FRACTIONAL STOKES-BOUSSINESQ-LANGEVIN EQUATION AND MITTAG-LEFFLER CORRELATION DECAY

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This contribution is dedicated to the 85th anniversary of Professor Mykhailo Iosipovych Yadrenko

Abstract. This paper presents some stationary processes which are solutions of the fractional Stokes-Boussinesq-Langevin equation. These processes have reflection positivity and their correlation functions, which may exhibit the Alder-Wainwright effect or long-range dependence, are expressed in terms of the Mittag-Leffler functions. These properties are established rigorously via the theory of KMO-Langevin equation and a combination of Mittag-Leffler functions and fractional derivatives. A relationship to fractional Riesz-Bessel motion is also investigated. This relationship permits to study the effects of long-range dependence and second-order intermittency simultaneously.

1. Introduction

In a computer experiment of molecular dynamics, Alder and Wainwright [2, 3] found the tail behavior $t^{-3/2}$ as $t \to \infty$ for the autocorrelation function of a stationary process of non-Markovian type. This behavior is known as the Alder-Wainwright effect. The usual Langevin equation for the velocity $\xi(t)$ of a Brownian particle of mass m at position $x(t)$ in a fluid, which neglects the effect of the fluid flow around the particle, is not adequate to capture this behavior. Taking into account the hydrodynamic drag force, which is due the acceleration of the particle, the Langevin equation becomes the Stokes-Boussinesq-Langevin equation :

$$\frac{d\xi(t)}{dt} = -\frac{1}{\sigma^*}\xi(t) - \frac{a}{\sigma^*\sqrt{\pi\nu}}\int_{-\infty}^{t}\frac{1}{\sqrt{t-s}}\frac{d\xi(s)}{ds}ds + \frac{1}{m^*}W(t), \tag{1.1}$$

where $m^* = m\left(1+\frac{\rho}{2\rho_0}\right)$ is the effective mass, ρ being the density of the fluid, ρ_0 being the density of the particle, $\sigma^* = m^*\mu$ is the modified relaxation time, μ is the mobility coefficient, ν is the kinematic viscosity of the fluid, and $W(t)$ denotes the random force arising from rapid thermal fluctuations (see Appendix A for the derivation of Eq. (1.1)). It was shown in Widom [64], for example, that the autocorrelation function of the random process $\xi(t)$ defined by Eq. (1.1) has the tail $t^{-3/2}$ as $t \to \infty$, which agrees with the result of the Alder-Wainwright experiment.

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Many works on physical models of anomalous diffusion reported a Mittag-Leffler decay for the autocorrelation function:

$$\rho(t) = E_\alpha\left(-b|t|^\alpha\right), \ t \in \mathbb{R}, \ 0 < \alpha \le 1, \ b \ge 0, \tag{1.2}$$

where E_α is the one-parameter Mittag-Leffler function (defined in Section 2). This formula covers a complete range from the exponential decay of Ornstein-Uhlenbeck processes to the hyperbolic decay of strongly dependent processes, and includes the Alder-Wainwright effect. Metzler *et al.* [44] introduced a fractional Fokker-Planck equation using fractional derivatives to describe subdiffusive behavior of a system close to thermal equilibrium. Based on this equation, they showed that the mean square displacement of a particle has a Mittag-Leffler decay as $t \to \infty$, hence implying the long-range dependence (LRD) for its velocity. Metzler and Klafter [43] and Barkai and Silbey [9] investigated a fractional Klein-Kramers equation, from which the fractional Fokker-Planck equation is deduced, and again established the Mittag-Leffler relaxation.

Lutz [40] described another pathway to anomalous diffusion using random matrix theory. This approach considers a system coupled to a fractal heat bath with a random-matrix interaction. In the limit of weak coupling, the following fractional Langevin equation is obtained:

$$m\ddot{x}(t) + m\int_0^t \gamma(t-s)\,\dot{x}(s)\,ds = W(t), \tag{1.3}$$

where $W(t)$ is a Gaussian random force with mean zero and covariance function $R_W(t) = \mathrm{E}\left(W(t)W(0)\right) \sim 2A_0\Gamma(\alpha)\cos\left(\frac{\alpha\pi}{2}\right)t^{-\alpha}, 0 < \alpha < 2$, in the limit of large bandwidth, A_0 being the strength of the coupling, and $\gamma(t)$ is a response kernel that obeys the second fluctuation-dissipation theorem $m\kappa T\gamma(t) = R_W(t), \ \kappa \equiv$ Boltzmann constant, $T \equiv$ absolute temperature (Kubo [36]). Using this equation, the Mittag-Leffler decay of the autocorrelation function is obtained. Kou and Xie [35] and Min *et al.* [46] used the fractional Langevin equation (1.3) to investigate subdiffusion $(0 < \alpha < 1)$ within a single protein molecule. Fa [25], Lim and Teo [39], Eab and Lim [23] extended the fractional Langevin equation (1.3) to the case where the response kernel $\gamma(t)$ is given in terms of a Mittag-Leffler function and the time derivative of $\dot{x}(t)$ is replaced by a Caputo fractional time derivative. The resulting equation is called a fractional generalized Langevin equation (FGLE). Camargo *et al.* [12] considered a two-parameter Mittag-Leffler function in the response kernel $\gamma(t)$ for the FGLE, while Sandev *et al.* [58], [59] considered a three-parameter Mittag-Leffler function for $\gamma(t)$. The paper [59] provides a review of works in this direction. It should be noted that these works used the fluctuation-dissipation theorem and followed the Laplace transform method applied to the FGLE, which is a random equation, to obtain a formal expression for the displacement $x(t)$.

From a different angle, Okabe [50, 51] introduced and gave a rigorous treatment of the linear stochastic delay equation

$$\dot{X}(t) = -\beta X(t) - \int_0^t \gamma(t-s)\,\dot{X}(s)\,ds + \alpha I(t), \tag{1.4}$$

in which the solution $X(t)$ is defined as a random tempered distribution, and $\dot{X}(t)$ is its derivative. Here, α and β are positive numbers, the delay kernel $\gamma : (0,\infty) \to [0,\infty)$ has the representation

$$\gamma(t) = \int_0^\infty e^{-t\lambda}\rho(d\lambda),\ t > 0, \tag{1.5}$$

ρ being a Borel measure on $(0,\infty)$ such that $\int_0^\infty \left(\lambda^{-1}+\lambda\right)\rho(d\lambda) < \infty$, and $I(t)$ is a stationary Gaussian random tempered distribution associated with $X(t)$, called the Kubo noise of the process $X(t)$ (this concept comes from Kubo's linear response theory detailed in Kubo [36], Kubo *et al.* [37]). The Kubo noise $I(t)$ is needed for a fluctuation-dissipation theorem to hold. Eq. (1.4) has a physical meaning by considering $X(t)$ to be the x-component of the velocity of a particle as described in Appendix A. A key feature of Eq. (1.4) is that it describes the time evolution of a stationary Gaussian process with reflection positivity (defined in the next section); this concept arises from an axiom of the quantum field theory. Under the conditions on the measure $\rho(d\lambda)$, the diffusion coefficient $D = \int_0^\infty R_X(t)dt$ is finite for Eq. (1.4). Inoue [33] extended Okabe's work by considering the case $D = \infty$. In this latter work, Eq. (1.4) is also established, but with $\beta = 0$. A key result obtained is that the solution of Eq. (1.4) (with $\beta = 0$) possesses both long-range dependence and reflection positivity. A causality condition (defined in (2.18) of Section 2) is needed for uniqueness of the solution.

In Section 2, we apply the theory of Okabe [50, 51] and Inoue [33] to a fractional generalization of the Stokes-Boussinesq-Langevin equation:

$$\dot{X}(t) = -\lambda X(t) - b\mathcal{D}^{1-\alpha}X(t) + W(t),\ t\in\mathbb{R}, \lambda \geq 0, b\geq 0, \tag{1.6}$$

where the fractional derivative $\mathcal{D}^{1-\alpha}, 0 \leq \alpha \leq 1$, is defined in (2.15) below, and $W(t)$ is Kubo noise with a certain spectral density. In this application, the delay kernel $\gamma(t)$ of (1.5) takes the specific form of the fractional derivative $\mathcal{D}^{1-\alpha}$ and the Kubo noise has two specifications in Theorems 2.1 and 2.2 respectively. In Theorem 2.1 we confirm analytically the Mittag-Leffler decay in the autocorrelation function of the solution process, while in Theorem 2.2, for $\alpha = 1/2$ the asymptotic behaviour of the correlation function is $\rho_X(t) = O\left(t^{-3/2}\right),\ t\to\infty$, which is the Alder-Wainwright effect. A new aspect of Theorems 2.1 and 2.2 is that the results are given in an explicit form using the Mittag-Leffler functions, rather than asymptotic results as given in Inoue [33]. These exact results highlight the important role played by a combination of Mittag-Leffler functions and fractional derivatives, which takes advantage of the availability of nice formulae of the Laplace transform in terms of Mittag-Leffler functions.

