On uniqueness in some physical systems

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A thesis submitted in partial fulfilment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

APRIL 2017

SCHOOL OF MATHEMATICS

CARDIFF UNIVERSITY

Abstract

In this work we present some uniqueness and cloaking results for a related pair of inverse problems. The first concerns recovering the parameter q in the Bessel-type operator pencil, over $L^2(0,1;r\mathrm{d}r)$, formally given by

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}u(r;\lambda)\right)+q(r)u(r;\lambda)=\lambda w(r)u(r;\lambda)\quad(0<r<1)$$

from (a generalisation of) the Weyl–Titchmarsh boundary m-function

$$m(\lambda)=-\frac{u'(1;\lambda)}{u(1;\lambda)}.$$

We assume that both w and q are singular at 0. We prove q is uniquely determined by the sequence $m(-n^2)$ $(n=1,2,3,\ldots)$, using asymptotic and spectral analysis and m-function interpolation results. For corollary we find, in a half-disc with a singular "Dirichlet-point" boundary condition on the straight edge, a singular radial Schrödinger potential is uniquely determined by Dirichlet-to-Neumann boundary measurements on the semi-circular edge.

The second result concerns recovery of three things—a Schrödinger potential in a planar domain, a Dirichlet-point boundary condition on part of the boundary, and a self-adjointness-imposing condition—from Dirichlet-to-Neumann measurements on the remaining boundary. With modern approaches to the inverse conductivity problem and a solution-space density argument we show the boundary condition cloaks the potential and *vice versa*. Appealing to negative eigen-value asymptotics we find the full-frequency problem has full uniqueness.

Dedication

Writing a thesis, even so modest as this example, is no mean feat, as anybody who has done so will undoubtedly attest. Yet without the help I have received the result would have been indubitably meaner and cruder. For that reason I will remain indebted to many people. Firstly, I thank Professors Marco Marletta and Malcolm Brown, my supervisors, for their patience with my endless questions, their invaluable feedback on my work, their encouragement and knowledge, their continually insightful ideas, and their consistently incising questions, which separated the wheat from the chaff.

I thank my parents Bridgit and Nick, my step-parents Elke and Henry, my brother Henry and all the rest of my closer and extended family, for their boundless support, in the good times and bad, as well as an upbringing that led me to question everything with which I am presented and only to believe that which I have personally verified. Moreover the endless laypersons' questions are always deeply appreciated! I will be forever grateful to my housemates, friends and colleagues, present and past—you are too many for me to name all and not forget one or two—for the stimulating and fun environments shared, knowledge and jokes exchanged, and life advice freely offered.

I thank the Engineering and Physical Sciences Research Council for funding my studies these four years, and Cardiff University and the School of Mathematics for hosting me as I conducted my research. I also thank my lecturers and teachers, for exciting and challenging me, thereby stoking the flame of my passion, as well as the School administrative staff for their kind help at all hours and at the shortest notice, especially Catherine, Vicky, Alastair, Jonathan, Rob and Paul.

I offer deep gratitude to Professors Karl Michael Schmidt and Sasha Pushnitski, my examiners, for the many improvements due to their valuable comments, and I give many thanks to my proof-readers: in particular to Dr. Christoph Fischbacher for his detailed and indispensable suggestions, as well as the many mathematical and philosophical conversations shared over more than one beer.

And finally, I thank Chiara, for her infinite kindness and wisdom, her patience and optimism, her inspiring yet pragmatic ideas, the laughter and love shared, and especially the physical and soul food that kept me writing through many a long evening.

Contents

‘Τηλέμαχ᾽, ἄλλα μὲν αὐτὸς ἐνὶ φρεσὶ σῇσι νοήσεις, ἄλλα δὲ καὶ δαίμων ὑποθήσεται.’

– Ὅμηρος

‘Telemachus, some things you will think of yourself, and others the gods will put into your mind.’

– Homer[1]

[1] Ἀθήνη, or Athena, in Ὀδύσσεια, or *Odyssey*, Book III, Line 26 (ca. C8 BCE)

1

Introduction to inverse problems

'Can one hear the shape of a drum?'

– Mark Kac[1.1]

"Definition". *We call two problems* inverses *of one another if the formulation of each involves all or part of the solution of the other. Often, for historical reasons, one of the two problems has been studied extensively for some time, while the other is newer and not so well understood. In such cases, the former is called the* direct problem*, while the latter is called the* inverse problem*.*

This oft-quoted definition[1.2] was first put down by Keller [76]. Examples of inverse problems arise naturally in physics, though only within the last century has their framework become mathematically well formalised.

Such physical examples range from the famous (X-ray tomography, ultrasound computer tomography) to the lesser-known (electrical impedance tomography (EIT), land-mine detection, inverse wave scattering) or the abstract ("hear-

[1.1] *The American Mathematical Monthly*, (1966) [74].
[1.2] 'Definition' is intended here with a looser, not-necessarily-mathematical meaning.

ing" shapes of resonators). Since the first X-ray images of internal physiology, there has been great medical interest in developing techniques for non-invasive imaging. More generally, the determination of material properties or object locations from scattered radiation or far-/near-field effects has manifold applications, including but not limited to astronomy, flight control and military systems, oceanography and geological surveying[1.3]. Much time and resource has been invested in developing methods of recovery of such material properties, usually involving numerical procedures applied to measured data, which return mathematical models describing the desired properties.

Of paramount importance, however, is assurance of the *uniqueness* of the recovered solution. Without such a guarantee the recovery efforts may turn out to be in vain. If the recovered "solution" is radically different from the real answer, yet both return the same measured data, there could be inconvenient or even disastrous consequences such as faulty prediction of extreme weather, malfunctioning flight equipment or failure to locate a malignant tumour. On the other hand, there has been a recent surge of interest in the concept of *invisibility cloaking* and its applications, which is an example exhibiting precisely the non-uniqueness that could cause so many problems in the aforementioned cases. A great deal of effort has recently been made to realise the theory of cloaking in practical terms, and the interested reader is directed to the substantial and detailed review of this, and the underlying theory, available in [59].

As worthwhile and popular as recovery procedures and practical cloaking devices are, in this thesis we will focus almost entirely on the mathematical issue of uniqueness, in the case of some newly formulated and connected inverse

[1.3] A selection of detailed review articles or books covering these particular topics is as follows: X-ray tomography [97], ultrasound tomography [43], EIT [105, Ch. 12], landmine detection [96], inverse scattering [1], hearing the shape of a drum [74], astronomy [39], flight control/simulation [106], oceanography [138] and gravimetry [141]. Of course these topics are not the focus of this thesis, and are simply included for the reader sufficiently interested to read further.

problems. By way of a preliminary introduction we will now describe in Section 1.1 one of the earliest inverse problems mathematically related to our new work. This will illustrate the concerns faced by those working on problems of this nature, motivating a general definition of solvability. Subsequently, in Section 1.2 we will outline the novel problems worked on in this thesis. To conclude the chapter we will describe in Section 1.3 the structure of the rest of the thesis and provide a list of symbols defined.

1.1 The inverse conductivity problem

In the 1940s the engineer A. P. Calderón worked for the Argentine state oil company, Yacimientos Petrolíferos Fiscales. Whilst there he conceived the problem of determining the conductivity in a medium from measurements of current and voltage made at the boundary of said medium. His original aim was the location of oil reserves, but the ideas generated in response to his problem have found numerous other applications, notably in EIT[1.4], for example in locating malignant tumours, known to have a different conductivity to surrounding healthy tissue. For a recent detailed survey of Calderón's problem, focusing on analytic approaches to uniqueness, see [129].

Consider a volume Ω of material in the Earth's crust, with surface boundary denoted Γ, a subset of the full boundary $\partial\Omega$ of Ω. Calderón envisaged embedding electrodes at a grid of points on Γ, passing an input set of currents through the electrodes into the crust, and then measuring the resulting potential differences between pairs of points in an interspersed grid on the surface. From these measurements he hoped to determine the material conductivity in the entire volume.

[1.4] It is worth pointing out for the attentive reader that strictly speaking tomography involves reconstruction from "slices" through the medium, by which definition EIT is a misnomer. The name has stuck by historical convention. For a brief discussion of this issue and a detailed survey of some numerical approaches, see [93].

This set-up is a discrete approximation to inputting a current function all over the continuous surface, and measuring on the same surface the voltage function. This is mathematically formalised as follows.

The input current function h is set to be the normalised normal derivative on Γ of the voltage density in the volume Ω, and the output voltage then represents the restriction of this voltage density back to the surface. Making such a measurement is equivalent to solving the conductivity boundary-value problem

$$\begin{cases} -\nabla \cdot (\gamma \nabla u) &= 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega \backslash \Gamma, \\ \gamma \partial_\nu u &= h \quad \text{on} \quad \Gamma \end{cases} \tag{1.1.1}$$

for the voltage density u in all of Ω, then restricting to $u \restriction_\Gamma$. Repeating this process for every (admissible) h produces in this case what is called the *current-to-voltage map* on Γ; in general such a map is referred to as a *Neumann-to-Dirichlet map*, as it maps the normal derivative, called the Neumann data-set, to the trace, called the Dirichlet data-set. In modern notation, this is usually denoted $N_{\gamma,\Gamma} : h \mapsto u \restriction_\Gamma$.

More generally, Ω is taken to be a bounded domain in $\mathbb{R}^n$, with a sufficiently smooth boundary $\partial\Omega$ (usually, though not always, Lipschitz) containing the connected and (relatively) open subset Γ, referred to as the *accessible boundary*. The conductivity γ is assumed to be positive and uniformly bounded from above and below. For mathematical reasons one usually prefers to work with the negative of the inverse map, called Dirichlet-to-Neumann, and denoted $\Lambda_{\gamma,\Gamma}$. With these considerations $\Lambda_{\gamma,\Gamma}$ is well defined on the domain $H^{1/2}(\Gamma)$ and has range the dual space $H^{-1/2}(\Gamma)$. Calderón's problem, then, is thus:

Inverse Problem. *Given the map $\Lambda_{\gamma,\Gamma}$, recover the conductivity γ everywhere in Ω.*

Whilst there has been prolonged and deep study of this problem and the ram-

ifications of results concerning it, for brevity of this introduction we save our survey of the state-of-the-art for Section 2.4. Instead we outline two key features. The first is explained by the following example.

Example 1 (Alessandrini, 2007 [5]). *Let $\Omega = B_1(0)$ be the ball in $\mathbb{R}^n$ with centre 0 and radius 1, and set $\gamma_\varepsilon(\mathbf{x}) = 1 + \chi_{B_{1/2}(\varepsilon e_1)}(\mathbf{x})$ for every $0 \leq \varepsilon < 1/4$, where $e_1 = (1, 0, \ldots, 0)^T$. Then*

$$\|\gamma_0 - \gamma_\varepsilon\|_{L^\infty(\Omega)} = 1 \quad (0 < \varepsilon < 1/4).$$

For each $0 \leq \varepsilon < 1/4$ define $\Lambda_{\gamma_\varepsilon,\partial\Omega} : H^{1/2}(\partial\Omega) \ni h \mapsto -\partial_\nu u_\varepsilon \in H^{-1/2}(\partial\Omega)$ where u_ε solves

$$\begin{cases} -\nabla \cdot (\gamma_\varepsilon \nabla u_\varepsilon) &= 0 \quad in \quad \Omega, \\ u_\varepsilon &= h \quad on \quad \partial\Omega. \end{cases}$$

As is proven in the reference, one can show that there are constants $C > 0$ and $0 < \delta < 1$ such that

$$\|\Lambda_{\gamma_0,\partial\Omega} - \Lambda_{\gamma_\varepsilon,\partial\Omega}\|_{\mathcal{L}(H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega))} \leq C\varepsilon^\delta \quad (0 < \varepsilon < 1/4).$$

The above example demonstrates there can be instability in determining γ from $\Lambda_{\gamma,\Gamma}$ in the L^∞-norm, even when we take data on the full boundary. However, we have the following pair of theorems, describing the second aforementioned feature.

Theorem 1.1 (Caro–Rogers, 2016 [36]). *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ a bounded domain so that $\partial\Omega$ is Lipschitz. Suppose γ_1 and $\gamma_2 \in C^{0,1}(\overline{\Omega})$, and are both bounded from below by $c > 0$. If $\Lambda_{\gamma_1,\partial\Omega} = \Lambda_{\gamma_2,\partial\Omega}$ then $\gamma_1 = \gamma_2$.*

Theorem 1.2 (Astala–Päivärinta, 2006 [9]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Suppose γ_1 and $\gamma_2 \in L^\infty(\Omega)$ and both are bounded from below by a constant $c > 0$. If $\Lambda_{\gamma_1,\partial\Omega} = \Lambda_{\gamma_2,\partial\Omega}$ then $\gamma_1 = \gamma_2$.*

These theorems[1.5] establish uniqueness of the conductivity from knowledge

[1.5] They are, at the time of writing, the best results yet for full-boundary uniqueness. For a

of the Dirichlet-to-Neumann map on the whole boundary.

Calderón's problem illustrates a property common to many physically inspired questions, namely failure of the solution to satisfy one or more of the three given criteria: existence, uniqueness and stability. In particular inverse problems frequently act in this disagreeable fashion. We now give a definition to formalise these concepts. We use a general classification of many problems (e.g., mathematical or physical in nature) given succinctly by Hadamard [64] and which can be written as follows:

Definition 1.1. *Suppose U and V are topological spaces, and $T : U \to V$ is a continuous mapping between them. We consider the problem of finding $u \in U$ given some $v \in V$ (called the* data*) such that $T(u) = v$. We say that this problem is* ill-posed *if any of the following three conditions are violated:*

1. *a solution always exists, i.e., for each $v \in V$ there is $u \in U$ such that $T(u) = v$;*
2. *any solution is unique, i.e., if $T(u_1) = T(u_2)$ for some u_1 and $u_2 \in U$ then $u_1 = u_2$;*
3. *any solution u depends continuously on the data v, i.e., if there are $u \in U$ and $v \in V$ with $T(u) = v$, then for any open set $O \subseteq U$ containing u there is an open $\tilde{O} \subseteq V$ containing v such that $\tilde{O} \subseteq T(O)$.*

If all are satisfied then we say the problem is well-posed.

Whilst well-posed problems are the easiest to understand and solve, commonly—as observed above, in at least one key example—in physics and applied mathematics one encounters ill-posed problems. There are various methods of approaching the latter, usually involving some techniques for restoration of well-posedness. We will not study these techniques, however the methods can be found in the literature, e.g., [48, 104, 126].

review see Section 2.4.

1.2 Singular boundary conditions and pencils

We now outline, briefly and roughly, the original problems of this work, which are linked to Calderón's in the following way. Consider again the conductivity boundary-value problem (1.1.1), and let γ be a twice continuously differentiable conductivity on Ω. Then with the substitutions

$$u = \gamma^{-1/2} v, \quad q = \gamma^{-1/2} \Delta \gamma^{1/2}, \quad g = \gamma^{1/2} h \tag{1.2.1}$$

we may derive the Schrödinger form of (1.1.1):

$$\begin{cases} (-\Delta + q)v &= 0 \quad \text{in} \quad \Omega, \\ v &= 0 \quad \text{on} \quad \Gamma_c, \\ v &= g \quad \text{on} \quad \Gamma. \end{cases}$$

The Dirichlet-to-Neumann operator on Γ may then be realised as the map

$$\Lambda_{q,\Gamma} : H^{1/2}(\Gamma) \ni g \mapsto -\partial_\nu v \restriction_\Gamma \in H^{-1/2}(\Gamma).$$

In 2008 Berry and Dennis [21] considered the Helmholtz equation

$$-\Delta u = k^2 u \quad \text{on } \Omega$$

where the two-dimensional Ω is one of two domains: the unit disc D or the upper half-plane H. They imposed the following boundary condition, written using, respectively, polar and Cartesian coordinates:

$$\begin{cases} \kappa(\vartheta) u(1,\vartheta) + \partial_r u(1,\vartheta) &= 0 \quad \big(\vartheta \in [0, 2\pi)\big), \quad \Omega = D, \\ \kappa(x) u(x,0) - \partial_y u(x,0) &= 0 \qquad (x \in \mathbb{R}), \qquad \Omega = H, \end{cases}$$

where κ is real-valued and 2π-periodic. Then they made a crucial observation: when κ is allowed to have a (non-empty, discrete set of) simple pole(s), the spectrum of the associated operator[1.6]

$$D(L_{\kappa,\Omega}) := \{u \in L^2(\Omega) \mid \Delta u \in L^2(\Omega), (\kappa u + \partial_\nu u) \restriction_{\partial\Omega} = 0\},$$
$$L_{\kappa,\Omega} u := -\Delta u,$$

fills the real line, and entirely comprises eigen-values of $L_{\kappa,\Omega}$. This would be contradictory if $L_{\kappa,\Omega}$ were self-adjoint[1.7], since the eigen-functions of a self-adjoint operator must be countable.

Marletta and Rozenblum offered a resolution to the "paradox" in 2009 [99], showing that a related operator is not actually self-adjoint by working roughly as follows. Consider the special geometry of the unit half-disc[1.8]

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1, x > 0\}$$

with boundary $\partial\Omega_1$ divided into the straight portion Γ_1 and semi-circular Γ_i:

$$\Gamma_1 = \{(0, y) \mid y \in (-1, 1)\};$$
$$\Gamma_i = \{(x, y) \mid x^2 + y^2 = 1, x > 0\}.$$

Then, for a fixed $\varepsilon > 0$, the operator

$$D(\ell) := \{U \in L^2(\Omega_1) \mid \Delta U \in L^2(\Omega_1), U \restriction_{\Gamma_i} = 0, (U + \varepsilon y \partial_\nu U) \restriction_{\Gamma_1} = 0\},$$
$$\ell U := -\Delta U, \tag{1.2.2}$$

admits decomposition, *via* separation of variables, into the orthogonal sum of

[1.6] We will more precisely define what we mean by operators on Hilbert space in Section 2.1.
[1.7] See Section 2.1.
[1.8] The notation is for consistency with Chapter 4.

ordinary differential operators

$$
\begin{aligned}
D(\ell_0) &:= \left\{u \in L^2(0,1;r\mathrm{d}r) \;\middle|\; r^{-1}\Big(ru'(r)\Big)' + \varepsilon^{-2}r^{-2} \in L^2(0,1;r\mathrm{d}r), u(1) = 0\right\},\\
\ell_0 u(r) &:= -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}(r)\right) - \frac{1}{\varepsilon^2 r^2}u(r),\\
D(\ell_n) &:= \left\{u \in L^2(0,1;r\mathrm{d}r) \;\middle|\; r^{-1}\Big(ru'(r)\Big)' - n^2r^{-2} \in L^2(0,1;r\mathrm{d}r), u(1) = 0\right\},\\
\ell_n u(r) &:= -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}(r)\right) + \frac{n^2}{r^2}u(r) \quad (n \geq 1).
\end{aligned}
$$

Their key observation is that of all the ℓ_n, only ℓ_0 is not self-adjoint, possessing a one-dimensional deficiency space[1.9].

To find self-adjoint restrictions of ℓ they added a boundary condition to the definition of ℓ_0. Taking the function $u_0(r) = \sin\left(\varepsilon^{-1}\log(r)\right)$ from the kernel of ℓ_0 they formed the new operator

$$
\begin{aligned}
D(\ell_0') &:= D(\ell_0) \cap \{u \in L^2(0,1;r\mathrm{d}r) \mid [u,u_0](r) \to 0 \ (r \to 0)\},\\
\ell_0' u(r) &:= -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}(r)\right) - \frac{1}{\varepsilon^2 r^2}u(r).
\end{aligned} \tag{1.2.3}
$$

Here $[u,v](r) := r(uv' - u'v)(r)$ is the associated Lagrange bracket. Then the orthogonal sum of ℓ_0' and ℓ_n $(n \geq 1)$ generates the self-adjoint partial differential operator

$$
\begin{aligned}
D(\ell') &= \left\{U \in L^2(\Omega_1) \;\middle|\; \Delta U \in L^2(\Omega_1), U\restriction_{\Gamma_i} = 0 = (U + \varepsilon y\partial_\nu U)\restriction_{\Gamma_1}, \int_{\Omega_1} u_0 \Delta U = 0\right\},\\
\ell' U &= -\Delta U.
\end{aligned}
$$

We may add a real-valued Schrödinger potential[1.10] q to form the boundary-

[1.9] The reasoning behind this and the other claims in this section will be given in Section 3.5.

[1.10] It will be taken from a class of functions to be defined more precisely in Chapter 3; in particular, we will allow q to be singular as its argument approaches 0.

value problem

$$\begin{cases} (-\Delta + q)U &= 0 \quad \text{in} \quad \Omega_1, \\ U + \varepsilon y \partial_\nu U &= 0 \quad \text{on} \quad \Gamma_1, \\ [U, u_0] &= 0 \quad \text{at} \quad 0, \\ U &= g \quad \text{on} \quad \Gamma_i. \end{cases} \tag{1.2.4}$$

Provided 0 is not in the spectrum of the operator underlying this problem[1.11], (1.2.4) uniquely determines a solution u for each $g \in H^{1/2}(\Gamma)$, since said operator is a self-adjoint perturbation of ℓ'. Thus we may form the Dirichlet-to-Neumann operator $\Lambda_{q,\Gamma_i} : H^{1/2}(\Gamma_i) \ni g \mapsto -\partial_\nu u \restriction_{\Gamma_i} \in H^{-1/2}(\Gamma_i)$.

Now suppose this Schrödinger potential is radially symmetric, $q(\mathbf{x}) = q(|\mathbf{x}|)$. Then the separation of variables decomposition may again be performed, and one finds that in the basis of angular eigen-functions on Γ_i the Dirichlet-to-Neumann operator Λ_{q,Γ_i} takes the form of the diagonal matrix

$$\begin{pmatrix} m(-\lambda_0) & 0 & 0 & \cdots \\ 0 & m(-\lambda_1) & 0 & \cdots \\ 0 & 0 & m(-\lambda_2) & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

in which the diagonal terms are all point evaluations of (a suitable generalisation of) the Weyl–Titchmarsh m-function for the problem

$$\begin{cases} -\frac{1}{r}\big(ru'(r;\lambda)\big)' + q(r)u(r;\lambda) &= \lambda \frac{1}{r^2} u(r;\lambda) \quad \big(r \in (0,1)\big), \\ \mathrm{SA}[u](r) &\to 0 \qquad (r \to 0). \end{cases} . \tag{1.2.5}$$

Here $\mathrm{SA}[u]$ denotes the Lagrange bracket $[u, u_0](r) = r(uu_0' - u'u_0)(r)$ if the ordi-

[1.11] For the class of q considered in this work, this operator has discrete spectrum. Moreover regarding the inverse problems considered, it does not matter whether we recover q or $q - \lambda$ for some fixed λ, meaning for our purposes we can translate 0 out of the spectrum of this operator; see the hypothesis on p. 114.

nary differential equation has two dimensions to its $L^2(0,1;rdr)$-solution space, whilst it is identically 0 if not[1.12]; the λ_n are angular eigen-values for the operator underlying (1.2.4) and are given by the sequence $\lambda_0 = -\varepsilon^{-2}, \lambda_n = n^2$ $(n \geq 1)$. The m-function is then defined (for any $\lambda \in \mathbb{C}\backslash\mathbb{R}$) to be $m(\lambda) = -u'(1;\lambda)/u(1;\lambda)$ for any non-trivial solution $u(\cdot\,;\lambda)$.

As we shall see, the ordinary differential problem (1.2.5) may be written using a pencil of operators. Roughly, the operator L, whose action on $u(r)$ is formally given by $-\frac{1}{r}\Big(ru'(r)\Big)' + q(r)u(r)$, and P_0, whose action is the multiplicative $\frac{1}{r^2}u(r)$, densely defined over the Hilbert space $L^2(0,1;rdr)$, may be assembled into

$$(L - \lambda P_0)u(\cdot\,;\lambda) = 0,$$

which is formally identical to the differential equation in (1.2.5). If we could determine q from knowledge of $m(-\lambda_n)$ for every n then we would have solved the inverse Schrödinger problem in this case of special geometry and singular boundary condition.

More generally, we can consider the multiplicative operator formally prescribed by $Pu(r) = w(r)u(r)$ where $w(r) = r^{-\nu}\Big(1 + o(1)\Big)$ and ν is some non-negative parameter. The pencil it generates, $L - \lambda P$, turns out to behave similarly over $L^2(0,1;rdr)$ to $L - \lambda P_0$. Inspired thus, the first inverse problem we will consider is as follows:

Inverse Problem 1. *Let $m(-\lambda_n)$ be given for each $n \geq 0$, and $w(r)$ be known to satisfy asymptotic equivalence with $r^{-\nu}$ for some $\nu \geq 0$ (in particular, w is not necessarily known exactly, although its growth parameter ν is). Determine q in the interval $(0,1)$.*

The following is then related to Inverse Problem 1, so we will also examine it.

[1.12] These cases are traditionally referred to as *limit-circle* and *limit-point*. A classical introduction to the terms is in Section 2.2, whilst the details of the case at hand may be found in Chapter 3.

Inverse Problem 1′. *Given Λ_{q,Γ_i}, recover q in Ω_1.*

Another class of problems we will consider is thus. We take the boundary-value problem (1.2.4) and "embed" its spatial domain Ω_1 into the larger Ω, with q now only being radially symmetric in Ω_1 and more arbitrarily specified elsewhere. Again, we take $\partial\Omega$ to be sufficiently smooth, and Γ a connected subset of $\partial\Omega$. We require moreover that $\Gamma \cap \Gamma_1 = \emptyset$, and we extend $f(y) = \varepsilon y$ to the rest of $\partial\Omega\backslash\Gamma$ so that it presents no simple zeros. Next we take not only u_0 from earlier but also $v_0(r) = \cos\left(\varepsilon^{-1}\log(r)\right)$, and let $\beta \in \mathbb{R}$ parameterise the boundary condition $[u, u_0 + \beta v_0](r, \vartheta) \to 0$ $(r \to 0)$. Now our Dirichlet-to-Neumann operator takes the form $\Lambda_{q,f,\beta} : H^{1/2}(\Gamma) \ni g \mapsto -\partial_\nu u \restriction_\Gamma \in H^{-1/2}(\Omega)$ where u uniquely solves

$$\begin{cases} (-\Delta + q)u &= 0 \quad \text{in} \quad \Omega, \\ u + f\partial_\nu u &= 0 \quad \text{on} \quad \partial\Omega\backslash\Gamma, \\ [u, u_0 + \beta v_0] &= 0 \quad \text{at} \quad 0, \\ u &= g \quad \text{on} \quad \Gamma. \end{cases}$$

Thus, the second inverse inverse problem we will examine is as follows:

Inverse Problem 2. *Let $\Lambda_{q,f,\beta}$ be given. Determine the potential q, the singular boundary condition f and the self-adjointness-imposing condition β.*

Most of the purpose of this thesis is to investigate the question of uniqueness for both Inverse Problems 1 and 2. Without giving away the spoiler of our methods, we simply state now that we will find an affirmative answer for the first and a conditional answer for the second. The latter we interpret as being a cloaking result, in which choosing f appropriately may "hide" the q to which it is adjoined. The final inverse problem considered in this work aims to restore full uniqueness to the triple (q, f, β). We achieve this by introducing a spectral parameter to the

problem, thus modifying our input data and consequently Inverse Problem 2. Consider the full-frequency spectral Dirichlet-to-Neumann map $\Lambda(\lambda) := \Lambda_{q-\lambda,f,\beta}$, mapping $H^{1/2}(\Gamma) \ni g \mapsto -\partial_\nu u(\cdot\,;\lambda) \restriction_\Gamma \in H^{-1/2}(\Gamma)$ with $u(\cdot\,;\lambda)$ uniquely solving

$$\begin{cases} (-\Delta + q)u(\cdot\,;\lambda) &= \lambda u(\cdot\,;\lambda) & \text{in} & \Omega, \\ -f\partial_\nu u(\cdot\,;\lambda) &= u(\cdot\,;\lambda) & \text{on} & \partial\Omega\backslash\Gamma, \\ [u(\cdot\,;\lambda), u_0 + \beta v_0] &= 0 & \text{at} & 0, \\ u(\cdot\,;\lambda) &= g & \text{on} & \Gamma. \end{cases}$$

In the final stages of the thesis we will prove a full uniqueness theorem for the following:

Inverse Problem 2′. *Let $\Lambda(\lambda)$ be given for every $\lambda \in \mathbb{R}$. Determine q, f and β.*

1.3 Outline

The remaining work is roughly divided into three parts. The first, Chapter 2, contains the introduction to the necessary preliminary material for understanding the other two parts. We will briefly run through the parts of the spectral theory of self-adjoint operators in a Hilbert space that we will need later on. The focus is on the finicky question of self-adjointness, and in particular the realisation of self-adjoint restrictions of symmetric operators. A great rôle will also be played by the spectral theorem for diagonalisation of self-adjoint operators. Following this, we will need to describe roughly the techniques for analysing one-dimensional second-order formally symmetric boundary-value problems, usually referred to as Sturm–Liouville type, since we will need to adapt such techniques to our situations. Specifically we will briefly cover the ideas developed by Weyl, Kodaira and Titchmarsh [134, 135, 136, 78, 127] for classifying such equations by the number of linearly independent solutions.

A natural and useful continuation of this line of thought is into the inverse spectral theory of Sturm–Liouville problems. We will need to run through the state-of-the-art regarding the uniqueness theorems for this, which are named after Borg and Marčenko [23, 101]. In particular we will focus on the recent approach by Avdonin, Mikhaylov and Rybkin [14] using boundary control methods, as it will play an important part in our own proofs.

We conclude the preliminary work with a survey of the work on uniqueness for Calderón's problem, starting in three dimensions and moving down to two, in which our last two inverse problems lie. In two dimensions we will concentrate on the recent work of Imanuvilov, Uhlmann and Yamamoto [67], who examine recovery from partial data, and to whose work our own problem is linked.

The second part of the three, in Chapter 3, is a detailed exposition of original work on the aforementioned second-order pencils, namely a proof of uniqueness for Inverse Problem 1. We develop a classification, analogous to that of Weyl–Kodaira–Titchmarsh, by which we may classify such pencils by the dimension of their solution-space. This allows us to define our boundary data rigorously, represented here as a single m-function.

As mentioned in our introductory descriptions, we want to recover the coefficient of a differential equation from discrete values of the m-function. Our method is *via* interpolation of the m-function from these values; in a certain classical case an interpolation formula exists, and we will simply derive another version of it for a different case. The final proof of uniqueness will follow by appropriately transforming our differential equation and then applying the Borg–Marčenko theorem. To conclude the chapter we will link the result back to the inverse problem inspired by the Berry–Dennis–Marletta–Rozenblum work.

In Chapter 4 we will disseminate the third and final main part. To start with

we will define precisely the underlying boundary-value problem for which Inverse Problems 2 and 2′ are defined. Subsequently we will define special oscillatory solutions to the boundary-value problem, and we will need to show density of these in a "fuller" set of solutions. These results, along with an adapted version of some of the Imanuvilov–Uhlmann–Yamamoto theorems, will prove conditional uniqueness—or, alternatively, cloaking—for Inverse Problem 2.

To draw a satisfying close to this conditional uniqueness, we will spend the remainder of that chapter developing an argument involving the asymptotics of negative eigen-values. This will establish a proper uniqueness result for Inverse Problem 2′, in the case of full-frequency data. Finally, in Chapter 5 we will offer our concluding thoughts on the work as well as ideas for future research.

In the Appendices we will collect some important results that would have otherwise detracted from the coherence of the presentation in this thesis. Some of these are novel and necessary to a complete understanding of the original work here, whilst others are simply existing results, either with the proof omitted and referred to, or included in a hopefully clearer style. Also attached is a collection of MATLAB codes that were utilised to illustrate the cloaking in Chapter 4.

We assume the reader is familiar with basic concepts in complex analysis (holomorphy, contour integration, residues, etc.), spectral theory on Hilbert spaces (spectrum, adjoint, compactness, resolvent sets and operators, extensions, spectral theorem), the theory of ordinary differential equations (unique continuation, fundamental systems, Wronskians, transformations) and the theory of partial differential operators (in particular Sobolev spaces of integer and non-integer order). Although we mention in passing some concepts—for example, *pseudo-differential operators*—familiarity with them will not be necessary. Since there is often variability in different authors' notation, and since we also introduce some new notation, for the rest of this introduction we fix a list of symbols and notation.

Dramatis Personae

$\mathbb{N}$ - the natural numbers $1, 2, 3, \ldots$;

$\mathbf{x}$ - a point $(x_1, \ldots, x_n)^T$ in $\mathbb{R}^n$;

$\mathbf{z}$ - an n-tuple $(z_1, \ldots, z_n)^T$ of complex numbers;

$\mathbf{x}^T, A^T$ - the transpose of a vector $\mathbf{x}$ or matrix A;

$\mathbf{z} \cdot \mathbf{w}$ - the scalar product of two complex n-tuples $\mathbf{z}$ and $\mathbf{w}$, given by $z_1 w_1 + \ldots + z_n w_n$;

$\overline{A}$ - the topological closure of $A \subseteq \mathbb{R}^n$, i.e., the intersection of all closed sets containing A;

$\mathrm{int}(A)$ - the topological interior of $A \subseteq \mathbb{R}^n$, i.e., the union of all open subsets of A;

∂A - the boundary of $A \subset \mathbb{R}^n$ defined to be $\overline{A} \backslash \mathrm{int}(A)$;

Ω - a bounded domain, i.e., an open, simply connected, bounded subset of $\mathbb{R}^n$; in $\mathbb{R}^2$ also a domain in $\mathbb{C}$ *via* $(x_1, x_2) \leftrightarrow x_1 + ix_2$;

$\Omega', \Omega_0, \Omega_1, \Omega_2$ - subdomains of Ω, i.e., subsets that are domains;

$\Gamma, \Gamma', \Gamma_j$ - connected subsets of boundaries $\partial\Omega, \partial\Omega'$, etc., taken to be open with respect to the $(n-1)$-dimensional manifold topology of the boundary;

Γ_c - the relative interior of the complement in $\partial\Omega$ of Γ;

$\overline{z}, \overline{f}$ - the complex conjugate of the complex number z or function f;

$\sqrt{\cdot}$ - the square root in $\mathbb{C}$, with branch cut on $(0, +\infty)$;

$\mathrm{d}\rho, \rho; \mathrm{d}x, \mathrm{d}\mathbf{x}$ - a measure on $\mathbb{R}$, or increasing function; Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^n$;

$\frac{\mathrm{d}}{\mathrm{d}x}, \frac{\mathrm{d}}{\mathrm{d}r}$, etc. - one-dimensional derivatives with respect to x, r, etc.;

$\partial_j, \partial_r, \partial_\vartheta$ - partial derivatives: in $\mathbb{R}^n$ with respect to the Cartesian variable x_j, and in $\mathbb{R}^2$ with respect to the polar variables r and ϑ;

$\partial_z, \partial_{\bar{z}}$ - derivative operations in $\mathbb{C}$: given, in terms of Cartesian coordinates x_1 and x_2, by $(\partial_1 - i\partial_2)/2$ and $(\partial_1 + i\partial_2)/2$;

∇, Δ - the del operator in $\mathbb{R}^n$, returning $(\partial_1, \ldots, \partial_n)^T$, and the Laplacian in $\mathbb{R}^n$, given by $\nabla \cdot \nabla$;

∂_ν - the normal derivative on a C^1 part of the boundary of a spatial domain, defined to be $\nu \cdot \nabla$ where ν is the outward-directed unit normal to said boundary;

$\Lambda_{q,\Gamma}, \breve{\Lambda}_{\gamma,\Gamma}$ - Dirichlet-to-Neumann operators on Γ;

$\Lambda_{q,f,\beta}(\lambda)$ - the spectral Dirichlet-to-Neumann operator on Γ for Berry–Dennis-type and self-adjointness-imposing boundary conditions on Γ_c;

H - usually a Hilbert space;

$\langle \cdot, \cdot \rangle_H, \| \cdot \|_H$ - the inner product and norm on H, the former linear and conjugate linear in respectively the first and second slots;

T^* - the Hilbert-space adjoint of a linear operator T;

λ - spectral parameter;

$\sigma(T)$ - the spectrum of a linear operator T;

$\sigma_{\mathrm{d}}(T)$ - the isolated eigen-values of T with finite multiplicity;

$\varrho(T)$ - the resolvent set of T.

2

Preliminaries and existing work

'The beginner [...] should not be discouraged if he finds that he does not have the prerequisites for reading the prerequisites.'

– Paul Halmos[2.1]

We will deal with inverse problems of both a spectral and non-spectral nature. In this chapter we will present much of the material requisite to a broad understanding of some differential problems in these areas. Broadly speaking, this will include a brief reminder of some elements of spectral theory, including the famous spectral theorem for self-adjoint operators, von Neumann's theory of self-adjoint realisations, as well as the concrete example of these that is the analysis of second-order ordinary differential operators. The first two are covered in Section 2.1 and the third in Section 2.2. To maintain a light tone in these sections we will focus on illustrating the results with examples. In Section 2.3 we review developments in the inverse spectral theory of Sturm–Liouville operators, with a focus on some recent results. We conclude the chapter in Section 2.4 by examining developments regarding the aforementioned inverse conductivity problem.

[2.1] *Measure Theory* (1950).

2.1 Spectral theory

Everything in this section can be found in any "good" reference on the topic. In the author's opinion one of the most lucid texts is Reed and Simon's treatise on mathematical physics, and in particular the first two volumes [116, 115]. Other good options include [92, 47, 19].

First, we remind the reader what we mean by spectrum, and the various ways it may be decomposed. Consider a linear operator T, closed, densely defined and mapping between Banach spaces X and Y, i.e., $T : D(T) \to Y$ with $\overline{D(T)} = X$. The *spectrum* of T is the set $\sigma(T)$ comprising all $\lambda \in \mathbb{C}$ such that $T - \lambda$ fails to have a bounded inverse. We also refer to $\mathbb{C}\backslash\sigma(T)$ as the *resolvent set* of T, and denote it

$$\varrho(T) := \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is boundedly invertible on all of } Y\}.$$

Spectral theory consists largely in the study of the relationship between T and $\sigma(T)$, alongside certain other spectral quantities. There are many different types of spectrum that can be defined, all of which have different qualitative properties and arise, roughly speaking, from consideration of how "badly" T fails to have a bounded inverse. We will describe a simple decomposition, parts of which play some rôle in later calculations.

We say $\lambda \in \mathbb{C}$ is an *eigen-value* for T if there is $v \in X$ with $Tv = \lambda v$ or equivalently, if $T - \lambda$ is not injective (it has non-trivial kernel). Such a v is called an *eigen-vector* for T. Corresponding to each eigen-value λ there is a subspace of X whose elements are all the eigen-vectors corresponding to λ. The dimension of this subspace is termed the (geometric) *multiplicity* of λ; an eigen-value with multiplicity one is called *simple*. The set of all eigen-values λ of T, with finite multiplicity and a neighbourhood containing no other spectrum of T, is called

the *discrete spectrum* and is labelled $\sigma_{\mathrm{d}}(T)$.

For its utility, we also refer to

$$\sigma_{\mathrm{ess}}(T) := \sigma(T) \backslash \sigma_{\mathrm{d}}(T)$$

as the *essential spectrum*. By mutual exhaustion, we have the disjoint unions

$$\varrho(T) \cup \sigma_{\mathrm{d}}(T) \cup \sigma_{\mathrm{ess}}(T) = \varrho(T) \cup \sigma(T) = \mathbb{C}.$$

We restrict our attention to Banach spaces possessing an inner product, known as Hilbert spaces. Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$, by convention conjugate linear in the second entry, and corresponding norm $\|h\|_H := \langle h, h \rangle_H^{1/2}$. Further, let T be a linear operator densely defined on H. Consider pairs (g, g^*) of elements of H that satisfy[2.2]

$$\langle Tf, g \rangle_H = \langle f, g^* \rangle_H \quad \big(f \in D(T)\big).$$

By density of $D(T)$, to each $g \in H$ there is at most one such corresponding g^*; this can also be concluded from Riesz's representation theorem. Hence this defines a linear map $T^* : g \mapsto g^*$ on the subspace $D(T^*)$ comprising all g for which there exists such a g^*. T^* is called the *adjoint* of T.

T is called *self-adjoint* if it equals its adjoint, i.e., $T = T^*$, and in particular $D(T) = D(T^*)$. This is a stronger condition than *symmetry*—usually written as $T = T^* \restriction_{D(T)}$, or $T \subseteq T^*$—and in general is a difficult property to verify. All self-adjoint operators have wholly real spectrum [115, Thm. X.1]. The same might not be true of symmetric operators.

Example 2. *Consider the Hilbert space* $H = L^2(0, 1)$, *and the operator* $T = i\frac{\mathrm{d}}{\mathrm{d}x}$ *acting on*

[2.2] There is always at least the pair $(0, 0)$.

the densely-defined domain[2.3] $D(T) = \{u \in L^2(0,1) \mid u' \in L^2(0,1), u(0) = 0 = u(1)\}$. *This operator is symmetric, since for each* u *and* v *in* $D(T)$ *we have the integration-by-parts formula*

$$\langle Tu, v\rangle_H = \int_0^1 (iu')\overline{v} = i[uv]_0^1 - i\int_0^1 u\overline{v}' = \langle u, Tv\rangle_H.$$

Now assume $\lambda \in \varrho(T)$ *and consider* $f \in L^2(0,1)$. *Then there is* $u \in D(T)$ *satisfying* $(T-\lambda)u = f$. *Multiplying by the integrating factor* $e^{i\lambda x}$ *we see that this* u *must satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{i\lambda x}u(x)\right) = -ie^{i\lambda x}f(x).$$

From this and $u(0) = 0$ *we may deduce that* $u(x) = -ie^{-i\lambda x}\int_0^x e^{i\lambda t}f(t)\mathrm{d}t$. *But then attempting to apply* $u(1) = 0$ *results in the requirement* $\int_0^1 e^{i\lambda t}f(t)\mathrm{d}t = 0$. *Clearly* f *must be chosen to depend on* λ *in some way, but then we are not free to choose arbitrary* $f \in L^2(0,1)$. *It follows that* $\lambda \notin \varrho(T)$. *Hence* $\varrho(T) = \emptyset$, *and consequently* $\sigma(T) = \mathbb{C}$.

Non-symmetric operators also need not have real spectrum.

Example 3. *Now let* $H = L^2(0,\pi)$, *and consider the operator*[2.4] $T_{\max} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ *acting on* $D(T_{\max}) = \{u \in L^2(0,\pi) \mid u'' \in L^2(0,\pi), u(0) = 0\}$. *This operator fails to be symmetric, since integrating by parts for any* $u, v \in D(T_{\max})$ *yields*

$$\begin{aligned}\langle T_{\max}u, v\rangle_H - \langle u, T_{\max}v\rangle_H &= \int_0^\pi (u''\overline{v} - u\overline{v}'')\\ &= [u'\overline{v} - u\overline{v}']_0^\pi\\ &= u'(\pi)\overline{v(\pi)} - u(\pi)\overline{v'(\pi)},\end{aligned}$$

which a priori *need not vanish. Moreover any* $\lambda \in \mathbb{C}$ *is an eigen-value, with eigen-*

[2.3] Here, and throughout this thesis, to keep the exposition as simple as possible, we take the liberty of assuming that when we write, e.g., $u', u'', \ldots$, the functions in question possess sufficient regularity to well-define these expressions. Here, for example, in defining $D(T)$ we abbreviate the pair of conditions $u \in \mathrm{AC}_{\mathrm{loc}}[0,1]$ and $u' \in L^2(0,1)$ solely by the second condition, since by our convention simply writing u' forces the first condition.

[2.4] The reason for the notation "$T_{\max}$" will become apparent shortly when we discuss self-adjoint realisations.

function $\sin(\sqrt{\lambda}x) \in D(T_{\max})$.

The operator $T_{\max}$ is an example of a *formally symmetric differential operator*, i.e., an operator for which the formal adjoint agrees formally with the original operator. The following example illustrates how the full adjoint of a formally symmetric operator can be symmetric, and how "between" the two lies a self-adjoint realisation of the formally symmetric operator.

Example 4. *Take the operator* $T_{\max}$ *from Example 3. We wish to calculate its adjoint. Since it is formally symmetric (its adjoint has identical action on functions) we simply need to calculate the domain of its adjoint. This can be expressed as*

$$\begin{aligned} D(T_{\max}^*) &= \left\{ v \in H \;\middle|\; \langle T_{\max} u, v\rangle_H = \langle u, T_{\max}^* v\rangle_H \left(u \in D(T_{\max})\right) \right\} \\ &= \left\{ v \in L^2(0,\pi) \;\middle|\; v'' \in L^2(0,\pi), \int_0^\pi (-u''v + uv'') = 0 \left(u \in D(T_{\max})\right) \right\}, \end{aligned}$$

and performing the integration by parts we see that $v \in D(T_{\max}^*)$ *if and only if* $v, v'' \in L^2(0,\pi)$ *and the boundary terms* $[uv' - u'v]_0^1$ *are* 0 *for every* $u \in D(T_{\max})$*. Since such* u *satisfy* $u(0) = 0$ *we see that* v *must satisfy* $v(0) = v(\pi) = v'(\pi) = 0$*, i.e.,*

$$D(T_{\max}^*) = \left\{ v \in L^2(0,\pi) \;\middle|\; v'' \in L^2(0,\pi), v(0) = 0 = v(\pi) = v'(\pi) \right\}.$$

As in Example 2 we may integrate by parts (twice, here) to see that $T_{\max}^*$ *is symmetric. Additionally* $T_{\max}^*$ *also has empty resolvent, though the argument*[2.5] *to establish this fact is more involved than that used in Example 2. In fact, the adjoint* $(T_{\max}^*)^* = T_{\max}$*, so by symmetricity we have* $T_{\max}^* \subset T_{\max}$*. We denote* $T_{\max}^* = T_{\min}$*.*

[2.5] One way is to attempt to calculate, using variation of parameters (Proposition C2), the solution u to $(T_{\max}^* - \lambda)u = f$ when f comes from a dense set in $L^2(0,\pi)$, e.g., the Fourier basis e^{ijx} $\left(x \in (0,\pi), j \in \mathbb{Z}\right)$.

Now consider $T_h \supset T_{\min}$ *with domain*

$$D(T_h) = \{v \in L^2(0,\pi) \mid v'' \in L^2(0,\pi), v(0) = 0 = v'(\pi) - hv(\pi)\},$$

for a given fixed $h \in \mathbb{R} \cup \{\infty\}$ *(with* $h = \infty$ *representing the Dirichlet condition* $v(\pi) = 0$*). Integrating by parts twice we easily observe that its adjoint is* T_h*, and clearly* $T_h \subset T_{\max}$*. So we have*

$$T_{\min} \subset T_h = T_h^* \subset T_{\max}.$$

The operator T_h *is called a* **self-adjoint realisation** *of the differential expression*

$$\tau u = -u'',$$

and $T_{\min}$ *and* $T_{\max}$ *are, respectively,* **minimal** *and* **maximal** *operators generated by the expression.*

Remark. *There are additional possible self-adjoint realisations of* τ*. Some involve non-local boundary conditions (periodic, anti-periodic or further mixed; for a treatment of all types of boundary condition in one formalism see, e.g., [81]), corresponding to different choices of minimal operator and extensions. We shall see in Example 5 a concrete proof that this* $T_{\min}$ *has a one-parameter family of self-adjoint extensions, necessarily possessing a separated boundary condition at* π*.*

The following theorem is due to von Neumann [131], though it was essentially proved in a simple case for a second-order differential operator by Weyl [136]. It provides an abstract representation of a method for constructing all self-adjoint realisations of a given symmetric minimal operator, generalising the above example.

Theorem 2.1 (von Neumann)**.** *Let* $T_{\min}$ *be symmetric, closed and densely-defined on a Hilbert space, and denote by* $T_{\max}$ *the adjoint* $T_{\min}^*$*. Then* $T \subseteq T_{\max}$ *is a self-adjoint*

extension of $T_{\min}$ *if and only if there is an* $n \in \mathbb{N} \cup \{0, \infty\}$ *and an* $n \times n$ *unitary matrix* U *(bijectively) mapping*

$$\ker(T_{\max} - i) \to \ker(T_{\max} + i).$$

Proof. A proof of the theorem may be found in any comprehensive text on the spectral theory of operators on Hilbert spaces, for example [109, 133, 19]. □

Remark. *The kernels* $\ker(T_{\max} \mp i)$ *are referred to as* deficiency spaces *and their respective dimensions* $n_\pm$ *are the* deficiency indices*. Then* $T_{\min}$ *has self-adjoint extensions if and only if* $n_- = n_+$*.*

Example 5. *For* $T_{\min}$ *as in Example 4 we find that* $\ker(T_{\max} \mp i)$ *are the one-dimensional spaces* $\operatorname{lin}\{(\sinh \cdot \cos \pm i \cos \cdot \sinh)(x/\sqrt{2})\}$*, between the two of which we may define the unitary* 1×1 *map* $e^{i\varphi}$ *for some* $\varphi \in [0, 2\pi)$*. Corresponding to each such* φ *there is precisely one* $h = \tan(\varphi/2) \in \mathbb{R} \cup \{\infty\}$ *parameterising a self-adjoint boundary condition. Hence every self-adjoint extension of* $T_{\min}$ *is of the form* T_h *as in Example 4.*

In the next section we will apply von Neumann's extension theory to a general class of second-order ordinary differential operators. The other main result we will need from the spectral theory of self-adjoint operators gives a means by which one may "diagonalise" an unbounded linear operator on a Hilbert space. In the case of an operator with purely discrete spectrum, the theorem provides an orthonormal Hilbert-space basis comprising the system of eigen-vectors of the operator, yielding a natural isomorphism of the space with $l^2(\mathbb{N})$. For operators with non-discrete spectrum, the theorem gives a continuous generalisation of this diagonalisation.

