Intermittency of trawl processes

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Abstract: We study the limiting behavior of continuous time trawl processes which are defined using an infinitely divisible random measure of a time dependent set. In this way one is able to define separately the marginal distribution and the dependence structure. One can have long-range dependence or short-range dependence by choosing the time set accordingly. We introduce the scaling function of the integrated process and show that its behavior displays intermittency, a phenomenon associated with an unusual behavior of moments.

1 Introduction

Trawl processes form a class of stationary infinitely divisible processes that allow the marginal distribution and dependence structure to be modelled independently from each other (see Barndorff-Nielsen (2011), Barndorff-Nielsen et al. (2014) and Barndorff-Nielsen et al. (2015)). They are defined by

$$X(t) = \Lambda(A_t), \quad t \in \mathbb{R}, \tag{1}$$

where Λ is a homogeneous infinitely divisible independently scattered random measure (*Lévy basis*) and $A_t = A + (0,t)$ for some Borel subset A of $\mathbb{R} \times \mathbb{R}$ of finite Lebesgue measure. The set A is called the *trawl* and is usually specified using the *trawl function* $g : [0,\infty) \to [0,\infty)$ as

$$A = \left\{(\xi, s) : 0 \leq \xi \leq g(-s),\ s \leq 0\right\},$$

so that

$$A_t = \left\{(\xi, s) : 0 \leq \xi \leq g(t-s),\ s \leq t\right\}.$$

As explained in Barndorff-Nielsen et al. (2015) the trawl A can be regarded as a fishing net dragged along the sea, so that at time t it is in position A_t. A similar structure can be

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found in Wolpert & Taqqu (2005). To any Lévy basis Λ there corresponds a Lévy process $L = \{L(t),\, t \geq 0\}$ referred to as the *Lévy seed*. The choice of the Lévy seed determines the marginal law of the trawl process, while the shape of the trawl set A controls the dependence structure. In particular, taking the trawl function to be $-(\alpha+1)$-regularly varying at infinity for some $\alpha \in (0,1)$, one obtains long-range dependence of the resulting trawl process. See Section 2 for details.

A discrete time analog of the trawl process (1) has been defined in Doukhan et al. (2016) as a process

$$Y(k) = \sum_{j=0}^{\infty} Z^{(k-j)}(a_j), \quad k \in \mathbb{Z}, \tag{2}$$

where $Z^{(k)} = \{Z^{(k)}(u),\, u \in \mathbb{R}\}$, $k \in \mathbb{Z}$ are i.i.d. copies of some process $Z = \{Z(u),\, u \in \mathbb{R}\}$ stochastically continuous at zero and $(a_j)_{j\in\mathbb{N}}$ is a sequence of constants such that $a_j \to 0$ as $j \to \infty$. The long-range dependent case in the discrete time setting corresponds to choosing a sequence $a_j = L(j)j^{-\alpha-1}$ where L is some slowly varying function.

The correspondence of $Y(k)$ in (2) with the continuous time trawl process (1) is the following. Suppose on one hand that $\{Y_k,\, k \in \mathbb{Z}\}$ is a discrete time trawl process with trawl sequence $(a_j)_{j\in\mathbb{N}_0}$ and such that Z is some two-sided Lévy process $L = \{L(t),\, t \in \mathbb{R}\}$. On the other hand, let $\{X(t),\, t \in \mathbb{R}\}$ be a trawl process with Lévy seed process L and trawl specified by the function

$$g(x) = \sum_{j=0}^{\infty} a_j \mathbf{1}_{(-j-1,-j]}(x).$$

Then $\{Y_k,\, k \in \mathbb{Z}\}$ is equal in law to a discretized process $\{X(k),\, k \in \mathbb{Z}\}$ (Doukhan et al. (2016)). While the marginal distribution of the trawl process $X(t)$ in (1) is necessarily infinitely divisible, the discrete time setting allows for rather general seed processes.