In Section 3, we consider the fractional Stokes-Boussinesq-Langevin equation (1.6) in the context of fractional Gaussian noise $\dot{B}_H(t)$. Eq. (1.6) will then take the form

$$\dot{X}(t) = -\lambda X(t) - b\mathcal{D}^{1-2H}X(t) + \dot{B}_H(t),\quad t\in\mathbb{R}, \lambda\geq 0, b\geq 0. \tag{1.7}$$

We will look at two cases of interest: $\lambda = 0$ and then $b = 0$ separately. For $\lambda = 0$ in (1.7), existence and uniqueness of a stationary solution for (1.7) with long-range dependence is confirmed for $0 < H < 1/2$. For the case $1/2 < H < 1$, we will consider Eq. (1.7) with $b = 0$ in the Itô approach. The corresponding equation is the Ornstein-Uhlenbeck equation driven by fractional Brownian motion $B_H(t),\ 1/2 < H < 1$. Existence and uniqueness of a stationary solution with long-range dependence is also obtained, as well as an explicit form for its spectral density.

In the approach of Okabe [50, 51] and Inoue [33], the noise term (Kubo noise) is associated with the underlying process via its spectral decomposition. If we are able to obtain long-range dependence or the Alder-Wainwright effect in the solution of the fractional Stokes-Boussinesq-Langevin equation under the scenario of system-independent noise, we may use the noise term to represent other effects such as intermittency (see Frisch [27] for example). There may also be more flexibility in defining the response function $\gamma(t)$. We demonstrate these possibilities in Section 4, where a stationary process related to fractional Riesz-Bessel motion (Anh *et al.* [5, 7]) is derived. This permits to study the effects of long-range dependence and second-order intermittency simultaneously. These effects are known to be important features of data in geophysics, turbulence and finance.

2. Stationary processes governed by the fractional Stokes-Boussinesq-Langevin equation

2.1. **Reflection positivity.** Let $X = \{X(t), t \in \mathbb{R}\}$ be a real-valued, measurable, mean-square continuous, stationary (in the wide sense) random process with mean $\mathrm{E}X(t) = const$, covariance function $R_X(t) = \mathrm{Cov}(X(t), X(0))$, $t \in \mathbb{R}$, and spectral density $f_X(\omega)$, $\omega \in \mathbb{R}$, that is,

$$R_X(t-s) = \int_{\mathbb{R}} \cos\{\omega(t-s)\} f_X(\omega)\, d\omega. \tag{2.1}$$

The concept of reflection positivity arises in the axiomatic quantum field theory (Osterwalder and Schrader [53], Nelson [49], Hegerfeldt [29], Glimm and Jaffe [28], pp. 90-92). Following Osterwalder and Schrader [53], we say that the process X has reflection positivity if its covariance function (2.1) satisfies

$$\sum_{j,k=1}^{n} z_j R_X(t_j + t_k)\, \bar{z}_k \geq 0,\ t_j \in [0,\infty),\ j = 1,...,n,$$

for any $n \geq 1$, $z_j \in \mathbb{C}$, $j = 1,...,n$. Hida and Streit [31] showed that a Gaussian process X has reflection positivity if and only if there exists uniquely a bounded non-negative Borel measure σ on $[0,\infty)$ such that

$$\frac{R_X(t)}{R_X(0)} = \rho_X(t) = \sigma(\{0\}) + \int_{(0,\infty)} \rho_\lambda(t)\, \sigma(d\lambda), \tag{2.2}$$

where

$$\rho_\lambda(t) = e^{-|t|\lambda},\ t \in \mathbb{R},\ \lambda > 0 \tag{2.3}$$

is the correlation function of the stationary Gaussian Ornstein-Uhlenbeck (OU) process $\xi(t)$ defined by the equation

$$d\xi(t) = -\lambda\xi(t)\, dt + \gamma dB(t),\ t \in \mathbb{R}, \lambda > 0, \gamma > 0. \tag{2.4}$$

Here, $B = \{B(t), t \in \mathbb{R}\}$ is a one-dimensional Brownian motion or Wiener process such that

$$\mathrm{E}B(t) = 0,\ \mathrm{Var}B(t) = |t|. \tag{2.5}$$

The stationary Gaussian solution of (2.4) has the following covariance function and spectral density:

$$R_\xi(t) = \frac{\gamma^2}{2\lambda} e^{-\lambda|t|},\ t \in \mathbb{R};\ f_\xi(\omega) = \frac{A}{\omega^2 + \lambda^2},\ A = \frac{\gamma^2}{2\pi},\ \omega \in \mathbb{R}. \tag{2.6}$$

By Bernstein's Theorem (see Feller [26], p. 426) we obtain that the condition (2.2) is equivalent to the complete monotonicity of the function $\rho_X(t)$ on $(0,\infty)$, that is,

$$(-1)^k \frac{d^k}{dt^k}\rho_X(t) \geq 0,\ t>0,\ k=0,1,2,.... \tag{2.7}$$

The following functions on $(0,\infty)$ are known to be completely monotone:

$$\begin{aligned}
&\exp\{-ct^{\gamma}\},\ c>0, 0<\gamma\leq 1;\\
&(2^{v-1}\Gamma(\nu))^{-1}(c\sqrt{t})^{\nu}K_{\nu}(c\sqrt{t}),\ c>0,\nu>0;\\
&(1+ct^{\gamma})^{\nu},\ c>0, 0<\gamma\leq 1, \nu>0;\\
&2^{\nu}(e^{c\sqrt{t}}+e^{-c\sqrt{t}})^{-\nu},\ c>0,\nu>0;\\
&E_{\alpha,\beta}(-t),\ \ (0<\alpha<1,\beta=1) \text{ or } (0<\alpha\leq 1,\beta\geq\alpha);\\
&E_{\alpha,1}(-t^{\gamma}),\ 0<\alpha<1, 0<\gamma<1.
\end{aligned}$$

In this list, K_ν is the modified Bessel function and $E_{\alpha,\beta}(-x)$, $x>0$ is the Mittag-Leffler function of the negative real argument (see formula (2.32) below).

If we assume that

$$\sigma(\{0\})=0,\ 0<\sigma([0,\infty))<\infty,\ \int_0^\infty \lambda^2\sigma(d\lambda)<\infty,$$

then the spectral density $f_X(\omega)$ is given by

$$f_X(\omega)=\frac{1}{\pi}\int_0^\infty \frac{\lambda}{\omega^2+\lambda^2}\sigma(d\lambda),\ \ \omega\in\mathbb{R}-\{0\}, \tag{2.8}$$

and

$$f_X(\omega)\in L_1(\mathbb{R}).$$

Example 2.1. *If we define the Borel measure σ on $(0,\infty)$ according to the gamma distribution:*

$$\sigma(d\lambda)=\lambda^{\alpha-1}e^{-\lambda}/\Gamma(\alpha),\ \alpha>0, \tag{2.9}$$

then from (2.2) we obtain

$$R_X(t)=\frac{1}{(1+|t|)^{\alpha}},\ t\in\mathbb{R}, \tag{2.10}$$

and the corresponding spectral density function is given by

$$f_X(\omega)=\frac{1}{\pi}\operatorname{Im}\int_0^\infty \frac{e^{-y\omega}dy}{(1+e^{-i\pi/2}y)^{\alpha}},\ \omega>0, \tag{2.11}$$

which is also known as the probability density of the generalized Linnik distribution (Erdogăn and Ostrovskii [24]). Analytic and asymptotic properties of the function (2.11) has been studied by Erdogăn and Ostrovskii [24]. In particular, for $0<\alpha<1$, $\omega\downarrow 0$,

$$f_X(\omega)=\frac{1}{2\Gamma(\alpha)\cos\left\{\frac{\alpha\pi}{2}\right\}}\frac{1}{|\omega|^{1-\alpha}}(1-\theta(\omega)),\ \theta(\omega)\to 0, \tag{2.12}$$

while for $\alpha=1$, $\omega\downarrow 0$

$$f_X(\omega)=\frac{1}{\pi}\log\frac{1}{|\omega|}-\frac{\gamma}{\pi}+\frac{1}{2}|\omega|^2\log\frac{1}{|\omega|}+O\left(\omega^2\right),$$

where $\gamma=\Gamma'(1)$ is Euler constant. Thus, for $\alpha\in(0,1]$ the process X with covariance function (2.10) and spectral density (2.11) displays long-range dependence.

2.2. **Fractional Stokes-Boussinesq-Langevin equation.** Let us recall some definitions of fractional derivatives (see Caputo [13], Caputo and Mainardi [14], Miller and Ross [45], Samko *et al.* [56], Djrbashian [17], Podlubny [55] among others).

