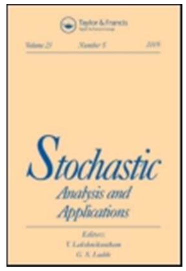

Isotropic random fields with infinite divisible marginal distributions

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Isotropic random fields with infinitely divisible marginal distributions

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ABSTRACT

A simple but efficient approach is proposed in this paper to construct the isotropic random field in $\mathbb{R}^d$ ($d \geq 2$), whose univariate marginal distributions may be taken as any infinitely divisible distribution with finite variance. The three building blocks in our building tool box are a second-order Lévy process on the real line, a d-variate random vector uniformly distributed on the unit sphere, and a positive random variable that generates a Pólya-type function. The approach extends readily to the multivariate case and results in a vector random field in $\mathbb{R}^d$ with isotropic direct covariance functions and with any specified infinitely divisible marginal distributions. A characterization of the turning bands simulation feature is also derived for the covariance matrix function of a Gaussian or elliptically contoured random field that is isotropic and mean square continuous in $\mathbb{R}^d$.

KEYWORDS

Covariance matrix function; cross covariance; direct covariance; elliptically contoured random field; Gaussian random field; infinitely divisible; Lévy process; Pólya-type function; turning bands method

MATHEMATICS SUBJECT CLASSIFICATION
60G60; 60G51; 60E07

1 Introduction

Spatial or spatio-temporal data are frequently modeled as realizations of random fields in spatial statistics [11], [14], [15], [26], a fundamental characterization of which would be the underlying finite-dimensional distributions. When a random field is assumed to be of second-order, its correlation structure is often of crucial importance. The Gaussian random field model is among the most popular choices, mostly due to the fact that its correlation structure is one of the richest structures, in the sense that any positive definite function could be employed as its covariance function. Including the Gaussian one as a special case, the set of second-order elliptically contoured random fields is one of the largest, if not the largest, sets that allow any positive definite function to be a covariance function [36]. On the other hand, non-Gaussian models are called for and are encountered in various natural and applied science fields, such as agriculture, astronomy, economics, environment, finance, geophysics, hydrology, and other areas [29]. Occasionally, a positive definite function is adopted in the literature as the covariance function of a non-Gaussian or non-elliptically-contoured random field, as is pointed out in [44], without awareness that positive definiteness is a necessary condition for the covariance function of a second-order random field to be satisfied but whether it is sufficient must be checked on a case-by-case basis. The primary objectives of this paper are to connect a subset of isotropic positive definite functions in $\mathbb{R}^d$ to a class of non-Gaussian or non-elliptically-contoured random fields in $\mathbb{R}^d$ ($d \geq 2$), and to propose a simple but efficient approach to construct isotropic random fields in $\mathbb{R}^d$, whose univariate marginal distributions could be an arbitrary infinitely divisible distribution, and whose covariance functions are of the form (4) below that is an important particular case of (1) or (3).

Given an even and continuous function $C(x)$ on $\mathbb{R}$, $C(\|\mathbf{x}\|)$ is a positive definite function in $\mathbb{R}^d$ ($d \geq 2$) if and only if it possesses an integral representation [47]

$$C(\|\mathbf{x}\|) = \int_0^\infty \Omega_d(\|\mathbf{x}\|u) dF(u), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{1}$$

where $F(u)$ is an increasing and bounded function on $[0, \infty)$, $\|\mathbf{x}\|$ is the usual Euclidean norm of $\mathbf{x} \in \mathbb{R}^d$,

$$\Omega_d(\omega) = \Gamma\left(\frac{d}{2}\right)\left(\frac{2}{\omega}\right)^{\frac{d}{2}-1} J_{\frac{d}{2}-1}(\omega), \quad \omega \in \mathbb{R}, \tag{2}$$

and $J_\nu(x)$ stands for a Bessel function of order ν. In other words, $C(\|\mathbf{x}\|)$ is the covariance function of an isotropic Gaussian or elliptically contoured random field in $\mathbb{R}^d$ if and only if (1) holds [35], [41], [54], [55]. The so-called turning bands method was introduced by Matheron [43] to simulate an isotropic random field in $\mathbb{R}^d$, after observing that (1) is equivalent to

$$C(\|\mathbf{x}\|) = \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \int_0^1 C_1(\|\mathbf{x}\|u)(1-u^2)^{\frac{d-3}{2}} du, \qquad \mathbf{x} \in \mathbb{R}^d, \tag{3}$$

where $C_1(x)$ is a positive definite function on $\mathbb{R}$. The mapping $C_1 \to C(\|\cdot\|)$ is one-to-one from the set of positive definite functions on $\mathbb{R}$ onto that in $\mathbb{R}^d$, in that, for every isotropic positive definite function $C(\|\mathbf{x}\|)$ in $\mathbb{R}^d$, there exists a unique positive definite function $C_1(x)$ on $\mathbb{R}$ such that (3) holds. In Section 3 we restrict our attention to the cases that $C_1(x)$ are Pólya-type functions.

Section 3 proposes a simple but efficient approach to generate isotropic random fields in $\mathbb{R}^d$, whose marginal distributions could be an arbitrary infinitely divisible distribution, and whose covariance functions take the form

$$C(\|\mathbf{x}\|) = \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \int_0^1 g(\|\mathbf{x}\|u)(1-u^2)^{\frac{d-3}{2}} du, \qquad \mathbf{x} \in \mathbb{R}^d, \tag{4}$$

where $g(x)$ is a Pólya-type function on $\mathbb{R}$ that is even, continuous, and nonnegative, and is convex on $(0, \infty)$, with $g(0) = 1$ and $\lim_{x\to\infty} g(x) = 0$. A Pólya-type function $g(x)$ can be treated either as the covariance function of a stationary Gaussian or elliptically contoured stochastic process on $\mathbb{R}$ by Bochner's theorem [21], or as the characteristic function of an absolutely continuous distribution function [12], [21], [34], whose density function is an even function and is continuous everywhere except possibly at the point $x = 0$. Properties of Gaussian processes with Pólya-type covariance function are studied in [7] and [31]. Probabilistic constructions are given in [21] for time series reformulations of Pólya's theorem on characteristic functions, with the marginal distributions of the process to be any infinitely divisible distribution with finite variance. A class of stationary Gaussian or elliptically contoured vector stochastic processes on $\mathbb{R}$ is formulated by [17], with Pólya-type direct and cross covariance functions. Another class of stationary vector stochastic processes on $\mathbb{R}$ is constructed by [23], whose marginal distributions are infinitely divisible distributions with finite variance, and whose direct covariance functions are of Pólya type. A stationary stochastic process on $\mathbb{R}$ is built in [39], of which the covariance function is of Pólya-type and the marginal distributions may be taken as any infinitely divisible distribution with finite variance, just like those in [21]. A stationary random field in $\mathbb{R}^d$ or $\mathbb{Z}^d$ is constructed and characterized by [24] that can take any (univariate) infinitely divisible distribution with finite variance and has the covariance function expressed as a product of Pólya-type functions. See also [33], which constructs classes of homogeneous random fields on $\mathbb{R}^3$ that take values in linear spaces of tensors of a fixed rank and are isotropic with respect to a fixed orthogonal representation of the group of orthogonal matrices.

A vector random field in $\mathbb{R}^d$ is built in Section 4, whose marginal distributions could be any infinitely divisible distribution with finite variance, and whose direct covariances are of the form (4). For the covariance matrix function of a Gaussian or elliptically contoured vector random field in $\mathbb{R}^d$, we also obtain a matrix version of (3), which may be regarded as a turning bands method to simulate the vector random field just as Matheron [43] did. Some preliminary results are given in Section 2, and all the proofs are presented in Section 5.

2 Preliminary results

This section recalls some basic properties of second-order Lévy processes, uniform distribution on the unit sphere $\mathbb{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d, \|\mathbf{x}\| = 1\}$, and Pólya-type functions, which constitute the three building blocks in our tool box in Sections 3 and 4. For Lévy processes and infinitely divisible distributions we refer the reader to [1], [9], [46], and [50], to [19] and [20] for the uniformly distributed random vector on $\mathbb{S}^{d-1}$, and to [21] and [34] for the Pólya-type function on $\mathbb{R}$.

An m-variate Lévy process $\{\mathbf{Y}(x), x \in \mathbb{R}\}$ is a real stochastic process that possesses the following properties:

(i) $P(\mathbf{Y}(0) = \mathbf{0}) = 1$,

(ii) (independent increments) $\mathbf{Y}(x_2) - \mathbf{Y}(x_1), \ldots, \mathbf{Y}(x_n) - \mathbf{Y}(x_{n-1})$ are independent for every positive integer n and any $x_1 < x_2 < \ldots < x_n$,

(iii) (stationary increments) for any $x_1 < x_2$, $\mathbf{Y}(x_2) - \mathbf{Y}(x_1)$ and $\mathbf{Y}(x_2 - x_1)$ have the same distribution,

(iv) it is stochastically continuous.