Theorem 2.2 (Spectral theorem of self-adjoint operators)**.** *Let* H *be a separable Hilbert space. Then* T *is a self-adjoint operator, closed and densely-defined on* H*, if and only if there is a right-continuous* projection-valued measure E_S*, for each measurable* $S \subseteq \mathbb{R}$*, defined by the conditions*

(i) *for each S the operator E_S is an orthogonal projection in H,*

(ii) *$E_\emptyset = 0$ and $E_{\mathbb{R}} = \mathbb{1}$,*

(iii) *if $S = \cup_{n=1}^{\infty} S_n$ and $S_n \subseteq S_{n+1}$ then for each $h \in H$ we have $\|E_S h - E_{S_n} h\|_H \to 0$ as $n \to \infty$, and*

(iv) *for each S and R the product $E_S E_R = E_{S \cap R}$,*

for which

$$T = \int_{\mathbb{R}} t \mathrm{d}E_{(-\infty,t]}.$$

We interpret this integral in the sense that for any Borel-measurable function f we have the functional calculus

$$\langle f(T)h, h\rangle_H = \int_{\mathbb{R}} f(t) \mathrm{d}\langle E_{(-\infty,t]}h, h\rangle_H,$$

where $\mathrm{d}\langle E_{(-\infty,t]}h, h\rangle_H$ is a complex-valued measure.

Proof. See, for example, [116, Thms. VIII.4–6], or any other good text on Hilbert space spectral theory. □

Remark. *If f is real-valued then $f(T)$ is also self-adjoint. A real number $\lambda \in \sigma(T)$ if and only if $E_{(\lambda-\varepsilon,\lambda+\varepsilon)} \neq 0$ ($\varepsilon > 0$), and is an isolated eigen-value of finite multiplicity if and only if $E_{(-\infty,t]}$ "jumps" at λ, i.e., $E_{(-\infty,\lambda]} - E_{(-\infty,\lambda-\varepsilon]} \nrightarrow 0$ ($\varepsilon \to 0$). Thus, in general*

$$f(T) = \int_{\sigma(T)} f(t) \mathrm{d}E_{(-\infty,t)},$$

and if $\sigma(T) = \sigma_d(T) = \{\lambda_n\}_{n=1}^{\infty}$ then

$$f(T) = \sum_{n=1}^{\infty} f(\lambda_n) E_{\lambda_n},$$

where $E_t := \lim_{\varepsilon\to 0}(E_{(-\infty,t]} - E_{(-\infty,t-\varepsilon]})$. *If* λ *is an eigen-value for* T *then* E_λ *is the projection onto the corresponding eigen-space. Then the final formula above provides the famous Fourier-type-series when applied to* $f(t) = 1 = t^0$, *whilst* $f(t) = t$ *yields the* diagonalisation *of* T. *In Appendix A we outline some key connections between this theory and the theory for Herglotz functions.*

Example 6. *Let* T_∞ *be as in Example 4. We can easily calculate that* T_∞ *has eigen-values* $\lambda_n = n^2 \in \mathbb{N}$ *and corresponding eigen-functions* $\varphi_n(x) = \frac{2}{\pi}\sin(nx)$ $\left(n \in \mathbb{N}, x \in (0,\pi)\right)$ *normalised so their* $L^2(0,\pi)$*-norm is* 1. *Then the eigen-spaces are the spans of the eigen-vectors, so each* E_{λ_n} *projects onto* $\operatorname{lin}\{\varphi_n\}$. *Since* E_t *are all orthogonal projections, we may calculate that* $E_{\lambda_n}h = \frac{2}{\pi}\int_0^\pi h(x)\sin(nx)\mathrm{d}x = \langle h, \varphi_n\rangle_{L^2(0,\pi)}$. *Thus we have the standard Fourier-series representation* $h = \sum_{n=1}^\infty \langle h, \varphi_n\rangle_{L^2(0,\pi)}\varphi_n$. *We will use such series decompositions later.*

A useful corollary to the spectral theorem is that it implies the normalised sequence of eigen-functions of a self-adjoint operator is in fact an orthonormal basis for the Hilbert space. This can be written in the following way:

Corollary 2.1. *Suppose* T *is self-adjoint over the Hilbert space* H, *and* $\sigma(T) = \sigma_d(T)$. *Denote the eigen-values of* T *as the sequence* $(\lambda_n)_{n=1}^\infty$ *and the corresponding normalised eigen-functions as* $(\varphi_n)_{n=1}^\infty$. *Then for any* $u \in H$ *and* $\lambda \in \varrho(T)$ *we have*

$$u = \sum_{n=1}^\infty \langle u, \varphi_n\rangle_H \varphi_n,$$

which converges in norm.

2.2 Weyl–Kodaira–Titchmarsh theory

We now relate some of the abstract theory of the last section to the second-order ordinary differential equations named after Sturm and Liouville, in parts roughly

following the recent formulation of Schmidt [119]. Any good text [37, 92, 133, for example] on the topic will do equally well as an introduction, and a well-researched review and collection of the proofs was also put together by Zettl [140]. The original analysis was conducted by Weyl [134, 135, 136] and furthers the results of Sturm and Liouville from their series of papers (the lengthy list of references can be found in the bibliography of [95]). A comprehensive exposition of the history from Weyl onwards was given by Everitt [50], to which the reader is directed for references.

Let $-\infty < a < b \leq +\infty$ and suppose we have functions $p, w > 0$ and q on (a, b) with $1/p, q, w \in L^1_{\mathrm{loc}}[a, b)$. Then the *Sturm–Liouville equation*

$$-\big(pu'(\cdot\,;\lambda)\big)' + qu(\cdot\,;\lambda) = \lambda w u(\cdot\,;\lambda) \text{ on } (a, b) \tag{2.2.1}$$

is called *regular at* b if $1/p, q, w \in L^1(a, b)$. If this is not the case it is called *singular at* b. If λ is non-real one can show that in the regular case the space of solutions of (2.2.1) that are in

$$L^2(a, b; w(x)\mathrm{d}x) := \left\{ v : (a, b) \to \mathbb{C} \;\middle|\; \int_a^b w(x)|v(x)|^2 \mathrm{d}x < \infty \right\}$$

is two-dimensional. In the singular case this space may have dimension two or less than two—independently[2.6] of $\lambda \in \mathbb{C}$—and the equation is then respectively referred to as *limit-circle* or *limit-point, at* b. Moreover we may classify the equation as *oscillatory* or *non-oscillatory* at b according to whether its solutions have, respectively, infinitely or finitely many zeros in any given neighbourhood of b.

Remark. *The terms limit-point and limit-circle originate in the considerations of Weyl as to wherein lies the soon-to-be-defined coefficient* $m(\lambda)$*, with a fixed* λ *from the upper half-plane* $\mathbb{C}^+$ *and a varied boundary condition at* b*. For a full explanation the reader*

[2.6] See [92, Thm. II.2.2].

should consult any of the aforementioned appropriate references, e.g., [92, Sec. II.2]. It is important we also note that, whilst regular equations always have bounded continuous solutions, their singular counterparts, as the name suggests, can have solutions becoming arbitrarily large or oscillating arbitrarily rapidly in a neighbourhood of the singular end-point [92, Ch. I & II].

Suppose $\alpha, \beta \in [0, \pi)$, and in either of the two-dimensional cases (regular or limit-circle) affix to (2.2.1) the boundary condition

$$[u, u_0](b^-; \lambda) = 0, \tag{2.2.2}$$

where $u_0 \in L^2(a, b; w(x)\mathrm{d}x)$ is any given pre-specified solution to $-(pu_0')' + qu_0 = 0$. Additionally in all cases apply the boundary condition

$$u(a; \lambda)\cos(\alpha) = [pu'](a; \lambda)\sin(\alpha). \tag{2.2.3}$$

The conditions together are an example of the separated boundary conditions mentioned in the previous section. It is not hard to prove the classical result:

Lemma 2.1. *The differential expression*

$$\ell_{p,q,w}(x, \cdot) := \frac{1}{w(x)}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + q(x)\right) \quad \big(x \in (a, b)\big) \tag{2.2.4}$$

has a self-adjoint realisation given by the operator

$$\begin{aligned} D(L) := \Big\{ u \in L^2\big(a, b; w(x)\mathrm{d}x\big) \,\Big|\, \big(-(pu')' + qu\big)/w \in L^2\big(a, b; w(x)\mathrm{d}x\big), &\text{ (2.2.3) holds} \\ \text{and, if } \ell_{p,q,w} \text{ is not limit-point at } b, &\text{ (2.2.2) holds} \Big\}, \\ Lu := \frac{1}{w}\big(-(pu')' + qu\big). \end{aligned}$$

Proof. See, for example, [37, Sec. 9.3] or [92, Sec. II.2]. □

Remark. *Applying these boundary conditions is precisely an example in action of von Neumann's theory for self-adjoint extensions.*

Now suppose we drop the boundary condition at a. In all three cases—regular, limit-circle and limit-point—when $\mathrm{Im}(\lambda) \neq 0$ we are left with just one $L^2(a,b;w(x)\mathrm{d}x)$ solution of $Lu = \lambda u$, up to multiplication by a constant. Consider for such a solution $u(\cdot\,;\lambda)$ the fraction

$$m(\lambda) := \frac{\cos(\alpha)[pu'](a;\lambda) - \sin(\alpha)u(a;\lambda)}{\sin(\alpha)[pu'](a;\lambda) + \cos(\alpha)u(a;\lambda)} \tag{2.2.5}$$

called the *Weyl–Titchmarsh m-function*. One can show that m is in the Herglotz class[2.7] of functions that analytically map the upper and lower half-planes $\mathbb{C}^{\pm}$ to themselves. Thanks to (A.2) it admits the Stieltjes integral representation [92, Thm. II.5.2]

$$m(\lambda) = A + B\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right)\mathrm{d}\rho(t) \tag{2.2.6}$$

for some increasing ρ satisfying the growth condition $\int(1+t^2)^{-1}\mathrm{d}\rho(t) < \infty$. By (A.3), Stieltjes' inversion formula [92, Thm. II.5.1], we have

$$\rho(y) - \rho(x) = \lim_{\varepsilon \searrow 0} \frac{1}{\pi i} \int_x^y \Big(m(\nu + i\varepsilon) - m(\nu - i\varepsilon)\Big)\mathrm{d}\nu. \tag{2.2.7}$$

Moreover m has analytic continuation from $\mathbb{C}^+$ to $\mathbb{C}^-$ through any interval $I \subseteq \mathbb{R}\backslash\sigma(L)$. This is owing to Theorem A3 and the fact that m can be analytically extended to the full resolvent set of L.

Furthermore one can show that m has poles at the eigen-values of L—this is done in a particular case in Section 3.3, but is also crudely observed from the fact that, as λ approaches an eigen-value λ_0, $m(\lambda)$ has denominator that is exactly $O(1/|\lambda - \lambda_0|)$ as well as bounded numerator. Now suppose we are given such an

[2.7] See Appendix A for a discussion of some key properties of these functions. In Section 3.3 we show briefly the standard calculation proving the Herglotz property.

isolated eigen-value λ. Then we may find $\varepsilon, \delta > 0$ such that

$$2\pi i \operatorname{Res}(m;\lambda) = -\int_{\lambda-\delta}^{\lambda+\delta} \Big(m(s+i\varepsilon)-m(s-i\varepsilon)\Big)\mathrm{d}s - i\int_{-\varepsilon}^{\varepsilon} \Big(m(\lambda-\delta+i\sigma)-m(\lambda+\delta-i\sigma)\Big)\mathrm{d}\sigma.$$

The second integral is across the real line, but crosses subintervals of the resolvent set. Since m is continuous on this contour it is also bounded, so taking the limit as $\varepsilon \to 0$ we find

$$0 \neq \operatorname{Res}(m;\lambda) = -\Big(\rho(\lambda+\delta) - \rho(\lambda-\delta)\Big).$$

The residue was independent of δ all along, so we see that ρ has a jump discontinuity at λ. In addition the same argument applied to real points in the resolvent set shows that ρ is constant at such points. These facts will prove crucial in Section 3.3.

Not only does the m-function have poles at the eigen-values, but it turns out that further spectral information is encoded in the function. Namely the residues of m at the eigen-values are the so-called *norming constants*, where $\alpha := \|u\|_{L^2(a,b)}$ is a norming constant for an eigen-value λ if u solves the Sturm–Liouville problem

$$\begin{cases} -\big(pu'\big)' + qu &= \lambda w u \quad \text{on} \quad (a,b), \\ [u,u_0](b^-;\lambda) &= 0, \\ u(a) &= 1. \end{cases}$$

It is for this reason m is often taken as the starting data in inverse spectral theory for the Sturm–Liouville problem and its generalisations. This is discussed in more detail in Section 2.3. For a deeper discussion of some aspects to this topic see Appendix C, in which we mention asymptotic expansions of fundamental systems and of $m(\lambda)$, as well as Appendix A where we detail general properties of Herglotz functions.

2.3 Inverse spectral theory: one-dimensional review

An excellent illustration of the utility of spectral theory for differential operators is in the well-developed inverse theory for Sturm–Liouville equations. We will review some classical and more recent uniqueness results for this class of inverse problems. To start with we will define the inverse problem, then state and offer the ideas in one of the first proofs of the first uniqueness theorem offered for it, independently established by Borg [23] and Marčenko [101]. Following that we will explain how the result has been improved, finishing with the current strongest version: a local uniqueness result also independently due to Simon [120] and Bennewitz [20]. To round off the section we will mention a key result in one of the proofs of this current version, which provides an alternative representation for the m-function that we will need to use later.

Let $0 < b \leq \infty$, and take any real-valued and locally integrable q on $(0, b)$ that allows the differential expression

$$\ell_q(x, \cdot) := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x) \quad \big(x \in (0, b)\big),$$

to be regular at 0. Then define the boundary expression

$$\mathbb{C} \ni \mathrm{SA}_q[u] := \begin{cases} 0 & \text{if } \ell_q \text{ is limit-point at } b, \\ [u, u_0](b^-) & \text{otherwise,} \end{cases} \tag{2.3.1}$$

where u_0 is non-trivial and solves $-u_0'' + qu_0 = 0$ on $(0, b)$. Thanks to the results of the previous section, for any $\lambda \in \mathbb{C}\backslash\mathbb{R}$ the problem

$$\begin{cases} \ell_q\big(x, u(x;\lambda)\big) &= \lambda u(x;\lambda) \quad \big(x \in (0, b)\big), \\ \mathrm{SA}_q[u(\cdot\,;\lambda)] &= 0, \\ u(\cdot\,;\lambda) &\in L^2(0, b) \end{cases} \tag{2.3.2}$$

has a one-dimensional space of solutions. This space defines the Weyl–Titchmarsh m-function. Let $u(\cdot\,;\lambda)$ be any non-trivial element of this space. Then

$$m(\lambda) := \frac{u'(0;\lambda)}{u(0;\lambda)} \tag{2.3.3}$$

is well defined for every $\lambda \in \mathbb{C}\backslash\mathbb{R}$, and is moreover a Herglotz function[2.8].

Inverse Problem 3. *Let $m(\cdot)$ be a Herglotz function. Recover the potential q (if it exists) for which the problem (2.3.2) generates $m(\cdot)$ in the above fashion.*

One of the most celebrated results was made public independently by Borg and Marčenko in the same year[2.9] [23, 101].

Theorem 2.3 (Borg–Marčenko, 1952)**.** *Given a Weyl–Titchmarsh m-function $m(\cdot)$ there is at most one such differential expression ℓ_q that may have given rise to it.*

Proof. We will present the ideas of the Gel'fand–Levitan proof [54]. In its entirety this approach can be found in, e.g., [91, Sec. 2.2]. The main tool used is the *transformation kernel*, which is the solution $K(\cdot,\cdot)$ to the wave-equation problem

$$\begin{cases} \partial_1^2 K(x,t) - \partial_2^2 K_(x,t) &= q(x)K(x,t) & (0<t<x<\infty), \\ \partial_1 K(x,x) + \partial_2 K(x,x) &= \frac{1}{2}q(x) & (0<x<\infty), \\ \partial_2 K(x,0) &= 0 & (0<x<\infty). \end{cases}$$

This is not a hyperbolic initial-boundary-value problem, so standard existence and uniqueness theorems fail[2.10] Instead one may guarantee the existence and

[2.8] For simplicity we restrict our attention to the m-function of the form (2.2.5) with $\alpha = 0$, although the results in this section would hold for any choice of α.

[2.9] Although strictly speaking Borg was the first in 1946, he published in German in a relatively little-known journal [22]. A linked result was due to Levinson in 1949 [90]. Marčenko first published his result in 1950, in a Soviet journal [100], and it was not until 1952 that his result was publicised globally.

[2.10] In fact the diagonal $x = t$ turns out to be a characteristic curve of the differential equation, on which one is not traditionally supposed to specify "boundary" data.

uniqueness of $K(x,t)$ $(0 < t < x < \infty)$ by *ad hoc* methods such as the analysis in [91, Ch. 1].

This function $K(\cdot,\cdot)$ owes its name to the rôle it plays in transforming the solutions of one Sturm–Liouville problem to another. Namely, if $c_0(\cdot\,;\lambda)$ solves

$$\begin{cases} -c_0''(x;\lambda) &= \lambda c_0(x;\lambda) \quad (0 < x < \infty), \\ c_0(0;\lambda) &= 1, \\ c_0'(0;\lambda) &= 0, \end{cases} \tag{2.3.4}$$

then the solution to the initial-value problem

$$\begin{cases} -c''(x;\lambda) + q(x)c(x;\lambda) &= \lambda c(x;\lambda) \quad (0 < x < \infty), \\ c(0;\lambda) &= 1, \\ c'(0;\lambda) &= 0 \end{cases}$$

—provided λ is not in the spectrum of the associated operator—is given by

$$c(x;\lambda) := c_0(x;\lambda) + \int_0^x K(x,t)c_0(t;\lambda)\mathrm{d}t. \tag{2.3.5}$$

To show that q is uniquely determined by $m(\lambda)$ one proceeds roughly as follows. Firstly construct a spectral measure $\mathrm{d}\rho(\lambda)$ *via* Stieltjes' inversion formula (A.3). Secondly show this spectral measure defines an integral equation for $K(x,t)$. Finally conclude K and therefore $q(x) = 2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x)$ are uniquely determined. □

This deep result not only establishes uniqueness of ℓ_q from $m(\cdot)$, it also provides through the above proof a means of reconstructing the Schrödinger potential $q(\cdot)$ from the data of this spectral function. Precursor results include Ambarzumian [6], who in 1929 showed the Neumann-boundary-condition eigen-

values $0^2, 1^2, 2^2, 3^2, \ldots$ uniquely determine the 0 potential[2.11] on the interval $(0, \pi)$, as well as Tikhonov, who in 1949 proved a limited version of the theorem, with certain restrictions on q [125]. As we saw above, the Gel'fand–Levitan approach of 1951 [54] uses the spectral measure, which thanks to the Herglotz representation and Stieltjes inversion formulae—(A.2) and (A.3)—is completely equivalent to the m-function. Kreĭn also published a contemporary pair of papers in 1951 and 1953, outlining his own method involving transfer functions [82, 83].

Since this pioneering work of the '50s, there was essentially no progress made until 1999. We will now present this leap forward. Elements of the new theory associated with it will inform the original work presented later in this thesis. The main improvement it offers is to relax the requirement of knowing the m-function everywhere, instead necessitating only negative-exponentially good knowledge of $m(\lambda)$ as $\lambda \to \infty$ on a ray in the upper half-plane $\mathbb{C}^+$.

Theorem 2.4 (Simon, 1999; Gesztesy–Simon, 2000; Bennewitz, 2001)**.** *Consider any real-valued $q_j \in L^1_{\mathrm{loc}}[0, b_j)$ $(j = 1, 2)$ and $b_1, b_2 \in (0, \infty]$, and take $m_j(\lambda)$ $(\lambda \in \mathbb{C}\backslash\mathbb{R}, j = 1, 2)$ to be the Dirichlet m-functions[2.12] associated respectively with the differential expressions*

$$\ell_{q_j}(x, \cdot) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q_j(x) \quad (x \in (0, b_j), j = 1, 2),$$

with self-adjoint boundary conditions at b_j if needed, i.e., we impose (2.3.1) with q replaced by q_1 or q_2. In addition let $0 < \vartheta < \pi/2$ and $0 < a < \min\{b_1, b_2\}$. Define $\mathcal{R}_\vartheta := \{te^{i\vartheta} \mid t \in (0, \infty)\}$, a ray in the upper half-plane. If, for every $\varepsilon > 0$, as $\mathcal{R}_\vartheta \ni \lambda \to \infty$ we have

$$m_1(\lambda) - m_2(\lambda) = O(e^{-2a(1-\varepsilon)\mathrm{Re}(\sqrt{\lambda})}), \tag{2.3.6}$$

[2.11] This is a curious result, since in general one needs more than just one set of eigen-values to ensure uniqueness, this being the original result of Levinson [90]. Much more recently, Davies [41] has published a substantial generalisation of Ambarzumian's theorem to compact Riemannian manifolds, compact quantum graphs and finite combinatorial graphs, all with Neumann boundary conditions.

[2.12] These are m-functions of the form (2.2.5) with $\alpha = 0$.

then $q_1 = q_2$ *almost everywhere on* $(0, a)$.

Proof. Simon's original version [120, Thm. 1.2] is less general, requiring knowledge of $m(\cdot)$ in a sector in the upper-half-plane. The proof involves writing the m-function in a novel form, discussed shortly in Theorem 2.5, and originally formulated in [120, Thm. 2.1]. The key object in this formulation is the so-called *A-amplitude*. Simon's method was continued by Simon and Gesztesy [56, Thm. 1.1]. The latest proof by Bennewitz [20, Thm. 1] is more direct, relying on asymptotic estimates for the m-function, special solutions of the differential equation, and some classical complex analysis. □

Subsequent work by Avdonin, Mikhaylov and Rybkin [14] links the A-amplitude with the *boundary control approach* towards the inverse spectral theory of the one-dimensional Schrödinger operator[2.13]. In particular, they show that the so-called *response operator* for the inverse problem has integral kernel in one-to-one constructive correspondence with the A-amplitude. The main outcome is a new way to compute the m-function, giving us a means to solve our own novel inverse problems later. We will briefly present their main results [14, Eq. (2.15), Thm. 2].

Theorem 2.5 (Avdonin–Mikhaylov–Rybkin, 2007)**.** *Suppose* $q \in l^\infty(L^1)(0,\infty)$, *i.e.*,

$$\|q\| := \|q\|_{l^\infty(L^1)(0,\infty)} := \sup_{x\geq 0} \int_x^{x+1} |q| < \infty.$$

Then for $m(\lambda)$ *defined in* (2.3.3), *there is* $A(\alpha)$ $(\alpha \geq 0)$ *called the* A-amplitude *such that*

$$m(-\kappa^2) = -\kappa + \int_0^\infty e^{-2\kappa\alpha} A(\alpha) \mathrm{d}\alpha, \tag{2.3.7}$$

[2.13] See, for example, [18] for an extensive review of the history of boundary control, and [13] for a further analysis of its connections with all approaches to the inverse spectral theory for Sturm–Liouville problems, as well as a healthy list of references.

and the integral is absolutely convergent when $\mathrm{Re}(\kappa) > 2\max\{\sqrt{2\|q\|}, e\|q\|\}$. *Moreover, for any* $\alpha \geq 0$, *the A-amplitude satisfies*

$$|A(\alpha) - q(\alpha)| \leq \frac{1}{2}\left(\int_0^\alpha |q|\right)^2 \left(e^{2\sqrt{2\|q\|}\alpha} + \frac{e^{2e\|q\|\alpha}}{\sqrt{2\pi}}\right). \tag{2.3.8}$$

Remark. *In fact they do not prove the result for* $A(\alpha)$, *but rather for the integral convolution kernel* $r(\alpha) = -A(\alpha/2)/2$ *of the response operator. See, for example, [12, 11]. Note that there is a deep link between the boundary control approach and the Gel'fand–Levitan method, which is carefully highlighted in [13].*

This theorem may be understood as an implicit connection between m and q that is in most situations more applicable or useful than the approach of directly solving the differential equation. It is this connection—and that in Simon's version, which has a different bound in place of the right-hand side of (2.3.8)—which largely underlies Simon's original proof of Theorem 2.4 [120] and his and Gesztesy's subsequent development of an alternative approach to this inverse spectral theory [55]. Moreover, the power and utility of this theorem is apparent from its appearance in the proof of new results for interpolation of m-functions. We will make use of these in Section 3.3.

2.4 Calderón's inverse conductivity problem

All the work described in Section 2.3 relates to one-dimensional Schrödinger operators, with a real-valued potential $q \in L^1_{\mathrm{loc}}[0, b)$ for some $0 < b \leq \infty$. These take the form

$$D(S_q) := \{u \in L^2(0,b) \mid -u'' + qu \in L^2(0,b), u(0) = 0, \mathrm{SA}_q[u] = 0\},$$
$$S_q u := -u'' + qu.$$

Here $\mathrm{SA}_q[u] = 0$ denotes the self-adjointness-imposing boundary condition (2.2.2) at b if the differential expression underlying S_q is limit-circle or regular there, or is the null condition $0 = 0$ if the differential expression is limit-point at b. The resulting m-function can be interpreted as a map in the following way. Let $c \in \mathbb{C}$ and suppose $u(\cdot\,;\lambda)$ is the square-integrable solution to

$$\begin{cases} -u''(x;\lambda) + q(x)u(x;\lambda) &= \lambda u(x;\lambda) \quad (0 < x < b), \\ \mathrm{SA}_q[u] &= 0, \\ u(0) &= c. \end{cases} \tag{2.4.1}$$

If $u(x;\lambda) = 0$ $(0 < x < b)$ then clearly c must have already been zero. If the Dirichlet datum $c \neq 0$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$ then there is a valid and non-trivial square-integrable solution to the problem. We may then make the Neumann "measurement" $u'(0;\lambda)$ and form the well-defined ratio

$$m(\lambda) = \frac{u'(0;\lambda)}{u(0;\lambda)},$$

which is clearly a multiplicative operator mapping $u(0;\lambda) \mapsto u'(0;\lambda)$. In the spirit of the discussion of Calderón's problem in Section 1.1, then, it seems reasonable to call it a Dirichlet-to-Neumann map. We will now spell out its link with as well as formalise rigorously the concepts from that section. Following this we will run through the developments in uniqueness for Calderón's problem.

As before, take any bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$ fixed) whose boundary is now piecewise C^1 and admits the decomposition into the non-empty, connected, *accessible* Γ and remaining, *inaccessible* $\Gamma_c := \partial\Omega\backslash\Gamma$. Suppose we have a measurable function $\gamma : \Omega \to (0,\infty)$ bounded away from 0 and ∞, called a *conductivity*. Let $f \in H^{1/2}(\Gamma)$ and take ∂_ν to denote the outward-directed normal derivative on

$\partial\Omega$. Then, as with the m-function and Section 1.1, the solution $u(\cdot\,;\lambda)$ to

$$\begin{cases} -\nabla\cdot\big(\gamma\nabla u(\cdot\,;\lambda)\big) &= \lambda\gamma u(\cdot\,;\lambda) \quad \text{in} \quad \Omega, \\ u(\cdot\,;\lambda) &= 0 \quad \text{on} \quad \Gamma_c, \\ u(\cdot\,;\lambda) &= f \quad \text{on} \quad \Gamma, \end{cases} \tag{2.4.2}$$

defines a map—provided λ is not in the spectrum of the $L^2(\Omega)$-operator associated with this problem—in the following way. We may make measurements $\gamma\partial_\nu u$ on Γ; this can be seen to be a map from $u(\cdot\,;\lambda)\restriction_\Gamma$ to $-\gamma\partial_\nu u(\cdot\,;\lambda)\restriction_\Gamma$, which we denote by

$$\breve{\Lambda}_{\gamma,\Gamma}(\lambda) : f \mapsto -\gamma\partial_\nu u(\cdot\,;\lambda) \in H^{-1/2}(\Gamma). \tag{2.4.3}$$

Definition 2.1. *The map $\breve{\Lambda}_{\gamma,\Gamma}(\lambda)$ is called the Dirichlet-to-Neumann map for the conductivity γ and accessible boundary Γ.*

Remark. *We map to negative Neumann data to ensure the so-called half-plane property, proved in Proposition 2.2. The same convention applies to $m(\lambda)$, since $u'(0;\lambda)$ is a negative normal derivative at the boundary of the interval $(0,b)$. In Chapter 3 we will examine an m-function at the other end of the interval $(0,1)$, and for the same reason we will map $u(1;\lambda) \mapsto -u'(1;\lambda)$.*

The Dirichlet-to-Neumann map can also be realised as a graph: the set

$$\breve{C}_{\gamma,\Gamma}(\lambda) := \Big\{\big(u, \gamma\partial_\nu u\big)(\cdot\,;\lambda)\restriction_\Gamma \;\Big|\; u(\cdot\,;\lambda), \nabla\cdot\big(\gamma\nabla u(\cdot\,;\lambda)\big)/\gamma \in L^2(\Omega), \\ \big(\nabla\cdot(\gamma\nabla) + \lambda\gamma\big)u(\cdot\,;\lambda)\restriction_\Omega = 0 = u(\cdot\,;\lambda)\restriction_{\Gamma_c}\Big\}$$

is referred to as the set of Cauchy boundary data, and is equivalent to $\breve{\Lambda}_{\gamma,\Gamma}(\lambda)$ when λ is not an element of the spectrum. For a fixed, given λ it can be more convenient to work with these data, though it is not necessary. In the more general case we will prefer to work with $\breve{\Lambda}_{\gamma,\Gamma}(\lambda)$ and its equivalent in the Schrödinger problem, because of the following results.

Proposition 2.1 (Analyticity of the Dirichlet-to-Neumann map)**.** *For each real-valued $\gamma \in C^2(\overline{\Omega})$ with $\gamma, 1/\gamma \in L^\infty(\Omega)$ the map $\lambda \mapsto \check{\Lambda}_{\gamma,\Gamma}(\lambda)$ is an analytic function on the resolvent set $\varrho(A)$ of the operator*

$$D(A) := \{u \in L^2(\Omega) \mid \nabla \cdot (\gamma \nabla u)/\gamma \in L^2(\Omega), u \restriction_{\partial\Omega} = 0\}$$
$$Au := -\frac{1}{\gamma}\nabla \cdot (\gamma \nabla u),$$

in the sense that there is an operator $\check{\Lambda}'_{\gamma,\Gamma}(\lambda) : H^{1/2}(\Gamma) \to H^{3/2}(\Gamma)$ $\big(\lambda \in \varrho(A)\big)$ for which

$$\left\| \frac{\check{\Lambda}_{\gamma,\Gamma}(\lambda) - \check{\Lambda}_{\gamma,\Gamma}(\mu)}{\lambda - \lambda_0} - \check{\Lambda}'_{\gamma,\Gamma}(\mu) \right\|_{B\left(H^{1/2}(\Gamma)\right)} \to 0 \quad \big(\mu \to \lambda \in \varrho(A)\big),$$

where $\|\cdot\|_{B\left(H^{1/2}(\Gamma)\right)}$ denotes the standard operator norm on $H^1/2(\Gamma)$.

Proof. Choosing any $w_0 \in L^2(\Omega)$ taking the value f on Γ, and satisfying $w_0 \restriction_{\Gamma_c} = 0$ and $\nabla \cdot (\gamma \nabla w_0)/\gamma \in L^2(\Omega)$, we see that the solution to the boundary-value problem (2.4.2) is given by

$$\left[\mathbb{1} - (A - \lambda)^{-1}\left(-\frac{1}{\gamma}\nabla \cdot (\gamma \nabla) - \lambda\right)\right] w_0.$$

Hence, in terms of the trace maps

$$\begin{aligned} D(\mathrm{tr}) &= \{v \in L^2(\Omega) \mid \nabla \cdot (\gamma \nabla v)/\gamma \in L^2(\Omega), v \restriction_{\Gamma_c} = 0\}, & \mathrm{tr} \restriction_{C^0(\Omega)} : v &\mapsto v \restriction_\Gamma, \\ D(\partial_\nu) &= \{v \in L^2(\Omega) \mid v \restriction_{\Gamma_c} = 0\}, & \partial_\nu \restriction_{C^1(\Omega)} : v &\mapsto \nu \cdot (\gamma \nabla v \restriction_\Gamma) \end{aligned}$$

where ν is the outward directed unit normal, we see that

$$\check{\Lambda}_{\gamma,\Gamma}(\lambda) = \partial_\nu \left[\mathbb{1} - (A - \lambda)^{-1}\left(-\frac{1}{\gamma}\nabla \cdot (\gamma \nabla) - \lambda\right)\right] \mathrm{tr}^{-1},$$

where by tr^{-1} we mean any right-inverse of tr. Thus we may calculate, applying the resolvent formula [116, Thm. VIII.2], that for every λ and $\mu \in \varrho(A)$ we have

$$\frac{\check{\Lambda}_{\gamma,\Gamma}(\lambda) - \check{\Lambda}_{\gamma,\Gamma}(\mu)}{\lambda - \mu} =$$

$$
\begin{aligned}
&= \partial_\nu \frac{(A-\lambda)^{-1}(\frac{1}{\gamma}\nabla\cdot(\gamma\nabla)+\lambda) - (A-\mu)^{-1}(\frac{1}{\gamma}\nabla\cdot(\gamma\nabla)+\mu)}{\lambda-\mu}\mathrm{tr}^{-1} \\
&= \partial_\nu \frac{\lambda(A-\lambda)^{-1} - \mu(A-\mu)^{-1} + \Big((A-\lambda)^{-1}-(A-\mu)^{-1}\Big)\Big(\frac{1}{\gamma}\nabla\cdot(\gamma\nabla)\Big)}{\lambda-\mu}\mathrm{tr}^{-1} \\
&= \partial_\nu (A-\lambda)^{-1}\Big[\mathbb{1} - (A-\mu)^{-1}\Big(-\tfrac{1}{\gamma}\nabla\cdot(\gamma\nabla)-\mu\Big)\Big]\mathrm{tr}^{-1} \qquad (2.4.4)
\end{aligned}
$$

has order -1 on the scale of Sobolev spaces on Γ, since it is a product (from right to left) of operators with order $1/2$, 0, -2 and $1/2$. By Sobolev embedding [47, Thm. V.4.18] this Newton quotient is compact, and in particular is bounded. As $\mu \to \lambda$, its norm limit is the compact operator

$$
\check{\Lambda}'_{\gamma,\Gamma}(\lambda) = \partial_\nu (A-\lambda)^{-1}\left[\mathbb{1} - (A-\lambda)^{-1}\left(-\frac{1}{\gamma}\nabla\cdot(\gamma\nabla)-\lambda\right)\right]\mathrm{tr}^{-1}.
$$

By construction, the resulting operator is independent of the left-inverse tr^{-1}. □

Proposition 2.2 (Half-plane property for the Dirichlet-to-Neumann form)**.** *For each $\gamma, 1/\gamma \in L^\infty(\Omega)$ and $f \in H^{1/2}(\Gamma)$ the map $\lambda \mapsto \langle \check{\Lambda}_{\gamma,\Gamma}(\lambda)f, f\rangle_{L^2(\Gamma)}$ maps each half-plane $\mathbb{C}^\pm$ into itself.*

Proof. Without a loss of generality take f to be defined on $\partial\Omega$, extending it by 0 into Γ_c. If the given solution u of (2.4.2)—defining $\check{\Lambda}_{\gamma,\Gamma}(\lambda)$—is in $H^2(\Omega)$ and its quasi-derivative $\gamma\nabla u$ is in $H^1(\Omega)$, we can use Green's formula to show the usual identity

$$
\int_\Omega \gamma\left(\nabla u\cdot\overline{\nabla u} - \lambda u\overline{u}\right) = \int_{\partial\Omega}\gamma(\partial_\nu u)\overline{u}. \qquad (2.4.5)
$$

However, since $\gamma \in L^\infty(\Omega)$ we must in general interpret the differential equation in (2.4.2) in its weak formulation. Moreover this forces a weaker definition of normal derivative. We shall use that specified in [9, Eq. (1.3)] and [7, p. 2101]: the normal quasi-derivative $\gamma\partial_\nu v$ of a given $v \in H^1(\Omega)$ with $\nabla\cdot(\gamma\nabla v) \in L^2(\Omega)$ is the

unique $\psi \in H^{-1/2}(\partial\Omega)$ defined by the duality relation

$$\langle \psi, w \rangle = \int_\Omega \Big(\gamma \nabla v \cdot \nabla w + \nabla \cdot (\gamma \nabla v) w \Big) \quad \Big(w \in H^1(\Omega) \Big).$$

In this case we *define*

$$\int_{\partial\Omega} \gamma (\partial_\nu v) w := \langle \gamma \partial_\nu v, w \rangle \quad \Big(v, w \in H^1(\Omega), \nabla \cdot (\gamma \nabla v) \in L^2(\Omega) \Big),$$

which agrees with Green's formula. Replacing v by the solution u to (2.4.2) and w by $\overline{u}$ we arrive immediately at (2.4.5). Taking the imaginary parts of either side of the latter equation concludes the proof. □

Put together, the two results above imply that $\check{\Lambda}_{\gamma,\Gamma}(\lambda)$ is what the literature refers to as a *generalised* or *operator-valued Herglotz function*. These considerations, especially in light of the transformation (1.2.1) to Schrödinger form, clearly show the direct analogue between the Dirichlet-to-Neumann map $\check{\Lambda}_{\gamma,\Gamma}(\lambda)$ and the Weyl–Titchmarsh m-function $m(\lambda)$.

The natural analogue to Inverse Problem 3 for this higher-dimensional situation is simply Calderón's; see Section 1.1. We re-phrase it here.

Inverse Problem 4 (Calderón). *Given the fixed-frequency Dirichlet-to-Neumann operator $\check{\Lambda}_{\gamma,\Gamma}(0)$ recover the conductivity γ on Ω which gave rise to it.*

Remark. *We are able to consider $\check{\Lambda}_{\gamma,\Gamma}$ at but one frequency owing to the fact that this contains enough independent information to perform the recovery. Indeed, in dimension $n = 3$ or greater, there is "surplus" information, overdetermining γ, whilst at $n = 2$ both $\check{\Lambda}_{\gamma,\Gamma}(0)$ and γ contain the same amount of information. Roughly speaking, calculating the Schwartz kernel[2.14] of $\check{\Lambda}_{\gamma,\Gamma}(0)$ shows it is a function of $2(n-1)$ linearly independent variables, whilst γ takes n variables as argument.*

[2.14] See, e.g., [124, Sec. 0.2].

Originally presenting his work at a conference in 1980, Calderón proved that in the linearised version of the problem there are complex-exponential oscillatory solutions of the form $e^{i\zeta\cdot\mathbf{x}}$ ($\zeta \in \mathbb{C}, \zeta\cdot\zeta = 0, \mathbf{x} \in \Omega$) that may be used to determine $\gamma = 1$ uniquely from global boundary data. These are simply Dirichlet-to-Neumann data on the whole boundary $\partial\Omega$. Calderón was preceded by Langer [87] who formulated the problem on a half-plane and considered conductivities that were analytic and only dependent on depth, and later by Cannon, Douglas and Jones [34] who worked on cylindrical domains and conductivities independent of height. The problem may also be rephrased in Schrödinger form, precisely the formal higher-dimensional analogue of Inverse Problem 3:

Inverse Problem 4'. *Given* $\Lambda_{q,\Gamma}(0)$ *recover the potential* $q(\cdot)$ *on* Ω *giving rise to it.*

The earliest efforts to establish stronger uniqueness results than Calderón's focused on dimensions strictly greater than 2; as remarked earlier, this was initially found to be easier, until alternative techniques were developed. We will devote the remainder of this section to understanding some of the general principles used to prove uniqueness from full boundary data in all dimensions n and reviewing the progress made, distinguishing first the case $n \geq 3$ from $n = 2$, presented second, except where the two overlap.

Calderón was, more precisely, able to show analyticity of $\gamma \mapsto \check{\Lambda}_{\gamma,\partial\Omega}(0)$ in the following sense [33]. Define the quadratic form $Q_\gamma(f) = \int_\Omega \gamma|\nabla u|^2$ $\left(f \in H^{1/2}(\partial\Omega)\right)$ where $u \in H^1(\Omega)$ uniquely solves the Dirichlet problem (2.4.2) with $\Gamma_c = \emptyset$ and $\lambda = 0$. Then we may apply Green's formula to integrate by parts and see

$$Q_\gamma(f) = -\int_{\partial\Omega} \overline{u}\gamma\partial_\nu u = \langle \Lambda_{\gamma,\partial\Omega}(0)f, f\rangle_{L^2(\partial\Omega)}.$$

Now define the norms

$$\|f\|_\Omega^2 = \int_\Omega |\nabla v|^2 \quad \left(f \in H^{1/2}(\partial\Omega)\right),$$

where v solves

$$\begin{cases} \Delta v &= 0 \quad \text{in} \quad \Omega, \\ v &= f \quad \text{on} \quad \partial\Omega, \end{cases}$$

and

$$\|Q\|_{\text{sup}} = \sup_{\|f\|_\Omega = 1} |Q(f)|.$$

Then $\gamma \mapsto \check{\Lambda}_{\gamma,\partial\Omega}(0)$ is analytic on real-valued functions γ satisfying $\gamma, 1/\gamma \in L^\infty(\Omega)$ in the sense that we have $\mathrm{d}Q_\gamma$ such that for any $\varepsilon > 0$ there is $\delta > 0$ with

$$\frac{\|Q_\gamma - Q_{\gamma_0} - \mathrm{d}Q_{\gamma-\gamma_0}\|_{\text{sup}}}{\|\gamma - \gamma_0\|_{L^\infty(\Omega)}} \leq \varepsilon \text{ if } \|\gamma - \gamma_0\|_{L^\infty(\Omega)} < \delta, \frac{1}{\gamma_0} \in L^\infty(\Omega, \mathbb{R}).$$

Moreover he showed that $\gamma \mapsto \mathrm{d}Q_\gamma$ is injective at constant conductivities γ. This means that the linearised problem has an affirmative answer for uniqueness. But $\gamma \mapsto \mathrm{d}Q_\gamma$ has an image that is not closed, or in other words its left inverse is not bounded. Hence he was unable to conclude injectivity of $\gamma \mapsto Q_\gamma$ in a non-linear neighbourhood of constant conductivities.

A rough outline of subsequent progress is as follows. Kohn and Vogelius [79] proved in $n \geq 2$ dimensions that all partial derivatives of smooth γ, to arbitrary non-negative order, are uniquely determined in some neighbourhood of a smooth boundary $\partial\Omega$, by the values of $Q_\gamma(f)$ $\left(f \in H^{1/2}(\Gamma)\right)$ for any fixed, connected and non-empty Γ. Hence an analytic, real-valued γ is fully determined by Q_γ; they later generalised this to include piece-wise analytic conductivities [80]. Sylvester and Uhlmann utilised the *complex geometric optics* solutions [122] introduced by Faddeev in scattering theory [52]. With a conductivity in $C^\infty(\overline{\Omega})$ and smooth

boundary, they linked the large complex frequency asymptotics of these solutions to the Fourier transform in every frequency of the conductivity [123]; their trick in finding this link, which influenced much of the later work on the problem, limited them to no less than three dimensions.

Soon after, Alessandrini generalised both [80] and [123]. Assuming a Lipschitz-regular boundary he showed that, when $n = 2$ a Lipschitz conductivity γ is determined uniquely by $\Lambda_{\gamma,\partial\Omega}(0)$ amongst its piece-wise analytic perturbations $\gamma + \varphi$, whilst when $n \geq 3$ the conductivity is determined uniquely in $W^{2,\infty}(\Omega)$ [3, 4]. He used a related result of Nachman, Sylvester and Uhlmann [108] in his proof. Later Brown [28] showed that when γ has $3/2 + \varepsilon$ derivatives then it is uniquely recoverable from $\Lambda_{\gamma,\partial\Omega}(0)$. The first main improvements to this result came in 2003 both from Päivärinta, Panchenko and Uhlmann [113] and Brown and Torres [29]. Both worked on a spatial domain with Lipschitz boundary $\partial\Omega$; the former established uniqueness when γ is *a priori* from the Sobolev class $W^{3/2,\infty}(\Omega)$ and is strictly positive on $\overline{\Omega}$, whilst the latter generalised this slightly to include all $\gamma \in W^{3/2,2n+\varepsilon}(\Omega)$ for some fixed $\varepsilon > 0$. See also [60] for techniques developed on manifolds.

Uhlmann conjectured at the International Congress of Mathematicians of 1998 that Lipschitz conductivities are the optimal case for dimension $n \geq 3$. This was later established for conductivities that are also sufficiently close[2.15] to 1, by Haberman and Tataru [63], who introduced the idea of the geometric optics remainder term decaying in average by applying theory for spaces of Bourgain type, and subsequently with that condition relaxed in dimensions $n = 3$ and 4 by Haberman [62]. Shortly thereafter, Caro and Rogers finally reached Uhlmann's hypothesis, showing that any Lipschitz conductivity with $n \geq 3$ is uniquely determined. Currently this is the best result from full data in three or more dimen-

[2.15] The "closeness" is measured by the norm $\|\nabla\gamma\|_{L^\infty(\Omega)}$.

sions; the question of its optimality still remains open.

In two dimensions, since the trick of Sylvester and Uhlmann has no utility, no progress was made for the decade after Kohn and Vogelius until Nachman [107] was able to show that a twice continuously differentiable conductivity is uniquely determined. He used what has become referred to as a $\overline{\partial}$-method[2.16], relating the scattering transform of q to the Dirichlet-to-Neumann map, then showing this transform uniquely determines q and hence γ. Brown and Uhlmann [30] showed soon afterwards that, adapting first-order system scattering methods of Beals and Coifman [16, 17], one may relax the regularity to $\gamma \in W^{1,p}(\Omega)$ for any fixed $p > 2$.

It took almost another decade for this result to be improved upon. Astala and Päivärinta, in a mathematical tour de force, resolved the problem completely [9]. Briefly, we will present this.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply-connected domain. Suppose γ_1 and γ_2 are measurable in Ω and that there is a constant $c > 0$ such that $c^{-1} \leq \gamma_j \leq c$ $(j = 1, 2)$. If $\tilde{\Lambda}_{\gamma_1,\partial\Omega} = \tilde{\Lambda}_{\gamma_2,\partial\Omega}$ then $\gamma_1 = \gamma_2$.*

Remark. *The lack of any boundary regularity does not prevent one from specifying $\tilde{\Lambda}_{\gamma,\partial\Omega}$. Its domain is defined by $H^{1/2}(\partial\Omega) = H^1(\Omega)/H_0^1(\Omega)$ in the sense that elements of $H^1(\Omega)$ are equivalent—and therefore correspond to the same element of $H^{1/2}(\partial\Omega)$—if and only if they differ by an element of $H_0^1(\Omega)$. This extends the usual Sobolev definition. Then the range is simply the dual $H^{-1/2}(\partial\Omega) := \left(H^{1/2}(\partial\Omega)\right)^*$, so the range elements are defined distributionally instead of differentiably. The resulting Dirichlet-to-Neumann map is well defined and bounded.*

Their proof avoids the standard transformation to Schrödinger form, by instead constructing the real-valued *γ-harmonic conjugate* $v \in H^1(\Omega)$ of the real-

[2.16] Pronounced "dee bar".

valued solution $u \in H^1(\Omega)$ to

$$\begin{cases} -\nabla \cdot (\gamma \nabla u) &= 0 \quad \text{in } \Omega, \\ u &= h \quad \text{on } \partial\Omega. \end{cases}$$

This is defined up to a constant so that $f := u + iv$ satisfies the Beltrami equation

$$\partial_{\bar{z}} f = \mu \overline{\partial_z f} \quad \text{in } \Omega,$$

where $\partial_z := (\partial_1 - i\partial_2)/2$ and $\partial_{\bar{z}} := \overline{\partial_z}$, whilst $\mu = (1-\gamma)/(1+\gamma)$. They then find complex geometric optics solutions to the Beltrami equation, with the crux of the reasoning being a careful analysis of the subtle asymptotics of the remainder term in these solutions by use of *quasi-conformal* maps. The transformation back to the conductivity equation necessitates the boundedness of $1/\gamma$ as well as γ, and the proof is concluded by establishing certain properties of a so-called *transport matrix* for the problem.