Under certain natural conditions on the real-valued function $f(t)$, the Caputo fractional derivative of order $\beta \in [n-1, n)$, $n = 1, 2, ...$, is defined as

$$ {}_a\mathcal{D}_C^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t \frac{(d^n/d\tau^n) f(\tau)}{(t-\tau)^{\beta+1-n}} d\tau, \tag{2.13}$$

while the Riemann-Liouville fractional derivative of order $\beta \in [n-1, n)$, $n = 1, 2, ...$, is defined as

$$ {}_a\mathcal{D}_{RL}^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\beta+1-n}} d\tau. \tag{2.14}$$

The main advantage of Caputo's definition is that the fractional derivative of a constant C is equal to zero: ${}_0\mathcal{D}_C^\beta C = 0$, while in the Riemann-Liouville definition we have

$$ {}_0\mathcal{D}_{RL}^\beta C = Ct^{-\beta}/\Gamma(1-\beta), \ 0 \le \beta < 1.$$

Putting $a = -\infty$ in both definitions (2.13) and (2.14) and requiring reasonable behavior of $f(t)$ and its derivatives for $t \to -\infty$, we arrive at the same formula

$$ \mathcal{D}^\beta f(t) = {}_{-\infty}\mathcal{D}_C^\beta f(t) = {}_{-\infty}\mathcal{D}_{RL}^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_{-\infty}^t \frac{(d^n/d\tau^n) f(\tau)}{(t-\tau)^{\beta+1-n}} d\tau, \tag{2.15}$$

where $n-1 \le \beta < n$, $n = 1, 2, ...$ The fractional derivative (2.14) is also called Weyl's fractional derivative (see Samko *et al.* [56], p. 356).

In order to obtain exact formulae for Eq. (1.6) instead of asymptotic expressions, we will widely use the one-parameter and two-parameter Mittag-Leffler functions (see, for example, Djrbashian [17]). In particular, the entire function of order $1/\alpha$ of type 1

$$ E_\alpha(z) = \sum_{j=0}^\infty \frac{z^j}{\Gamma(\alpha j+1)}, \ z \in \mathbb{C}, \ \alpha > 0$$

is known as the one-parameter Mittag-Leffler function. For real $x \ge 0$ the function

$$ E_\alpha(-x) = \sum_{j=0}^\infty \frac{(-1)^j x^j}{\Gamma(\alpha j+1)}, \ x \ge 0, \ 0 < \alpha \le 1 \tag{2.16}$$

is infinitely differentiable and completely monotone. It follows from the definition that

$$ E_1(-x) = e^{-x}, \ E_{1/2}(-x) = e^{x^2}\left(1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt\right), \ x \ge 0.$$

From Djrbashian [17], p. 5, we obtain the following asymptotic formula:

$$ E_\alpha(-x) = -\sum_{k=1}^N \frac{(-1)^k x^{-k}}{\Gamma(1-\alpha k)} + O\left(|x|^{-N-1}\right), \ 0 < \alpha < 1, \tag{2.17}$$

as $x \to \infty$.

The next theorem is concerned with the fractional Stokes-Boussinesq-Langevin equation (1.6) for the case $\lambda = 0$. The uniqueness of its stationary solution is obtained under the causality condition

$$ \Sigma_t(X) = \Sigma_t(W) \tag{2.18}$$

for any $t \in \mathbb{R}$, where $\Sigma_t(Y)$ denotes the closed linear hull of $\{Y(\phi), \phi \in \mathcal{D}(\mathbb{R}), \text{ supp}\{\phi\} \subset (-\infty, t]\}$ in $L_2(\Omega, \mathcal{F}, P)$, $(\Omega, \mathcal{F}, P)$ being the underlying complete probability space and $\mathcal{D}(\mathbb{R})$ being the space of all $\phi \in C^\infty(\mathbb{R})$ with compact support (see Appendix B for further details). It should be noted that Eq. (1.6) is not an Itô stochastic differential equation in general because the fractional operator (2.15) is not local. For $\lambda \geq 0$, $b > 0$, Eq. (1.6) is a particular case of the second KMO-Langevin equation (Okabe [51], Inoue [33]).

Theorem 2.1. *There exists a unique stationary solution X (in the sense of random distributions) of the fractional Stokes-Boussinesq-Langevin equation (1.6) with $\lambda = 0$ and spectral density of the Kubo noise of the form*

$$f_W(\omega) = \gamma b \sqrt{\frac{2}{\pi}} \cos\left\{\frac{(1-\alpha)\pi}{2}\right\} |\omega|^{1-\alpha}, \; 0 < \alpha < 1, \; \gamma > 0, \; b > 0, \tag{2.19}$$

under the causality condition (2.18). The solution X is a purely nondeterministic zero-mean stationary Gaussian process having the following properties:

$$R_X(0) = \gamma\sqrt{2\pi}; \tag{2.20}$$

$$\rho_X(t) = R_X(t)/R_X(0) = E_\alpha(-b|t|^\alpha), \; t \in \mathbb{R}, \; 0 < \alpha < 1; \tag{2.21}$$

for $\alpha \in (0,1)$ and large $t > 0$

$$\rho_X(t) = \frac{1}{|t|^\alpha b\Gamma(1-\alpha)} + O\left(\frac{1}{|t|^{2\alpha}}\right); \tag{2.22}$$

for $\alpha \in (0,1)$, X has reflection positivity, that is, (2.2) holds with

$$\sigma(d\lambda) = \frac{\sin\{\pi\alpha\}}{\pi} \frac{\lambda^{\alpha-1}}{1 + 2\cos\{\alpha\pi\}\lambda^\alpha + \lambda^{2\alpha}} d\lambda; \tag{2.23}$$

its spectral density is

$$f_X(\omega) = \frac{\sin\{\pi\omega\}}{\pi^2} \int_0^\infty \frac{u^\alpha du}{(\omega^2 + u^2)(1 + 2\cos\{\alpha\pi\}u^\alpha + u^{2\alpha})}, \; \omega > 0, \; 0 < \alpha < 1; \tag{2.24}$$

the correlation function (2.21) is the unique solution of the Cauchy problem for the fractional differential equation

$${}_0\mathcal{D}_C^\alpha \rho_X(t) + b\rho_X(t) = 0, \; b > 0, \; \rho_X(0) = 1; \tag{2.25}$$

the process X has the time-domain representation as a random distribution:

$$\begin{aligned} X(t) &= \lim_{M\to\infty} \mathbf{1}_{(0,M)} E_\alpha(-b|.|^\alpha) * W(t) \\ &= \int_{-\infty}^t E_\alpha(-b|t-s|^\alpha) W(s)\, ds, \; 0 < \alpha < 1. \end{aligned} \tag{2.26}$$

Proof. The existence and uniqueness of the stationary solution of equation (1.6) with $\lambda = 0$ follows from Theorems 1.1 and 1.2 of Inoue [33]. The expression (2.19) of the Kubo noise is given in Example 5.11 of Inoue [33]). The result (2.20) is part of Theorem 5.10 of Inoue [33]). Moreover, from (ii) of Theorem 1.2 of Inoue [33] we obtain

$$\int_0^\infty e^{i\zeta t} (R_X(t)/R_X(0))\, dt = \left(-i\zeta - i\zeta \int_0^\infty e^{i\zeta t} \gamma(t)\, dt\right)^{-1},$$

$\operatorname{Im} \zeta > 0$, which, with

$$\gamma(t) = bt^{\alpha-1}/\Gamma(\alpha), \ 0 < \alpha < 1, \tag{2.27}$$

and $\zeta = ip$, reduces to the following equation:

$$\int_0^\infty e^{-pt} \rho_X(t)\, dt = \frac{1}{p + bp^{1-\alpha}}, \ p > 0. \tag{2.28}$$

It is known (see for example Samko *et al.* [56], p. 21) that the Laplace transform of the Mittag-Leffler function (2.21) is

$$\int_0^\infty e^{-pt} E_\alpha(-b\,|t|^\alpha)\, dt = \frac{p^{\alpha-1}}{p^\alpha + b}, \ \operatorname{Re} p > |b|^{1/\alpha}, \tag{2.29}$$

$0 < \alpha < 1$. Thus, (2.21) follows from (2.27) and (2.28). The formula (2.22) then follows from (2.17), and (2.23) is a particular case of Theorems 1.3-5 of Djrbashian [17]. The formula (2.24) follows from (2.8) and (2.23). The fractional differential equation (2.25) is solved by (2.21) in Djrbashian and Nersesian [18], which uses the same definition of fractional derivatives as Caputo's. The representation (2.26) is a particular case of Theorem 1.2 of Inoue [33], while (2.19) follows from (5.17) of Inoue [33] with appropriate choice of $\gamma(t)$ according to (2.27). □

Remark 2.1. *If $\alpha = 1$, the corresponding equation is (2.4) and the covariance function of its stationary solution is given by (2.6). The correlation function (2.21) formally reduces to (2.6) (up to constants) in this case. For $\alpha \in (0,1)$ the process X given in Theorem 2.1 displays LRD, that is,*

$$\int_0^\infty \rho_X(\tau)\, d\tau = \infty. \tag{2.30}$$

The exact formula (2.21) in terms of the Mittag-Leffler function thus gives a complete interpolation between the exponential covariance function of OU processes and the hyperbolic covariance function of LRD processes. The representation (2.26) of the process itself also interpolates the moving-average representations of OU processes and LRD processes.