An important and interesting question regarding trawl processes are limit theorems for cumulative processes arising from them. Assuming the trawl process has zero mean, in the discrete time setup, the cumulative process would be a partial sum process $S_n(t) = \sum_{k=1}^{[nt]} Y(k)$ while in the continuous time it is natural to consider the integrated process $X^*(t) = \int_0^t X(u)du$. However, as we show in this paper, the limiting behavior of moments seems to be unexpected.

Doukhan et al. (2016) have interesting results. In their paper, a limit theorem is proved with convergence to fractional Brownian motion for the partial sum process formed from a zero mean long-range dependent discrete time trawl process (Doukhan et al. 2016, Theorem 1.(i)). The crucial condition for this result is the following small time moment asymptotics of the seed process: for some $\delta > 0$, one has

$$\mathbb{E}|Z(t)|^{2+\delta} = O(|t|^{\frac{2+\delta}{2}}), \quad \text{as } t \to 0. \tag{3}$$

One may wonder whether the proof of (Doukhan et al. 2016, Theorem 1.(i)) could be extended to the continuous time trawl processes. The following argument shows that the condition (3) excludes the possibility that the seed process is any Lévy process except

Brownian motion. Indeed, suppose Z is a Lévy process with Lévy measure ν such that $\mathbb{E}Z(1) = 0$. By (Asmussen & Rosiński 2001, Lemma 3.1) for any $\delta \geq 0$ such that $\mathbb{E}|Z(1)|^{2+\delta} < \infty$, one has

$$\lim_{n\to\infty} n\mathbb{E}|Z(1/n)|^{2+\delta} = \int_{\mathbb{R}} |x|^{2+\delta}\nu(dx).$$

Hence, $\mathbb{E}|Z(t)|^{2+\delta} \sim C_\delta t$ as $t \to 0$ for any $\delta > 0$ and (3) cannot hold unless $\nu = 0$ and Z is a Brownian motion. Since Brownian motion is self-similar with self-similarity parameter $1/2$, condition (3) holds for Brownian motion but not for any other Lévy process. Hence, the conditions of (Doukhan et al. 2016, Theorem 1(i)) cannot be adapted to obtain a limit theorem for a continuous time trawl process (1) when generated by a non-Gaussian seed process.

Our focus in this paper is on the convergence of moments. We prove that the integrated long-range dependent non-Gaussian trawl processes satisfying certain regularity assumptions on the trawl, have a specific limiting behavior called *intermittency*. A precise definition is given in Section 3. Such a property has so far been established for a partial sum and integrated process of superpositions of Ornstein-Uhlenbeck type processes (see Grahovac et al. (2016) and Grahovac et al. (2017)). This result sheds a new light on the limiting behaviour related to trawl processes.

2 Trawl processes

In this section we define trawl processes following Barndorff-Nielsen (2011), Barndorff-Nielsen et al. (2014) and Barndorff-Nielsen et al. (2015).

2.1 Preliminaries

Let

$$\kappa_Y(\zeta) = C\{\zeta \ddagger Y\} = \log \mathbb{E}e^{i\zeta Y}$$

denote the cumulant (generating) function of a random variable Y and, assuming it exists, $\kappa_Y^{(m)}$ for $m \in \mathbb{N}$ will denote the m-th cumulant of Y, that is

$$\kappa_Y^{(m)} = (-i)^m \frac{d^m}{d\zeta^m}\kappa_Y(\zeta)\big|_{\zeta=0}.$$

If $\kappa_Y(\cdot)$ is analytic around the origin, then

$$\kappa_Y(\zeta) = \sum_{m=1}^{\infty} \frac{(i\zeta)^m}{m!}\kappa_Y^{(m)}. \tag{4}$$

For a stochastic process $Y = \{Y(t)\}$ we write $\kappa_Y(\zeta, t) = \kappa_{Y(t)}(\zeta)$, and by suppressing t we mean $\kappa_Y(\zeta) = \kappa_Y(\zeta, 1)$, that is the cumulant function of the random variable $Y(1)$. Similarly, for the cumulants of $Y(t)$, we use the notation $\kappa_Y^{(m)}(t)$, and $\kappa_Y^{(m)}$ for $\kappa_Y^{(m)}(1)$.

