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INCREASING DOMAIN ASYMPTOTICS FOR THE FIRST MINKOWSKI FUNCTIONAL OF SPHERICAL RANDOM FIELDS

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This contribution is dedicated to the 85th anniversary of Professor Myhailo Iosypovych Yadrenko

Abstract. The restriction to the sphere of an homogeneous and isotropic random field defines a spherical isotropic random field. This paper derives central and non-central limit results for the first Minkowski functional subordinated to homogeneous and isotropic Gaussian and chi-squared random fields, restricted to the sphere in $\mathbb{R}^3$. Both scenarios are motivated by their interesting applications in the analysis of the Cosmic Microwave Background (CMB) radiation.

Key words and phrases. Central and non-central limit theorems, chi-squared random fields, Gaussian random fields, Karhunen–Loéve expansion, spherical Rosenblatt distribution.

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1. Introduction

The main purpose of this article is two-fold. First, we give a survey of some key results on central and non-central limit theorems for Minkowski functionals of spherical random fields. Second, we discuss some new results on the spherical Rosenblatt-type distribution, in particular we present an explicit expression of its characteristic function and series representation. These results highlight some of Professor Yadrenko's pioneering contribution in the area spectral theory of spherical random fields and to study their spherical averages, see [33] and [27].

Recent year have witnessed an enormous amount of attention, in the astrophysical and cosmological literature, on testing for Gaussianity for spherical random fields. The empirical motivation for these studies can be explained as follows. The NASA satellite mission WMAP and the nearly finished (data collection will be released in 2018) ESA mission Planck [25, 26] will probe Cosmic Microwave Background radiation (CMB) to an unprecedented accuracy. CMB can be viewed as a signature of the distribution of matter and radiation in the very early universe, and as such it is expected to yield very tight constraints on physical models for the Big Bang. For the density fluctuations of this field, the highly popular inflationary scenario predicts a Gaussian distribution, whereas alternative cosmological theories, such as topological defects or non-standard inflationary models, predict otherwise. Non-Gaussianities may also have a non-physical origin, i.e. they might be generated by systematic errors in the CMB map, such as noise which has not been properly removed, contamination from the galaxy or distortions in the optics of the telescope. A proper understanding of the density distributions of fluctuations is also instrumental for correct inference on the physical constants which can be estimated from CMB radiation.

For these reasons, many different Gaussianity tests were considered in the recent cosmological literature, some of them based upon the topological properties of Gaussian

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fields ([1, 5, 21, 29], see also their references), others on higher-order cumulants spectra (see [19]). In particular, the so-called Minkowski functionals, which have a simple and intuitive geometrical meaning, were introduced in [1, 5]. Choosing a threshold ν, we can divide the sphere into two parts: hot regions where the random field T passes the threshold, and cold regions where $T < \nu$. The hot region is also called the excursion set of the field T over the threshold ν, or the spherical measure of excess of level $\nu \geq 0$.

In two dimensions three Minkowski functionals are:

1) *Area:* $M_0(\nu)$ is the total area of all hot regions, that is points on a sphere $s_2(r) = \left\{x \in \mathbb{R}^3 : \|x\| = x_1^2 + x_2^2 + x_3^2 = r^2\right\}$, where $\widetilde{T}(x) = T(r, \theta, \varphi) > \nu$, where $x = (x_1, x_2, x_3)' \in \mathbb{R}^3$, and $u = (\theta, \varphi) \in s_2(1)$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$, where (θ, φ) are the spherical polar coordinates, and $r = |x|$.
2) *Boundary length:* $M_1(\nu)$ is proportional to the total length of the boundary between cold and hot regions
3) *Euler characteristic:* $M_2(\nu)$, a purely topological quantity, counts the number of isolated hot regions minus the number of isolated cold regions.

Note that the morphological analysis of random fields using Minkowski functionals are widely used in many other areas of applications. The results derived in this paper on Central and Non-Central Limit theorems for first Minkowski functional $M_0(\nu)$, under increasing domain asymptotics, constitute a methodological contribution to this area, as preamble of its fixed domain asymptotics counterpart, which is the most interesting case in cosmological applications. But this last subject constitutes the topic of a subsequent paper.

2. Isotropic random fields

This section reviews a number of mostly known results from the monograph [33] (see, also [6–8, 19]).

Consider a sphere in three-dimensional Euclidean space

$$s_2(r) = \left\{x \in \mathbb{R}^3 : \|x\| = r\right\} \subset \mathbb{R}^3$$

with the Lebesgue measure (the area element on the sphere)

$$\widetilde{\sigma}_r(du) = \sigma_r(d\theta.d\varphi) = r^2 \sin\theta d\theta d\varphi, \quad (\theta, \varphi) \in s_2(1), \quad r = \|x\| > 0.$$

A spherical random field on a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$, denoted by

$$T = \{T(r, \theta, \varphi) = T_\omega(r, \theta, \varphi) : 0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi,\ r > 0,\ \omega \in \Omega\},$$

or $T = \left\{\widetilde{T}(x)\ ,\ x \in s_2(r)\right\}$, is a stochastic function defined on the sphere $s_2(r)$. We consider a real-valued spherical random field T, with finite second-order moments, and being continuous in the mean-square sense. Note that [20] proved that the covariance function of a measurable finite-variance isotropic random field on the sphere is necessarily everywhere continuous.

Under these conditions, the field T can be expanded in the mean-square sense as a Laplace series [33, p. 73]:

$$T(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(\theta, \varphi) a_{lm}(r), \tag{1}$$

where $Y_l^m(\theta, \varphi)$ represent the spherical harmonics. The spectral representation (1) can be viewed as Karhunen–Loève expansion, which converges in the Hilbert space

$L_2\big(\Omega \times s_2(r), r^2 \sin\theta d\theta d\varphi\big)$; that is,

$$\lim_{L\to\infty} \mathsf{E}\left(\int\limits_{s_2(r)} \left(T(r,\theta,\varphi) - \sum_{l=0}^{L}\sum_{m=-l}^{l} Y_l^m(\theta,\varphi)a_{lm}(r)\right)^2 r^2 \sin\theta d\theta d\varphi\right) = 0.$$

According to Peter–Weyl Theorem (see [20, p. 69]), the expansion (1) also converges in the Hilbert space $L_2(\Omega)$, for every $x \in s_2(r)$; that is, for each $x \in s_2(r)$,

$$\lim_{L\to\infty} \mathsf{E}\left(\widetilde{T}(x) - \sum_{l=0}^{L}\sum_{m=-l}^{l} \widetilde{Y}_l^m(x)a_{lm}(r)\right)^2 = 0.$$

Recall that for $-l \le m \le l$,

$$\widetilde{Y}_l^m(x) = Y_l^m(\theta,\varphi) = c_{lm}\exp(im\varphi)P_l^m(\cos\theta), \quad c_{lm} = (-1)^m\left[\frac{2l+1}{4\pi}\frac{(l-k)!}{(l+k)!}\right]^{1/2}, \tag{2}$$

and $P_l^m(\cos\theta)$ denotes the associated Legendre polynomial of degree l, m, i. e.

$$P_l^m(x) = (-1)^m\big(1-x^2\big)^{m/2}\frac{d^m}{dx^m}P_l(x), \quad P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}\big(x^2-1\big)^l. \tag{3}$$

The spherical harmonics have the following properties

$$\int_0^{\pi}\int_0^{2\pi} \overline{Y_l^m}(\theta,\varphi)Y_{l'}^{m'}(\theta,\varphi)\sin\theta d\theta d\varphi = \delta_l^{l'}\delta_m^{m'}, \tag{4}$$

$$\overline{Y_l^m}(\theta,\varphi) = (-1)^m Y_l^{-m}(\theta,\varphi),$$

$$Y_l^m(\pi-\theta,\varphi+\pi) = (-1)^l Y_l^m(\theta,\varphi),$$

where $\delta_l^{l'}$ represent the Kronecker delta. The random coefficients in the Laplace series (1) can be obtained through inversion arguments in the form of mean-square stochastic integrals

$$\begin{aligned} a_l^m(r) &= \int_0^{\pi}\int_0^{2\pi} T(r,\theta,\varphi)\overline{Y_l^m}(\theta,\varphi)r^2\sin\theta d\theta d\varphi = \\ &= \int_{s_2(1)} \widetilde{T}(ru)\overline{Y_l^m}(u)\widetilde{\sigma}_1(du), \quad u = \frac{x}{\|x\|} \in s_2(1) \quad r = \|x\|. \end{aligned} \tag{5}$$

The field $T(r,\theta,\varphi) = \widetilde{T}(x)$ is said to be isotropic (in the weak sense) on a sphere $s_2(r)$ if $\mathsf{E}\widetilde{T}(x)^2 < \infty$, and its first and second-order moments are invariant with respect to the group of rotations on the sphere, i.e.

$$\mathsf{E}\widetilde{T}(x) = \mathsf{E}\widetilde{T}(gx), \quad \mathsf{E}\widetilde{T}(x)\widetilde{T}(y) = \mathsf{E}\widetilde{T}(gx)\widetilde{T}(gy),$$

for every $g \in SO(3)$, the group of rotations in $\mathbb{R}^3$. This is equivalent to saying that the mean $\mathsf{E}T(r,\theta,\varphi) = c =$ constant (we assume $c = 0$), and that the covariance function $\mathsf{E}T(r,\theta,\varphi)T(r,\theta',\varphi')$ depends only on the angular distance $\theta = \theta_{PQ}$ between the points $P = (\theta,\varphi)$ and $Q = (\theta',\varphi')$ on $s_2(r)$. The field is isotropic if and only if

$$\mathsf{E}a_l^m(r)\overline{a_{l'}^{m'}}(r) = \delta_l^{l'}\delta_m^{m'}C_l(r), \quad -l \le m \le l, \quad -l' \le m' \le l', \tag{6}$$

or

$$\mathsf{E}|a_l^m(r)|^2 = C_l(r), \quad m = 0, \pm 1, \dots, \pm l. \tag{7}$$

The functional series $\{C_1(r), C_2(r), \dots, C_l(r), \dots\}$, $r > 0$, is called the angular power spectrum of the isotropic random field $T(r,\theta,\varphi)$. From (1), (5) and (6) we deduce that

$$\Gamma_r(\cos\theta) = \mathsf{E}T(r,\theta,\varphi)T(r,\theta',\varphi') = \frac{1}{4\pi}\sum_{l=1}^{\infty}(2l+1)C_l(r)P_l(\cos\theta), \tag{8}$$

where

$$\sum_{l=1}^{\infty}(2l+1)C_l(r)<\infty, \tag{9}$$

for every fixed $r>0$. If $T(r,\theta,\varphi)$ is an isotropic Gaussian field, then the coefficients $a_l^m(r)$, $m=-l,\dots,l$, $l\geq 1$, are complex-valued independent Gaussian random processes unless $m=-m'$, with

$$\mathsf{E}a_l^m(r)=0,\quad \mathsf{E}a_l^m(r)\overline{a_{l'}^{m'}}(r)=\delta_m^{m'}\delta_l^{l'}C_l(r),$$

if $C_l(r)>0$, or degenerate at zero if $C_l(r)=0$. This does not deny that they are uncorrelated for every m,m', including $m=-m'$.