The distribution of $\mathbf{Y}(x)$ is infinitely divisible, for each $x \in \mathbb{R}$. It may or may not have first or second-order moments. When second-order moment exists, a general form of the covariance matrix function is given in Lemma 1 for a second-order Lévy process.

Lemma 1. *If* $\{\mathbf{Y}(x), x \in \mathbb{R}\}$ *is an* m*-variate second-order Lévy process, then its covariance matrix function is of the form*

$$\mathrm{cov}(\mathbf{Y}(x_1), \mathbf{Y}(x_2)) = \left(\frac{|x_1| + |x_2|}{2} - \frac{|x_1 - x_2|}{2} \right) \boldsymbol{\Sigma}, \qquad x_1, x_2 \in \mathbb{R}, \tag{5}$$

where $\boldsymbol{\Sigma} = \mathrm{cov}(\mathbf{Y}(1), \mathbf{Y}(1))$ *is a positive definite matrix.*

A d-variate random vector $\mathbf{V} = (V_1, \ldots, V_d)'$ uniformly distributed on $\mathbb{S}^{d-1}$ has a stochastic representation [19], [20],

$$\mathbf{V} = \left(\frac{W_1}{\left(\sum_{k=1}^{d} W_k^2 \right)^{\frac{1}{2}}}, \ldots, \frac{W_d}{\left(\sum_{k=1}^{d} W_k^2 \right)^{\frac{1}{2}}} \right)', \tag{6}$$

where $W_1, \ldots, W_d$ are independent standard normal random variables. The joint density of $(V_1, \ldots, V_k)'$ is

$$\begin{cases} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-k}{2}\right)\pi^{\frac{k}{2}}} \left(1 - \sum_{i=1}^{k} v_i^2 \right)^{\frac{d-k}{2}-1}, & \sum_{i=1}^{k} v_i^2 < 1, \\ 0, & \text{otherwise}, \end{cases}$$

for $1 \le k < d$. In particular, the density function of V_1 is given by

$$f_{V_1}(v) = \begin{cases} \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \left(1 - v^2\right)^{\frac{d-3}{2}}, & |v| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The function $\Omega_d(\|\boldsymbol{\omega}\|)$ appearing in (2) is nothing but $\mathbf{V}$'s characteristic function, namely,

$$\Omega_d(\|\boldsymbol{\omega}\|) = \mathrm{E} \exp(\imath \boldsymbol{\omega}' \mathbf{V}) = \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} \exp(\imath \boldsymbol{\omega}' \mathbf{v}) d\mathbf{v}, \qquad \boldsymbol{\omega} \in \mathbb{R}^d, \tag{7}$$

where $\omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$ is the surface area of the unit sphere $\mathbb{S}^{d-1}$ and $\imath$ is the imaginary unit. If $F(u)$ in (1) is assumed to be a cumulative distribution function of a nonnegative random variable, U_0, say, that is independent of $\mathbf{V}$, then (1) becomes

$$C(\|\mathbf{x}\|) = \mathrm{E} \exp(\imath \mathbf{x}' \mathbf{V} U_0), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{8}$$

and (3) is identical to

$$C(\|\mathbf{x}\|) = \mathrm{E}\exp(\imath\|\mathbf{x}\|V_1U_0), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{9}$$

since $\mathbf{x}'\mathbf{V}$ and $\|\mathbf{x}\|V_1$ have the same distribution, according to Theorem 2.4 of [19].

Note that a Pólya-type function $g(x)$ on $\mathbb{R}$ can be represented in the form

$$g(x) = \int_0^\infty \left(1 - \frac{|x|}{u}\right)_+ dF_U(u), \qquad x \in \mathbb{R}, \tag{10}$$

where $x_+ = \max(x, 0)$, and $F_U(u)$ is the distribution function of a positive random variable U with $P(U \leq 0) = 0$. See, e.g., Theorem 4.3.3 of [34]. For this reason [39] we say that the random variable U generates the Pólya-type function $g(x)$. If $g(x)$ is given, then the density or distribution function of U could be obtained, as shown in Lemmas 2 and 3, by solving the equation (10) when $g(x)$ possesses a second-order derivative on $(0, \infty)$ or $F_U(u)$ is continuous.

Lemma 2. *If $g(x)$ is a Pólya-type function on $\mathbb{R}$ and is twice differentiable on $(0, \infty)$, then it can be generated by a positive random variable U having a density*

$$f_U(x) = \begin{cases} xg''(x), & x > 0, \\ 0, & x \leq 0. \end{cases} \tag{11}$$

Lemma 3. *If the distribution function $F_U(u)$ is continuous, then the Pólya-type function $g(x)$ is continuously differentiable on $(0, \infty)$ and $F_U(u)$ can be recovered from $g(x)$ with*

$$F_U(x) = \begin{cases} 1 + xg'(x) - g(x), & x > 0, \\ 0, & x \leq 0. \end{cases} \tag{12}$$

Let V_0 be a random variable with density function $\frac{1}{2\pi}\left(\frac{\sin(v/2)}{v/2}\right)^2$, $v \in \mathbb{R}$, and be independent of U that generates $g(x)$. Then (10) can be alternatively written as

$$g(x) = \mathrm{E}\exp\left(\imath\frac{V_0}{U}x\right), \qquad x \in \mathbb{R}, \tag{13}$$

by noticing that the characteristic function of V_0 is $(1 - |x|)_+$. Furthermore, $g(x)$ may be interpreted as the covariance function of the following stationary process,

$$Z(x) = \cos\left(\frac{V_0}{U}x + \Theta\right), \qquad x \in \mathbb{R},$$

where Θ is a random variable uniformly distributed on $[0, 2\pi]$ and independent of (U, V_0). As a result, (4) is a special case of (1) or (3).

3 Isotropic random fields with infinitely divisible marginal distributions

This section introduces a new class of isotropic random fields in $\mathbb{R}^d$ ($d \geq 2$) with infinitely divisible marginal distributions and their covariance functions take the form (4). Our three building blocks are a second-order Lévy process on $\mathbb{R}$, a random variable U that generates a Pólya-type function on $\mathbb{R}$, and a random vector $\mathbf{V}$ uniformly distributed on $\mathbb{S}^{d-1}$.

Theorem 1. *Suppose that $g(x)$ is a Pólya-type function on $\mathbb{R}$ and U is a positive random variable generating $g(x)$, and that $\mathbf{V}$ is a d-variate random vector uniformly distributed on $\mathbb{S}^{d-1}$. If $\{Y(x), x \in \mathbb{R}\}$ is a second-order Lévy process with covariance*

$$\mathrm{cov}(Y(x_1), Y(x_2)) = \frac{|x_1| + |x_2|}{2} - \frac{|x_1 - x_2|}{2}, \qquad x_1, x_2 \in \mathbb{R},$$

and $\{Y(x), x \in \mathbb{R}\}$, U, and $\mathbf{V}$ are independent of each other, then

$$Z(\mathbf{x}) = Y\left(\frac{\mathbf{x}'\mathbf{V}}{U} + 1\right) - Y\left(\frac{\mathbf{x}'\mathbf{V}}{U}\right), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{14}$$

is an isotropic random field. Moreover,

(i) *for each fixed $\mathbf{x} \in \mathbb{R}^d$, $Z(\mathbf{x})$ follows the same infinitely divisible distribution as $Y(1)$;*

(ii) *the covariance function of $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\}$ is*

$$\begin{aligned} \mathrm{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) &= \mathrm{E}g(\|\mathbf{x}_1 - \mathbf{x}_2\| V_1) \\ &= \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \int_0^1 g(\|\mathbf{x}_1 - \mathbf{x}_2\| u)(1-u^2)^{\frac{d-3}{2}} du, \end{aligned} \tag{15}$$

or, equivalently,

$$\mathrm{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) = \int_0^\infty \Omega_d(\|\mathbf{x}_1 - \mathbf{x}_2\| u) dF(u), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d, \tag{16}$$

where

$$F(u) = \begin{cases} 2P\left(\frac{V_0}{U} \leq u\right), & u \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

and V_0 is a random variable with characteristic function $(1-|x|)_+$ and is independent of U;

(iii) *The spectral distribution function of* (14) *is identical to the distribution function of a d-variate isotropic random vector* $\frac{V_0}{U}\mathbf{V}$*, where* V_0 *is a random variable with characteristic function* $(1-|x|)_+$ *and is independent of* U *and* $\mathbf{V}$*; and, whenever it exists, the density function of* $\frac{V_0}{U}\mathbf{V}$ *is the spectral density function of* (14).

As a remark, the covariance function of the isotropic random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\}$ gets the random variable V_0U^{-1} involved through either the characteristic function $g(x)$ as in (15) or half its distribution function $F(u)$ as in (16). Interestingly, the isotropic covariance function (4) or (15) enjoys a Pólya-type property similar to that of $g(x)$, as is described below.