Recall that the cumulant function of infinitely divisible random variable Y has the Lévy-Khintchine representation

$$\kappa(\zeta) = C\{\zeta \ddagger Y\} = ia\zeta - \frac{b}{2}\zeta^2 + \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta \mathbf{1}_{[-1,1]}(x)\right) \nu(dx), \quad \zeta \in \mathbb{R} \tag{5}$$

where $a \in \mathbb{R}$, $b > 0$, and the *Lévy measure* ν is a deterministic Radon measure on $\mathbb{R}\backslash\{0\}$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \min\{1, x^2\} \nu(dx) < \infty$. The triplet (a, b, ν) is referred to as the *characteristic triplet*. A stochastic process $\{L(t),\, t \geq 0\}$ with stationary, independent increments and continuous in probability ($L(t) \to^P 0$ as $t \to 0$) has a càdlàg modification which we refer to as a *Lévy process*. For any infinitely divisible random variable Y, there is a corresponding Lévy process $\{L(t),\, t \geq 0\}$ such that $Y =^d L(1)$.

Next, we review some basic facts about (homogeneous) *Lévy bases* on $\mathbb{R}^d$, $d \in \mathbb{N}$. A Lévy basis on $\mathbb{R}^d$ is an infinitely divisible independently scattered random measure, that is, a collection of random variables $\Lambda = \left\{\Lambda(A),\, A \in \mathcal{B}_b(\mathbb{R}^d)\right\}$ where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the family Borel subsets of $\mathbb{R}^d$ with finite Lebesgue measure. That Λ is independently scattered random measure means that for every sequence $\{A_n\}$ of disjoint sets in $\mathcal{B}_b(\mathbb{R}^d)$, the random variables $\Lambda(A_n)$, $n = 1, 2, ...$ are independent and

$$\Lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \Lambda(A_n) \quad a.s.$$

whenever $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}_b(\mathbb{R}^d)$. Moreover, Λ is infinitely divisible in the sense that for any collection of sets $A_1, \ldots, A_n \in \mathcal{B}_b(\mathbb{R}^d)$ the random vector $(\Lambda(A_1), \ldots, \Lambda(A_n))$ is infinitely divisible. We will be dealing only with *homogeneous* Lévy bases which have the property that for every $A \in \mathcal{B}_b(\mathbb{R}^d)$ the cumulant function of $\Lambda(A)$ is given by

$$C\{\zeta \ddagger \Lambda(A)\} = Leb(A)\kappa(\zeta)$$

where Leb denotes the Lebesgue measure and κ is the cumulant function of some infinitely divisible law having the Lévy-Khintchine representation (5) with $a \in \mathbb{R}$, $b > 0$, and Lévy measure ν. A Lévy process $\{L(t),\, t \geq 0\}$ such that $C\{\zeta \ddagger L(1)\} = \kappa_L(\zeta) = \kappa(\zeta)$ is called the *Lévy seed* of Λ. In the more general context, (a, b, ν, Leb) is referred to as the *characteristic quadruple* and Leb as the *control measure*. Note that to any infinitely divisible distribution there corresponds a homogeneous Lévy basis on $\mathbb{R}^d$. The integration of deterministic functions with respect to the Lévy basis can be defined first for real simple functions, then as a limit in probability of such integrals. More details can be found in Rajput & Rosinski (1989).

2.2 Trawl processes

Suppose Λ is a homogeneous Lévy basis on $\mathbb{R}^d \times \mathbb{R}$, $d \in \mathbb{N}$, with characteristic quadruple (a, b, ν, Leb) and let $\kappa = \kappa_L$ denote the cumulant function (5) of the Lévy seed process $L = \{L(t),\, t \geq 0\}$.