A random field $\widetilde{T}(x)$, $x\in\mathbb{R}^3$, with $\mathsf{E}\widetilde{T}(x)^2<\infty$ is called homogenous (in the weak sense) if its first two moments are invariant with respect to the Abelian group of shifts in $\mathbb{R}^3$. An isotropic field $\widetilde{T}(x)$, $x\in\mathbb{R}^3$, is homogenous if and only if [33, p. 89]

$$\mathsf{E}a_l^m(r)\overline{a_{l'}^{m'}}(s)=\delta_l^{l'}\delta_m^{m'}C_l(r,s) \tag{10}$$

with

$$C_l(r,s)=2\pi^2\int_0^{\infty}\frac{J_{l+\frac{1}{2}}(\mu r)}{(\mu r)^{1/2}}\frac{J_{l+\frac{1}{2}}(\mu s)}{(\mu s)^{1/2}}G(d\mu), \tag{11}$$

$l=1,2,\dots$, where G is a finite measure on the Borel sets of $[0,\infty)$ such that

$$\sigma^2=\mathrm{Var}\left\{\widetilde{T}(0)\right\}=\int_0^{\infty}G(d\mu)<\infty,$$

and $J_\nu(z)$ is the Bessel function of the first kind of order ν.

The covariance function $\mathrm{Cov}\left\{\widetilde{T}(x),\widetilde{T}(y)\right\}$ of a mean-square continuous isotropic random field $\widetilde{T}(x)$ depends only on the Euclidean distance

$$r=|x-y|=\sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos\gamma},\quad \cos\gamma=\frac{\langle x,y\rangle}{\rho_1\rho_2},\quad x=(\rho_1,u_1),\quad y=(\rho_2,u_2),$$

with $\rho_1=\|x\|$, and $\rho_2=\|y\|$. Moreover, by the addition theorem for Bessel functions (see, for example, [33, p. 6]) the covariance function can be represented as

$$\begin{aligned}B(r)&=\int_0^{\infty}\frac{\sin(\mu r)}{\mu r}G(d\mu)=\\&=2\pi^2\sum_{l=1}^{\infty}\sum_{m=-l}^{l}Y_l^m(u_1)\overline{Y_l^m}(u_2)\int_0^{\infty}\frac{J_{l+\frac{1}{2}}(\mu\rho_1)}{(\mu\rho_1)^{1/2}}\frac{J_{l+\frac{1}{2}}(\mu\rho_2)}{(\mu\rho_2)^{1/2}}G(d\mu).\end{aligned} \tag{12}$$

By Karhunen's Theorem (see, for example, [33, p. 10]), a mean-square continuous homogenous isotropic random field with zero mean has a spectral representation

$$\widetilde{T}(x)=T(r,\theta,\varphi)=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}Y_l^m(\theta,\varphi)a_l^m(r), \tag{13}$$

where, in this case,

$$a_l^m(r)=\pi\sqrt{2}\int_0^{\infty}\frac{J_{l+\frac{1}{2}}(\mu r)}{\sqrt{(\mu r)}}Z_l^m(d\mu), \tag{14}$$

with Z_l^m, $-l\leq m\leq l$, $l=1,2,\dots$, being the family of complex-valued random measures on Borel sets of $[0,\infty)$ such that

$$\mathsf{E}Z_l^m(A)=0,\quad \mathsf{E}Z_l^m(A)\overline{Z_{l'}^{m'}}(B)=\delta_l^{l'}\delta_m^{m'}G(A\cap B). \tag{15}$$

If there exists an isotropic spectral density $g(\mu)\geq 0$ such that

$$\frac{G(d\mu)}{d\mu}=|s_2(1)|\mu^2 g(\mu),\quad \mu^2 g(\mu)\in L_1([0,\infty)), \tag{16}$$

where $|s_2(1)|$ denotes the Lebesgue measure of the unit sphere in $\mathbb{R}^3$, then (13) holds with

$$a_l^m(r) = (2\pi)^{3/2} \int_0^\infty \sqrt{\mu} J_{l+\frac{1}{2}}(\mu r)\sqrt{g(\mu)} W_l^m(d\mu), \tag{17}$$

and

$$\mathsf{E} W_l^m(A)\overline{W_{l'}^{m'}}(B) = \delta_l^{l'} \delta_m^{m'} |A \cap B|,$$

being W_l^m, $-l \leq m \leq l$, $l = 1, 2, \ldots$, a family of white noise random measures. The restriction of an homogenous and isotropic random field $\widetilde{T}(x)$, $x \in \mathbb{R}^3$, to the sphere $s_2(r)$ is an isotropic random field on the sphere. In this particular case, the covariance function of this isotropic random field T on $s_2(r)$ is representable in the form (8) with the angular power spectrum

$$C_l(r) = 2\pi^2 \int_0^\infty \frac{J^2_{l+\frac{1}{2}}(\mu r)}{\mu r} G(d\mu), \quad l = 1, 2, \ldots, \tag{18}$$

or if (16) holds,

$$C_l(r) = (2\pi)^3 \int_0^\infty \frac{J^2_{l+\frac{1}{2}}(\mu r)}{\mu r} \mu^2 g(\mu) d\mu. \tag{19}$$

For example, if (18) is satisfied and

$$g(\mu) = \frac{h(\mu)}{\mu^{1-\kappa}}, \quad -2 < \kappa < 0, \tag{20}$$

where $h(\mu)$ is continuous and positive in a neighborhood of zero and bounded everywhere on $[0, \infty)$, then by the Tauberian theorem [33, p. 32], from (19) and (20) we obtain the following asymptotic result:

$$C_l(r) = (2\pi)^3 h(0) k_1(l, \kappa) r^{2-\kappa}\{1 + o(1)\}, \quad \text{as } r \to \infty,$$

where

$$k_1(l, \kappa) = \int_0^\infty J^2_{l+\frac{1}{2}}(z) z^\kappa dz = \left\{\Gamma^2\left(\frac{1-\kappa}{2}\right)\Gamma\left(\frac{2l+2-\kappa}{2}\right)\right\}^{-1} \Gamma(-\kappa)\Gamma(2l+2-\kappa)2^{-\kappa}.$$

Note that the convergence as $r \to \infty$, is not uniform over l, and the sequence $k_1(l, \kappa)$ diverges (for every fixed κ). This does not contradict to (9), since the convergence is not uniform. It follows that if the field is homogeneous the series (9) does not depend on r.

According to the standard terminology, for $\kappa \in (-2, -1)$, the random field has a radial long range dependence, while for $\kappa \in [-1, 0)$ the random field has a radial singularity at zero.

In general, the covariance function on sphere Γ_r can be originated from covariance function of some homogenous and isotropic random fields on Euclidean spaces B as follows [33, p. 76]:

$$\Gamma_r(\cos\theta) = \mathrm{Cov}\big(T(r, \theta, \varphi), T(r, \theta', \varphi')\big) = B\big(2r\sin(\theta/2)\big) = \int_0^\infty \frac{\sin\big(2\mu r \sin\frac{\theta}{2}\big)}{2r\mu\sin\frac{\theta}{2}} G(d\mu). \tag{21}$$

In this case we consider two locations $P = (r, \theta, \varphi)$ and $Q = (r, \theta', \varphi')$ on the sphere $s_2(r)$ with angle $\theta \in [0, \pi]$, then the Euclidean distance between them in terms of the angle is $2r\sin(\theta/2)$, which gives a direct correspondence between the original covariance function $B(\rho)$ and the covariance function $\Gamma_r(\cos\theta)$ on sphere depending only on the angular distance $\theta = \theta_{PQ}$ between the points on the sphere $s_2(r)$.

Examples of valid covariance functions, based on equation (21), can be found in [16].

A Gaussian random field $\widetilde{T}(x)$, $x \in \mathbb{R}^3$, is called a fractional Brownian field with the Hurst parameter H, if it satisfies the conditions

$$\mathsf{E}\Big\{\widetilde{T}(x) - \widetilde{T}(y)\Big\} = 0; \quad \mathsf{E}\Big\{\widetilde{T}(x) - \widetilde{T}(y)\Big\}^2 = \|x-y\|^H, \quad x, y \in \mathbb{R}^3, \quad 0 \leq H \leq 2.$$

We may assume $\widetilde{T}(0) = 0$; it is well-known that fractional Brownian motion is an isotropic and self-similar Gaussian random field, i.e.

$$\widetilde{T}(x) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} Y_l^m(\theta, \varphi) \tilde{a}_{lm}(\|x\|), \quad x \in \mathbb{R}^3, \tag{22}$$

where

$$a_{lm}(r) = \int_{s_2(1)} \widetilde{T}(ru) Y_l^m(u) \widetilde{\sigma}_1(du), \quad u = (\theta, \varphi),$$

with

$$\mathsf{E}\tilde{a}_{lm}(r)\overline{\tilde{a}}_{l'm'}(s) = \delta_l^{l'} \delta_m^{m'} C_l(r, s), \tag{23}$$

and the self-similarity property holds:

$$\widetilde{T}(\lambda x) \stackrel{d}{=} \widetilde{T}(x)\lambda^{H/2}, \quad \lambda > 0, \quad \text{for every } H \in [0, 2].$$

The covariance function $C_l(r, s)$, in (23), satisfies the self-similarity condition:

$$C_l(\lambda r, \lambda s) = \lambda^H C_l(r, s).$$

By Lamperti's transformation [6, p. 117] the process

$$\overline{a}_{l1}(t) = e^{-Ht} a_{l1}\big(e^{2t}\big), \quad t \in \mathbb{R},$$

is stationary with covariance function

$$R(t, s) = e^{-H(t+s)} C_l\big(e^{2t}, e^{2t}\big) = e^{H(t-s)} C_l\Big(1, e^{-2(t-s)}\Big), \quad t, s \in \mathbb{R}.$$

If $\overline{a}_{l1}(t)$ is purely non-deterministic then it has a canonical representation

$$\overline{a}_{l1}(t) = \int_{-\infty}^{t} A(t-s) W_0(ds), \quad A \in L_2(\mathbb{R}),$$

where $W_0(\cdot)$ is a complex Gaussian white noise random measure on the Borel sets of $\mathbb{R}$, and then $a_{l1}(r)$ is representable as

$$a_{l1}(r) = \int_0^r \frac{1}{\sqrt{2}} A\Big(\log\sqrt{r/s}\Big) r^{H/2} s^{-1/2} W(ds), \tag{24}$$

with $W(\cdot)$ being a complex Gaussian white noise random measure on $[0, \infty)$. The coefficient processes $\overline{a}_{lm}(t)$ in (22) are independent copies of the process (24). The restriction of a field $\widetilde{T}(x)$ to $s_2(r)$ is an isotropic random field on $s_2(r)$, with spectral representation (22), and angular power spectrum

$$C_l(r) = C_l(r, r) = \frac{1}{2} \int_0^r \Big| A\Big(\log\Big(\frac{r}{s}\Big)\Big)\Big|^2 r^H s^{-1} ds,$$

for some function $A(u)$ such that $|A(u)|^2 \in L_2([0, \infty))$.

