\int_\Omega u_\varepsilon^\alpha\right)^{\frac{p}{\alpha}} &= (t_2-t_1)\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left(\int_\Omega u_\varepsilon^\alpha\right)^{\frac{p}{\alpha}}\\ &\le (t_2-t_1)\frac{1}{(t_2-t_1)^{\frac{p}{\alpha}}}\left(\int_{t_1}^{t_2}\int_\Omega u_\varepsilon^\alpha\right)^{\frac{p}{\alpha}}\\ &\le (t_2-t_1)^{\frac{\alpha-p}{\alpha}}\left[\left(\frac{a|\Omega|(t_2-t_1)}{b}\right)^{\frac{p}{\alpha}}+\left(\frac{M_\varepsilon(t_1)}{b}\right)^{\frac{p}{\alpha}}\right],\end{aligned} \tag{19}$$

where we have also used relations (14c) and (12) with $\mathfrak{p} = \alpha/p > 1$.

On the other hand, since $9/5 < \alpha < 2$, the Sobolev embedding $W^{2,\alpha} \hookrightarrow W^{1,4}$ infers a positive constant $\hat{C}$ such that

$$\left(\frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{\alpha}{4}} \le \hat{C}\int_\Omega ({v_\varepsilon}^\alpha + |\nabla v_\varepsilon|^\alpha + |\Delta v_\varepsilon|^\alpha). \tag{20}$$

The linear (Cauchy) problem extracted from the second equation of (13), that is $v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon$, $x \in \Omega, t > 0$, with $\partial v_\varepsilon/\partial \nu = 0, x \in \partial\Omega, t > 0$ and $v_\varepsilon(x, 0) = v_0(x)$, has a unique classical and global solution, so that we can apply relation (12) of Proposition 2 of [33] in the interval $[t_1, t_2]$. Precisely, in line with the nomenclature used in such a proposition, setting $p = q = \alpha > 1$, using that for any $A, B \ge 0$ the relation $(A + B)^\alpha \le 2^\alpha(A^\alpha + B^\alpha)$ holds and estimating the interpolation norm $\|v_\varepsilon(\cdot, t_1)\|_{\alpha,1-\frac{1}{\alpha}}$ with $\|v_\varepsilon(\cdot, t_1)\|_{W^{2,\alpha}(\Omega)}$, we can find a positive constant $\tilde{C}$, independent on ε, such that

$$\int_{t_1}^{t_2}\int_\Omega ({v_\varepsilon}^\alpha + |\nabla v_\varepsilon|^\alpha + |\Delta v_\varepsilon|^\alpha) \le \tilde{C}\left(||v_\varepsilon(\cdot, t_1)||^\alpha_{W^{2,\alpha}(\Omega)} + \int_{t_1}^{t_2}\int_\Omega u_\varepsilon^\alpha\right). \tag{21}$$

In this way, by integrating (20) between t_1 and t_2, and in view of (14c), relation (21) infers

$$\int_{t_1}^{t_2}\left(\frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{\alpha}{4}} \le C_\Omega M_\varepsilon(t_1) + C_\Omega\frac{a|\Omega|(t_2-t_1) + M_\varepsilon(t_1)}{b}, \tag{22}$$

where $C_\Omega = \hat{C}\tilde{C}$. The sum of the expressions (18) and (22), once the bound (19) is also taken into account, concludes the proof. □

The following lemma provides, for some $p > 1$ and beyond some time, an ε-independent boundedness for $\|u_\varepsilon\|_{L^p(\Omega)}$; this estimate is attained by establishing an absorptive differential inequality for the time dependent energy function $\Phi_\varepsilon(t) = \frac{1}{p}\int_\Omega u_\varepsilon^p + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^4$, associated to $(u_\varepsilon, v_\varepsilon)$.

Lemma 5.2. *Let Ω be a convex smooth and bounded domain of $\mathbb{R}^3$ and $(u_\varepsilon, v_\varepsilon)$ the solution of problem (13) provided by Proposition 1. Then, for any $9/5 < p < \alpha < 2$ and $\tau > 0$, it is possible to find two positive and ε-independent real numbers $\delta(\tau)$ and $C(\tau)$ with the property that if there exists $t_1 \ge 0$ such that for all $\varepsilon \in (0, 1)$*

$$M_\varepsilon(t_1) = \max\left\{\left(\frac{a}{b}\right)^{\frac{1}{\alpha}}|\Omega|, ||u_\varepsilon(\cdot, t_1)||_{L^1(\Omega)}, ||v_\varepsilon(\cdot, t_1)||^\alpha_{W^{2,\alpha}(\Omega)}\right\} < \delta(\tau), \tag{23}$$

then

$$||u_\varepsilon(\cdot, t)||_{L^p(\Omega)} \le C(\tau) \quad \textit{for all } t \ge t_1 + \frac{\tau}{2}.$$

Proof. For any $t_1 \geq 0$, let us define the function

$$\Phi_\varepsilon(t) := \frac{1}{p}\int_\Omega u_\varepsilon^p + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^4 \quad \text{for all } t \geq t_1.$$

By retracing the proof of Lemma 5.2 in [34], and using the same constants therein introduced, one can show that Φ_ε satisfies this absorptive differential inequality for any $t > t_1$

$$\Phi_\varepsilon'(t) \leq -c_6\Phi_\varepsilon(t) + c_7\Phi_\varepsilon(t)^{\frac{3}{2}} + c_8\Phi_\varepsilon(t)^3 + c_9\Phi_\varepsilon(t)^k + c_5(M_\varepsilon(t_1)^{p-1} + M_\varepsilon(t_1)^3),$$

with $k = 3\,(p-1)\,/\,(5p-9) > 3$. Let us assign for any $\xi \geq 0$ and $M_\varepsilon := M_\varepsilon(t_1) \geq 0$ the function

$$\Theta_{M_\varepsilon}(\xi) := -c_6\xi + c_7\xi^{\frac{3}{2}} + c_8\xi^3 + c_9\xi^k + c_5(M_\varepsilon^{p-1} + M_\varepsilon^3);$$

it is seen that $\Theta_0(\xi) = -c_6\xi + c_7\xi^{\frac{3}{2}} + c_8\xi^3 + c_9\xi^k$ is such that

$$\Theta_0(0) = 0 \quad \text{and} \quad \lim_{\xi\to+\infty}\Theta_0(\xi) = +\infty,$$

as well as

$$\Theta_0'(0) = -c_6 < 0, \quad \lim_{\xi\to+\infty}\Theta_0'(\xi) = +\infty \quad \text{and} \quad \Theta_0''(\xi) > 0.$$