All of the above are results for an isotropic medium, utilising data from the full boundary. We make no mention of anisotropic media, since we are not immediately concerned with such a case in our original work. We will discuss data on the partial boundary in greater depth in the opening portion of Chapter 4.

3

Second-order linear pencils and discrete data

'The art of doing mathematics consists in finding that special case which contains all the germs of generality.'

– David Hilbert[3.1]

In any physically inspired problem, a simple improvement to a given solution is that of using fewer starting data. In the case of the classical inverse spectral problem described in Section 2.3 our starting data in all cases were an m-function, considered as measurements of boundary values at one end-point of the interval on which the differential operator is defined. These measurements were assumed to be made for all λ for which they make sense—i.e., not associated eigen-values or spectrum. Being able to perform a similar recovery for measurements from a smaller set of λ—or indeed for a discrete, countable such set—would represent a substantial strengthening of the result. Moreover, practically one never has access to measurements over a continuous or infinite set of spectral parameters,

[3.1] *Hilbert*, Constance Reid (1970).

so avoiding such a restriction could provide greater applicability of the relevant uniqueness results.

It is with this in mind that we approach this chapter. We will consider a linear pencil of ordinary differential operators, with highest order 2. In this situation we can formulate an inverse problem, and for it we prove a uniqueness theorem. The data we allow ourselves are discrete, comprising the sequence of values $m(-n^2)$ $(n = 1, 2, 3, \ldots)$ where m is a suitably-generalised Weyl–Titchmarsh-type m-function for the pencil of operators. As outlined in Section 1.3, this second-order pencil is a generalisation of a specific pencil arising from separating the variables in a certain Schrödinger singular-boundary-value problem in two dimension. To conclude the chapter we will formulate this problem, show the link with the pencil, and prove uniqueness for it from our results for the pencil. A concise version of most of the material in this chapter was published in 2016 [27]. Here we have expanded on much of the details and arguments involved.

Pencils of second-order differential operators are not new, though so far it is mostly the quadratic pencils which have seen attention. These are usually written

$$-\nabla \cdot \big(p \nabla u(\cdot\,;\lambda)\big) + q u(\cdot\,;\lambda) = (\lambda w + \lambda^2) u(\cdot\,;\lambda)$$

in some domain of one or more dimensions, sometimes with a manifold instead of a linear structure. The works [85, 88] in particular focus on manifolds. There are also many examples of such work on a one-dimensional interval, both finite and half-infinite. These are not of interest in our situation, which is a linear pencil and involves highly singular coefficients. The interested reader is directed, for example, to [73, 139] or [46] and its substantial list of references, or the book chapter [103] for a list of applications.

3.1 Problem definition and outline

We recall from Section 2.1 or, for example, [115, Thm. X.1] that a sufficient condition for the spectrum of an operator to be real is that the operator be self-adjoint; this is important, since to avoid dealing with unnecessary technical complications a meaningful, non-trivial spectrum is necessary to constitute part of the input data for associated inverse spectral problems. Indeed it is usually in this case that one defines a boundary map associated to a differential operator[3.2]. The issue is the same when dealing with pencils of operators.

For reasons outlined shortly, the Hilbert space we consider in this chapter is

$$H := L^2(0,1;r\mathrm{d}r) = \left\{ u : (0,1) \to \mathbb{C} \;\middle|\; \int_0^1 r|u(r)|^2 \mathrm{d}r < \infty \right\}.$$

Let us take both $q, w \in L^\infty_{\mathrm{loc}}(0,1]$, with additionally q real-valued and w almost everywhere positive. We construct the following pencil of operators over H:

$$Lu(r;\lambda) = \lambda P u(r;\lambda) \quad \big(r \in (0,1)\big). \tag{3.1.1}$$

The differential expression

$$\ell[u](r) = -\frac{1}{r}\Big(ru'(r)\Big)' + q(r)u(r) \tag{3.1.2}$$

defines the action of the operator L. The domain of L will be defined precisely shortly. The multiplication operator

$$Pu(r) = w(r)u(r) \tag{3.1.3}$$

[3.2] Of course inverse problems for some non-self-adjoint operators have been considered, e.g., Simon and Bennewitz's work [120, 20] allows for a complex-valued Schrödinger potential, and the Weyl–Kodaira–Titchmarsh theory of Section 2.2 has more recently been extended similarly to allow complex coefficients [24].

possessing domain

$$D(P) = \left\{ u \in H \;\middle|\; \int_0^1 w(r)^2 |u(r)|^2 r\mathrm{d}r < \infty \right\} = L^2(0,1; w(r)^2 r\mathrm{d}r)$$

making it self-adjoint, is unbounded—except when $\nu = 0$—since w is assumed to satisfy

$$w(r) = \frac{1}{r^\nu}\Big(1 + o(1)\Big) \quad (r \to 0),$$

with a fixed $\nu \geq 0$.

In Section 2.2 we presented a version of the approach developed by Weyl and later Titchmarsh for analysing the singular behaviour of Sturm–Liouville problems. This analysis formed the means by which we constructed an m-function. This will be our standard choice of boundary spectral data for any one-dimensional problem. In their framework we would write (3.1.1) in the form

$$\frac{1}{rw(r)}\Big(ru'(r)\Big)' + \frac{q(r)}{w(r)}u(r) = \lambda u(r).$$

We would then note that, over $L^2(0,1; rw(r)dr)$, the expression on the left hand side is formally symmetric, and perform our analysis using this as our underlying Hilbert space. However it turns out that from a certain physically motivated perspective the most natural space is instead H, which we explain now.

Letting $w(r)$ be precisely r^{-2}—i.e., $\nu = 2$—and replacing λ by the sequence $-\lambda_n$ generates from (3.1.1) the Bessel-type system of equations

$$-\frac{1}{r}\Big(ru_n'(r;\mu)\Big)' + q(r)u_n(r;\mu) + \frac{\lambda_n}{r^2}u_n(r;\mu) = \mu u_n(r;\mu), \tag{3.1.4}$$

in the case where $\mu = 0$. In a natural way, we may pre-specify λ_n as the angular eigen-values of a boundary-value problem for a spherically-symmetric Schrödinger equation in any sub-domain of $\mathbb{R}^2$. Separation of the variables for this equation,

in polar coordinates, yields precisely the system (3.1.4) with u_n playing the part of the radial component of the solutions for the boundary-value Schrödinger problem. Under such a separation of variables, the Lebesgue area measure $\mathrm{d}\mathbf{x}$ becomes $r\mathrm{d}r\mathrm{d}\vartheta$, the radial part of which we take as the measure for our L^2-type Hilbert space H.

Indeed, the Bessel equation arises naturally in such scenarios as the radial equation, hence a natural choice for λ_n are the aforementioned eigen-values. Of course the domain and boundary conditions are needed to determine the eigen-values. As previously mentioned, later in the chapter we will apply our one-dimensional uniqueness result to a partial differential operator. For this operator we will provide a domain and boundary conditions, and explain how these λ_n arise in this particular case.

We must now precisely define the domain of L, or, rather, the domain of the pencil $L - \lambda P$. To do so in a consistent way we need to develop a classification, similar to that of the Weyl–Titchmarsh limit-point or limit-circle[3.3], which we may apply to our pencil of operators. This is explained in the following definition.

Definition 3.1. *The equation*

$$-\frac{1}{r}\Big(ru'(r)\Big)' + q(r)u(r) = \lambda w(r)u(r) \tag{3.1.5}$$

is said to be in pencil-limit-circle *at* 0 *whenever, for all* $\lambda \in \mathbb{C}$*, the space of its solutions that lie in* $L^2(0,1;r\mathrm{d}r)$ *is two-dimensional. Otherwise—i.e., if for all* $\lambda \in \mathbb{C}$ *this solution-space is one- or zero-dimensional—we say* (3.1.5) *is in* pencil-limit-point *at* 0*. We abbreviate these two cases, respectively, by PLC and PLP. Sometimes we will also refer to the differential expression* $\ell - \lambda w$ *using this classification.*

Remark. *In Section 3.2 we prove that, when* $w(r) = r^{-\nu}\big(1 + o(1)\big)$ $(r \to 0)$ *and satisfies*

[3.3] These were defined in Section 2.2.

certain technical conditions, we have the following classification. Equation (3.1.1) *is in PLC at* 0 *for* $\nu \in [0, 2)$*, it is in PLP at* 0 *for* $\nu \in (2, \infty)$*, and it is in a "mixed", λ-dependent situation when* $\nu = 2$*. Indeed, when* $\nu = 2$ *and* $\operatorname{Im}(\lambda) \neq 0$*, if* $\operatorname{Im}(\sqrt{\lambda}) \geq 1$ *then* (3.1.1) *is in PLP at* 0*, whilst if* $\operatorname{Im}(\sqrt{\lambda}) < 1$ *then the equation is in PLC at* 0*. In other words, for* $\nu = 2$*, we may divide* $\mathbb{C}\backslash\mathbb{R}$ *into the components* Ω_p *and* Ω_c *either side of the parabola* $\operatorname{Im}(\sqrt{\lambda}) = 1$*, where* $\lambda \in \Omega_p$ *puts the equation in PLP and* $\lambda \in \Omega_c$ *puts it in PLC, at* 0*. For a visual representation of this classification, see Figure* 3.1.

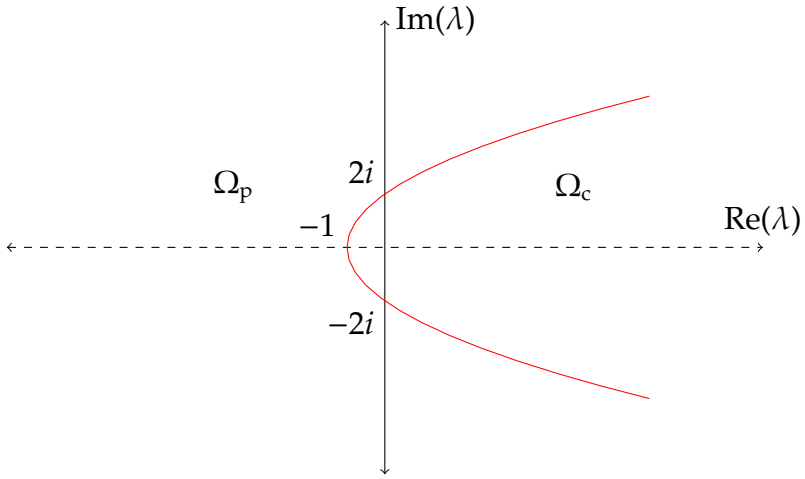

Figure 3.1: λ-plane for equation (3.1.5) *with* $\nu = 2$

It is well known [37, 133, 15] that for Sturm–Liouville problems that are regular or classical limit-circle at a given end-point of the interval, one may use a solution of the equation for any real λ in the resolvent set of the associated Sturm–Liouville operator—possibly, for example, $\lambda = 0$—to define a boundary condition at the given end-point. This owes to the fact that the limit-circle/-point classification may be extended from $\mathbb{C}\backslash\mathbb{R}$ to all of this resolvent set.

The same extension and boundary condition may be utilised here, with care taken to choose λ from $\mathbb{R}$ to ensure that, in cases where the pencil has mixed classification, e.g., $\nu = 2$, the expression $\ell - \lambda w$ has a two-dimensional $L^2_{\mathrm{loc}}[0, 1)$-solution space. We may assume with no loss of generality that this condition is automatically satisfied, for the pencil (3.1.5) when $w(r) \sim r^{-\nu}$ $(r \to 0)$, by the choice

$\lambda = 0$, thanks to the discreteness of the spectrum of $L - \lambda P$ and the fact that we may translate q by any real λ without affecting uniqueness or its properties in the coming theorems. Define the Lagrange bracket, $[u, v](r) = r(uv' - u'v)(r)$, for the differential expression ℓ. This is a compact notation for the boundary term that emerges from a double integration by parts, as follows:

$$\int_r^1 (v\ell u - u\ell v) = [u, v](r) \quad \Big(r \in (0, 1], u, v \in C^2(0, 1], u(1) = v(1) = 0\Big).$$

Definition 3.2. *Suppose $\ell - \lambda w$ is in PLC at 0, and that the not-everywhere-zero, real-valued function U solves $\ell U = 0$. Then a function u satisfies* the boundary condition parameterised by U *if and only if*

$$[u, U](0^+) = \lim_{r \searrow 0} r\Big(u(r)U'(r) - u'(r)U(r)\Big) = 0. \tag{3.1.6}$$

Remark. *Conversely, it can be shown that any self-adjoint boundary condition for $\ell - \lambda w$ at the PLC end-point 0 may be written in the form (3.1.6). By* self-adjoint boundary condition *we mean any boundary condition that would restrict the operator generated by $\ell - \lambda w$ and a homogeneous Dirichlet condition at 1, to a self-adjoint realisation.*

Definition 3.3 (Domain of $L - \lambda P$)**.** *Let $\mathrm{Im}(\lambda) \neq 0$. When $\ell - \lambda w$ is in PLP at 0, we define*

$$D(L - \lambda P) = \{u \in H \,|\, \ell u - \lambda w u \in H, \;\; u(1; \lambda) = 0\},$$

and when it is in PLC at 0, we define

$$D(L - \lambda P) = \{u \in H \,|\, \ell u - \lambda w u \in H, \;\; u(1; \lambda) = 0, \; [u, U](0^+; \lambda) = 0\}.$$

Owing to the Remark on page 53 we make the following definition.

Definition 3.4. *Let $\mathrm{Im}(\lambda) \neq 0$. If $\ell - \lambda w$ is in PLP at 0, define $u(\cdot\,; \lambda)$ to be the unique solution in H, up to a constant multiple, of equation (3.1.5). If $\ell - \lambda w$ is in PLC*

at 0, denote by $u(\cdot\,;\lambda)$ the unique (up to constant multiple) solution in H, of (3.1.5), satisfying some self-adjoint boundary condition $[u,U](0^+;\lambda)=0$ for some non-trivial U solving $\ell U=0$. We then define the Dirichlet m-function by

$$m(\lambda)=-\frac{u'(1;\lambda)}{u(1;\lambda)}\quad \big(\mathrm{Im}(\lambda)\neq 0\big). \tag{3.1.7}$$

Remark. *This function may be extended analytically to the resolvent set of $L-\lambda P$. We remind the reader that it has a "$-$" to ensure the correct sign for the Herglotz property, i.e., so that it maps $\mathbb{C}^{\pm}\to\mathbb{C}^{\pm}$—otherwise it would be anti-Herglotz, mapping $\mathbb{C}^{\pm}\to\mathbb{C}^{\mp}$.*

When $\nu=2$ there is no reason to expect the above-defined m to be continuous across the curve $\mathrm{Im}(\sqrt{\lambda})$ separating Ω_p from Ω_c. It is not hard to convince oneself of the following proposition, though since it is not necessary for the purposes of this work we omit the proof.

Proposition 3.1. *When $\nu=2$, there are analytic continuations m_p and m_c of, respectively $m\restriction_{\Omega_p}$ and $m\restriction_{\Omega_c}$, whose domains both include $\mathbb{C}\backslash\mathbb{R}$. Moreover,*

$$m_p\restriction_{\mathbb{C}\backslash\mathbb{R}}\neq m_c\restriction_{\mathbb{C}\backslash\mathbb{R}},$$

and the boundary condition required for $\ell-\lambda w$ to generate m_p in the region $\Omega_c\backslash\mathbb{R}$ is not of a self-adjoint type.

Moreover, in the vein of the discussion in Section 2.2 one can show that the following holds.

Proposition 3.2. *The pencil $L-\lambda P$ is boundedly invertible wherever $m(\lambda)$ is analytic, and its eigen-values are the poles of $m(\lambda)$. The multiplicities of the poles are exactly the multiplicities of the eigen-values.*

As we stated in the beginning of the chapter, we will prove a (pair of) unique-

ness theorem(s) for a particular inverse problem, which we may now formulate.

Inverse Problem I. *Consider a locally bounded $w : (0,1] \to (0,+\infty)$, so that (3.1.1) is in PLP at 0, or in PLC at 0 equipped with the self-adjoint boundary condition from Definition 3.3. Suppose we have specified the admissible sequence $\mathcal{S} := \left((-n^2, m_n)\right)_{n=1}^{\infty}$ of points from the graph of some generalised Titchmarsh–Weyl m-function for (3.1.5). From $\mathcal{S}$ and w recover the potential q.*

To establish uniqueness for this problem, we will proceed as follows. First we transform the differential expression $\ell - \lambda w$ to Liouville normal form. In our PLP case—i.e., $\nu \geq 2$; we will establish in Section 3.2 that the "mixed" case $\nu = 2$ turns out to be treatable as PLP—the spatial domain transforms from $(0,1)$ to $(0,\infty)$, with the new differential expression chosen to be regular at 0 and in PLP at ∞. In our PLC case—$0 \leq \nu < 2$—the transformation maps $(0,1)$ to some finite interval, which we scale to be $(0,1)$, and whose orientation we choose to ensure that the new expression is regular at 1 and in PLC at 0.

Remark. *Intuitively, we may interpret this difference in transformation as owing to the PLC "singularity" at 0 being too "weak" for the associated transformation to map the unit interval to the half-line, whereas the PLP expression is "sufficiently" singular.*

Next we note that the transformation to Liouville normal form correspondingly transforms the m-function, in the sense that the new equation has an m-function associated with it that is a (non-linear) transformation of the original m-function (3.1.7). This means that we may interpolate the original m-function by instead interpolating its transformed version. Moreover, because the classical limit-point and limit-circle m-functions are formally identical to, respectively, the PLP and PLC m-functions, we see that we simply need interpolation results for the classical counterparts. Such a result already exists [118] for the classical limit-

point case, and we present it in Section 3.3. However an analysis of the available literature indicates there is no interpolation result for a classical limit-circle m-function, and a study of the proof for limit-point interpolation offers further indication that the methods involved are not adaptable to the limit-circle case. In the same section we discuss this issue, and present a novel proof of interpolation of a particular classical limit-circle m-function, which corresponds exactly to our transformed PLC m-function. These results, together, allow us to move from knowing the discrete sequence $\mathcal{S}$ to knowing the full (PLP or PLC) m-function.

The final step is application of the generalised Borg–Marčenko theorem. This classical and far-reaching result was discussed in depth in Section 2.3.

The remaining chapter is divided into the following sections. We analyse the solution-space dimensions for equation (3.1.5) in Section 3.2, proving concretely the pencil-limit-point/-circle classification of $\ell - \lambda w$ alluded to earlier. In Section 3.3 we discuss the problem of interpolating classical limit-point m-functions and present our interpolation of a classical limit-circle m-function. We then pull all the results together in Section 3.4 and round off the proofs of two uniqueness theorems for Inverse Problem I. Finally, in Section 3.5 we apply this uniqueness of a pencil of ordinary differential operators to a problem involving partial differential operators, singular partial boundary conditions and partial boundary data of Cauchy type, which we call the Berry–Dennis inverse problem.

3.2 Pencil-limit-point and -limit-circle behaviour

We will analyse here the dimension of the solution space of (3.1.5) with $w(r) \sim r^{-\nu}$ and $\nu \geq 0$. It will be helpful to treat the two cases $\nu \geq 2$ and $0 \leq \nu < 2$ separately, respectively in Lemmata 3.2 and 3.3. The first analysis is *via* transforming

the problem to Liouville normal form on the half-line and using known large-x asymptotics of solutions. The second follows a different approach, using asymptotic analysis and variation of parameters to build recursion formulae that can be used to construct a pair of linearly independent solutions to the original equation. In this second case we could, equivalently, work with the Liouville normal form on the finite interval, but it is just as convenient here to use (3.1.5) in its original form.

The understanding of x-asymptotics is deeper for equations in Liouville normal form when they are presented on the half-line rather than on a finite interval. The asymptotics we will use for the half-line problem arise from analyses collected by Eastham [45]. They and related asymptotics all emerge naturally as corollaries or generalisations of the Asymptotic Levinson Theorem [89]. This theorem states the following, where we follow the style of [45, Thm. 1.3.1].

Proposition 3.3 (The Asymptotic Levinson Theorem). *Let* $Y'(x) = \big(D(x)+R(x)\big)Y(x)$ $\big(x \in (0,\infty)\big)$ *be an* $n \times n$ *system with coefficients satisfying*

- *the matrix* $D(x)$ *is diagonal with diagonal entries labelled* $d_k(x)$ $(k = 1, \dots, n)$,
- *uniformly in* i, j, t *and* x *the integrated difference* $\int_t^x \mathrm{Re}(d_i - d_j)$ *is either bounded from above or below,*
- *whilst the remainder term* $R \in L^1(0,\infty;\mathbb{C}^{n\times n})$.

Then there is an $n \times n$ *fundamental matrix* $Y(x)$ *whose components satisfy*

$$Y_{jk}(x) = (\delta_{jk} + o(1))e^{\int_0^x d_k} \quad (x \to \infty), \tag{3.2.1}$$

where δ_{jk} *is the Kronecker-*δ*, taking the value* 1 *when* $j = k$ *and* 0 *otherwise.*

One may then prove the following lemma, presented as an example in [45,

Ex. 1.9.1], that describes the asymptotic form of solutions to the one-dimensional Schrödinger problem with square-integrable potential.

Remark. *The potential being in $L^2(0,\infty)$ is sufficient to ensure that the one-dimensional Schrödinger problem is in classical limit-point at $+\infty$.*

Lemma 3.1 (Eastham)**.** *Let c be non-zero and $R \in L^2(a,\infty)$. Then the differential equation*

$$-y'' + Ry = c^2 y \quad \text{on } (a,\infty)$$

has a linearly independent pair of solutions $y_\pm$ asymptotically given, as $x \to \infty$, by

$$y_\pm(x) = \exp\left(\pm i\Big(cx - \frac{1}{2c}\int_a^x R\Big)\right)\Big(1 + o(1)\Big).$$

Eastham's Lemma is the key ingredient in the proof of the following classification, the first main result of this section.

Lemma 3.2. *Suppose $\nu \geq 2$ and $\alpha > \frac{\nu-2}{2}$, and furthermore let $q, w \in L^\infty_{\mathrm{loc}}(0,1]$ be real-valued and satisfy, as $r \to 0$,*

(i) $w(r) = \dfrac{1}{r^\nu}(1 + O(r^\alpha))$,

(ii) $w(r) \geq \omega > 0$ *almost everywhere, and*

(iii) $q(r) = w(r)O(r^\alpha)$

Then equation (3.1.5) *is*

1. *in PLP at 0 when $\nu > 2$ or $\operatorname{Im}\sqrt{\lambda} \geq 1$, and*
2. *in PLC at 0 when $\nu = 2$ and $1 > \operatorname{Im}\sqrt{\lambda} > 0$.*

Proof. We write

$$w(r) = \frac{1}{r^\nu}\Big(1 + \varepsilon_1(r)\Big) \text{ and}$$
$$\frac{q(r)}{w(r)} = \varepsilon_2(r) \quad (r \in (0,1]),$$

where

$$\varepsilon_j(r) = O(r^\alpha)\ (r \to 0,\ j = 1,2). \tag{3.2.2}$$

By performing a Liouville–Green transformation [45, Sec. 2.5], with

$$t = \tau(r) = \int_r^1 \rho^{-\nu/2} \mathrm{d}\rho \quad (r \in (0,1)), \tag{3.2.3}$$
$$z(t) = \tau^{-1}(t)^{\frac{2-\nu}{4}} u(\tau^{-1}(t)) \quad (t \in (0,\infty)),$$

we arrive at the equation

$$-z''(t;\lambda) + Q(t;\lambda)z(t;\lambda) = \lambda z(t;\lambda), \tag{3.2.4}$$
$$Q(t;\lambda) := \Big(\varepsilon_1\varepsilon_2 - \lambda\varepsilon_1 + \varepsilon_2\Big)\Big(\tau^{-1}(t)\Big) - \left(\frac{\nu-2}{4}\right)^2 \tau^{-1}(t)^{\nu-2}, \tag{3.2.5}$$

where $t \in (0,\infty)$. Note that if $\nu > 2$ we have $\tau(r) = \frac{\nu-2}{2}(r^{1-\nu/2} - 1)$, whereas if $\nu = 2$ then $\tau(r) = -\log(r)$.

Consider first the case $\nu > 2$. In order to apply Lemma 3.1 to (3.2.4) we need $Q(\cdot\,;\lambda) \in L^2(0,\infty)$. Firstly note that we have $Q(\cdot\,;\lambda) \in L^\infty_{\mathrm{loc}}[0,\infty)$ since, as $t \to 0$,

$$\begin{aligned}
\varepsilon_1\Big(\tau^{-1}(t)\Big) &= \tau^{-1}(t)^\nu w\Big(\tau^{-1}(t)\Big) - 1 \\
&= \left(\left(1 + \frac{2t}{\nu-2}\right)^{\frac{2}{2-\nu}}\right)^\nu w\Big(\tau^{-1}(t)\Big) - 1 \\
&= O(1), \qquad (3.2.6) \\
\varepsilon_2\Big(\tau^{-1}(t)\Big) &= \frac{q}{w}\Big(\tau^{-1}(t)\Big)
\end{aligned}$$

$$\leq \frac{q\left(\tau^{-1}(t)\right)}{\omega}$$
$$= O(1), \tag{3.2.7}$$
$$\tau^{-1}(t)^{\nu-2} = \left(1 + \frac{2t}{\nu-2}\right)^{-2}$$
$$= O(1). \tag{3.2.8}$$

Thus, from (3.2.5) we see that only the large-t behaviour of $Q(t;\lambda)$ determines its square integrability. Moreover, since $\tau^{-1}(t)^{\nu-2}$ is square-integrable for t in a neighbourhood of ∞ it is clear from (3.2.5) that $Q(\cdot\,;\lambda) \in L^2(0,\infty)$ if and only if

$$\infty > \int_0^\infty \left|\left(\varepsilon_1\varepsilon_2 - \lambda\varepsilon_1 + \varepsilon_2\right)(\tau^{-1}(t))\right|^2 \mathrm{d}t = \int_0^1 r^{-\nu/2}\left|\left(\varepsilon_1\varepsilon_2 - \lambda\varepsilon_1 + \varepsilon_2\right)(r)\right|^2 \mathrm{d}r. \tag{3.2.9}$$

Recall that (3.2.2) tells us $\varepsilon_j(r) = O(r^\alpha)$ $(r \to 0,\ j = 1,2)$ where, by the hypothesis of the lemma, $\alpha > \nu/2 - 1$. This implies that the integrand of the right-hand side of (3.2.9) is asymptotically bounded, as $r \to 0$, by a constant times

$$r^{-\nu/2} r^{2\alpha} < r^{-\nu/2+\nu-2} < r,$$

since $\nu > 2$. Thus by Eastham's Lemma 3.1 we have a linearly independent pair of solutions $z_\pm(\cdot\,;\lambda)$ for equation (3.2.4) given, as $t \to \infty$, by

$$z_\pm(t;\lambda) = \exp\left(\pm i\left(\sqrt{\lambda}t - \frac{1}{2\sqrt{\lambda}}\int_0^t Q(\cdot\,;\lambda)\right)\right)\left(1 + o(1)\right). \tag{3.2.10}$$

Now, by a change of variables, the integral in the argument of this exponential is easily seen to be

$$\int_{\tau^{-1}(t)}^1 \left\{\left(\varepsilon_1\varepsilon_2 - \lambda\varepsilon_1 + \varepsilon_2\right)(\rho) - \left(\frac{\nu-2}{4}\right)^2 \rho^{\nu-2}\right\} \rho^{-\nu/2}\mathrm{d}\rho. \tag{3.2.11}$$

For sufficiently large t we may split the first part of the integral (3.2.11) so that its absolute value satisfies

$$\left|\int_{\tau^{-1}(t)}^{1}\Big(\varepsilon_1\varepsilon_2-\lambda\varepsilon_1+\varepsilon_2\Big)(\rho)\rho^{-\nu/2}\mathrm{d}\rho\right|\leq\int_{\tau^{-1}(t)}^{\delta}|\varepsilon_1\varepsilon_2-\lambda\varepsilon_1+\varepsilon_2|(\rho)\rho^{-\nu/2}\mathrm{d}\rho + \int_{\delta}^{1}|\varepsilon_1\varepsilon_2-\lambda\varepsilon_1+\varepsilon_2|(\rho)\rho^{-\nu/2}\mathrm{d}\rho.$$

By the hypotheses w, $1/w$ and $q\in L^{\infty}_{\mathrm{loc}}(0,1]$ the second integral here is bounded. Moreover since both $\varepsilon_j(\rho)=O(\rho^{\alpha})$ and $\alpha>\frac{\nu-2}{2}$ the first integral is also bounded. Hence (3.2.11) converges as $t\to\infty$, pointwise for each $\lambda\in\mathbb{C}$ and $\nu\geq 2$. We denote by $C(\lambda)$ the limit of (3.2.11).

Now $\nu>2$ implies $\nu-2-\nu/2>-1$, so the second part of the integral in (3.2.11) is clearly convergent as $t\to\infty$, possessing integrand asymptotically bounded near $\rho=0$ by a monomial whose order is strictly greater than -1. Thus we observe the leading-order asymptotics

$$\begin{aligned}z_{\pm}(t;\lambda)&=e^{\pm i\sqrt{\lambda}t}\underbrace{\exp\left(\mp\frac{iC(\lambda)+o(1)}{2\sqrt{\lambda}}\right)}_{\to:\,C_{\pm}(\lambda)}\Big(1+o(1)\Big)\\&=e^{\pm i\sqrt{\lambda}t}\Big(C_{\pm}(\lambda)+o(1)\Big).\end{aligned}\tag{3.2.12}$$

Now consider the case $\nu=2$. As remarked after (3.2.5) the transformed variable $t=\tau(r)$ is now given by $-\log(r)$, and as before, we need to show $Q(\cdot\,;\lambda)\in L^2(0,\infty)$ to apply Eastham's Lemma. Again, $Q(\cdot\,;\lambda)\in L^{\infty}_{\mathrm{loc}}[0,\infty)$, since, as $t\to 0$

$$\begin{aligned}\varepsilon_1\Big(\tau^{-1}(t)\Big)&=\tau^{-1}(t)^2w\Big(\tau^{-1}(t)\Big)-1\\&=e^{-2t}w(e^{-t})-1\\&=O(1),\end{aligned}$$

$$\begin{aligned}\varepsilon_2\Big(\tau^{-1}(t)\Big) &= \frac{q}{w}\Big(\tau^{-1}(t)\Big) \\ &\leq \frac{q\Big(\tau^{-1}(t)\Big)}{\omega} \\ &= O(1),\end{aligned}$$

and $Q(t;\lambda) = (\varepsilon_1\varepsilon_2 - \lambda\varepsilon_1 + \varepsilon_2)\Big(\tau^{-1}(t)\Big)$ precisely. Thus, by the same reasoning as before, $Q(\cdot\,;\lambda)$ is indeed in $L^2(0,\infty)$, allowing Eastham's Lemma 3.1 to provide us with the same linearly independent solution-pair as in (3.2.10). In turn, the same reasoning shows that these solutions satisfy the simpler relation in (3.2.12).

Now, in both cases $\nu > 2$ and $\nu = 2$, for any solution $u(\cdot\,;\lambda)$ of equation (3.1.5) and its corresponding transformed solution $z(\cdot\,;\lambda)$ of (3.2.4) we have

$$\int_0^1 r|u(r;\lambda)|^2\mathrm{d}r = \int_0^\infty \tau^{-1}(t)^2|z(t;\lambda)|^2\mathrm{d}t.$$

But the leading-order asymptotics (3.2.12) show that

$$\int_0^\infty \tau^{-1}(t)^2|z_\pm(t;\lambda)|^2\mathrm{d}t < \infty$$

if and only if

$$\infty > \int_0^\infty \tau^{-1}(t)^2|e^{\pm i\sqrt{\lambda}t}|^2\mathrm{d}t = \int_0^\infty \tau^{-1}(t)^2 e^{\pm 2\mathrm{Im}\,\sqrt{\lambda}t}\mathrm{d}t. \tag{3.2.13}$$

When $\nu > 2$ and $\mathrm{Im}(\lambda) \neq 0$, the transformation $\tau^{-1}(t) = \left(1 - \frac{2-\nu}{2}t\right)^{\frac{2}{2-\nu}}$, so the growth of the integrand of (3.2.13) is governed by real-exponential large-t asymptotics. Since the integrand for $u_+(\cdot\,;\lambda)$ grows and that for $u_-(\cdot\,;\lambda)$ decays, only the latter solution of equation (3.1.5) (up to scaling by a constant) is in $L^2(0,1;r\mathrm{d}r)$. In other words, for $\nu > 2$ and $\mathrm{Im}(\lambda) \neq 0$, (3.1.5) is in PLP at 0.

On the other hand, when $\nu = 2$, we find $\tau^{-1}(t)^2 = e^{-2t}$. When multiplied

with the other exponential factor $e^{\pm 2\mathrm{Im}\sqrt{\lambda}t}$ in (3.2.13) we see that if $\mathrm{Im}\sqrt{\lambda} \geq 1$ and $\mathrm{Im}(\lambda) \neq 0$ then (3.1.5) is PLP at 0, whilst if $\mathrm{Im}\sqrt{\lambda} < 1$ the latter is PLC at 0. □

Remark. *When $\nu = 2$ we may represent graphically the $L^2(0,1;r\mathrm{d}r)$ nature of the solutions of (3.1.5); see Figure 3.1. Here, $\Omega_{\mathrm{p}} := \{\lambda \in \mathbb{C} \mid \mathrm{Im}\sqrt{\lambda} \geq 1\}$ and $\Omega_{\mathrm{c}} := \{\lambda \in \mathbb{C} \mid \mathrm{Im}\sqrt{\lambda} < 1\}$, so that if $\lambda \in \Omega_{\mathrm{p}}$ or Ω_{c} then equation (3.1.5) is respectively PLP or PLC.*

We now proceed with the proof of the classification of (3.1.5) as pencil limit-circle when $0 \leq \nu < 2$. Since we cannot apply Eastham's Lemma, we will instead analyse directly the asymptotics and integrability of the solutions of the equation.

Lemma 3.3. *Consider equation (3.1.5) with real-valued $w, q \in L^\infty_{\mathrm{loc}}(0,1]$. Suppose that $0 \leq \nu < 2$, and that*

$$\begin{aligned} w(r) &= \frac{1}{r^\nu}\Big(1 + \varepsilon_1(r)\Big), \\ q(r) &= w(r)\varepsilon_2(r), \\ \varepsilon_j(r) &= o(1) \ (r \to 0, j = 1,2). \end{aligned} \tag{3.2.14}$$

Then there is a fundamental system $\{u_1(\cdot\,;\lambda), u_2(\cdot\,;\lambda)\}$ satisfying $u_1(r;\lambda) \to 1$ and $u_2(r;\lambda) \sim \log(r)$ as $r \to 0$, and both $u_1(\cdot\,;\lambda)$ and $u_2(\cdot\,;\lambda)$ are in $L^2(0,1;r\mathrm{d}r)$.

Proof. Transform by $v(r) = r^{1/2}u(r)$, so that (3.1.5) becomes

$$\begin{aligned} -v''(r;\lambda) - \frac{1}{4r^2}v(r;\lambda) &= \Big(\lambda w - q\Big)(r)v(r;\lambda) \\ &= r^{-\nu}\Big(1 + \varepsilon_1(r)\Big)\Big(\lambda - \varepsilon_2(r)\Big)v(r;\lambda). \end{aligned} \tag{3.2.15}$$

Consider the sequences $\Big(v_k(\cdot\,;\lambda)\Big)_{k=0}^\infty$ and $\Big(y_k(\cdot\,;\lambda)\Big)_{k=0}^\infty$ defined by

$$-v''_{k+1}(r;\lambda) - \frac{1}{4r^2}v_{k+1}(r;\lambda) = \Big(\lambda w - q\Big)(r)v_k(r;\lambda),$$

and an equation of the same form for y_k, satisfying $v_0(r;\lambda) = r^{1/2}$ and $y_0(r;\lambda) = r^{1/2}\log(r)$. After imposing the boundary conditions

$$\left.\begin{matrix} v_k(r) \\ y_k(r) \end{matrix}\right\} = o(r^{1/2}) \quad (r \to 0, k \in \mathbb{N}),$$

we find that both sequences are well-defined, as each step simply involves solving a linear second-order equation with an inhomogeneity; this is be variation of parameters, and shortly we will do the explicit calculation.

We will now drop the λ-dependency to simplify notation. If we can show that the series (note, starting from $k = 1$) $V := \sum_{k=1}^{\infty} v_k$ and $Y := \sum_{k=1}^{\infty} y_k$ converge uniformly near 0, and satisfy the asymptotics

$$V(r) = o(r^{1/2}), Y(r) = o\Big(r^{1/2}\log(r)\Big) \ (r \to 0), \tag{3.2.16}$$

then $v = v_0 + V$ and $y = y_0 + Y$, after division by $r^{1/2}$, is the required solution pair.

Note that v_0, y_0 form a fundamental system in the kernel of the left-hand side of (3.2.15), and their Wronskian is 1. Therefore, by the variation of parameters given in Proposition C2, v_k (and y_k in place of v_k) must satisfy

$$\begin{aligned} v_{k+1}(r) &= \int_0^r \Big(v_0(r)y_0(s) - y_0(r)v_0(s)\Big)\Big(\lambda w - q\Big)v_k(s)\mathrm{d}s \\ &= r^{1/2}\int_0^r s^{1/2-\nu}\Big(\log(s) - \log(r)\Big)\Big(1 + \varepsilon_1(s)\Big)\Big(\lambda - \varepsilon_2(s)\Big)v_k(s)\mathrm{d}s. \end{aligned} \tag{3.2.17}$$

We want to estimate this integrand. By (3.2.14), this is straight-forward. Since $\varepsilon_j(r) = o(1)$ $(r \to 0)$, for each fixed λ there is $\delta_1 > 0$ such that

$$\Big|\Big(1 + \varepsilon_1(r)\Big)\Big(\lambda - \varepsilon_2(r)\Big)\Big| < 2|\lambda| \quad (0 < r < \delta_1). \tag{3.2.18}$$

Furthermore there is $0 < \delta_2 < 1$ with

$$|\log(s) - \log(r)| < |\log(r)| < r^{-1+\nu/2} \quad (0 < s < r < \delta_2). \tag{3.2.19}$$

Hence define $\delta(\varepsilon_0) = \min\{\delta_1, \delta_2\}$ and $\varepsilon_0 = 1 - \nu/2 > 0$.

We will focus on v_k for now. By the triangle inequality, (3.2.18) and (3.2.19) we have, for $0 < r < \delta$ and $k = 0, 1, 2, \ldots$, the estimate

$$|v_{k+1}(r)| \leq 2|\lambda| r^{1/2} \left\{ \int_0^r s^{\varepsilon_0 - 3/2} |v_k(s)| \mathrm{d}s + r^{-\varepsilon_0} \int_0^r s^{2\varepsilon_0 - 3/2} |v_k(r)| \mathrm{d}s \right\}.$$

From this and $|v_0(r)| \leq r^{1/2}$ we derive inductively that

$$|v_k(r)| \leq \frac{(3/2)_k}{(k+1)!k!} r^{1/2} \left(\frac{4|\lambda| r_0^{\varepsilon}}{\varepsilon_0} \right)^k \quad (0 < r < \delta, k = 1, 2, 3, \ldots), \tag{3.2.20}$$

where $(z)_k := z(z+1)\cdots(z+k-1)$ is the *Pochhammer symbol*. But for all $j \geq 0$ we have

$$\frac{3/2 + j}{2 + j} < 1 \implies \frac{(3/2)_k}{(k+1)!} < 1 \quad (k \in \mathbb{N}).$$

Thus (3.2.20) simplifies to

$$|v_k(r)| < r^{1/2} \frac{1}{k!} \left(\frac{4|\lambda| r_0^{\varepsilon}}{e} \right)^k \quad (0 < r < \delta, k \in \mathbb{N}), \tag{3.2.21}$$

implying, by Weierstraß' M-test for convergence of functional series, that V is uniformly convergent on the interval $(0, \delta)$. Furthermore, by (3.2.21), all terms in V are $O(r^{1/2+\varepsilon_0}) = o(r^{1/2})$, so one half of (3.2.16) is satisfied; it follows that $v(r) = r^{1/2}\Big(1 + O(r^{\varepsilon_0})\Big)$ $(r \to 0)$, as required.

We appeal to a similar argument in the case of y, using (3.2.18) alongside the

slightly different estimates

$$
\begin{aligned}
|\log(r)| &< r^{-\varepsilon_0/2} \quad \Big(0 < r < \delta(\varepsilon_0/2)\Big),\\
|y_1(r)| &\le \frac{10|\lambda| r^{1/2+\varepsilon_0}}{3\varepsilon_0} \quad \Big(0 < r < \delta(\varepsilon_0/2)\Big),\\
|y_{k+1}(r)| &\le 2|\lambda| r^{1/2} \left\{ \int_0^r s^{\varepsilon_0 - 3/2} |y_k(s)| \mathrm{d}s + r^{-\varepsilon_0} \int_0^r s^{2\varepsilon_0 - 3/2} |y_k(s)| \mathrm{d}s \right\} \quad \Big(0 < r < \delta(\varepsilon_0), k \in \mathbb{N}\Big).
\end{aligned}
$$

These can be used inductively to show that

$$
|y_k(r)| \le \frac{2(3/2)_k}{(k+1)!k!} r^{1/2} \left(\frac{4|\lambda| r_0^{\varepsilon}}{\varepsilon_0} \right)^k \quad \Big(0 < r < \delta(\varepsilon_0/2), k \in \mathbb{N}\Big).
$$

Thus, as with v_k, the series Y is uniformly convergent on $\Big(0, \delta(\varepsilon_0/2)\Big)$, and the estimates prove the second part of (3.2.16): $Y(r) = O(r^{1/2+\varepsilon_0}) = o\Big(r^{1/2}\log(r)\Big)$.

The last claim is that both $u_1(r) = r^{-1/2}v(r)$ and $u_2(r) = r^{-1/2}y(r)$ are in $L^2(0,1;r\mathrm{d}r)$. Clearly, for any $\delta > 0$, on the interval $(\delta, 1)$ the equation (3.1.5) is regular, so its solutions are all continuous. We now see that $u_1(r) \to 1, u_2(r) \sim \log(r)$ as $r \to 0$, so the claim follows immediately. □

Remark. *These results are the realisation of an analogue to Weyl's alternative ([134, 136, 135]; see, e.g., [133, Thm. 5.6]) in the case of the pencil (3.1.1). The curious distinction between the classical case and ours is that for pencils the dimension of the solution space can depend on λ, as when $\nu = 2$. This is in contrast with Weyl's original result, where if the Sturm–Liouville equation (2.2.1) is classically limit-circle for one fixed $\lambda = \lambda_0$, then it is limit-circle for every $\lambda \in \mathbb{C}$.*

3.3 Interpolation of m-functions

The problem of exact interpolation of complex-valued analytic functions is classical. In the early 20th century Pick [114] and Nevanlinna [110, 111] independently

examined the following.

Problem. *Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$, $n \in \mathbb{N}$ and $z_1, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{D}$. Can we find an analytic $f : \mathbb{D} \to \overline{\mathbb{D}}$ such that $f(z_j) = w_j$ $(j = 1, \ldots, n)$?*

They each obtained necessary and sufficient conditions for the existence of such an f, but uniqueness of the function fails since one may add to f any non-trivial function whose zeros lie at the set of z_j and which has arbitrarily small bound over $\mathbb{D}$, for example a polynomial.[3.4] More generally, one may consider interpolation from an infinite sequence of points.

Problem. *Let $(z_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ be complex sequences, and suppose $|z_n| \to \infty$ $(n \to \infty)$. Can we find an analytic f such that $f(z_n) = w_n$ $(n \in \mathbb{N})$?*

In this case existence is automatically guaranteed by the theorems of Weierstraß and Mittag-Leffler [98, Thms. 10.1 and 10.10] (indeed, this is a special case of [117, Thm. 15.13]). Once again uniqueness fails[3.5], since Weierstraß' theorem guarantees the existence of a non-trivial entire function that vanishes at all the z_n.

As outlined in Section 3.1, we would like to interpolate[3.6] the m-function (3.1.7) uniquely from its values m_n at the sequence of points $z_n = -n^2$. It is apparent that the methods of complex analysis are likely insufficient of themselves to achieve the uniqueness aspect to this. However a recent result [118, Thm. 5] in the spectral theory of Sturm–Liouville equations provides just such a unique interpolation.

[3.4] Note, however, that requiring $\|f\|_{L^\infty(\mathbb{D})}$ be minimised, the solution is uniquely found to be a constant multiple of a Blaschke product; see, e.g., [121] and the references therein.

[3.5] The Blaschke product is still the unique *minimal* analytic interpolant, but there is a necessary and sufficient condition on the interpolation sequence to ensure convergence of the now infinite product; again, see [121].

[3.6] Whilst it might seem that constructive interpolation is unnecessarily stronger than mere unique determination of (3.1.7) from the interpolation sequence, it appears from the author's calculations that the latter is not an easy result to establish without resorting to these constructive methods.

Theorem 3.1 (Rybkin–Tuan, 2009). *Let $Q \in l^\infty(L^1)(0,\infty)$ be real-valued, where this class comprises all $f : [0,\infty) \to \mathbb{C}$ such that*

$$\|f\| := \|f\|_{l^\infty(L^1)(0,\infty)} := \sup_{x\geq 0} \int_x^{x+1} |f| < \infty.$$

Suppose m_S is the Weyl–Titchmarsh m-function associated with the limit-point one-dimensional Schrödinger operator[3.7]

$$S(x,\cdot) := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + Q(x) \quad (x \in (0,\infty))$$

on $L^2(0,\infty)$, i.e., $m_S(\lambda) := u'(0,\lambda)/u(0,\lambda)$ $(\mathrm{Im}(\lambda) \neq 0)$ for any non-trivial square-integrable solution $u(\cdot\,;\lambda)$ of $Su(\cdot\,;\lambda) = \lambda u(\cdot\,;\lambda)$. If λ is from the concave parabolic region with $(\mathrm{Im}\lambda)^2 > 4\beta_0^2 \mathrm{Re}\lambda + 4\beta_0^4$ then

$$m_S(\lambda) - i\sqrt{\lambda} = \sum_{n\geq 0} c_n(-i\sqrt{\lambda} - \beta_0 + 1) \sum_{k=0}^{n} a_{nk}\Big(m_S(-\omega_k^2) + \omega_k\Big),$$

where

$$\begin{aligned}
\beta_0 &:= \max\Big\{\sqrt{2\|Q\|}, e\|Q\|\Big\} + \frac{1}{2} + \varepsilon_0,\\
&\varepsilon_0 > 0 \text{ is any fixed number},\\
c_n(z) &:= (2n+1)\frac{(1/2-z)_n}{(1/2+z)_{n+1}} \text{ (for a.e. } z \in \mathbb{C}),\\
(z)_n &:= z(z+1)\cdots(z+n-1)\ (z \in \mathbb{C}),\\
a_{nk} &:= \frac{(-n)_k(n+1)_k}{(k!)^2}\ (n,k \geq 0), \text{ and}\\
\omega_k &:= k + \beta_0 - \frac{1}{2}.
\end{aligned}$$

Remark. *We may write the parabolic region of convergence more concisely as* $\mathrm{Im}(\sqrt{\lambda}) > \beta_0$. *It is the concave part of the plane, and its intersection with any non-real ray through*

[3.7] This is the m-function defined in (2.2.5) with $a = 0, b = \infty$ and $\alpha = 0$.

the origin is an infinite complex interval.

This result does not include, and cannot—to the best of our knowledge—be immediately generalised to, a classical limit-circle case. Indeed, the proof in [118] relies on the A-amplitude representation of $m_S(\lambda)$, discussed in Section 2.3.

As we desire a limit-circle version to complete our proofs of uniqueness, it will be instructive at this point to discuss the proof of Theorem 3.1. Two main ingredients are needed, one of which we already mentioned as Theorem 2.5. The other is the following lemma [118, Thm. 4]:

Lemma 3.4 (Interpolation of a Laplace transform, Rybkin–Tuan)**.** *Suppose*

$$F(z) = \mathscr{L}[f](z) = \int_0^\infty e^{-zt} f(t)\mathrm{d}t \quad (\mathrm{Re}(z) > 0)$$

is the Laplace transform of measurable f, and that f satisfies

$$\int_0^\infty e^{-\varepsilon_0 t}|f(t)|\mathrm{d}t < \infty \quad (\varepsilon_0 > 0). \tag{3.3.1}$$

Then, for any $\varepsilon_0 > 0$, uniformly[3.8] *in any compact subset of the half-plane $\mathrm{Re}(z) > \varepsilon_0 + 1/2$ we have*

$$F(z) = \sum_{n=0}^{\infty} c_n \left(z + \frac{1}{2} - \varepsilon_0\right) \sum_{k=0}^{n} a_{nk} F(k + \varepsilon_0), \tag{3.3.2}$$

where c_n, a_{nk} are defined in Theorem 3.1.