3. The first Minkowski functional for Gaussian random fields

3.1. Direct formulae. In this section we examine the first Minkowski functional for a Gaussian random field with SRD or LRD. We use some ideas from [3, 6, 8, 33]. Let $\widetilde{T}(x)$, $x \in \mathbb{R}^3$, be a measurable, mean-square continuous homogenous isotropic Gaussian random field, with zero mean, and covariance function $\widetilde{B}(x) = \operatorname{Cov}\big(\widetilde{T}(x), \widetilde{T}(0)\big) = B(\|x\|)$, $x \in \mathbb{R}^3$; in the sequel, we assume $B(0) = 1$. Consider now an isotropic random field $T(r, \theta, \varphi)$, which is the restriction to the sphere $s_2(r)$ of the random field $\widetilde{T}(x)$. The first Minkowski functional can be represented as

$$\begin{aligned} M_0(\nu) &= \sigma(\{(r,\theta,\varphi) \in s_2(r) : T(r,\theta,\varphi) > \nu\}) = \\ &= \int_{s_2(r)} 1_{\{T>\nu\}}(r,\theta,\varphi)\sigma_r(d\theta, d\varphi) = \\ &= \int_{s_2(r)} 1_{\{\widetilde{T}>\nu\}}(x)\widetilde{\sigma}_r(dx), \end{aligned} \tag{25}$$

with $1_{\{\cdot\}}$ denoting the indicator function. Now let $N(\cdot)$ represents any, real measurable function such that $\mathsf{E}\Big[N\Big(\widetilde{T}(0)\Big)\Big]^2 < \infty$. The function $N(\cdot)$ can be expanded in the series

$$N(u) = \sum_{k=0}^{\infty} \frac{K_k}{k!} H_k(u), \qquad K_k = \int_{\mathbb{R}} N(u) H_k(u) \phi(u) du, \tag{26}$$

which converges in the Hilbert space $L_2(\mathbb{R}, \phi(u)du)$. In (26), the function $\phi(u) = (2\pi)^{-1/2} e^{-u^2/2}$, $u \in \mathbb{R}$, is a standard Gaussian density, and

$$H_k(u) = (-1)^k \exp\left(\frac{u^2}{2}\right) \frac{d^k}{du^k} \exp\left(-\frac{u^2}{2}\right), \quad u \in \mathbb{R},$$

is the kth Hermite polynomial. It is well-known that such polynomials form a complete orthonormal system in the Hilbert space $L_2(\mathbb{R}, \phi(u)du)$, and $\mathsf{E}H_k(\xi)H_{k'}(\eta) = \delta_k^{k'} q!\{\mathsf{E}\xi\eta\}^k$, where (ξ, η) is a zero-mean Gaussian vector. In particular, for the indicator function

$$1_{T>\nu} = \sum_{k=0}^{\infty} \frac{K_k(\nu)}{k!} H_k(T), \tag{27}$$

where

$$K_k(\nu) = \begin{cases} 1 - \Phi(\nu), & k = 0, \\ \phi(\nu) H_{k-1}(\nu), & k \geq 1, \end{cases} \tag{28}$$

and $\Phi(\nu) = \int_{-\infty}^{\nu} \phi(u) du$.

Thus, the first Minkowski functional can be expanded in the Hilbert space $L_2(\Omega)$ as follows:

$$\begin{aligned} M_0(\nu) &= \sum_{k=0}^{\infty} \frac{K_k(\nu)}{k!} \int_{s_2(r)} H_k\Big(\widetilde{T}(x)\Big)\widetilde{\sigma}_r(dx) = \\ &= \mathsf{E}M_0(\nu) + \sum_{k=1}^{\infty} \frac{K_k(\nu)}{k!} \int_{s_2(r)} H_k\Big(\widetilde{T}(x)\Big)\widetilde{\sigma}_r(dx), \end{aligned} \tag{29}$$

where

$$\mathsf{E}M_0(\nu) = \{1 - \Phi(\nu)\} r^2 |s(1)| = 4\pi r^2 \{1 - \Phi(\nu)\}, \tag{30}$$

and

$$\mathsf{E}\left\{\int_{s_2(r)} H_k\Big(\widetilde{T}(x)\Big)\widetilde{\sigma}_r(dx) \int_{s_2(r)} H_{k'}\Big(\widetilde{T}(y)\Big)\widetilde{\sigma}_r(dy)\right\} = \delta_k^{k'} d_k^2(r),$$

$$d_k^2(r) = k! \int_{s_2(r)} \int_{s_2(r)} B^k(\|x-y\|) \widetilde{\sigma}_r(dx) \widetilde{\sigma}_r(dy). \tag{31}$$

Let $s_2(r)$ be a sphere in $\mathbb{R}^3$ and consider two independent vectors β and γ; we assume β, γ are uniformly distributed on $s_2(r)$, that is

$$\mathsf{P}(\beta \in \Delta) = \mathsf{P}(\gamma \in \Delta) = \int_{\Delta \cap s(r)} \frac{\widetilde{\sigma}_r(dx)}{r^2 |s(1)|}, \quad \Delta \subset s_2(r).$$

It can then be shown (see for instance [33, p. 28]) that the probability density function of the Euclidean distance $\rho(\beta, \gamma) = \|\beta - \gamma\|$ is of the form

$$u_r\big(\rho(\beta,\gamma)\big) = u_r(u) = \frac{1}{2} \cdot \frac{u}{r^2}, \quad 0 \le u \le 2r. \tag{32}$$

Then, from (31) and (32), we obtain

$$d_k^2(r) = r^4 |s(1)|^2 k! \mathsf{E} B^k(\|\beta - \gamma\|) = 2^3 \pi^2 r^2 k! \int_0^{2r} z B^k(z)\, dz. \tag{33}$$

Thus,

$$\mathrm{Var}\{M_0(\nu)\} = \sum_{k=1}^{\infty} \frac{K_k^2(\nu)}{(k!)^2} d_k^2(r) = 2^3 \pi^2 r^2 \sum_{k=1}^{\infty} \frac{K_k^2(\nu)}{k!} \int_0^{2r} z B^k(z)\, dz. \tag{34}$$

Now, consider the bivariate Gaussian density

$$\phi(x,y;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ -\frac{x^2+y^2-2xy\rho}{2(1-\rho^2)} \right\}, \quad (x,y) \in \mathbb{R}^2, \quad |\rho| \le 1; \tag{35}$$

using the well-known formula

$$\int_\nu^\infty \int_\nu^\infty \phi(x,y;\rho) dx dy = \left\{ \int_\nu^\infty \phi(x) dx \right\} + \frac{1}{2\pi} \int_0^\rho \exp\left\{ -\frac{\nu^2}{1+\rho} \right\} \frac{dz}{\sqrt{1-\rho^2}},$$

we obtain the following alternative expression for the Minkowski functional itself and its variance:

$$\begin{aligned} M_0(\nu) &= \{1 - \Phi(\nu)\} r^2 |s_2(1)| + \phi(\nu) \int_{s_2(r)} \widetilde{T}(x) \widetilde{\sigma}_r(dx) + \\ &\quad + \phi(\nu) \sum_{k=2}^{\infty} K_k(\nu) \frac{1}{k!} \int_{s_2(r)} H_k\Big(\widetilde{T}(x)\Big) \widetilde{\sigma}_r(dx) = \\ &= \mathsf{E} M_0(\nu) + \eta_\nu(r) + R_\nu(r), \end{aligned} \tag{36}$$

where $\eta_\nu(r)$ is a Gaussian random field with zero mean and variance

$$\mathsf{E} \eta_\nu(r)^2 = \phi^2(\nu) 2^3 \pi^2 r^2 \int_0^{2r} z B(z)\, dz,$$

and

$$\mathrm{Var}\{R_\nu(r)\} = 2^3 \pi^2 r^2 \phi^2(\nu) \sum_{k=2}^{\infty} \frac{K_k^2(\nu)}{k!} \int_0^{2r} z B^k(z)\, dz.$$

Also one can obtain the following direct formula:

$$\mathrm{Var}\{M_0(v)\} = 2\pi r^2 \int_0^{2r} z \int_0^{B(z)} \exp\left\{ -\frac{\nu^2}{1+w} \right\} \frac{dw}{\sqrt{1-w^2}}\, dw, \tag{37}$$

since

$$\begin{aligned} \mathrm{Var}\left\{ \int_{s_2(r)} 1_{\{\widetilde{T} > \nu\}}(x) \widetilde{\sigma}_r(dx) \right\} &= \int_{s_2(r)} \int_{s_2(r)} \widetilde{\sigma}_r(dx) \widetilde{\sigma}_r(dy) \times \\ \times \iint_{\mathbb{R}^2} 1_{\{u > \nu\}} 1_{\{w > \nu\}} &\big[\phi\big(u, w; B(\|x-y\|)\big) - \phi(u)\phi(w) \big] du dw = \end{aligned}$$

$$= \int_{s_2(r)} \int_{s_2(r)} \widetilde{\sigma}_r(dx)\widetilde{\sigma}_r(dy) \int_{\nu}^{\infty}\int_{\nu}^{\infty} \big[\phi\big(u,w;B(\|x-y\|)\big) - \phi(u)\phi(w)\big]\,du dw =$$

$$= 2^3\pi^2 r^2 \left[\int_0^{2r} z dz\, \frac{1}{2\pi}\int_0^{B(z)} \exp\left\{-\frac{\nu^2}{1+\rho}\right\}\frac{d\rho}{\sqrt{1-\rho^2}}\right]. \tag{38}$$

For a general isotropic Gaussian random field T with zero mean and covariance function $\Gamma_r(\cos\theta)$, we get:

$$\mathsf{E}M_0(\nu) = 4\pi r^2\{1-\Phi(\nu)\},$$

$$\operatorname{Var} M_0(v) = \frac{1}{2\pi}\int_{s_2(r)}\int_{s_2(r)} \widetilde{\sigma}_r(dx)\widetilde{\sigma}_r(dy)\int_0^{\Gamma_r(\cos\theta)} \exp\left\{-\frac{\nu^2}{1+\rho}\right\}\frac{d\rho}{\sqrt{1-\rho^2}}.$$

The last formula seems new and computational friendly.