Subsequently, $\Theta_0(\xi)$ admits exactly two roots: 0 and another one, which is strictly positive. Moreover, since $\Theta_{M_\varepsilon}(\xi)$ is obtained by translating $\Theta_0(\xi)$ in the positive direction of the Θ-axis by the amount of $c_5(M_\varepsilon^{p-1} + M_\varepsilon^3)$, let us consider the set

$$S_{M_\varepsilon} = \{\xi \geq 0 \text{ such that } \Theta_{M_\varepsilon}(\xi) = 0\}.$$

Through a continuous dependence argument, regardless the size of the positive root of $\Theta_0(\xi)$, it is always possible to find a value $M_{\lim} > 0$ with the property that the equation $\Theta_{M_{\lim}}(\xi) = 0$ possesses two positive roots (let us say ξ_1 and ξ_2, with $\xi_1 < \xi_2$) such that $\xi_{\lim} = \xi_1 < 1$. In this sense $S_{M_{\lim}} \equiv \{\xi_{\lim}, \xi_2\}$. Hence for any $\xi \geq 0$ we have

$$0 < M_\varepsilon < M_{\lim} \Rightarrow \begin{cases} \Theta_0(\xi) < \Theta_{M_\varepsilon}(\xi) < \Theta_{M_{\lim}}(\xi), \\ S_{M_\varepsilon} = \{\xi_{\min}, \xi_{\max}\}, \quad \text{with} \quad \xi_{\min} < \xi_{\lim} < 1. \end{cases} \tag{24}$$

Additionally, $\Phi_\varepsilon(t) \equiv \xi_{\lim}$ satisfies this initial problem

$$\begin{cases} \Phi_\varepsilon'(t) = \Theta_{M_{\lim}}(\Phi_\varepsilon(t)) \quad t > 0, \\ \Phi_\varepsilon(0) = \xi_{\lim}. \end{cases} \tag{25}$$

Successively, for any $\tau > 0$ let us define

$$\delta(\tau) := \min\left\{\begin{matrix} \left(\frac{\xi_{\lim}|\Omega|^{\alpha-1}}{4C_\Omega}\right)^{\frac{1}{\alpha}} \\ \frac{\tau}{2}\left(\frac{p\xi_{\lim}b^{\frac{p}{\alpha}}}{4|\Omega|^{\frac{\alpha-p}{p}}}\right)^{\frac{\alpha}{p}} \\ \frac{\tau b\xi_{\lim}}{8C_\Omega(b+1)} \\ \left(\frac{p\xi_{\lim}|\Omega|^{p-1}}{4}\right)^{\frac{1}{p}} \\ M_{\lim} \end{matrix}\right\}. \tag{26}$$

From now on our aim is to justify the existence of a $\hat{t} \in (t_1, t_1 + \tau/2)$ and show that $\Phi_\varepsilon(t)$ satisfies the initial problem

$$\begin{cases} \Phi_\varepsilon'(t) \le \Theta_{M_\varepsilon}(\Phi_\varepsilon(t)) & t > \hat{t}, \\ \Phi_\varepsilon\left(\hat{t}\right) = \frac{1}{p}\int_\Omega u_\varepsilon^p\left(\cdot, \hat{t}\right) + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon\left(\cdot, \hat{t}\right)|^4 < \xi_{\lim}. \end{cases} \tag{27}$$

Indeed, in view of (26), assumption (23) would imply relations (24), which in turn would allow us to apply an ODE comparison principle to problems (25) and (27) and conclude that $\Phi_\varepsilon(t) \le \xi_{\min}$ for all $t > \hat{t}$, $\xi_{\min}$ being some zero of $\Theta_{M_\varepsilon}(\xi)$, with $0 < \xi_{\min} < \xi_{\lim} < 1$.

The average theorem establishes the existence of a time $\hat{t}$ belonging to $(t_1, t_1 + \frac{\tau}{2})$ such that

$$\frac{1}{p}\int_\Omega u_\varepsilon^p(\cdot, \hat{t}) + \left(\frac{1}{2}\int_\Omega |\nabla v_\varepsilon(\cdot, \hat{t})|^4\right)^{\frac{\alpha}{4}} = \frac{2}{\tau}\int_{t_1}^{t_1+\frac{\tau}{2}} \left[\frac{1}{p}\int_\Omega u_\varepsilon^p + \left(\frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^4\right)^{\frac{\alpha}{4}}\right].$$

In this way, an application of (15) from Lemma 5.1 with $t_2 = t_1 + \frac{\tau}{2}$ yields

$$\begin{aligned} \frac{1}{p}\int_\Omega u_\varepsilon^p(\cdot, \hat{t}) + \left(\frac{1}{2}\int_\Omega |\nabla v_\varepsilon(\cdot, \hat{t})|^4\right)^{\frac{\alpha}{4}} \le & \frac{|\Omega|}{p}\left(\frac{a}{b}\right)^{\frac{p}{\alpha}} + \left(\frac{\tau}{2}\right)^{\frac{-p}{\alpha}} \frac{|\Omega|^{\frac{\alpha-p}{\alpha}} M_\varepsilon(t_1)^{\frac{p}{\alpha}}}{p b^{\frac{p}{\alpha}}} \\ & + \frac{2}{\tau} C_\Omega M_\varepsilon(t_1)\left(\frac{b+1}{b}\right) + |\Omega| C_\Omega \frac{a}{b}. \end{aligned}$$

Through (23) and (26) we derive that

$$\frac{1}{p}\int_\Omega u_\varepsilon^p(\cdot, \hat{t}) + \left(\frac{1}{2}\int_\Omega |\nabla v_\varepsilon(\cdot, \hat{t})|^4\right)^{\frac{\alpha}{4}} \le \xi_{\lim} < 1,$$

so that, in particular, $\left(\frac{1}{2}\int_\Omega |\nabla v_\varepsilon(\cdot, \hat{t})|^4\right)^{\frac{\alpha}{4}} < 1$. Subsequently, in view of $\alpha/4 < 1$, we finally get

$$\frac{1}{p}\int_\Omega u_\varepsilon^p(\cdot, \hat{t}) + \frac{1}{2}\int_\Omega |\nabla v_\varepsilon(\cdot, \hat{t})|^4 \le \frac{1}{p}\int_\Omega u_\varepsilon^p(\cdot, \hat{t}) + \left(\frac{1}{2}\int_\Omega |\nabla v_\varepsilon(\cdot, \hat{t})|^4\right)^{\frac{\alpha}{4}} \le \xi_{\lim} < 1.$$

As claimed, this implies that $\Phi_\varepsilon(t) \le \xi_{\min}$ for all $t > \hat{t}$, for some $\xi_{\min} = \xi_{\min}(\tau) < 1$; thereafter, since $||u_\varepsilon||_{L^p(\Omega)} \le p^{\frac{1}{p}}\Phi_\varepsilon(t)^{\frac{1}{p}}$, the proof is complete upon the choice $C(\tau) = (p\xi_{\min})^{\frac{1}{p}}$. □

Now, the next conclusion proves that the boundedness of $||u_\varepsilon||_{L^p(\Omega)}$ is sufficient to show that for some $3 < q < 3p/(p-3)$ the $W^{1,q}(\Omega)$-norm of the component v_ε is controlled by a certain ε-independent and uniform-in-time positive constant; more exactly, we show that if $u_\varepsilon \in L^p(\Omega)$ for some $t \ge t_1$, with $t_1 \ge 0$, than v_ε belongs to some $W^{1,\infty}(\Omega)$ for t beyond t_1. We will use also some ideas presented in [3].