Let $A = A_0 \in \mathcal{B}_b(\mathbb{R}^d \times \mathbb{R})$ be a Borel set of finite Lebesgue measure and for $t \in \mathbb{R}$ put $A_t = A + (\mathbf{0}, t)$. The *trawl process* associated with Lévy basis Λ and *trawl* A is defined as

$$X(t) = \Lambda(A_t) = \int_{\mathbb{R}^d \times \mathbb{R}} \mathbf{1}_A(\boldsymbol{\xi}, s-t)\Lambda(d\boldsymbol{\xi}, ds), \quad t \in \mathbb{R}.$$

The process $\{X(t),\, t \in \mathbb{R}\}$ is strictly stationary (Barndorff-Nielsen et al. (2014)) and

$$C\{\zeta \ddagger X(t)\} = Leb(A)\kappa_L(\zeta)$$

The cumulants, if they exist, are given by

$$\kappa_X^{(m)} = Leb(A)\kappa_L^{(m)} \tag{6}$$

where $\kappa_L^{(m)}$ denotes the m-th order cumulant of $L(1)$.

While specifying the infinitely divisible law of the Lévy basis controls the marginal distribution of the trawl process, the choice of the trawl set A determines the dependence structure of the process. For simplicity, we will assume in the following that $d = 1$ so that $A \in \mathcal{B}_b(\mathbb{R} \times \mathbb{R})$ and

$$X(t) = \Lambda(A_t) = \int_{\mathbb{R} \times \mathbb{R}} \mathbf{1}_A(\xi, s-t)\Lambda(d\xi, ds), \quad t \in \mathbb{R}. \tag{7}$$

The typical way to specify the trawl $A \in \mathcal{B}_b(\mathbb{R} \times \mathbb{R})$ is to put

$$A = \{(\xi, s) : 0 \le \xi \le g(-s),\ s \le 0\},$$

where $g : [0, \infty) \to [0, \infty)$ is a measurable function such that $Leb(A) < \infty$. Then, clearly

$$A_t = \{(\xi, s) : 0 \le \xi \le g(t-s),\ s \le t\}$$

and we can write

$$X(t) = \int_{\mathbb{R} \times (-\infty, t]} \mathbf{1}_{[0, g(t-s)]}(\xi)\Lambda(d\xi, ds), \quad t \in \mathbb{R}.$$

We will refer to g as the *trawl function* and in the following we always assume g is nonincreasing and hence $g(-s)$, $s \in (-\infty, 0]$ is nondecreasing.

By using (Barndorff-Nielsen et al. 2015, Proposition 5.), one can show that for $\zeta_1, \zeta_2 \in \mathbb{R}$ and $h \ge 0$

$$\log \mathbb{E}e^{i(\zeta_1 X(0) + \zeta_2 X(h))} = \int_{\mathbb{R} \times \mathbb{R}} \kappa_L\left(\zeta_1 \mathbf{1}_A(\xi, s) + \zeta_2 \mathbf{1}_A(\xi, s-h)\right) d\xi ds. \tag{8}$$

Now if $EX(t)^2 < \infty$, then taking derivative with respect to ζ_1 and ζ_2 in (8) and letting $\zeta_1, \zeta_2 \to 0$ we obtain

$$\mathbb{E}X(t)X(t+h) = \int_{\mathbb{R} \times \mathbb{R}} \mathbf{1}_A(\xi, s)\mathbf{1}_A(\xi, s-h) d\xi ds = \int_{-\infty}^{0} \int_0^{g(h-s)} d\xi ds = \int_h^\infty g(x)dx.$$

Hence, the correlation function of the trawl process for $h \ge 0$ is

$$r(h) = \operatorname{Corr}\left(X(t), X(t+h)\right) = \frac{\int_h^\infty g(x)dx}{\int_0^\infty g(x)dx}. \tag{9}$$

This shows how the choice of g affects the dependence.