Remark. *Noting that*

$$e^2\sqrt{2\|Q\|\alpha} + \frac{e^{2e\|Q\|\alpha}}{\sqrt{2\pi}} \leq \left(1 + \frac{1}{\sqrt{2\pi}}\right) e^{2\gamma x}$$

with $\gamma := \max\{\sqrt{2\|Q\|}, e\|Q\|\}$ we start to see where the constant $\beta_0 = \gamma + 1/2 + \varepsilon_0$ comes

[3.8] The authors also provide the error $O(N^{-\mathrm{Re}(z)+1/2+\varepsilon_0})$ upon truncation of the series at its N-th term, which is superfluous to our needs, but is useful when implementing the series in a numerical procedure.

from in Theorem 3.1 (the $1/2 + \varepsilon_0$ part lies in Lemma 3.4). Clearly from (2.3.7) we see that $m_S(-\kappa^2) + \kappa$ is of the form $F(\kappa)$ to be plugged into Lemma 3.4. One only needs to check that it may be written in a way that involves an appropriate f. This is achieved by carefully splitting up the exponential term, using part as the Laplace transform integral kernel and the other part to ensure the negative-exponentially weighted L^1-convergence required by the Lemma. The appropriate way to do this emerges from Theorem 2.5.

A further remark is that, since the proof of Theorem 3.1 relies fundamentally on the exact formula 2.3.7, one would need an equivalent version for the classical limit-circle case on the finite interval $(0, b)$. Such an exact formula does not (yet) exist. The best found yet comes from Simon's original work on the matter, [120, Eq. (1.24)]. In our case, it amounts to truncating the integral to the interval $(0, a) \subset (0, b)$, and noting the resulting error is of order $O(e^{-2a(1-\delta)\mathrm{Re}(\kappa)})$ for every $\delta > 0$.

We also observe that this is quite different from the classical interpolation result of, say, Lagrange[3.9] [86]. Classically to interpolate (formally) from an infinite sequence one truncates the sequence at the N-th term, constructs the (at most) N-th-order Lagrange polynomial interpolant, then takes the limit as $N \to \infty$. This differs from Lemma 3.4, since it produces a sequence of polynomials where the latter instead has partial summands that are rational functions. Moreover the convergence of the Lagrange interpolants takes place uniformly in any compact subset of $\mathbb{C}$, without the restriction $\mathrm{Re}(z) > \varepsilon_0 + 1/2$.

The remainder of this section is dedicated to establishing an analogous result to Theorem 3.1, for a particular finite-interval classical limit-circle Sturm–Liouville problem. The methods we will use are of a more elementary, *ad hoc* nature than those underlying Rybkin and Tuan's deep result. Nevertheless, we hope the result can be generalised in method and scope.

Proposition 3.4. *Suppose $0 < b < \infty$ and $Q \in L^1(0, b)$ is real-valued. At 0 the differen-*

[3.9] Actually, it was originally due to Waring [132]; for a detailed review of the topic, see [102].

tial equation

$$-z''(x;\lambda)+\Big(Q(x)-\frac{1}{4x^2}\Big)z(x;\lambda)=\lambda z(x;\lambda)\quad\Big(x\in(0,b)\Big)\tag{3.3.3}$$

is classical limit-circle non-oscillatory over $L^2(0,b)$.

Proof. If we were to restrict the problem to the interval $(0,1)$ it is almost exactly of the form in Lemma 3.3, with $w(x)=1$ (i.e., $\nu=0$ and $\varepsilon_1(x)=0$). The major difference is that to apply the lemma directly we would require at least $Q(x)=o(1)$ $(x\to 0)$. Whilst this is clearly not necessarily true, we can easily adapt the methods of proof.

As before we seek a pair of solutions, which we denote $z(\cdot\,;\lambda)$ and $y(\cdot\,;\lambda)$. Define the sequences

$$\begin{cases} -z''_{k+1}(x;\lambda)-\frac{1}{4x^2}z_{k+1}(x;\lambda)=\Big(\lambda-Q(x)\Big)z_k(x;\lambda), & z_0(x;\lambda)=x^{1/2},\\ -y''_{k+1}(x;\lambda)-\frac{1}{4x^2}y_{k+1}(x;\lambda)=\Big(\lambda-Q(x)\Big)y_k(x;\lambda), & y_0(x;\lambda)=x^{1/2}\log(x),\end{cases}$$

and impose the boundary conditions

$$\left.\begin{matrix}z_k(x)\\ y_k(x)\end{matrix}\right\}=o(x^{1/2})\quad(x\to 0,k\in\mathbb{N}).$$

Firstly note that z_0 and y_0 are always a fundamental system in the kernels of the left-hand sides of the differential equations defining the sequences, and their Wronskian is 1. Hence by variation of parameters

$$\begin{aligned}z_{k+1}(x;\lambda)&=x^{1/2}\int_0^x s^{1/2}\Big(\log(s)-\log(x)\Big)\Big(\lambda-Q(s)\Big)z_k(s;\lambda)\mathrm{d}s,\\ y_{k+1}(x;\lambda)&=x^{1/2}\int_0^x s^{1/2}\Big(\log(s)-\log(x)\Big)\Big(\lambda-Q(s)\Big)y_k(s;\lambda)\mathrm{d}s,\end{aligned}$$

and if we choose $x < 1$ we may estimate

$$|z_{k+1}(x;\lambda)| \leq x^{1/2} \int_0^x s^{1/2} |\log(s)| |\lambda - Q(s)| |z_k(s;\lambda)| \mathrm{d}s,$$
$$|y_{k+1}(x;\lambda)| \leq x^{1/2} \int_0^x s^{1/2} |\log(s)| |\lambda - Q(s)| |y_k(s;\lambda)| \mathrm{d}s.$$

Clearly $\int_0^x |\lambda - Q| \leq \|\lambda - Q\|_{L^1(0,1)}$, and for each $\varepsilon_0 > 0$ there is some $0 < \delta < 1$ such that $x\,|\log(x)| < x\,|\log(x)|^2 \leq x^{1-\varepsilon_0}$ whenever $0 < x < \delta$. Working inductively we are then able to show that, when $x < \delta$, both

$$|z_k(x;\lambda)|, |y_k(x;\lambda)| \leq x^{1/2} (x^{1-\varepsilon_0} \|\lambda - Q\|_{L^1(0,1)})^k \quad (k \geq 1).$$

Hence for $0 < x < \min\{\delta, \|\lambda - Q\|\}$ we see that the series

$$z(x;\lambda) := \sum_{k=0}^{\infty} z_k(x;\lambda),$$
$$y(x;\lambda) := \sum_{k=0}^{\infty} y_k(x;\lambda)$$

are absolutely and uniformly convergent.

In a neighbourhood of 0 both sums formally solve (3.3.3). Moreover we may estimate the tails of the sums and write

$$z(x;\lambda) = x^{1/2}\Big(1 + O(x^{1-\varepsilon_0})\Big) \text{ and } y(x;\lambda) = x^{1/2} \log(x)\Big(1 + O(x^{1-\varepsilon_0})\Big).$$

By unique continuation these may be extended to solutions of (3.3.3) on the whole of $(0, b)$. Since the differential equation is regular on $(\delta/2, b)$, all solutions are in $L^2(\delta/2, b)$. It follows from the asymptotics of the series that both $z(\cdot\,;\lambda)$ and $y(\cdot\,;\lambda) \in L^2(0, b)$. Finally, the last pair of asymptotic relations demonstrates that neither solution has any zeros in a sufficiently small neighbourhood of 0. □

Since the problem is limit-circle at 0 we require a boundary condition there so we may define our m-function at the other end-point. We will use the *Friedrichs* or *principal* condition. Let U_p be a principal solution of (3.3.3), i.e., U_p is non-trivial and for any linearly independent solution V we have $U_p(x;\lambda) = o\big(V(x;\lambda)\big)$ as $x \to 0$. The Friedrichs boundary condition at 0 is the requirement that a solution z satisfy

$$\mathrm{Fr}[z](0^+;\lambda) := [z, U_p](0^+;\lambda) = 0. \tag{3.3.4}$$

Remark. *This boundary condition forms part of a characterisation of the so-called* Friedrichs extension. *The details may be found in, e.g., [112, Thm. 4.3].*

We write (3.3.3) and (3.3.4) as the partial boundary-value problem

$$\begin{cases} -z''(x;\lambda) + \left(Q(x) - \frac{1}{4x^2}\right) z(x;\lambda) &= \lambda z(x;\lambda) \quad \big(x \in (0,b)\big), \\ \mathrm{Fr}[z](x;\lambda) &\to 0 \qquad (x \to 0), \end{cases} \tag{3.3.5}$$

which, up to a constant multiple, uniquely specifies a solution $z(\cdot\,;\lambda)$ thanks to Proposition 3.4. Taking such a non-trivial solution, we choose a purely Robin (-to-Robin) m-function, i.e., for $h \neq H$ both real, the unique $m_{h,H}(\lambda)$ satisfying[3.10]

$$z'(b;\lambda) - Hz(b;\lambda) = m_{h,H}(\lambda)\big(z'(b;\lambda) - hz(b;\lambda)\big). \tag{3.3.6}$$

We will interpolate this m-function, in the style of Theorem 3.1. One portion of the work is the application of the interpolation result for Laplace transforms, Lemma 3.4, in the same way it was used for Theorem 3.1.

The other portion is as follows. We will find a Laplace transform representation of $m_{h,H}(\lambda)$ by first constructing its so-called *Mittag-Leffler series expansion*

[3.10] Whilst this is not simply a constant times any m-function of the form (2.2.5), it is clearly a fractional linear transformation of one. By Corollary A1, if $h > [<]H$ then $m_{h,H}$ is [anti-]Herglotz. We will provide an alternative more direct proof of this latter fact on page 79.

(see, e.g., [38, Ch. 8]) then algebraically relating it to a Laplace transform. We then prove that condition (3.3.1) holds, allowing us to apply Lemma 3.4.

First we mention some preliminary results. The following is well known:

Lemma 3.5. *A self-adjoint operator associated with a classical limit-circle non-oscillatory Sturm–Liouville expression and separated boundary conditions on a finite interval possesses a spectrum comprising simple eigen-values accumulating only at $+\infty$.*

Proof. One way to observe this is to use the Niessen–Zettl transformation [112] of such a problem to a regular problem on the same interval, then recall that spectra of regular separated Sturm–Liouville problems also comprise simple eigen-values only accumulating at $+\infty$; see, e.g., [37]. Alternatively the eigen-value fact is proved in general in, e.g., [92, Ch. 2]. □

Define the operator

$$\begin{aligned} D(L_h) = \{z \in L^2(0,b) \mid & -z''(x) + (Q(x) - 1/4x^2)z(x) \in L^2(0,b;\mathrm{d}x), \\ & [z, U_p](0^+) = 0, z'(b) = hz(b)\}, \\ L_h z(x) = -z''(x) + \big(Q(x) - 1/4x^2\big)z(x) & \quad \big(x \in (0,b)\big). \end{aligned}$$

Lemma 3.6. *The Weyl–Titchmarsh m-function $m_{h,H}$ is meromorphic in $\mathbb{C}$. Its poles and zeros are simple, interlace on $\mathbb{R}$, and are found at the eigen-values of, respectively, L_h and L_H, accumulating only at $+\infty$.*

Proof. The differential expression is classical limit-circle at b, so with the given boundary conditions, L_h and L_H are self-adjoint owing to Lemma 2.1. Since the boundary conditions are separated the eigen-values must be simple, by Lemma 3.5. Classical interlacing theorems [92, Thm. I.3.1-2] guarantee that when $h \neq H$

the eigen-values for L_h and L_H interlace. That they accumulate only at $+\infty$ follows from Lemma 3.5.

Finally we observe that the poles of $m_{h,H}$ occur at λ for which the non-trivial solution $z(\cdot\,;\lambda)$ to (3.3.5) satisfies

$$z'(b;\lambda) = hz(b;\lambda), \tag{3.3.7}$$

i.e., when this solution is an eigen-function of L_h. Similarly the zeros occur when $z'(b;\lambda) = Hz(b;\lambda)$. □

In Lemma B1 we show that in the classical limit-circle non-oscillatory case, enumerating the eigen-values in increasing order as λ_n, $n = 1, 2, 3, \ldots$, we have

$$\begin{aligned} \lambda_n &= (n + 1/4)^2\pi^2/b^2 + O(1), \\ \sqrt{\lambda}_n &= (n + 1/4)\pi/b + O(1/n) \quad (n \to \infty). \end{aligned} \tag{3.3.8}$$

For each n, the eigen-function φ_n corresponding to λ_n is defined by

$$\varphi_n = \varphi(\cdot\,;\lambda_n),$$

where φ solves (3.3.3) with initial conditions $\varphi(b;\lambda) = 1, \varphi'(b;\lambda) = h$. Suppose ψ is the linearly independent solution with $\psi(b;\lambda) = 1, \psi'(b;\lambda) = H$, and

$$\Phi(\lambda) := \varphi(0^+;\lambda), \quad \Psi(\lambda) := \psi(0^+;\lambda).$$

Then, by checking that $f(x;\lambda) := \psi(x;\lambda) + m_{h,H}(\lambda)\varphi(x;\lambda)$ satisfies the "boundary condition" in (3.3.6), it follows that

$$m_{h,H}(\lambda) = -\frac{\Psi(\lambda)}{\Phi(\lambda)}. \tag{3.3.9}$$

Therefore, $\Phi(\lambda_n)$ being 0 implies *via* integration by parts that,

$$\begin{aligned} H - h &= f'(b;\lambda)\varphi_n(b) - f(b;\lambda)\varphi_n'(b) \\ &= (\lambda - \lambda_n)\int_0^b f(\cdot\,;\lambda)\varphi_n \\ &= (\lambda - \lambda_n)\int_0^b \psi(\cdot\,;\lambda)\varphi_n - (\lambda - \lambda_n)\frac{\Psi(\lambda)}{\Phi(\lambda) - \Phi(\lambda_n)}\int_0^b \varphi(\cdot\,;\lambda)\varphi_n \\ &\to -\frac{\Psi(\lambda_n)}{\Phi'(\lambda_n)}\int_0^b \varphi_n^2 \ \text{ as } \lambda \to \lambda_n. \end{aligned} \tag{3.3.10}$$

If we denote the norming constants associated with λ_n by $\alpha_n := \int_0^b \varphi_n^2$ then we see from (3.3.9), (3.3.10) and the standard formula for evaluating residues at simple poles that the residue of the m-function at its poles is given by

$$\operatorname{Res}(m_{h,H};\lambda_n) = -\frac{\Psi(\lambda_n)}{\Phi'(\lambda_n)} = \frac{H-h}{\alpha_n}. \tag{3.3.11}$$

In Lemma B2 we show that the norming constants satisfy

$$\alpha_n = b/2 + O(1/n). \tag{3.3.12}$$

We may now state the Mittag-Leffler series result for $m_{h,H}$:

Lemma 3.7. *Uniformly for λ in any compact set that is non-intersecting with $\{\lambda_n\}_{n=0}^{\infty}$, we have a Mittag-Leffler series representation for the Robin m-function given by*

$$m_{h,H}(\lambda) - \frac{H}{h} = \sum_{n=1}^{\infty} \frac{h-H}{\alpha_n(\lambda_n - \lambda)}. \tag{3.3.13}$$

Proof. The asymptotics (3.3.12) and (3.3.8) immediately imply that, uniformly for λ in any compact set bounded away from $\{\lambda_n\}_{n=1}^{\infty}$, as $n \to \infty$ we have

$$\alpha_n(\lambda - \lambda_n) = \left(\frac{b}{2} + O\left(\frac{1}{n}\right)\right)\left(\left(n + \frac{1}{4}\right)^2 \frac{\pi^2}{b^2} + O(1)\right) \sim \frac{n^2\pi^2}{2b}.$$

Thus $\sum_{n=1}^{\infty} \frac{1}{\alpha_n(\lambda-\lambda_n)}$ is convergent, uniformly for the same choice of λ. Moreover,

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n(\lambda-\lambda_n)} \to 0 \quad (\mathrm{Im}(\lambda) \to \infty). \tag{3.3.14}$$

We need to link the series with the m-function. We do this *via* Herglotz-type properties of $m_{h,H}$. For completeness we briefly repeat here the following well-known calculation, showing that $m_{h,H}$ is (anti-)Herglotz (see Section 2.2 and Appendix A). Observe that for any non-trivial solution $z(\cdot\,;\lambda)$ of (3.3.3) and (3.3.4) we have

$$(\lambda-\bar{\lambda})\int_0^b z(\cdot\,;\lambda)z(\cdot\,;\bar{\lambda}) = (h-H)\Big(m_{h,H}(\lambda)-m_{h,H}(\bar{\lambda})\Big),$$

so subtracting the complex conjugate of this whilst noting $z(\cdot\,;\bar{\lambda}) = \overline{z(\cdot\,;\lambda)}$ shows that

$$\int_0^b |z(\cdot\,;\lambda)|^2 = (h-H)\frac{\mathrm{Im}\Big(m_{h,H}(\lambda)\Big)}{\mathrm{Im}(\lambda)}.$$

Hence, if $h > H$ then $m_{h,H}$ is in the Herglotz class of functions that map the upper and lower half-planes to themselves, whilst if $h < H$ then $m_{h,H}$ is the negative of such a function, known as anti-Herglotz.

As remarked in equation (2.2.6) and Appendix A, all (anti-)Herglotz functions have a Stieltjes integral representation; here

$$m_{h,H}(\lambda) = A + B\lambda + \int_{\mathbb{R}}\left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right)\mathrm{d}\rho(t),$$

where ρ is the spectral measure associated with the problem (3.3.3), and

$$A = \mathrm{Re}\Big(m_{h,H}(i)\Big), \quad B = \lim_{\tau\to+\infty}\frac{m_{h,H}(i\tau)}{i\tau}.$$

Note that ρ is increasing if and only if $h > H$. Furthermore as a measure it assigns "mass" only at points in the spectrum of the Sturm–Liouville operator associated

with (3.3.3), i.e., for any $\mathrm{d}\rho$-integrable g,

$$\int_{\mathbb{R}} g(t)\mathrm{d}\rho(t) = \sum_{n=1}^{\infty} \gamma_n g(\lambda_n),$$

γ_n being the mass at λ_n. Thus

$$m_{h,H}(\lambda) = A + B\lambda + \sum_{n=1}^{\infty} \gamma_n \left(\frac{1}{\lambda_n - \lambda} - \frac{\lambda_n}{1+\lambda_n^2} \right).$$

Integrating anti-clockwise on a sufficiently small, simple, closed contour around λ_n and comparing with (3.3.11) shows that $\gamma_n = -\mathrm{Res}(m_{h,H}; \lambda_n) = \frac{h-H}{\alpha_n}$. Hence we may split up the sum and write

$$m_{h,H}(\lambda) = \tilde{A} + B\lambda + \sum_{n=1}^{\infty} \frac{h-H}{\alpha_n(\lambda_n - \lambda)}. \tag{3.3.15}$$

To proceed, we need large-$\mathrm{Im}(\lambda)$ asymptotics of $m_{h,H}(\lambda)$. Expressing $m_{h,H}$ in terms of the Neumann m-function $m_N(\lambda) := z(b;\lambda)/z'(b;\lambda)$ and using Lemma C3, we see

$$m_{h,H}(\lambda) = \frac{1 - Hm_N(\lambda)}{1 - hm_N(\lambda)} \sim \frac{1 + iH\sqrt{\lambda}}{1 + ih\sqrt{\lambda}} \to \frac{H}{h} \quad (\mathrm{Im}(\lambda) \to +\infty).$$

Applying this and (3.3.14) to (3.3.15) we see[3.11] that $\tilde{A} = H/h$ and $B = 0$. □

Remark. *Our calculations proving this result are adapted from parts of a calculation in [91, Ch. 3] for a regular Sturm–Liouville problem in normal form.*

Lemma 3.7 gives us enough to deduce a Laplace transform representation of $m_{h,H}$, and hence our interpolation result. For the reader's convenience we state the theorem in full.

Theorem 3.2 (Classical limit-circle m-function interpolation)**.** *Under the hypothesis*

[3.11] This corrects the minor error "$\tilde{A} = 1$" in [27].

that $Q \in L^2(0,b)$ is real-valued, the Robin m-function

$$m_{h,H}(\lambda) = \frac{z'(b;\lambda) - Hz(b;\lambda)}{z'(b;\lambda) - hz(b;\lambda)}$$

for any non-trivial square-integrable solution $z(\cdot\,;\lambda)$ of the limit-circle non-oscillatory problem

$$\begin{cases} -z''(x;\lambda) + \Big(Q(x) - \dfrac{1}{4x^2}\Big)z(x;\lambda) &= \lambda z(x;\lambda) \quad \Big(x \in (0,b)\Big), \\ [U_p, z](0^+;\lambda) &= 0, \end{cases}$$

satisfies the interpolation formula

$$m_{h,H}(\lambda) - \frac{H}{h} = \sum_{n=0}^{\infty} c_n(1/2 - \beta - i\sqrt{\lambda}) \sum_{k=0}^{n} a_{nk} \Big(m_{h,H}\Big(-(k+\beta)^2\Big) - \frac{H}{h}\Big).$$

Here $\beta > 0$ is fixed,

$$\begin{aligned} c_n(z) &:= (2n+1)\frac{(1/2 - z)_n}{(1/2 + z)_{n+1}} \text{ (for a.e. } z \in \mathbb{C}), \\ (z)_n &:= z(z+1)\cdots(z+n-1) \ (z \in \mathbb{C}), \\ a_{nk} &:= \frac{(-n)_k (n+1)_k}{(k!)^2} \ (n,k \geq 0), \end{aligned}$$

and the convergence of the series is uniform in any compact subset of $\mathrm{Im}(\sqrt{\lambda}) > 1/2 + \beta$.

The proof uses Lebesgue's dominated convergence theorem. To apply it we will need the following lemma:

Lemma 3.8. *Let $\varepsilon_0 > 0$ and denote $\rho_n = \sqrt{\lambda_n}$. Then $g_N(t) := e^{-\varepsilon_0 t} \sum_{n=1}^{N} \frac{\sin(\rho_n t)}{\alpha_n \rho_n}$ is uniformly bounded, in $t \in (0,\infty)$ and $N \in \mathbb{N}$, by a fixed integrable function.*

Proof. Firstly note that the asymptotic expansion (3.3.8) may be written as $\rho_n =$

$(n + 1/4)\pi + \varepsilon_n$, where $\varepsilon_n = O(1/n)$. Then, for each fixed $t \geq 0$,

$$\sin(\rho_n t) = \sin\Big((n + 1/4)\pi t\Big)\cos(\varepsilon_n t) + \cos\Big((n + 1/4)\pi t\Big)\sin(\varepsilon_n t). \tag{3.3.16}$$

Write $\varepsilon_0 = 2\sigma$. It would be enough to find an $L^1(0, 2\pi)$ function that bounds, uniformly in N, the expression

$$s_N(t) := e^{-\sigma t}\sum_{n=1}^{N}\frac{\sin(\rho_n t)}{\alpha_n \rho_n} \quad \Big(t \in (0, 2\pi)\Big),$$

so that $g_N(t) = e^{-\sigma t}s_N(t)$ $(t \in (0, \infty))$ is dominated by an $L^1(0, \infty)$ function, thanks to the exponential decay of $e^{-\sigma t}$. So, notice that

$$\begin{cases} |\varepsilon_n t| &= O(\log(n)/n) \qquad \Big(0 \leq t < \sigma^{-1}\log(n)\Big), \\ e^{\sigma t} &\leq e^{-\sigma\sigma^{-1}\log(n)} = 1/n, \qquad \Big(t \geq \sigma^{-1}\log(n)\Big), \end{cases}$$

so that $e^{-\sigma t}\sin(\varepsilon_n t) = O(1/\sqrt{n})$; by a similar argument $e^{-\sigma t}\Big(\cos(\varepsilon_n t) - 1\Big) = O(1/n)$. Both estimates are uniform in $t \geq 0$. With (3.3.16), these are enough to ensure a constant bound for

$$s_N(t) - e^{-\sigma t}\sum_{n=1}^{N}\frac{\sin\Big((n + 1/4)\pi t\Big)}{\alpha_n \rho_n} \quad \Big(t \in (0, 2\pi)\Big).$$

Hence, substituting the asymptotic expansions (3.3.8) and (3.3.12) into the second sum in the above expression means the following: if we can show that both $\sum_{n=1}^{N}\cos(nx)/n$ and $\sum_{n=1}^{N}\sin(nx)/n$ $\Big(x \in (0, 2\pi)\Big)$ are bounded, uniformly in N, by some fixed element of $L^1(0, 2\pi)$, then it will follow that so is $s_N(t)$ $\Big(t \in (0, 2\pi)\Big)$, proving the lemma.

We will prove the uniform $L^1(0, 2\pi)$ bound for the cos-series; the same approach produces a similar bound for the sin-series. Denote by $c_N(x)$ the partial sum $\sum_{n=1}^{N}\cos(nx)/n$. This is an anti-derivative of the conjugate Dirichlet kernel

sum [31, Sec. 15.2]—for the sin-series use the usual Dirichlet kernel—meaning

$$c_N'(x) = -\sum_{n=1}^{N} \sin(nx) = \frac{\cos\big((N+1/2)x\big) - \cos(x/2)}{2\sin(x/2)}.$$

Thus $c_N'(x)$ is bounded by $1/\sin(x/2)$. Noting $|c_N(\pi)| \le 1$, we see $|c_N(x)| \le 1 + \int_\pi^x |c_N'| \le 1 + 2\log|\cot(x/4)|$, which is certainly integrable over $(0, 2\pi)$ since to leading-order it is $-\log(x)$ for x near 0 and $-\log(2\pi - x)$ near 2π. □

Proof of Theorem 3.2. We first observe that the Mittag-Leffler series (3.3.13) may be written as

$$\begin{aligned} m_{h,H}(-\kappa^2) - \frac{H}{h} &= \sum_{n=1}^{\infty} \frac{h-H}{\alpha_n(\rho_n^2 + \kappa^2)} \\ &= (h-H)\sum_{n=1}^{\infty} \int_0^\infty e^{-\kappa t} \frac{\sin(\rho_n t)}{\alpha_n \rho_n} \mathrm{d}t \quad (\mathrm{Re}(\kappa) > 0). \end{aligned} \tag{3.3.17}$$

Assuming that integration and summation may be interchanged (we show this below) we see that $\big(m_{h,H}(-\kappa^2) - H/h\big)/(h-H)$ is the Laplace transform $\mathscr{L}[f](\kappa)$ of

$$f(t) := \sum_{n=1}^{\infty} \frac{\sin(\rho_n t)}{\alpha_n \rho_n} \quad (t \ge 0). \tag{3.3.18}$$

We now prove the convergence of (3.3.18) and justify the interchange of summation and integration in (3.3.17).

From (3.3.16) we have $\sin(\rho_n t) = \cos(\pi t/4)\sin(n\pi t) + \sin(\pi t/4)\cos(n\pi t) + O(1/n)$. Hence, by (3.3.8) and (3.3.12), the pointwise convergence of (3.3.18) is determined by that of the series $\sum_{j=1}^{\infty} e^{ijx}/j$. But this is simply the Fourier series for the 2π-periodic extension of the expression $-\log|2\sin(x/2)| + i(\pi - x)/2$ $\big(x \in (-\pi, \pi)\big)$ so the pointwise convergence of (3.3.18) is immediate.

We may now simply apply Lemma 3.8 to see that $g_N(t) := e^{-\mathrm{Re}(\kappa)t}\sum_{n=1}^{N} \frac{\sin(\rho_n t)}{\alpha_n \rho_n}$ is

dominated by an integrable function. Dominated convergence follows, so

$$m_{h,H}(-\kappa^2) - \frac{H}{h} = (h - H)\int_0^\infty e^{-\kappa t} f(t)\mathrm{d}t \quad (\mathrm{Re}(\kappa) < 0).$$

All that remains is to check condition (3.3.1). But that it holds is obvious, since, by dominated convergence, $e^{-\delta t}|f(t)|$ is integrable for every $\delta > 0$. Therefore, by application of the interpolation result (3.3.2) to $F(\kappa) = m_{h,H}(-\kappa^2) - H/h$, the theorem follows, with uniform convergence in any compact subset of the parabolic λ-region $\mathrm{Im}(\sqrt{\lambda}) > 1/2 + \beta$. □

3.4 Uniqueness theorems for the inverse problem

The main result of this chapter is a pair of uniqueness theorems for Inverse Problem I. We will state and prove these here, by means of Theorem 3.1 and our interpolation result in Theorem 3.2. The uniqueness theorems are kept separate due to certain technical conditions in both being similar in representation but fundamentally different in structure.

Theorem 3.3 (Uniqueness in the PLP case)**.** *Fix $\nu \geq 2$, $c > 0$ and $\alpha > \nu/2 - 1 \geq 0$, and let $w, q \in L^\infty_{\mathrm{loc}}(0,1]$, with q real-valued and $w \geq c$. Suppose that $w', w'' \in L^\infty_{\mathrm{loc}}(0,1]$. Suppose also that, as $r \to 0$,*

(i) $w(r) = \dfrac{1}{r^\nu}\Big(1 + O(r^\alpha)\Big)$,

(ii) $q(r) = w(r)O(r^\alpha)$,

(iii) $\Big(w(r)r^\nu\Big)' = O(r^{-\nu/2})$, *and* $\Big(w(r)r^\nu\Big)'' = O(r^{-\nu})$.

If w is known, then the interpolation sequence $\mathcal{S} = \Big((-n^2, m_n)\Big)_{n=1}^\infty$, of values (in the graph) of the PLP Dirichlet m-function (3.1.7) for (3.1.5), uniquely determines the potential q.

Proof. We need to transform the differential equation to Liouville normal form, as it is for m-functions associated with such problems that we have interpolation formulae. In this case we set

$$t = \tau(r) = \int_r^1 \sqrt{w},$$
$$z\Big(t(r)\Big) = r^{1/2} w(r)^{1/4} u(r) \quad \Big(r \in (0,1)\Big).$$

This leads to the corresponding solution space $L^2(0,\infty;\tau^{-1}(t)^\nu \mathrm{d}t)$ in which we seek $z(\cdot\,;\lambda)$ and over which the equation is PLP at $t = +\infty$ (or, in the case $\nu = 2$, has the PLP/PLC behaviour outlined in Section 3.2). This is not problematic since, although the classical limit-point m-function is defined using solutions from $L^2(0,\infty)$, by their definition the two m-functions are linearly related: if m_S denotes the Schrödinger m-function—as defined in, e.g., Theorem 3.1—then

$$m_S(\lambda) = -\frac{1}{2} w(1)^{-1/2} - \frac{1}{4} w(1)^{-3/2} w'(1) + w(1)^{-1/2} m(\lambda). \tag{3.4.1}$$

That the domain in which t lies is $(0,\infty)$ follows from the fact that $\tau(r) \sim \int_r^1 s^{-\nu/2} \mathrm{d}s \to \infty$ as $r \to 0$. The equation satisfied by z is

$$-z''(t;\lambda) + Q(t) z(t;\lambda) = \lambda z(t;\lambda) \quad \Big(t \in (0,\infty)\Big), \tag{3.4.2}$$

where

$$\begin{aligned} Q\Big(\tau(r)\Big) &:= \frac{q(r)}{w(r)} - r^{-1/2} w(r)^{-3/4} \frac{\mathrm{d}}{\mathrm{d}r} \left\{ r \frac{\mathrm{d}}{\mathrm{d}r} (r^{-1/2} w(r)^{-1/4}) \right\} \\ &= -\left(\frac{\nu-2}{4}\right)^2 r^{\nu-2} \Big(1 + \zeta(r)\Big) + \varepsilon_2(r) \quad \Big(r \in (0,1)\Big), \end{aligned} \tag{3.4.3}$$

and

$$\varepsilon_1(r) = w(r)r^\nu - 1, \tag{3.4.4}$$

$$\varepsilon_2(r) = \frac{q(r)}{w(r)}, \tag{3.4.5}$$

$$\begin{aligned}\zeta(r) = &-\frac{\varepsilon_1(r)}{1+\varepsilon_1(r)} - \left(\frac{2}{\nu-2}\right)^2 \frac{r^2\varepsilon_1''(r)}{(1+\varepsilon_1(r))^2} \\ &+ \frac{5}{(\nu-2)^2}\frac{r^2\varepsilon_1'(r)^2}{(1+\varepsilon_1(r))^3} - \frac{2\nu}{(\nu-2)^2}\frac{r\varepsilon_1'(r)}{(1+\varepsilon_1(r))^2}.\end{aligned} \tag{3.4.6}$$

We now want to apply Theorem 3.1 to the m-function of equation (3.4.2); for this we need $\int_x^{x+1}|Q|$ to be a bounded expression in $x \in (0,\infty)$, i.e., $Q \in l^\infty(L^1)(0,\infty)$. It would suffice that $Q \in L^\infty(0,\infty)$. Notice that

$$\int_x^{x+1}|Q(t)|\mathrm{d}t = \int_x^{x+1}\left|\left(\frac{\nu-2}{4}\right)^2\tau^{-1}(t)^{\nu-2}\Big(1+\zeta\big(\tau^{-1}(t)\big)\Big) - \varepsilon_2\big(\tau^{-1}(t)\big)\right|\mathrm{d}t.$$

By applying the hypotheses (i) and (iii) to (3.4.4) we easily observe that

$$\varepsilon_1'(r) = O(r^{-\nu/2}) \subset O(r^{1-\nu}) \text{ and } \varepsilon_1''(r) = O(r^{-\nu}) \quad (r \to 0).$$

Thus $\zeta(r) \in L^\infty_{\mathrm{loc}}(0,1]$ and is $O(r^{2-\nu})$ as $r \to 0$. Further $w, q \in L^\infty_{\mathrm{loc}}(0,1]$ implies that $\varepsilon_2(r)$ is bounded. Therefore $Q \in L^\infty(0,\infty) \subset l^\infty(L^1)(0,\infty)$, so (3.4.2) is in classical limit-point at ∞. Since the integral hypothesis of Theorem 3.1 is satisfied, m_S can be interpolated from its values at the points $(-n^2)_{n=1}^\infty$. Then we may transform back to m by (3.4.1).

In particular, given any non-real ray through the origin and the sequence of interpolation pairs

$$\big((-n^2, m_n)\big)_{n=1}^\infty, \tag{3.4.7}$$

for any λ on this ray we can calculate the value of $m(\lambda)$. Choosing any such ray in the first quadrant and applying Theorem 2.4 we immediately have that Q and

hence q are uniquely determined by the sequence $\mathcal{S}$. □

Remark. *For this proof to work it is crucial to know w everywhere in order to be able to reverse the transformations (3.4.1) and (3.4.3).*

Theorem 3.4 (Uniqueness in the PLC case)**.** *Let $0 \leq \nu < 2$, $c > 0$ and $\alpha > 0$, and fix $w, q \in L^\infty_{\mathrm{loc}}(0,1]$, with q real-valued and $w \geq c$ a.e.. Suppose that $w', w'' \in L^\infty_{\mathrm{loc}}(0,1]$, and that, as $r \to 0$,*

(i) $w(r) = \dfrac{1}{r^\nu}\Big(1 + O(r^\alpha)\Big)$,

(ii) $q(r) = w(r)O(r^{\alpha-2})$,

(iii) $\Big(w(r)r^\nu\Big)' = O(r^{\alpha-1})$, *and* $\Big(w(r)r^\nu\Big)'' = O(r^{\alpha-2})$.

If w is known, then the interpolation sequence $\mathcal{S} = \Big((-n^2, m_n)\Big)_{n=1}^\infty$, of values (in the graph) of the PLC Dirichlet m-function (3.1.7) for (3.1.5) with boundary condition (3.1.6), uniquely determines the potential q.

Remark. *The hypotheses of this theorem differ from those of Theorem 3.3 by the range of possible ν's and the technical condition (iii).*

To prove this result we will need the following lemma:

Lemma 3.9. *Let $0 \leq \nu < 2$, $c > 0$ and $c \leq w \in L^\infty_{\mathrm{loc}}(0,1]$. If $w(r) = r^{-\nu}\Big(1 + o(1)\Big)$ as $r \to 0$ then*

$$\tau(r) := \int_0^r \sqrt{w} \tag{3.4.8}$$

is invertible everywhere on $\Big(0, b := \int_0^1 \sqrt{w}\Big)$ and satisfies

$$\tau^{-1}(t) = \left(\frac{2-\nu}{2}t\right)^{\frac{2}{2-\nu}}\Big(1 + o(1)\Big) \quad (t \to 0). \tag{3.4.9}$$

Proof. First note that $\sqrt{w}$ is integrable. Thus the function τ is clearly continuous and strictly monotone increasing, since $\tau' \geq \sqrt{c} > 0$, implying its invertibility. By the asymptotic hypothesis of w, for any $\varepsilon_0 > 0$ there is $\delta > 0$ such that

$$(1+\varepsilon_0)\frac{2}{2-\nu}r^{\frac{2-\nu}{2}} > \tau(r) > (1-\varepsilon_0)\frac{2}{2-\nu}r^{\frac{2-\nu}{2}} \quad (0 < r < \delta),$$

Now observe that for any two strictly increasing invertible functions f and g, sharing a domain $D \subset \mathbb{R}$ and satisfying $f < g$ everywhere, we have $f^{-1} > g^{-1}$ everywhere in $f(D) \cap g(D)$, since if there were y with $f^{-1}(y) \leq g^{-1}(y)$ then

$$y = f\Big(f^{-1}(y)\Big) \leq f\Big(g^{-1}(y)\Big) < g\Big(g^{-1}(y)\Big) = y.$$

It follows that

$$\left(\frac{2-\nu}{2(1+\varepsilon_0)}t\right)^{\frac{2}{2-\nu}} < \tau^{-1}(t) < \left(\frac{2-\nu}{2(1-\varepsilon_0)}t\right)^{\frac{2}{2-\nu}} \quad \left(0 < t < (1-\varepsilon_0)\frac{2}{2-\nu}\delta^{\frac{2-\nu}{2}}\right).$$

It is now simple to deduce (3.4.9). □

Proof of Theorem 3.4. In what follows all asymptotic estimates are as r or $t \to 0$. Since we want to apply Theorem 3.2 we must transform (3.1.5) in the following way:

$$t = \tau(r) = \int_0^r \sqrt{w},$$
$$z\Big(\tau(r)\Big) = r^{1/2}w(r)^{1/4}u(r) \quad \Big(r \in (0,b)\Big).$$

This gives rise to

$$-z''(t;\lambda) + \tilde{Q}(t)z(t;\lambda) = \lambda z(t;\lambda) \quad \Big(t \in (0,b)\Big),$$

Here the potential is given by

$$\tilde{Q}\big(\tau(r)\big) = -\left(\frac{2-\nu}{4}\right)^2 r^{-(2-\nu)}\big(1+\zeta(r)\big) + \varepsilon_2(r),$$

with ε_2 and ζ given in (3.4.5) and (3.4.6). We calculate from Lemma 3.9 that

$$-\left(\frac{2-\nu}{4}\right)^2 \tau^{-1}(t)^{-(2-\nu)} = -\frac{1}{4t^2}\big(1+\varepsilon_3(t)\big),$$

where $\varepsilon_3(t) = o(1)$, and is also continuous, implying it is bounded.

To apply Theorem 3.2 we need $Q(t) := \tilde{Q}(t) + 1/4t^2 \in L^2(0,b;\mathrm{d}t)$. Recalling (3.4.4), we use condition (iii) to observe $\varepsilon_1'(r) = O(r^{\alpha-1})$ and $\varepsilon_1''(r) = O(r^{\alpha-2})$. Thus, by (3.4.6), $\zeta(r) = O(r^{\alpha}) = o(1)$. By hypothesis all of the functions ε_2, ε_3 and ζ are bounded except possibly in a neighbourhood of 0, and by their asymptotic behaviour at 0 they are all square integrable. Since $f(t) \in L^2(0,b;\mathrm{d}t)$ if and only if $f\big(\tau(r)\big) \in L^2(0,b;\sqrt{w(r)}\mathrm{d}r)$ we see $Q \in L^2(0,b)$, as required.

Hence, by Theorem 3.2 the Robin m-function (and by a fractional linear transformation, any m-function) is uniquely determined by the sequence $\mathcal{S}$. Theorem 2.4 concludes the proof. □

Corollary 3.1. *In either theorem, any finite number of values $m(-n^2)$ in the interpolation sequence may be discarded yet the m-function, and hence the potential, will still be uniquely determined.*

Proof. Since, in (3.3.2), the parameter $\varepsilon_0 > 0$ may be chosen freely, one may take ε_0 to be any positive integer. The resulting interpolation formula does not require the values $F(1),\dots,F(\varepsilon_0-1)$ and so the values $m(-1),\dots,m(-(\varepsilon_0-1)^2)$ are not needed. □

Remark. *One final curious point to note is as follows. Firstly, these uniqueness theorems rely on the interpolation results in Section 3.3. The interpolation results are, however,*

stronger than mere unique prescription of an m-function from the given sequence, since they are in addition constructive. One might hope that an answer to the question of uniqueness of an m-function interpolant could be separated from the constructive nature of the interpolation theorems. To the author's best understanding, this has not yet been done, and appears to be a difficult problem.

3.5 The Berry–Dennis problem

At the beginning of this chapter and in the introductory chapter we mentioned that we can achieve uniqueness in inverse recovery of a partial differential operator in two dimensions as a corollary of Theorem 3.3, which as written above applies to an ordinary differential operator. We will prove this now. For the reader's convenience we elaborate on the set-up outlined in Section 1.3. The analysis was originally conducted by Marletta and Rozenblum [99] to resolve a seemingly paradoxical non-self-adjointness result of Berry and Dennis [21]. We repeat it here in the framework defined earlier in this chapter.

Take a planar domain $\Omega_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1, x > 0\}$ in the shape of a half-disc. Its boundary $\partial\Omega_1$ is assumed to be divided $\big($up to the point set comprising $(0, \pm 1)\big)$ into the straight portion Γ_1 and semi-circular portion Γ_i:

$$\Gamma_1 = \{(0, y) \mid y \in (-1, 1)\};$$
$$\Gamma_i = \{(x, y) \mid x^2 + y^2 = 1, x > 0\}.$$

Define the operator given by the negative Laplacian in Ω_1 equipped with a Berry–Dennis-type boundary condition on Γ_1 and a homogeneous Dirichlet condition

on Γ_i:

$$D(\ell) := \{U \in L^2(\Omega_1) \mid \Delta U \in L^2(\Omega), U \restriction_{\Gamma_1} = 0, (U + \varepsilon y \partial_\nu U) \restriction_{\Gamma_i} = 0\},$$
$$\ell U := -\Delta U, \tag{3.5.1}$$

In polar coordinates $\mathbf{x} = (r, \vartheta)$ the Laplacian is given by $\Delta = \partial^2/\partial r^2 + (1/r)\partial/\partial r + (1/r^2)\partial^2/\partial\vartheta^2$, meaning the action of ℓ is that of the negative Laplacian

$$-\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\vartheta^2.$$

Since on Γ_1 the normal derivative is given by $-\partial_x = \pm r^{-1}\partial_\vartheta$ ($\vartheta = \pm\pi/2$), we find that the singular boundary condition in the definition of ℓ becomes

$$U(\mathbf{x}) + \varepsilon\partial_\vartheta U(\mathbf{x}) = 0 \quad \left(r \in (0,1), \vartheta = \pm\frac{\pi}{2}\right).$$

Hence, after performing the separation of variables $U(r, \vartheta) = u(r)\Theta(\vartheta)$ we arrive at the angular eigen-value problem

$$\begin{cases} -\Theta''(\vartheta) &= \lambda\Theta(\vartheta) \quad \left(\vartheta \in (-\pi/2, \pi/2)\right), \\ -\Theta(\vartheta) &= \varepsilon\Theta'(\vartheta) \qquad (\vartheta = \pm\pi/2), \end{cases}$$

where λ is the parameter introduced by the separation. The operator associated with this is easily calculated to possess eigen-values and eigen-functions

$$\lambda_0 = -\frac{1}{\varepsilon^2},$$
$$\lambda_n = n^2 \quad (n \in \mathbb{N});$$
$$\Theta_n(\vartheta) = \begin{cases} e^{-\vartheta/\varepsilon} & (n = 0), \\ \cos(n\vartheta) - (n\varepsilon)^{-1}\sin(n\vartheta) & (n \text{ even}), \\ \cos(n\vartheta) + n\varepsilon\sin(n\vartheta) & (n \text{ odd}). \end{cases}$$

Formally writing $U(r,\vartheta)$ as the sum $\sum_{n=0}^{\infty}\Theta_n(\vartheta)u_n(r)$ and then requiring this formally satisfy the differential equation and boundary conditions in ℓ produces the countable collection of systems

$$\begin{cases} -\frac{1}{r}\Big(ru_n'(r)\Big)' + \frac{\lambda_n}{r^2}u_n(r) &= 0 \quad (0<r<1), \\ u_n(1) &= 0, \end{cases}$$

which generates the collection of ordinary differential operators

$$\begin{aligned} D(\ell_n) &:= \Big\{u \in L^2(0,1;r\mathrm{d}r) \;\Big|\; r^{-1}\Big(ru'(r)\Big)' + \lambda_n r^{-2} \in L^2(0,1;r\mathrm{d}r), u(1)=0\Big\}, \\ \ell_n u(r) &:= -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}(r)\right) - \frac{\lambda_n}{r^2}u(r) \quad (n=0,1,2,\ldots). \end{aligned}$$

Now we are in the regime of the pencil classification system developed in Section 3.1. We have $w(r)=r^{-2}$ so ℓ_n is from the "mixed" PLP/PLC class. Since $\lambda=-\lambda_0,-\lambda_1,\ldots$, we find from Lemma 3.2 that the ℓ_n are all PLP for $n\geq 1$, whilst ℓ_0 is PLC. To find a self-adjoint restriction of ℓ_0 we need to adjoin to it a boundary condition. As *per* Definition 3.2, we choose a u_0 in the kernel of ℓ_0—in this case taking $u_0(r)=\sin\left(\varepsilon^{-1}\log(r)\right)$—to parameterise such a boundary condition. Then the operator

$$\begin{aligned} D(\ell_0') &:= D(\ell_0) \cap \{u \in L^2(0,1;r\mathrm{d}r) \mid [u,u_0](r) \to 0 \ (r\to 0)\}, \\ \ell_0' u(r) &:= -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}(r)\right) - \frac{1}{\varepsilon^2 r^2}u(r) \end{aligned} \tag{3.5.2}$$

is a self-adjoint restriction of ℓ_0. It follows that the orthogonal sum of operators $\ell' = \ell_0' \oplus \oplus_{n=1}^{\infty}\ell_n$, written more precisely as

$$\begin{aligned} D(\ell') &:= \left\{U \in L^2(\Omega_1) \;\middle|\; \Delta U \in L^2(\Omega), U\restriction_{\Gamma_1}=0, (U+\varepsilon y\partial_\nu U)\restriction_{\Gamma_1}=0, \int_{\Omega_1} u_0\Delta U = 0\right\}, \\ \ell' U &:= -\Delta U, \end{aligned} \tag{3.5.3}$$

is a self-adjoint restriction of ℓ.

Remark. *The condition* $\int_{\Omega_1} u_0 \Delta U = 0$ *is written with a degree of informality, since, of course,* u_0 *is a function on* $(0,1)$*, not* Ω_1*. In fact we mean* $u_0(r,\vartheta) = \Theta_0(\vartheta)u_0(r) = e^{-\vartheta/\varepsilon}\sin\left(\varepsilon^{-1}\log(r)\right)$*. Intuitively, one may take the condition to mean that either* U *oscillates in phase with* u_0 *in a neighbourhood of* 0 *or else it decays rapidly enough that the Lagrange bracket at this point, of its radial component with* u_0*, is of* $o(1)$*. By applying Green's formula to* $\int_{\Omega_1\setminus\delta\Omega_1} u_0\Delta U$ *one can easily calculate that it is completely equivalent to the Lagrange bracket requirement in the definition of* ℓ'_0*.*

As remarked in [99] the self-adjoint ℓ' *has discrete spectrum accumulating at* $\pm\infty$ *alone.*

The above Marletta–Rozenblum version of the Berry–Dennis model inspires an inverse problem with a curious method of resolution, which we are now able to explain. Let $q : \Omega_1 \to \mathbb{R}$ be a Schrödinger potential from some class of functions left, for now, unspecified. Define the map $\Lambda_{q,\varepsilon} : h \to -\partial_\nu U \restriction_{\Gamma_i} \left(h \in H^{1/2}(\Gamma_i)\right)$ where U is the $L^2(\Omega)$ solution to

$$\begin{cases} \left(-\Delta + q(\mathbf{x})\right)U(\mathbf{x}) &= 0 \quad (\mathbf{x}\in\Omega_1), \\ (1+\varepsilon y\partial_\nu)U(\mathbf{x}) &= 0 \quad (\mathbf{x}\in\Gamma_1), \\ U(\mathbf{x}) &= h(\mathbf{x}) \quad (\mathbf{x}\in\Gamma_i), \\ \int_{\Omega_1} u_0\Delta U &= 0. \end{cases} \tag{3.5.4}$$

Inverse Problem I′. *From a given Dirichlet-to-Neumann operator* $\Lambda_{q,\varepsilon}$ *recover the potential* q *which gave rise to it.*

Remark. *This inverse problem is well defined, since once again the operator associated with the above boundary-value problem is self-adjoint with discrete spectrum, which may without loss of generality be translated so as not to include* 0 *by addition of a suitable constant to* q*. Without the final integral boundary condition* ℓ' *would be the adjoint of*

a symmetric operator, yet have spectrum $\mathbb{C}$, meaning 0 would not be in its resolvent set. Therefore the differential equation and boundary conditions would well-define neither a solution U nor, consequently, the above Dirichlet-to-Neumann map.