Lemma 5.3. *Let Ω be a convex smooth and bounded domain of $\mathbb{R}^3$ and $(u_\varepsilon, v_\varepsilon)$ the solution of problem (13) provided by Proposition 1. Let also assume that for any $1 < p < 3$ there exists $t_1 \ge 0$ such that*

$$||u_\varepsilon(\cdot, t)||_{L^p(\Omega)} \le C(t_1) \quad \forall t \ge t_1, \tag{28}$$

with $C(t_1)$ a positive constant independent on ε. Then, for all $q > p$, with $3 < q < 3p/(p-3)$, and $\tau^* > 0$ it is possible to find a positive and ε-independent constant $C(\tau^*)$ such that

$$||v_\varepsilon(\cdot, t)||_{W^{1,q}(\Omega)} \le C(\tau^*) \quad \textit{for all } t \ge t_1 + \tau^*. \tag{29}$$

Moreover, there exists also another positive and ε-independent constant $C_\infty(\tau^*)$ such that

$$||u_\varepsilon(\cdot,t)||_{L^\infty(\Omega)} + ||v_\varepsilon(\cdot,t)||_{W^{1,\infty}(\Omega)} \leq C_\infty(\tau^*) \quad \text{for all} \quad t \geq t_1 + 2\tau^*. \tag{30}$$

Proof. For any $t_1 \geq 0$, let us fix $\tau^* > 0$. For the first part of the theorem, we rely on the proof of Lemma 4.1 of [9]. Precisely, adapting their work to our notation, one can see that it is only necessary to check that $C(\tau^*)$, in this present manuscript, is a ε-independent constant. This is justified by the ε-independence of $||u_\varepsilon(\cdot,t)||_{L^p(\Omega)}$ for $t \geq t_1$ and of $||v_\varepsilon(\cdot,t_1)||_{L^1(\Omega)}$. The former, which takes the place of assumption $||u(t)||_{L^\gamma(\Omega)} \leq c_1\, \forall\, t \in [\tau, T_{max})$ appearing in the mentioned Lemma 4.1, is ensured by our hypothesis (28); the latter, corresponding to the boundedness of $||v(\tau)||_{L^1(\Omega)}$ of the same Lemma in [9], is obtained as follows: by integrating over Ω the second equation of (13), and using the boundary conditions, give

$$\frac{d}{dt}\int_\Omega v_\varepsilon = -\int_\Omega v_\varepsilon + \int_\Omega u_\varepsilon.$$

By considering expression (14b) applied with $t_1 = 0$, which infers $\int_\Omega u_\varepsilon(\cdot,t) \leq \max\{m, \|u_0\|_{L^1(\Omega)}, \|v_0\|^\alpha_{W^{2,\alpha}(\Omega)}\} =: M$ for all $t > 0$, the solution of the previous ODE allows us to obtain

$$\begin{aligned}\int_\Omega v_\varepsilon(\cdot,t) = \int_\Omega u_\varepsilon + e^{-t}\left(\int_\Omega v_0(x)dx - \int_\Omega u_\varepsilon\right) &\leq M + e^{-t}\left(\int_\Omega v_0(x)dx - \int_\Omega u_\varepsilon\right)\\ &\leq \max\left\{M, \|v_0\|_{L^1(\Omega)}\right\} \quad \forall\, t \geq 0;\end{aligned}$$

hence, we also have

$$||v_\varepsilon(\cdot,t_1)||_{L^1(\Omega)} \leq \max\{M, \|u_0\|_{L^1(\Omega)}\} \quad \forall\, t_1 \geq 0.$$

Now, assumption $(H1_\alpha)$ ensures that for any $x \in \Omega$ and $t > 0$ he following relation holds,

$$u_{\varepsilon t} \leq \Delta u_\varepsilon - \chi\nabla\cdot(u_\varepsilon \nabla v_\varepsilon) + a.$$

Hence, for any $t \geq t_1 + \tau^*$, we set $t_0 := t - \tau^*$, so that the representation formula for u_ε yields

$$\begin{aligned}u_\varepsilon(\cdot,t) \leq e^{(t-t_0)\Delta}u_\varepsilon(\cdot,t_0) - \chi\int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon(\cdot,s)\nabla v_\varepsilon(\cdot,s))ds\\ + \int_{t_0}^t e^{(t-s)\Delta}a ds =: u_{\varepsilon 1}(\cdot,t) + u_{\varepsilon 2}(\cdot,t) + u_{\varepsilon 3}(\cdot,t).\end{aligned} \tag{31}$$

Since $t - t_0 = \tau^*$, and $t_0 \geq t_1$, an application of (9) infers this estimate

$$\|u_{\varepsilon 1}(\cdot,t)\|_{L^\infty(\Omega)} \leq e^{-\tau^*}C_S(\tau^*)^{\frac{-3}{2}}\|u_\varepsilon(\cdot,t_0)\|_{L^1(\Omega)} \leq c_0(\tau^*), \tag{32}$$

with $c_0(\tau^*) = e^{-\tau^*}C_S M(\tau^*)^{\frac{-3}{2}}$ and where, as for a previous step, we have introduced $M = \max\{m, \|u_0\|_{L^1(\Omega)}, \|v_0\|^\alpha_{W^{2,\alpha}(\Omega)}\}$ and used that $\|u_\varepsilon(\cdot,t_0)\|_{L^1(\Omega)} \leq M$, exactly in view of expression (14b) with the choice $t_1 = 0$.