Corollary 1. *The isotropic covariance function (15) possesses the following properties:*

(i) $C(\|\mathbf{x}\|)$ *is a decreasing function of* $\|\mathbf{x}\|$*, and takes nonnegative values;*

(ii) $C(\|\mathbf{x}\|)$ *is a convex function in* $\mathbb{R}^d$*, in the sense that the inequality*

$$C(\|\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2\|) \le \lambda C(\|\mathbf{x}_1\|) + (1-\lambda)C(\|\mathbf{x}_2\|), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d,$$

holds for every $\lambda \in [0,1]$.

Indeed, these properties follow directly from (15), in view of the fact that $g(x)$ is decreasing, nonnegative, and convex on $[0,\infty)$, and $\|\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2\|) \le \lambda\|\mathbf{x}_1\| + (1-\lambda)\|\mathbf{x}_2\|$. The condition $\lim_{x\to\infty} g(x) = 0$ in a Pólya-type function may be substituted [12], [21] by $\lim_{x\to\infty} g(x) = a_0 \ge 0$. Theorem 1 can be modified appropriately to cover this case as well.

The set of second-order Lévy processes is rich. Familiar special cases include Brownian motion, Poisson process, negative binomial process, gamma process, inverse Gaussian process, normal inverse Gaussian process [3], inverse Gamma process [25], variance Gamma process [40], [22], and second-order Student process [30].

To simulate the isotropic random field (14), we need the simulators of the second-order Lévy process $\{Y(x), x \in \mathbb{R}\}$, the uniformly distributed random vector $\mathbf{V}$ on $\mathbb{S}^{d-1}$, and the random variable U that generates $g(x)$. Methods for simulating Lévy processes are available in the literature [48], [49], [53]. Simulation of $\mathbf{V}$ can be made through that of d independent standard normal random variables and the stochastic representation (6). What remains is to simulate the random variable U, which would be simple if its distribution is known. For a given $g(x)$, the distribution function of U may be found by solving the equation (10), to which solutions are given in Lemmas 2 and 3. The former deals with the scenario when $g(x)$ possesses a second-order derivative on $(0,\infty)$, while the latter requires only $g(x)$ to be continuously differentiable on $(0,\infty)$.

Corollary 2. *In the particular case $d = 3$, the isotropic covariance function (15) reduces to*

$$C(\|\mathbf{x}_1 - \mathbf{x}_2\|) = \frac{\int_0^{\|\mathbf{x}_1-\mathbf{x}_2\|} g(u)du}{\|\mathbf{x}_1 - \mathbf{x}_2\|}, \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3, \tag{17}$$

and, conversely, $g(x)$ can be recovered from (17),

$$g(x) = \frac{d}{dx}(xC(x)), \qquad x > 0.$$

Examples of Pólya-type functions are given next, with which we will explain how to find the associated generator U and the resulting isotropic covariance functions.

Example 1. For $\tau \geq 1$, $g(x) = (1 - |x|)_+^\tau$ is a Pólya-type function on $\mathbb{R}$ generated from a beta random variable U, and (17) becomes

$$\text{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) = \frac{1 - (1 - \|\mathbf{x}_1 - \mathbf{x}_2\|)_+^{\tau+1}}{(\tau + 1)\|\mathbf{x}_1 - \mathbf{x}_2\|}, \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3,$$

which is an isotropic and compactly supported covariance function in $\mathbb{R}^3$. In (17) taking another Pólya-type function

$$g(x) = \begin{cases} e^{-\alpha|x|}, & |x| \leq 1, \\ e^{-\alpha}(2 - |x|), & 1 < |x| \leq 2, \\ 0, & |x| > 2, \end{cases}$$

where $\alpha > 0$, we obtain an isotropic and power-law decaying covariance function

$$\text{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) = \begin{cases} \frac{1-\exp(-\alpha\|\mathbf{x}_1-\mathbf{x}_2\|)}{\alpha\|\mathbf{x}_1-\mathbf{x}_2\|}, & \|\mathbf{x}_1 - \mathbf{x}_2\| \leq 1, \\ \left(2 - \frac{\|\mathbf{x}_1-\mathbf{x}_2\|}{2}\right) e^{-\alpha} + \left(\frac{1}{\alpha} - \left(\frac{3}{2} + \frac{1}{\alpha}\right) e^{-\alpha}\right) \frac{1}{\|\mathbf{x}_1-\mathbf{x}_2\|}, & 1 < \|\mathbf{x}_1 - \mathbf{x}_2\| \leq 2, \\ \frac{1}{\alpha\|\mathbf{x}_1-\mathbf{x}_2\|} + \left(\frac{1}{2} - \frac{1}{\alpha}\right) \frac{e^{-\alpha}}{\|\mathbf{x}_1-\mathbf{x}_2\|}, & \|\mathbf{x}_1 - \mathbf{x}_2\| > 2, \end{cases}$$

for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$.

Example 2. For a positive constant α, $g(x) = \exp(-\alpha|x|)$ is a Pólya-type function on $\mathbb{R}$, which serves as the covariance function of the Ornstein-Uhlenbeck process. From (11) we obtain that U is a Gamma random variable with density

$$f_U(x) = \begin{cases} \alpha^2 x \exp(-\alpha x), & x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Using (14) one may build many non-Gaussian random fields of Ornstein-Uhlenbeck type [4]. We next derive the covariance function of the resulting isotropic random field.

Noticing that (see, for example, formula (3.621.5) of [28])

$$\int_0^1 u^k (1-u^2)^{\frac{d-3}{2}} du = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}{2\Gamma\left(\frac{k+d}{2}\right)}, \quad k \in \mathbb{N}, \tag{18}$$

the isotropic covariance function (15) becomes

$$\begin{aligned} C(\|\mathbf{x}\|) &= \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \int_0^1 \exp(-\|\mathbf{x}\|u)\left(1-u^2\right)^{\frac{d-3}{2}} du \\ &= \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \sum_{k=0}^{\infty} \frac{(-\|\mathbf{x}\|)^k}{k!} \int_0^1 u^k (1-u^2)^{\frac{d-3}{2}} du \\ &= \Gamma\left(\frac{d}{2}\right) \pi^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(\frac{k+1}{2}\right)}{k!\Gamma\left(\frac{k+d}{2}\right)} \|\mathbf{x}\|^k, \qquad \mathbf{x} \in \mathbb{R}^d. \end{aligned}$$

For an odd d, $d = 2d_0 + 1$ ($d_0 \in \mathbb{N}$), say, $C(\|\mathbf{x}\|)$ can be expressed as a linear combination of Mittag-Leffler functions $E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^n}{\Gamma(\alpha k+\beta)}$, $x \in \mathbb{R}$, for $\alpha > 0$ and $\beta > 0$. To see this, we write $C(\|\mathbf{x}\|)$ as

$$C(\|\mathbf{x}\|) = \Gamma\left(\frac{d}{2}\right) \pi^{-\frac{1}{2}} 2^{\frac{d-1}{2}} \sum_{k=0}^{\infty} \frac{(-\|\mathbf{x}\|)^k}{(k+d-2)(k+d-4)\cdots(k+1)k!},$$

by virtue of the following property

$$\Gamma\left(\frac{k+d}{2}\right) = 2^{-\frac{d-1}{2}} (k+d-2)(k+d-4)\cdots(k+1)\Gamma\left(\frac{k+1}{2}\right),$$

if d is odd.

Note that $\frac{1}{(k+d-2)(k+d-4)\cdots(k+1)k!}$ can be expressed as a linear combination of $\frac{1}{(k+2d_0-1)!}$, $\frac{1}{(k+2d_0-2)!}$, ..., $\frac{1}{(k+d_0)!}$. In other words,

$$\frac{1}{(k+d-2)(k+d-4)\cdots(k+1)k!} = \sum_{j=0}^{d_0-1} \frac{a_j}{(k+2d_0-1-j)!} \tag{19}$$

for some constants a_j that can be fully determined and their values are pertinent to d. The proof of (19) is deferred to Subsection 5.4. As a result, the covariance function becomes

$$C(\|\mathbf{x}\|) = \Gamma\left(\frac{d}{2}\right) \pi^{-\frac{1}{2}} 2^{\frac{d-1}{2}} \sum_{j=0}^{(d-3)/2} a_j E_{1,d-1-j}(-\|\mathbf{x}\|).$$

In particular, when $d = 3$,

$$C(\|\mathbf{x}\|) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \|\mathbf{x}\|^k = E_{1,2}(-\|\mathbf{x}\|), \qquad \mathbf{x} \in \mathbb{R}^3,$$

and, when $d = 5$,

$$\begin{aligned} C(\|\mathbf{x}\|) &= 3 \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+3)(k+1)k!} \|\mathbf{x}\|^k, \\ &= 3 \sum_{k=0}^{\infty} (-\|\mathbf{x}\|)^k \left(\frac{1}{(k+2)!} - \frac{1}{(k+3)!} \right) \\ &= 3 \left(E_{1,3}(-\|\mathbf{x}\|) - E_{1,4}(-\|\mathbf{x}\|) \right), \qquad \mathbf{x} \in \mathbb{R}^5. \end{aligned}$$

See also [5] for a stationary process on $\mathbb{R}$ of Ornstein-Uhlenbeck type with covariance $E_{\alpha,\beta}(-|x|)$, and [37] for Mittag-Leffler vector random fields in $\mathbb{R}^d$ with Mittag-Leffler direct and cross covariance functions.