3.2. Asymptotic formulae as $r \to \infty$. We introduce

Assumption AI. The homogeneous isotropic Gaussian random field $\widetilde{T}(x)$, $x \in \mathbb{R}^3$, has the covariance function $B(\|x\|)$, $x \in \mathbb{R}^3$, such that

$$\int_0^{\infty} z|B(z)|\,dz < \infty, \qquad \int_0^{\infty} zB(z)\,dz \neq 0.$$

Under the assumption AI, as $r \to \infty$

$$\operatorname{Var} M_0(v) = k_2(\nu) r^2\{1+o(1)\},$$

where

$$k_2(\nu) = 2^3\pi^2\phi^2(\nu)\sum_{k=1}^{\infty}\frac{K_k^2(\nu)}{k!}\int_0^{\infty} zB^k(z)\,dz \,\in (0,\infty).$$

Assumption AII.

i) The homogeneous isotropic random field $\widetilde{T}(x)$, $x \in \mathbb{R}^3$, is a zero-mean mean-square continuous Gaussian random field with $\mathsf{E}[\widetilde{T}^2(x)] = 1$, for all $x \in \mathbb{R}^3$, and has the covariance function

$$B(\|x\|) = \frac{L(\|x\|)}{\|x\|^{\alpha}}, \quad x \in \mathbb{R}^3, \quad 0 < \alpha < 2, \tag{39}$$

where $L(\|x\|)$ is a slowly varying function, as $\|x\| \to \infty$, and

ii) the Gaussian random field $\widetilde{T}$ has absolutely continuous spectrum, with spectral density $f_0(\|\lambda\|)$ defined on $\mathbb{R}^3$, and being a decreasing function for $\|\lambda\| \in (0,\epsilon]$, $\epsilon > 0$.

Under the assumption AII i), as $r \to \infty$

$$\operatorname{Var} M_0(\nu) = k_3(\alpha,\nu) r^{4-\alpha} L(r)\{1+o(1)\}, \tag{40}$$

where

$$k_3(\alpha,\nu) = 2^{4-\alpha}\pi^2\phi^2(\nu)\frac{\Gamma\left(\frac{3-\alpha}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)} \in (0,\infty), \tag{41}$$

while for $\alpha = 2$, and $r \to \infty$

$$\operatorname{Var} M_0(\nu) = k_4(\nu) r^2 L(r)\log(2r)\{1+o(1)\},$$

with

$$k_4(\nu) = k_2(\nu)/4 \in (0,\infty).$$

We can now formulate Theorem 1 about the asymptotic normality of the first Minkowski functional of a Gaussian random field under increasing domain asymptotics. The following lemma is required in the proof of Theorem 1.

Lemma 1. *Under the assumption AII ii) for* $0<\alpha<2$, *the following identities hold:*

$$\int_{\mathbb{R}^2}|Y_3(\|\lambda\|)|^2\left[\frac{1}{(\|\lambda\|)^{2-\alpha/2}}\right]^2 d\lambda = k_3(\alpha,\nu)\left[\frac{\gamma(\alpha/2)}{|s_2(1)|}\right]^2<\infty,$$

where

$$\gamma(\beta)=\frac{\pi 2^{\beta}\Gamma(\beta/2)}{\Gamma\left(\frac{2-\beta}{2}\right)}=\frac{1}{c(2,\beta)},\quad 0<\beta<2, \tag{42}$$

and

$$Y_3(r\|\lambda\|)=\frac{1}{|s_2(r)|}\int_{s_2(r)}\exp(i\langle\lambda,x\rangle)\widetilde{\sigma}_r(dx). \tag{43}$$

Proof. The proof is based in the following identity:

$$\int_{\mathbb{R}^2}|Y_3(\|\lambda\|)|^2\left[\frac{1}{(\|\lambda\|)^{2-\alpha/2}}\right]^2 d\lambda=\left[\frac{\gamma(\alpha/2)}{|s_2(1)|}\right]^2\int_{s_2(1)}\int_{s_2(1)}\frac{1}{\|x-y\|^{\alpha}}\widetilde{\sigma}(dx)\widetilde{\sigma}(dy),$$

which holds from the weak-sense definition of the Fourier transform of the Riesz potential, and its associated convolution properties (see, for example, Lemma 1 in [30, p. 117]), as well as from the inner product induced by the kernel defining such a potential (see also Lemma 2, and Theorem 3 (i) in [13], for $d=2$ and $D=s_2(1)$). □

Theorem 1. *Under Assumption AI and* $r\to\infty$

$$\left[M_0(\nu)-4\pi r^2\{1-\Phi(\nu)\}\right]/r\to^{dist}\mathcal{N}_1\sim \mathrm{N}\big(0,k_2(\nu)\big). \tag{44}$$

Under Assumption AII i) for $0<\alpha<2$ *and* $r\to\infty$

$$\left[M_0(\nu)-4\pi r^2\{1-\Phi(\nu)\}\right]\Big/\left[r^{2-\frac{\alpha}{2}}\sqrt{L(r)}\right]\to^{dist}\mathcal{N}_2\sim \mathrm{N}(0,k_3(\alpha,\nu)), \tag{45}$$

while for $\alpha=2$

$$\left[M_0(\nu)-4\pi r^2\{1-\Phi(\nu)\}\right]\Big/\left[r\sqrt{L(r)\log(2r)}\right]\to^{dist}\mathcal{N}_3\sim \mathrm{N}(0,k_4(\nu)), \tag{46}$$

where "$\to^{dist}$*" stands for convergence in distributions.*

Remark 1. Under Assumption AII i), ii), for $0<\alpha<2$, as $r\to\infty$, the limit Gaussian random variable $\mathcal{N}_2$ admits the following stochastic integral representation, in the mean-square sense:

$$\mathcal{N}_2=\frac{K_1(\nu)|s_2(1)|}{\gamma(\alpha/2)}\int_{\mathbb{R}^2}Y_3(\|\lambda\|)\,\frac{Z(d\lambda)}{\|\lambda\|^{2-\frac{\alpha}{2}}},$$

where Z denotes complex Gaussian white noise.

Proof. The proof used some ideas from different sources. Namely, (44) and (46) follow from [14], while (45) can also be obtained from [9], taking into account that the Hermite rank of the function $N(u)=1_{\{u>\nu\}}$ is equal to one.

We now present a summary of the main steps of the proof. Specifically, under Assumption AII, we retsrict our attention to the case $0<\alpha<2$, and detail the main steps in the derivation of the Gaussian limit distribution when $r\to\infty$.

First, for $0<\alpha<2$, we have to note that, in view of

$$\frac{1}{L(r)r^{4-\alpha}}\mathsf{E}\left[M_0(\nu)-4\pi r^2\{1-\Phi(\nu)\}-K_1(\nu)\int_{s_2(r)}H_1\Big(\widetilde{T}(x)\Big)\widetilde{\sigma}_r(dx)\right]^2\leq$$

$$\leq\frac{1}{L(r)r^{4-\alpha}}\int_{s_2(r)}\int_{s_2(r)}B^2(\|x-y\|)\widetilde{\sigma}_r(dx)\widetilde{\sigma}_r(dy)\sum_{j=2}^{\infty}K_j^2\to 0,\quad r\to\infty, \tag{47}$$

a *Reduction Principle* can be applied, and the limit, in the mean-square sense, as $r \to \infty$, of

$$\frac{1}{L(r)r^{2-\alpha/2}}\, M_0(\nu) - 4\pi r^2\{1-\Phi(\nu)\}$$

coincides with the mean-square limit of

$$\frac{1}{L(r)r^{2-\alpha/2}}\, K_1(\nu) \int_{s_2(r)} H_1\Big(\widetilde{T}(x)\Big)\widetilde{\sigma}_r(dx).$$

Hence, as $r \to \infty$ both functionals has the same limit in distribution sense. Note that inequality (47) directly follows from Assumption AII, since

$$B^j(\|x\|) \le B^2(\|x\|), \quad j \ge 2,$$

and, considering r sufficiently large, there exists and $M(\epsilon)$ such that $B(\|x\|) < \epsilon$, for $\|x\| > M(\epsilon)$, $\epsilon \to 0$, with $B(\|x\|) \le 1$, for $\|x\| \le M(\epsilon)$, keeping in mind that $\mathsf{E}\Big[\widetilde{T}^2(x)\Big] = 1$, for all $x \in \mathbb{R}^3$. In particular,

$$\int_{s_2(r)}\int_{s_2(r)} B^2(\|x-y\|)\widetilde{\sigma}_r(dx)\widetilde{\sigma}_r(dy) \le \widetilde{M}\left[\frac{r^2}{L(r)r^{4-\alpha}} + \epsilon\frac{r^{4-\alpha}L(r)}{L(r)r^{4-\alpha}}\right], \quad r\to\infty,\ \epsilon\to 0.$$

We now study the limit, in the mean-square sense, of

$$\frac{1}{L(r)r^{2-\alpha/2}} \int_{s_2(r)} H_1\Big(\widetilde{T}(x)\Big)\widetilde{\sigma}_r(dx).$$

Let us consider an homogeneous isotropic Gaussian random field $\mathcal{T}$ on $\mathbb{R}^2$ having covariance function:

$$\mathrm{Cov}\big(\mathcal{T}(0),\mathcal{T}(x)\big) = \widetilde{B}_{\mathcal{T}}(x) = \frac{\mathcal{L}(x)}{\|x\|^{\alpha}} = B_{\mathcal{T}}(\|x\|) = \frac{L(\|x\|)}{\|x\|^{\alpha}}, \quad x\in\mathbb{R}^2, \tag{48}$$

where $\mathcal{L}(x) = L(\|x\|) = L(r)$, $r>0$, is a slowly varying function, as $r = \|x\| \to \infty$, such that the Gaussian random field $\mathcal{T}$ has absolutely continuous spectrum, with spectral density $f_0^{\mathcal{T}}(\|\lambda\|)$ defined on $\mathbb{R}^2$, and being a decreasing function for $\|\lambda\| \in (0,\epsilon]$, $\epsilon > 0$.