In addition, for all $t \geq t_1 + \tau^*$, we have

$$\|u_{\varepsilon 3}(\cdot,t)\|_{L^\infty(\Omega)} \leq \int_{t_0}^t \|e^{(t-s)\Delta}a\|_{L^\infty(\Omega)}ds = \tau^* a. \tag{33}$$

Furthermore, for any $3 < r < q$ we apply (10) and arrive at, that for $t \geq t_1 + \tau^*$,

$$
\begin{aligned}
\|u_{\varepsilon 2}(\cdot,t)\|_{L^\infty(\Omega)} &\leq \chi \int_{t_0}^t \|e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon(\cdot,s)\nabla v_\varepsilon(\cdot,s))\|_{L^\infty(\Omega)} ds \\
&\leq \chi C_S \int_{t_0}^t (1+(t-s)^{-\frac{1}{2}-\frac{3}{2r}})e^{-\mu_1(t-s)}\|u_\varepsilon(\cdot,s)\nabla v_\varepsilon(\cdot,s)\|_{L^r(\Omega)} ds.
\end{aligned} \tag{34}
$$

Now, for any given $t' > t_1 + \tau^*$ we consider the ε-sequence defined by

$$
A_\varepsilon(t') := \sup_{t\in(t_1+\tau^*,t')} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}, \tag{35}
$$

which is bounded in view of the properties of u_ε. Hence, the Hölder inequality, assumption (28) and estimate (29) provide for all $s \in (t_1+\tau^*, t')$

$$
\begin{aligned}
\|u_\varepsilon(\cdot,s)\nabla v_\varepsilon(\cdot,s)\|_{L^r(\Omega)} &\leq \|u_\varepsilon(\cdot,s)\|_{L^{\frac{rq}{q-r}}(\Omega)} \|\nabla v_\varepsilon(\cdot,s)\|_{L^q(\Omega)} \\
&\leq \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}^{1-\frac{\alpha(q-r)}{rq}} \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)}^{\frac{\alpha(q-r)}{rq}} \|\nabla v_\varepsilon(\cdot,s)\|_{L^q(\Omega)} \\
&\leq (A_\varepsilon(t'))^{1-\frac{\alpha(q-r)}{rq}} C(t_1)^{\frac{\alpha(q-r)}{rq}} C(\tau^*) = (c_2(\tau^*))A_\varepsilon^l(t'),
\end{aligned}
$$

with $c_2(\tau^*) = C(t_1)^{\frac{\alpha(q-r)}{rq}} C(\tau^*)$ and $0 < l = 1-\alpha(q-r)/rq < 1$. Further, by defining $t - t_0 = \tau^*$ and by using in (34) the estimate for the cross-diffusive term $u_\varepsilon(\cdot,s)\nabla v_\varepsilon(\cdot,s)$, then by direct integration we have that for all $t \geq t_1+\tau^*$,

$$
\begin{aligned}
\|u_{\varepsilon 2}(\cdot,t)\|_{L^\infty(\Omega)} &\leq \chi C_S \frac{1}{\mu_1}\left(1-e^{-\mu_1\tau^*}\right) + \mu_1^{-\frac{1}{2}+\frac{3}{2r}}\Gamma\left(\frac{1}{2}-\frac{3}{2r}\right) c_2(\tau^*)A_\varepsilon^l(t') \\
&=: c_3(\tau^*)A_\varepsilon^l(t'),
\end{aligned} \tag{36}
$$

where the Gamma function, Γ, has also been employed.

From expression (31), by collecting (32), (33) and (36) we infer

$$
\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq (c_4(\tau^*))(A_\varepsilon^l(t')+1) \quad \text{for all} \quad t \in (t_1+\tau^*, t'),
$$

where $c_4(\tau^*) = \max\{c_0(\tau^*) + \tau^* a, c_3(\tau^*)\}$. Therefore recalling (35)

$$
\sup_{t\in(t_1+\tau^*,t')} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} := A_\varepsilon(t') \leq c_4(\tau^*)(A_\varepsilon^l(t')+1) \quad \text{for all} \quad t' > t_1+\tau^*,
$$

which through Lemma 3.3 yields this uniform ε-independent bound for u_ε:

$$
\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \max\left\{1, (2c_4(\tau^*))^{\frac{1}{1-l}}\right\} =: C_1(\tau^*) \quad \text{for all} \quad t \geq t_1+\tau^*. \tag{37}
$$

Now (recall $\Omega \subset \mathbb{R}^3$), since $q > 3$ implies $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$, inequality (29) infers some positive constant $C_2(\tau^*)$ such that for any $\varepsilon \in (0,1)$

$$
\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq C_2(\tau^*) \quad \forall\, t \geq t_1+\tau^*. \tag{38}
$$

Additionally, for any $t \geq t_1 + 2\tau^*$, boundedness of ∇v_ε can be achieved by applying to both sides of the representation formula for v_ε,

$$
v_\varepsilon(\cdot,t) = e^{(t-t_0)(\Delta-1)}v_\varepsilon(\cdot,t_0) + \int_{t_0}^t e^{(t-s)(\Delta-1)}u_\varepsilon(\cdot,s)ds \quad \text{for all} \quad t \geq t_0,
$$

the gradient operator ∇; we obtain, again for $t_0 = t - \tau^*$, through estimate (11), the support of expressions (37) and (38), and the fact that $t_0 \geq t_1 + \tau^*$

$$\begin{aligned}
\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq & \|\nabla e^{(t-t_0)(\Delta-1)} v_\varepsilon(\cdot,t_0)\|_{L^\infty(\Omega)} \\
& + \int_{t_0}^t \|\nabla e^{(t-s)(\Delta-1)} u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} ds \\
\leq & C_S(1+(t^*)^{-\frac{1}{2}}) e^{-\tau^*(1+\mu_1)} \|v_\varepsilon(\cdot,t_0)\|_{L^\infty(\Omega)} \\
& + C_S \int_{t_0}^t \left[1+(t-s)^{-\frac{1}{2}}\right] e^{-\mu_1(t-s)} \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)} ds \\
\leq & C_S C_2(\tau^*)(1+(t^*)^{-\frac{1}{2}}) e^{-\tau^*(1+\mu_1)} \\
& + C_S C_1(\tau^*) \int_{t_0}^t \left[1+(t-s)^{-\frac{1}{2}}\right] e^{-\mu_1(t-s)} ds \leq C_3(\tau^*),
\end{aligned} \tag{39}$$

with

$$\begin{aligned}
C_3(\tau^*) :=& C_S C_2(\tau^*)(1+(t^*)^{-\frac{1}{2}}) e^{-\tau^*(1+\mu_1)} \\
& + C_S C_1(\tau^*) \left(\frac{1}{\mu_1} \left(1 - e^{-\mu_1 \tau^*}\right) + \mu_1^{\frac{-1}{2}} \Gamma\left(\frac{1}{2}\right)\right).
\end{aligned}$$

Thereafter, expressions (37), (38) and (39) justify the claim with $C_\infty(\tau^*) = C_1(\tau^*) + C_2(\tau^*) + C_3(\tau^*)$. □

After these preparations, the proof of our main results consists in demonstrating higher regularity for $(u_\varepsilon, v_\varepsilon)$; precisely, we first apply to the classical solution $(u_\varepsilon, v_\varepsilon)$ of problem (13) regularity results which enable us to derive ε-independent bounds of this solution in some Hölder space. Then, by properly interpreting the two equations of the same (13) as Neumann boundary value problems with Hölder and bounded sources and coefficients, we gain higher regularity for $(u_\varepsilon, v_\varepsilon)$. Subsequently, passing to the limit and organizing the statements, our claims follow.