Example 3. For $\tau \in (0, 1]$, $g(x) = (1 + |x|^\tau)^{-1}$ is a Pólya-type function on $\mathbb{R}$. From (11) we obtain U's density

$$f_U(x) = \begin{cases} \frac{\tau x^\tau (1-\tau+(1+\tau)x^\tau}{(1+x^\tau)^3}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

The covariance function (15) is

$$\begin{aligned} C(\|\mathbf{x}\|) &= \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \int_0^1 (1 + (\|\mathbf{x}\|u)^\tau)^{-1} \left(1 - u^2\right)^{\frac{d-3}{2}} du \\ &= \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \sum_{k=0}^{\infty} (-\|\mathbf{x}\|^\tau)^k \int_0^1 u^{\tau k} (1 - u^2)^{\frac{d-3}{2}} du \\ &= \Gamma\left(\frac{d}{2}\right) \pi^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(\frac{\tau k+1}{2}\right)}{\Gamma\left(\frac{\tau k+d}{2}\right)} \|\mathbf{x}\|^k, \qquad \mathbf{x} \in \mathbb{R}^d. \end{aligned}$$

When d is odd, $d = 2d_0 + 1$, say, we have

$$\Gamma\left(\frac{\tau k + d}{2}\right) = 2^{-\frac{d-1}{2}} (\tau k + d - 2)(\tau k + d - 4) \cdots (\tau k + 1) \Gamma\left(\frac{\tau k + 1}{2}\right),$$

and

$$C(\|\mathbf{x}\|) = \Gamma\left(\frac{d}{2}\right)\pi^{-\frac{1}{2}}2^{\frac{d-1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k\|\mathbf{x}\|^k}{(\tau k+d-2)(\tau k+d-4)\cdots(\tau k+1)}, \qquad \mathbf{x}\in\mathbb{R}^d.$$

It reduces to, when $d=3$ and $\tau=1$,

$$C(\|\mathbf{x}\|) = \sum_{k=0}^{\infty}\frac{(-1)^k\|\mathbf{x}\|^k}{k+1} = \frac{\ln(1+\|\mathbf{x}\|)}{\|\mathbf{x}\|}, \qquad \mathbf{x}\in\mathbb{R}^d.$$

Example 4. For a positive constant τ, $g(x) = (1+|x|)^{-\tau}$ is a Pólya-type function on $\mathbb{R}$ generated from a positive random variable with density function

$$f_U(x) = \tau(\tau+1)x(1+x)^{-\tau-2}, \quad x\geq 0.$$

In this case, the covariance function of the isotropic random field (14) decays in a power law and

$$C(\|\mathbf{x}\|) = \frac{2\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\pi^{\frac{1}{2}}}\int_0^1(1+\|x\|v)^{-\tau}\left(1-v^2\right)^{\frac{d-3}{2}}dv.$$

Note that

$$(1+x)^{-\tau} = \sum_{k=0}^{\infty}\frac{(-1)^k\tau^{[k]}}{k!}x^k, \qquad |x|<1,$$

where $\tau^{[k]}$ is the rising factorial, i.e., $\tau^{[k]} = \frac{\Gamma(\tau+k)}{\Gamma(\tau)}$. In view of (18), we have

$$C(\|\mathbf{x}\|) = \frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(-1)^k\tau^{[k]}}{k!}\|\mathbf{x}\|^k\frac{\Gamma(1/2+k/2)}{\Gamma(d/2+k/2)}.$$

When $\tau=1$, $\tau^{[k]} = k!$ and thus

$$C(\|\mathbf{x}\|) = \frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{1}{2}}}\sum_{k=0}^{\infty}(-1)^k\|\mathbf{x}\|^k\frac{\Gamma(1/2+k/2)}{\Gamma(d/2+k/2)}.$$

Example 5. The function

$$g(x) = \begin{cases} 1-|x|, & |x|\leq\frac{1}{2}, \\ \frac{1}{4|x|}, & |x|>\frac{1}{2}, \end{cases}$$

is a Pólya-type function on $\mathbb{R}$ and is generated by the distribution function

$$F_U(x) = \begin{cases} 1 - \frac{1}{2x}, & x \geq \frac{1}{2}, \\ 0, & x \leq \frac{1}{2}. \end{cases}$$

The isotropic covariance function (17) becomes

$$C(\|\mathbf{x}\|) = \begin{cases} 1 - \frac{\|\mathbf{x}\|}{2}, & \|\mathbf{x}\| \leq \frac{1}{2}, \\ \\ \frac{3+2\ln\|\mathbf{x}\|+2\ln 2}{8\|\mathbf{x}\|}, & \|\mathbf{x}\| > \frac{1}{2}, \ \mathbf{x} \in \mathbb{R}^3. \end{cases}$$

If its domain is restricted on $\mathbb{S}^{d-1}$, (14) becomes an isotropic random field on the sphere, on which the spherical (angular, or geodesic) distance of two points $\mathbf{x}_1$ and $\mathbf{x}_2$ is the distance between $\mathbf{x}_1$ and $\mathbf{x}_2$ on the largest circle on $\mathbb{S}^{d-1}$ that passes through them; more precisely,

$$\vartheta(\mathbf{x}_1, \mathbf{x}_2) = \arccos(\mathbf{x}_1'\mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^{d-1},$$

or

$$\vartheta(\mathbf{x}_1, \mathbf{x}_2) = \arccos\left(1 - \frac{1}{2}\|\mathbf{x}_1 - \mathbf{x}_2\|^2\right), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^{d-1},$$

where $\mathbf{x}_1'\mathbf{x}_2$ is the inner product between $\mathbf{x}_1$ and $\mathbf{x}_2$. Evidently, $0 \leq \vartheta(\mathbf{x}_1, \mathbf{x}_2) \leq \pi$, and the Euclidean and spherical distances are closely connected on $\mathbb{S}^{d-1}$, with

$$\|\mathbf{x}_1 - \mathbf{x}_2\| = (2 - 2\mathbf{x}_1'\mathbf{x}_2)^{\frac{1}{2}} = (2 - 2\cos\vartheta(\mathbf{x}_1, \mathbf{x}_2))^{\frac{1}{2}} = 2\sin\left(\frac{\vartheta(\mathbf{x}_1, \mathbf{x}_2)}{2}\right), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^{d-1}.$$

For properties of isotropic random fields on spheres see [38] and the references therein.

Corollary 3. *Suppose that $g(x)$ is a Pólya-type function on $\mathbb{R}$ and U is a positive random variable generating $g(x)$, and that $\mathbf{V}$ is a d-variate random vector uniformly distributed on $\mathbb{S}^{d-1}$. If $\{Y(x), x \in \mathbb{R}\}$ is a second-order Lévy process with covariance*

$$\operatorname{cov}(Y(x_1), Y(x_2)) = \frac{|x_1| + |x_2|}{2} - \frac{|x_1 - x_2|}{2}, \quad x_1, x_2 \in \mathbb{R},$$

and $\{Y(x), x \in \mathbb{R}\}$, U, and $\mathbf{V}$ are independent of each other, then

$$Z(\mathbf{x}) = Y\left(\frac{\mathbf{x}'\mathbf{V}}{U} + 1\right) - Y\left(\frac{\mathbf{x}'\mathbf{V}}{U}\right), \quad \mathbf{x} \in \mathbb{S}^{d-1}, \tag{20}$$

is an isotropic random field, and

(i) *for each fixed* $\mathbf{x} \in \mathbb{S}^{d-1}$, $Z(\mathbf{x})$ *follows the same infinitely divisible distribution as* $Y(1)$*;*

(ii) *the covariance function of* $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{S}^{d-1}\}$ *is*

$$\begin{aligned} & \operatorname{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) \\ = \ & \mathrm{E}g\left(2V_1 \sin\left(\tfrac{\vartheta(\mathbf{x}_1,\mathbf{x}_2)}{2}\right)\right) \\ = \ & \tfrac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \int_0^1 g\left(2u \sin\left(\tfrac{\vartheta(\mathbf{x}_1,\mathbf{x}_2)}{2}\right)\right)(1-u^2)^{\frac{d-3}{2}} du, \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^{d-1}. \end{aligned}$$

It is of interest to compare the subclass of isotropic covariance functions of the form (15) with others constructed earlier in the literature, for instance, those in [2], [8], [27], [45]. To this end, define

$$C_d(x) = \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \int_0^1 g(xu)(1-u^2)^{\frac{d-3}{2}} du, \quad x \geq 0.$$

Then $C_d(\|\mathbf{x}\|), \mathbf{x} \in \mathbb{R}^d$, is identical to the covariance function (15). For $d \geq 3$, it can be verified the following recursive relationship between $C_d(x)$ and $C_{d-2}(x)$,

$$C_{d-2}(x) = C_d(x) + \frac{xC'_d(x)}{d-2}, \qquad x \geq 0. \tag{21}$$

For $d \geq 4$, equation (21) differs from equation (25) in [27], or equation (3.9) in [45], and thus the subclass of isotropic covariance functions of the form (15) differs from that constructed in [8], [27], [45].