Under Assumption AII, from equations (40) and (41),

$$\lim_{r\to\infty}\frac{1}{L(r)r^{4-\alpha}}\mathsf{E}\left[\int_{s_2(r)} H_1\Big(\widetilde{T}(x)\Big)\widetilde{\sigma}_r(dx) - \int_{s_2(r)}\mathcal{T}(x)\widetilde{\sigma}_r(dx)\right]^2 = 0 \tag{49}$$

Hence, as $r\to\infty$,

$$\frac{1}{L(r)r^{4-\alpha}}\int_{s_2(r)} H_1\Big(\widetilde{T}(x)\Big)\widetilde{\sigma}_r(dx) \qquad \text{and} \qquad \frac{1}{L(r)r^{4-\alpha}}\int_{s_2(r)}\mathcal{T}(x)\widetilde{\sigma}_r(dx)$$

have the same limit in distribution sense.

Furthermore, denoting by $W(d\lambda)$ the Wiener measure on $\mathbb{R}^2$,

$$\mathsf{E}\left[\frac{1}{L(r)r^{2-\alpha/2}}\int_{s_2(r)}\mathcal{T}(x)\widetilde{\sigma}_r(dx) - \frac{|s_2(1)|}{\gamma(\alpha/2)}\int_{\mathbb{R}^2} Y_3(\|\lambda\|)\frac{W(d\lambda)}{\|\lambda\|^{2-\frac{\alpha}{2}}}\right]^2 =$$

$$= \int_{\mathbb{R}^2} |Y_3(\|\lambda\|)|^2\left[\frac{|s_2(1)|}{\gamma(\alpha/2)}\right]^2 Q_r(\lambda)\,\frac{d\lambda}{\|\lambda\|^{4-\alpha}},$$

where

$$Q_r(\lambda) = \left(\left[\frac{\|\lambda\|^{2-(\alpha/2)}\gamma(\alpha/2)}{\sqrt{L(r)}r^{2-(\alpha/2)}}\left[f_0^{\mathcal{T}}(\|\lambda\|/r)\right]^{1/2}\right] - 1\right)^2.$$

By Tauberian theorems (see, for example, [9]; [10]), Q_r converges to zero pointwise, as $r\to\infty$. From Lemma 1, we can apply Dominated Convergence Theorem to obtain the mean-square convergence to zero, and hence, in distribution sense. □

Note that using the Malliavin–Stein method, the rate of convergence in the central limit theorem (44) is investigated in [23, 24].

4. The first Minkowski functional for χ^2-square random fields

4.1. Direct formulae. Consider the chi-square random field of the form

$$\widetilde{S}_q(x) = \frac{1}{2}\Big(\widetilde{T}_1^2(x) + \cdots + \widetilde{T}_q^2(x)\Big), \quad x \in \mathbb{R}^3, \quad q \geq 1, \tag{50}$$

where $\widetilde{T}_1(x), \ldots, \widetilde{T}_q(x)$ are independent copies of zero-mean homogeneous isotropic Gaussian field $\widetilde{T}(x)$, $x \in \mathbb{R}^3$, such that

$$\mathsf{E}\Big[\widetilde{T}^2(x)\Big] = 1, \quad \mathrm{Cov}\Big(\widetilde{T}(0), \widetilde{T}(x)\Big) = \widetilde{B}(x) = B(\|x\|), \quad x \in \mathbb{R}^3.$$

Note that

$$\mathsf{E}\widetilde{S}_q(x) = \frac{q}{2}, \quad \mathrm{Var}\,\widetilde{S}_q(x) = \frac{q}{2}, \quad \mathrm{Cov}\Big(\widetilde{S}_q(0), \widetilde{S}_q(x)\Big) = \frac{q}{2}B^2(\|x\|). \tag{51}$$

The χ^2-random fields belong to the Lancaster–Sarmanov class (see [12]), thus, the marginal and bivariate densities are of the form:

$$p(u) = p_{q/2}(u) = \frac{d}{du}\,\mathsf{P}\Big\{\widetilde{S}_q(x) \leq u\Big\} = \frac{u^{(q/2)-1}e^{-u}}{\Gamma(q/2)}, \quad u \in (0,\infty), \tag{52}$$

$$\begin{aligned}
p_{q/2}\big(u, w, \gamma(\|x-y\|)\big) &= \frac{\partial^2}{\partial u \partial w}\,\mathsf{P}\Big\{\widetilde{S}_q(x) \leq u,\ \widetilde{S}_q(y) \leq w\Big\} = \\
&= p(u)p(w)\left(1 + \sum_{k=1}^{\infty}\gamma^k(\|x-y\|)e_k(u)e_k(w)\right) = \\
&= \left(\frac{uw}{\gamma}\right)^{(\frac{q}{2}-1)/2}\exp\left(-\frac{u+w}{1-\gamma}\right)\, I_{(q/2)-1}\left(2\frac{\sqrt{uw\gamma}}{1-\gamma}\right)\frac{1}{\Gamma(q/2)(1-\gamma)}, \quad (u,w) \in (0,\infty)^2,
\end{aligned} \tag{53}$$

where $I_\mu(z)$ is the modified Bessel function of the first kind of order μ,

$$\gamma = \gamma(\|x-y\|) = \mathrm{Corr}\Big(\widetilde{S}_q(x), \widetilde{S}_q(y)\Big) = B^2(\|x\|),$$

$$e_k(u) = e_k^{(q/2)}(u) = L_k^{((q/2)-1)}(u)\,\left(\frac{k!\Gamma(q/2)}{\Gamma((q/2)+k)}\right)^{1/2}, \ k = 0, 1, 2, \ldots,$$

with $L_k^{(b)}(u)$ being the generalized Laguerre polynomials of index b, for $k \geq 0$. These functions are orthogonal with respect to the density $p_{q/2}(u)$, $u > 0$. Using the representation

$$L_k^{(b)}(u) = \frac{u^{-b}e^u}{k!}\frac{d^k}{du^k}\big(e^{-u}u^{b+k}\big),$$

one can derive the first few polynomials:

$$e_0^{(q/2)}(u) \equiv 1, \quad e_1^{(q/2)}(u) = \big((q/2) - u\big)(q/2)^{-1/2}.$$

Note that

$$\mathsf{E}e_k^{(q/2)}\Big(\widetilde{S}_q(x)\Big) = 0, \quad k \geq 1,$$

$$\mathsf{E}\Big(e_m^{(q/2)}\Big(\widetilde{S}_q(x)\Big)e_k^{(q/2)}\Big(\widetilde{S}_q(y)\Big)\Big) = \delta_m^k\gamma^m(\|x-y\|) = \delta_m^k B^{2m}(\|x-y\|).$$

In this construction of χ^2-random fields, the correlation function $B^2(\|x-y\|) \geq 0$ must be not only non-negative definite, but also nonnegative.

Consider now an isotropic random field $S_q(r,\theta,\varphi)$, which is the restriction to the sphere $s_2(r)$ of the random field $\widetilde{S}_q(x)$. The first Minkowski functional can be represented as

$$
\begin{aligned}
M_0^q(\nu) = \sigma\{(r,\theta,\varphi)\in s_2(r) : S_q(r,\theta,\varphi) > \nu\} = \\
= \int_{s_2(r)} 1_{\{S_q>\nu\}}(r,\theta,\varphi)\sigma_r(d\theta,d\varphi) = \\
= \int_{s_2(r)} 1_{\{\widetilde{S}_q>\nu\}}(x)\widetilde{\sigma}_r(dx). \qquad (54)
\end{aligned}
$$

Now let $N(\cdot)$ denote any, real measurable function such that $\mathsf{E}N\left(\widetilde{S}_q(0)\right)^2 < \infty$. The function $N(\cdot)$ can be expanded in the series

$$
N(u) = \sum_{k=0}^{\infty} K_k^L e_k^{(q/2)}(u), \quad K_k^L = \int_0^{\infty} N(u)e_k^{(q/2)}(u)p_{q/2}(u)du, \qquad (55)
$$

which converges in the Hilbert space $L_2\big((0,\infty),p_{q/2}(u)du\big)$, and in view of the Parseval equality: $\sum_{k=0}^{\infty}\left[K_k^L\right]^2 < \infty$. In particular, for the indicator function $1_{\{u>\nu\}}$

$$
K_k^L = \int_{\nu}^{\infty} e_k^{(q/2)}(u)p(u)du. \qquad (56)
$$

We denote the incomplete Gamma function

$$
\Gamma(q/2,w) = \int_w^{\infty} \frac{u^{(q/2)-1}e^{-u}}{\Gamma(q/2)}du,
$$

which has the property:

$$
\Gamma(\beta+1,w) = \beta\Gamma(\beta,w) + e^{\beta}e^{-u}, \quad \beta>0, \quad w\geq 0,
$$

and thus, from (56), we get:

$$
K_1^L(\nu) = -\sqrt{q/2}\,p_{(q/2)+1}(\nu), \qquad (57)
$$

where

$$
\frac{\partial p_{q/2}}{\partial\gamma} = \frac{q}{2}\frac{\partial^2 p_{(q/2)+1}}{\partial u\partial w}. \qquad (58)
$$

Then, the first Minkowski functional can be expanded in the Hilbert space $L_2\big((0,\infty),p_{q/2}(u)du\big)$ as follows:

$$
\begin{aligned}
M_0^q(\nu) = \sum_{k=0}^{\infty} K_k^L(\nu)\int_{s_2(r)} e_k^{(q/2)}\left(\widetilde{S}_q(x)\right)\widetilde{\sigma}_r(dx) = \\
= \mathsf{E}M_0^q(\nu) + \sum_{k=1}^{\infty} K_k^L(\nu)\int_{s_2(r)} e_k^{(q/2)}\left(\widetilde{S}_q(x)\right)\widetilde{\sigma}_r(dx), \qquad (59)
\end{aligned}
$$

where

$$
\begin{aligned}
\operatorname{Var} M_0^q(\nu) = \sum_{k=1}^{\infty}\left[K_k^L(\nu)\right]^2 \int_{s_2(r)}\int_{s_2(r)} \gamma^k(\|x-y\|)\widetilde{\sigma}_r(dx)\widetilde{\sigma}_r(dy) = \\
= 2^3\pi^2 r^2 \sum_{k=1}^{\infty}\left[K_k^L(\nu)\right]^2 \int_0^{2r} z\gamma^k(z)dz.
\end{aligned}
$$

By Kinematic Formula (see [1]) we have

$$
\mathsf{E}M_0^q(\nu) = 4\Gamma(q/2,w)r^2\pi, \qquad (60)
$$

and using the differential equation (58), for function $p_{(q/2)}(u,w,\gamma)$ (see [2]), one can obtain the following direct expression for the variance

$$\mathrm{Var}\, M_0^q(\nu) = 4\pi^2 r^2 q\left[\int_0^{2r} z dz \int_0^{\gamma(z)} p_{(q/2)+1}(\nu,\nu,s)ds\right] =$$
$$= 4\pi^2 r^2 q \frac{\nu^{q/2}}{\Gamma(q/2+1)}\int_0^{2r} z dz \int_0^{\gamma(z)} \exp\left(-\frac{2\nu}{1-s}\right) I_{q/2}\left(2\nu\frac{\sqrt{s}}{1-s}\right)\frac{1}{s^{q/4}\,(1-s)}\,ds.$$