Proof of Theorem 2.1. Let $\tau > 0$. Thanks to assumption $9/5 < \alpha < 2$, we can pick $9/5 < p < \alpha < 2$ such that, in view of relation (5), Lemma 5.2 for $t_1 = 0$ implies $\|u_\varepsilon\|_{L^p(\Omega)} \leq C(\tau)$ for any $t \geq \tau/2$; hence, for $3 < q < 3p/(3-p)$, with $q > p$, by choosing in Lemma 5.3 the value of t_1 as $\tau/2$ and $\tau^* = \tau/4$, the ε-uniform boundedness of $u_\varepsilon, v_\varepsilon$ and ∇v_ε in $L^\infty(\Omega \times [\tau,\infty))$ are given by (30).

Now, writing the first equation of (13) as

$$u_{\varepsilon t} - \nabla \cdot \mathcal{A}(x,t,u_\varepsilon,\nabla u_\varepsilon) = \mathcal{B}(x,t) \quad x \in \Omega,\ t > 0,$$

where $\mathcal{A}(x,t,u_\varepsilon,\nabla u_\varepsilon) := \nabla u_\varepsilon - \chi u_\varepsilon \nabla v_\varepsilon$ and $\mathcal{B}(x,t) = g(u_\varepsilon) - \varepsilon u_\varepsilon^\beta$, with $x \in \Omega,\ t > 0$, we note that

$$\mathcal{A}(x,t,u_\varepsilon,\nabla u_\varepsilon) \cdot \nabla u_\varepsilon = |\nabla u_\varepsilon|^2 - \chi u_\varepsilon \nabla v_\varepsilon \cdot \nabla u_\varepsilon \geq C_0 \Phi(|u_\varepsilon|)|\nabla u_\varepsilon|^2 - \psi_0,$$

with $C_0 = 1/2$, $\Phi(|u_\varepsilon|) \equiv 1$ and $\psi_0 := \chi^2 u_\varepsilon^2 |\nabla v_\varepsilon|^2/2$. Moreover, for $C_1 = C_2 = 1$, $\psi_1 := \chi u_\varepsilon |\nabla v_\varepsilon|$ and $\psi_2 := |g(u_\varepsilon)| + \varepsilon u_\varepsilon^\beta$ we also have

$$\begin{cases}
|\mathcal{A}(x,t,u_\varepsilon,\nabla u_\varepsilon)| \leq C_1 \Phi(|u_\varepsilon|)|\nabla u_\varepsilon| + \Phi(|u_\varepsilon|)^{\frac{1}{2}} \psi_1, \\
|\mathcal{B}(x,t)| \leq \psi_2 \leq C_2 \Phi(|u_\varepsilon|)|\nabla u_\varepsilon|^2 + \psi_2.
\end{cases}$$

Hence, time-uniform ($L^\infty(\Omega)$-)bounds for ψ_0, ψ_1 and ψ_2 are also attained; subsequently conditions (A_1)-(A_6) and (A_{11}) of [26] are verified so that Theorem 1.3 of this same paper applied for any $t \geq \tau$ to the (classical and hence also weak) solution u_ε of problem

$$u_{\varepsilon t} - \nabla \cdot (\nabla u_\varepsilon - \chi u_\varepsilon \nabla v_\varepsilon) = g(u_\varepsilon) - \varepsilon u_\varepsilon^\beta \text{ in } \Omega \times [t, t+2], \quad \frac{\partial u_\varepsilon}{\partial \nu} = 0 \text{ on } \partial\Omega \times [t, t+2],$$

infers constants $\alpha_1 \in (0,1)$ and $C_{\alpha_1} > 0$ such that $u_\varepsilon \in C^{\alpha_1, \frac{\alpha_1}{2}}\left(\bar{\Omega}, [\tau + \frac{1}{2}, \infty)\right)$ and, also, such that for any $t \geq \tau$

$$\|u_\varepsilon\|_{C^{\alpha_1, \frac{\alpha_1}{2}}\left(\bar{\Omega} \times [t+\frac{1}{2}, t+2]\right)} \leq C_{\alpha_1}.$$

Similarly, a straightforward reasoning carried out for the solution v_ε to the problem

$$v_{\varepsilon t} - \nabla \cdot \nabla v_\varepsilon = u_\varepsilon - v_\varepsilon \text{ in } \Omega \times \left[t + \frac{1}{2}, t+2\right], \quad \frac{\partial v_\varepsilon}{\partial \nu} = 0 \text{ on } \partial\Omega \times \left[t + \frac{1}{2}, t+2\right],$$

justifies that for any $t \geq \tau$,

$$\|v_\varepsilon\|_{C^{\alpha_2, \frac{\alpha_2}{2}}\left(\bar{\Omega} \times [t+1, t+2]\right)} \leq C_{\alpha_2},$$

for some $\alpha_2 \in (0,1)$ and $C_{\alpha_2} > 0$, as well as $u_\varepsilon \in C^{\alpha_2, \frac{\alpha_2}{2}}\left(\bar{\Omega}, [\tau + 1, \infty)\right)$.

Letting $\theta = \min\{\alpha_3, \alpha_4\}$ we have for some $C_\theta > 0$ and any $t \geq \tau$

$$\|u_\varepsilon\|_{C^{\theta, \frac{\theta}{2}}\left(\bar{\Omega} \times [t+1, t+2]\right)} + \|v_\varepsilon\|_{C^{\theta, \frac{\theta}{2}}\left(\bar{\Omega} \times [t+1, t+2]\right)} \leq C_\theta. \tag{40}$$

Thereafter, in order to apply classical regularity results, we construct the *cutting function ζ for the domain* $(1/2, 1)$; ζ is an increasing function, belongs to $C^0([1/2, 1])$, has first-order piecewise-continuous bounded derivatives and is such that $\zeta_{|(-\infty, 1/2]} \equiv 0$ and $\zeta_{|[1,\infty)} \equiv 1$. In this way, for any $\tilde{t} > \tau + 1$, the function $(x,t) \mapsto \zeta(t - \tilde{t}) v_\varepsilon(x,t)$ satisfies this parabolic initial boundary problem

$$\begin{cases} (\zeta v_\varepsilon)_t - \nabla \cdot \nabla(\zeta v_\varepsilon) = \zeta' v_\varepsilon + \zeta u_\varepsilon - \zeta v_\varepsilon \quad \text{in} \quad \Omega \times (\tilde{t}, \tilde{t}+2), \\ \frac{\partial(\zeta v_\varepsilon)}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (\tilde{t}, \tilde{t}+2), \\ (\zeta v_\varepsilon)(\cdot, \tilde{t}) = 0 \quad \text{in} \quad \Omega, \end{cases}$$