For $d = 2$, we have

$$C_2(x) = \frac{2}{\pi} \int_0^1 g(xu)(1-u^2)^{-\frac{1}{2}} du, \quad x \geq 0,$$

and

$$xC'_2(x) \ = \ -\frac{2}{\pi} \int_0^1 (1-u^2)^{-\frac{3}{2}} g(xu) du, \qquad x \geq 0.$$

Since $xC'_2(x)$ doesn't converge to 0 when $x \to 0$, condition (b) in Theorem 2 of [45] is not fulfilled, and, consequently, $C_2(\|\mathbf{x}\|)$ differs from that constructed in [45].

Interestingly, for $d = 3$, $C_3(x)$ satisfies conditions (a), (b), and (c) of Theorem 1 of [45], if $g(x)$ is twice differentiable on $(0, \infty)$ and $x^{-1}g''(x)$ is non-increasing in $x > 0$. In fact, condition (a) is satisfied since

$$C_3(x) = \int_0^1 g(xu) du, \quad x \geq 0,$$

is continuous, convex, $C_3(0) = 1$, and $C_3(x) = o(1)$ for $x \to \infty$ (due to the properties of $g(x)$ and Corollary 1). Condition (b) is fulfilled, because

$$xC_3'(x) = g(x) - C_3(x), \qquad x \geq 0,$$

$C_3'(x)$ is absolutely continuous on $[\epsilon, \infty)$ for every $\epsilon > 0$, and $\lim_{x\to 0} xC_3'(x) = \lim_{x\to\infty} xC_3'(x) = 0$. Moreover, for $x > 0$,

$$\frac{1}{x}C_3''(x) = \frac{1}{x}\int_0^1 g''(xu)u^2 du,$$

is non-negative and non-increasing in x, in that $g(x)$ is convex and $x^{-1}g''(x)$ is non-increasing in x, and thus condition (c) of [45] is also satisfied. More precisely, $C_3(\|\mathbf{x}\|)$ is an isotropic covariance function in $\mathbb{R}^3$ given by (1.2) of [45] for the density function

$$f\left(\frac{x}{2}\right) = 2xC_3''(x) = 2\int_0^1 f_U(xu)u^2 du,$$

where the second equality is due to Lemma 2. The covariance function considered in Example 2, i.e., $C(\|\mathbf{x}\|) = E_{1,2}(-\|\mathbf{x}\|)$, fulfills these conditions.

As another comparison, consider Askey's isotropic covariance functions [2]. For an odd integer $d \geq 3$, let

$$\tilde{C}_d(x) = (1-x)_+^{\frac{d+1}{2}}, \quad x \geq 0.$$

As is shown in [2], $\tilde{C}_d(\|\mathbf{x}\|)$ is an isotropic covariance function in $\mathbb{R}^d$. In such a case, the recursive relationship between $\tilde{C}_d(x)$ and $\tilde{C}_{d-2}(x)$ is

$$\tilde{C}_{d-2}(x) = \tilde{C}_d(x) - \frac{2}{d+1}x\frac{d}{dx}\tilde{C}_d(x), \qquad x \geq 0,$$

but differs from (21). It implies that the subclass of isotropic covariance functions of the form (15) differs from that constructed in [2].

For further investigation, a question of interest would be to construct the isotropic random field in $\mathbb{R}^d$ $(d \geq 2)$ with infinitely divisible marginals, whose covariance function is of the form in [2], [8], [27], [45]. Another question would be to characterize the isotropic covariance function (15), just as the subclass in [45] is generated as the scale mixture of Euclid's hat [27].

4 Vector random fields with isotropic direct covariance functions

This section contains two results: a vector version of (14) where the vector random field has infinitely divisible marginal distributions and isotropic direct covariance functions, and a characterization of the covariance matrix function of a Gaussian or elliptically contoured vector random field that is isotropic and mean square continuous in $\mathbb{R}^d$. The latter may be used as the turning bands method to simulate an isotropic random field in $\mathbb{R}^d$.

Theorem 2. *Suppose that the Pólya-type functions $g_1(x), \ldots, g_m(x)$ are generated by positive random variables $U_1, \ldots, U_m$, respectively, and $\mathbf{V}$ is a d-variate random vector uniformly distributed on $\mathbb{S}^{d-1}$. Let $\{\mathbf{Y}(x), x \in \mathbb{R}\}$ be an m-variate second-order Lévy process with covariance matrix function (5) and write $\mathbf{\Sigma}$ as $(\sigma_{ij})_{m\times m}$. If $\{\mathbf{Y}(x), x \in \mathbb{R}\}$, $\mathbf{V}$, and $U_1, \ldots, U_m$ are independent, then*

$$\mathbf{Z}(\mathbf{x}) = \left(Y_1\left(\frac{\mathbf{x}'\mathbf{V}}{U_1}+1\right) - Y_1\left(\frac{\mathbf{x}'\mathbf{V}}{U_1}\right), \ldots, Y_m\left(\frac{\mathbf{x}'\mathbf{V}}{U_m}+1\right) - Y_m\left(\frac{\mathbf{x}'\mathbf{V}}{U_m}\right)\right)', \quad \mathbf{x} \in \mathbb{R}^d, \tag{22}$$

is an m-variate random field with direct covariance functions

$$\mathrm{cov}(Z_i(\mathbf{x}_1), Z_i(\mathbf{x}_2)) = \sigma_{ii}\mathrm{E}g_i(\|\mathbf{x}_1 - \mathbf{x}_2\|V_1), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d,$$

and cross covariance functions

$$\mathrm{cov}(Z_i(\mathbf{x}_1), Z_j(\mathbf{x}_2)) = \sigma_{ij}\int_0^\infty\int_0^\infty \mathrm{E}\left(1 - \left\|\frac{\mathbf{x}_1}{u_i} - \frac{\mathbf{x}_2}{u_j}\right\| V_1\right)_+ dF_{U_i}(u_i)dF_{U_j}(u_j), \quad i \neq j.$$

Furthermore, $\mathbf{Z}(\mathbf{x})$ follows the same infinitely divisible distribution as $\mathbf{Y}(1)$ for each fixed $\mathbf{x} \in \mathbb{R}^d$.

In the particular case $m = 1$, (22) reduces to (14). The vector random field (22) is nonstationary, but each of its components is stationary and isotropic in $\mathbb{R}^d$. Random variables $U_1, \ldots, U_m$ in Theorem 2 may be relaxed to be dependent, provided that each individually is the generator of a Pólya-type function.

The covariance matrix function of an isotropic Gaussian or second-order elliptically contoured vector random field in $\mathbb{R}^d$ is characterized in [52], with a matrix version of (1) derived there. The following theorem gives a matrix version of (3).

Theorem 3. *Let $\mathbf{C}(x), x \in \mathbb{R}$, be an $m \times m$ matrix function whose entries are even and continuous on $\mathbb{R}$.*

(i) *If $\mathbf{C}(\|\mathbf{x}\|)$ is the covariance matrix function of an m-variate isotropic random field in $\mathbb{R}^d$, then it can be expressed as*

$$\mathbf{C}(\|\mathbf{x}\|) = \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}\int_0^\infty \mathbf{C}_1(\|\mathbf{x}\|u)(1-u^2)^{\frac{d-3}{2}}du, \qquad \mathbf{x} \in \mathbb{R}^d, \tag{23}$$

where $\mathbf{C}_1(x)$ *is the covariance matrix function of an* m*-variate stationary process on* $\mathbb{R}$.

(ii) *Conversely, if* $\mathbf{C}(\|\mathbf{x}\|)$ *adopts the representation* (23)*, then it is the covariance matrix function of an* m*-variate isotropic elliptically contoured random field in* $\mathbb{R}^d$.

The relationship (23) enables one to use the turning bands method to simulate an isotropic vector random field in $\mathbb{R}^d$, with scalar examples in [10], [16], [18], [32], [42], [51]. Examples of stationary covariance matrix functions on $\mathbb{R}$ may be found in, for instance, [17] and [39].

