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# A note on the wallet game with discrete bid levels[☆]

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## HIGHLIGHTS

- We analyse a Japanese–English auction with fixed bid levels for the wallet game.
- We prove that bidding twice the value of the private signal is never an equilibrium.
- We show that there is a discontinuity in the equilibrium set from discrete to continuous.

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## ABSTRACT

It is well-known that in the wallet game with two bidders, bidding twice the (individual) signal is an equilibrium. We prove that this strategy is never an equilibrium in a Japanese–English auction once discrete bid levels are introduced; we also discuss the implications of this result.



## 1. Introduction

Milgrom and Weber (1982) analysed a particular version of the English auction, the so-called Japanese–English Auction (henceforth JEA), commonly known as a clock auction, in which the price of the object increases continuously and the bidders must keep on pressing a button whilst they are interested in buying the object at the posted price; the auction ends when all but one bidder release the button. Later, Klemperer (1998) focused on a particular common value auction with two bidders, popularly known as the wallet game (in which the common value is simply the sum of two private signals, the "wallets"), as a special case of the above model and illustrated that bidding twice the (individual) signal forms the unique symmetric (Bayesian–Nash) equilibrium in this game.

In real world examples of auctions of different formats, the price actually increases in discrete increments. In the recent past, English auctions with predefined discrete bid levels have been analysed (Rothkopf and Harstad, 1994; Yu, 1999; Sinha and Greenleaf, 2000; Cheng, 2004; David et al., 2007; Isaac et al., 2007); in these studies, bidders have to choose among the exogenously fixed bid levels when it is their turn to bid (Rothkopf and Harstad, 1994; David et al., 2007) or at the very least, increase the going bid by a minimum increment (Isaac et al., 2007). Yu (1999) observes that English auctions with discrete bids are likely to yield different equilibrium strategies from their continuous counterpart.

Following the seminal experiment by Avery and Kagel (1997) on a continuous-bid JEA based on the wallet game, not much further theoretical and experimental literature has emerged on this issue, with the exception of Gonçalves and Hey (2011). One should note that the existing (above-mentioned) literature on discrete bids,

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in the context of single object auctions, has focussed almost entirely on private value environments (for example, Rothkopf and Harstad, 1994); virtually nothing has been done for the common value model. There is a vast literature on both multi-object and multi-unit auctions, some of which considers discrete bidding. However, this literature also mainly refers to private values; for example, Brusco and Lopomo (2002) look at multi-object simultaneous ascending auctions with private values and complementarities across objects. Ausubel (2004) proposes a novel multi-unit ascending bid auction for homogeneous goods, both with private as well as with interdependent values (a generalisation of both the private and common value models) and models the auction through a price clock with either integer (steps) or continuous increments; interestingly and of relevance to our work, in Ausubel (2004), discrete increments are only used in the private values case while the ascending auction with interdependent values is analysed under continuous bid increments.

We, in this note, consider a set-up similar to the usual JEA, except that the price goes up in discrete commonly known bid levels. As in the usual JEA, if a bidder wants to drop out, all he has to do is release the button. The final auction price is equal to the highest bid level at which at least one bidder was active. We focus on the wallet game. To the best of our knowledge, nobody so far has theoretically analysed the equilibria of a common value environment like the wallet game, using JEA with exogenously specified discrete bid levels.

We prove that one cannot construct a symmetric equilibrium using bids that are twice the private signal (as in the case of continuous bid levels illustrated by Klemperer, 1998). Our result is somewhat similar to the one in Isaac et al. (2007) which, in a private values setting, shows that the equilibrium strategies in a continuous bid setting do not extend to a discrete bid environment, focusing on jump bidding equilibria in their set-up. Further research in this area should follow this note, as our result implies that the equilibria in this environment are inherently more complex, with obvious implications on the design of such auctions.