In what follows we need the two-parameter Mittag-Leffler function (see again Djrbashian [17], p. 1-6) which can be defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha > 0, \ \beta > 0, \ z \in \mathbb{C}. \tag{2.31}$$

It is clear that $E_{\alpha,1}(z) = E_\alpha(z)$, and $E_{1,1}(z) = e^z$, $E_{2,1}(z) = \cosh\sqrt{z}$, $E_{2,2}(z) = (\sinh\sqrt{z})/\sqrt{z}$, $E_{1,2}(z) = (e^z - 1)/z$, $E_{1,3}(z) = (e^z - 1 - z)/z^2$. If $\alpha < 1$, $\beta \geq \alpha$ the function

$$E_{\alpha,\beta}(-u) = \sum_{k=0}^\infty \frac{(-1)^k u^k}{\Gamma(\alpha k + \beta)}, \ u \geq 0, \tag{2.32}$$

is completely monotone, that is,

$$E_{\alpha,\beta}(-u) = \int_0^\infty e^{-u\tau} q_{\alpha,\beta}(\tau)\, d\tau, \tag{2.33}$$

where

$$q_{\alpha,\beta}(\tau) = -\frac{1}{\pi} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \Gamma(1 - \beta + \alpha(k+1)) \sin\{\pi(\alpha(k+1) - \beta)\}\, \tau^k \geq 0.$$

Theorem 2.2. *There exists a unique stationary solution X (in the sense of distributions) of the fractional Stokes-Boussinesq-Langevin equation (1.6) with $\lambda > 0$, $b > 0$ and the spectral density of the Kubo noise of the form*

$$f_W(\omega) = \gamma\sqrt{\frac{2}{\pi}}\left(\lambda + b\int_0^\infty \frac{\omega^2}{u^2+\omega^2}\frac{du}{u^\alpha}\right),\ \omega \in \mathbb{R}-\{0\},\ 0<\alpha<1. \tag{2.34}$$

The solution X is a stationary Gaussian process with reflection positivity and correlation function of the form

$$\rho_X(t) = R_X(t)/R_X(0) = \sum_{k=0}^\infty \frac{(-1)^k}{k!}(\lambda t)^k E^{(k)}_{\alpha,1+k-\alpha\kappa}(-bt^\alpha), \tag{2.35}$$

where $E^{(0)}_{\alpha,\beta}(x) = E_{\alpha,\beta}(x)$ is defined by (2.32) and

$$E^{(k)}_{\alpha,\beta}(x) = \frac{d^k}{dx^k}E_{\alpha,\beta}(x) = \sum_{j=0}^\infty \frac{x^j(j+k)!}{j!\Gamma(\alpha j+\alpha k+\beta)},\ k=1,2,... \tag{2.36}$$

Proof. The existence and uniqueness of a stationary solution of (1.6) with $\lambda > 0$, $b > 0$ is given in Okabe [51]. The causality condition (2.18) follows in this solution. The expression (2.34) is the spectral density for Kubo noise of model (6.6) of Inoue [33] corresponding to the response function $\gamma(t)$ of (2.27). With this choice of $\gamma(t)$, we obtain from (6.1) of Inoue [33] that

$$\int_0^\infty e^{-pt}\rho_X(t)\,dt = \left(\lambda + p + bp^{1-\alpha}\right)^{-1},\ p>0. \tag{2.37}$$

From Djrbashian [17], Podlubny [55], we can obtain the following expression for the Laplace transform of the function (2.36):

$$\int_0^\infty e^{-pt}t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(\pm bt^\alpha)\,dt = \frac{k!p^{\alpha-\beta}}{(p^\alpha \mp b)^{k+1}},\ \mathrm{Re}\,p > |b|^{1/\alpha},\ k=0,1,... \tag{2.38}$$

and for $0<\alpha<1$ the function (2.37) can be written as

$$\frac{1}{\lambda}\frac{\lambda p^{\alpha-1}}{p^{1-\alpha}+b}\left[1+\frac{\lambda p^{\alpha-1}}{p^{1-\alpha}+b}\right] = \frac{1}{\lambda}\sum_{k=0}^\infty (-1)^k\lambda^{k+1}p^{\alpha-1+k(\alpha-1)}/\left(p^{1-\alpha}+b\right)^{k+1}. \tag{2.39}$$

Term-by-term inversion of (2.39) based on the general expansion theorem for the Laplace transform using (2.38) produces (2.35). □

Remark 2.2. *Putting $b = 0$ formally in (2.35) we obtain*

$$\rho_X(t) = \sum_{k=0}^\infty \frac{(-1)^k}{k!}(\lambda t)^k = e^{-\lambda t},\ t\geq 0,$$

which is the correlation function (2.3) of the OU process (2.4). Moreover, putting $\lambda = 0$ formally in (2.35) with $b > 0$ we obtain

$$\rho_X(t) = E_{\alpha,1}(-bt^\alpha),\ t\geq 0,$$

which coincides with the correlation function (2.21) of the stationary solution to (1.6) with $\lambda = 0$, $\alpha \in (0,1]$.

Remark 2.3. *The correlation function (2.35) is the inverse Laplace transform of the function*

$$g_\alpha(p) = \frac{1}{\lambda + p + bp^{1-\alpha}},\ p > 0,\ 0 < \alpha < 1.$$

The behaviour of $g_\alpha(p)$ as $p \to 0$ is $c\left(1 - O\left(p^{1-\alpha}\right)\right)$, c being a constant. This yields, via Watson's lemma, the behaviour $\rho_X(t) = O\left(t^{2-\alpha}\right)$ as $t \to \infty$. Thus, for $\alpha = 1/2$ the asymptotic behaviour of the correlation function is $\rho_X(t) = O\left(t^{-3/2}\right)$, $t \to \infty$, which is the Alder-Wainwright effect; but the exact expression (2.35) for the correlation function is more informative.

3. Fractional Stokes-Boussinesq-Langevin equation driven by fractional Gaussian noise

3.1. **A fractional Stokes-Boussinesq-Langevin equation.** Fractional Brownian motion (FBM), $B_H = \{B_H(t), t \in \mathbb{R}\}$, with Hurst parameter $H \in (0,1)$, is a Gaussian, mean-zero and H-self-similar process with $B_H(0) = 0$ and stationary increments. By H-self-similarity we mean that, for $a > 0$, $\{B_H(at), t \in \mathbb{R}\} \stackrel{d}{=} \{a^H B_H(t), t \in \mathbb{R}\}$, where $\stackrel{d}{=}$ stands for equality in finite-dimensional distributions. The FBM B_H with $H = \frac{1}{2}$ is the usual Brownian motion $B = \{B(t), t \in \mathbb{R}\}$.

Samorodnitsky and Taqqu [57] provided an introduction to FBM. For a detailed treatment of FBM we refer to Mishura [47]. We note that the covariance function of FBM is

$$\operatorname{Cov}(B_H(t), B_H(1)) = \frac{c}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}, \quad t, s \in \mathbb{R}, \tag{3.1}$$

where $c = \operatorname{Var} B_H(1)$, while the covariance function of an increment of FBM is given by

$$\begin{aligned}
& \operatorname{Cov}(B_H(t+h) - B_H(h), B_H(t+s+h) - B_H(s+h)) \\
= \ & \operatorname{Cov}(B_H(t), B_H(t+s) - B_H(s)) \\
= \ & c \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \left(\prod_{k=0}^{2n-1} (2H-k) \right) s^{2H-2n} \\
= \ & c \sum_{n=1}^{N} \frac{t^{2n}}{(2n)!} \left(\prod_{k=0}^{2n-1} (2H-k) \right) s^{2H-2n} + O\left(s^{2H-2N-2}\right),\ s \to \infty,
\end{aligned}$$

for every $N \in \{1, 2, \ldots\}$ and all $0 < t < s, h \in \mathbb{R}$. If we denote $R_H(n) = \operatorname{Cov}(B_H(t), B_H((n+1)t) - B_H(nt))$, then, for $H \in \left(0, \frac{1}{2}\right)$, $\sum_{n=-\infty}^{\infty} R_H(n) = 0$ (negative correlation property) and, for $H \in \left(\frac{1}{2}, 1\right)$, $\sum_{n=-\infty}^{\infty} |R_H(n)| = \infty$ (long-range dependence).