Remark. It would be of interest to construct a non-Gaussian or non-elliptically-contoured random field with the covariance matrix function of the form (23), while we have worked out a part of them in Theorems 1 and 2. Note that the vector random field defined in (22) is not isotropic, though it enjoys an isotropic direct covariance function. Thus, how to construct a vector isotropic random field is left for future research.

Example 6. For distinct positive constants $\alpha_1, \ldots, \alpha_m$, the $m \times m$ matrix functions

$$C_{ij,1}(x) = \begin{cases} 2\alpha_i \exp\left(-\frac{|x|}{2\alpha_i}\right), & i = j, \\ (\alpha_i + \alpha_j) \exp\left(-\frac{|x|}{\alpha_i+\alpha_j}\right) - |\alpha_i - \alpha_j| \exp\left(-\frac{|x|}{|\alpha_i-\alpha_j|}\right), & i \neq j, \end{cases}$$

for $i, j = 1, \ldots, m$ and $x \in \mathbb{R}$, form a stationary covariance function on $\mathbb{R}$, as Example 2 of [17] illustrates. In (23) taking $d = 3$ yields an isotropic covariance matrix function with direct/cross covariances

$$C_{ij}(\|\mathbf{x}\|) = \begin{cases} 4\alpha_i^2 E_{1,2}\left(-\frac{\|\mathbf{x}\|}{2\alpha_i}\right), & i = j, \\ \\ (\alpha_i + \alpha_j)^2 E_{1,2}\left(-\frac{\|\mathbf{x}\|}{\alpha_i+\alpha_j}\right) - (\alpha_i - \alpha_j)^2 E_{1,2}\left(-\frac{\|\mathbf{x}\|}{|\alpha_i-\alpha_j|}\right), & i \neq j,\ \mathbf{x} \in \mathbb{R}^3, \\ & i, j = 1, \ldots, m. \end{cases}$$

5 Proofs

5.1 Proof of Lemma 1

Denote by $\mathbf{\Sigma}(x)$ the variance-covariance matrix of $\mathbf{Y}(x)$ for each $x \in \mathbb{R}$, i.e., $\mathbf{\Sigma}(x) = \mathrm{cov}(\mathbf{Y}(x), \mathbf{Y}(x))$. It is an even function, since, by the stationary increment property, $\mathbf{Y}(0) - \mathbf{Y}(-x) = -\mathbf{Y}(-x)$ and $\mathbf{Y}(0 - (-x)) = \mathbf{Y}(x)$ have the same distribution, so that

$$\mathbf{\Sigma}(-x) = \mathrm{cov}(\mathbf{Y}(-x), \mathbf{Y}(-x)) = \mathrm{cov}(\mathbf{Y}(x), \mathbf{Y}(x)), \qquad x \geq 0.$$

In terms of $\mathbf{\Sigma}(x)$, the covariance matrix function of $\{\mathbf{Y}(x), x \in \mathbb{R}\}$ is given by, for $x_1, x_2 \geq 0$,

$$\mathrm{cov}(\mathbf{Y}(x_1), \mathbf{Y}(x_2)) = \mathbf{\Sigma}(\min(x_1, x_2)) = \min(x_1, x_2)\mathbf{\Sigma},$$

as is shown by [39], where $\mathbf{\Sigma} = \mathrm{cov}(\mathbf{Y}(1), \mathbf{Y}(1))$ is a positive definite matrix. For x_1 and x_2 having opposite signs, we have

$$\begin{aligned} & \mathrm{cov}(\mathbf{Y}(x_1), \mathbf{Y}(x_2)) \\ = & -\mathrm{cov}(\mathbf{Y}(\max(x_1, x_2)) - \mathbf{Y}(0), \mathbf{Y}(0) - \mathbf{Y}(\min(x_1, x_2))) \\ = & \ 0, \end{aligned}$$

since $\mathbf{Y}(\max(x_1, x_2)) - \mathbf{Y}(0)$ and $\mathbf{Y}(0) - \mathbf{Y}(\min(x_1, x_2))$ are independent of each other, by noticing that $\min(x_1, x_2) \leq 0 \leq \max(x_1, x_2)$. For $x_1, x_2 \leq 0$, $\mathbf{Y}(0) - \mathbf{Y}(\max(x_1, x_2))$ and $\mathbf{Y}(\max(x_1, x_2)) - \mathbf{Y}(\min(x_1, x_2))$ are independent of each other, so that

$$\begin{aligned} & \mathrm{cov}(\mathbf{Y}(x_1), \mathbf{Y}(x_2)) \\ = & \ \mathrm{cov}(\mathbf{Y}(\min(x_1, x_2)), \mathbf{Y}(\max(x_1, x_2)) \\ = & \ \mathrm{cov}\{(\mathbf{Y}(0) - \mathbf{Y}(\max(x_1, x_2))) + (\mathbf{Y}(\max(x_1, x_2)) - \mathbf{Y}(\min(x_1, x_2))), \ \mathbf{Y}(0) - \mathbf{Y}(\max(x_1, x_2))\} \\ = & \ \mathrm{cov}(\mathbf{Y}(\max(x_1, x_2)), \mathbf{Y}(\max(x_1, x_2))) \\ & + \mathrm{cov}(\mathbf{Y}(\max(x_1, x_2)) - \mathbf{Y}(\min(x_1, x_2)), \mathbf{Y}(0) - \mathbf{Y}(\max(x_1, x_2))) \\ = & \ \mathbf{\Sigma}(\max(x_1, x_2)) \\ = & \ \mathbf{\Sigma}(-\max(x_1, x_2)) \\ = & \ \mathbf{\Sigma}(\min(-x_1, -x_2)) \\ = & \ \min(-x_1, -x_2)\mathbf{\Sigma}, \end{aligned}$$

where the fifth equality is due to the even property of $\mathbf{\Sigma}(x)$. Finally, (5) is confirmed.

5.2 Proof of Lemma 2

Assuming that U has a density function $f_U(x)$, (10) becomes

$$\int_x^\infty \left(1 - \frac{x}{u}\right) f_U(u)du = g(x), \qquad x \geq 0,$$

or

$$\int_x^\infty f_U(u)du - x\int_x^\infty \frac{f_U(u)}{u}du = g(x), \qquad x \geq 0.$$

Both sides taking derivatives yields

$$-\int_x^\infty \frac{f_U(u)}{u}du = g'(x),$$

from which (11) is established. It can verified that $f_U(x)$ is a solution of the equation (10).

5.3 Proof of Lemma 3

Note that

$$g(x) = \int_{|x|}^{\infty} \left(1 - \frac{|x|}{u}\right) dF_U(u)$$
$$= 1 - |x| \int_{|x|}^{\infty} \frac{1}{u^2} F_U(u) du. \tag{24}$$

Therefore, $g(x)$ is continuously differentiable on $(0, \infty)$ and

$$g'(x) = -\int_{x}^{\infty} \frac{1}{u^2} F_U(u) du + \frac{1}{x} F_U(x), \quad x > 0.$$

In view of (24), we have

$$F_U(x) = 1 + xg'(x) - g(x), \quad x > 0. \tag{25}$$

Moreover, note that $g(x)$ has the right-hand derivative at $x = 0$ and we write it as $g'(0)$. Thus the $F_U(x)$ determined in (25) is continuous at $x = 0$.

5.4 Derivation of (19)

Note that when $d = 3$ or $d_0 = 1$, the left-hand side of (19) is $\frac{1}{(k+1)!}$. Assume (19) holds when $d_0 = n \geq 1$, i.e.,

$$\frac{1}{(k + 2n - 1)(k + 2n - 3) \cdots (k + 1)k!} = \sum_{j=0}^{n-1} \frac{a_{n,j}}{(k + 2n - 1 - j)!},$$

for some constants $a_{n,j}$. We next show the following holds

$$\frac{1}{(k + 2n + 1)(k + 2n - 1) \cdots (k + 1)k!} = \sum_{j=0}^{n} \frac{a_{n+1,j}}{(k + 2n + 1 - j)!}, \tag{26}$$

for some constants $a_{n+1,j}$.