Really,

$$\mathrm{Var}\left\{\int_{s_2(r)} 1_{\{\widetilde{S}_q>\nu\}}(x)\widetilde{\sigma}_r(dx)\right\} = \int_{s_2(r)}\int_{s_2(r)} \widetilde{\sigma}_r(dx)\widetilde{\sigma}_r(dy)\times$$
$$\times\iint_{\mathbb{R}^2} 1_{\{u>\nu\}}1_{\{w>\nu\}}[p_{q/2}(u,w;\gamma(\|x-y\|)-p(u)p(w)]dudw =$$
$$= \int_{s_2(r)}\int_{s_2(r)} \widetilde{\sigma}_r(dx)\widetilde{\sigma}_r(dy)\int_\nu^\infty\int_\nu^\infty\int_0^{\gamma(\|x-y\|}\frac{\partial}{\partial t}[p_{q/2}(u,w;t)]dtdudw =$$
$$= \frac{q}{2}\int_{s_2(r)}\int_{s_2(r)} \widetilde{\sigma}_r(dx)\widetilde{\sigma}_r(dy)\int_0^{\gamma(\|x-y\|} ds\int_\nu^\infty\int_\nu^\infty\frac{\partial^2 p_{(q/2)+1}(u,w,s)}{\partial u\partial w}dudw =$$
$$= \frac{q}{2}\int_{s_2(r)}\int_{s_2(r)} \widetilde{\sigma}_r(dx)\widetilde{\sigma}_r(dy)\int_0^{\gamma(\|x-y\|} p_{(q/2)+1}(\nu,\nu,s)ds =$$
$$= 4\pi^2 r^2 q\left[\int_0^{2r} z dz\int_0^{\gamma(z)} p_{(q/2)+1}(\nu,\nu,s)ds\right].$$

For a general isotropic χ^2-random field T with the covariance function $\Gamma_r(\cos\theta)$, one can show that

$$\mathsf{E}M_0^q(\nu) = 4\Gamma(q/2,w)r^2\pi,\ \mathrm{Var}\{M_0^q(\nu)\} =$$
$$= \frac{q\nu^{q/2}}{\Gamma(q/2+1)}\int_{s_2(r)}\int_{s_2(r)} \widetilde{\sigma}_r(dx)\widetilde{\sigma}_r(dy)\times$$
$$\times\int_0^{\Gamma_r^q(\cos\theta)} \exp\left(-\frac{2\nu}{1-s}\right) I_{q/2}\left(2\nu\frac{\sqrt{s}}{1-s}\right)\frac{1}{s^{q/4}\,(1-s)}\,ds,$$

where $\Gamma_r^q(\cos\theta) = [\Gamma_r(\cos\theta)]^2$, and $\Gamma_r(\cos\theta) = \mathsf{E}T(r,\theta,\varphi)T(r,\theta',\varphi')$.

The last formula seems new and computational friendly.

4.2. Asymptotic formulae as $r\to\infty$.

The following assumptions will be considered.

Assumption BI. The homogeneous isotropic χ^2-random random field $\widetilde{S}_q(x)$, $x\in\mathbb{R}^3$, has the correlation function $\gamma(\|x\|)$, $x\in\mathbb{R}^3$, such that

$$\int_0^\infty z|\gamma(z)|\,dz<\infty,\quad \int_0^\infty z\gamma(z)\,dz\neq 0.$$

Under the Assumption BI, as $r\to\infty$

$$\mathrm{Var}\, M_0^q(\nu) = k_5(\nu)r^2\{1+o(1)\},$$

where

$$k_5(\nu) = 2^3\pi^2\sum_{k=1}^\infty\left[K_k^L(\nu)\right]^2\int_0^\infty z\gamma^k(z)\,dz\ \in(0,\infty).$$

Assumption BII. The homogeneous isotropic χ^2-random field $\widetilde{S}_q(x)$, $x \in \mathbb{R}^3$, has correlation function

$$\gamma(\|x\|) = \left[\frac{L(\|x\|)}{\|x\|^{\alpha}}\right]^2, \quad x \in \mathbb{R}^3, \quad 0 < \alpha < 1, \tag{61}$$

where $L(\|x\|)$ is a slowly varying function, as $\|x\| \to \infty$.

Condition BII is satisfied by the correlation function

$$\gamma(\|z\|) = \frac{1}{(1 + \|z\|^{\delta})^{a}}, \quad 0 < \delta \leq 2, \quad a > 0, \tag{62}$$

where $L(\|z\|) = \|z\|^{\delta a}/\big(1 + \|z\|^{\delta}\big)^{a}$.

Under Assumption BII, for $0 < \alpha < 1$, and $r \to \infty$,

$$\operatorname{Var} M_0^q(\nu) = k_6(\alpha, \nu) r^{4-2\alpha} L(r)\{1 + o(1)\}, \tag{63}$$

where

$$k_6(\alpha, \nu) = \frac{\big[K_1^L(\nu)\big]^2 2^{4-\alpha}\pi^2\Gamma\big(\frac{4-\alpha}{2}\big)}{\Gamma\big(\frac{2-\alpha}{2}\big)} \in (0, \infty), \tag{64}$$

and $J_1(\nu)$ is the Bessel function of the first kind of order 1, while for $\alpha = 1$, and $r \to \infty$

$$\operatorname{Var} M_0^q(\nu) = k_7(\nu) r^2 L(r) \log(2r)\{1 + o(1)\},$$

with

$$k_7(\nu) = k_5(\nu)/4 \in (0, \infty).$$

Assumption BIII. The slowly varying function L, appearing in equation (61), in Assumption BII, is such that, for every $m \geq 2$ there exists a constant $C > 0$, satisfying

$$\begin{aligned}
&\int_{s_2(1)} \overset{\cdots}{(m)} \int_{s_2(1)} \frac{L(r\|x_1 - x_2\|)}{L(r)\|x_1 - x_2\|^{\alpha}} \cdot \frac{L(r\|x_2 - x_3\|)}{L(r)\|x_2 - x_3\|^{\alpha}} \times \\
&\quad \times \cdots \times \frac{L(r\|x_m - x_1\|)}{L(r)\|x_m - x_1\|^{\alpha}} \, \widetilde{\sigma}(dx_1)\widetilde{\sigma}(dx_2)\ldots\widetilde{\sigma}(dx_m) \leq \\
&\leq C \int_{s_2(1)} \overset{\cdots}{(m)} \int_{s_2(1)} \frac{\widetilde{\sigma}(dx_1)\widetilde{\sigma}(dx_2)\ldots\widetilde{\sigma}(dx_m)}{\|x_1 - x_2\|^{\alpha}\|x_2 - x_3\|^{\alpha}\ldots\|x_m - x_1\|^{\alpha}}.
\end{aligned}$$

In the following result, we will use the Fredholm determinant of an operator A, which is a complex-valued function generalizing the determinant of a matrix, as given in the next definition.

Definition 1 (see, for example, [28], Ch. 5, pp. 47–48, equation (5.12))**.** Let A be a trace operator on a separable Hilbert space H. The Fredholm determinant of A is

$$\mathcal{D}(\omega) = \det(I - \omega A) = \exp\left(-\sum_{k=1}^{\infty} \frac{\operatorname{Tr} A^k}{k}\omega^k\right) = \exp\left(-\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}[\lambda_l(A)]^k \frac{\omega^k}{k}\right), \tag{65}$$

for $\omega \in \mathbb{C}$, and $|\omega| \cdot \|A\|_1 < 1$. Note that $\|A^m\|_1 \leq \|A\|_1^m$, for A being a trace operator.

We can now formulate the theorem about asymptotic distributions of the first Minkowski functional of χ^2-random field under increasing domain asymptotics.

Theorem 2. *Under Assumption BI, as* $r \to \infty$

$$\big[M_0^q(\nu) - 4\Gamma(q/2, w) r^2\pi\big]/r \to^{dist} \mathrm{N}\ (0, k_5(\nu)). \tag{66}$$

Under Assumptions BII–BIII, for $0 < \alpha < 1$, *and* $r \to \infty$,

$$\big[M_0^q(\nu) - 4\Gamma(q/2, w) r^2\pi\big]/\big[r^{2-\alpha}L(r)\big] \to^{dist} \mathrm{R}, \tag{67}$$

where the random variable R *has the Rosenblatt-type distribution with characteristic function*

$$\psi(z) = \mathsf{E}e^{\mathrm{i}z\mathrm{R}} = \exp\left(\frac{q}{2}\sum_{m=2}^{\infty}\frac{\left(2\mathrm{i}z\sqrt{q/2}\,p_{(q/2)+1}(\nu)/\sqrt{2q}\right)^m}{m}c_m\right) \tag{68}$$

with c_m, $m \geq 2$, *being defined as*

$$c_m = \int_{s_2(1)} \underset{(m)}{\dots} \int_{s_2(1)} \frac{1}{\|x_1 - x_2\|^{\alpha}} \cdot \frac{1}{\|x_2 - x_3\|^{\alpha}} \cdots \frac{1}{\|x_m - x_1\|^{\alpha}}\,\widetilde{\sigma}(dx_1)\dots\widetilde{\sigma}(dx_m).$$

For $\alpha = 1$

$$\left[M_0^q(\nu) - 4\Gamma(q/2, w)r^2\pi\right]\Big/\left[r\sqrt{L_1(r)\log(2r)}\right] \to^{dist} \mathrm{N}\big(0, k_7(\nu)\big). \tag{69}$$

Remark 2. Note that, for the covariance function (62), Theorem 2 holds under Assumption BII, and clearly, Assumption BIII is not needed.

Proof. Again, we pay attention to the proof of the non-central limit result derived under Assumptions BII–BIII (for more details, we refer to the reader to [12–15, 31]). We now summarize the main steps of the proof of such a result.

First, a *Reduction Principle* can be applied, since the following inequality holds:

$$\left[\frac{1}{L(r)r^{2-\alpha}}\right]^2 \mathsf{E}\left[M_0^q(\nu) - \mathsf{E}[M_0^q(\nu)] - K_1^L(\nu)\int_{s_2(r)} e_1^{q/2}\left(\widetilde{\chi_q^2}(x)\right)\widetilde{\sigma}_r(dx)\right]^2 \leq$$
$$\leq \left[\frac{1}{L(r)r^{2-\alpha}}\right]^2 \int_{s_2(r)}\int_{s_2(r)} B^3(\|x-y\|)\widetilde{\sigma}(dx)\widetilde{\sigma}(dy)\sum_{j=2}^{\infty}[K_j^L(\nu)]^2 \to 0, \quad r \to \infty,$$

where the convergence to zero of the last integral can be proved in a similar way to equation (47), considering

$$B^j(\|x\|) \leq B^3(\|x\|), \quad j \geq 3,$$

and similar inequalities for $B^3(\|x\|)$ and $B^2(\|x\|)$ (instead of $B^2(\|x\|)$ and $B(\|x\|)$).