Example 2.1. Suppose for some $\alpha > 0$, g is $-(\alpha+1)$-regularly varying at infinity so that $g(x) = L(x)x^{-\alpha-1}$, with L slowly varying at infinity, i.e. for every $x > 0$, $L(tx)/L(t) \to 1$ as $t \to \infty$. Then from (9) by Karamata's theorem (Bingham et al. 1989, Proposition 1.5.10.) we have that

$$r(h) \sim \frac{1}{\alpha \int_0^\infty g(x)dx} L(h)h^{-\alpha}, \qquad \text{as } h \to \infty.$$

In particular, by taking $\alpha \in (0, 1)$ we can obtain a trawl process with non-integrable correlation function, a property well known as the *long-range dependence.* The next example is a particular case.

Example 2.2. Suppose $\{X(t),\ t \in \mathbb{R}\}$ is a trawl process with finite second moment specified by the trawl function

$$g(x) = (1 + x)^{-\alpha-1}, \tag{10}$$

for some $\alpha > 0$. From (9) it follows that the correlation function is

$$r(h) = (1 + h)^{-\alpha}, \quad h \geq 0.$$

In Barndorff-Nielsen et al. (2014), the same example is obtained indirectly as a special case of the so-called superposition trawl. The general superposition trawl is specified by the trawl function

$$\widetilde{g}(x) = \int_0^\infty e^{-\lambda x} \pi(d\lambda), \qquad x \geq 0,$$

where π is some probability measure on $(0, \infty)$ such that $\int_0^\infty \lambda^{-1}\pi(d\lambda) < \infty$. Taking π to be the Gamma distribution $\Gamma(1 + \alpha, 1)$ distribution, defined by the density

$$f(\lambda) = \frac{1}{\Gamma(1 + \alpha)} \lambda^\alpha e^{-\lambda} \mathbf{1}_{(0,\infty)}(\lambda),$$

we obtain a trawl specified by (10). Such a modelling framework is motivated by the similar approach used in superpositions of Ornstein-Uhlenbeck type processes (see Barndorff-Nielsen (2001)).

2.3 Integrated process

Given a trawl process $\{X(t),\ t \in \mathbb{R}\}$ we will denote by $\{X^*(t),\ t \geq 0\}$ the integrated process

$$X^*(t) = \int_0^t X(u)du. \tag{11}$$

The following lemma expresses cumulants of the integrated process $\kappa_{X^*}^{(m)}(t)$ in terms of the cumulants $\kappa_L^{(m)}$ of the Lévy seed. We will assume that the cumulant function κ_L of the Lévy seed process is analytic in a neighborhood of the origin. A sufficient condition for the analyticity of κ_L in the neighborhood of the origin is that there exists $a > 0$

such that $\mathbb{E}e^{a|L(1)|} < \infty$ (Lukacs 1970, Theorem 7.2.1). This implies in particular that all the moments and cumulants of $X(t)$ exist. Many infinitely divisible distributions satisfy this condition, for example, inverse Gaussian, normal inverse Gaussian, gamma, variance gamma, tempered stable (see Grahovac et al. (2017) for details).

Lemma 2.1. *Suppose $\{X(t),\ t \in \mathbb{R}\}$ is a trawl process (7) such that the cumulant function κ_L of the Lévy seed process $\{L(t)\}$ is analytic in a neighborhood of the origin. The cumulants of $X^*(t)$ are then given by*

$$\kappa_{X^*}^{(m)}(t) = \kappa_L^{(m)} \int_{\mathbb{R}\times\mathbb{R}} (h_A(\xi, s, t))^m \, d\xi ds, \quad m \geq 1, \tag{12}$$

where $\kappa_L^{(m)}$ is the m-th order cumulant of the Lévy seed process L and

$$h_A(\xi, s, t) = \int_0^t \mathbf{1}_A(\xi, s-u)du = \int_0^t \mathbf{1}_{(-\infty, g(u-s)]}(\xi)\mathbf{1}_{(-\infty,u]}(s)du. \tag{13}$$