Note that

$$\begin{aligned}
&\frac{1}{(k+2n+1)(k+2n-1)\cdots(k+3)(k+1)!}\\
=&\frac{1}{(k+2n)}\sum_{j=0}^{n-1}\frac{a_{n,j}}{(k+2n-1-j)!}-\frac{1}{(k+2n+1)(k+2n)}\sum_{j=0}^{n-1}\frac{a_{n,j}}{(k+2n-1-j)!}\\
=&-\frac{a_{n,0}}{(k+2n+1)!}+\frac{a_{n,0}}{(k+2n)!}+\sum_{j=1}^{n-1}\frac{(k+2n-1)\cdots(k+2n-j)a_{n,j}}{(k+2n)!}\\
&-\sum_{j=1}^{n-1}\frac{(k+2n-1)\cdots(k+2n-j)a_{n,j}}{(k+2n+1)!}.
\end{aligned}$$

For $j=1$,

$$\begin{aligned}
\frac{(k+2n-1)}{(k+2n)!}&=-\frac{1}{(k+2n)!}+\frac{1}{(k+2n-1)!},\\
\frac{(k+2n-1)}{(k+2n+1)!}&=-\frac{2}{(k+2n+1)!}+\frac{1}{(k+2n)!}.
\end{aligned}$$

When $j>1$,

$$\begin{aligned}
&\frac{(k+2n-1)\cdots(k+2n-j)}{(k+2n)!}\\
=&\frac{(k+2n)(k+2n-1)\cdots(k+2n-j+1)}{(k+2n)!}-j\frac{(k+2n-1)\cdots(k+2n-j+1)}{(k+2n)!}\\
=&\frac{1}{(k+2n-j)!}-j\frac{(k+2n-1)\cdots(k+2n-j+1)}{(k+2n)!}.
\end{aligned}$$

Thus $\frac{(k+2n-1)\cdots(k+2n-j)}{(k+2n)!}$ can be written as a linear combination of $\frac{1}{(k+2n-j)!}$, $\frac{1}{(k+2n-j+1)!}$, $\cdots$, $\frac{1}{(k+2n)!}$. Moreover, note that

$$\begin{aligned}
&\frac{(k+2n-1)\cdots(k+2n-j)}{(k+2n+1)!}\\
=&\frac{(k+2n-1)\cdots(k+2n-j+1)(k+2n+1-(j+1))}{(k+2n+1)!}\\
=&\frac{(k+2n-1)\cdots(k+2n-j+1)}{(k+2n)!}-(j+1)\frac{(k+2n-1)\cdots(k+2n-j+1)}{(k+2n+1)!}.
\end{aligned}$$

We have $\frac{(k+2n-1)\cdots(k+2n-j)}{(k+2n+1)!}$ can be expressed as a linear combination of $\frac{1}{(k+2n-j+1)!}$, $\cdots$, $\frac{1}{(k+2n)!}$, $\frac{1}{(k+2n+1)!}$.

As a result, $\sum_{j=1}^{n-1} \frac{(k+2n-1)\cdots(k+2n-j)a_{n,j}}{(k+2n)!}$ can be written as a linear combination of $\frac{1}{(k+n+1)!}, \frac{1}{(k+n+2)!}, \cdots, \frac{1}{(k+2n)!}$, while $\sum_{j=1}^{n-1} \frac{(k+2n-1)\cdots(k+2n-j)a_{n,j}}{(k+2n+1)!}$ is a linear combination of $\frac{1}{(k+n+2)!}, \cdots, \frac{1}{(k+2n)!}, \frac{1}{(k+2n+1)!}$. Thus (26) holds. In other words, (19) holds true.

5.5 Proof of Theorem 1

(i) Since U, $\mathbf{V}$, and $\{Y(x), x \in \mathbb{R}\}$ are independent and, for each fixed $\mathbf{x} \in \mathbb{R}^d$, $Y\left(\frac{\mathbf{x}'\mathbf{v}}{u}+1\right) - Y\left(\frac{\mathbf{x}'\mathbf{v}}{u}\right)$ and $Y(1)$ or $Y(-1)$ have the same distribution, we obtain the characteristic function of $Z(\mathbf{x})$,

$$\begin{aligned}
\mathrm{E}\exp(\imath Z(\mathbf{x})\omega) &= \mathrm{E}\exp\left\{\imath\omega\left(Y\left(\frac{\mathbf{x}'\mathbf{V}}{U}+1\right) - Y\left(\frac{\mathbf{x}'\mathbf{V}}{U}\right)\right)\right\} \\
&= \frac{1}{\omega_{d-1}}\int_{\mathbb{S}^d}\int_0^\infty \mathrm{E}\exp\left\{\imath\omega\left(Y\left(\frac{\mathbf{x}'\mathbf{v}}{u}+1\right) - Y\left(\frac{\mathbf{x}'\mathbf{v}}{u}\right)\right)\right\} dF_U(u)d\mathbf{v} \\
&= \frac{1}{\omega_{d-1}}\int_{\mathbb{S}^d}\int_0^\infty \mathrm{E}\exp\left(\imath\omega Y(1)\right) dF_U(u)d\mathbf{v} \\
&= \mathrm{E}\exp\left(\imath\omega Y(1)\right); \qquad \mathbf{x} \in \mathbb{R}^d,
\end{aligned}$$

that is, the distribution of $Z(\mathbf{x})$ is identical to that of $Y(1)$, and is thus an infinitely divisible distribution. As a consequence, $\mathrm{E}Z(\mathbf{x}) = \mathrm{E}Y(1)$.

(ii) It is easy to check that

$$|x+1| + |x-1| - 2|x| = 2(1-|x|)_+, \qquad x \in \mathbb{R}, \tag{27}$$

from which we obtain the the covariance function of $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{R}^d\}$,

$$\begin{aligned}
& \mathrm{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) \\
= & \mathrm{cov}\left(Y\left(\frac{\mathbf{x}_1'\mathbf{V}}{U}+1\right) - Y\left(\frac{\mathbf{x}_1'\mathbf{V}}{U}\right),\ Y\left(\frac{\mathbf{x}_2'\mathbf{V}}{U}+1\right) - Y\left(\frac{\mathbf{x}_2'\mathbf{V}}{U}\right)\right) \\
= & \frac{1}{\omega_d}\int_{\mathbb{S}^d}\int_0^\infty \mathrm{cov}\left(Y\left(\frac{\mathbf{x}_1'\mathbf{v}}{u}+1\right) - Y\left(\frac{\mathbf{x}_1'\mathbf{v}}{u}\right),\ Y\left(\frac{\mathbf{x}_2'\mathbf{v}}{u}+1\right) - Y\left(\frac{\mathbf{x}_2'\mathbf{v}}{u}\right)\right) dF_U(u)d\mathbf{v} \\
= & \frac{1}{2\omega_d}\int_{\mathbb{S}^d}\int_0^\infty \left\{\left|\frac{(\mathbf{x}_1-\mathbf{x}_2)'\mathbf{v}}{u}+1\right| + \left|\frac{(\mathbf{x}_1-\mathbf{x}_2)'\mathbf{v}}{u}-1\right| - 2\left|\frac{(\mathbf{x}_1-\mathbf{x}_2)'\mathbf{v}}{u}\right|\right\} dF_U(u)d\mathbf{v} \\
= & \frac{1}{\omega_d}\int_{\mathbb{S}^d}\int_0^\infty \left(1 - \frac{|(\mathbf{x}_1-\mathbf{x}_2)'\mathbf{v}|}{u}\right)_+ dF_U(u)d\mathbf{v} \\
= & \mathrm{E}g((\mathbf{x}_1-\mathbf{x}_2)'\mathbf{V}), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d,
\end{aligned}$$

where the last equality is due to the formula (10). Since $\mathbf{V}$ is uniformly distributed on $\mathbb{S}^{d-1}$, it follows from Theorem 2.4 of [19] that $(\mathbf{x}_1-\mathbf{x}_2)'\mathbf{V}$ and $\|\mathbf{x}_1-\mathbf{x}_2\|V_1$ have the same distribution. Consequently,

$$\begin{aligned}\mathrm{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) &= \mathrm{E}g((\mathbf{x}_1-\mathbf{x}_2)'\mathbf{V}) \\ &= \mathrm{E}g(\|\mathbf{x}_1-\mathbf{x}_2\|V_1) \\ &= \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}\int_0^1 g(\|\mathbf{x}_1-\mathbf{x}_2\|u)(1-u^2)^{\frac{d-3}{2}}du, \qquad \mathbf{x}_1,\mathbf{x}_2\in\mathbb{R}^d,\end{aligned}$$

and $\{Z(\mathbf{x}), \mathbf{x}\in\mathbb{R}^d\}$ is an isotropic random field with covariance function (4).

To derive an alternative form of (15), from (13) we obtain

$$\begin{aligned}\mathrm{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) &= \mathrm{E}g((\mathbf{x}_1-\mathbf{x}_2)'\mathbf{V}) \\ &= \mathrm{E}\exp\left(\imath(\mathbf{x}_1-\mathbf{x}_2)'\mathbf{V}\frac{V_0}{U}\right) \\ &= \int_{-\infty}^{\infty}\mathrm{E}\exp\left(\imath(\mathbf{x}_1-\mathbf{x}_2)'\mathbf{V}u\right)dP\left(\frac{V_0}{U}\le u\right) \\ &= \int_0^{\infty}\Omega_d(\|\mathbf{x}_1-\mathbf{x}_2\|u)dF(u), \qquad \mathbf{x}_1,\mathbf{x}_2\in\mathbb{R}^d,\end{aligned}$$

where $F(u) = 2P\left(\frac{V_0}{U}\le u\right), u\ge 0$.