From equation (50),

$$e_1^{(q/2)}\big(\chi_q^2(x)\big) = -\frac{1}{\sqrt{2q}}\sum_{j=1}^{q} H_2\left(\widetilde{T}_j(x)\right). \tag{70}$$

Let us denote by $\widetilde{\mathcal{T}}_j$, $j = 1, \dots, q$, q independent copies of a zero-mean Gaussian isotropic random field $\widetilde{\mathcal{T}}$ on $\mathbb{R}^2$ with covariance function

$$\frac{L(\|x\|)}{\|x\|^{\alpha}}, \quad x \in \mathbb{R}^2.$$

Under BII, from equation (63), for $j = 1, \dots, q$,

$$\lim_{r\to\infty}\frac{1}{L(r)r^{4-2\alpha}}\mathsf{E}\left[\int_{s_2(r)} H_2\left(\widetilde{T}_j(x)\right)\widetilde{\sigma}_r(dx) - \int_{s_2(r)}\left(\widetilde{\mathcal{T}}_j^2(x) - 1\right)\widetilde{\sigma}_r(dx)\right]^2 = 0. \tag{71}$$

Hence, as $r \to \infty$, for $j = 1, \dots, q$,

$$\frac{1}{L(r)r^{2-\alpha}}\int_{s_2(r)} H_2\left(\widetilde{T}_j(x)\right)\widetilde{\sigma}_r(dx) \quad \text{and} \quad \frac{1}{L(r)r^{2-\alpha}}\int_{s_2(r)}\left(\widetilde{\mathcal{T}}_j^2(x) - 1\right)\widetilde{\sigma}_r(dx)$$

have the same limit in distribution sense. Therefore, it is sufficient to compute the limit, as $r \to \infty$, of the following characteristic function:

$$\begin{aligned}
\phi_r(z) &= \mathsf{E}\left[\exp\left(\frac{\mathrm{i}K_1^L(\nu)z}{r^{2-\alpha}L(r)}\int_{s_2(r)}\left(-\frac{1}{\sqrt{2q}}\sum_{j=1}^{q}H_2(\widetilde{\mathcal{T}}_j(x))\right)\widetilde{\sigma}_r(dx)\right)\right] = \\
&= \prod_{j=1}^{q}\mathsf{E}\left[\exp\left(\frac{\sqrt{q/2}\,p_{(q/2)+1}(\nu)\mathrm{i}z}{\sqrt{2q}r^{2-\alpha}L(r)}\int_{s_2(r)}(\widetilde{\mathcal{T}}_j^2(x)-1)\widetilde{\sigma}_r(dx)\right)\right] = \\
&= \prod_{j=1}^{q}\exp\left(\frac{1}{2}\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{2\mathrm{i}z\sqrt{q/2}\,p_{(q/2)+1}(\nu)}{\sqrt{2q}r^{2-\alpha}L(r)}\right)^m\mathrm{Tr}\left(R^m_{\widetilde{\mathcal{T}}_j,s_2(r)}\right)\right) = \\
&= \exp\left(\frac{q}{2}\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{2\mathrm{i}z\sqrt{q/2}\,p_{(q/2)+1}(\nu)}{\sqrt{2q}r^{2-\alpha}L(r)}\right)^m\mathrm{Tr}\left(R^m_{\widetilde{\mathcal{T}},s_2(r)}\right)\right), \qquad (72)
\end{aligned}$$

where, in its computation, we have applied Fredholm determinant formula (see Definition 1), and the identity

$$K_1^L(\nu) = -\sqrt{q/2}\,p_{(q/2)+1}(\nu).$$

Here, for $m \geq 2$, $\mathrm{Tr}\left(R^m_{\widetilde{\mathcal{T}},s_2(r)}\right)$, and $\mathrm{Tr}\left(R^m_{\widetilde{\mathcal{T}}_j,s_2(r)}\right)$ denote the trace of the mth power of the autocovariance operator $R_{\widetilde{\mathcal{T}},s_2(r)}$ of $\widetilde{\mathcal{T}}$, and of the autocovariance operator $R_{\widetilde{\mathcal{T}}_j,s_2(r)}$ of $\widetilde{\mathcal{T}}_j$ on $s_2(r)$, for $j = 1,\ldots,q$. Futhermore, the following pointwise convergence holds

$$\lim_{r\to\infty}\left[\frac{1}{L(r)r^{2-\alpha}}\right]^m\mathrm{Tr}\left(R^m_{\widetilde{\mathcal{T}},s_2(r)}\right) = \mathrm{Tr}\left(R^m_{\widetilde{\mathcal{T}},s_2(1)}\right), \quad m \geq 2. \qquad (73)$$

To apply Dominated Convergence Theorem, under Assumptions BII-BIII, consider

$$\begin{aligned}
&\left|\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{2\mathrm{i}z\sqrt{q/2}\,p_{(q/2)+1}(\nu)}{\sqrt{2q}r^{2-\alpha}L(r)}\right)^m\mathrm{Tr}\left(R^m_{\widetilde{\mathcal{T}},s_2(r)}\right)\right| \leq \\
&\quad\leq C\sum_{m=2}^{\infty}\frac{1}{m}\left|\frac{2\sqrt{q/2}\,p_{(q/2)+1}(\nu)z}{\sqrt{2q}}\right|^m\mathrm{Tr}\left(R^m_{\widetilde{\mathcal{T}},s_2(1)}\right) \leq \\
&\quad\leq -C\ln\left(\mathcal{D}_{R_{\widetilde{\mathcal{T}},s_2(1)}}\left(\left|\frac{2\sqrt{q/2}\,p_{(q/2)+1}(\nu)z}{\sqrt{2q}}\right|\right)\right), \qquad (74)
\end{aligned}$$

where $\mathcal{D}_{R_{\widetilde{\mathcal{T}},s_2(1)}}\left(\left|\frac{2\sqrt{q/2}\,p_{(q/2)+1}(\nu)z}{\sqrt{2q}}\right|\right)$ denotes the Fredholm determinant of $R_{\widetilde{\mathcal{T}},s_2(1)}$, at point $\left|\frac{2\sqrt{q/2}\,p_{(q/2)+1}(\nu)z}{\sqrt{2q}}\right|$, which is finite for

$$\left|\frac{2\sqrt{q/2}\,p_{(q/2)+1}(\nu)z}{\sqrt{2q}}\right|\mathrm{Tr}\left(R_{\widetilde{\mathcal{T}},s_2(1)}\right) < 1.$$

From equations (73) and (74),

$$\lim_{r\to\infty}\phi_r(z) = \exp\left(\frac{q}{2}\sum_{m=2}^{\infty}\frac{1}{m}\left(\frac{2\mathrm{i}z\sqrt{q/2}\,p_{(q/2)+1}(\nu)}{\sqrt{2q}}\right)^m\mathrm{Tr}\left(R^m_{\widetilde{\mathcal{T}},s_2(1)}\right)\right) = \psi(z), \qquad (75)$$

for all z such that

$$\left|2\mathrm{i}z\frac{\sqrt{q/2}\,p_{(q/2)+1}(\nu)}{\sqrt{2q}}\right|\mathrm{Tr}\left(R_{\widetilde{\mathcal{T}},s_2(1)}\right) < 1,$$

where

$$\operatorname{Tr}\left(R^{m}_{\widetilde{\mathcal{T}},s_2(1)}\right) = \int_{s_2(1)} \overset{\cdots}{{}_{(m)}} \int_{s_2(1)} \frac{1}{\|x_1 - x_2\|^{\alpha}} \frac{1}{\|x_2 - x_3\|^{\alpha}} \cdots \frac{1}{\|x_m - x_1\|^{\alpha}} \times \\ \times \widetilde{\sigma}(dx_1)\dots\widetilde{\sigma}(dx_m) = c_m.$$

An analytic continuation argument (see [17, Th. 7.1.1]) guarantees that ψ defines the unique limit characteristic function for all real values of z. □

Alternatively, an isonormal representation of R in (67) can be obtained, in the mean-square sense, as follows from the following result, applying the above reduction principle. The following additional assumption is considered

Assumption BIV. The slowly varying function L, appearing in equation (61), in Assumption BII, is such that, the chi-squared random field $\widetilde{S}_q = 1/2\sum_{j=1}^{q}\widetilde{T}_j^2$, introduced in equation (50), has absolutely continuous spectra. Specifically, for $j = 1,\dots,q$, $\widetilde{T}_j$ has spectral density $f_{0j}(\|\lambda\|)$, being a decreasing functions for $\|\lambda\| \in (0,\epsilon]$, $\epsilon > 0$.

The following lemma will be applied in the proof of Theorem 3 below.

Lemma 2. *For* $0 < \alpha < 1$, *the following identities hold:*

$$\int_{\mathbb{R}^4} |Y_3(\|\lambda_1 + \lambda_2\|)|^2 \frac{d\lambda_1 d\lambda_2}{(\|\lambda_1\|\|\lambda_2\|)^{2-\alpha}} = k_6(\alpha,\nu)\left[\frac{\gamma(\alpha)}{|s_2(1)|}\right]^2 < \infty, \tag{76}$$

where Y_3 *is defined in (43), constant* $k_6(\alpha,\nu)$ *has been computed in (64), and* $\gamma(\alpha)$ *has been introduced in equation (42).*

The proof of Lemma 2 can be derived in a similar way to Theorem 3(i) in [13], considering $d = 2$, and $D = s_2(1)$.