with smooth coefficients and source in $C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times [\tilde{t}, \tilde{t}+2])$, due to the uniform bounds for ζ, ζ' (by definition), and the Hölder continuity of u_ε and v_ε (relation (40)). As a result, Theorem IV.5.3 of [12] (together with Theorem III.5.1 of [12], concerning the existence question of the problem above) ensures that there exists $\tilde{C}_\theta > 0$ such that

$$\begin{aligned} \|v_\varepsilon\|_{C^{2+\theta, 1+\frac{\theta}{2}}(\bar{\Omega} \times [\tilde{t}+1, \tilde{t}+2])} &= \|\zeta v_\varepsilon\|_{C^{2+\theta, 1+\frac{\theta}{2}}\left(\bar{\Omega} \times \left[\tilde{t}+1, \tilde{t}+2\right]\right)} \\ &\leq \|\zeta v_\varepsilon\|_{C^{2+\theta, 1+\frac{\theta}{2}}\left(\bar{\Omega} \times \left[\tilde{t}, \tilde{t}+2\right]\right)} \leq \tilde{C}_\theta. \end{aligned}$$

Analogously, the function $(x,t) \mapsto \zeta(t - \tilde{t}) u_\varepsilon(x,t)$ solves the problem

$$\begin{cases} (\zeta u_\varepsilon)_t - \nabla \cdot (\nabla(\zeta u_\varepsilon) - \chi(\zeta u_\varepsilon) \nabla v_\varepsilon) = \zeta' u_\varepsilon + \zeta \left(g(u_\varepsilon) - \varepsilon u_\varepsilon^\beta\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad \Omega \times (\tilde{t} + \frac{1}{2}, \tilde{t}+2), \\ \frac{\partial(\zeta u_\varepsilon)}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (\tilde{t} + \frac{1}{2}, \tilde{t}+2), \\ (\zeta u_\varepsilon)(\cdot, \tilde{t} + \frac{1}{2}) = 0 \quad \text{in} \quad \Omega, \end{cases}$$

so that it is possible to find $\hat{C}_\theta > 0$ such that

$$\begin{aligned} \|u_\varepsilon\|_{C^{2+\theta, 1+\frac{\theta}{2}}\left(\bar{\Omega} \times \left[\tilde{t}+1, \tilde{t}+2\right]\right)} &= \|\zeta u_\varepsilon\|_{C^{2+\theta, 1+\frac{\theta}{2}}\left(\bar{\Omega} \times \left[\tilde{t}+1, \tilde{t}+2\right]\right)} \\ &\leq \|\zeta u_\varepsilon\|_{C^{2+\theta, 1+\frac{\theta}{2}}\left(\bar{\Omega} \times \left[\tilde{t}+\frac{1}{2}, \tilde{t}+2\right]\right)} \leq \hat{C}_\theta. \end{aligned}$$

Hence, the two previous estimates yield for any $t > \tau$

$$\|u_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}\left(\bar{\Omega}\times[t+3,t+4]\right)}+\|v_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}\left(\bar{\Omega}\times[t+3,t+4]\right)}\leq \tilde{C}_\theta + \hat{C}_\theta. \tag{41}$$

From all of the above, Proposition 2 provides us a very weak solution (u, v) of (4), as the limit with $\varepsilon \to 0$ of the sequence $(u_\varepsilon, v_\varepsilon)$; it represents the classical solution of (13), that is uniformly bounded in (τ, ∞), and satisfies (41). In view of the properties that this sequence transmits to its limit, also in terms of the compact embeddings for Hölder spaces, (u, v) improves its characteristics and whence is a very weak solution to problem (4), which is bounded in (τ, ∞) and, additionally, satisfies relation (6).

Proof of Theorem 2.2. Let us fix $\tau = 1$ in Lemma 5.2, so that the corresponding values of $C(1)$ and $\delta(1)$ therein introduced are also given. We set

$$\tilde{\delta} := \min\left\{\frac{\delta(1)}{4|\Omega|\tilde{C}}, \left(\frac{\delta(1)}{|\Omega|}\right)^\alpha\right\}, \tag{42}$$

where $\tilde{C}$ is the positive constant from Lemma 5.1 and, as for our hypothesis, let us assume that $a/b < \tilde{\delta}$ and, thus, $m := (a/b)^{\frac{1}{\alpha}}\,|\Omega| < \delta(1)$.

From inequality (14a) with $t_1 = 0$, we have that

$$\int_\Omega u_\varepsilon(\cdot, t) \leq m + e^{-\alpha a^{\frac{\alpha-1}{\alpha}} b^{\frac{1}{\alpha}} t}\left(||u_0||_{L^1(\Omega)} - m\right),$$

so that there exists $t_* \geq 0$ such that

$$\int_\Omega u_\varepsilon(\cdot, t_*) < \delta(1). \tag{43}$$

Additionally, inequalities (14c) and (21), with $t_1 = 0$, as well as the average theorem, infer a positive value $t^* \in \left(\frac{t_2}{2}, t_2\right)$ such that

$$\begin{aligned}
\|v_\varepsilon(\cdot, t^*)\|^\alpha_{W^{2,\alpha}(\Omega)} &= \frac{2}{t_2}\int_{\frac{t_2}{2}}^{t_2} \|v_\varepsilon\|^\alpha_{W^{2,\alpha}(\Omega)} \leq \frac{2}{t_2}\int_0^{t_2} \|v_\varepsilon\|^\alpha_{W^{2,\alpha}(\Omega)} \\
&= \frac{2}{t_2}\int_0^{t_2}\int_\Omega ({v_\varepsilon}^\alpha + |\nabla v_\varepsilon|^\alpha + |\Delta v_\varepsilon|^\alpha), \\
&\leq \frac{2\tilde{C}}{t_2}\left(||v_0||^\alpha_{W^{2,\alpha}(\Omega)} + \int_0^{t_2}\int_\Omega u_\varepsilon^\alpha\right), \\
&\leq \frac{2\tilde{C}}{t_2}\left(||v_0||^\alpha_{W^{2,\alpha}(\Omega)} + \frac{M}{b}\right) + \frac{2a\tilde{C}}{b}|\Omega|,
\end{aligned} \tag{44}$$

with, again, $M = \max\{m, \|u_0\|_{L^1(\Omega)}, \|v_0\|^\alpha_{W^{2,\alpha}(\Omega)}\}$. Now we choose $t_2 > 0$ so large that the corresponding t^* is such that

$$\frac{2\tilde{C}}{t^*}\left(||v_0||^\alpha_{W^{2,\alpha}(\Omega)} + \frac{M}{b}\right) < \frac{\delta(1)}{2}.$$

Let $\mathring{t} = \max\{t_*, t^*\}$; recalling assumption $a/b < \tilde{\delta}$ and definition (42), as well as bounds (43) and (44), we have that hypothesis (23) of Lemma 5.2 holds for $t_1 = \mathring{t}$, so that $\|u_\varepsilon\|_{L^p(\Omega)} \leq C(1)$ for $t \geq \mathring{t}+1/2$. In turn, we apply Lemma 5.3 with $t_1 = \mathring{t}+1/2$ and $\tau^* = 1/4$. In view of the details given in the proof of the previous theorem, the existence and boundedness questions and validity of relation (7) are seen to be true upon the choice $T = \mathring{t} + 1$.