FBM admits a time-domain representation in the form of Itô stochastic integral with respect to standard Brownian motion $B(t)$:

$$B_H(t) = \frac{1}{\Gamma(H + \frac{1}{2})} \int_{\mathbb{R}} [g_H(t-s) - g_H(-s)] dB(s), \quad t \in \mathbb{R},$$

where $g_H(s) = s^{H-\frac{1}{2}} 1_{(0,\infty)}(s)$. We consider the spectral representation

$$B(t) = \int_{\mathbb{R}} \frac{e^{-it\omega} - 1}{-i\omega} Z(d\omega),$$

where $Z(d\omega)$ is a complex Gaussian random measure with

$$\mathrm{E}|Z(d\omega)|^2 = \sigma^2 d\omega.$$

Then the spectral representation of FBM is

$$B_H(t) = \int_{\mathbb{R}} \frac{e^{-it\omega} - 1}{-i\omega} \frac{1}{(-i\omega)^{H-\frac{1}{2}}} Z(d\omega), \tag{3.2}$$

from which we get a formal representation of the derivative process, which exists only in the sense of random distributions:

$$\frac{d}{dt} B_H(t) = \dot{B}_H(t) = \int_{\mathbb{R}} e^{-it\omega} (-i\omega)^{\frac{1}{2}-H} Z(d\omega), \tag{3.3}$$

where

$$(-i\omega)^{\frac{1}{2}-H} = \lim_{\eta \downarrow 0} (-i\zeta)^{\frac{1}{2}-H}, \ \zeta = \omega + i\eta$$

and we choose the branch of $(-i\zeta)^{\frac{1}{2}-H}$ such that $(-i\zeta)^{\frac{1}{2}-H}\Big|_{\zeta=i} = 1$. The above formulae show that we can consider the fractional noise $\dot{B}_H = \{\dot{B}_H(t), t \in \mathbb{R}\}$ as a random distribution with spectral density

$$\sigma^2 |\omega|^{1-2H}, \ H \in (0,1). \tag{3.4}$$

Remark 3.1. *The formulae (3.2)-(3.3) correspond to the definition of the outer function $h(\xi)$ as the boundary value of an analytic function $h(\zeta)$ in the upper half-plane Im $\zeta > 0$ (see Appendix C). More often (see, for instance, Samorodnitsky and Taqqu* [57] *or Igloi and Terdik* [32]*) the expressions such as*

$$B(t) = \int_{\mathbb{R}} \frac{e^{it\omega} - 1}{i\omega} Z(d\omega)$$

are used. These expressions correspond to considering the Hardy functions in the lower half-plane Im $\zeta < 0$. Thus, to use these latter expressions, we must change the definition of the outer function (C.1) such that this function becomes analytical in Im $\zeta < 0$.

Remark 3.2. *The variance of $B_H(t)$ has the spectral representation*

$$\mathrm{Var} B_H(t) = 4 \int_0^\infty (1 - \cos \omega t) f_{B_H}(\omega) d\omega,$$

where, in view of (3.4), $f_{B_H}(\omega) = \sigma^2 |\omega|^{-(2H+1)}$, $\omega \in \mathbb{R}$, is the spectral density of $B_H(t)$. We have denoted above that $\mathrm{Var} B_H(1) = c$*; thus the connection between σ^2 and c is* $\sigma^2 = \frac{c}{4\int_0^\infty (1-\cos\omega)|\omega|^{-(2H+1)} d\omega} = \frac{c\Gamma(2H+1)\sin(\pi H)}{2\pi}$.

For a nonrandom function f, integration with respect to FBM B_H can be based on formal calculations:

$$\begin{aligned} \int_{\mathbb{R}} f(t) dB_H(t) &= \int_{\mathbb{R}} f(t) \dot{B}_H(t) dt \\ &= \int_{\mathbb{R}} e^{-it\omega} \int_{\mathbb{R}} f(t) (-i\omega)^{\frac{1}{2}-H} Z(d\omega) dt \\ &= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} e^{-it\omega} f(t) dt \right\} (-i\omega)^{\frac{1}{2}-H} Z(d\omega). \end{aligned}$$

A precise meaning is given by the following definition.

Definition 1. *Let $f \in L_2(\mathbb{R})$ be a non-random real-valued function and*

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{-it\omega} f(t)dt \right|^2 |\omega|^{1-2H} d\omega < \infty.$$

Then

$$\int_{\mathbb{R}} f(t)dB_H(t) \stackrel{def}{=} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-it\omega} f(t)dt \right) (-i\omega)^{\frac{1}{2}-H} Z(d\omega) \tag{3.5}$$

for $H \in (0,1)$.

Note that because FBM is not a semimartingale, more advanced tools have been developed to handle integration with respect to FBM in both time and frequency domains (Igloi and Terdik [32], Alós, Mazet and Nualart [4], Pipiras and Taqqu [54]).

As an application of Theorem 2.1, we consider a fractional Stokes-Boussinesq-Langevin equation in the form

$$\dot{X}(t) = -b\mathcal{D}^{1-2H} X(t) + \dot{B}_H(t), \quad b \geq 0, t \in \mathbb{R}, \tag{3.6}$$

where the fractional derivative $\mathcal{D}^{1-2H}, H \in (0, \frac{1}{2}]$ is defined in Eq. (2.15). Then

$$R_X(0) = b\sqrt{2\pi} \tag{3.7}$$

and

$$\rho_X(t) = \frac{R_X(t)}{R_X(0)} = E_{2H}\left(-b|t|^{2H}\right), \quad t \in \mathbb{R}. \tag{3.8}$$

In view of (2.26) and (3.8), we consider a stationary solution of (3.6) in the form

$$X(t) = \frac{1}{R_X(0)} \int_{-\infty}^{t} R_X(t-s)\dot{B}_H(s)ds. \tag{3.9}$$

We write the spectral representation of the noise $\dot{B}_H$ in the form of stochastic integral with transfer function $h_{\dot{B}}$:

$$\dot{B}_H(t) = \int_{\mathbb{R}} e^{-it\omega} h_{\dot{B}}(\omega) Z(d\omega). \tag{3.10}$$

Since Eq. (3.9) is of convolution type, we obtain by Parseval 's identity and 3.10) the spectral representation of $X(t)$ as

$$X(t) = \int_{\mathbb{R}} e^{-it\omega} \left(\frac{1}{R_X(0)} \int_0^{\infty} e^{is\omega} R_X(s)ds \right) h_{\dot{B}}(\omega) Z(d\omega),$$

that is,

$$X(t) = \int_{\mathbb{R}} e^{-it\omega} h_X(\omega)\, Z(d\omega), \tag{3.11}$$

with

$$h_X(\omega) = \frac{1}{R_X(0)} \left(\int_0^{\infty} e^{is\omega} R_X(s)ds \right) h_{\dot{B}}(\omega).$$

Then, by Eqs. (3.6), (3.10) and (3.11) we get

$$\int_{\mathbb{R}} e^{-it\omega} \left\{ -i\omega + b(-i\omega)^{1-2H} \right\} h_X(\omega) Z(d\omega) = \int_{\mathbb{R}} e^{-it\omega} h_{\dot{B}}(\omega) Z(d\omega),$$

or

$$\left[-i\omega + b(-i\omega)^{1-2H} \right] h_X(\omega) = h_{\dot{B}}(\omega),$$

which yields

$$|-i\omega + b(-i\omega)^{1-2H}|^2 f_X(\omega) = f_{\dot{B}}(\omega).$$

The spectral density of $X(t)$ is then given by

$$f_X(\omega) = \frac{f_{\dot{B}}(\omega)}{|-i\omega + b(-i\omega)^{1-2H}|^2}\ .$$

Using (3.4), we obtain for $H \in (0, \frac{1}{2}]$ that

$$f_X(\omega) = \frac{\sigma^2 \omega^{1-2H}}{\omega^2 + b^2\omega^{2(1-2H)} + 2b\omega^{2(1-H)} \sin\left(\pi\left(\frac{1}{2} - H\right)\right)}, \quad \omega \in \mathbb{R}. \tag{3.12}$$

3.2. **Ornstein-Uhlenbeck equation driven by fractional Brownian motion.** We have seen in the subsection above that the linear response theory with Kubo noise works for the case $0 < H < 1/2$. In this subsection we pay attention to the case $1/2 < H < 1$ of strongly correlated noise. We consider the linear stochastic differential equation driven by FBM:

$$dX(t) = -\lambda X(t)dt + dB_H(t), \quad \lambda > 0, t \in \mathbb{R}. \tag{3.13}$$