Theorem 3. *Under Conditions BII–BIV, for* $0 < \alpha < 1$, *the limit random variable* R *in (67) admits the following integral representation:*

$$\mathrm{R} = -\frac{|s(1)|K_1^L(\nu)}{\gamma(\alpha)\sqrt{2q}} \sum_{j=1}^{q} \int_{\mathbb{R}^4}' Y_3(\|\lambda_1 + \lambda_2\|) \frac{Z_{1j}(d\lambda_1) Z_{2j}(d\lambda_2)}{\|\lambda_1\|^{\frac{2-\alpha}{2}} \|\lambda_2\|^{\frac{2-\alpha}{2}}},$$

where, as before, $\gamma(\alpha) = \frac{\pi 2^{\alpha}\Gamma(\alpha/2)}{\Gamma\left(\frac{2-\alpha}{2}\right)}$, $0 < \alpha < 1$, Z_{ij}, $i = 1,2$, $j = 1,\dots,r$, *are independent complex Gaussian white noise measures,* $\int_{\mathbb{R}^4}'$ *means that one does not integrate on the hyperdiagonals* $\lambda_1 = \pm\lambda_2$, *and* Y_3 *denotes, as in Lemma 2, the spherical Bessel function.*

Proof. First a reduction principle is applied as in Theorem 2. Secondly, as in such a theorem, apply the relationship between the first Laguerre polynomial and the second-Hermite polynomial, to obtain

$$\left[\frac{K_1^L}{L(r)r^{2-\alpha}}\right] \int_{s_2(r)} e_1^{q/2}(\chi_q^2(x))\widetilde{\sigma}_r(dx) = \\ = \left[-\frac{K_1^L}{\sqrt{2q}L(r)r^{2-\alpha}}\right] \left[\sum_{j=1}^{q} \int_{s_2(r)} H_2(\widetilde{T}_j(x))\widetilde{\sigma}_r(dx)\right]. \tag{77}$$

Under Assumption BII, we can also consider here the asymptotic mean-square identity (71), for $r \to \infty$, between the functionals

$$-\frac{1}{L(r)r^{2-\alpha}} \sum_{j=1}^{q} \int_{s_2(r)} H_2(\widetilde{T}_j(x))\widetilde{\sigma}_r(dx) \quad \text{and} \quad \frac{1}{L(r)r^{2-\alpha}} \sum_{j=1}^{q} \int_{s_2(r)} (\widetilde{\mathcal{T}}_j^2(x) - 1)\widetilde{\sigma}_r(dx),$$

where, as before, $\widetilde{\mathcal{T}}_j$, $j = 1, \dots, q$, are independent copies of a zero-mean Gaussian isotropic random field $\widetilde{\mathcal{T}}$ on $\mathbb{R}^2$ with covariance function

$$\frac{L(\|x\|)}{\|x\|^{\alpha}}, \quad x \in \mathbb{R}^2, \quad 0 < \alpha < 2.$$

Thus, for $0 < \alpha < 1$, we now study the limit, in mean-square sense, as $r \to \infty$, of

$$\frac{1}{L(r)r^{2-\alpha}} \sum_{j=1}^{q} \int_{s_2(r)} \left(\widetilde{\mathcal{T}}_j^2(x) - 1\right) \widetilde{\sigma}_r(dx).$$

Denote, for $j = 1, \dots, q$, by $f_0^{\widetilde{\mathcal{T}}_j}(\|\lambda\|)$ the spectral density of the isotropic and homogeneous random field $\widetilde{\mathcal{T}}_j$, being a decreasing function of $\|\lambda\| \in (0, \epsilon]$, $\epsilon > 0$. Using the self-similarity of Gaussian white noise, and the Itô formula (see, for example, [4] and [18]), we obtain

$$\begin{aligned}
&\left[-\frac{K_1^L(\nu)}{\sqrt{2q}L(r)r^{2-\alpha}}\right] \left[\sum_{j=1}^{q} \int_{s_2(r)} H_2\left(\widetilde{\mathcal{T}}_j(x)\right) \widetilde{\sigma}_r(dx)\right] = -\frac{K_1^L}{\sqrt{2q}} \sum_{j=1}^{q} S_{rj} = \\
&\quad = -\frac{K_1^L(\nu)|s_2(r)|}{\gamma(\alpha)\sqrt{2q}r^{2-\alpha}L(r)} \times \\
&\quad \times \sum_{j=1}^{q} \int_{\mathbb{R}^4}'' Y_3(\|\lambda_1 + \lambda_2\|) \left(\gamma(\alpha) \prod_{k=1}^{2} \left[f_0^{\widetilde{\mathcal{T}}_j}\right]^{1/2} (\|\lambda_k\|)\right) Z_{1j}(d\lambda_1) Z_{2j}(d\lambda_2) \underset{d}{=} \\
&\quad \underset{d}{=} -\frac{K_1^L(\nu)|s_2(1)|}{\gamma(\alpha)\sqrt{2q}r^{2-\alpha}L(r)} \times \\
&\quad \times \sum_{j=1}^{q} \int_{\mathbb{R}^4}'' Y_3(\|\lambda_1 + \lambda_2\|) \left(\gamma(\alpha) \prod_{k=1}^{2} \left[f_0^{\widetilde{\mathcal{T}}_j}\right]^{1/2} (\|\lambda_k\|/r)\right) Z_{1j}(d\lambda_1) Z_{2j}(d\lambda_2). \qquad (78)
\end{aligned}$$

By the isometry property of multiple Wiener–Itô stochastic integrals, for $j = 1, \dots, q$,

$$\begin{aligned}
&\mathsf{E}\left[S_{rj} - \frac{|s_2(1)|}{\gamma(\alpha)} \int_{\mathbb{R}^4}'' Y_3(\|\lambda_1 + \lambda_2\|) \frac{Z_{1j}(d\lambda_1) Z_{2j}(d\lambda_2)}{\|\lambda_1\|^{\frac{2-\alpha}{2}} \|\lambda_2\|^{\frac{2-\alpha}{2}}}\right]^2 = \\
&\quad = \int_{\mathbb{R}^4} |Y_3(\|\lambda_1 + \lambda_2\|)|^2 \left[\frac{|s_2(1)|}{\gamma(\alpha)}\right]^2 Q_{rj}(\lambda_1, \lambda_2) \frac{d\lambda_1 d\lambda_2}{\|\lambda_1\|^{2-\alpha} \|\lambda_2\|^{2-\alpha}}, \qquad (79)
\end{aligned}$$

where, for $j = 1, \dots, q$, S_{rj} has been introduced in (78), and

$$Q_{rj}(\lambda_1, \lambda_2) = \left(\left[\frac{\|\lambda_1\|^{(2-\alpha)/2} \|\lambda_2\|^{(2-\alpha)/2} \gamma(\alpha)}{r^{2-\alpha} L(r)} \prod_{k=1}^{2} \left[f_0^{\widetilde{\mathcal{T}}_j}\right]^{1/2} (\|\lambda_k\|/r)\right] - 1\right)^2. \qquad (80)$$

Under Assumptions BII and BIV, by Tauberian theorems (see, for example, [10]), for $j = 1, \dots, q$, $Q_{rj}(\lambda_1, \lambda_2)$ converges to 0, pointwise as $r \to \infty$. From Lemma 2, we can apply Lebesgue's Dominated Convergence Theorem, to prove the convergence to zero of the integral in equation (79), for $j = 1, \dots, q$. $\square$

Remark 3. Note that, in [11], one can find results on the limit distributions of the first Minkowski functional for Student and Fisher–Snedecor random fields, in terms of multiple Wiener–Itô integral representations, but the spherical random field case has not been addressed yet.

From Theorem 3, a series expansion of the limit spherical Rosenblatt-type random variable R can be derived, as given in the following corollary.

Corollary 1. *Assume that the conditions of Theorem 3 hold. Then, the limit random variable* R, *appearing in such a theorem, admits the following series representation:*

$$\mathrm{R} \underset{d}{=} -\frac{|s_2(1)|}{\sqrt{2q}\gamma(\alpha)} \sum_{j=1}^{q} \sum_{n\geq 1} \mu_n(Y_3)\big(\varepsilon_n^2 - 1\big) = \sum_{j=1}^{q} \sum_{n\geq 1} \xi_n(\mathrm{R})\big(\varepsilon_n^2 - 1\big), \tag{81}$$

where $\{\varepsilon_n\}$ *are independent and identically distributed standard Gaussian random variables, and*

$$\mu_n(Y_3) = -\frac{\sqrt{2q}\gamma(\alpha)\xi_n(\mathrm{R})}{|s_2(1)|}, \quad n \geq 1,$$

is a sequence of positive real numbers, which are the eigenvalues of the self-adjoint Hilbert–Schmidt operator, given by, for all $h \in L_2\big(\mathbb{R}^2, G_\alpha(dx)\big) = L_2\Big(\mathbb{R}^2, \frac{1}{\|x\|^{2-\alpha}}dx\Big)$,

$$\mathcal{Y}_3(h)(\lambda_1) = \int_{\mathbb{R}^2} Y_3(\|\lambda_1 - \lambda_2\|)h(\lambda_2)G_\alpha(d\lambda_2), \tag{82}$$

with

$$G_\alpha(dx) = \frac{1}{\|x\|^{2-\alpha}}dx, \tag{83}$$

and $L_2(\mathbb{R}^2, G_\alpha(dx))$ *denoting the collection of linear combinations, with real-valued coefficients, of complex-valued and Hermitian functions, that are square integrable with respect to* $G_\alpha(dx) = \frac{1}{\|x\|^{2-\alpha}}dx$. *Note that* $L_2(\mathbb{R}^2, G_\alpha(dx))$ *is a real Hilbert space, endowed with the scalar product*

$$\langle \psi_1, \psi_2 \rangle_{G_\alpha} = \int_{\mathbb{R}^2} \psi_1(x)\overline{\psi_2(x)}G_\alpha(dx)$$

(see [22, pp. 159–161]).

The proof is similar to the proof of Corollary 1 in [13], considering $d = 2$ and $D = s_2(1)$ (see also [12]).

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ЗБІЛЬШУВАНІ РАЗОМ З ОБЛАСТЮ АСИМПТОТИКИ ДЛЯ ПЕРШОГО ФУНКЦІОНАЛА МІНКОВСЬКОГО ВІД СФЕРИЧНИХ ВИПАДКОВИХ ПОЛІВ

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Анотація. Однорідне та ізотропне випадкове поле, звужене на сферу, визначає сферичне ізотропне випадкове поле. У цій статті доводяться центральні та нецентральні граничні теореми для першого функціонала Мінковського, підпорядкованого гауссівському або хі-квадрат однорідному випадковому полю, звуженому на сферу в $\mathbb{R}^3$. Обидва сценарії мотивовані цікавими застосуваннями до аналізу космічного реліктового мікрохвильового випромінювання.

УВЕЛИЧИВАЮЩИЕСЯ ВМЕСТЕ С ОБЛАСТЬЮ АСИМПТОТИКИ ДЛЯ ПЕРВОГО ФУНКЦИОНАЛА МИНКОВСКОГО ОТ СФЕРИЧЕСКИХ СЛУЧАЙНЫХ ПОЛЕЙ

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Аннотация. Однородное и изотропное случайное поле, суженное на сферу, определяет сферическое изотропное случайное поле. В данной статье доказываются центральные и нецентральные предельные теоремы для первого функционала Минковского, подчиненного гауссовскому или хи-квадрат однородному случайному полю, суженному на сферу в $\mathbb{R}^3$. Оба сценария мотивированы интересными применениями к анализу космического реликтового микроволнового излучения.