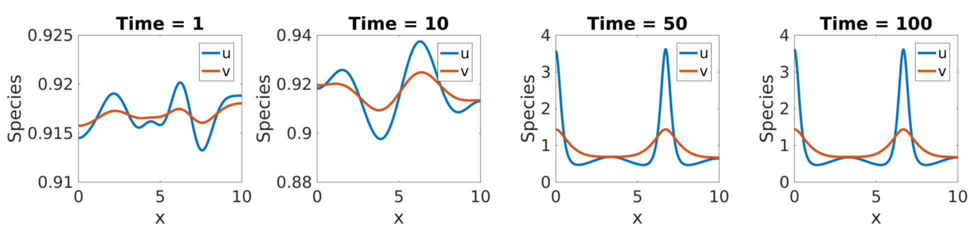
6. **Numerical simulations.** In this section we numerically test the presented results by simulating the chemotaxis systems in one, two and three dimensions. Further, we investigate whether the global solutions are bounded and stationary, or whether they have complex temporal dynamics, such as moving peaks, oscillations, or chemotactic blow-up.

Specifically, we use finite element methods to simulate

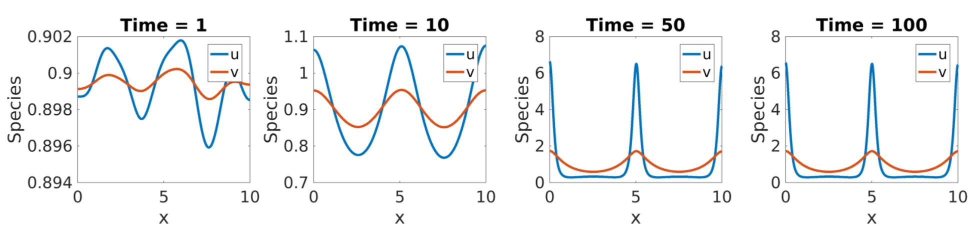
$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + a - b u^{\alpha}, \\ v_t = \Delta v - v + u, \end{cases} \tag{45}$$

on domains of different dimension, employing Neumann conditions on the domain boundaries, where applicable. Further, (unless otherwise stated) the initial conditions are uniformly randomly generated with mean equal to the homogeneous steady state, $(a/b)^{1/\alpha}$ and with range $(a/b)^{1/\alpha}/100$. Domain sizes, domain discretisations and parameter values are given in the captions of each figure.

Figure 1 illustrates the one-dimensional evolution of system (45) over time and over multiple values of α. As highlighted in Remark 1 of Proposition 2, we expect that when $1 < \alpha < 2$ the solution converges to a bounded solution, which we note is heterogeneous and stationary (see Figure 1(a), and, in particular, compare the figures as times 50 and 100). However, although the theory ensures existence of global (and bounded) solutions only for $\alpha > 1$, we can see that this is not a bifurcation point since in Figure 1(b) the system also converges to a bounded, heterogeneous, stationary solution when $\alpha = 0.9$.



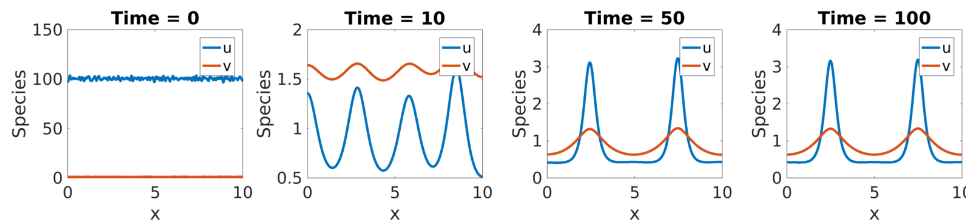
(a) $\alpha = 1.1$



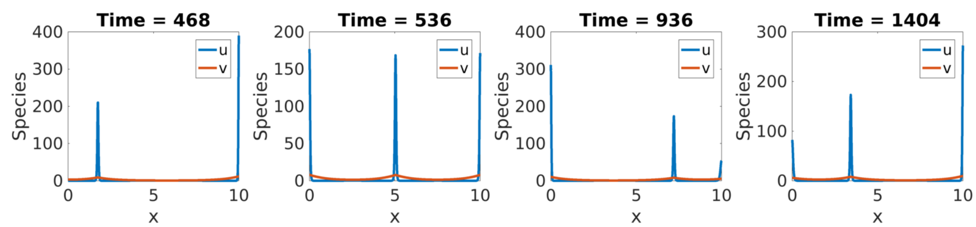
(b) $\alpha = 0.9$

FIGURE 1. Simulations of system (45) in one dimension with varying value of α, given beneath each subfigure. Each subfigure contains the system evaluated at the time points $t = 1$, 10, 50 and 100. The remaining parameters values are $a = 1$, $b = 1.1$ and $\chi = 6$. The domain was discretised into 1000 equally spaced points.

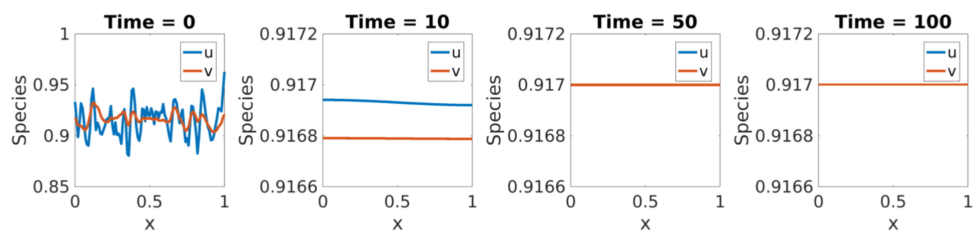
In Figure 2 we further illustrate the coherence of Remark 1 in this paper as it suggests that for all $1 < \alpha < 2$ the one-dimensional simulations should be bounded no matter the initial conditions on u and no matter the size of b. Figure 2(a) illustrates that even after increasing the average initial value of u to 100 the solution still tends to a stable, heterogeneous, bounded solution. Equally, in Figure 2(b), where the parameter b has been reduced to 0.2 we see that the solution is bounded. However, here we see a new dynamic of continuously evolving peaks. Namely, peaks appear approximately in the centre of the domain and then travel towards the boundaries $x = 0$, or $x = 10$. The direction of travel depends on which side of $x = 5$ the peak first appears. Further, the peaks appear to alternate in directions, with the first peak travelling to the left.