Our proof of uniqueness for this inverse problem uses a novel technique, related directly to the results earlier in this chapter. Using the Marletta–Rozenblum decomposition we may reduce the problem to a family of one-dimensional problems in the case of a spherically-symmetric potential. This family may then be analysed as a single one-dimensional inverse problem with a spectral parameter, and, as may already have been guessed by the reader, this final inverse problem is of the type tackled in the earlier sections of the chapter. The remainder of the section, and chapter, details this process.

The crucial assumption needed is spherical symmetry of the potential. So, let $q \in L^\infty_{\mathrm{loc}}(0,1]$, and consider (3.5.4) with this q, i.e., the system

$$\begin{cases} \big(-\Delta + q(|\mathbf{x}|)\big)U(\mathbf{x}) &= 0 \quad (\mathbf{x} \in \Omega_1), \\ (1 + \varepsilon y \partial_\nu)U(\mathbf{x}) &= 0 \quad (\mathbf{x} \in \Gamma_1), \\ U(\mathbf{x}) &= h(\mathbf{x}) \quad (\mathbf{x} \in \Gamma_i), \\ \int_{\Omega_1} u_0 \Delta U &= 0. \end{cases}$$

With $h = 0$, the system defines the self-adjoint operator

$$D(L') := D(\mathcal{L}) \cap \left\{ U \in L^2(\Omega_1) \;\middle|\; \int_{\Omega_1} u_0 \Delta U = 0 \right\},$$
$$L'U(\mathbf{x}) := \big(-\Delta + q(|\mathbf{x}|)\big)U(\mathbf{x}) \quad (\mathbf{x} \in \Omega_1),$$

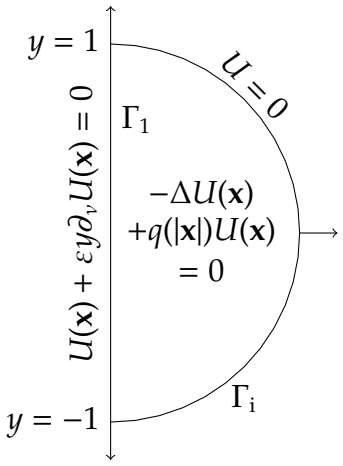

Figure 3.2: A diagram of Ω_1 and a non-self-adjoint boundary-value problem

which is a restriction of the operator

$$D(L) := \{U \in L^2(\Omega_1) \mid (-\Delta + q)U \in L^2(\Omega_1), (1 + f\partial_\nu)U \restriction_{\Gamma_1} = 0 = U \restriction_{\Gamma_i}\},$$
$$LU(\mathbf{x}) := \Big(-\Delta + q(|\mathbf{x}|)\Big)U(\mathbf{x}),$$

where $f(0, y) = \varepsilon y$.

One may prove, working in the same way as with ℓ, that L' is isometrically equal to the orthogonal direct sum of the ordinary differential operators L_0' and L_n ($n \geq 1$) given by

$$D(L_0') := \Big\{u \in L^2(0,1;r\mathrm{d}r) \;\Big|\; -r^{-1}\Big(ru'(r)\Big)' + (q(r) - \varepsilon^{-2}r^{-2})u(r) \in L^2(0,1;r\mathrm{d}r), \\ u(1) = 0 = [u_0, u](0^+)\Big\},$$
$$L_0'u(r) := -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}(r)\right) + q(r)u(r) - \frac{1}{\varepsilon^2 r^2}u(r), \tag{3.5.5}$$
$$D(L_n) := \Big\{u \in L^2(0,1;r\mathrm{d}r) \;\Big|\; -r^{-1}\Big(ru'(r)\Big)' + (q(r) + n^2r^{-2})u(r) \in L^2(0,1;r\mathrm{d}r), \\ u(1) = 0\Big\},$$
$$L_nu(r) := -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}(r)\right) + q(r)u(r) + \frac{n^2}{r^2}u(r). \tag{3.5.6}$$

Being concerned with an associated inverse problem, we will consider the

generalisation for $\lambda \in \mathbb{C}$ of L_0' and L_n ($n \geq 1$) given by

$$D(\mathcal{L}_\lambda) := \Big\{ u \in L^2(0,1;r\mathrm{d}r) \;\Big|\; -r^{-1}\big(ru'(r)\big)' + (q(r) - \lambda r^{-2})u(r) \in L^2(0,1;r\mathrm{d}r), \\ u(1) = 0, \chi_{\Omega_c}(\lambda)[u_0,u](0^+) = 0 \Big\},$$

$$\mathcal{L}_\lambda := -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}(r)\right) + q(r)u(r) - \frac{\lambda}{r^2}u(r),$$

where χ_{Ω_c} is the characteristic function of the set Ω_c, defined in the remark after Definition 3.1. Then for $n \geq 1$ each $\mathcal{L}_{\lambda_n} = L_n$ whilst $\mathcal{L}_{\lambda_0} = L_0'$, and in fact $\mathcal{L}_\lambda$ is precisely the pencil $L - \lambda P$ defined in (3.1.1). The differential equation given in $\mathcal{L}_\lambda u = 0$ is simply (3.1.5) with $w(r) = 1/r^2$.

Remark. *As we can see, even the domains match up, for the choice $\lambda = 1/\varepsilon^2$ puts us into the complex parabolic region Ω_c in which $L - \lambda P$ is PLC and thereby requires the extra boundary condition $[u_0, u](0^+) = 0$ to be self-adjoint.*

This differential equation, then, displays the PLP/PLC behaviour outlined in Lemma 3.2 and Figure 3.1. We define the Dirichlet m-function $m(\lambda)$ as in (3.1.7).

Consider the solution U to (3.5.4) with $q(\mathbf{x}) = q(|\mathbf{x}|)$. We may write any such solution using the decomposition[3.12]

$$U(r,\vartheta) = \sum_{n=0}^{\infty} u_n(r)\Theta(\vartheta). \tag{3.5.7}$$

By differentiating it follows that, in the basis Θ_n for $L^2(\Gamma_i)$, the Dirichlet-to-Neumann

[3.12] Each u_n now satisfies the differential equation $-r^{-1}(ru_n'(r))' + q(r)u_n(r) + \lambda_n r^{-2}u_n(r) = 0$.

map $\Lambda_{q,\varepsilon}$ takes the form of the diagonal matrix

$$\begin{pmatrix} m(-\lambda_0) & 0 & 0 & \cdots \\ 0 & m(-\lambda_1) & 0 & \cdots \\ 0 & 0 & m(-\lambda_2) & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Uniqueness for Inverse Problem I′ is immediate from Theorem 3.3, under the conditions $q(\mathbf{x}) = q(|\mathbf{x}|)$, $q \in L^\infty_{\text{loc}}(0,1]$ and $q(r) = O(r^{\alpha-2})$ $(r \to 0)$ for some fixed $\alpha > 0$. The uniqueness follows since for positive n the restrictions on q make the type (3.1.5) pencil, associated with each operator L_n, be PLP at 0 (see [99]), whilst the sequences $-\lambda_n = -n^2$ and $m(-n^2)$ form the interpolation sequence required in Theorem 3.3. Thus we have proved uniqueness as follows:

Theorem 3.5 (Uniqueness for the Berry–Dennis inverse problem)**.** *Any given Dirichlet-to-Neumann map $\Lambda_{q,\varepsilon}$ for the system (3.5.4) may have arisen from at most one radially symmetric potential $q \in L^\infty_{\text{loc}}(0,1] \cap O(r^{\alpha-2}; r \to 0)$.*

The 0-th term $\left(1/\varepsilon^2, m(1/\varepsilon^2)\right)$ is superfluous. Moreover, we can go farther. Following Corollary 3.1, we may discard arbitrarily many of the diagonal terms in the matrix representation of $\Lambda_{q,\varepsilon}$ and *still* retain uniqueness of q.

Remark. *Theorem 3.5 is markedly different from existing results for inverse problems involving partial-boundary Dirichlet-to-Neumann measurements in planar domains. Such existing results, e.g., [67, 69, 70] all deal with problems in which the portion of the boundary where the measurements are not made, Γ_1, has a homogeneous Dirichlet or Neumann condition assigned; the Berry–Dennis set-up has a singular boundary condition here. In the next chapter, we will examine generalisations of such domains and boundary conditions.*

4

Partial data in two-dimensional domains with boundary singularities

'The shortest path between two truths in the real domain passes through the complex domain.'

– Jacques Hadamard[4.1]

The problem of inverse recovery from partial boundary data in a two-dimensional domain presents many subtleties when compared with that in three or more dimensions. As described in the introductory chapter, we aim to prove a uniqueness theorem in a particular two-dimensional set-up, from data on a part of the boundary and with a singular boundary condition on the remaining boundary. Before we detail, in Sections 4.1–4.5, our own problem and the uniqueness results we have developed towards its resolution, we will run through the "state-of-the-art" for the existing classes of partial-data problems. Firstly recall that at a single frequency, the three-dimensional problems are over-determined, and one may exploit this by applying the Sylvester–Uhlmann "trick" [123] to pro-

[4.1] *The Mathematical Intelligencer 13*, 1 (1991).

duce knowledge of every Fourier-transform (of the difference of two Schrödinger potentials) from specially chosen rapidly-oscillating *complex geometric optics* solutions (to the corresponding pair of differential equations with equated boundary data). For more details on the three-dimensional Calderón problem see Section 2.4.

The aforementioned trick cannot be used in two dimensions, but instead one may still find cunningly chosen geometric optics solutions that can be combined appropriately to form a sufficient argument. One has to utilise all the degrees of freedom of the problem as it is no longer over-determined. But, there is also the "toolkit" of complex analysis now at the disposal of the researcher.

Partial data necessitate different methods and the results are weaker. The best known to the author, from 2015, is due to Imanuvilov, Uhlmann and Yamamoto [68, Thm. 1.1]. For smooth $\partial\Omega$ the Neumann-to-Dirichlet map $\Lambda_{q,\Gamma}^{-1} : H^{-1/2}(\Gamma) \ni g \mapsto u \restriction_\Gamma \in H^{1/2}(\Gamma)$, where

$$\begin{cases} (-\Delta + q)u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_c, \\ \partial_\nu u &= g \quad \text{on } \Gamma, \end{cases}$$

uniquely determines q, for each $p > 2$, from the Sobolev space $W^{1,p}(\Omega)$. This, by the transformation (1.2.1), determines γ from $W^{3,p}(\Omega)$. The proof adapts techniques from one of their earlier papers [67], in which their starting data are the Dirichlet-to-Neumann map.

Remark. *The Neumann-to-Dirichlet operator is, as was mentioned in Chapter 1, the negative of the inverse of the Dirichlet-to-Neumann operator $\Lambda_{q,\Gamma}$. For the Schrödinger problem, the latter was defined in Section 1.2. For convenience we repeat here that it maps*

$H^{1/2}(\Gamma) \ni g \mapsto -\partial_\nu u \restriction_\Gamma \in H^{-1/2}(\Gamma)$ *where*

$$\begin{cases} (-\Delta + q)u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_c, \\ u &= g \quad \text{on } \Gamma. \end{cases}$$

We will also need to adapt such techniques, and for simplicity of the exposition we will use this earlier work in our proofs. After reviewing the other literature we will explain the basic methods involved in these techniques.

Other partial data results for the inverse Schrödinger or conductivity problems are of a weaker nature. As previously mentioned, in the mid-'80s Kohn and Vogelius [79, 80] were able to reconstruct a piecewise analytic conductivity γ from partial boundary data, in 2 or more dimensions. In 1988 Isakov [71] used Runge-type arguments in dimension no less than 3 to prove a conductivity γ, known in a neighbourhood of the boundary[4.2], is determined by the Dirichlet-to-Neumann map on $\Gamma \subset \partial\Omega$ when it is of the class $C^2(\overline{\Omega})$, except for possessing a single discontinuity across the interior boundary of said neighbourhood.

In a domain of dimension $n \geq 3$ the map of Dirichlet data on the full C^2 boundary to Neumann data on a certain sufficiently large subset of the boundary was shown by Bukhgeim and Uhlmann in 2002 [32] to specify a bounded Schrödinger potential uniquely, by use of a linear phase function. In the same-dimensional case with smooth boundary, Kenig, Sjöstrand and Uhlmann [77] found in 2007 that with quadratic phase functions knowledge of Dirichlet data on Γ mapped to Neumann data on a boundary-ε-neighbourhood of Γ_c determines $q \in L^\infty(\Omega)$ uniquely, for arbitrarily small, non-empty Γ. In the same year Isakov [72] found that, in precisely three dimensions, when Γ_c is either a portion of a plane or a sphere and Ω is contained in the corresponding half-space or ball then a bounded

[4.2] According to [26] this is a not-so-unreasonable requirement in many applications.

potential is uniquely determined. His methods extended those of Sylvester and Uhlmann [123] and involved reflection of the complex geometric optics solutions across the planar Γ_c. The spherical case can then be solved for using a Kelvin transform of the ball to the half-space.

Finally we mention the recent work of Brown, Marletta and Reyes [26] who considered in 2016 the related inverse problem for Maxwell's coefficients in the time-harmonic electromagnetism equations. They proved that the coefficients, if known in a neighbourhood of the $C^{1,1}$ boundary, are determined uniquely from the class $C^{1,1}(\overline{\Omega}) \cap W^{2,\infty}(\Omega)$—with some small technical restrictions—by the Cauchy data-set on Γ, equivalent in the conductivity, low-frequency limit to the Dirichlet-to-Neumann map. In all of the above it is assumed that homogeneous Dirichlet or Neumann conditions are applied on the portion of the boundary where measurements are not made.

None of the above will help in our original work, except some results from [26] which we will adapt in Section 4.2, and the ideas in [67, 68]. The latter were developed for two-dimensional domains, and we describe them now. The main result of [67] is as follows.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, with smooth boundary $\partial\Omega$ containing the non-empty and connected Γ. Suppose q_1 and $q_2 \in C^{2+\alpha}(\overline{\Omega})$ for some fixed $\alpha > 0$. If $\Lambda_{q_1,\Gamma} = \Lambda_{q_2,\Gamma}$ then $q_1 = q_2$.*

The main ingredients in their proof are the following results. Firstly, let any $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ be represented by the complex number $z = x_1 + ix_2$. Note that $\Phi(z)$ $(z \in \Omega)$ is holomorphic if and only if the Cauchy–Riemann equations $\partial_{\bar{z}}\Phi(z) = 0$ $(z \in \Omega)$ are satisfied.

Definition 4.1. *We say that a holomorphic $\Phi = \varphi + i\psi$ on Ω, with continuous extension to $\overline{\Omega}$ and with φ and ψ real-valued is an* admissible phase function *if the following*

criteria are met:

(i) its set of critical points $\mathcal{H} := \{z \in \overline{\Omega} \mid \partial_z \Phi(z) = 0\}$ *does not intersect* $\overline{\Gamma}$*;*

(ii) its critical points are non-degenerate, i.e., $\partial_z^2 \Phi(z) \neq 0$ $(z \in \mathcal{H})$*;*

(iii) its imaginary part ψ *vanishes on* Γ_c*.*

Remark. *The admissible phase function will be a crucial ingredient in constructing the complex geometric optics solutions. By holomorphicity, since its critical points are non-degenerate, an admissible phase function can only have a finite number of critical points, which we label* z_k $(k = 1, \ldots, l)$*. Moreover, by the Cauchy–Riemann equations,* z *is a critical point of* Φ *if and only if it is also a critical point for* φ *or* ψ*.*

Let $z_k \in \overline{\Omega} \backslash \overline{\Gamma}$ $(k = 1, \ldots, l)$ and denote by $\mathbf{z}^T$ the vector $(z_k)_{k=1}^l$ and by $\mathbf{c}^T = (\mathbf{c}_0^T, \mathbf{c}_1^T, \mathbf{c}_2^T)$ an element of $\mathbb{C}^{3l}$ where $\mathbf{c}_j \in \mathbb{C}^l$. Suppose $b \in C^2\left(\overline{\Gamma_c}\right)$ is complex-valued, and consider the following Cauchy problem for $\Phi = \varphi + i\psi$, involving the Cauchy–Riemann equations and partial boundary and point conditions:

$$\begin{cases} \partial_{\bar{z}} \Phi(z) &= 0 \quad (z \in \Omega), \\ \Phi(z) &= b(z) \quad (z \in \Gamma_c), \\ \begin{pmatrix} \Phi(\mathbf{z}) \\ \partial_z \Phi(\mathbf{z}) \\ \partial_z^2 \Phi(\mathbf{z}) \end{pmatrix} &= \mathbf{c}, \end{cases} \tag{4.1}$$

where by $\Phi(\mathbf{z})^T$ we mean the vector $\left(\Phi(z_k)\right)_{k=1}^l$. We consider the triplet $(\mathbf{z}, b, \mathbf{c})$ as the Cauchy data-set for (4.1).

Lemma 4.1 (Imanuvilov–Uhlmann–Yamamoto, 2010, Prop. 5.1)**.** *For each* $\mathbf{z} \in (\overline{\Omega} \backslash \overline{\Gamma})^l$ *there is a dense subset* $\mathcal{O}$ *of* $C^2(\overline{\Gamma_c}) \times \mathbb{C}^{3l}$*. This set satisfies that for each of its elements* $(b, \mathbf{c})$ *the Cauchy problem* (4.1) *with data-set* $(\mathbf{z}, b, \mathbf{c})$ *has at least one solution* $\Phi \in C^2(\overline{\Omega})$*.*

Remark. *The intuitive consequence is that given any* $z_k \in \overline{\Omega}\backslash\overline{\Gamma}$ $(k = 1, \dots, l)$ *we can "approximately" find a (not-yet-admissible) phase function with critical points* z_k *and with arbitrarily specified non-zero values and second derivatives at each* z_k*. In fact this can be improved on the part* Γ *of the boundary.*

Corollary 4.1 (Imanuvilov–Uhlmann–Yamamoto, 2010, Prop. 4.2)**.** *Let* $y_0, \dots, y_m \in \Omega$ *and* $\eta_1, \dots, \eta_n \in \Gamma_c$*, and consider* $\mathcal{R} : D(\mathcal{R}) \subseteq C_0^\infty(\Gamma) \to \mathbb{C}^{3m} \times \mathbb{R}^{2n}$ *given by* $\mathcal{R} = (\mathbf{r}_1, \dots, \mathbf{r}_m, \mathbf{s}_1, \dots, \mathbf{s}_n)$ *where*

$$\begin{aligned}\mathbf{r}_j g &= \Big(\Phi(y_j), \partial_z\Phi(y_j), \partial_z^2\Phi(y_j)\Big)^T \quad (j = 1, \dots, m),\\ \mathbf{s}_j g &= \Big(\mathrm{Re}\Big(\Phi(\eta_j)\Big), \partial_\tau \mathrm{Re}\Big(\Phi(\eta_j)\Big)\Big)^T \quad (j = 1, \dots, n),\end{aligned}$$

and Φ *(uniquely) solves the Cauchy problem*

$$\begin{cases} \partial_{\bar z}\Phi(z) &= 0 \quad (z \in \Omega),\\ \mathrm{Im}\Big(\Phi(z)\Big) &= 0 \quad (z \in \Gamma_c),\\ \mathrm{Im}\Big(\Phi(z)\Big) &= g \quad (z \in \Gamma),\\ \mathrm{Re}\Big(\Phi(y_0)\Big) &= 0. \end{cases} \tag{4.2}$$

The map $\mathcal{R}$ *is surjective, i.e., for each initial data-set from* $\mathbb{C}^{3m} \times \mathbb{R}^{2n}$ *there is at least one* $g \in C_0^\infty(\Gamma)$ *for which the above Cauchy problem may be solved.*

The next result is not explicitly stated in the original work [67], so we present it here in a self-contained format. Firstly we need to define the operators

$$\begin{aligned}\partial_{\bar z}^{-1} g(z) &:= -\frac{1}{\pi}\int_\Omega \frac{g(\xi_1 + i\xi_2)}{\xi_1 + i\xi_2 - z}\mathrm{d}\xi_2\mathrm{d}\xi_1,\\ &=: \overline{\partial_z^{-1}\overline{g(z)}}.\end{aligned}$$

Lemma 4.2 (Imanuvilov–Uhlmann–Yamamoto, 2010, Sec. 3)**.** *Let* q_1 *be as in The-*

orem 4.1, and take any admissible phase function Φ. Then, for each $\tau > 0$, there is a solution to

$$\begin{cases} (-\Delta + q_1)u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_c \end{cases}$$

given by

$$v_1(z;\tau) = e^{\tau\Phi(z)}\big(a(z) + a_0(z)/\tau\big) + e^{\tau\overline{\Phi(z)}}(\overline{a(z)} + \overline{a_1(z)}/\tau) + e^{\tau \mathrm{Re}[\Phi](z)}\tilde{v}_1(x;\tau), \tag{4.3}$$

where the following conditions hold.

(i) The amplitude function *$a(\cdot) \in C^2(\overline{\Omega})$ is non-trivial, holomorphic on Ω, its real part vanishes on Γ_c and $a = \partial_z a = 0$ in $\mathcal{H} \cap \partial\Omega$; such an $a(\cdot)$ is called* admissible.

(ii) The remainder $\|\tilde{v}_1(\cdot\,;\tau)\|_{L^2(\Omega)} = o(1/\tau)$ as $\tau \to +\infty$.

(iii) The functions a_0 and a_1 are holomorphic and satisfy the τ-independent boundary condition

$$(a_0 + a_1)\restriction_{\Gamma_c} = \frac{\tilde{M}_1}{4\partial_z\Phi} + \frac{\tilde{M}_3}{4\overline{\partial_z\Phi}}. \tag{4.4}$$

The functions $\tilde{M}_1 := \partial_{\bar{z}}^{-1}[aq_1] - M_1$ and $\tilde{M}_3(z) := \partial_z^{-1}[\overline{a(z)}q_1(z)] - M_3(\bar{z})$, where M_1 and M_3 are any polynomials satisfying, for $j = 0, 1$ and 2,

$$\partial_z^j(\partial_{\bar{z}}^{-1}[a(z)q_1(z)] - M_1(z)) = 0, \tag{4.5}$$

$$\partial_{\bar{z}}^j(\partial_z^{-1}[\overline{a(z)}q_1(z)] - M_3(\bar{z})) = 0. \tag{4.6}$$

Remark. *The amplitude function $a(\cdot)$ is independent of q.*

Corollary 4.2. *For any admissible phase function Φ and $\tau > 0$ we can find the same amplitude function $a(\cdot)$ as in Lemma 4.2 so that*

$$v_2(x;\tau) = e^{-\tau\Phi(z)}(a(z) + b_0(z)/\tau) + e^{-\tau\overline{\Phi(z)}}(\overline{a(z)} + \overline{b_1(z)}/\tau) + e^{-\tau \mathrm{Re}[\Phi](z)}\tilde{v}_2(x;\tau) \tag{4.7}$$

solves $(-\Delta + q_2)u = 0, u \restriction_{\Gamma_c} = 0$, *with*

(i) $\|\tilde{v}_2(\cdot\,;\tau)\|_{L^2(\Omega)} = o(1/\tau)$ *as* $\tau \to +\infty$, *and*

(ii) holomorphic b_0 *and* b_1 *satisfying*

$$(b_0 + b_1) \restriction_{\Gamma_c} = -\frac{\tilde{M}_2}{4\partial_z \Phi} - \frac{\tilde{M}_4}{4\overline{\partial_z \Phi}}. \tag{4.8}$$

Here $\tilde{M}_2 := \partial_{\bar{z}}^{-1}[aq_2] - M_2$ *and* $\tilde{M}_4(z) := \partial_z^{-1}[\overline{a(z)}q_2(z)] - M_4(\bar{z})$, *where* M_2 *and* M_4 *are any polynomials satisfying, for* $j = 0, 1$ *and* 2,

$$\partial_z^j(\partial_{\bar{z}}^{-1}[a(z)q_2(x)] - M_2(z)) = 0, \tag{4.9}$$

$$\partial_{\bar{z}}^j(\partial_z^{-1}[\overline{a}(z)q_2(x)] - M_4(\bar{z})) = 0. \tag{4.10}$$

The final key result relates the values of $q := q_1 - q_2$ at the points in $\mathcal{H}$ to a special integral linear in q.

Lemma 4.3. *Suppose we have an admissible phase function* Φ *and functions* a, a_0, a_1, b_0 *and* b_1 *satisfying (i),*(4.4) *and* (4.8), *where* M_1, M_2, M_3 *and* M_4 *satisfy* (4.5), (4.6), (4.9) *and* (4.10). *Denote by* H_Φ *the Hessian matrix, and by* $\mathcal{H}$ *the set of critical points, of* Φ. *Let* $q_1, q_2 \in C^{2+\alpha}(\overline{\Omega})$ *for some* $\alpha > 0$, *set* $q = q_1 - q_2$, *and suppose that the Dirichlet-to-Neumann maps* $\Lambda_{q_1,\Gamma}$ *and* $\Lambda_{q_2,\Gamma}$ *are equal. Then, for any* $\tau > 0$,

$$\sum_{z\in\mathcal{H}} \frac{|a(z)|^2 \cos(2\tau \mathrm{Im}[\Phi(z)])}{|\det(\mathrm{Im}[H_\Phi(z)])|^{1/2}} q(z) = \frac{1}{8\pi}\int_\Omega \left[\left(\frac{\tilde{M}_1 - \tilde{M}_2}{\partial_z\Phi} - 4(a_0 + b_0)\right)a + \left(\frac{\tilde{M}_3 - \tilde{M}_4}{\overline{\partial_z\Phi}} - 4\overline{(a_1 + b_1)}\right)\overline{a}\right] q.$$

Remark. *This last lemma is established firstly by proving the identity* $\int_\Omega q v_1 v_2 = 0$ *for any complex geometric optics solutions of the form in Lemma 4.2 and Corollary 4.2, then carefully analysing the resulting integrals after substituting from* (4.3) *and* (4.7).

In light of these lemmata the proof of Theorem 4.1 proceeds roughly as follows. Fix a point $x_0 \in \Omega$, and using Corollary 4.1 and Lemma 4.1 find a phase function Φ_0 with a non-generate critical point at x_0. Since Φ_0 might have some critical points on $\overline{\Gamma}$, one needs to correct it, by adding a certain $\varepsilon\Phi_1$, chosen using Lemma 4.1. This introduces new critical points $\mathcal{H}_\varepsilon = \{z_{\varepsilon,1}, \ldots, z_{\varepsilon,l(\varepsilon)}\}$ in Ω, and by the implicit function theorem, for sufficiently small ε, one of these, $x_\varepsilon \to x_0$, is the only critical point of $\Phi_\varepsilon = \Phi_0 + \varepsilon\Phi_1$ in a certain neighbourhood of x_0 and is moreover non-degenerate. It can be seen that Φ_ε is an admissible phase function. To apply Lemma 4.3 one then needs to construct an admissible amplitude function a_ε, which is achieved again by Corollary 4.1. Then applying Lemma 4.3 one finds that $q = q_1 - q_2$ satisfies a weighted sum of the form

$$\sum_{k=1}^{l(\varepsilon)} c_{\varepsilon,k} \cos(\tilde{c}_{\varepsilon,k}\tau) q(z_{\varepsilon,k}) = \tilde{c}_{\varepsilon,0}.$$

In particular, the above process ensures that $c_{\varepsilon,\hat{k}} \neq 0$ where $\hat{k}$ is the index of $z_{\varepsilon,\hat{k}} = x_\varepsilon$. Varying $\tau > 0$ then shows that the only way the sum can hold is if $q(x_\varepsilon) = 0$, implying $q(x_0) = 0$.

In concluding Chapter 3 we discussed the application of Theorem 3.3 to establish a uniqueness result for the geometrically symmetric Inverse Problem I′ based on a Berry–Dennis-type boundary condition and the Marletta–Rozenblum decomposition. In this chapter we will explain a different approach to a related class of inverse problems. These will involve the same type of boundary singularity as in the previous symmetric geometry. The underlying differential expression is of Schrödinger form, and we will adapt the Imanuvilov–Uhlmann–Yamamoto techniques to establish the first results for our operator.

We will outline the problem in Section 4.1, and in the following three sections we will prove a partial uniqueness result at one frequency. This is a con-

ditional uniqueness result: starting in either of the following two cases with the Dirichlet-to-Neumann map on part of the boundary, (i) knowledge of the singular boundary condition forces uniqueness of the Schrödinger potential, and (ii) *vice versa*. Our proof utilises unique continuation and density arguments—developed in Section 4.2—applied alongside an analogue to Lemma 4.3, proved from complex geometric optics considerations in Section 4.3.

We prove the partial uniqueness in Section 4.4; it may alternatively be interpreted as a so-called *cloaking* result at a single frequency. We also illustrate there the cloaking of the potential (by a suitable choice of boundary condition) in a simple numerical example, coded in MATLAB. Finally, in Section 4.5, we prove that by moving to full frequency spectral Dirichlet-to-Neumann data we are able to establish total uniqueness, of the boundary conditions and the potential.

4.1 Problem definition

Throughout the rest of this chapter the set-up is as follows.

Definition 4.2 (Specially decomposable domain)**.** *We say that a bounded two-dimensional simply connected open set Ω is a specially decomposable domain if it has $C^{2,1}$ boundary and can be written as*

$$\Omega = \mathrm{int}\Big(\overline{\Omega_1 \cup \Omega_0}\Big),$$

where Ω_1 is a half-disc of radius 1, $\Omega_1 \cap \Omega_0 = \emptyset$ and the straight portion Γ_1 of the boundary of Ω_1 is contained entirely in the boundary of Ω, i.e., $\Gamma_1 \subset \partial\Omega$.

Remark. *This decomposition will underlie a special case of Glazman decomposition [58] that we will use. The operator* int *returns the interior of a subset of $\mathbb{R}^n$. See Figure 4.1.*

Definition 4.3 (Boundary accessibility)**.** *Letting Ω be a specially decomposable domain, specify the further boundary decomposition $\partial\Omega = \overline{\Gamma \cup \Gamma_c}$, $\Gamma \cap \Gamma_c = \emptyset$ such that*

$\Gamma_1 \subset \Gamma_c$, *and* Γ *and* Γ_c *are both relatively open. The boundary portions* Γ *and* Γ_c *are called, respectively,* accessible *and* inaccessible. *We identify*[4.3] Ω_1 *with the set of the same name defined in Section 3.5 [p. 90], so that the mid-point of* Γ_1 *is the point* 0.

Definition 4.4 (Singular boundary condition)**.** *The Berry–Dennis boundary condition on* Ω *is the requirement on a given* $u : \Omega \to \mathbb{C}$ *that*

$$\mathrm{BD}_f[u] := (u - f\partial_\nu u) \restriction_{\Gamma_c} = 0,$$

where f *is a bounded, real-valued, a.e. absolutely continuous function possessing one strictly simple zero at the point* 0 *and satisfying that for every* $\varepsilon_0 > 0$ *we have* 0 *either not in the range of* $f \restriction_{\Gamma_c \setminus \{\mathbf{x} \,:\, |\mathbf{x}| < \varepsilon_0\}}$, *or an isolated point in this range. For the same technical reasons as [99] and in Section 3.5 we will need* f *to be linear on the straight edge* Γ_1*: under the same coordinate convention as in that section, we require there be* $\varepsilon > 0$ *such that*

$$f(y) = -\varepsilon y \quad \Big(y \in (-1, 1)\Big). \tag{4.1.1}$$

Such an f *is called an* admissible boundary function.

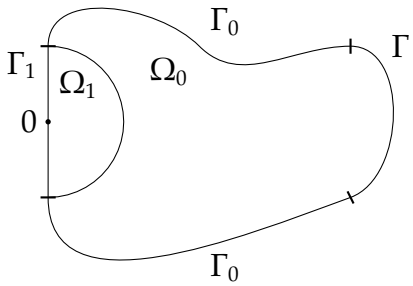

Figure 4.1: an example domain $\Omega = int(\overline{\Omega_1 \cup \Omega_0})$

Remark. *Here,* ∂_ν *denotes the outward-oriented normal derivative on* $\partial\Omega$*. Note we do allow* f *to "jump" to* 0*, or be identically* 0 *in a relatively open portion of* Γ_c*. As explained*

[4.3] This might require rescaling the coordinates.

in Section 3.5, this type of boundary condition was first examined by Berry and Dennis [21]. When paired with a boundary condition on Γ it is seen to produce a non-self-adjoint operator from a second-order differential expression; this operator possesses adjoint that is symmetric and has a one-dimensional deficiency space.

Notice that we denote by Γ_0 the (possibly disjoint) portion of the boundary given by $\Gamma_c \backslash \Gamma_1$, and by Γ_i the *interface* between Ω_0 and Ω_1, i.e., setting relint to mean the interior relative to the topology on the line,

$$\Gamma_i := \operatorname{relint}\left(\overline{\Omega_1} \cap \overline{\Omega_0}\right).$$

Definition 4.5 (Admissible potential)**.** *A function $q \in L^\infty(\Omega)$ is called an admissible potential (for a Schrödinger differential operator) if it is locally radially symmetric about 0, i.e., there is a $\delta > 0$ such that in the ball $r < \delta$ we have $\partial_\vartheta q(r, \vartheta) = 0$.*

We briefly recapitulate (and rewrite more conveniently) the key points of the analysis from [99] given in Section 3.5. Consider an admissible potential q, and over the Hilbert space $L^2(\Omega_1)$ define the operator

$$\begin{aligned} D(L) &:= \{u \in L^2(\Omega_1) \mid (-\Delta + q)u \in L^2(\Omega_1), \mathrm{BD}_f[u] \restriction_{\Gamma_1} = 0 = u \restriction_{\Gamma_i}\}, \\ Lu &:= (-\Delta + q)u \restriction_{\Omega_1}. \end{aligned}$$

This operator is the adjoint of a symmetric operator (see [99, Sec. 3] and (1.2.2)), and one may impose an extra boundary condition at 0 to yield a self-adjoint restriction of L, in the following fashion:

Definition 4.6 (Self-adjoint boundary condition)**.** *Take*[4.4]

$$u_0(r, \vartheta) = e^{-\vartheta/\varepsilon} \sin\left(\varepsilon^{-1} \log(r)\right), \tag{4.1.2}$$

[4.4] Hopefully without forsaking the reader's sympathy, we continue the notational abuse mentioned in the Remark on page 93, namely $u_0(r, \vartheta) = u_0(r)e^{-\vartheta/\varepsilon}$ and $v_0(r, \vartheta) = v_0(r)e^{-\vartheta/\varepsilon}$.

which is a solution to

$$\begin{cases} -\Delta u &= 0 \quad \textit{in } \Omega_1, \\ u - f\partial_\nu u &= 0 \quad \textit{on } \Gamma_1, \\ u &= 0 \quad \textit{on } \Gamma_i. \end{cases}$$

The boundary condition defined at 0 *in [99, Eq. (22)] is the requirement on* u *that its Lagrange bracket with* u_0 *vanishes at* 0*:*

$$[u, u_0](r; \vartheta) := r\Big(u\partial_r u_0 - (\partial_r u)u_0\Big)(r, \vartheta) \to 0 \quad (r \to 0).$$

In fact, we will impose a generalised version. Consider the solution to the differential equation with only the Berry–Dennis boundary condition on Γ_1*—i.e., without the Dirichlet condition on* Γ_i*—which is linearly independent to* u_0*, given by:*

$$v_0(r, \vartheta) = e^{-\vartheta/\varepsilon} \cos\left(\varepsilon^{-1} \log(r)\right).$$

We set an admissible self-adjoint boundary condition*, with real parameter* β*, as follows. With an abuse of notation,*

$$\beta[u] := \lim_{r\to 0}[u, u_0 + \beta v_0](r, \vartheta) = \lim_{r\to 0} r\Big(u\partial_r(u_0 + \beta v_0) - (\partial_r u)(u_0 + \beta v_0)\Big)(r, \vartheta) = 0. \quad (4.1.3)$$

It can be seen $\Big($see [99, Eq. (24)] and (3.5.3)$\Big)$ that application of a boundary condition of the form (4.1.3) is enough to restrict the operator L to the self-adjoint realisation

$$D(L') := D(L) \cap \{u \in L^2(\Omega_1) \mid \beta[u] = 0\}, \quad (4.1.4)$$

$$L'u := (-\Delta + q)u \restriction_{\Omega_1}, \quad (4.1.5)$$

whose spectrum is simple and accumulates only at $\pm\infty$. By exactly the same reasoning as in [99], explained here on Page 92, we have an orthogonal decomposition of L':

Proposition 4.1. *Over the Hilbert space $L^2(\Omega_1)$, the operator L' is given by the orthogonal sum*

$$L' = L'_0 \oplus \bigoplus_{n=1}^{\infty} L_n,$$

where L'_0 and L_n $(n \geq 1)$ are the ordinary differential operators defined respectively in (3.5.5) and (3.5.6).

Remark. *The presence of the potential q in L' is not problematic for either self-adjointness of the operator or discreteness of the spectrum. The former is easily established by repeating the direct sum trick of [99, Eq. (24)] explained in Section 3.5—or appealing to the abstract result [75, Thm. V.4.3] which states that a bounded, symmetric perturbation leaves self-adjointness unchanged. The latter follows from Weyl's Theorem [75, Thm IV.5.35], which states that adding a relatively compact operator leaves the essential spectrum unchanged; since the spectrum of L' with $q = 0$ comprises isolated eigen-values of finite multiplicities, its essential spectrum is empty. Since $q \in L^\infty(\Omega)$ multiplication by q is automatically L'-compact.*

One may then define the self-adjoint operator

$$\begin{aligned} D(T) &:= \{u \in L^2(\Omega) \mid (-\Delta + q)u \in L^2(\Omega), \mathrm{BD}_f[u] \restriction_{\Gamma_c} = u \restriction_{\Gamma} = \beta[u] = 0\}, \\ Tu &:= (-\Delta + q)u, \end{aligned} \tag{4.1.6}$$

possessing purely discrete spectrum accumulating only at $\pm\infty$ [99, Sec. 5]. For convenience, define the set $\Omega^* := \overline{\Omega}\backslash\{0\}$.

Proposition 4.2. *Any $u \in D(T)$ is given by $v + w$ with $v \in D(L')$ and $\overline{\mathrm{supp}(w)} \subseteq \Omega^*$.*

Proof. Define the smooth cut-off function μ to be 1 in $\frac{1}{2}\Omega_1$ and 0 in Ω_0, to transition

smoothly between the two regions, and to be radially symmetric in Ω_1. Then set $v := \mu u$ and $w := (1 - \mu)u$ and check the appropriate boundary conditions. □

We may now define the input and output data for our inverse problem. Let Ω be specially decomposable, f an admissible boundary function that specifies a Berry–Dennis boundary condition, q an admissible potential and β a self-adjoint boundary condition. Consider the Dirichlet-to-Neumann map

$$\Lambda_{q,f,\beta} : h \mapsto -\partial_\nu v \restriction_\Gamma \tag{4.1.7}$$

where h is any admissible function on Γ such that v solves

$$\begin{cases} (-\Delta + q)v &= 0 \quad \text{in } \Omega, \\ v - f\partial_\nu v &= 0 \quad \text{on } \Gamma_c, \\ \beta[v] &= 0 \quad (\text{at } 0), \\ v &= h \quad \text{on } \Gamma. \end{cases}$$

Inverse Problem II. *Given the Dirichlet-to-Neumann map $\Lambda_{q,f,\beta}$, recover the potential q everywhere in Ω, the function f on Γ_c and the value of β that parameterises the boundary condition* (4.1.3) *at* 0.

The problem of uniqueness we wish to examine is therefore the following. For $j = 1, 2$ let q_j, f_j and β_j be, respectively, admissible potentials, boundary functions, and self-adjoint boundary conditions. Then we want to determine under which conditions

$$\Lambda_{q_1,f_1,\beta_1} = \Lambda_{q_2,f_2,\beta_2} \tag{4.1.8}$$

implies that

$$q_1 = q_2, \ f_1 = f_2 \text{ and } \beta_1 = \beta_2.$$

Resolvent Hypothesis. *Without a loss of generality, when considering the above question of uniqueness we may assume that both of the operators T_1 and T_2—corresponding respectively to the triples (q_1, f_1, β_1) and (q_2, f_2, β_2)—have 0 in their resolvent set. To see why, suppose either (or both) has 0 in their spectrum. Then we may simply shift their spectra, which are discrete, by adding the same sufficiently small constant to both q_1 and q_2 so that the resulting operators now both have 0 in their resolvent set. The question of uniqueness is left unchanged by the shift. We will assume the hypothesis holds for the rest of the chapter.*

4.2 Unique continuation and density

We will need certain analogous results to those in [26], adapted to our singular case. The following lemmata firstly detail useful conditions for when one may extend solutions of the differential equation uniquely, and secondly establish density, in the full space of solutions, of those satisfying the singular boundary condition.

Lemma 4.4 (Unique continuation principles for a Schrödinger-type equation)**.** *Let q be admissible and $\Omega' \subset \Omega$, with Ω' a non-empty, bounded domain in $\mathbb{R}^2$ such that $\partial\Omega' \in C^2$ and $\Omega\backslash\overline{\Omega'}$ is connected. Recall that $\Omega^* = \Omega\backslash\{0\}$.*

(i) If $u \in H^2_{\mathrm{loc}}(\Omega^*)$ satisfies $(-\Delta+q)u = 0$ in Ω, and there is an open ball B with $\overline{B} \subset \Omega$ and $u \restriction_B = 0$, then $u = 0$.

(ii) If Γ is smooth, $u \in H^2_{\mathrm{loc}}(\Omega^*\backslash\Omega')$ and $(-\Delta + q)u \restriction_{\Omega\backslash\overline{\Omega'}} = 0$, and the boundary conditions

$$u \restriction_\Gamma = 0 = \partial_\nu u \restriction_\Gamma$$

are satisfied, then $u \restriction_{\Omega\backslash\Omega'} = 0$.

Proof. Part (i) follows standard methods, for example [84, Cor. 1.1]. Part (ii) is consequential to extending u by 0 through Γ and applying (i). □

Lemma 4.5. *Under the hypotheses of Lemma 4.4 define the sets*

$$K := \{v \in H^2_{\text{loc}}(\Omega^*) \mid (-\Delta + q)v \restriction_\Omega = 0\},$$
$$\tilde{K} := \{g \in K \mid \text{BD}_f[g] \restriction_{\Gamma_c} = 0 = \beta[g]\}.$$

Then $\tilde{K}$ is dense in K under the topology induced by $\|\cdot\|_{L^2(\Omega')}$.

Proof. We adapt the proofs of [26, Prop. 5.1-2]. Let $v \in K$ such that $\langle g, v\rangle_{\Omega'} = 0$ for every $g \in \tilde{K}$; we aim to show $v = 0$. By the Resolvent Hypothesis we may uniquely define $V \in D(T)$ to solve $TV = \chi_{\Omega'}\overline{v}$, where $\chi_A(x) = 1$ $(x \in A)$; 0 $(x \notin A)$.

Now we make some technical definitions (see Fig. 4.2): the sub-domain $\Omega_2 \subset \Omega\backslash\overline{\Omega'}$ is taken to have boundary $\partial\Omega_2 = \overline{\Gamma \cup \Gamma' \cup \tilde{\Gamma}} \in C^{2,1}$ such that Γ, Γ' and $\tilde{\Gamma}$ are all disjoint. Here $\tilde{\Gamma}$ continuously extends Γ in $\partial\Omega_2 \cap \partial\Omega$ at both its endpoints, Γ' is relatively open and entirely contained in $\Omega\backslash\overline{\Omega'}$, and $0 \notin \partial\Omega_2$. This means Ω_2 is a neighbourhood of Γ, and its complement is separated from Γ, i.e., $\overline{\Omega\backslash\Omega_2} \cap \overline{\Gamma}$. Take a mollifier μ on Ω_2 to be 1 in a neighbourhood of Γ, 0 in a neighbourhood of Γ' and transition smoothly between these neighbourhoods (including part of $\tilde{\Gamma}$). Assume without loss of generality that the level curves of μ are orthogonal to $\tilde{\Gamma}$.

Consider $g \in \tilde{K} \subset H^2(\Omega_2)$. We wish to make a decomposition of g into two parts: $g_0 \in D(T)$, and $g_1 \in H^2(\Omega_2)$ which is supported in Ω_2 and whose "behaviour" on Γ is equivalent to that of g. Firstly extend f to a bounded, a.e. continuous function in the interior of Ω. By the trace theorem [137, Thm. 8.7] $(g - f\partial_\nu g) \restriction_\Gamma$ can be extended by 0 to $\tilde{F}_2 := (g - f\partial_\nu g) \restriction_{\partial\Omega} \in H^{1/2}(\partial\Omega)$. Take any $F_2 \in H^{1/2}(\partial\Omega_2)$ which agrees with $\tilde{F}_2$ on $\Gamma \cup \tilde{\Gamma}$ and is 0 on Γ'. Define $F_1 = \mu g \restriction_{\partial\Omega_2} \in H^{3/2}(\partial\Omega_2)$.

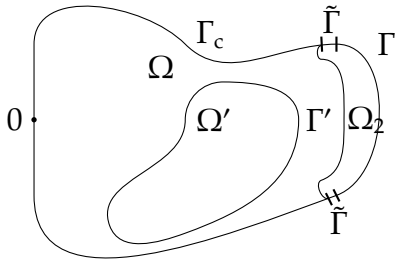

Figure 4.2: an example admissible domain Ω containing Ω' and Ω_2

The inverse trace theorem [137, Thm. 8.8] guarantees the existence of $g_1 \in H^2(\Omega_2)$ such that $g_1 \restriction_{\partial\Omega_2} = F_1 \in H^{3/2}(\partial\Omega)$. Furthermore, since

$$
\begin{aligned}
(g_1 - F_2)/f \restriction_{\partial\Omega_2} &= (\mu g - F_2)/f \\
&= \begin{cases} \big(g - (g - f\partial_\nu g)\big)/f & \text{on } \Gamma; \\ \mu g/f & \text{on } \tilde{\Gamma}; \\ 0 & \text{on } \Gamma' \end{cases} \\
&= \begin{cases} \partial_\nu g \in H^{1/2}(\Gamma) & \text{on } \Gamma; \\ \mu g/f \in H^{3/2}(\tilde{\Gamma}) & \text{on } \tilde{\Gamma}; \\ 0 \in H^{1/2}(\Gamma') & \text{on } \Gamma' \end{cases} \\
&\in H^{1/2}(\partial\Omega_2),
\end{aligned}
$$

we can choose this g_1 to satisfy $\partial_\nu g_1 \restriction_{\partial\Omega_2} = (g_1 - F_2)/f \in H^{1/2}(\partial\Omega_2)$. This ensures that $\partial_\nu g_1 \restriction_{\Gamma'} = 0$; since $g_1 \restriction_{\Gamma'} = F_1 \restriction_{\Gamma'} = 0$ we may extend this g_1 by 0 into Ω, and note that, by construction, $(g_1 - f\partial_\nu g_1) \restriction_\Gamma = (g - f\partial_\nu g) \restriction_\Gamma$, i.e., it is a candidate g_1 for encoding the behaviour of g on Γ.