## 2. Model

We consider the wallet game in which there are two symmetric risk-neutral bidders $i \in \{1, 2\}$ who compete for the purchase of one single good, whose value, $\tilde{V}$, is common but *ex ante* unknown to both bidders. Each bidder receives an independent and uniformly distributed[1] private signal $x_i \sim U(0, 1)$, $i = 1, 2$. The (ex ante) unknown common value of the good is simply the sum of the two signals: $\tilde{V} = x_1 + x_2$.

We make use of the JEA with some exogenously fixed discrete bids. In our set up, as in the usual JEA, the price increases; however the bid levels are discrete (rather than continuous) and are fixed exogenously. Formally, the bid levels are the elements of the set $A = \{a_1, \ldots, a_k\}$, with $0 < a_1 < \cdots < a_k < 2$, $k \geq 2$ a finite integer; the set $A$ is common knowledge to the bidders. We will denote a typical bid level by $a_j$, for $j = 1, \ldots, k$, with the implicit assumption that $a_0 = 0$ and $a_{k+1} = 2$, for notational convenience whenever required in this paper.

The commonly-known (publicly displayed) auction price goes up in discrete bid levels in the set $A$ starting from $a_1$ and ending at $a_k$. The bidders have to keep pressing a button at each bid level to be actively bidding; if a bidder wants to drop out of the auction at any stage, all he has to do is release the button. The final auction price is equal to the highest bid level in which at least one bidder was active. This rule implies that, for any $j = 1, \ldots, k-1$, if one bidder is active at $a_j$ but not at $a_{j+1}$ while his opponent is active at $a_{j+1}$, then the latter wins the auction and pays a price equal to $a_{j+1}$; by contrast, if both bidders are active at $a_j$, but not at $a_{j+1}$, then the auction winner is decided at random with equal probabilities and the final price is $a_j$; finally, if both bidders are active at the last bid level $a_k$, the winner will be chosen at random with equal probabilities and will pay the price $a_k$. The net payoff to the (selected) winner in each of the above cases is the realised value of $x_1 + x_2$ minus the price to pay while the payoff to the loser is 0. If no bidder is active at $a_1$, then the auction ends immediately and the payoff to either bidder is 0.

A strategy in this Bayesian game is therefore to choose (as in the standard JEA) a drop out bid level as a function of the signal. Given a signal $x \in (0, 1)$, a bidding strategy for a player thus chooses 0 (which implies that the bidder is not active even at $a_1$) or a bid level $a_j$ so that the bidder will be active at $a_j$ but not at $a_{j+1}$, where $j = 1, \ldots, k$ (with $a_{k+1} = 2$). We denote a typical strategy by $\sigma$ which is a function dented by $b(x) \in \{0, a_1, \ldots, a_k\}$, which implies that the player with signal $x$ is active until $b(x)$.

The JEA for the wallet game with $k$ bid levels $(a_1, \ldots, a_k)$ as described above will henceforth be called $G_k$.

## 3. Results

As it is well-known, the symmetric (Bayesian–Nash) equilibrium for the JEA with continuous bids is given by bid functions $b_i^*(x_i) = 2x_i$, $i = 1, 2$, as derived by Milgrom and Weber (1982), in a general model, and later specifically for the wallet game by Klemperer (1998) and Avery and Kagel (1997). A relevant question, therefore, is whether these equilibrium strategies also form an equilibrium in the JEA with discrete bids for the wallet game or not.

A direct translation of the above (continuous) JEA bidding strategies into our setting would yield the following bidding strategy: each bidder $i$ should stay active in the auction until the bid reaches $b_i^*(x_i) = 2x_i$ and drop out after that, that is, each bidder $i$ will choose 0 if $2x_i < a_1$ (equivalent to $x_i < \frac{a_1}{2}$) and will choose $a_j \in A$ (active at bid level $a_j$ but would drop out at bid level $a_{j+1}$) if $a_j \leq 2x_i < a_{j+1}$ (equivalent to $\frac{a_j}{2} \leq x_i < \frac{a_{j+1}}{2}$), for $j = 1, \ldots, k-1$ and will choose $a_k$ if $2x_i > a_k$ (equivalent to $x_i > \frac{a_k}{2}$). Let us call this strategy the "twice-signal bidding". We show that this bidding strategy in our setting is not an equilibrium.