(a)



(b)

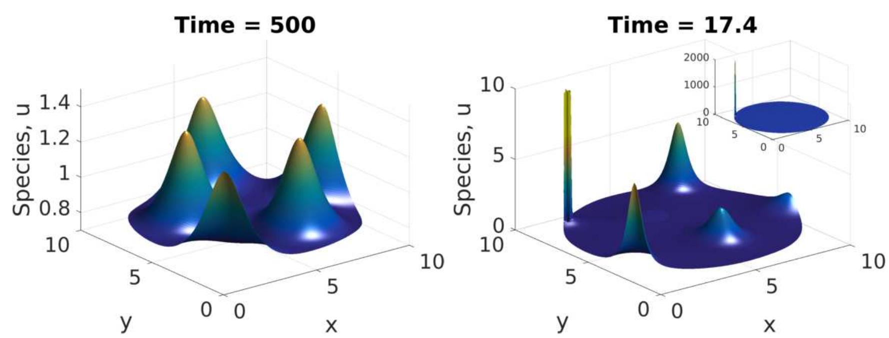


(c)

FIGURE 2. Simulations of system (45) in one dimension. The simulations are nearly identical to those seen in Figure 1(a). However, each simulation involves a single parameter change. Specifically, in (a) a larger initial condition for u was used (100 was added to the mean); in (b) the parameter b was reduced to 0.2; Finally, in (c) the spatial solution domain has been reduced from 10 to 1.

Up until now all of the solutions have presented bounded, spatially heterogeneous solutions. However, stationary, bounded, uniform solutions are also possible and these are illustrated in Figure 2(c). Specifically, using spectral analysis it is well known [19, 41] that concentration heterogeneity arises through a symmetry breaking of the balance between diffusion and chemotaxis. Hence, patterning of the solution domain requires a minimal spatial scale before instability of the uniform steady state can occur [15, 42]. Since the solution domain in Figure 2(c) has been reduced below this critical spatial scale the system simply tends to a uniform steady state.

Figure 3 illustrates the two-dimensional evolution of system (45) for $\alpha = 1.6$ and 1.1. The solutions are simulated on a circular domain of radius 5. The two simulations contain values of α either side of the critical value of $3/2$ (recall again Proposition 2), which states that when α is above the critical value the solutions exist, are global and (possibly) bounded. In the specific case of Figure 3(a) the solutions are, once again, bounded, heterogeneous and stationary. However, for

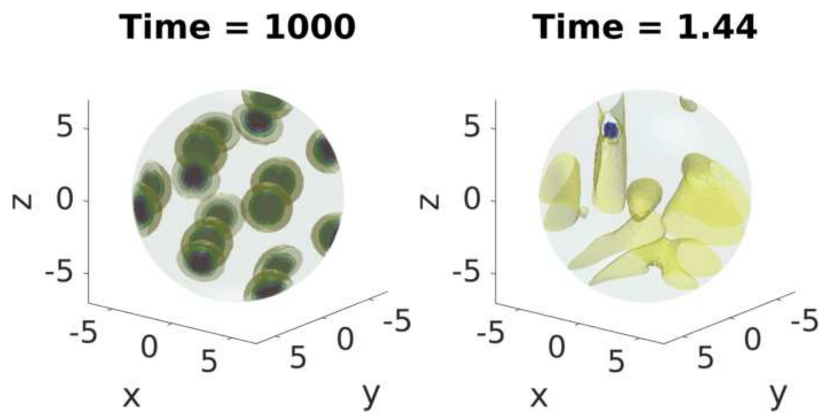


(a) $\alpha = 1.6$ (b) $\alpha = 1.1$

FIGURE 3. Simulations of system (45) in two dimensions with varying value of α, given beneath each subfigure. Evolution time shown above each subfigure. The remaining parameters values are $a = 1$, $b = 1.1$ and $\chi = 6$. The domain was triangulated into 24,968 finite elements. The figure inset of (b) shows the full extent of the peak, which is growing without bound.

$\alpha = 1.1$, which is below this critical value, we are able to produce solutions that are prone to blow up. Note that this difference in convergence is purely a property of domain dimension because, apart for this difference, figures 1(a) and 3(b) have the same parameters values. Further, we note that the exploding solution appears to occur foremost on the boundary and that the blow-up occurs very quickly since the maximum value of the solution is over 10^3 (see inset of Figure 3(b)) in just over 17.4 time points. By $t \approx 17.43$ the peak is over 10^{12} and the solution fails to converge.

Finally, Figure 4 illustrates the three-dimensional evolution of system (45) for $\alpha = 1.8$ and 1.3, which are values either side of the critical value of $5/3$. The solutions are simulated within a solid sphere of radius 7. Once again, we see that Proposition 2 holds in Figure 4(a) as, for suitable initial data, the system evolves to a stationary solution, which is both bounded and heterogeneous. On the other hand, we see that the simulation within Figure 4(b) blows up within 1.44 time units. In both cases, we can see that the balls become darker towards their centre. This means that the population u becomes denser there because the isosurfaces corresponding to darker colours correspond to larger values of u (see caption of Figure 4 for more details). Whilst Figure 4(a) remains finite over the entire region, Figure 4(b) contains a region that begins to undergo chemotactic collapse (see the dark spot in Figure 4(b)). At the illustrated time point the density is over 10^6 and grows to over 10^{12}, before the simulation fails to converge. In this case, in contrast to Figure 3(b), the unstable growth occurs within the domain, rather than on the boundary.



(a) $\alpha = 1.8$ (b) $\alpha = 1.3$

FIGURE 4. Simulations of system (45) illustrating the density of u in three dimensions with varying value of α, given beneath each subfigure. Evolution time shown above each subfigure. The remaining parameters values are $a = 1$, $b = 1.1$ and $\chi = 6$. The domain was discretised into 1,139,254 voxel elements. Apart from the light grey ball illustrating the boundary of the solution domain the images illustrate isosurfaces of the solution (i.e. surface that represent points of a constant value, thus, they are the three-dimensional analogue of contours). In Figure (a) there are five isosurfaces of value 1, 1.25, 1.5 1.75 and 2, coloured, yellow, green, blue, red and black, respectively. In Figure (b) there are three isosurfaces of value 1, 10, and 10^6, coloured, yellow, blue and black, respectively.

In summary, these simulations illustrate the veracity of the proof contained within this paper. Specifically, global boundedness of a chemotactic system depends on the spatial dimension we are considering, as well as the kinetic parameters of the system.

Acknowledgments. Giuseppe Viglialoro is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and gratefully acknowledges the Italian Ministry of Education, University and Research (MIUR) for the financial support of Scientific Project "Smart Cities and Communities and Social Innovation - ILEARNTV anywhere, anytime - SCN_00307".

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Received xxxx 20xx; revised xxxx 20xx.

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