Note that Eq. (3.13) was discussed by a number of authors including Comte and Renault [16], Igloi and Terdik [32], Cheridito *et al.* [15]. One can show that there exists a unique continuous solution of Eq. (3.13) in the form

$$X(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} dB_H(s) \tag{3.14}$$

or, in the frequency domain,

$$X(t) = \int_{\mathbb{R}} e^{-it\omega} \frac{1}{-i\omega + \lambda} (-i\omega)^{-H+\frac{1}{2}} Z(d\omega), \quad t \in \mathbb{R}, \tag{3.15}$$

for a complex Gaussian random measure $Z(d\omega)$ with

$$\mathrm{E}|Z(d\omega)|^2 = \sigma^2 d\omega.$$

Thus, the stationary process (3.14) has spectral density

$$f_X(\omega) = \frac{\sigma^2}{\omega^2 + \lambda^2} |\omega|^{1-2H}, \quad \omega \in \mathbb{R}, \tag{3.16}$$

and covariance function

$$R_X(t) = \frac{1}{2}\sigma^2 \sum_{n=1}^{N} \lambda^{-2n} \left(\prod_{k=0}^{2n-1} (2H - k) \right) t^{2H-2n} + O\left(t^{2H-2N-2}\right)$$

as $t \to \infty$, for any $N = 1, 2, \ldots$ and $H \neq \frac{1}{2}$. In particular, for $H \in \left(\frac{1}{2}, 1\right)$ the process (3.14) is stationary and exhibits long-range dependence, that is,

$$\int_{\mathbb{R}} R_X(s) ds = \infty,$$

while, for $H \in \left(0, \frac{1}{2}\right)$, it has the negative correlation property

$$\int_{\mathbb{R}} R_X(s) ds = 0.$$

We assume for simplicity that $\sigma^2 = 1, \lambda = 1$. In the former case, the outer function and canonical representation kernel can be obtained explicitly (see Inoue and Kasahara [34]). By applying Exercises 2.3.4 and 2.7.2 of Dym and McKean [21] to the rational functions

$$\begin{aligned} \frac{1}{1-i\zeta} &= \exp\left\{\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1+\omega\zeta}{\omega-\zeta}\frac{\log\left(1+\omega^2\right)^{-1}}{1+\omega^2}d\omega\right\}, \quad \operatorname{Im}\zeta > 0, \\ -i\zeta &= \exp\left\{\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1+\omega\zeta}{\omega-\zeta}\frac{\log\omega^2}{1+\omega^2}d\omega\right\}, \quad \operatorname{Im}\zeta > 0 \end{aligned}$$

(noting that both $\frac{1}{(1-i\zeta)}$ and $-i\zeta$ are positive on the upper imaginary axis), we obtain for the outer function (C.1) an explicit formula:

$$h(\zeta) = \frac{(-i\zeta)^{\frac{1}{2}-H}}{1-i\zeta}, \quad \operatorname{Im}\zeta > 0. \tag{3.17}$$

For the function (3.17) we have

$$h(\zeta) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{i\zeta t}E(t)dt \tag{3.18}$$

with

$$E(t) = \frac{\sqrt{\pi}}{\Gamma\left(H-\frac{1}{2}\right)}\int_0^t e^{s-t}s^{H-\frac{3}{2}}ds, \quad t>0. \tag{3.19}$$

Thus, the covariance function of the process (3.14) with $\sigma^2 = 1, \lambda = 1$ has the remarkable representation

$$R(t) = \frac{1}{2\pi}\int_0^\infty E\left(|t|+s\right)E(s)ds \tag{3.20}$$

with the function E given by (3.19). To our knowledge, this is the only case where the canonical representation (C.3) can be written explicitly, unless the process is an Ornstein-Uhlenbeck process.

4. Stationary processes related to fractional Riesz-Bessel motion

Fractional Riesz-Bessel motion (FRBM) was introduced in Anh *et al.* [5] and further investigated in Anh *et al.* [7]. Its model is governed by the operator $-\left(I-\Delta\right)^{\beta/2}\left(-\Delta\right)^{\gamma/2}$, where Δ is the Laplace operator, the fractional operators $\left(I-\Delta\right)^{\beta/2}$ and $\left(-\Delta\right)^{\gamma/2}$ are the inverses of the Bessel and Riesz potentials respectively. Formally, a real-valued Gaussian process $X(t)$ which has (i) $X(0) = 0$ a.s., (ii) stationary increments, and (iii) spectral density of the form

$$f_X(\omega) = \frac{const}{|\omega|^{2\gamma}\left(1+\omega^2\right)^\beta}, \; 1/2 < \gamma < 3/2, \beta \geq 0, \omega \in \mathbb{R},$$

is called a fractional Riesz-Bessel motion (FRBM). Fractional Brownian motion can be deduced from fractional Riesz-Bessel motion by putting $\beta = 0, \gamma = H + 1/2$. If $\gamma \in (1, 3/2)$, the process displays long-range dependence (as $|\omega| \to 0$). The sum $\gamma + \beta$ describes clustering of extreme values (as $|\omega| \to \infty$) of FRBM (see Anh *et al.* [6]).

The spectral density of the stationary increments of FRBM has the form

$$f(\omega) = \frac{const}{|\omega|^{2\delta}\left(1+\omega^2\right)^{\beta+1}}, \; \omega \in \mathbb{R}, \tag{4.1}$$

where $\delta = \gamma - 1 \in (-1/2, 1/2)$, $\beta > -1$. In this section we introduce a model related to stationary fractional Riesz-Bessel motion which has spectral density similar to (4.1). This model is based on the fractional Stokes-Boussinesq-Langevin equation with stationary random noise.

Consider again a fractional Stokes-Boussinesq-Langevin equation of the following form

$$\dot{X}(t) = -\lambda X(t) - b\mathcal{D}^{\delta} X(t) + W(t), \ t \in \mathbb{R}, \tag{4.2}$$

where $\lambda \geq 0$, $b \geq 0$, $\delta \in (0,1)$ and $W(t)$, $t \in \mathbb{R}$ is stationary random noise with spectral distribution $F_W(\omega)$ and spectral density $f_W(\omega)$. The fractional derivative $\mathcal{D}^{\delta}$ is defined in (2.15). As we cannot interpret Eq. (4.2) in the Itô sense, we will use the approach proposed by Wong and Hajek [66], pp. 78–116. In particular, every mean-square continuous second-order stationary process $\eta(t)$, $t \in \mathbb{R}$, with mean zero can be represented as

$$\eta(t) = \int_{\mathbb{R}} e^{-i\omega t} d\hat{\eta}(\omega), \tag{4.3}$$

where $\hat{\eta}(\omega)$ is the spectral process with orthogonal increments such that

$$\mathrm{E}\left[\hat{\eta}(a) - \hat{\eta}(b)\right] \overline{\left[\hat{\eta}(c) - \hat{\eta}(d)\right]} = F_{\eta}([a,b) \cap [c,d)).$$

Then

$$\int_{\mathbb{R}} g(\omega)\, d\hat{\eta}(\omega) = \int_{\mathbb{R}} \hat{g}(t)\, \eta(t)\, dt, \ g \in \mathcal{D}(\mathbb{R}),$$

where $\hat{g}$ denotes the Fourier transform of g, $\mathcal{D}(\mathbb{R})$ is the space of all functions in $C^{\infty}(\mathbb{R})$ with compact support (see Appendix B). In particular, $\eta(t)$ can be (generalized) white noise. The integral $\int_{\mathbb{R}} \hat{g}(t)\, \eta(t)\, dt$ can be handled by replacing it with the second-order stochastic integral $\int_{\mathbb{R}} g(\omega)\, Z_{\eta}(d\omega)$, where Z_{η} is a random measure such that $\mathrm{E}\left|Z_{\eta}([a,b))\right|^2 = F_{\eta}([a,b))$. Then

$$\mathcal{D}^{\delta}\eta(t) = \int_{\mathbb{R}} (-i\omega)^{\delta} e^{-it\omega} d\hat{\eta}(\omega) = \int_{\mathbb{R}} (-i\omega)^{\delta} e^{-it\omega} Z_{\eta}(d\omega). \tag{4.4}$$

We will write Z_X, Z_W for the random measures corresponding to $X(t)$ and $W(t)$ of (4.2) in this approach. Since (4.2) can be rewritten as

$$\left(\mathcal{D} + b\mathcal{D}^{\delta} + \lambda\right) X(t) = W(t), \tag{4.5}$$

we have

$$Z_X(d\omega) = \left[-i\omega + b(-i\omega)^{\delta} + \lambda\right]^{-1} Z_W(d\omega).$$

Thus, there exists a stationary solution of Eq. (4.2) with $\lambda > 0$, $b > 0$ (Gaussian if $W(t)$ is Gaussian) of the form

$$X(t) = \int_{\mathbb{R}} e^{-it\omega} \frac{1}{-i\omega + b(-i\omega)^{\delta} + \lambda} Z_W(d\omega), \ 0 < \delta < 1, \tag{4.6}$$

where

$$\mathrm{E}\left|Z_W(d\omega)\right|^2 = f_W(\omega)\, d\omega.$$

The spectral density of the process (4.6) is of the form

$$f_X(\omega) = \frac{f_W(\omega)}{\left|g(\omega)\right|^2}, \ \omega \in \mathbb{R}, \tag{4.7}$$

where

$$g(\omega) = -i\omega + b(-i\omega)^{\delta} + \lambda = \lambda + |\omega| e^{-\frac{i\pi}{2}} + b|\omega|^{\delta} e^{-\frac{i\pi\delta}{2}}.$$

A direct calculation yields

(4.8)

$$|g(\omega)|^2 = \lambda^2 + |\omega|^2 + 2b|\omega|^{1+\delta} \cos\frac{\pi(1-\delta)}{2} + 2\lambda b|\omega|^{\delta} \cos\frac{\pi\delta}{2} + b^2|\omega|^{2\delta}, \ 0 < \delta < 1, \ \omega \in \mathbb{R}.$$

Suppose now that the correlation function of the random noise $W(t)$ is of the form

$$\rho_W(t) = c_1(\beta)|t|^{\beta+1/2} K_{\beta+1/2}(|t|), \ t \in \mathbb{R}, \ \beta > -1/2, \tag{4.9}$$

where

$$c_1(\beta) = \left[\Gamma(\beta+1/2) 2^{\beta-1/2}\right]^{-1}.$$

Here, $K_{\mu}(z)$ is the modified Bessel function of the third kind of order μ (Abramowitz and Stegun [1]). Note that

$$K_{\mu}(z) = K_{-\mu}(z), \ z > 0; \ K_{\mu}(z) \sim \Gamma(\mu) 2^{\mu-1} z^{-\mu}, \ z \downarrow 0, \ \mu > 0. \tag{4.10}$$

From (4.9) and (4.10) we obtain $\rho_W(0) = 1$.