**Proposition 1.** *The twice-signal bidding strategy profile is not an equilibrium in* $G_k$.

**Proof.** We will prove Proposition 1 by showing that there are signal realisations for which there exists some profitable individual deviation for a bidder. We will illustrate this using bidder 1's strategy. Without loss of generality, suppose $x_1 > x_2$ where $x_1$ and $x_2$ are the signals of two respective bidders; further assume that for some $j$, $j = 1, \ldots, k$, $a_{j-1} \leq 2x_2 < a_j \leq 2x_1 < a_{j+1}$ (assume $a_0 = 0$ and $a_{k+1} = 2$ if required). Hence, following the twice-signal bidding strategy, bidder 2 would be active at bid level $a_{j-1}$ but would drop out at bid level $a_j$ while bidder 1 would be active at $a_j$ - the price that bidder 1, the winning bidder, would pay. Note that bidder 1 's expected payoff, conditional on winning at $a_j$, is given by:

$$\pi_1 = x_1 + E\left[X_2 | \frac{a_{j-1}}{2} \leq X_2 < \frac{a_j}{2}\right] - \frac{a_j}{2} = x_1 + \frac{\frac{a_{j-1}}{2} + \frac{a_j}{2}}{2} - a_j$$

which is equal to $x_1 + \frac{a_{j-1}}{4} - \frac{3}{4}a_j$.

Now suppose, bidder 1's signal realisation is 'too low' within the chosen interval $\frac{a_j}{2} \leq x_1 < \frac{a_{j+1}}{2}$, that is $x_1 = \frac{a_j}{2} + \varepsilon$ for some small $\varepsilon > 0$. In this case, he will find the expected value of the good to be lower than $a_j$, thus yielding negative profits. This is because in such a case, bidder 1's expected payoff $\pi_1$ will be $\frac{a_j}{2} + \varepsilon + \frac{a_{j-1}}{4} - \frac{3}{4}a_j = \frac{a_{j-1} - a_j}{4} + \varepsilon < 0$, for an appropriately chosen small $\varepsilon$. ■

[1] The uniform distribution is undoubtedly easier to compute solutions for, however, any other specific distribution should not matter in our analysis.

Clearly, there is a discontinuity in the fact that the twice-signal bidding strategy is an equilibrium in the continuous case but not so in the discrete case. We prove this formally.

**Proposition 2.** *The twice-signal bidding strategy profile is not an equilibrium in $G_k$, for any finite k, but is an equilibrium in the corresponding game with continuous bids.*

**Proof.** The proof follows immediately from the proof of Proposition 1. Consider any natural number $k$ and thus a JEA with $k$ many bid levels. Take the minimum of the distances between two successive bid levels and call it $\delta$; hence, for any $j$, $a_j - a_{j-1} \geq \delta$. One can now choose a small enough $\varepsilon$ such that $\varepsilon < \delta/4$. Now, in the case mentioned in last line of the proof of Proposition 1 above, bidder 1's expected payoff ($\pi_1 = \frac{a_{j-1}-a_j}{4} + \varepsilon$) will definitely be strictly negative. Hence, the twice-signal bidding strategy profile is not an equilibrium in the JEA with $k$ many discrete bids, for any finite $k$, while as we already know (Klemperer, 1998), twice-signal bidding does constitute an equilibrium in the continuous case. ■

## 4. Conclusion

We have shown that the standard equilibrium (of bidding twice the signal) in JEA with continuous bid levels is not an equilibrium in a setting where bid levels are discrete for the wallet game.

The intuition behind our result is clear. In the continuous equilibrium (Klemperer, 1998), the pivotal event on which a bidder's decision to bid up to a price conditional on winning is that the rival's signal equals half the price. However, by contrast, with discrete bid levels, this pivotal event is that upon winning the auction, the rival's signal only belongs to an interval (between two discrete bid levels) and the winning bidder's expected value of the wallet thus is lower than the price he would pay by bidding twice the signal.

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