Now define $g_0 = g - g_1$. By checking the boundary conditions on $\Gamma, \tilde{\Gamma}$ and $\Gamma_c \backslash \tilde{\Gamma}$, we see that $g_0 \in D(T)$. Therefore, for any $g \in \tilde{K}$,

$$0 = \langle g, v \rangle_{L^2(\Omega')}$$

$$
\begin{aligned}
&= \langle g_0, T\overline{V}\rangle_{L^2(\Omega)} + \langle g_1, T\overline{V}\rangle_{L^2(\Omega)} \\
&= \langle Tg_0, \overline{V}\rangle_{L^2(\Omega)} + \int_{\Omega_2} g_1(-\Delta + q)V \\
&= \langle (-\Delta + q)g, \overline{V}\rangle_{L^2(\Omega)} + \int_{\Gamma\cup\Gamma'\cup\tilde{\Gamma}} (-g_1\partial_\nu V + V\partial_\nu g_1) \\
&= -\int_{\Gamma} g\partial_\nu V. \qquad (4.2.1)
\end{aligned}
$$

Here, to reach the third line we applied the self-adjointness of T, whilst for the fourth line we applied Green's formula twice to the integral term, noting g_1 and $V \in H^2(\Omega_2)$, then recomposed g from g_0 and g_1 inside the inner product. The final line was achieved by noting that $\mathrm{BD}_f[V] = 0 = \mathrm{BD}_f[g]$ on $\tilde{\Gamma}$, $g = \partial_\nu g = 0$ on Γ', and $V \in D(T)$ means that $V = 0$ on Γ.

Now observe that the $L^2(\Gamma)$-closure of $\{g \restriction_\Gamma \mid g \in \tilde{K}\}$ is in fact $L^2(\Gamma)$. This is because for any given basis ψ_n of $L^2(\Gamma)$ we can solve

$$
\begin{cases}
(-\Delta + q)g &= 0 \quad \text{in } \Omega, \\
\mathrm{BD}_f[g] &= 0 \quad \text{on } \Gamma_c, \\
\beta[g] &= 0 \quad \text{at } 0, \\
g &= \psi_n \quad \text{on } \Gamma,
\end{cases}
$$

as $0 \in \varrho(T)$. Thus, from (4.2.1), we see $\partial_\nu V \restriction_\Gamma = 0$. As $V \restriction_\Gamma = 0$, the unique continuation from Lemma 4.4(ii) implies $V \restriction_{\Omega\setminus\overline{\Omega'}} = 0$ since $(-\Delta + q)V \restriction_{\Omega\setminus\overline{\Omega'}} = 0$.

In particular $V \restriction_{\partial\Omega'} = \partial_\nu V \restriction_{\partial\Omega'} = 0$, and so

$$
\begin{aligned}
\langle v, v\rangle_{L^2(\Omega')} &= \int_{\Omega'} v(-\Delta + q)V \\
&= \int_{\Omega'} V(-\Delta + q)v + \int_{\partial\Omega'} (-v\partial_\nu V + V\partial_\nu v) \\
&= 0.
\end{aligned}
$$

By the unique continuation in Lemma 4.4(i) we deduce that $v \restriction_\Omega = 0$. □

4.3 A weighted sum of q-values

In this section we will prove an analogue to Lemma 4.3 [67, Prop. 4.1] in the case of the Dirichlet-to-Neumann map (4.1.7)—with Berry–Dennis and Marletta–Rozenblum boundary conditions—as opposed to the more standard map from Section 2.4 and the introduction of this chapter, which has a homogeneous Dirichlet condition on Γ_c. The complex geometric optics solutions from Lemma 4.2 and Corollary 4.2 will fit our needs, as we can "approach" such solutions using those with the singular boundary conditions, thanks to the density from Lemma 4.5.

As in Lemmata 4.4 and 4.5, we consider $\Omega' \subset \Omega$ to be non-empty, bounded domains in $\mathbb{R}^2$ with $\Omega\backslash\overline{\Omega'}$ connected and $\partial\Omega' \in C^2$. However we are forced, for the same technical reasons as in [67], to accept the restriction $\partial\Omega \in C^\infty$. Let q_1, q_2 be admissible potentials, fix $\alpha > 0$, and assume the additional restrictions $q_1, q_2 \in C^{2+\alpha}(\overline{\Omega})$ and $q_1 = q_2$ in $\Omega\backslash\Omega'$.

Proposition 4.3. *Suppose we have an admissible phase function Φ and functions a, a_0, a_1, b_0 and b_1 satisfying (i),(4.4) and (4.8), where M_1, M_2, M_3 and M_4 satisfy (4.5), (4.6), (4.9) and (4.10). Denote by H_Φ the Hessian matrix and by $\{x_1, \ldots, x_l\}$ the critical points of Φ. Let $q_1, q_2 \in C^{2+\alpha}(\overline{\Omega})$ for some $\alpha > 0$ such that $q_1 = q_2$ in $\Omega\backslash\Omega'$, set $q = q_1 - q_2$, and suppose that the Dirichlet-to-Neumann maps (defined the same way as in (4.1.7)) are equal, but with the same f and β, i.e., $\Lambda_{q_1,f,\beta} = \Lambda_{q_2,f,\beta}$. Then, for any $\tau > 0$,*

$$\sum_{z\in\mathcal{H}} \frac{|a(z)|^2 \cos(2\tau \mathrm{Im}[\Phi(z)])}{|\det(\mathrm{Im}[H_\Phi(z)])|^{1/2}} q(z) = \frac{1}{8\pi}\int_\Omega \left[\left(\frac{\tilde{M}_1 - \tilde{M}_2}{\partial_z\Phi} - 4(a_0 + b_0)\right)a + \left(\frac{\tilde{M}_3 - \tilde{M}_4}{\overline{\partial_z\Phi}} - 4\overline{(a_1 + b_1)}\right)\overline{a}\right] q. \tag{4.3.1}$$

To prove Proposition 4.3 we will need an integration-by-parts formula. Since elements of $D(T)$ are not necessarily in $H^2(\Omega)$—in fact, expanded in polar co-

ordinates, the function $\sin(\varepsilon^{-1}\log r) + \beta\cos(\varepsilon^{-1}\log r)$ may be C^2-extended to an element of $D(L')$, and this linear combination clearly fails to be in H^2 at 0—we see that Green's formula cannot be applied. We circumvent this by analysing a certain line integral, which turns out to be equivalent to a Lagrange bracket.

Lemma 4.6. *As before, take admissible domain Ω and subset $\Omega' \subset \Omega$. Suppose $\Omega\backslash\overline{\Omega'}$ is connected and $\partial\Omega' \in C^2$. For $j = 1, 2$ let q_j be admissible potentials and f_j admissible boundary functions, and suppose that the latter have the same behaviour on Γ_1, i.e., $\varepsilon_1 = \varepsilon_2$. Choose an admissible self-adjoint boundary condition β. Let u_j be any respective solutions to*

$$\begin{cases} (-\Delta + q_j)u_j &= 0 \quad \text{in } \Omega, \\ \mathrm{BD}_{f_j}[u_j] &= 0 \quad \text{on } \Gamma_c, \\ \beta[u_j] &= 0 \quad \text{at } 0. \end{cases} \tag{4.3.2}$$

Then, if $\Lambda_{q_1,f_1,\beta} = \Lambda_{q_2,f_2,\beta}$, we have

$$\int_\Omega (q_1 - q_2)u_1u_2 = \int_{\partial\Omega} (u_2\partial_\nu u_1 - u_1\partial_\nu u_2). \tag{4.3.3}$$

Proof. Take a δ-radius half-disc $\Omega_\delta \subset \Omega_1$ for some $0 < \delta < 1$, centred at 0, and define $\Omega_{0,\delta} = \Omega\backslash\overline{\Omega_\delta}$. Set Γ_δ and $\Gamma_{1,\delta}$ to be, respectively, the straight and semi-circular parts of the boundary of Ω_δ. Then

$$\begin{aligned} \int_\Omega (q_1 - q_2)u_1u_2 &= \underbrace{\int_{\Omega_\delta} (q_1 - q_2)u_1u_2}_{=0} + \int_{\Omega_{0,\delta}} (-u_1\Delta u_2 + u_2\Delta u_1) \\ &= \int_{\partial\Omega_{0,\delta}} (u_2\partial_\nu u_1 - u_1\partial_\nu u_2) \\ &= \int_{\partial\Omega} (u_2\partial_\nu u_1 - u_1\partial_\nu u_2) - \int_{\partial\Omega_\delta} (u_2\partial_\nu u_1 - u_1\partial_\nu u_2) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial\Omega} (u_2\partial_\nu u_1 - u_1\partial_\nu u_2) + \int_{\Gamma_\delta} \left(\frac{1}{f_1} - \frac{1}{f_2}\right) u_1 u_2 \\
&\qquad - \int_{\Gamma_{1,\delta}} (u_2\partial_\nu u_1 - u_1\partial_\nu u_2) \\
&= \int_{\partial\Omega} (u_2\partial_\nu u_1 - u_1\partial_\nu u_2) - \int_{\Gamma_{1,\delta}} (u_2\partial_\nu u_1 - u_1\partial_\nu u_2), \qquad (4.3.4)
\end{aligned}
$$

where we applied identity of the q_j outside Ω', identity of f_j on Γ_1, and equality of the Dirichlet-to-Neumann maps (4.1.8) to eliminate various terms, and Green's formula over $\Omega_{0,\delta}$ to achieve the second line. Thus the lemma follows if the second integral on the right-hand side of (4.3.4) converges to 0 as $\delta \to 0$.

Observe that

$$
\begin{aligned}
\int_{\Gamma_{1,\delta}} (u_1\partial_\nu u_2 - u_2\partial_\nu u_1) &= \int_{-\pi/2}^{\pi/2} \delta(u_1\partial_r u_2 - u_2\partial_r u_1)(\delta, \vartheta)\mathrm{d}\vartheta \\
&= \int_{-\pi/2}^{\pi/2} [u_1, u_2](\delta, \vartheta)\mathrm{d}\vartheta.
\end{aligned}
$$

Moreover, by expanding the determinant and calculating $[v_0, u_0](r, \vartheta) = \varepsilon^{-1}e^{-2\vartheta/\varepsilon}$, one can easily see that

$$
[u_1, u_2](r, \vartheta) = \varepsilon e^{2\vartheta/\varepsilon} \begin{vmatrix} [u_1, v_0] & [u_1, u_0] \\ [u_2, v_0] & [u_2, u_0] \end{vmatrix} (r, \vartheta). \qquad (4.3.5)
$$

Now apply $[u_j, u_0 + \beta v_0](r, \vartheta) \to 0$ as $r \to 0$ to see that the columns in the right-hand side of (4.3.5) become collinear as $r \to 0$. The lemma follows. □

Remark. *The identity* (4.3.5) *is usually written for solutions of ordinary differential equations; see, for example, [53, (2.8-9)], [94] or [81, Lem. 1].*

Proof of Proposition 4.3. We consider all solutions u_j $(j = 1, 2)$ of (4.3.2) with $f_j = f$,

$\beta_j = \beta$. By Lemma 4.6 we see

$$\int_\Omega (q_1 - q_2) u_1 u_2 = 0.$$

Note the hypothesis $q_1 - q_2 = 0$ outside Ω'. Clearly $v_j \in K$—see (4.3) and (4.7)—and $u_j \in \tilde{K}$, so using Proposition 4.5 we deduce from (4.3.3) that

$$\int_\Omega (q_1 - q_2) v_1(\cdot\,;\tau) v_2(\cdot\,;\tau) = 0 \qquad (\tau > 0).$$

We have now arrived at precisely [67, Eq. (4.3)]. From here on our proof exactly follows that of [67, Prop. 4.1]. □

4.4 Single-frequency cloaking

We establish uniqueness for potentials already known in a neighbourhood of the boundary.

Theorem 4.2 (Uniqueness of potential)**.** *If $q_1, q_2 \in C^{2+\alpha}(\overline{\Omega})$ for some fixed $\alpha > 0$, $q_1 = q_2$ in $\Omega \backslash \Omega'$, f is an admissible boundary function, $\beta \in \mathbb{R}$, and the Dirichlet-to-Neumann maps are equal—i.e., $\Lambda_{q_1,f,\beta} = \Lambda_{q_2,f,\beta}$—then in fact $q_1 = q_2$ in all of Ω.*

Proof. The proof is now exactly analogous to the proof of Theorem 4.1 [67, Thm. 1.1], with the only difference being replacing Lemma 4.3 [67, Prop. 4.1] by our Proposition 4.3. □

Theorem 4.3 (Uniqueness of boundary condition)**.** *Consider admissible boundary functions f_1 and f_2. Let $q \in C^{2+\alpha}(\overline{\Omega})$ for fixed $\alpha > 0$, let $\beta \in \mathbb{R}$, and suppose the Dirichlet-to-Neumann maps are equal: $\Lambda_{q,f_1,\beta} = \Lambda_{q,f_2,\beta} = \Lambda$. Then $f_1 = f_2$.*

Proof. Let $g \in H^{1/2}(\Gamma)$, and choose functions u_1 and u_2 solving

$$\begin{cases} (-\Delta + q)u_j &= 0 \quad \text{in } \Omega, \\ u_j - f_j \partial_\nu u_j &= 0 \quad \text{on } \Gamma_c, \\ \beta[u_j] &= 0 \quad \text{at } 0, \\ u_j &= g \quad \text{on } \Gamma. \end{cases}$$

By hypothesis $\partial_\nu u_1 = \partial_\nu u_2 = -\Lambda g$. Then $u_j \in H^2_{\text{loc}}(\Omega^*)$, so with the definitions $u = u_1 - u_2$ and $f = f_1 - f_2$ we see

$$\begin{cases} (-\Delta + q)u &= 0 & \text{in } \Omega, \\ u = \partial_\nu u &= 0 & \text{on } \Gamma, \\ u - f\partial_\nu u &= f_1 \partial_\nu u_2 - f_2 \partial_\nu u_1 & \text{on } \Gamma_c. \end{cases}$$

The unique continuation principle Lemma 4.4(ii) immediately implies $u = 0$ in Ω or, in other words, $u_1 = u_2$, whence $\partial_\nu u_1 = \partial_\nu u_2$ on Γ_c. Thus, along Γ_c,

$$f_1 = u_1/\partial_\nu u_1 = u_2/\partial_\nu u_2 = f_2,$$

as required. □

We observe, thus, that fixing either the potential q or boundary condition f uniquely determines the other, respectively f or q, from the Dirichlet-to-Neumann map, provided β is *a priori* known. However, a crucial component in the proof of the q-uniqueness is Proposition 4.3. Applying this without the condition $f_1 = f_2$ is insufficient to achieve the uniqueness. Similarly, without the condition $q_1 = q_2$, the proof of the f-uniqueness fails. Thus, in fact, we have the following result:

Corollary 4.3 (Single-frequency cloaking)**.** *Suppose we are given a Dirichlet-to-Neumann map of type* (4.1.7) *for fixed* $\beta \in \mathbb{R}$*, and we know the admissible pairs* (q_1, f_1) *and* (q_2, f_2) *gave rise to it, with both* $q_j \in C^{2+\alpha}(\overline{\Omega})$*. Then* $q_1 = q_2$ *if and only if* $f_1 = f_2$*.*

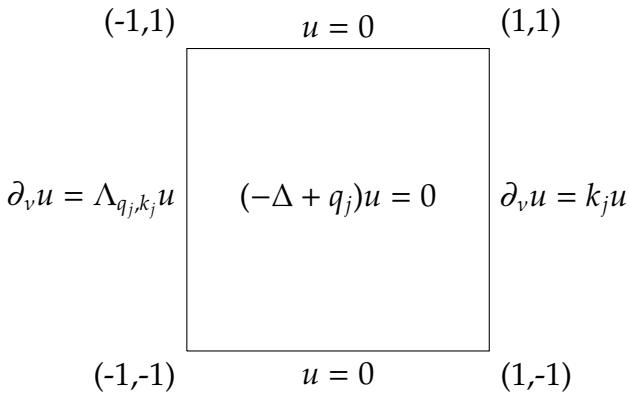

Figure 4.3: the underlying problem for the numerical example in Appendix D

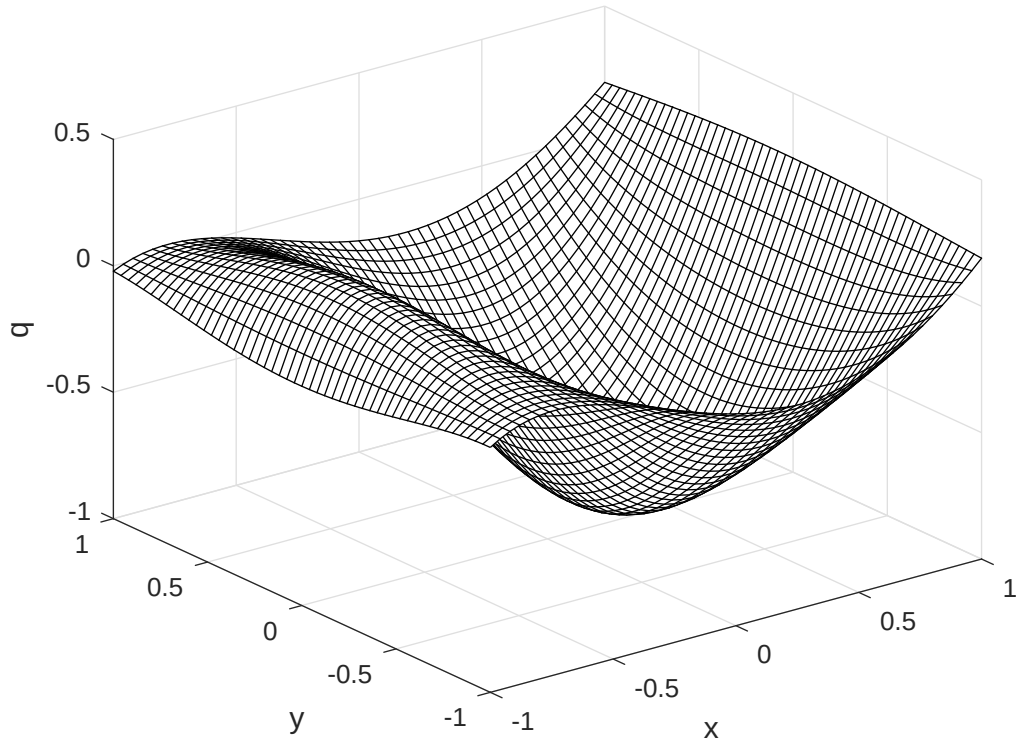

Figure 4.4: a MATLAB *plot of q_2 from the code in Appendix D*

Figure 4.3 summarises an example two-dimensional problem for which we have performed numerical calculations to illustrate Corollary 4.3. In fact, we do not even require that f produce a Dirichlet-singularity, and the example prob-

lem shows this in action. In the figure, Dirichlet-to-Neumann measurements are made on the West edge of the square, Dirichlet conditions are applied to the North and South edges, and a Robin condition of the form $u = f\partial_\nu u$ is applied on the East edge, with constant $f(y) = k_j^{-1}$. Although we lack the singular boundary condition of the Berry–Dennis regime, we are able to construct from the pair $(q_1, k_1) = (0, 0)$ a distinct pair (q_2, k_2), both of which, by our calculations, yield the same Dirichlet-to-Neumann map $\Lambda_{q,k}$. In Figure 4.4 we have plotted q_2.

4.5 Full-frequency uniqueness

The cloaking result in Section 4.4 is, we emphasise, valid only at a single frequency: we proved it at $\lambda = 0$, but a non-zero, real λ can easily be translated away in the potential. In this section we will establish uniqueness of q, f and β when one has access to full-frequency Dirichlet-to-Neumann data.

Consider the spectral Schrödinger problem

$$\begin{cases} (-\Delta + q)u &= \lambda u \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \\ u - f\partial_\nu u &= 0 \quad \text{on } \Gamma_c, \\ [u, u_0 + \beta v_0] &= 0 \quad \text{at } 0. \end{cases} \tag{4.5.1}$$

As in (4.1.7) this defines the Dirichlet-to-Neumann operator $\Lambda_{q-\lambda,f,\beta} : g \mapsto -\partial_\nu u \restriction_\Gamma$. For notational simplicity we will write this as $\Lambda(\lambda) = \Lambda_{q-\lambda,f,\beta}$. We now reformulate Inverse Problem II to include full-spectrum data.

Inverse Problem II′. *Given $\Lambda(\lambda)$ at every frequency $\lambda \in \mathbb{R}$, recover the potential q, singular boundary condition f and self-adjointness-imposing β.*

In the case of symmetric geometry $\Omega = \Omega_1$ knowledge of the Dirichlet-to-

Neumann operator is sufficient to prove a full uniqueness theorem for inverse problem II′. This is because, owing to the poles of Λ being eigen-values of the associated operator, these data include the following negative eigen-value asymptotics:

Lemma 4.7. *Take the operator L' defined in (4.1.4), and label its eigen-values by λ_n^1 so that $\lambda_{-1}^1 < 0 \leq \lambda_0^1$. Then as $n \to -\infty$ we have*

$$\lambda_n^1 \sim -e^{-2\varepsilon(\vartheta_0 + \tan^{-1}\beta)} e^{-2\varepsilon n\pi}. \tag{4.5.2}$$

Here $\vartheta_0 = \tan^{-1}(A/B) \in (-\pi/2, \pi/2]$ is known, where

$$A = \lim_{t\to+\infty} e^{-t} w_s(t), \; B = \lim_{t\to+\infty} e^{-t} w_c(t), \tag{4.5.3}$$

and the non-trivial functions w_s and w_c satisfy, as $t \to 0$,

$$\left[w_s(t), t^{1/2}\sin\left(\varepsilon^{-1}\log(t)\right)\right] \to 0, \; \left[w_c(t), t^{1/2}\cos\left(\varepsilon^{-1}\log(t)\right)\right] \to 0 \tag{4.5.4}$$

and solve $-w''(t) - (1/4 + 1/\varepsilon^2)t^{-2}w(t) = -w(t)$ on the interval $(0, |\lambda|^{1/2})$.

Proof. We slightly sharpen the proof of [99, Eq. (41)], correcting some small typographical errors and stating more precisely the asymptotics in the last few steps. For the convenience of the reader we present our proof here in its entirety. The first crucial consideration is that we may assume, without a loss of generality, that $q = 0$, since by [75, Thm. V.4.10] a non-zero q perturbs the spectrum by at most $\|q\|_{L^\infty(\Omega_1)}$, leaving (4.5.2) unchanged, whilst Weyl's Theorem [75, Thm. IV.5.35] ensures the spectrum with a non-zero q is still discrete.

From Proposition 4.1 we note the orthogonal decomposition $L' = L_0' \oplus \oplus_{n=1}^{\infty} L_n$. The spectrum of L' is the union of the spectra of its constituent ordinary differential operators. Since all the L_n have positive spectrum, the only negative eigen-

values of L' are those of L'_0. The latter are generated by the eigen-value problem

$$\begin{cases} -\frac{1}{r}\Big(ru'(r;\lambda)\Big)' - \frac{1}{\varepsilon^2 r^2}u(r;\lambda) &= \lambda u(r;\lambda) \quad \Big(r \in (0,1)\Big), \\ u(1) &= 0, \\ [u(\cdot\,;\lambda), u_0 + \beta v_0](0^+) &= 0. \end{cases}$$

So, let $\lambda < 0$. We transform the above problem by setting $\kappa = |\lambda|^{1/2}$, $t = \kappa r$ and $w(t) = r^{1/2}u(r;\lambda)$. Moreover define $\tilde{u}_0(t) = r^{1/2}u_0(r) = r^{1/2}\sin\Big(\varepsilon^{-1}\log(r)\Big)$ and $\tilde{v}_0(t) = r^{1/2}v_0(r) = r^{1/2}\cos\Big(\varepsilon^{-1}\log(r)\Big)$. Then we easily calculate

$$\begin{cases} -w''(t) - \Big(\frac{1}{4} + \frac{1}{\varepsilon^2}\Big)\frac{1}{t^2}w(t) &= -w(t) \quad \Big(t \in (0,\kappa)\Big), \\ w(\kappa) &= 0, \\ [w, \tilde{u}_0 + \beta\tilde{v}_0](0^+) &= 0, \end{cases} \tag{4.5.5}$$

where $[w,\tilde{w}](t) = w(t)\tilde{w}'(t) - w'(t)\tilde{w}(t)$ is the Lagrange bracket for the new problem. We have $[w,\tilde{w}](t) = \kappa^{-1}[u,\tilde{u}](r)$, where $\tilde{w}(t) = r^{1/2}\tilde{u}(r)$.

Observe that, because $\log(\kappa^{-1}t) = \log(t) - \log(\kappa)$, we have

$$\begin{aligned} \tilde{u}_0(t) &= \kappa^{-1/2}t^{1/2}\Big(v_0(\kappa)u_0(t) - u_0(\kappa)v_0(t)\Big), \\ \tilde{v}_0(t) &= \kappa^{-1/2}t^{1/2}\Big(v_0(\kappa)v_0(t) + u_0(\kappa)u_0(t)\Big). \end{aligned}$$

Thus any eigen-function $w(t)$ for the operator generated by (4.5.5) must satisfy, as $t \to 0$,

$$(v_0 + \beta u_0)(\kappa)[w(t), t^{1/2}u_0(t)] - (u_0 - \beta v_0)(\kappa)[w(t), t^{1/2}v_0(t)] \to 0.$$

Therefore, by the hypotheses of the lemma and the fact w_s and w_c are a fundamental system for the differential equation in (4.5.5), such an eigen-function w is given by the linear combination

$$(v_0 + \beta u_0)(\kappa)w_s - (u_0 - \beta v_0)(\kappa)w_c. \tag{4.5.6}$$

Moreover, to be an eigen-function, w must satisfy $w(\kappa) = 0$. So we seek large, positive zeros of (4.5.6).

One can calculate without great difficulty that the differential equation in (4.5.5) is non-oscillatory at $+\infty$—crudely, the t^{-2}-term will play a negligible part when t is large—so there are constants A and B such that as $t \to +\infty$ we have

$$w_s(t) = Ae^t\Big(1 + o(1)\Big),$$
$$w_c(t) = Be^t\Big(1 + o(1)\Big).$$

Thus, asymptotically, the desired zeros approach the zeros in $\tilde{\kappa}$ of the function

$$A(v_0 + \beta u_0)(\tilde{\kappa}) - B(u_0 - \beta v_0)(\tilde{\kappa}). \tag{4.5.7}$$

Rearranging, we find that such $\tilde{\kappa}$ solve

$$(A + B\beta)\cos\Big(\varepsilon^{-1}\log(\tilde{\kappa})\Big) + (A\beta - B)\sin\Big(\varepsilon^{-1}\log(\tilde{\kappa})\Big) = 0,$$

i.e., they can be enumerated as $\tilde{\kappa}_n$ ($n < 0$) and satisfy

$$\varepsilon^{-1}\log(\tilde{\kappa}_n) = -n\pi + \tan^{-1}\left(\frac{A + B\beta}{B - A\beta}\right) = n\pi + \tan^{-1}\left(\frac{A}{B}\right) + \tan^{-1}(\beta).$$

By interlacing there is precisely one zero of w between each $\tilde{\kappa}_n$, so we label the former zeros as κ_n in such a way that κ_{-1} is the smallest negative zero of w. As $n \to -\infty$ we have $\kappa_n = \tilde{\kappa}_n\Big(1 + o(1)\Big)$. Thus

$$\lambda_n^1 = -\kappa_n^2 = -e^{-2\varepsilon(A/B + \tan^{-1}\beta)}e^{-2\varepsilon n\pi}\Big(1 + o(1)\Big),$$

which concludes the proof. □

Remark. *In fact, repeating the above calculation for the domain $\Omega = R\Omega_1$ shows that*

the negative eigen-value asymptotics are independent of the choice of $R > 0$, which is to be expected, since the negative eigen-values arise from the singular boundary condition, not the shape of the domain. Intuitively, the eigen-functions increasingly "concentrate" their oscillations near 0 as the spectral parameter becomes more negative[4.5].

We now prove that in the case of a general domain $\Omega = \mathrm{int}\left(\overline{\Omega_1 \cup \Omega_0}\right)$, as defined in the opening paragraphs of Section 4.1, the very same negative eigen-value asymptotics hold. For the technical reason of ensuring a definite sign for a particular Dirichlet-to-Neumann operator we will require that $f \restriction_{\Gamma_0} = 0$. Our approach is to show that the counting functions of the eigen-values asymptotically agree. To avoid ambiguity we specify the enumeration of the eigen-values λ_n and λ_n^1 of, respectively, T and L' to be such that λ_0 and λ_0^1 are the smallest non-negative eigen-values of each operator.

To prove our negative eigen-value asymptotics we will need upper and lower bounds on the difference between the counting functions of the negative eigen-values for, respectively, T and L'. The lower bound will follow from a "pseudo-mode" argument involving a fairly elementary asymptotic analysis of functions—constructed from the true eigen-functions for L'—that approximate some eigen-functions of T. Our proof of the upper bound requires us to consider a pencil of Dirichlet-to-Neumann operators on Γ_i, the terminology and underlying results for which we will set up in two lemmata. The interface Dirichlet-to-Neumann operators are defined as follows: for $j = 0, 1$,

$$\Lambda_j(\lambda) : h \mapsto -\partial_{\nu_j} w_j,$$

[4.5] We will make this statement more rigorous in the proof of Theorem 4.4.

where w_j solve the boundary-value problems

$$\begin{cases} -\Delta w_0 &= \lambda w_0 & \text{in } \Omega_0, \\ w_0 &= 0 & \text{on } \Gamma_0 \cup \Gamma, \\ w_0 &= h & \text{on } \Gamma_{\mathrm{i}}, \end{cases} \tag{4.5.8}$$

$$\begin{cases} -\Delta w_1 &= \lambda w_1 & \text{in } \Omega_1, \\ w_1 &= f\partial_{\nu_1} w_1 & \text{on } \Gamma_1, \\ \beta[w_1] &= 0 & \text{at } 0, \\ w_1 &= h & \text{on } \Gamma_{\mathrm{i}}, \end{cases} \tag{4.5.9}$$

and ∂_{ν_j} denotes the outward directed normal derivative for the subdomain Ω_j (i.e., $\partial_{\nu_1} \restriction_{\Gamma_1} = \partial_\nu = \partial_{\nu_0} \restriction_{\Gamma_0 \cup \Gamma}$, but $\partial_{\nu_0} \restriction_{\Gamma_{\mathrm{i}}} = -\partial_{\nu_1} \restriction_{\Gamma_{\mathrm{i}}}$).

Remark. *It is clear that a real number λ is an eigen-value for T if and only if the pencil of operators $\Lambda_1(\lambda) + \Lambda_0(\lambda)$ has a non-trivial kernel $\mathcal{K}(\lambda)$, since any function in this kernel will correspond to a pair (w_0, w_1) solving, respectively,* (4.5.8) *and* (4.5.9)*, for which the normal derivatives match on the interface, i.e., $\partial_{\nu_0} w_0 = -\partial_{\nu_1} w_1$ on Γ_{i}.*

To conduct our analysis we bring into play the $L^2(-\pi/2, \pi/2)$-basis Θ_n on Γ_{i}, defined by

$$\Theta_n(\vartheta) = \begin{cases} \dfrac{e^{-\vartheta/\varepsilon}}{2\varepsilon \sinh(\pi/2\varepsilon)} & (n = 0), \\ k_n(n\varepsilon \cos(n\vartheta) - \sin(n\vartheta)) & (n \text{ even}), \\ k_n(\cos(n\vartheta) + n\varepsilon \sin(n\vartheta)) & (n \text{ odd}). \end{cases}$$

These Θ_n are normalised versions of those in Section 3.5, so that their $L^2(-\frac{\pi}{2}, \frac{\pi}{2})$-norms are 1. Since the sufficiently negative eigen-values of T can only arise from the presence of the singular boundary condition (regular boundary conditions would yield a spectrum that is bounded below), any function $h = \sum_{n=0}^{\infty} h_n \Theta_n \in \mathcal{K}(\lambda)$ for $\lambda \ll 0$ is either identically 0 or has non-trivial first component. It follows that for $\lambda \ll 0$ such that $\mathcal{K}(\lambda) \neq \{0\}$ we may normalise the non-trivial kernel

element h so that $h_0 = 1$.

In the basis Θ_n on Γ_i the map $\Lambda_1(\lambda)$ takes the form of an infinite diagonal matrix, represented in the block partitioned form

$$\left(\begin{array}{c|c} m_0(\lambda) & \mathbf{0}^T \\ \hline \mathbf{0} & M(\lambda) \end{array}\right). \tag{4.5.10}$$

Here $m_0(\lambda) := -\varphi_0'(1;\lambda)/\varphi_0(1;\lambda)$ is the Weyl–Titchmarsh m-function for the $L^2(0,1)$-classical-limit-circle ordinary differential problem

$$\left\{\begin{array}{rcll} -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\varphi_0}{\mathrm{d}r}(r;\lambda)\right) - \frac{1}{\varepsilon^2 r^2}\varphi_0(r;\lambda) & = & \lambda\varphi_0(r;\lambda) & \big(r\in(0,1)\big), \\ r\Big(\varphi_0\partial_r(u_0+\beta v_0) - (\partial_r\varphi_0)(u_0+\beta v_0)\Big)(r;\lambda) & \to & 0 & (r\to 0), \end{array}\right.$$

the infinite column-vector of zeros is denoted by $\mathbf{0}$, and $M(\lambda)$ is the diagonal submatrix whose n-th diagonal term ($n = 1, 2, 3, \ldots$) is the $L^2(0,1)$-classical-limit-point Weyl–Titchmarsh m-function $m_n(\lambda) := -\varphi_n'(1;\lambda)/\varphi_n(1;\lambda)$ for the ordinary differential problem

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\varphi_n}{\mathrm{d}r}(r;\lambda)\right) + \frac{n^2}{r^2}\varphi_n(r;\lambda) = \lambda\varphi_n(r;\lambda) \quad \big(r\in(0,1)\big). \tag{4.5.11}$$

In the same basis, $\Lambda_0(\lambda)$ lacks this diagonal structure. Nonetheless we represent it in the block partitioned form

$$\left(\begin{array}{c|c} a(\lambda) & \mathbf{b}(\lambda)^T \\ \hline \mathbf{b}(\lambda) & C(\lambda) \end{array}\right).$$

Lemma 4.8. *The pencil of Dirichlet-to-Neumann maps $\Lambda_1 + \Lambda_0$ is an analytic, operator-valued function on $\mathbb{C}\backslash\mathbb{R}$, and the quadratic form $\langle(\Lambda_1+\Lambda_0)(\lambda)h, h\rangle_{L^2(\Gamma_i)}$ has imaginary part of the same sign as* $\mathrm{Im}(\lambda)$. *The derivative of the pencil is a compact operator on any Sobolev space $H^k(\Gamma_i)$.*

Remark. *We can think of* $\Lambda_1 + \Lambda_0$ *as an operator-valued Herglotz function.*

Proof. Owing to Proposition 2.1, we know that any Dirichlet-to-Neumann map, for the conductivity problem (2.4.2) with Dirichlet boundary conditions, is analytic. Applying exactly the same argument to a general Schrödinger problem with any self-adjoint boundary condition on part of the boundary and a Dirichlet-to-Neumann map defined on the remaining boundary shows that Λ_0 and Λ_1 are both analytic, whence so is their sum.

To establish the half-plane property, we need to examine the imaginary parts of both

$$\langle \Lambda_1(\lambda)h, h\rangle_{\Gamma_i} \text{ and } \langle \Lambda_0(\lambda)h, h\rangle_{\Gamma_i}.$$

The latter involves straight-forward integration by parts and proceeds like the proof of Proposition 2.2. Using Green's formula and the solution w_0 to (4.5.8),

$$\begin{aligned}\langle \Lambda_0(\lambda)h, h\rangle_{\Gamma_i} &= \int_{\Gamma_i} (-\partial_{\nu_0} w_0)\overline{w_0} \\ &= -\int_{\Omega_0} \left((\Delta w_0)\overline{w_0} + |\nabla w_0|^2\right) \\ &= \lambda \int_{\Omega_0} |w_0|^2 - \int_{\Omega_0} |\nabla w_0|^2. \end{aligned} \tag{4.5.12}$$

This clearly has imaginary part of the same sign as that of λ.

On the other hand, integration by parts fails for the solutions of (4.5.9) (recall the solutions u_0 and v_0 when $\lambda = 0$). Instead, decompose the solution as in the proof of Lemma 4.6, effectively treating $-\lambda$ and $-\overline{\lambda}$ as, respectively, the potentials[4.6] q_1 and q_2. Letting w_1 solve (4.5.9), there are real constants ω_n ($n = 0, 1, 2, \ldots$) so that $W_n(r, \vartheta) = \omega_n U_n(r)\Theta_n(\vartheta)$ form the terms in a series expansion: $w_1 = \sum_{n=0}^{\infty} W_n =: W_0 + W$. The term W arises from the regular part of the problem,

[4.6] The key difference between these two proofs is that before, we had $q_1 = q_2$ in some neighbourhood of the boundary.

and is in $H^2(\Omega_1)$ [99], and moreover all the W_n are pairwise orthogonal in $L^2(\Omega_1)$. Hence, by Green's formula,

$$\begin{aligned}(\lambda-\overline{\lambda})\int_{\Omega_1} w_1\overline{w_1} &= (\lambda-\overline{\lambda})\int_{\Omega_1} W_0\overline{W_0} - \int_{\Omega_1}\Big((\Delta W)\overline{W} - W\Delta\overline{W}\Big)\\ &= \omega_0^2(\lambda-\overline{\lambda})\int_0^1 r\mathrm{d}r\,|U_0(r)|^2\int_{-\pi/2}^{\pi/2}\mathrm{d}\vartheta\,|\Theta_0(\vartheta)|^2\\ &\quad - \int_{\partial\Omega_1}\Big((\partial_{\nu_1}W)\overline{W} - W\partial_{\nu_1}\overline{W}\Big)\\ &= \omega_0^2(\lambda-\overline{\lambda})\int_0^1 r\mathrm{d}r\,|U_0(r)|^2 + \int_{\Gamma_i}\Big(W\partial_{\nu_1}\overline{W} - (\partial_{\nu_1}W)\overline{W}\Big). \qquad (4.5.13)\end{aligned}$$

Without loss of generality we scale ω_0 to be 1, set $0<\delta<1$ and examine

$$\begin{aligned}(\lambda-\overline{\lambda})\int_\delta^1 r\mathrm{d}r\,U_0(r)\overline{U_0(r)} &= U_0(1)\overline{U_0'(1)} - U_0'(1)\overline{U_0(1)} - \delta\Big(U_0(\delta)\overline{U_0'(\delta)} - U_0'(\delta)\overline{U_0(\delta)}\Big)\\ &= \int_{\Gamma_i}\Big(W_0\partial_1\overline{W_0} - (\partial_1 W_0)\overline{W_0}\Big) - [U_0,\overline{U_0}](\delta). \qquad (4.5.14)\end{aligned}$$

Clearly the lemma will follow if we can show that $[U_0,\overline{U_0}](\delta)$ vanishes as $\delta\to 0$, since by monotone or dominated convergence the integral on the left-hand side of (4.5.14) tends to $\int_0^1 r|U_0(r)|^2\mathrm{d}r$; combining this with (4.5.13) yields

$$\mathrm{Im}(\lambda)\int_{\Omega_1}|w_1|^2 = \mathrm{Im}\Big(\langle\Lambda_1(\lambda)h,h\rangle_{L^2(\Gamma_i)}\Big). \qquad (4.5.15)$$

Similarly to Lemma 4.6, we relate the boundary behaviour of U_0 to that of the solutions u_0 and v_0 for $\lambda=0$ by applying the elementary identity

$$[U_0,\overline{U_0}] = -\varepsilon\begin{vmatrix}[U_0,u_0] & [U_0,v_0]\\ [\overline{U_0},u_0] & [\overline{U_0},v_0]\end{vmatrix}, \qquad (4.5.16)$$

since u_0 and $-\varepsilon v_0$ form a fundamental system satisfying $[u_0,v_0]=-\varepsilon^{-1}$. Thus, since both U_0 and its conjugate in its place satisfy $[U_0,u_0+\beta v_0](0^+)=0$, we see that the columns in the right-hand side of (4.5.16) become collinear as its argument

approaches 0. The lemma is proved. □

Lemma 4.9. *Let λ be less than the infima of the spectra of each L_j ($j = 1, 2, 3, \ldots$) and of the Laplace operator in Ω_0 with homogeneous Dirichlet boundary conditions; let z be from the cut plane $\mathbb{C}_c := \mathbb{C}\backslash[0, +\infty)$. Then both $(M + C)(\lambda)$ and $(M + C)(\lambda + z)$ are invertible, and the inverse $(M + C)(\lambda + z)^{-1}$ is given by*

$$\left(\mathbb{1} + z(M + C)(\lambda)^{-1} \int_{[\lambda,\lambda+z]} (M + C)'\right)^{-1} (M + C)(\lambda)^{-1}$$

Proof. Lemma 4.8 implies that $M + C$ is differentiable anywhere in $\lambda + \mathbb{C}_c$, and its derivative is compact. By the fundamental theorem of calculus[4.7],

$$(M + C)(\lambda + z) = (M + C)(\lambda) + \int_{[\lambda,\lambda+z]} (M + C)'.$$

Hence, the lemma will be established upon showing that $(M + C)(\lambda)$ and subsequently $\mathbb{1} + (M + C)(\lambda)^{-1} \int_{[\lambda,\lambda+z]} (M + C)'$ are invertible. The first will be achieved by checking the definiteness of the sign of $(M + C)(\lambda)$, the second will be a consequence of the analytic Fredholm theorem [116, p. 201].

By (4.5.12) we see that $\Lambda_0(\mu) \leq 0$ for any $\mu \leq 0$, from which $C(\lambda) \leq 0$ follows immediately. Furthermore, the diagonal entries of $M(\lambda)$ are precisely $m_n(\lambda)$; see (4.5.11) and the preceding discussion. Owing to a remark in [99, p. 4], for $n = 1, 2, 3, \ldots$, we have $m_n(\lambda) = -i\sqrt{-\lambda} J_n'(i\sqrt{-\lambda})/J_n(i\sqrt{-\lambda})$, to which we may apply [44, Eqs. 10.6.2, 10.19.1] to show by an algebraic calculation that with fixed $\lambda < 0$, as $n \to \infty$,

$$m_n(\lambda) \sim -n.$$

We deduce that $(M + C)(\lambda) < 0$, and its invertibility follows.

[4.7] The version for operator-valued analytic functions, of course.

Consider, now, the analytic operator-valued function

$$\mathscr{A}(z) := (M + C)(\lambda)^{-1} \int_{[\lambda,\lambda+z]} (M + C)' \quad (z \in \mathbb{C}_c).$$

Since $\int_{[\lambda,\lambda+z]}(M+C)' = (M+C)(\lambda+z)-(M+C)(\lambda)$ we may apply the reasoning that led to equation (2.4.4) to see that this integral is compact. Hence, since $(M+C)(\lambda)^{-1}$ is bounded, we observe that $\mathscr{A}(z)$ is compact for every $z \in \mathbb{C}_c$. Furthermore, if $z \in \mathbb{C}\backslash\mathbb{R}$ then $\ker\Big((M + C)(\lambda + z)\Big) = \{0\}$, since if this were not the case we would have a non-trivial function on the interface Γ_i, meaning (by the Remark on page 129) there would be an eigen-function for T with a non-real eigen-value, which is forbidden by the self-adjointness of T. Indeed, this is also contradictory for any $z < 0$ since neither M nor C can give rise to eigen-values less than λ. Therefore, by the analytic Fredholm theorem [116, p. 201], the following two cases are mutually exhaustive:

(i) $\Big(\mathbb{1} + \mathscr{A}(z)\Big)^{-1}$ exists for *no* $z \in \mathbb{C}_c$;

(ii) $\Big(\mathbb{1} + \mathscr{A}(z)\Big)^{-1}$ exists for *every* $z \in \mathbb{C}_c$.

Clearly

$$\begin{aligned}(M + C)(\lambda + z) &= (M + C)(\lambda)\left(\mathbb{1} + (M + C)(\lambda)^{-1} \int_{[\lambda,\lambda+z]} (M + C)'\right) \\ &= (M + C)(\lambda)\Big(\mathbb{1} + \mathscr{A}(z)\Big),\end{aligned}$$

so if $z \leq 0$ then both sides are invertible. This excludes case (i), and the lemma is proved. □

Theorem 4.4 (Negative eigen-value asymptotics for general domain)**.** *Let* Ω *be specially decomposable,* q *an admissible potential,* f *an admissible boundary function that is additionally* 0 *outside* Γ_1*, and* $\beta \in \mathbb{R}$ *parameterise a self-adjoint boundary condition*

at 0. *Then the operator* T *defined in* (4.1.6) *has discrete spectrum accumulating only at* $\pm\infty$, *and—with* ϑ_0 *defined above equation* (4.5.3)*—its negative eigen-values possess the asymptotic expansion*

$$\lambda_n \sim -e^{-2\varepsilon(\vartheta_0+\tan^{-1}\beta)}e^{-2n\pi\varepsilon} \quad (n \to -\infty).$$

Proof. For discreteness and accumulation points of the spectrum of T we refer to [99, Sec. 5]. The remainder of the proof is split into two parts, in which we asymptotically bound the counting function for the negative eigen-values λ_n of T from, in turn, above and below by that for the negative eigen-values λ_n^1 of L'. We may again without loss of generality take $q = 0$ in our calculations.

1. Bound from below. Take any smooth cut-off function μ on Ω that is supported and radially symmetric in Ω_1, and takes value 1 in $\frac{1}{2}\Omega_1$ (in particular, $\partial_\nu\mu \restriction_{\Gamma_1} = 0$). Note that the partial derivatives of μ are supported in the half-annulus $\mathcal{A} := \Omega_1\backslash\frac{1}{2}\Omega_1$. Let $n < 0$, and choose eigen-function φ_n for L' at the eigen-value λ_n^1, such that it has L^2-norm $\|\varphi_n\|_{L^2(\Omega_1)} = 1$. We will show that these φ_n are "pseudo-modes" for T, i.e.,

$$\frac{\left\|(T-\lambda_n^1)\mu\varphi_n\right\|_{L^2(\Omega)}}{\|\mu\varphi_n\|_{L^2(\Omega)}} =: \varepsilon_n \to 0 \quad (n \to -\infty). \tag{4.5.17}$$

By the spectral theorem, denoting the eigen-values of T as λ_j and the corresponding normalised eigen-functions as ψ_j ($j \in \mathbb{Z}$), we may write

$$\|(T-\lambda_n^1)\mu\varphi_n\|^2 = \sum_{j\in\mathbb{Z}}(\lambda_j-\lambda_n^1)^2|\langle\mu\varphi_n,\psi_j\rangle_{L^2(\Omega)}|^2.$$

Suppose $|\lambda_j - \lambda_n^1| > \varepsilon_n$ for every $j \in \mathbb{N}$. Then

$$\varepsilon_n^2\|\mu\varphi_n\|_{L^2(\Omega)}^2 > \varepsilon_n^2\sum_{j\in\mathbb{Z}}|\langle\mu\varphi_n,\psi_j\rangle_{L^2(\Omega)}|^2 = \varepsilon_n^2\|\mu\varphi_n\|_{L^2(\Omega)}^2,$$

a contradiction. Thus there is a subsequence λ_{j_n} satisfying $|\lambda_{j_n} - \lambda_n^1| \leq \varepsilon_n$ ($n = -1, -2, -3, \ldots$). Hence if $\varepsilon_n \to 0$ ($n \to -\infty$) we will have established the lower bound.

Firstly observe that, by choice of μ and the fact $\varphi_n \in D(L')$, we have

$$\mu\varphi_n - f\partial_\nu(\mu\varphi_n) = \mu(\varphi_n - f\partial_\nu\varphi_n) - f(\partial_\nu\mu)\varphi_n = 0,$$

implying $\mu\varphi_n$ is indeed in $D(T)$. Next we calculate that, clearly,

$$(T - \lambda_n^1)\mu\varphi_n = \begin{cases} 0 & \text{in } \Omega\backslash\mathcal{A}, \\ -(\Delta\mu)\varphi_n - 2\nabla\mu\cdot\nabla\varphi_n & \text{in } \mathcal{A}. \end{cases}$$

If we can show that φ_n and $\nabla\mu\cdot\nabla\varphi_n$ go to 0 uniformly in $\mathcal{A}$ then (4.5.17) will follow.