(iii) By Bochner's theorem, $C(\|\mathbf{x}\|)$ can be expressed as

$$C(\|\mathbf{x}\|) = \int_{\mathbb{R}^d}\exp(\imath\mathbf{x}'\boldsymbol{\omega})d\mathbf{F}(\boldsymbol{\omega}) = \mathrm{E}\exp(\imath\mathbf{x}'\mathbf{W}), \qquad \mathbf{x}\in\mathbb{R}^d.$$

where $\mathbf{W}$ is a d-variate random vector with distribution function $\mathbf{F}(\boldsymbol{\omega})$. On the other hand, it follows from (13) that

$$C(\|\mathbf{x}\|) = \mathrm{E}g(\mathbf{x}'\mathbf{V}) = \mathrm{E}\exp\left(\imath\mathbf{x}'\frac{V_0}{U}\mathbf{V}\right), \qquad \mathbf{x}\in\mathbb{R}^d.$$

By the unique theorem, $\mathbf{W}$ and $\frac{V_0}{U}\mathbf{V}$ have the same distribution, so that the latter's distribution function is the spectral distribution function of (14) and the latter's density function, if it exists, is the spectral density function of (14).

5.6 Proof of Theorem 2

The distribution of $\mathbf{Z}(\mathbf{x})$ is infinitely divisible, because it can be verified to be identical to that of $\mathbf{Y}(1)$, in a way analogous to the proof of Theorem 1. Also, the direct covariance function $\mathrm{cov}(Z_i(\mathbf{x}_1), Z_i(\mathbf{x}_2))$ can be derived in the same way as the proof of Theorem 1.

For $i \neq j$, we obtain the cross covariance function $\text{cov}(Z_i(\mathbf{x}_1), Z_j(\mathbf{x}_2))$ from identity (27) as follows,

$$\begin{aligned}
& \text{cov}(Z_i(\mathbf{x}_1), Z_j(\mathbf{x}_2)) \\
= & \text{cov}\left(Y_i\left(\frac{\mathbf{x}_1'\mathbf{V}}{U_i}+1\right) - Y_i\left(\frac{\mathbf{x}_1'\mathbf{V}}{U_i}\right),\ Y_j\left(\frac{\mathbf{x}_2'\mathbf{V}}{U_j}+1\right) - Y_j\left(\frac{\mathbf{x}_2'\mathbf{V}}{U_j}\right)\right) \\
= & \frac{1}{\omega_{d-1}}\int_{\mathbb{S}^{d-1}}\int_0^\infty\int_0^\infty \text{cov}\left(Y_i\left(\frac{\mathbf{x}_1'\mathbf{v}}{u_i}+1\right) - Y_i\left(\frac{\mathbf{x}_1'\mathbf{v}}{u_i}\right),\right. \\
& \qquad \left. Y_j\left(\frac{\mathbf{x}_2'\mathbf{v}}{u_j}+1\right) - Y_j\left(\frac{\mathbf{x}_2'\mathbf{v}}{u_j}\right)\right) dF_{U_i}(u_i)dF_{U_j}(u_j)d\mathbf{v} \\
= & \frac{\sigma_{ij}}{\omega_{d-1}}\int_{\mathbb{S}^{d-1}}\int_0^\infty\int_0^\infty \left(1-\left|\frac{\mathbf{x}_1'\mathbf{v}}{u_i}-\frac{\mathbf{x}_2'\mathbf{v}}{u_j}\right|\right)_+ dF_{U_i}(u_i)dF_{U_j}(u_j)d\mathbf{v} \\
= & \sigma_{ij}\int_0^\infty\int_0^\infty \mathrm{E}\left(1-\left|\frac{\mathbf{x}_1'\mathbf{V}}{u_i}-\frac{\mathbf{x}_2'\mathbf{V}}{u_j}\right|\right)_+ dF_{U_i}(u_i)dF_{U_j}(u_j) \\
= & \sigma_{ij}\int_0^\infty\int_0^\infty \mathrm{E}\left(1-\left\|\frac{\mathbf{x}_1}{u_i}-\frac{\mathbf{x}_2}{u_j}\right\|V_1\right)_+ dF_{U_i}(u_i)dF_{U_j}(u_j), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d,
\end{aligned}$$

where the last equality follows from the fact that $\left(\frac{\mathbf{x}_1}{u_i}-\frac{\mathbf{x}_2}{u_j}\right)'\mathbf{V}$ and $\left\|\frac{\mathbf{x}_1}{u_i}-\frac{\mathbf{x}_2}{u_j}\right\|V_1$ have the same distribution, according to Theorem 2.4 of [19].

5.7 Proof of Theorem 3

(i) Since $\mathbf{C}(\|\mathbf{x}\|), \mathbf{x} \in \mathbb{R}^d$, is an isotropic covariance matrix function, it adopts an integral expression, by Theorem 3.1 of [52],

$$\mathbf{C}(\|\mathbf{x}\|) = \int_0^\infty \Omega_d(\|\mathbf{x}\|\omega)d\mathbf{F}(\omega), \qquad \mathbf{x} \in \mathbb{R}^d,$$

where $\mathbf{F}(\omega), \omega \in [0, \infty)$, is an $m \times m$ right-continuous, bounded matrix function with $\mathbf{F}(0) = \mathbf{0}$, and $\mathbf{F}(\omega_2) - \mathbf{F}(\omega_1)$ is positive definite for every pair of ω_1 and ω_2 with $0 \leq \omega_1 \leq \omega_2$. It can be rewritten as, via the identity (7) and Theorem 2.4 of [19],

$$\begin{aligned}
\mathbf{C}(\|\mathbf{x}\|) &= \int_0^\infty \mathrm{E}\exp(\imath\mathbf{x}'\mathbf{V}\omega)d\mathbf{F}(\omega) \\
&= \int_0^\infty \mathrm{E}\exp(\imath\|\mathbf{x}\|V_1\omega)d\mathbf{F}(\omega) \\
&= \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}\int_0^\infty\int_0^1 \exp(\imath\|\mathbf{x}\|u\omega)(1-u^2)^{\frac{d-3}{2}}dud\mathbf{F}(\omega)
\end{aligned}$$

$$= \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \int_0^1 \mathbf{C}_1(\|\mathbf{x}\|u)(1-u^2)^{\frac{d-3}{2}} du, \qquad \mathbf{x} \in \mathbb{R}^d,$$

where

$$\mathbf{C}_1(x) = \int_0^\infty \exp(\imath x\omega) d\mathbf{F}(\omega), \qquad x \in \mathbb{R},$$

is an $m \times m$ stationary covariance matrix function on $\mathbb{R}$, as is shown in Section 8.1 of [13].

(ii) For $\mathbf{C}(\|\mathbf{x}\|)$ being of the form (23), since $\mathbf{C}_1(x)$ is an $m \times m$ stationary covariance matrix function on $\mathbb{R}$, it follows from Section 8.1 of [13] that

$$\mathbf{C}_1(x) = \int_0^\infty \exp(\imath x\omega) d\mathbf{F}(\omega), \qquad x \in \mathbb{R},$$

where $\mathbf{F}(\omega), \omega \in [0, \infty)$, is an $m \times m$ right-continuous, bounded matrix function with $\mathbf{F}(0) = \mathbf{0}$, and $\mathbf{F}(\omega_2) - \mathbf{F}(\omega_1)$ is positive definite for every pair of ω_1 and ω_2 with $0 \le \omega_1 \le \omega_2$. In terms of the identity (7), (23) can be rewritten as

$$\begin{aligned}
\mathbf{C}(\|\mathbf{x}\|) &= \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \int_0^1 \mathbf{C}_1(\|\mathbf{x}\|u)(1-u^2)^{\frac{d-3}{2}} du \\
&= \frac{2\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)} \int_0^\infty \int_0^1 \exp(\imath\|\mathbf{x}\|u\omega)(1-u^2)^{\frac{d-3}{2}} du d\mathbf{F}(\omega) \\
&= \int_0^\infty \mathrm{E}\exp(\imath\|\mathbf{x}\|V_1\omega) d\mathbf{F}(\omega) \\
&= \int_0^\infty \mathrm{E}\exp(\imath\mathbf{x}'\mathbf{V}\omega) d\mathbf{F}(\omega) \\
&= \int_0^\infty \Omega_d(\|\mathbf{x}\|\omega) d\mathbf{F}(\omega), \qquad \mathbf{x} \in \mathbb{R}^d.
\end{aligned}$$

By Theorem 3.1 of [52], there exists an m-variate elliptically contoured random field with $\mathbf{C}(\|\mathbf{x}\|)$ as its covariance matrix function.

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