It is known (see Donoghue [19], p. 293) that the spectral density which corresponds to (4.9) has the form

$$f_W(\omega) = \frac{c_2(\beta)}{(1+\omega^2)^{\beta+1}}, \ \omega \in \mathbb{R}, \ \beta > -1/2, \tag{4.11}$$

where

$$c_2(\beta) = \frac{\Gamma(\beta+1)}{\sqrt{\pi}\Gamma(\beta+1/2)}. \tag{4.12}$$

From (4.7)-(4.11) we obtain

Theorem 4.1. *There exists a stationary solution of the fractional Stokes-Boussinesq-Langevin equation (4.2) with* $\delta \in (0,1)$, $\lambda \geq 0$, $b > 0$ *and stationary random noise* $W(t)$, $t \in \mathbb{R}$ *with correlation function (4.9). The spectral density of this solution is given by*

(4.13)

$$f_X(\omega) = \frac{c_2(\beta)/2\pi}{(1+\omega^2)^{\beta+1}\left[\lambda^2 + \omega^2 + 2b|\omega|^{1+\delta}\cos\frac{\pi(1-\delta)}{2} + 2\lambda b|\omega|^{\delta}\cos\frac{\delta\pi}{2} + b^2|\omega|^{2\delta}\right]}, \ \omega \in \mathbb{R},$$

where $\delta \in (0,1)$, $\beta > -1/2$ *and* $c_2(\beta)$ *is defined by (4.12). The process* $X(t)$ *is Gaussian if* $W(t)$ *is Gaussian.*

Remark 4.1. *If the parameter* $\lambda = 0$ *but* $b > 0$, *the spectral density (4.13) reduces to*

$$f_X(\omega) = \frac{c_2(\beta)/2\pi}{|\omega|^{2\delta}(1+\omega^2)^{\beta+1}\left(b^2 + |\omega|^{2(1-\delta)} + 2b|\omega|^{1-\delta}\cos\frac{\pi(1-\delta)}{2}\right)}, \tag{4.14}$$

where $\delta \in (0,1/2)$, $\beta > -1/2$. *This spectral density displays LRD for* $\delta \in (0,1/2)$. *The asymptotic properties of the spectral density (4.14) is similar to those of (4.1). We therefore conclude that the stationary process described in Theorem 4.1 with* $\lambda = 0$ *provides a dynamic model of fractional Riesz-Bessel motion in the same way as the OU process providing a dynamic model of Brownian motion.*

Remark 4.2. *A general form of Eq. (4.5) is*

$$\left(A_n\mathcal{D}^{\beta_n}y\left(t\right)+...+A_1\mathcal{D}^{\beta_1}y\left(t\right)+A_0\mathcal{D}^{\beta_0}\right)X\left(t\right)=W\left(t\right) \tag{4.15}$$

with constant coefficients $A_n,...,A_1,A_0$ *and* $\beta_n>\beta_1>...>\beta_1>\beta_0,\ n\geq 1$. *The spectral density of the stationary solution* $X\left(t\right)$ *then takes the form (4.7) with*

$$\left|g\left(\omega\right)\right|^2=A_0^2+\sum_{j=1}^{n}A_j^2\left|\omega\right|^{2\beta_j}+2\sum_{1\leq i<j\leq n}A_iA_j\left|\omega\right|^{\beta_i+\beta_j}\cos\frac{\beta_j-\beta_i}{2}\pi+2A_o\sum_{j=1}^{n}A_i\left|\omega\right|^{\beta_j}\cos\frac{\beta_j\pi}{2}.$$

This spectral density belongs to $L_1\left(\mathbb{R}\right)$ *if* $\beta_n>1/2$ *or* $\beta_i+\beta_j>1,\ i,j\in\{1,...,n\}$. *If* $A_0=0$, *this spectral density displays LRD of the form* $O\left(\left|\omega\right|^{-2\beta_1}\right)$ *as* $\left|\omega\right|\rightarrow 0$ *for* $\beta_1\in(0,1/2)$ *and second-order intermittency of the form* $O\left(\left|\omega\right|^{-2\beta_n}\right)$ *as* $\left|\omega\right|\rightarrow\infty$.

Appendix A. Dynamic models of Brownian motion and related processes

This appendix is based on Kubo [36], Nelson [48], Hauge and Martin-Löf [30], Kubo *et al.* [37], Okabe [52], Mainardi and Pironi [41]. Our aim is to recall a few historical facts which clarify our considerations in Sections 2 and 3. The term "Brownian particle" refers to a body of microscopically visible size suspended in a fluid. Its motion is caused by a molecular bombardment of the fluid and is called Brownian motion because it was first described by Robert Brown, a botanist, in 1827 (see Nelson [48] for some interesting historical facts).

The first mathematical theory of Brownian motion was proposed by Einstein [22] and Smoluchowski [61] based on the kinetic theory of heat. Einstein derived the diffusion equation or heat equation for the transition probability density of the position of a Brownian particle as

$$\frac{\partial p}{\partial t}=\mathcal{D}\frac{\partial^2 p}{\partial x^2},$$

where $\mathcal{D}$ is a positive constant, called the diffusion coefficient. The second part of Einstein's argument relates $\mathcal{D}$ to other physical quantities (see Einstein's relation (A.7) below). A more rigorous theory was developed by Wiener [65]. Therefore Brownian motion is also known as the Wiener process.

Langevin [38] initiated, and Uhlenbeck and Ornstein [63] developed the equation of the motion of a Brownian particle of mass m. This theory is derived from Newton's second law: $F=ma$, which in this special case reads

$$m\frac{d^2x\left(t\right)}{dt^2}=F\left(t\right)+W\left(t\right), \tag{A.1}$$

where $x\left(t\right)$ is the position of the particle, $F\left(t\right)$ is the frictional force (due to the fluid) and $W\left(t\right)$ denotes the random force arising from rapid thermal fluctuations. Eq. (A.1) can be equivalently rewritten as the following system of two equations:

$$\frac{dx\left(t\right)}{dt}=\xi\left(t\right),\ m\frac{d\xi\left(t\right)}{dt}=F\left(t\right)+W\left(t\right), \tag{A.2}$$

where $\xi\left(t\right)$ is the velocity of the particle. Assuming for the frictional force the Stokes expression for the drag of a spherical particle of radius a, it is given by

$$F=-\frac{1}{\mu}\xi\left(t\right),\ \frac{1}{\mu}=6\pi a\rho\nu,$$

where μ denotes the mobility coefficient and ρ and ν are the density and the kinematic viscosity of the fluid, respectively. The constant $\sigma = m\mu$ is called the friction characteristic time. Thus, the Langevin equation (A.1) reads

$$\frac{d\xi(t)}{dt} = -\frac{1}{\sigma}\xi(t) + \frac{1}{m}W(t). \tag{A.3}$$

We assume that the Brownian particle has been kept for a sufficiently long time in the fluid at absolute temperature T. Then, for any time t, the equilibrium law for the energy distribution requires that

$$m\mathrm{E}\left(\xi^2(t)\right) = \kappa T,$$

where κ is the Boltzmann constant (a knowledge of κ is equivalent to a knowledge of Avogadro's number and hence of molecular sizes). If we assume that there exists a Gaussian stationary solution to the Langevin equation (A.3), then the previous assumptions lead to the following expressions for the covariance functions of the velocity of the Brownian particle and the noise term:

$$\mathrm{E}(\xi(t)\xi(s)) = \mathrm{E}\left(\xi^2(0)\right)e^{-|t-s|/\sigma} = \frac{\kappa T}{m}e^{-|t-s|/\sigma}, \tag{A.4}$$

$$\mathrm{E}(W(t)W(s)) = \frac{m^2}{\sigma}\mathrm{E}\left(\xi^2(0)\right)\delta(t-s) = \frac{m\kappa T}{\sigma}\delta(t-s), \tag{A.5}$$

where $t, s \in \mathbb{R}$ and $\delta(\tau)$ denotes the Dirac distribution. The constant (finite of infinite)

$$\mathcal{D} = \sigma\mathrm{E}\left(\xi^2(0)\right) = \int_0^\infty \mathrm{E}(\xi(\tau)\xi(0))\,d\tau = \lim_{t\to\infty}\frac{\mathrm{E}X^2(t)}{2t} \tag{A.6}$$

is known as the diffusion coefficient and the Einstein relation

$$\mathcal{D} = \frac{\sigma\kappa T}{m} = \mu\kappa T \tag{A.7}$$

holds.