Set $\kappa_n := \sqrt{-\lambda_n^1}$, and define the (more conveniently notated) Hankel functions $H_z^\pm = J_z \pm iY_z$, where, as is standard, J_z and Y_z are the Bessel functions of, respectively, the first and second kind. Since $\lambda_n^1 < 0$ we know that the eigen-functions φ_n for L' come from the ℓ_0' operator in the decomposition (3.5.7). For any eigen-value λ of ℓ_0', the corresponding eigen-function is a constant times the sum

$$J_{i/\varepsilon}(r\sqrt{\lambda})Y_{i/\varepsilon}(\sqrt{\lambda}) - Y_{i/\varepsilon}(r\sqrt{\lambda})J_{i/\varepsilon}(\sqrt{\lambda});$$

see the last equation of [99, p. 4]. Thus φ_n is some constant multiple of

$$\phi_n(r,\vartheta) := e^{-\vartheta/\varepsilon}\Big(H_{i/\varepsilon}^+(ir\kappa_n)H_{i/\varepsilon}^-(i\kappa_n) - H_{i/\varepsilon}^-(ir\kappa_n)H_{i/\varepsilon}^+(i\kappa_n)\Big) =: e^{-\vartheta/\varepsilon}\mathcal{H}_n(r).$$

It follows that, to normalise ϕ_n asymptotically, we need to know the leading-

order behaviour, as $n \to -\infty$, of

$$\frac{1}{\varepsilon \sinh\left(\frac{\pi}{\varepsilon}\right)} \int_{\Omega_1} |\phi_n|^2 = \int_0^1 |\mathcal{H}_n(r)|^2 r \mathrm{d}r$$
$$= \int_0^{\kappa_n} \left| H^+_{i/\varepsilon}(it) H^-_{i/\varepsilon}(i\kappa_n) - H^-_{i/\varepsilon}(it) H^+_{i/\varepsilon}(i\kappa_n) \right|^2 \frac{t \mathrm{d}t}{\kappa_n^2}. \tag{4.5.18}$$

One may easily calculate from [44, Eq. 10.17.5-6] that, for $x > 0$,

$$H^\pm_{i/\varepsilon}(ix) \sim \sqrt{\frac{2}{\pi}} e^{-i(1\pm1)\pi/4 \pm \pi/2\varepsilon} e^{\mp x} x^{-1/2} \quad (x \to +\infty). \tag{4.5.19}$$

Expanding the absolute value in the right-hand side of (4.5.18), we see

$$\kappa_n^2 \frac{\int_0^1 |\mathcal{H}_n(r)|^2 r \mathrm{d}r}{|H^-_{i/\varepsilon}(i\kappa_n)|^2} =$$
$$= \int_0^{\kappa_n} |H^+_{i/\varepsilon}(it)|^2 \left\{ 1 + \left| \frac{H^-_{i/\varepsilon}(it)}{H^-_{i/\varepsilon}(i\kappa_n)} \right|^2 \left| \frac{H^+_{i/\varepsilon}(i\kappa_n)}{H^+_{i/\varepsilon}(it)} \right|^2 - 2\mathrm{Re}\left(\frac{H^-_{i/\varepsilon}(it)}{H^-_{i/\varepsilon}(i\kappa_n)} \frac{H^+_{i/\varepsilon}(i\kappa_n)}{H^+_{i/\varepsilon}(it)} \right) \right\} t \mathrm{d}t.$$

By (4.5.19), as $n \to -\infty$, the term in { }'s converges pointwise to 1, and moreover for sufficiently large n—denoting the greatest such n by $n_0 < 0$—this term is bounded by 4. By the latter it follows that for $n \le n_0$ the integrand of the right-hand side is bounded by $4t|H^+_{i/\varepsilon}(it)|^2$, which, owing to (4.5.19), is certainly integrable over $(0, \infty)$. Hence dominated convergence applies, yielding, as $n \to -\infty$,

$$\int_0^1 |\mathcal{H}_n(r)|^2 r \mathrm{d}r \sim \left| H^-_{i/\varepsilon}(i\kappa_n) \right|^2 \int_0^{\kappa_n} \left| H^+_{i/\varepsilon}(it) \right|^2 \frac{t \mathrm{d}t}{\kappa_n^2}$$
$$\sim \int_0^\infty \left| H^+_{i/\varepsilon}(it) \right|^2 t \mathrm{d}t \cdot \kappa_n^{-2} \left| H^-_{i/\varepsilon}(i\kappa_n) \right|^2$$
$$\sim \frac{2 \int_0^\infty \left| H^+_{i/\varepsilon}(it) \right|^2 t \mathrm{d}t}{\pi e^{\pi/\varepsilon}} \kappa_n^{-3} e^{2\kappa_n},$$

from which we find

$$\int_{\Omega_1} |\phi_n|^2 \sim \underbrace{\frac{2\varepsilon(1-e^{-2\pi/\varepsilon})\int_0^\infty \left|H^+_{i/\varepsilon}(it)\right|^2 t\mathrm{d}t}{\pi}}_{=:\eta^2} \cdot \kappa_n^{-3} e^{2\kappa_n} =: c_n^2.$$

According to this definition of c_n we have—up to sign—the pointwise asymptotics $\varphi_n \sim c_n^{-1}\phi_n$ as $n \to -\infty$. More explicitly, we can calculate from (4.5.19) that, pointwise, as $n \to \infty$, we have

$$e^{\vartheta/\varepsilon}\phi(r,\vartheta) \sim -\frac{2i}{\pi} r^{-1/2}\kappa_n^{-1} e^{(1-r)\kappa_n}.$$

Hence

$$e^{\vartheta/\varepsilon}\varphi_n(r,\vartheta) \sim -\frac{2i}{\pi\eta} r^{-1/2}\kappa_n^{1/2} e^{-r\kappa_n},$$

and since within $\mathcal{A}$ we have $1/2 < r < 1$ it is clear that $\varphi_n \restriction_{\mathcal{A}} \to 0$ uniformly as $n \to -\infty$.

Now we examine

$$\begin{aligned}
\nabla\mu(r,\vartheta)\cdot\nabla\varphi_n(r,\vartheta) &= |\nabla\mu(r,\vartheta)||\nabla\varphi_n(r,\vartheta)|\cos\Big(\arg\nabla\mu(r,\vartheta) - \arg\nabla\varphi_n(r,\vartheta)\Big)\\
&= |\partial_r\mu(r,\vartheta)\partial_r\varphi_n(r,\vartheta)|\cos\left(\frac{1}{r}(\partial_\vartheta\mu(r,\vartheta) - \partial_\vartheta\varphi_n(r,\vartheta)\right)\\
&= |\partial_r\mu(r,\vartheta)\partial_r\varphi_n(r,\vartheta)|\cos\left(\frac{\partial_\vartheta\varphi_n(r,\vartheta)}{r}\right).
\end{aligned}$$

It is clear from our prior calculations that $\partial_r\varphi_n \sim c_n^{-1}\partial_r\phi_n$. Now recall [44, Eq. 10.6.2] that $\frac{\mathrm{d}}{\mathrm{d}z}H^\pm_\zeta(z) = \frac{\zeta}{z}H^\pm_\zeta(z) - H^\pm_{\zeta+1}(z)$, from which we derive

$$\begin{aligned}
e^{\vartheta/\varepsilon}\partial_r\phi_n(r,\vartheta) = \frac{i}{r\varepsilon}&\left(H^+_{i/\varepsilon}(ir\kappa_n)H^-_{i/\varepsilon}(i\kappa_n) - H^-_{i/\varepsilon}(ir\kappa_n)H^+_{i/\varepsilon}(i\kappa_n)\right)\\
&- i\kappa_n\left(H^+_{1+i/\varepsilon}(ir\kappa_n)H^-_{i/\varepsilon}(i\kappa_n) - H^-_{1+i/\varepsilon}(ir\kappa_n)H^+_{i/\varepsilon}(i\kappa_n)\right). \qquad (4.5.20)
\end{aligned}$$

Again it is easy to calculate [44, Eq. 10.17.5-6] that

$$H^{\pm}_{1+i/\varepsilon}(ix) \sim \sqrt{\frac{2}{\pi}} e^{-i(1\pm 3)\pi/4 \pm \pi/2\varepsilon} e^{\mp x} x^{-1/2},$$

from which we may show that the first and second terms in the right-hand side of (4.5.20) are asymptotically equivalent to—i.e., they "~"—respectively,

$$\frac{2}{r\pi\varepsilon} e^{(1-r)\kappa_n} r^{-1/2} \kappa_n^{-1} \text{ and } -\frac{2}{\pi} e^{(1-r)\kappa_n} r^{-1/2}.$$

It follows, after substituting these into (4.5.20) then dividing by c_n, that

$$e^{\vartheta/\varepsilon} \partial_r \varphi_n(r, \vartheta) \sim -\frac{2}{\pi\eta} r^{-1/2} \kappa_n^{3/2} e^{-r\kappa_n} \quad (n \to -\infty),$$

and therefore, as desired, $\nabla\mu \cdot \nabla\varphi_n$ must go to 0 uniformly in $\mathcal{A}$. The lower bound on the difference between the counting functions for the negative eigen-values of T and L' is immediate.

2. Bound from above. Our technique here is quite different from the asymptotic analysis of part 1. Instead we will use the pencil of the Dirichlet-to-Neumann operators on Γ_i from the sub-domains either side of the interface, and analyse non-triviality of its kernel, which occurs precisely when T has an eigen-value. Counting zeros of an associated function will then provide us with the upper bound.

The normalisation $h_0 = 1$ ensures that $\lambda \ll 0$ is an eigen-value if and only if there is $\mathbf{h}$ such that

$$\left(\begin{array}{c|c} m_0(\lambda) + a(\lambda) & \mathbf{b}(\lambda)^T \\ \hline \mathbf{b}(\lambda) & M(\lambda) + C(\lambda) \end{array}\right) \left(\begin{array}{c} 1 \\ \hline \mathbf{h} \end{array}\right) = 0.$$

After expanding the product we find that this can only happen if $\mathbf{b}(\lambda) = -\big(M(\lambda) + C(\lambda)\big)\mathbf{h}$. Thanks to Lemma 4.9 we see that $M(\lambda) + C(\lambda)$ is invertible for $\lambda \ll 0$. Defining $\mathbf{h}(\lambda) = -\big(M(\lambda) + C(\lambda)\big)^{-1}\mathbf{b}(\lambda)$ and $h = (1, \mathbf{h}^T)^T$ we observe that λ is a sufficiently negative eigen-value for T if the following expression vanishes:

$$\begin{aligned}\mathcal{E}(\lambda) &:= \Big\langle \big(\Lambda_1(\lambda) + \Lambda_0(\lambda)\big)h(\lambda), h(\lambda)\Big\rangle_{L^2(\Gamma_i)} \\ &= m_0(\lambda) + a(\lambda) - \mathbf{b}(\lambda)^T\big(M(\lambda) + C(\lambda)\big)^{-1}\mathbf{b}(\lambda).\end{aligned}$$

Thanks to Lemma 4.8 both Λ_j are Herglotz in quadratic form, and analytic as operator-valued functions; in particular any sub-block is analytic. Furthermore Λ_0 arises from a problem on a bounded domain with regular boundary conditions. Therefore we see that $a(\lambda)$, $\mathbf{b}(\lambda)$ and $C(\lambda)$ lack poles when λ is sufficiently negative or non-real. Moreover for the same λ the coefficient $a(\lambda)$ is never 0, $\mathbf{b}(\lambda)$ is not identically the zero vector—though it could have some zero entries—and $C(\lambda)$ is never null. Moreover the Herglotz property of $\Lambda_1 + \Lambda_0$ ensures that $\mathrm{Im}\big(\mathcal{E}(\lambda)\big)\mathrm{Im}(\lambda) \geq 0$.

The invertibility of $M(\lambda) + C(\lambda)$ for non-real λ follows from Lemma 4.9, and is enough to ensure analyticity of $\mathcal{E}$ away from $\mathbb{R}$, and establish that $\mathcal{E}$ is, like $\Lambda_1 + \Lambda_0$, Herglotz. Now, for $\lambda \ll 0$ we know $M(\lambda)$ is of fixed sign, so we see that the poles of $\mathcal{E}$ and m_0 are identical; by the interlacing of the poles and zeros of Herglotz functions[4.8], we have shown that between any two sufficiently negative poles of m_0 there is a zero of $\mathcal{E}$ and hence at most one eigen-value of T. The upper bound on the difference between the eigen-value-counting functions for T and L' follows immediately. □

We are now in a position to prove our full-spectral-data uniqueness theorem.

[4.8] See the final remarks of Appendix A.

Theorem 4.5. *Let Ω be specially decomposable, Ω' a sub-domain of Ω, q_1 and q_2 admissible potentials satisfying $q_1 = q_2$ in $\Omega/\backslash\Omega'$, f_1 and f_2 admissible boundary functions for Berry–Dennis boundary conditions supported in the straight edge Γ_1, and $\beta_1, \beta_2 \in \mathbb{R}$ self-adjoint boundary conditions at 0. Consider the full-spectrum Dirichlet-to-Neumann operators $\Lambda_j(\lambda) : h \mapsto -\partial_\nu u_j$ ($h \in H^{1/2}(\Gamma)$, $\lambda \in \mathbb{R}$, $j = 1, 2$) defined by*

$$\begin{cases} (-\Delta + q_j)u_j(\cdot\,;\lambda) &= \lambda u_j(\cdot\,;\lambda) & \text{in } \Omega, \\ u_j - f_j\partial_\nu u_j &= 0 & \text{on } \Gamma_c, \\ [u_j, u_0 + \beta v_0](0) &= 0, & \\ u_j &= h & \text{on } \Gamma. \end{cases}$$

If $\Lambda_1 = \Lambda_2$ then $q_1 = q_2$, $f_1 = f_2$ and $\beta_1 = \beta_2$.

Proof. Denote by Λ the equal maps $\Lambda_1 = \Lambda_2$. Since we know the behaviour of Λ on the real line, we know where its poles lie, *ergo* we know where the eigenvalues λ_n of T are. In particular, by Theorem 4.4 we may deduce that, as $n \to -\infty$, we have

$$-\frac{2n\pi}{\log(-\lambda_n)} \to \varepsilon_1^{-1} = \varepsilon_2^{-1} =: \varepsilon^{-1},$$

which determines $f_1 = f_2$ completely on Γ_1, where the latter are supported. The constant ϑ_0 is fixed and may in principle be calculated (see Lemma 4.7), so in turn we may calculate from Theorem 4.4 that, as $n \to -\infty$,

$$-\frac{\log(-\lambda_n) + 2\varepsilon(n\pi + \vartheta_0)}{2\varepsilon} \to \tan^{-1}(\beta_1) = \tan^{-1}(\beta_2) =: \tan^{-1}(\beta).$$

Finally, we apply Theorem 4.2 to the triples (q_1, f, β) and (q_2, f, β), with equality of $\Lambda_1(\lambda)$ with $\Lambda_2(\lambda)$ for any fixed $\lambda \in \mathbb{R}$, to deduce that $q_1 = q_2$ in Ω. □

5

Further work

'I was born not knowing and have only had a little time to change that here and there.'

– Richard P. Feynman[5.1]

The work carried out above has explored some curious and interconnected issues. We highlight these in this final chapter, and simultaneously discuss ideas for directions in which the work could be continued.

Firstly, for one-dimensional pencils, how does the choice of underlying Hilbert space affect the boundary data we are allowed? In Chapter 3 we considered the differential equation

$$-\frac{1}{r}\Big(ru'(r;\lambda)\Big)' + q(r)u(r;\lambda) = \lambda w(r)u(r;\lambda),$$

and observed that over the Hilbert space $L^2(0,1;r\mathrm{d}r)$ the dimension of its solution space can change when λ is varied. This is in juxtaposition to the more standard

[5.1] *Perfectly Reasonable Deviations from the Beaten Track: The Letters of Richard P. Feynman*, Michelle Feynman (2005).

choice of Hilbert space with weight $rw(r)\mathrm{d}r$, over which the solution space dimension is invariant in λ. The concepts can easily be generalised to more arbitrary operators T_0 and T_1 over some Hilbert space, and one could consider applying the techniques of boundary triples[5.2] to determine more abstract formulations of boundary data for pencils of the form $T_0 + \lambda T_1$.

Secondly, the assumptions in Chapter 3 that both q and w must be locally bounded except near 0 seem somewhat artificial, though they do at least capture the essence of the singularity at 0. The author hopes that they could be lifted to at least $L^1_{\mathrm{loc}}(0,1]$, as this is the standard space for coefficients of Sturm–Liouville problems on $(0,1)$, regular at 1 and singular at 0.

Thirdly, the new interpolation theorem in Section 3.3 uses a technique specific to that particular classical limit-circle case to establish that the m-function has an appropriate Laplace transform representation for it to be interpolatable by the Rybkin–Tuan method. One would hope that such a specific m-function is not the only limit-circle candidate for such interpolation. Indeed, the author believes that the A-amplitude and boundary control methods would likely yield stronger interpolation results, if explored more deeply.

In Chapter 4 we considered a half-disc domain embedded at the boundary of some general domain. Moreover the Berry–Dennis Dirichlet-point boundary condition was linear and the Schrödinger potential was radial in a neighbourhood of the singularity. These three conditions have an artificial flavour, and were imposed only to make the given calculations possible. The author believes that the conditions can be relaxed. Perhaps by a perturbation-type argument, the Berry–Dennis condition should require at most a simple zero at the singular point and the potential could be simply continuous near the point. Moreover by taking a new coordinate system in a neighbourhood of the point one should, without too

[5.2] See, for example, [130, 42, 25] and their references for an introduction to these ideas.

much difficulty, be able to adapt the techniques of that chapter to a Dirichlet-point on a general C^1 part of the boundary.

Taking more Dirichlet-points could complicate matters, as the negative eigenvalues arising from each would become interspersed, rendering the eigen-value asymptotics argument from Theorem 4.4 useless as part of a proof of full uniqueness. However one could in principle consider the situation of a variable Dirichlet-point, and request that its location on the inaccessible boundary also be determined by the Dirichlet-to-Neumann measurements.

In proving the cloaking result of Chapter 4 we made use of the relatively recent techniques in the paper [67]. This was to keep the exposition simpler in this work, however in principle one ought to be able to improve the strength of the partial uniqueness of the potential by adapting the more recent methods in [68].

The full-frequency uniqueness result, Theorem 4.5, takes as input data the operator-valued function $\Lambda(\lambda)$, considered over $\lambda \in \mathbb{R}$. This probably exceeds necessity. If we do a simple variables count, we find that q is a function of two variables, f a function of one, and β a "function" of zero variables, since it is just a real number. For each fixed $\lambda \in \mathbb{R}$ (provided we avoid the Dirichlet spectrum, where Λ is singular) the operator $\Lambda(\lambda)$ has a Schwartz kernel[5.3] which is a function of two variables. Clearly its value for one choice of λ is insufficient to recover (q, f, β), but, at least in principle, one should be able to prove uniqueness from knowing merely $\Lambda \restriction_{\{\lambda_1,\lambda_2\}}$ for fixed $\lambda_1 \neq \lambda_2$; indeed, it should be overdetermined, as is Calderón's problem in three or higher dimensions.

In the entirety of this thesis, we have paid little attention to the question of stability—of course, it comes into play when considering numerical examples, such as that described in Section 4.4 and Appendix D—and none whatsoever to

[5.3] See, e.g., [124, Sec. 0.2].

the issue of existence of solutions to the inverse problems. Needless to say, these are not trivial issues, and a more complete solution of these inverse problems should include results regarding them. The author hopes this will be achieved in due course.

Finally, there is also the question of the physical significance of our version of the Dirichlet-point. Starting with admissible Ω, f, q and β, the set-up defining the operator T in (4.1.6) is given by

$$\begin{cases} (-\Delta + q)v(\cdot\,;\lambda) &= \lambda v(\cdot\,;\lambda) & \text{in } \Omega, \\ v(\cdot\,;\lambda) - f\partial_\nu v(\cdot\,;\lambda) &= 0 & \text{on } \Gamma_c, \\ [v(\cdot\,;\lambda), u_0 + \beta v_0] &= 0 & \text{at } \mathbf{x}_0, \\ v(\cdot\,;\lambda) &= 0 & \text{on } \Gamma. \end{cases}$$

Formally taking the inverse of the transformation (1.2.1), we write $q = \gamma^{-1/2}\Delta\gamma^{1/2}$, $u = \gamma^{-1/2}v$ and $g = \gamma^{1/2} - f\partial_\nu\gamma^{1/2}$, and find that

$$\begin{cases} -\nabla\cdot\big(\gamma\nabla u(\cdot\,;\lambda)\big) &= \lambda\gamma u(\cdot\,;\lambda) & \text{in } \Omega, \\ gu(\cdot\,;\lambda) - f\gamma^{1/2}\partial_\nu u(\cdot\,;\lambda) &= 0 & \text{on } \Gamma_c, \\ [\gamma^{1/2}u(\cdot\,;\lambda), u_0 + \beta v_0] &= 0 & \text{at } \mathbf{x}_0, \\ u &= 0 & \text{on } \Gamma. \end{cases}$$

A priori, there is no reason to believe g need not have simple zeros of Berry–Dennis type, although the so-called "Neumann-points"[5.4] are not problematic from the perspective of self-adjointness [21]. Assuming it is well behaved everywhere, then, we have the equivalent Berry–Dennis boundary condition $u - \tilde{f}\partial_\nu u$. Physically, this is a prescription of a relationship on Γ_c between the current and voltage, which decays (asymptotically linearly) at one point to a condition solely on the voltage. We are not aware of any physical examples of this sort of be-

[5.4] These are where the Robin condition degenerates to a Neumann condition.

haviour, but this could simply be because physicists and engineers have not yet looked for them. The author's opinion is that discovery of such examples is certainly a worthwhile endeavour, not only from his desire for mathematical completeness but also for the novel material properties that they might embody.

In conclusion, we hope firstly to have elucidated for the reader some interesting mathematics and described clearly for them some novel techniques in analysis. We also hope that in the last few paragraphs we have impressed on the reader that these ideas are but the first timid steps into another area of mathematics, with the potential for both physical application and to be a bountiful source of exciting and fascinating problems. Of course[5.5], if we have seen further, it is by standing on the shoulders of giants. Yet, whilst we but played on this grand sea-shore, we had and still have lying before us the great undiscovered ocean of truth.

[5.5] To paraphrase Newton.

Appendices

'I am a brain, Watson. The rest of me is a mere appendix.'

– Sir Arthur Conan Doyle[5.6]

A Herglotz functions

The classical theory of analytic functions in the upper half-plane $\mathbb{C}^+ = \{x + iy \mid x \in \mathbb{R}, y > 0\}$ with non-negative imaginary part has been ascribed to various people, notably Nevanlinna, Pick, and Herglotz. We follow a tradition in mathematical physics—to which the author feels a strong connection—in referring to such functions by the latter name, and intend no insult to the others who worked on them. In this appendix we collect some useful facts concerning these functions. The proofs can be found scattered in various works, though two detailed references on the topic are due to Gesztesy and Tsekanovskii [57], and (more classically) Aronszajn and Donoghue [8]. In what follows we refer more directly to [57].

Herglotz functions are usually extended by symmetry to $\mathbb{C}^- = \overline{\mathbb{C}^+}$, i.e., if h is a Herglotz function then its extension is defined by

$$h(z) = \overline{h(\overline{z})} \quad (z \in \mathbb{C}^-). \tag{A.1}$$

[5.6] Sherlock Holmes, in *The Adventure of the Mazarin Stone*, Strand Magazine (1921).

We note the following two results, which are both crucial yet trivial.

Proposition A1. *Let h_1 and h_2 be Herglotz functions. Then $h_1 + h_2$ is Herglotz, and if $h_2(\mathbb{C}^+) \subseteq \mathbb{C}^+$, then $h_1 \circ h_2$ is also Herglotz.*

Corollary A1. *As a consequence, for any complex a_{11}, a_{12}, a_{21} and a_{22}, defining $a = (a_{ij})_{i,j=1}^2$ and satisfying $a^* J a = J$ where $J = (0, -1; 1, 0)$, we have*

$$h_a(\lambda) := \frac{a_{21} + a_{22} h(\lambda)}{a_{11} + a_{12} h(\lambda)} \quad (\lambda \in \mathbb{C}^+)$$

defines a Herglotz function whenever h is Herglotz.

Remark. *In particular, a can be taken to be*

$$\begin{pmatrix} \sin(\alpha) & \cos(\alpha) \\ \cos(\alpha) & -\sin(\alpha) \end{pmatrix}.$$

Arguably the most important fact about Herglotz functions is their Lebesgue–Stieltjes integral representation:

Theorem A1. *If h is Herglotz then there is a real-valued Borel measure ρ on $\mathbb{R}$ satisfying the growth condition $\int_{-\infty}^{\infty} (1 + t^2)^{-1} \mathrm{d}\rho(t) < \infty$ such that*

$$h(\lambda) = A + B\lambda + \int_{-\infty}^{\infty} \left(\frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) \mathrm{d}\rho(t) \quad (\lambda \in \mathbb{C}^+), \tag{A.2}$$

and is absolutely convergent as an integral. The constants are given by

$$A = \mathrm{Re}\big(h(i)\big),$$
$$B = \lim_{r \to +\infty} \frac{h(ir)}{ir} \geq 0.$$

Conversely, any function h specified on $\mathbb{C}^+$ by $A \in \mathbb{R}$, $B \geq 0$ and an admissible measure ρ via the formula (A.2) is a Herglotz function.

Proof. See the references for [57, Thm. 2.2(iii)]. □

Remark. *The term* $\frac{t}{1+t^2}$ *simply increases the decay rate in the integrand. If* $\int_{-\infty}^{\infty} \frac{\mathrm{d}\rho(t)}{1+|t|} < \infty$ *then the formula collapses to*

$$h(\lambda) = \tilde{A} + B\lambda + \int_{-\infty}^{\infty} \frac{\mathrm{d}\rho(t)}{t-\lambda} \quad (\lambda \in \mathbb{C}^+),$$

where $\tilde{A} = A - \int_{-\infty}^{\infty} \frac{t\mathrm{d}\rho(t)}{1+t^2}$.

Part of the statement in Theorem A1 may be written in a constructive form:

Theorem A2. *The measure* ρ *may be calculated from its Herglotz function* h*:*

$$\rho([x,y]) + \rho((x,y)) = \frac{1}{i\pi} \lim_{\varepsilon \searrow 0} \int_x^y \Big(h(s+i\varepsilon) - h(s-i\varepsilon)\Big)\mathrm{d}s.$$

Proof. See the references for [57, Thm. 2.2(iv)]. □

Corollary A2. *If* ρ *is continuous in an interval* $I \subset \mathbb{R}$*, then for all* $x, y \in I$ *with* $x < y$ *we have, more explicitly,*

$$\rho(y) - \rho(x) = \frac{1}{i\pi} \lim_{\varepsilon \searrow 0} \int_x^y \Big(h(s+i\varepsilon) - h(s-i\varepsilon)\Big)\mathrm{d}s. \tag{A.3}$$

Remark. *We are thinking of* ρ *in two ways, both as a measure and as a real-valued function of bounded variation. In the former sense we interpret the integral in the manner of Lebesgue, whilst we view the integral in the latter sense as being of Riemann–Stieltjes type.*

Considering a closed and densely-defined self-adjoint operator[5.7] T on a Hilbert space H and $\lambda \in \mathbb{C}^+ \subset \mathbb{C}\backslash\sigma(T)$, for any $u \in H$ the form $h(\lambda) = \langle (T-\lambda)^{-1}u, u\rangle_H$ satisfies

$$\mathrm{Im}\Big(\langle (T-\lambda)^{-1}u, u\rangle_H\Big) = \int_{-\infty}^{\infty} \frac{\mathrm{Im}(\lambda)}{|t-\lambda|^2} \mathrm{d}\langle E_t u, u\rangle_H,$$

[5.7] See Section 2.1.

and is easily seen to be analytic in λ, making it a Herglotz function. It follows without much difficulty that $\mathrm{d}\rho(t) = \mathrm{d}\langle E_t u, u\rangle_H$ and $A - \int_{-\infty}^{\infty} \frac{t\mathrm{d}\rho(t)}{1+t^2} = 0 = B$. This links directly to Stone's formula [116, Thm. VII.3], which states

$$E_{[x,y]} + E_{(x,y)} = \lim_{\varepsilon \searrow 0} \frac{1}{i\pi} \int_x^y \Big((T - s - i\varepsilon)^{-1} - (T - s + i\varepsilon)^{-1} \Big) \mathrm{d}s,$$

where the limit is strong, i.e., in H-norm when acting on any fixed $u \in H$.

Another key result for these functions concerns when they may be analytically continued from $\mathbb{C}^+$ into $\mathbb{C}^-$:

Theorem A3. *The continuation by reflection in* (A.1) *is analytic through a real interval* (x, y) *if and only if* ρ *is not supported in* (x, y).

Proof. See [61, 57, Lem. 2.5]. □

In light of this we consider any Herglotz function h to be defined on its maximal domain. To close this appendix we list a few more important properties:

- If ρ as a function has a finite discontinuity at x and is continuous near x then h has a simple pole at x with negative residue [57, Thm. 2.2(vi)].
- If the function ρ increases continuously in the interval I then h is not defined on I and suffers a discontinuity approaching I from either side, i.e., either of $\mathbb{C}^\pm$ [57, Lem. 2.5].
- The zeros and poles of h interlace, i.e., if ρ is 0 between any two poles then in the same interval there is a zero.

The last is a simple consequence of the intermediate value theorem and noting that in any interval of continuity $h'(\lambda) = B + \int_{-\infty}^{\infty} (t - \lambda)^{-2} \mathrm{d}\rho(t) \geq 0$, i.e., h is increasing.

B Bessel functions

In this appendix we collect some necessary results on the large-n asymptotics of the eigen-values and norming constants defined in Section 3.3, as well as a result on asymptotics of the m-function, needed in the same section.

The eigen-values of the Bessel equation of zeroth order, with Dirichlet and Neumann boundary conditions at the left and right end-points respectively of $(0,1)$, are well-studied, and are algebraically equivalent to the positive zeros of the Bessel function J_1. This information is enough to determine the eigen-values λ_n for the boundary value problem (3.3.3), (3.3.4) and (3.3.7), asymptotically to order $1/n$. We calculate these first for the unperturbed equation, then use a result from [35] to move to the perturbed version.

Lemma B1. *Let $Q \in L^2(0,b)$, $h \in \mathbb{R}$ and denote by $U_p(\cdot\,;\lambda)$ the principal solution at 0 of (3.3.3),*

$$-u''(x;\lambda) + \left\{Q(x) - \frac{1}{4x^2}\right\} u(x;\lambda) = \lambda u(x;\lambda) \quad \Big(x \in (0,b)\Big),$$

i.e., U_p is non-trivial, and for all linearly independent solutions V we have $U_p(0^+) = o(V(0^+))$. When ordered by size and enumerated by $n = 1,2,3,\ldots$ the eigen-values λ_n of the above differential equation with the boundary conditions

$$\begin{cases} [u, U_p](0^+;\lambda) = 0, \\ u'(b;\lambda) = hu(b;\lambda), \end{cases}$$

satisfy the asymptotics

$$\sqrt{\lambda_n} = \left(n + \frac{1}{4}\right)\frac{\pi}{b} + O(1/n).$$

Proof. Suppose firstly that $Q \equiv 0$, and denote the corresponding eigen-values by

λ_n^0. The boundary condition at 0 allows us to choose any constant multiple of $x^{1/2}J_0(\sqrt{\lambda}x)$ as our solution. The condition at b then forces the eigen-values to be the positive zeros of

$$b\sqrt{\lambda}J_1(b\sqrt{\lambda}) + (h - 1/2)J_0(b\sqrt{\lambda}).$$

Thus, for each fixed c, we seek asymptotics for the zeros of

$$f(z) := zJ_1(z) - cJ_0(z).$$

Recall that J_0 and J_1 have only simple positive zeros [2, Sec. 9.5], and notice $f(j_{0,n}) = j_{0,n}J_1(j_{0,n})$ which alternates in sign as n is incremented because $j_{0,n}$ interlace with $j_{1,n}$. The intermediate value theorem then gives a zero $z_n \in (j_{0,n}, j_{0,n+1})$ for f, whilst the fact that J_0 and J_1 oscillate with asymptotically the same "period" [2, Sec. 9.2] means z_n is unique. Since $j_{0,n} = (n - 1/4)\pi + O(1/n)$ and $j_{1,n} = (n + 1/4)\pi + O(1/n)$ [2, Eq. 9.5.12] the positive zeros of f are

$$z_n = (n + 1/4)\pi + \varepsilon_n \sim n,$$

with the leading-order behaviour following from $|\varepsilon_n| \leq \pi/2 + O(1/n)$. We now use the asymptotic expansion [2, Eq. 9.2.1] of the first-order Bessel function $J_\mu(x) = \sqrt{(2/\pi x)}\Big(\cos(x - \mu\pi/2 - \pi/4) + O(1/x)\Big)$ $(x \to +\infty)$ to observe that

$$\begin{aligned} O(1/n) \ni \frac{c}{\sqrt{z_n}}J_0(z_n) &= \sqrt{z_n}J_1(z_n) \\ &= -\sqrt{\frac{2}{\pi}}\Big(\cos(z_n + \pi/4) + O(1/n)\Big). \end{aligned}$$

Taylor-expanding around the zeros of cosine implies $b\sqrt{\lambda_n^0} = z_n = (n + 1/4)\pi + O(1/n)$. Finally, the second equation of [35, p. 17] implies $\sqrt{\lambda_n} = \sqrt{\lambda_n^0} + O(1/n)$. □

The next lemma provides an asymptotic representation of the norming constants in Section 3.3. For its proof we will relate our notation to that of [35], then utilise some results from the same paper.

Lemma B2. *Let $Q \in L^2(0,b)$, $h \in \mathbb{R}$ and suppose $\varphi(\cdot\,;\lambda)$ solves (3.3.3) with initial conditions $\varphi(b;\lambda) = 1, \varphi'(b;\lambda) = h$. Then the norming constants $\alpha_n := \int_0^b \varphi(\cdot\,;\lambda_n)^2$ satisfy*

$$\alpha_n = b/2 + O(1/n).$$

Proof. We begin by transforming to the interval $(0,1)$, so that we may use some results from [35]: defining $\eta(x) = \varphi(bx)$ and $P(x) = b^2 Q(bx)$ $(0 < x < 1)$ we find

$$\begin{cases} -\eta''(x;\mu) + \left(P(x) - \frac{1}{4x^2}\right)\eta(x;\mu) &= \mu\eta(x;\mu) \quad \left(x \in (0,1)\right), \\ \eta(1;\mu) &= 1, \\ \eta'(1;\mu) &= bh, \end{cases} \tag{B.1}$$

where $\mu = b^2\lambda$. Then set $\mu_n = b^2\lambda_n$ and $\eta_n = \eta(\cdot\,;\mu_n)$. By checking the boundary conditions one may easily see that

$$\eta_n = y_2(\cdot\,;\mu_n)/y_2(1;\mu_n),$$

where $y_2(\cdot\,;\mu)$ is the solution of the differential equation in (B.1) satisfying the boundary condition $x^{-1/2}y_2(x;\mu) \to 1$ $(x \to 0)$. In the second-to-last equation of [35, p. 16] it is observed that, as $\rho \to +\infty$, we have

$$\int_0^1 y_2(\cdot\,;\rho^2)^2 = \frac{1}{\rho}\left[\frac{1}{2} + O\left(\frac{\log(\rho)}{\rho}\right)\right].$$

Since $\int_0^b \varphi(\cdot\,;\lambda_n)^2 = b\int_0^1 \eta_n^2$, if we set $\rho_n = \sqrt{\mu_n}$ then the lemma would follow if

$$y_2(1;\mu_n)^{-2} = \rho_n\left(1 + O(1/n)\right). \tag{B.2}$$

To justify (B.2) we appeal to [35, Lem. 3.2], which implies that

$$\left| y_2(1;\mu_n) - \sqrt{\frac{\pi}{2}} J_0(\rho_n) \right| \leq \frac{C}{\sqrt{n}} \left(e^{I(n)} - 1 \right), \tag{B.3}$$

where (using Cauchy–Schwarz for the third line)

$$\begin{aligned} 0 \leq I(n) &:= \int_0^1 \frac{x}{1+\rho_n x} \big(1 - \log(x)\big) |P(x)| \mathrm{d}x \\ &\leq \frac{1}{\rho_n} \int_0^1 \big(1 - \log(x)\big) |P(x)| \mathrm{d}x \\ &\leq \frac{1}{\rho_n} \left(\int_0^1 \big(1 - \log(x)\big)^2 \mathrm{d}x \right)^{1/2} \|P\|_{L^2(0,1)} \\ &< \frac{3\|Q\|_{L^2(0,b)}}{b\rho_n}. \end{aligned} \tag{B.4}$$

Thanks to (B.3), (B.4) and Lemma B1, we find

$$y_2(1;\mu_n) = \sqrt{\frac{\pi}{2}} J_0(\rho_n) + O(n^{-3/2}). \tag{B.5}$$

Lemma B1 shows furthermore that $\rho_n = j_{1,n} + O(1/n) = (n + 1/4)\pi + O(1/n)$ which, thanks to $J_0' = -J_1$, are asymptotically the local extrema of J_0. Hence by expanding the cosine part of [2, Eq. 9.2.1] in a first-order Taylor approximation around $n\pi$ it follows that

$$J_0(\rho_n) = \sqrt{\frac{2}{\pi\rho_n}} \left[(-1)^n + O\left(\frac{1}{n}\right) \right].$$

Upon substitution into (B.5) this yields (B.2), as desired. □

C Asymptotics for Sturm–Liouville problems

In Chapters 2 and 3 we have made free reference to various spectral asymptotic results for solutions of Sturm–Liouville problems, as well as for the resulting

Weyl–Titchmarsh m-functions. Such results are often classical, and well known to those familiar with the field. However the standard sources of their proofs can be arcane or difficult to follow. For this reason we include here the few key asymptotic results we will need, and our versions of their proofs, as well as references to more original sources.

Lemma C1 (Asymptotics in λ of a fundamental system)**.** *Let $p = w = 1$, suppose $0 < b \leq \infty$ and take a real-valued $q \in L^1_{\mathrm{loc}}[0, b)$. Then the fundamental system $\{s(\cdot\,;\lambda), c(\cdot\,;\lambda)\}$ for the one-dimensional Schrödinger equation*

$$-u''(x;\lambda) + qu(x;\lambda) = \lambda u(x;\lambda) \quad \Big(x \in (0,b)\Big), \tag{C.1}$$

defined by the initial conditions

$$\begin{cases} s(0;\lambda) &= 0, \\ s'(0;\lambda) &= 1, \end{cases} \quad \begin{cases} c(0;\lambda) &= 1, \\ c'(0;\lambda) &= 0, \end{cases}$$

satisfies the following asymptotic expansions. For any fixed $x \in (0, b)$ and $0 < \delta < \pi/2$, as $\lambda \to \infty$ in the sector $\mathcal{S}_\delta := \{\lambda \in \mathbb{C} \mid \delta < \arg(\lambda) < \pi - \delta\}$ we have

$$s(x;\lambda) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}\Big(1 + O(|\lambda|^{-1/2})\Big), \tag{C.2}$$

$$c(x;\lambda) = \cos(\sqrt{\lambda}x)\Big(1 + O(|\lambda|^{-1/2})\Big). \tag{C.3}$$

Proof. We explicitly show the calculation for $s(x;\lambda)$. By integrating by parts twice,

$$\begin{aligned} \frac{1}{\sqrt{\lambda}}\int_0^x \sin\Big(\sqrt{\lambda}(x-t)\Big)q(t)s(t;\lambda)\mathrm{d}t &= \frac{1}{\sqrt{\lambda}}\int_0^x \sin\Big(\sqrt{\lambda}(x-t)\Big)(s'' + \lambda s)(t;\lambda)\mathrm{d}t \\ &= -\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} + s(x;\lambda). \end{aligned} \tag{C.4}$$

Write $\sqrt{\lambda} = \mu + i\nu$ where $\mu, \nu > 0$ since $\lambda \in \mathcal{R}_\vartheta \subset \mathbb{C}^+$. Then, since $2i\sin(\sqrt{\lambda}x) =$

$e^{-\nu x}e^{i\mu x} - e^{\nu x}e^{-i\mu x}$, we see from (C.4) that

$$s(x;\lambda)e^{-\nu x} = \frac{e^{-2\nu x}e^{i\mu x} - e^{-i\mu x}}{2i\sqrt{\lambda}} + \int_0^x \frac{e^{-2\nu(x-t)}e^{i\mu(x-t)} - e^{-i\mu(x-t)}}{2i\sqrt{\lambda}} q(t)e^{-\nu t}s(t;\lambda)\mathrm{d}t \tag{C.5}$$

Now, on the interval $(0,x)$, (C.1) is regular and has only bounded continuous solutions[5.8]. Hence we may define $h(x;\lambda) := \sup_{t\in(0,x)} |s(t;\lambda)e^{-\nu t}|$, and we observe from (C.5) that

$$h(x;\lambda) \leq \frac{1}{|\sqrt{\lambda}|} + \frac{h(x;\lambda)}{|\sqrt{\lambda}|}\int_0^x |q|,$$

ergo, whenever $|\sqrt{\lambda}| - \int_0^x |q| \geq |\lambda|/2$ we have

$$|s(x;\lambda)| \leq h(x;\lambda) \leq \frac{1}{|\sqrt{\lambda}| - \int_0^x |q|} \leq \frac{2}{|\sqrt{\lambda}|}. \tag{C.6}$$

Finally, we may substitute this estimate into the integrand of (C.4). Since $-i\sqrt{\lambda} = \nu - i\mu$, the expressions $e^{-ix\sqrt{\lambda}}$ and $e^{ix\sqrt{\lambda}}$ are respectively exponentially growing and decaying in size as $\mathcal{S}_\delta \ni \lambda \to \infty$. Thus (C.4) is bounded by

$$2|\lambda|^{-1}\int_0^x |e^{\nu x}|.|e^{-2\nu x} - e^{-\nu t)}e^{-i2\mu(x-t)}|.|q(t)|\mathrm{d}t$$

Moreover $\sin(\sqrt{\lambda}x) = e^{\nu x}(e^{-i\mu x} - e^{-\nu x + i\mu x})/2i$, meaning that

$$\left|\frac{\sqrt{\lambda}s(x;\lambda)}{\sin(\sqrt{\lambda}x)} - 1\right| \leq \frac{4\int_0^x |q|}{|\sqrt{\lambda}|}.$$

This concludes the proof for $s(x;\lambda)$, whilst that for $c(x;\lambda)$ follows similarly. □

Remark. *These asymptotics were effectively first developed in 1958 by Titchmarsh [127, Sec. 1.7]; although his argument used a continuous potential, the exact same reasoning works with a potential from* $L^1_{\mathrm{loc}}[0,b)$. *The above calculation was adapted from that*

[5.8] See Section 2.2.

proving [92, Ch. I, Lem. 2.2].

Not only do $s(x;\cdot)$ and $c(x;\cdot)$ possess nice asymptotics, they are also entire functions. To prove this fact we will need to use the method of variation of parameters. Since this method is usually stated in general, abstract terms, we briefly note here the most general version of the result that we will require.

Proposition C2 (Variation of parameters; variation of constants)**.** *Let* $0 < b \leq \infty$, *and* $1/p, q$ *and* $f \in L^1_{\mathrm{loc}}(0,b)$. *Suppose* y_1 *and* y_2 *are linearly-independent solutions on* $(0,b)$ *of the homogeneous equation*

$$-(py_0')' + qy_0 = 0.$$

Then y *is a solution on* $(0,b)$ *of the inhomogeneous equation*

$$-(py')' + qy = f$$

if and only if there are constants c_1 *and* c_2 *such that*

$$y(x) = c_1y_1(x) + c_2y_2(x) - y_1(x)\int_0^x \frac{fy_2}{W(y_1,y_2)} + y_2(x)\int_0^x \frac{fy_1}{W(y_1,y_2)} \quad \Big(x \in (0,b)\Big),$$

where $W(y_1,y_2) = y_1py_2' - y_2py_1'$ *is the Wronskian.*

Lemma C2. *The fundamental system* $s(x;\lambda), c(x;\lambda)$ $\Big(x \in (0,b), \lambda \in \mathbb{C}\Big)$ *defined in Lemma C1 consists of functions that, for each fixed* $x \in (0,b)$, *are entire in* λ.

Proof. We present our version of the proof in [140, Thm. 3.8] for λ-differentiability. It is equivalent to show that *any* solution $y(\cdot\,;\lambda)$ of (C.1) with λ-independent initial conditions is entire in λ. Let $\lambda, \lambda_0 \in \mathbb{C}$ be not equal, and observe that, on $(0,b)$,

$$-y''(\cdot\,;\lambda) + qy(\cdot\,;\lambda) = (\lambda - \lambda_0)y(\cdot\,;\lambda) + \lambda_0 y(\cdot\,;\lambda).$$

Since $\{s(\cdot\,;\lambda_0), c(\cdot\,;\lambda_0)\}$ are a fundamental system, there are constants α_1 and α_2 such that $y(\cdot\,;\lambda_0) = \alpha_1 c(\cdot\,;\lambda_0) + \alpha_2 s(\cdot\,;\lambda_0)$, where $y(0;\lambda_0) = y(0;\lambda)$ and $y'(0;\lambda_0) = y'(0;\lambda)$ satisfy the same λ-independent initial conditions. Hence variation of parameters—Proposition C2—implies we may use the same fundamental system to write, for every $x \in (0, b)$,

$$y(x;\lambda) = y(x;\lambda_0) + (\lambda - \lambda_0)\int_0^x \Big(s(x;\lambda_0)c(t;\lambda_0) - s(t;\lambda_0)c(x;\lambda_0)\Big)y(t;\lambda)\mathrm{d}t, \qquad \text{(C.7)}$$

after noting that the Wronskian has constant value

$$c(0;\lambda_0)s'(0;\lambda_0) - c'(0;\lambda_0)s(0;\lambda_0) = 1.$$

Now $s(\cdot\,;\lambda)$ and $c(\cdot\,;\lambda)$ satisfy the unique continuation theorems for initial-value problems, meaning they are continuous for every fixed λ. This means that (C.7) shows that for each $x \in (0, b)$ the function $y(x;\cdot)$ is Lipschitz continuous at λ_0, implying that the same equation then proves $y(x;\cdot)$ is differentiable at λ_0. □

The next lemma is due to Everitt [49, Thm. p. 445], and provides one with leading-order asymptotics of the m-function for our particular Sturm–Liouville equation.

Lemma C3 (Asymptotics of an m-function; Everitt, 1972)**.** *Let $q \in L^1_{\mathrm{loc}}[0, b)$ be real-valued, and consider the differential equation*

$$-u''(x;\lambda) + q(x)u(x;\lambda) = \lambda u(x;\lambda) \quad \Big(x \in (0, b)\Big) \qquad \text{(C.8)}$$

If it is in limit-circle or regular at b, prescribe any self-adjoint boundary condition there. Then the Neumann m-function[5.9] $m(\lambda)$ associated with this equation (and possible bound-

[5.9] This is defined in (2.2.5) by $\alpha = \pi/2$.

ary condition) satisfies, as $\mathcal{S}_\delta \ni \lambda \to \infty$,

$$m(\lambda) = \frac{i}{\sqrt{\lambda}} + O(|\lambda|^{-1}), \tag{C.9}$$

where $\mathcal{S}_\delta$ *is the sector* $\{z \in \mathbb{C} \mid \delta < \arg(z) < \pi/\delta\}$ *for a given* $0 < \delta < \pi/2$.

Remark. *The first asymptotic formula of this type was due to Marčenko [101, Thm. 2.2.1], with remainder term* $o(|\lambda|^{-1/2})$. *Everitt and Halvorsen extended Lemma C3 [51]. If one imposes stronger regularity upon* q *then one may derive much more detailed asymptotics in place of the O-term in (C.9). For example, Atkinson [10] was able to show, by constructing the non-linear Riccati equation for* m, *that if* q *is absolutely continuous then*

$$m(\lambda) = \frac{i}{\sqrt{\lambda}} + \frac{iq(0)}{2\lambda^{3/2}} + o\left(\frac{1}{|\lambda|\mathrm{Im}(\sqrt{\lambda})}\right)$$

as $\mathrm{Re}(\sqrt{\lambda})$ *and* $\mathrm{Im}(\sqrt{\lambda})|\lambda|^{-1/2} \to \infty$. *Later Harris [65] was able to improve the remainder into a power sum in* $\lambda^{-1/2}$ *whose coefficients are polynomially dependent on* $q(0), q'(0), \ldots, q^{(k)}(0)$ *whenever* q *is* k*-times continuously differentiable in a neighbourhood of the origin. Danielyan and Levitan [40] slightly improved the remainder term of this representation using a Tauberian-type argument.*

To prove this we will first need the following cruder result. It was originally provided by Hille [66, Thm. 10.2.1], though his proof restricted one to the regular case with $q \in L^1(0, b)$; Everitt refined the proof [49, Lem. p. 447 & Sec. 12] extending the result to accommodate $q \in L^1_{\mathrm{loc}}[0, b)$. We present our version of Everitt's proof.