The Langevin equation (A.3) and Einstein relation (A.7) have been extremely useful in statistical physics and financial mathematics (see Shiryaev [60], for example). It is interesting to note that Bachelier [8] made the first attempt towards a mathematical description of the evolution of stock prices (on the Paris market) on the basis of probabilistic concepts analogous to Brownian motion.

In the theory of hydrodynamics, the Langevin equation (A.3) needs to be modified, since it ignores the effect of the added mass and retarded viscous force, which are due to the acceleration of the particle (see Hauge and Martin-Löf [30], for example). The added mass effect requires to substitute the mass of the particle m with the so-called effective mass given by

$$m^* = m\left[1 + \rho/(2\rho_0)\right],$$

where ρ_0 denotes the density of the particle. Keeping the Stokes drag law unmodified, the relaxation time changes from $\sigma = m\mu$ to $\sigma^* = m^*\mu$. The corresponding Langevin equation then has the form (A.3) with m replaced by m^* and σ by σ^*. Consequently, the diffusion coefficient is unmodified and turns out to be

$$\mathcal{D} = \sigma^*\mathrm{E}\left(\xi^2(0)\right) = \mu\kappa T,$$

so the Einstein relation (A.7) still holds.

The retarded viscous force effect is due to an additional term to the Stokes drag, which is related to the history of the particle acceleration. This additional drag force, proposed by Stokes [62], Boussinesq [11] and Basset [10], is referred to as the Basset history force (see Hauge and Martin-Löf [30] or Maxey and Riley [42], for example). In our notation,

$$F\left(t\right)=-\frac{1}{\mu}\frac{a}{\sqrt{\pi\nu}}\int_{b}^{t}\frac{d\xi\left(\tau\right)/d\tau}{\sqrt{t-\tau}}d\tau=-\frac{a}{\mu\sqrt{\nu}}\,{}_{b}\mathcal{D}_{C}^{1/2}\xi\left(t\right),\tag{A.8}$$

where ${}_{b}\mathcal{D}_{C}^{1/2}\xi\left(t\right)$ is the fractional derivative (2.13) of order $1/2$ in the Caputo sense (see Caputo [13], Caputo and Mainardi [14], Mainardi and Pironi [41]). Then using (A.3), (A.8), the generalised Langevin equation or Stokes-Boussinesq-Langevin equation turns out to be

$$\frac{d\xi\left(t\right)}{dt}=-\frac{1}{\sigma^{*}}\xi\left(t\right)-\frac{a}{\sigma^{*}\sqrt{\nu}}\,{}_{b}\mathcal{D}_{C}^{1/2}\xi\left(t\right)+\frac{1}{m^{*}}W\left(t\right).\tag{A.9}$$

It is worth noting that if the process is in thermodynamic equilibrium (at $b=0$), we would account for the long memory of hydrodynamic interaction, and thus it is correct to integrate Eq. (A.9) from $b=-\infty$. On the other hand Dufty [20] proposed in the case $b=0$ to modify the random force by replacing $W\left(t\right)$ by

$$W^{*}\left(t\right)=W\left(t\right)-\frac{a}{\mu\sqrt{\pi\nu}}\int_{-\infty}^{0}\frac{d\xi\left(\tau\right)/d\tau}{\sqrt{t-\tau}}d\tau.$$

In any case, the fluctuation-dissipation theory of Kubo [36] proposes to introduce a memory function, and one of the possible memory functions gives us the Stokes-Boussinesq-Langevin equation (A.9) or more general equation (1.6) or (4.2). This is the reason why we study in this paper the fractional version of the Stokes-Boussinesq-Langevin equation (1.6) or (4.2), which can also be called Langevin equation with Basset history force. Our consideration suggests useful models for financial mathematics in view of significant analogies (see again Shiryaev [60]) between Newtonian mechanics and stock price motions.

Appendix B. Random distributions

We denote by H the Hilbert space of $\mathbb{C}$-valued random variables, defined on a probability space $(\Omega,\mathcal{F},P)$, with zero expectation and finite variance: $(f,g)=\mathrm{E}\left[f\overline{g}\right],||f||=(f,f)^{\frac{1}{2}}$. By $\mathcal{D}(\mathbb{R})$ we denote the space of all $\phi\in C^{\infty}(\mathbb{R})$ with compact support, endowed with the usual topology. A *random distribution* is a linear and continuous map from $\mathcal{D}(\mathbb{R})$ to H. A random distribution X is *stationary* if $(X(\tau_{h}\phi),X(\tau_{h}\psi))=(X(\phi),X(\psi))$ for all $\phi,\psi\in\mathcal{D}(\mathbb{R})$ and $h\in\mathbb{R}$, where $\tau_{h}\phi(t)=\phi(t+h)$. We then denote by μ_{X} its *spectral measure*: $(X(\phi),X(\psi))=\int_{-\infty}^{\infty}\hat{\phi}(\xi)\hat{\psi}(\xi)d\mu_{X}(d\xi)$, where $\hat{\phi}$ is the Fourier transform of ϕ, namely $\hat{\phi}(\xi)=\int_{-\infty}^{\infty}e^{-it\xi}\phi(t)dt$. Any stationary random distribution X has the following *spectral representation*: $X(\phi)=\int_{-\infty}^{\infty}\hat{\phi}(\xi)Z_{X}(d\xi)$, where Z_{X} is the orthogonal measure corresponding to the spectral measure $\mu_{X}(d\xi)$: $\mathrm{E}\left|Z\left(d\xi\right)\right|^{2}=\mu_{X}\left(d\xi\right)$. We write $\dot{X}$ for the derivative of a random distribution X: $\dot{X}(\phi)=-X(\hat{\phi})$.

Let X and Y be random distributions. Then X is said to be *stationarily correlated* with Y if $(X(\tau_{h}\phi),Y(\tau_{h}\psi))=(X(\phi),Y(\psi))$ for all $\phi,\psi\in\mathcal{D}(\mathbb{R})$ and $h\in\mathbb{R}$; this is equivalent to $(X(t+s),Y(s))=(X(t),Y(0))$ for all $t,s\in\mathbb{R}$ if X and Y are both processes. We denote by $M(Y)$ the closed linear hull of $\{Y(\phi):\phi\in\mathcal{D}(\mathbb{R})\}$ in H. Then we have $M(Y)=\left\{\int_{-\infty}^{\infty}g(\xi)dZ_{Y}(\xi),\ g\in L^{2}(\mu_{Y})\right\}$. A stationary random distribution Y is said to be *purely non-deterministic* if $\bigcap_{t\in\mathbb{R}} M_t(Y) = \{0\}$, that is, the remote past does not contain any information at all.

Appendix C. Canonical representation

Suppose that X is a purely non-deterministic process, then X has a spectral density $f = \{f(\omega), \omega \in \mathbb{R}\}$ of the Hardy class:

$$(1+\omega^2)^{-1}\log f(\omega) \in L_1(\mathbb{R}).$$

Following Dym and McKean [21] we write h for the outer function of X :

$$h(\zeta) = \exp\left\{\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1+\omega\zeta}{\omega-\zeta}\,\frac{\log f(\omega)}{1+\omega^2}d\omega\right\}, \quad \mathrm{Im}\,\zeta > 0, \tag{C.1}$$

and E for the canonical representation kernel of X, that is, $E = \hat{h}$, where $\hat{h}$ is the Fourier transform of

$$h(\cdot) = \lim_{\eta\downarrow 0} h(\cdot + i\eta) \in L_2(\mathbb{R}),$$

i.e.,

$$\hat{h}(t) = \lim_{M\to\infty}\int_{-M}^{M} e^{-it\xi}h(\xi)d\xi.$$

We have

$$h(\zeta) = \frac{1}{2\pi}\int_0^\infty e^{i\zeta t}E(t)dt, \qquad \mathrm{Im}\,\zeta > 0, \tag{C.2}$$

and

$$X(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} E(t-s)dB(s), \qquad E \in L_2(\mathbb{R}), \tag{C.3}$$

or

$$R(t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty E(|t|+s)E(s)ds \tag{C.4}$$

is called the canonical representation of X, where $B = \{B(t), t \in \mathbb{R}\}$ is the standard Brownian motion. Note that the representations (C.3) and (C.4) play an important role in the prediction theory of stationary processes.

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