Lemma C4 (Hille, 1969; Everitt 1972)**.** *The Neumann m-function* $m(\lambda)$ *described in Lemma C3 satisfies, uniformly as* $\mathcal{S}_\delta \ni \lambda \to \infty$,

$$m(\lambda) = O(|\lambda|^{-1/2}).$$

Proof. Write $\sqrt{\lambda} = \mu + i\nu$ with $\mu, \nu > 0$. It is well known[5.10] [92, Sec. II.2] that

$$\mathrm{Im}\Big(m(\lambda)\Big) = 2\mu\nu \int_0^b |s(x;\lambda) - m(\lambda)c(x;\lambda)|^2 \mathrm{d}x,$$

which may be re-written as

$$\left|\frac{\lambda \mathrm{Im}\Big(m(\lambda)\Big)}{2\mu\nu}\right| = \int_0^b |\sqrt{\lambda}s(x;\lambda) - \sqrt{\lambda}m(\lambda)c(x;\lambda)|^2 \mathrm{d}x.$$

We may estimate from this that when, e.g., $\nu^2 > b^{-1}$ ($b < \infty$) or $\nu > 0$ ($b = \infty$), we have

$$\int_0^{\nu^{-1}\log(\nu)} |\sqrt{\lambda}s(x;\lambda) - \sqrt{\lambda}m(\lambda)c(x;\lambda)|^2 \mathrm{d}x \leq \left|\frac{\lambda \mathrm{Im}\Big(m(\lambda)\Big)}{2\mu\nu}\right|. \tag{C.10}$$

The reason for this choice of limit of integration will become clear shortly. Since $1 \leq |\lambda|/2\mu\nu \leq 1/\sin(\delta)$ ($\lambda \in \mathcal{S}_\delta$) we obtain that the right-hand side of this is $O\Big(|m(\lambda)|\Big)$. Note that for any c, s and m we have $\sqrt{\lambda}mc = \sqrt{\lambda}s - \sqrt{\lambda}(s - mc)$. Hence, by the triangle inequality in $L^2\Big(0, \nu^{-1}\log(\nu)\Big)$, we find

$$\begin{aligned} |m(\lambda)||\lambda|^{1/2}\left(\int_0^{\nu^{-1}\log(\nu)} |c(x;\lambda)|^2 \mathrm{d}x\right)^{1/2} - \left(\int_0^{\nu^{-1}\log(\nu)} |\sqrt{\lambda}s(x;\lambda)|^2 \mathrm{d}x\right)^{1/2} \\ \leq \left(\int_0^{\nu^{-1}\log(\nu)} |\sqrt{\lambda}s(x;\lambda) - \sqrt{\lambda}m(\lambda)c(x;\lambda)|^2 \mathrm{d}x\right)^{1/2} = O\Big(|m(\lambda)|^{1/2}\Big). \end{aligned}$$

Implementing Lemma C1 we calculate that, as $\lambda \to \infty$ in $\mathcal{S}_\delta$, we have

$$\int_0^{\nu^{-1}\log(\nu)} |c(x;\lambda)|^2 \mathrm{d}x \sim \frac{1}{4}\int_0^{\nu^{-1}\log(\nu)} e^{2\nu x} \mathrm{d}x = \frac{\nu^2 - 1}{8\nu} \sim \frac{\nu}{8}, \tag{C.11}$$

and similarly

$$\int_0^{\nu^{-1}\log(\nu)} |\sqrt{\lambda}s(x;\lambda)|^2 \mathrm{d}x \sim \frac{\nu}{8}. \tag{C.12}$$

[5.10] See Section 3.3.

Hence, as $\mathcal{S}_\delta \ni \lambda \to \infty$, we have

$$|\lambda|^{1/2}|m(\lambda)| = O(1) + O\Big(|\lambda|^{-1/4}|m(\lambda)|^{1/2}\Big) = O(1) + O\Big(|\lambda|^{1/4}|m(\lambda)|^{1/2}\Big)$$

Clearly if $m(\lambda) \notin O\Big(|\lambda|^{-1/2}\Big)$ then there is a contradiction, since the left-hand side would be growing slower than its own square root. This concludes the proof. □

We may now proceed with the proof of Lemma C3, presenting our version of the method of Everitt, which now follows easily from the previous proof.

Proof of Lemma C3. We start with (C.10) and apply Lemma C4 to observe that as $\lambda \to \infty$ in $\mathcal{S}_\delta$ we have

$$\int_0^{\nu^{-1}\log(\nu)} |i\sqrt{\lambda}s(x;\lambda) - i\sqrt{\lambda}m(\lambda)c(x;\lambda)|^2 \mathrm{d}x = O\Big(|\lambda|^{-1/2}\Big). \tag{C.13}$$

Then from the fact that for any c, s and m we have

$$(1 + i\sqrt{\lambda}m)c = (c + i\sqrt{\lambda}s) - i\sqrt{\lambda}(s - mc)$$

it is easily deduced from (C.13) and the triangle inequality in $L^2\Big(0, \nu^{-1}\log(\nu)\Big)$ that for the same limit of λ we have

$$\begin{aligned}\Bigg(\int_0^{\nu^{-1}\log(\nu)} \Big|\Big(1 - i\sqrt{\lambda}m(\lambda)\Big)c(\cdot\,;\lambda)\Big|^2\Bigg)^{1/2} \\ \leq \Bigg(\int_0^{\nu^{-1}\log(\nu)} |c(\cdot\,;\lambda) + i\sqrt{\lambda}s(\cdot\,;\lambda)|^2\Bigg)^{1/2} + O(|\lambda|^{-1/4}).\end{aligned} \tag{C.14}$$

By Lemma C1 the integral-term on the right-hand side is asymptotically equiva-

lent to

$$\begin{aligned}
&\left(\int_0^{\nu^{-1}\log(\nu)} |\cos(\sqrt{\lambda}x) + i\sin(\sqrt{\lambda}x)|^2 \mathrm{d}x\right)^{1/2} \\
&= \left(\int_0^{\nu^{-1}\log(\nu)} e^{i\sqrt{\lambda}x}\overline{e^{i\sqrt{\lambda}x}}\mathrm{d}x\right)^{1/2} \\
&= \left(\int_0^{\nu^{-1}\log(\nu)} e^{-2\nu x}\mathrm{d}x\right)^{1/2} \\
&\leq \nu^{-1/2}.
\end{aligned}$$

Moreover (C.11) tells us the left-hand side of (C.14) is asymptotically equivalent to

$$\left|1 + i\sqrt{\lambda}m(\lambda)\right| \frac{\sqrt{\nu}}{2\sqrt{2}}.$$

Therefore,

$$\left|1 + i\sqrt{\lambda}m(\lambda)\right| = O(|\lambda|^{-1/2}),$$

from which the lemma follows immediately. □

D Numerical procedures

We collect here the codes implemented for calculating approximately a pair (q_2, f_2) yielding the same Dirichlet-to-Neumann map as the pair $(q_1, f_1) = (0, 0)$, as illustrated in Figure 4.4. The problems in question are given by the following systems, for $j = 1, 2$, on the square $\Omega := [-1, 1]^2$:

$$\begin{cases}
(-\Delta + q_j)u &= 0, \\
u(\cdot\,, \pm 1) &= 0, \\
f_j\partial_\nu u(1, \cdot\,) - u(1, \cdot\,) &= 0, \\
\partial_\nu u(-1, \cdot\,) &= \Lambda_{q_j,k_j} u(-1, \cdot\,).
\end{cases}$$

The numerical computation is achieved in several steps.

(i) Fix f_2 to be the constant non-zero value $1/k$ (we choose $k = 10$ in our example).

(ii) Then calculate explicitly the Dirichlet-to-Neumann map for $(q_1, f_1) = (0, 0)$. In the one-dimensional basis $\varphi_\ell(y) = \sin(n\pi y)$ ($\ell = 1, 2, 3, \ldots$) on the edge $x = -1$, the Dirichlet-to-Neumann map $\Lambda_{0,0}$ is given by the diagonal matrix

$$\begin{pmatrix} \pi \tanh \pi & & & \\ & 2\pi \tanh 2\pi & & \\ & & 3\pi \tanh 3\pi & \\ & & & \ddots \end{pmatrix}.$$

(iii) Now, construct a functional, of possible q's, whose minimum lies at the appropriate q_2 corresponding to f_2 and Λ_{q_2,f_2}, the latter chosen to equal $\Lambda_{0,0}$. To do this, take any given q, and find the solutions u_ℓ and v_ℓ ($\ell = 1, 2, 3, \ldots$) of the following problems on the square $[-1, 1]^2$:

$$\begin{cases} (-\Delta + q)u_\ell &= 0, \\ u_\ell(\cdot\,, \pm 1) &= 0, \\ \partial_\nu u_\ell(1, \cdot\,) - k u_\ell(1, \cdot\,) &= 0, \\ u_\ell(-1, \cdot\,) &= \varphi_\ell, \end{cases} \qquad \begin{cases} (-\Delta + q)v_\ell &= 0, \\ v_\ell(\cdot\,, \pm 1) &= 0, \\ \partial_\nu v_\ell(1, \cdot\,) - k v_\ell(1, \cdot\,) &= 0, \\ \partial_\nu v_\ell(-1, \cdot\,) &= \Lambda_{0,0}\varphi_\ell. \end{cases}$$

The functional we use is formally

$$F(q) := \sum_{\ell=1}^{\infty} \int_\Omega \left\{ |u_\ell - v_\ell|^2 q + |\nabla(u_\ell - v_\ell)|^2 \right\}.$$

Clearly if the desired q_2 exists then $F(q_2) = 0 \leq F(q)$ for any admissible q, since when $q = q_2$ we have $u_\ell = v_\ell$ ($\ell \in \mathbb{N}$).

Remark. *In fact, the above series might diverge, in which case we would require a suitable weighting sequence to ensure convergence. But, since we will truncate the series, this will not be a problem for the numerics.*

(iv) Then encode q by its set of Chebyshev coefficients, i.e., the weightings c_{jk} such that

$$q(x,y) = \sum_{j,k=0}^{\infty} c_{jk} T_j(x) T_k(y),$$

where T_j is the j-th Chebyshev polynomial of the first kind:

$$T_j(x) = \cos\Big(j \arccos(x)\Big).$$

This allows us to write $F(q) = F\Big((c_{jk})_{j,k=0}^{\infty}\Big)$; after truncation we have the problem of minimising

$$F_{n,M}\Big((c_{jk})_{j,k=0}^{M}\Big) := \sum_{\ell=1}^{n} \int_{\Omega} \mathrm{d}x\mathrm{d}y \left\{ |u_\ell - v_\ell|(x,y) \sum_{j,k=0}^{M} c_{jk} T_j(x) T_k(y) + |\nabla(u_\ell - v_\ell)|^2(x,y) \right\}.$$

(v) The last major step in coding the procedure is to utilise a numerical differentiation scheme, allowing us to calculate ∇ of various functions as well as invert the differential operators associated with the above systems. We will implement the global method outlined by Trefethen [128] which comprises polynomial interpolation through the Chebyshev points then explicit differentiation of this polynomial. All functions involved will thus be described by their values on the grid of points

$$\Big(\cos(s\pi/N), \cos(t\pi/N)\Big) \quad (s,t = 0,1,2,\ldots,N), \tag{D.1}$$

and differentiation will involve multiplying by some explicitly known matrix D. The boundary conditions can then be incorporated into the differential operator by suitably tailoring certain entries of D^2. The whole scheme is

extremely efficient, exhibiting convergence no worse than finite differences and avoiding the undesirable Gibbs or Runge phenomena.

(vi) The above considerations yield a functional $F_{n,M,N}\left((c_{jk})_{j,k=0}^{M}\right)$ (whose form, for simplicity, we avoid writing explicitly here) which can be minimised over using any one of MATLAB's built-in minimisation routines. We use `fminunc`, as it avoids certain technical limitations inherent to other minimisers such as `fmincon`.

In the following pages we include the exact MATLAB code used to generate Figure 4.4. One may check that the q_2 produced gives rise to approximately the same Dirichlet-to-Neumann map—$\Lambda_{q_2,1/10} \approx \Lambda_{0,0}$, in the basis φ_ℓ, truncated to some $n \geq 1$, i.e., the entries of the matrices corresponding to either map are "close" in the L^∞ sense. Checking this involves inputting (the Chebyshev-grid version of) q_2 and f_2 into `DtoN.m`, and comparing results with the same applied to q_1 and f_1 for the same choice of Chebyshev parameter N and Λ-truncation n.

```
% MINIMISATION ROUTINE
NN = 30; % Number of Cheb. pts. to an edge, minus 1
MM = 6; % Degree of Cheb. approx., plus 1
nn = 6; % Number of rows of DN map considered
k0 = 0; % BC of first problem
kk = 10; % BC of second problem
err = 1e-5; % Desired error between iterations

[D,xx] = cheb(NN); % Cheb. diff. matrix and pts., 1D
[xg,yg] = meshgrid(xx,xx); % 2D grid of points
Q0 = zeros(NN+1,NN+1); % Potential of first problem
Q = polycheb(NN,MM-1,Cmin);   % Start-point of iteration

Cstart = polychebinv(NN,MM,Q); % Start-point of algorithm
Cstart = 0.01*round(100*Cstart); % Rounded start-point
Cstart = Cstart + 0.01*rand(MM,MM); % Perturbed start-point

Func = @(CC)FunctGener(NN,MM,nn,k0,kk,Q0,CC); % Define function
                                              % handle to minimise
CC = Cstart; % Initialise the coefficient matrix

options = optimoptions('fminunc','Algorithm','quasi-newton', ...
                       'MaxFunEvals',300*MM^2,'MaxIter',...
                       600,'TolFun',1e-10,'TolX',1e-10);

Qdiff = Q; % Initialise the inter-iteration error
Qdiffv = Qdiff(:); % Turn Q-array into list
maxQdiff = sqrt(max(Qdiffv.^2)); %  maximal value of Q-array
k = 1 % Start counter of minimisation iterations

while maxQdiff >= 1e-5 % Keep iterating the minimisation protocol
                       % until the desired error-bound is reached.
    tic
    [Cmin,Fval] = fminunc(Func,CC,options) % Perform minimisation
    toc                                    % starting from CC
    CC = Cmin; % Update the component matrix CC
    Qtally(1:NN+1,1:NN+1,k) = polycheb(NN,MM,CC); % Build Q from CC
    Qdiff = Qtally(:,:,k) - Q; % Calculate new error
    Qdiffv = Qdiff(:);
    maxQdiff = sqrt(max(Qdiffv.^2))
    Q = Qtally(:,:,k);
    k = k+1 % Increment counter
end

% Plotting
[xxx,yyy] = meshgrid(1:-.04:-1,1:-.04:-1); % Grid for plotting

QQ = interp2(xg,xg',Q,xxx,yyy,'spline'); % Interpolate Q over grid
figure(1), clf, mesh(xxx,yyy,QQ), colormap ([0 0 0]) % Plot Q
xlabel x, ylabel y, zlabel q

QQerr = interp2(xg,xg',Qdiff,xxx,yyy,'spline'); % Interpolate the
                                                % error in Q
figure(2), clf, mesh(xxx,yyy,QQerr), colormap([0 0 0]) % Plot error
xlabel x, ylabel y, zlabel q_error
```

```
function [FF] = FunctGener(NN,MM,nn,k0,kk,Q0,CM)

QQ = polycheb(NN,MM,CM); % calculate input potential from its
                         % Chebyshev expansion
[DD,xx] = cheb(NN); % Chebyshev differentiation matrix and points
Dx = kron(eye(NN+1),DD); % dee-by-dee-x
Dy = kron(DD,eye(NN+1)); % dee-by-dee-y
D2 = DD^2; % second-order derivative
D2x = kron(eye(NN+1),D2); % dee-two-by-dee-x-squared
D2y = kron(D2,eye(NN+1)); % dee-two-by-dee-y-squared
LL = D2x+D2y; % Laplacian
QQg = QQ'; % transpose potential on grid, for correct listed order
QQv = QQg(:); % list potential points
[UWM,DUWM] = DtoN(NN,nn,k0,Q0); % calculate the Dir.-to-Neu. operator
                                % for j = 1:nn
FF = 0; % initialise the functional value
for j = 1:nn
    UU = SchrDSol(NN,j,QQ,kk);
        [xg,DNPhig] = meshgrid(xx,DUWM(:,j));
        Vp = 0.25*((xg-1).^2.*DNPhig);
        Vg = Vp';
        Vv = Vg(:);
        Fv = (LL-diag(QQv))*Vv;
        Fg = reshape(Fv,NN+1,NN+1)';
        Fg = Fg(2:NN,1:NN+1)';
        Fg([1,NN+1],:) = zeros(2,NN-1);
        Fv = Fg(:);
        SS = SchrN(NN,QQ,kk);
        vv = SS\Fv;
        vg = reshape(vv,NN+1,NN-1)';
        Vh = zeros(NN+1,NN+1);
        Vh(2:NN,:) = vg;
    VV = Vh + Vp;
    UV = (UU - VV)';
    UVv = UV(:);
    DUV2v = (Dx*UVv).^2 + (Dy*UVv).^2;
    DUV2g = reshape(DUV2v,NN+1,NN+1)';
    Fg = QQ.*(UU-VV).^2 + DUV2g;
    for i = 1:NN+1
        Fxsec(i) = trapcheb(NN,Fg(:,i));
    end
    FF = FF + trapcheb(NN,Fxsec);
end
```

```
function [UU,Phi] = SchrDSol(NN,nn,QQ,kk)

%Chebyshev grid and spectral differentiation matrix
[DD,xx] = cheb(NN);                     %Cheb. 1st-order diff. matrix,
                                        %x-axis interpol. pts.
%Schrodinger operator
[xg,yg] = meshgrid(xx,xx);
% QQ = yg;
SS = SchrD(NN,QQ,kk);

%Inhomogeneity
[ff,up,Phi] = inhomgen(NN,nn,QQ,0); %builds PDE inhom. f from q, phi
fg          = ff(2:NN,1:NN)';  %ignore N, W and S edges
fg(1,:)     = zeros(1,NN-1);    %ignore E edge
fv          = fg(:);           %convert to list

%Solving the PDE
uv            = SS\fv;                %solve inh. PDE with hom. b.c.s
ug            = reshape(uv,NN,NN-1)'; %convert vector soln into grid
uu            = zeros(NN+1,NN+1);
uu(2:NN,1:NN) = ug;                   %embed ug into full grid
UU            = uu + up;              %add b.c. inhom.
```

```
function [UWM,DUWM] = DtoN(NN,nn,k0,Q0)

[DD,xx] = cheb(NN);
xg = meshgrid(xx,xx);
Dx = kron(eye(NN+1),DD);
UWM = zeros(NN+1,nn);
DUWM = zeros(NN+1,nn);
for j = 1:nn
    Ug = SchrDSol(NN,j,Q0,k0)';
    UWM(:,j) = Ug(NN+1,:)';
    Uv = Ug(:);
    DUv = Dx*Uv;
    DUg = reshape(DUv,NN+1,NN+1)';
    DUWM(:,j) = -DUg(:,NN+1);
end
end
```

```
function [S] = SchrD(N,Q,k)

D     = cheb(N);
IN    = eye(N);
e1    = IN(1,:);
INdoc = IN - diag(e1);

D2       = D^2; %Full (N+1)x(N+1) Laplacian matrix
D2y      = D2(2:N,2:N); %y-version with hom. Dir. BC at N&S
RBC      = D(1,1:N)-k*IN(1,1:N); %Robin BC at E
D2x      = D2(1:N,1:N); %x-version with hom. Dir. BC at W, no BC at E
D2x(1,:) = RBC; %Replace 1st row with Rob. BC at E

L        = kron(eye(N-1),D2x)+kron(D2y,INdoc); %Discrete Laplacian

Qg = Q(2:N,1:N);            %Ignore N, W, S for potential
Qg(1:N-1,1) = zeros(N-1,1); %Ignore E
Qg = Qg';                   %Transpose for ordering
Qv = Qg(:);                 %List

S = -L + diag(Qv); % Final form of Schrodinger operator, a square
                   % matrix with edge-length (N-1)*N
```

```
function [S] = SchrN(N,Q,k)

D      = cheb(N);
IN1    = eye(N+1);
e1     = IN1(1,:);
IN1doc = IN1 - diag(e1) - diag(flip(e1));

D2         = D^2; %Full (N+1)x(N+1) Laplacian matrix
D2y        = D2(2:N,2:N); %y-version with hom. Dir. b.c. at N&S
RBC        = D(1,:)-k*IN1(1,:); %Robin b.c. at E
D2x        = D2; %x-version
D2x(end,:) = D(end,:); %Replace last row with hom. Neu. b.c. at W
D2x(1,:)   = RBC; %Replace 1st row with Rob. BC at E

L        = kron(eye(N-1),D2x)+kron(D2y,IN1doc); %Discrete Laplacian

Qg = Q(2:N,:);            %Ignore N, S for potential
Qg(1:N-1,1) = zeros(N-1,1); %Ignore E
Qg(1:N-1,end) = zeros(N-1,1); %Ignore W
Qg = Qg';                   %Transpose for ordering
Qv = Qg(:);                 %List

S = -L + diag(Qv); % Final form of Schrodinger operator, a square
                   % matrix with edge-length (N-1)*(N+1)
```

```
function [F,Up,Phi] = inhomgen(N,n,Q,par)
%Turns an inhomogeneous Dir. b.c. into an inhomogeneity, and produces
%n-th y-direction e-fn, both for the following Schrodinger problem:
% [-1,1]^2: -Delta u = 0
% N&S:        u(x,1) = 0 = u(x,-1)
% E:     du(1,y)/dnu = u(1,y)
% W:    du(-1,y)/dnu = 0.5*n*pi*tanh(n*pi)*phi(y) = pn*phi(y)
%They are evaluated, respectively, on 2D & 1D grids of Chebyshev pts
%(Trefethen, 2000).

%Inputs: N = #{Chebyshev interpolation points per edge} - 1
%        n = enumeration of y-direction eigen-function
%        Q = (N+1)x(N+1) grid of values of potential at Cheb. pts.

%Outputs: F = (N+1)x(N+1) grid of values of the inhomogeneity
%        Up = (N+1)x(N+1) grid of values of a particular solution to
%              the b.c.s
%       Phi = (N+1)-vector of values of the y-direction e-fn

%Chebyshev grid and spectral differentiation matrix
[D,x] = cheb(N);              %Cheb. diff. matrix, x-axis interp. pts

%Eigen-function: grid and list form
Phi       = sin(n*pi*0.5*(x+1)); %y-direction e-fn.
[xg,Phig] = meshgrid(x,Phi);     %builds x-grid-pts and e-fn grid-pts

Up  = 0.25*(0.5*n*pi*tanh(n*pi))^par*(xg-1).^2.*Phig; %Particular
                      % solution to b.c.s up = -pn*(x-1)^2*Phi(y)/4
F = 0.5*(0.5*n*pi*tanh(n*pi))^par*Phig - 0.25*n^2*pi^2*Up - Q.*Up;
                                        %Inhomogeneity f=(Lap-Q)*up
```

```
function [Q] = polycheb(N,M,C)
% The function evaluates a linear combination of Chebyshev polys in
% two variables, with linear coeffs given by the array C, over an NxN
% grid of Chebyshev interpolation points. C must have edge lengths no
% greater than N+1.
for i = 1:M
    for j = 1:M
        for k = 1:N+1
            for l = 1:N+1
  COSM(k,l,i,j) = C(i,j)*cos((i-1)*(k-1)*pi/N)*cos((j-1)*(l-1)*pi/N);
            end
        end
    end
end
Q = zeros(N+1,N+1);
for i = 1:M
    for j = 1:M
        Q = Q + COSM(:,:,i,j);
    end
end
end
```

```
function [C] = polychebinv(N,M,Q)

% Q denotes the grid of values of a potential at the Chebyshev points
% on a two-dimensional, N+1xN+1 grid.
% M < N+2 is the order of polynomial approximation to Q.
% C is the MxM array of coefficients in a Chebyshev approxn to Q.
%N+1 is the number of Chebyshev points to an edge.
[D,x] = cheb(N);    %x is the vector consisting in these points.
for k = 1:N+1
    Cpar(1,k) = 0.5*onedwghtint(N,1,Q(k,:)');
    for j = 2:M
        Cpar(j,k) = onedwghtint(N,j,Q(k,:)');
    end
end
for j = 1:M
    C(1,j) = 0.5*onedwghtint(N,1,Cpar(j,:));
    for i = 2:M
        C(i,j) = onedwghtint(N,i,Cpar(j,:));
    end
end
```

```
function [I] = trapcheb(N,f)
% A simple trapezoidal quadrature rule
[D,x] = cheb(N);
for i = 1:N
    s(i) = (x(i)-x(i+1))*(f(i)+f(i+1));
end
I = 0.5*sum(s);
end
```

```
function [D,x] = cheb(N)
% Computes Chebyshev differentiation matrix D and Chebyshev
% interpolation points x
if N==0, D=0;
    x=1;
    return,
end
x = cos(pi*(0:N)*(N^(-1)))';
c = [2; ones(N-1,1); 2].*(-1).^(0:N)';
X = repmat(x,1,N+1);
dX = X-X';
D = (c*(1./c)')./(dX+(eye(N+1)));
D = D - diag(sum(D'));
```

Bibliography

[1] ABLOWITZ, M. J., AND CLARKSON, P. A. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, vol. 149 of *London Mathematical Society Lecture Note Series*. LMS, 1992.

[2] ABRAMOWITZ, M., AND STEGUN, I. *(Eds.) A handbook of mathematical functions with formulas, graphs and mathematical tables*. New York: Dover, 1972.

[3] ALESSANDRINI, G. Stable determination of conductivity by boundary measurements. *Appl. Anal. 27*, Issue 1-3 (1988), 153–172.

[4] ALESSANDRINI, G. Singular solutions of elliptic equations and the determination of conductivity by boundary measurements. *J. Differential Equations 84* (1990), 252–272.

[5] ALESSANDRINI, G. Open issues of stability for the Inverse Conductivity Problem. *J. Inv. Ill-Posed Probl. 15* (2007), 1–10.

[6] AMBARZUMIAN, V. A. Über eine Frage der Eigenwerttheorie. *Z. Physik 53* (1929), 690–695.

[7] ARENDT, W., AND TER ELST, A. F. M. The Dirichlet-to-Neumann operator on rough domains. *J. Differential Equations 251* (2011), 2100–2124.

[8] ARONSZAJN, N., AND DONOGHUE, W. F. (JR.). On exponential representations of analytic functions in the upper half-plane with positive imaginary part. *Anal. Math. 5* (1956), 321–388.

[9] ASTALA, K., AND PÄIVÄRINTA, L. Calderón's inverse conductivity problem in the plane. *Ann. Math. 163* (2006), 265–299.

[10] ATKINSON, F. V. On the location of the Weyl circles. *Proc. Roy. Soc. Edinburgh Sect. A 88*, 3-4 (1981), 345–356.

[11] AVDONIN, S. A., AND BELISHEV, M. I. Boundary control and dynamic inverse problem for nonselfadjoint Sturm–Liouville operator. *Control Cybern. 25* (1996), 429–440.

[12] AVDONIN, S. A., BELISHEV, M. I., AND IVANOV, S. A. Matrix inverse problem for the equation $u_{tt} - u_{xx} + Q(x)u = 0$. *Math. USSR Sbornik 72*, 2

(1992), 287–310. Engl. transl. from Russ., orig. publ. in *Matem. Sbornik 182*, 3 (1991).

[13] AVDONIN, S. A., AND MIKHAYLOV, V. The boundary control approach to inverse spectral theory. *Inverse Probl. 26*, 4 (2010), 045009.

[14] AVDONIN, S. A., MIKHAYLOV, V., AND RYBKIN, A. The boundary control approach to the Titchmarsh–Weyl m-function. I. The response operator and the A-amplitude. *Comm. Math. Phys. 275*, 3 (2007), 791–803.

[15] BAILEY, P. B., EVERITT, W. N., WEIDMANN, J., AND ZETTL, A. Regular approximations of singular Sturm–Liouville problems. *Results Math. 22* (1993), 3–22.

[16] BEALS, R., AND COIFMAN, R. R. Multidimensional inverse scattering and nonlinear partial differential equations. In *Pseudodifferential operators and applications*, F. Treves, Ed., vol. 43 of *Proceedings of Symposia in Pure Mathematics*. AMS, Verlag, 1985, pp. 45–70.

[17] BEALS, R., AND COIFMAN, R. R. The spectral problem for the Davey–Stewartson and Ishimori hierarchies. In *Nonlinear evolution equations: Integrability and spectral methods*. Manchester University Press, 1988, pp. 15–23.

[18] BELISHEV, M. I. Recent progress in the boundary control method. *Inverse Probl. 23* (2007), R1–R67. Topical review.

[19] BENNEWITZ, C. Spectral Theory in Hilbert space. Lecture notes from 2008. Available online at, e.g., https://www.researchgate.net/publication/44337185_Introduction_spectral_theory_in_Hilbert_space_Gilbert_Helmberg; link last checked on 07/08/2017.

[20] BENNEWITZ, C. A proof of the local Borg–Marčenko theorem. *Comm. Math. Phys. 218* (2001), 131–132.

[21] BERRY, M. V., AND DENNIS, M. R. Boundary-condition-varying circle billiards and gratings: the Dirichlet singularity. *J. Phys. A: Math. Theor. 41* (2008), 135203.

[22] BORG, G. Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte. *Acta Math. 78* (1946), 1–96.

[23] BORG, G. Uniqueness theorems in the spectral theory of $y'' + (\lambda - q)y = 0$. In *Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949* (1952), pp. 276–287.

[24] BROWN, B. M., EVANS, W. D., AND PLUM, M. Titchmarsh–Sims–Weyl theory for complex Hamiltonian systems. *P. Lond. Math. Soc. 87*, 3 (2003), 419–450.

[25] BROWN, B. M., MARLETTA, M., NABOKO, S., AND WOOD, I. Boundary triplets and m-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices. *J. Lond. Math. Soc. 77*, 2 (2008), 700–718.

[26] BROWN, B. M., MARLETTA, M., AND REYES, J. M. Uniqueness for an inverse problem in electromagnetism with partial data. *J. Differential Equations 260*, 8 (2016), 6525–6547.

[27] BROWN, B. M., MARLETTA, M., AND SYMONS, F. J. F. Uniqueness from discrete data in an inverse spectral problem for a pencil of ordinary differential operators. *J. Lond. Math. Soc. 94*, 3 (2016), 793–813.

[28] BROWN, R. M. Global uniqueness in the impedance-imaging problem for less regular conductivities. *SIAM J. Math. Anal. 27* (1996), 1049–1056.

[29] BROWN, R. M., AND TORRES, R. Uniqueness in the inverse conductivity problem for conductivities with 3/2 derivatives in $l^p, p > 2n$. *J. Fourier Anal. Appl. 9* (2003), 1049–1056.

[30] BROWN, R. M., AND UHLMANN, G. Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions. *Comm. Partial Differential Equations 22*, 5-6 (1997), 1009–1027.

[31] BRUCKNER, A. M., BRUCKNER, J. B., AND THOMSON, B. S. *Real Analysis*. Prentice Hall (Pearson), 1997.

[32] BUKHGEIM, A. L., AND UHLMANN, G. Recovering a potential from partial Cauchy data. *Comm. Partial Differential Equations 27*, 3-4 (2002), 653–668.

[33] CALDERÓN, A.-P. On an inverse boundary value problem. In *Seminar on Numerical Analysis and its Applications to Continuum Physics, Rio de Janeiro, Soc. Brasil. Mat.* (1980), pp. 65–73. Reprinted in *Comput. Appl. Math. 25*, 2-3 (2006), 133–138.

[34] CANNON, J. R., DOUGLAS, J., AND JONES, B. F. Determination of the diffusivity of an isotropic medium. *Internat. J. Engrg. Sci. 1* (1963), 453–455.

[35] CARLSON, R. A Borg–Levinson theorem for Bessel operators. *Pac. J. Math. 177* (1997), 1–26.

[36] CARO, P., AND ROGERS, K. M. Global uniqueness for the Calderón problem with Lipschitz conductivities. *Forum Math. Pi 4* (2016), e2.

[37] CODDINGTON, E., AND LEVINSON, N. *Theory of ordinary differential equations*. McGraw-Hill, 1955.

[38] CONWAY, J. B. *Functions of one complex variable, vol. I*. Springer-Verlag, 1978.

[39] CRAIG, I. J. D., AND BROWN, J. C. *Inverse problems in astronomy: a guide to inversion strategies for remotely sensed data*. Adam Hilger, Ltd., 1986.

[40] DANIELYAN, A. A., AND LEVITAN, B. M. On the asymptotic behavior of the Weyl–Titchmarsh *m*-function. *Math. USSR Izv. 36*, 3 (1991), 487–496. Engl. transl. from Russ., orig. publ. in *Izv. Akad. Nauk SSSR 54*, 3 (1990).

[41] DAVIES, E. B. An inverse spectral theorem. arxiv.org/abs/1010.0094 (2010); link last checked on 07/08/2017.

[42] DERKACH, V., HASSI, S., MALAMUD, M., AND DE SNOO, H. Boundary relations and their Weyl families. *Trans. Amer. Math. Soc. 358*, 12 (2006), 5351–5400.

[43] DEVANEY, A. J. Acoustic tomography. In *Inverse Problems of Acoustic and Elastic Waves*, F. Santosa, Ed., Proceedings in Applied Mathematics Series. SIAM, 1984.

[44] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.13 of 2016-09-16. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.; link last checked on 07/08/2017.

[45] EASTHAM, M. S. P. *The asymptotic solution of linear differential systems: Applications of the Levinson theorem*. Clarendon Press: LMS Monographs, 1989.

[46] ECKHARDT, J., AND KOSTENKO, A. Quadratic operator pencils associated with the conservative Camassa–Holm flow. *Bull. Soc. Math. Fr. 145*, 1 (2017), 47–95.

[47] EDMUNDS, D., AND EVANS, W. *Spectral theory and differential operators*. Clarendon Press, Oxford, 1987.

[48] ENGL, H. W., HANKE, M., AND NEUBAUER, A. *Regularization of inverse problems*. Kluwer Academic, Dordrecht, 1996.

[49] EVERITT, W. N. On a property of the *m*-coefficient of a second-order linear differential equation. *J. Lond. Math. Soc. s2-4*, 4 (1972), 443–457.

[50] EVERITT, W. N. A personal history of the *m*-coefficient. *J. Comput. Appl. Math. 171*, 1–2 (2004), 185–197.

[51] EVERITT, W. N., AND HALVORSEN, S. G. On the asymptotic form of the Titchmarsh–Weyl *m*-coefficient. *Applicable Anal. 8* (1978), 153–169.

[52] FADDEEV, L. D. Growing solutions of the Schrödinger equation. *Dokl. Akad. Nauk. SSSR 165* (1965), 514–517.

[53] FULTON, C. T. Parametrizations of Titchmarsh's $m(\lambda)$-Functions in the Limit Circle Case. *Trans. Amer. Math. Soc. 229* (1977), 51–63.

[54] GEL'FAND, I. M., AND LEVITAN, B. M. On the determination of a differential equation from its spectral function. *Amer. Math. Soc. Transl. Ser. 2 1* (1955), 253–304. Engl. transl. from Russ., orig. publ. in *Izv. Akad. Nauk SSSR 15*, (1951), 309–360.

[55] GESZTESY, F., AND SIMON, B. A new approach to inverse spectral theory, II. General real potentials and the connection to the spectral measure. *Ann. Math. 152* (2000), 593–643.

[56] GESZTESY, F., AND SIMON, B. On local Borg–Marchenko uniqueness results. *Comm. Math. Phys. 211* (2000), 273–287.

[57] GESZTESY, F., AND TSEKANOVSKII, E. On matrix-valued Herglotz functions. arXiv preprint availabe at arxiv.org/abs/funct-an/9712004; link last checked on 07/08/2017.

[58] GLAZMAN, I. M. On an application of the method of decomposition to multidimensional singular boundary problems. *Mat. Sb. (N.S.) 35(77)*, 2 (1954), 231–246.

[59] GREENLEAF, A., KURYLEV, B., LASSAS, M., AND UHLMANN, G. Invisibility and Inverse Problems. *Bull. Amer. Math. Soc. 46* (2009), 55–97.

[60] GREENLEAF, A., LASSAS, M., AND UHLMANN, G. The Calderón problem for conormal potentials: I Global uniqueness and reconstruction. *Comm. Pure Appl. Math. 56*, 3 (2003), 328–352.

[61] GREENSTEIN, D. S. On the analytic continuation of functions which map the upper half plane into itself. *J. Math. Anal. Appl. 1* (1960), 355–362.

[62] HABERMAN, B. Uniqueness in Calderón's problem for conductivities with unbounded gradient. *Comm. Math. Phys. 340*, 2 (2015), 639–659.

[63] HABERMAN, B., AND TATARU, D. Uniqueness in Calderón's problem with Lipschitz conductivities. *Duke Math. J. 162*, 3 (2013), 497–516.

[64] HADAMARD, J. Sur les problèmes aux dérivées partielles et leur signification physique. *Princeton University Bulletin 13* (1902), 49–52.

[65] HARRIS, B. J. The asymptotic form of the Titchmarsh–Weyl m-function. *J. Lond. Math. Soc. s2-30*, 1 (1984), 110–118.

[66] HILLE, E. *Lectures on ordinary differential equations*. Addison–Wesley Publishing Company, London, 1969.

[67] IMANUVILOV, O. YU., UHLMANN, G., AND YAMAMOTO, M. The Calderón problem with partial data in two dimensions. *J. Am. Math. Soc. 23*, 3 (2010), 655–691.

[68] IMANUVILOV, O. YU., UHLMANN, G., AND YAMAMOTO, M. The Neumann-to-Dirichlet map in two dimensions. *Adv. Math. 281* (2015), 578–593.

[69] IMANUVILOV, O. YU., AND YAMAMOTO, M. Inverse boundary value problem for Schrödinger equation in two dimensions. *SIAM J. Math. Anal. 44*, 3 (2012), 1333–1339.

[70] IMANUVILOV, O. YU., AND YAMAMOTO, M. Inverse problem by Cauchy data on an arbitrary sub-boundary for systems of elliptic equations. *Inverse Probl. 28*, 9 (2012), 095015.

[71] ISAKOV, V. On uniqueness of recovery of a discontinuous conductivity coefficient. *Comm. Pure Appl. Math 41* (1988), 865–877.

[72] ISAKOV, V. On uniqueness in the inverse conductivity problem with local data. *Inverse Probl. Imaging 1*, 1 (2006), 95–105.

[73] JAULENT, M., AND JEAN, C. The inverse *s*-wave scattering problem for a class of potentials depending on energy. *Comm. Math. Phys 28* (1972), 177–220.

[74] KAC, M. Can one hear the shape of a drum? *Am. Math. Mon. 73* (1966), 1–23.

[75] KATO, T. *Perturbation Theory for Linear Operators*, second ed. Springer-Verlag, Berlin Heidelberg, 1995. first publ. in 1980, vol. 132 in Grundlehren der mathematischen Wissenschaften.

[76] KELLER, J. B. Inverse Problems. *Am. Math. Mon. 83*, 2 (1976), 107–118.

[77] KENIG, C. E., SJÖSTRAND, J., AND UHLMANN, G. The Calderón problem with partial data. *Ann. Math. 165* (2007), 567–591.

[78] KODAIRA, K. The eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of S-matrices. *Amer. J. Math. 71* (1949), 921–945.

[79] KOHN, R., AND VOGELIUS, M. Determining conductivity by boundary measurements. *Comm. Pure Appl. Math. 37* (1984), 113–123.

[80] KOHN, R., AND VOGELIUS, M. Determining conductivity by boundary measurements II. Interior results. *Comm. Pure Appl. Math. 38* (1985), 643–667.

[81] KRALL, A. M., AND ZETTL, A. Singular self-adjoint Sturm–Liouville problems. *Differential Integral Equations 1*, 4 (1988), 423–432.

[82] KREĬN, M. G. Solution of the inverse Sturm–Liouville problem. *Doklady Akad. Nauk SSSR 76* (1951), 21–24.

[83] KREĬN, M. G. On the transfer function of a one-dimensional boundary problem of second order. *Doklady Akad. Nauk SSSR 88* (1953), 405–408.

[84] KURATA, K. A unique continuation theorem for the Schrödinger equation with singular magnetic field. *Proc. Am. Math. Soc. 125*, 3 (1997), 853–860.

[85] KURYLEV, YA., AND LASSAS, M. Gel'fand inverse problem for a quadratic operator pencil. *J. Funct. Anal. 176* (2000), 247–263.

[86] LAGRANGE, J. L. Leçons élémentaires sur les mathématiques données a l'école normale (1795). In *Œuvres de Lagrange*, J. Serret, Ed., vol. 7. Paris, France: Gauthier-Villars, 1877, pp. 183–287.

[87] LANGER, R. E. An inverse problem in differential equations. *Bull. Am. Math. Soc. 39* (1933), 814–820.

[88] LASSAS, M. The inverse boundary spectral problem for a hyperbolic equation with first order perturbation. *Appl. Anal. 70*, 3-4 (1998), 219–231.

[89] LEVINSON, N. The asymptotic nature of solutions of linear systems of differential equations. *Duke Math. J. 15*, 1 (1948), 111–126.

[90] LEVINSON, N. The inverse Sturm–Liouville problem. *Mat. Tidsskr. B* (1949), 25–30.

[91] LEVITAN, B. M. *Inverse Sturm–Liouville problems*. VNU Science Press, Utrecht, NLD., 1987. translated by O. Efimov.

[92] LEVITAN, B. M., AND SARGSJAN, I. S. *Introduction to spectral theory, Transl. Math. Monographs*, vol. 39. AMS, Providence, RI, 1975.

[93] LIONHEART, W. R. B. EIT reconstruction algorithms: pitfalls, challenges and recent developments. *Physiol. Meas. 25*, 1 (2004), 125–142.

[94] LITTLEJOHN, L. L., AND KRALL, A. M. Orthogonal polynomials and singular Sturm–Liouville systems. *Rocky Mountain J. Math. 16* (1986), 435–479.

[95] LÜTZEN, J. Sturm and Liouville's work on ordinary linear differential equations. The emergence of Sturm–Liouville theory. *Arch. Hist. Exact Sci. 29*, 4 (1984), 309–376.

[96] MACDONALD, J., LOCKWOOD, J. R., MCFEE, J., ALTSHULER, T., BROACH, T., CARIN, L., HARMON, R., RAPPAPORT, C., SCOTT, W., AND WEAVER, R. *Alternatives for landmine detection*. RAND, 2003.

[97] MAIRE, E., AND WITHERS, P. J. Quantitative X-ray tomography. *Int. Mater. Rev. 59*, 1 (2014), 1–43.

[98] MARKUSHEVICH, A. I. *Theory of functions of a complex variable*, vol. 2. Prentice-Hall, 1965. translated and edited by R. A. Silverman.

[99] MARLETTA, M., AND ROZENBLUM, G. A Laplace operator with boundary conditions singular at one point. *J. Phys. A: Math. Theor. 42* (2009), 125204.

[100] MARČENKO, V. A. Certain problems in the theory of second-order differential operators. *Doklady Akad. Nauk SSSR 72* (1950), 457–460.

[101] MARČENKO, V. A. Some questions in the theory of one-dimensional linear differential operators of the second order, I. *Trudy Moskov. Mat. Obšč. 1* (1952), 327–420. Eng. transl.: *Amer. Math. Soc. Transl. Ser. 2*, 101:1–104, 1973.

[102] MEIJERING, E. A chronology of interpolation: from ancient astronomy to modern signal and image processing. *Proc. IEEE 90*, 3 (2002), 319–342.

[103] MÖLLER, M., AND PIVOVARCHIK, V. Applications of Quadratic Operator Pencils. In *Spectral Theory of Operator Pencils, Hermite–Biehler Functions, and their Applications*. Springer International Publishing, 2015, pp. 33–67.

[104] MOROZOV, V. A. *Methods for solving incorrectly posed problems*. Springer-Verlag, New York, 1984.

[105] MUELLER, J. L., AND SILTANEN, S. *Linear and Nonlinear Inverse Problems with Practical Applications*. Computational Science and Engineering. SIAM, 2012.

[106] MURRAY-SMITH, D. J. The inverse simulation approach: a focused review of methods and applications. *Math. Comput. Simulat. 53* (2000), 239–247.

[107] NACHMAN, A. Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. Math. 143* (1996), 71–96.

[108] NACHMAN, A., SYLVESTER, J., AND UHLMANN, G. An n-dimensional Borg–Levinson theorem. *Comm. Math. Phys. 115* (1988), 593–605.

[109] NAIMARK, M. A. *Linear Differential Operators: II*. New York, Ungar, 1968.

[110] NEVANLINNA, R. Über beschränkte Funktionen, die in gegebenen Punkten vorgeschriebene Werte annehmen. *Ann. Acad. Sci. Fenn., Ser. A 13*, 1 (1919).

[111] NEVANLINNA, R. Über beschränkte analytische Funktionen. *Ann. Acad. Sci. Fenn., Ser. A 32*, 7 (1929).

[112] NIESSEN, H.-D., AND ZETTL, A. Singular Sturm–Liouville problems: the Friedrichs extension and comparison of eigenvalues. *P. Lond. Math. Soc. (3) 64* (1992), 545–578.

[113] PÄIVÄRINTA, L., PANCHENKO, A., AND UHLMANN, G. Complex geometrical optics for Lipschitz conductivities. *Rev. Mat. Iberoam. 19* (2003), 57–72.

[114] PICK, G. Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden. *Math. Ann. 77* (1916), 7–23.

[115] REED, M., AND SIMON, B. *Methods of Modern Mathematical Physics*, vol. II: Fourier Analysis, Self-Adjointness. Academic Press, 1975.

[116] REED, M., AND SIMON, B. *Methods of Modern Mathematical Physics*, vol. I: Functional Analysis. Academic Press, 1980. Revised and enlarged edition.

[117] RUDIN, W. *Real and complex analysis*. McGraw-Hill, 1966.

[118] RYBKIN, A., AND TUAN, V. K. A new interpolation formula for the Titchmarsh–Weyl m-function. *P. Am. Math. Soc. 137*, 12 (2009), 4177–4185.

[119] SCHMIDT, K. M. Spectral theory of ordinary differential operators. Lecture notes from 2011. Available online at https://maths-magic.ac.uk/admin/send/course_file.php?id=6327; link last checked on 07/08/2017.

[120] SIMON, B. A new approach to inverse spectral theory, I. Fundamental formalism. *Ann. Math. (2) 150*, 3 (1999), 1029–1057.

[121] STRAY, A. Minimal interpolation by Blaschke products. *J. Lond. Math. Soc. s2-32*, 3 (1985), 488–496.

[122] SYLVESTER, J., AND UHLMANN, G. A uniqueness theorem for an inverse boundary value problem in electrical prospection. *Comm. Pure Appl. Math. 39*, 1 (1986), 91–112.

[123] SYLVESTER, J., AND UHLMANN, G. A global uniqueness theorem for an inverse boundary value problem. *Ann. Math. 125* (1987), 153–169.

[124] TAYLOR, M. E. *Pseudodifferential operators and nonlinear PDE*. Birkhäuser Boston, 1991.

[125] TIKHONOV, A. N. On the uniqueness of the solution of the problem of electromagnetic sounding. *Doklady Akad. Nauk SSSR 69*, 6 (1949), 797–800.

[126] TIKHONOV, A. N., AND ARSENIN, V. YA. *Solution of ill-posed problems*. Wiley, New York, 1977.

[127] TITCHMARSH, E. C. *Eigenfunction expansions associated with second order differential equations: Part I*. Oxford University Press, 1962. 2nd ed.; 1st ed. publ. in 1958.

[128] TREFETHEN, L. N. *Spectral methods in* MATLAB. SIAM, Philadelphia, 2000.

[129] UHLMANN, G. Electrical impedance tomography and Calderón's problem. *Inverse Probl. 25*, 12 (2009), 123011.

[130] VISHIK, M. I. On general boundary problems for elliptic differential equations. *Am. Math. Soc., Transl. II Ser. 24* (1963), 107–172.

[131] VON NEUMANN, J. Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren. *Math. Ann. 102* (1929), 49–131.

[132] WARING, E. Problems concerning interpolation. *Phil. Trans. 69* (1779), 59–67.

[133] WEIDMANN, J. *Spectral Theory of Ordinary Differential Operators*. Lecture Notes in Mathematics. Springer-Verlag Berlin Heidelberg, 1987.

[134] WEYL, H. Über gewöhnliche lineare Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen. *Nachr. Acad. Wiss. Göttingen. Math. Phys. Kl.* (1909), 37–64.

[135] WEYL, H. Über gewöhnliche Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen. *Nachr. Acad. Wiss. Göttingen. Math. Phys. Kl.* (1910), 442–467.

[136] WEYL, H. Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Functionen. *Math. Annal. 68* (1910), 220–269.

[137] WLOKA, J. *Partial differential equations*. Cambridge University Press, 1987. Engl. transl. from German of *Partielle Differentialgleichungen*, orig. publ. by B.G. Teubner, Stuttgart, 1982.

[138] WUNSCH, C. *The Ocean Circulation Inverse Problem*. Cambridge University Press, 1996.

[139] YURKO, V. A. An inverse problem for differential operator pencils. *Sb. Math. 191* (2000), 1561–1586.

[140] ZETTL, A. Sturm–Liouville problems. In *Spectral theory and computational methods of Sturm–Liouville problems (Knoxville, TN, 1996)*, vol. 191 of *Lecture Notes in Pure and Appl. Math.* Dekker, New York, 1997, pp. 1–104.

[141] ZIDAROV, D. P. *Inverse Gravimetric Problems in Geoprospecting and Geodesy*. Developments in Solid Earth Geophysics, 19. Elsevier Science Ltd., 1990.