Proof. From (Barndorff-Nielsen et al. 2015, Proposition 5.) it follows that

$$C\left\{\zeta \ddagger X^*(t)\right\} = C\left\{\zeta \ddagger \int_0^t X(u)du\right\} = \int_{\mathbb{R}\times\mathbb{R}} C\left\{\zeta h_A(\xi, s, t) \ddagger L(1)\right\} d\xi ds.$$

with $h_A(\xi, s, t)$ given by (13). By the analyticity of κ_L we have

$$C\left\{\zeta \ddagger L(1)\right\} = \sum_{m=1}^{\infty} \kappa_L^{(m)} \frac{(i\zeta)^m}{m!}$$

and so

$$\begin{aligned} C\left\{\zeta \ddagger X^*(t)\right\} &= \int_{\mathbb{R}\times\mathbb{R}} \sum_{m=1}^{\infty} \kappa_L^{(m)} \frac{(i\zeta)^m}{m!} (h_A(\xi, s, t))^m \, d\xi ds \\ &= \sum_{m=1}^{\infty} \left(\kappa_L^{(m)} \int_{\mathbb{R}\times\mathbb{R}} (h_A(\xi, s, t))^m \, d\xi ds \right) \frac{(i\zeta)^m}{m!}. \end{aligned}$$

□

3 Intermittency

Intermittency is a property used to describe models exhibiting sharp fluctuations in time and a high degree of variability. The term has a precise definition in the theory of stochastic partial differential equations, where it is characterized by the Lyapunov exponents (see e.g. Carmona & Molchanov (1994), Chen & Dalang (2015), Khoshnevisan (2014), Zel'dovich et al. (1987)).

Here, we follow Grahovac et al. (2017) and define intermittency as a property which indicates that the stochastic process does not have a typical limiting behavior of moments.

Intermittency is characterized by the scaling function. The *scaling function* of the process $Y = \{Y(t),\, t \geq 0\}$ is defined in the range of finite moments $(0, \overline{q}(Y))$, $\overline{q}(Y) = \sup\{q > 0 : \mathbb{E}|Y(t)|^q < \infty \ \forall t\}$ as the limit

$$\tau_Y(q) = \lim_{t\to\infty} \frac{\log \mathbb{E}|Y(t)|^q}{\log t}, \tag{14}$$

assuming the limit exists and is finite. It can be shown that τ_Y is always convex and $q \mapsto \tau_Y(q)/q$ is non-decreasing (Grahovac et al. (2016)).

Definition 3.1. A stochastic process $Y = \{Y(t),\, t \geq 0\}$ is *intermittent* if there exist some $p, r \in (0, \overline{q}(Y))$ such that

$$\frac{\tau_Y(p)}{p} < \frac{\tau_Y(r)}{r}, \tag{15}$$

that is, $\tau_Y(q)/q$ is strictly increasing at some q.

Recall that the process Y is H-self-similar if for any $c > 0$, $\{Y(ct)\} \stackrel{d}{=} \{c^H Y(t)\}$, where $\{\cdot\} \stackrel{d}{=} \{\cdot\}$ denotes the equality of finite dimensional distributions. If Y is a H-self-similar process, then $\tau_Y(q) = Hq$, and $\tau_Y(q)/q$ is constant, therefore the process is not intermittent. Recall that by Lamperti's theorem (see, for example, (Embrechts & Maejima 2002, Theorem 2.1.1)), if as $n \to \infty$

$$\left\{ \frac{Y(nt)}{A_n} \right\} \stackrel{d}{\to} \{Z(t)\}, \tag{16}$$

where $\{\cdot\} \stackrel{d}{\to} \{\cdot\}$ means convergence of all finite-dimensional distributions, $Z(t)$ is always a self-similar process and the normalizing sequence must be of the form $A_n = L(n)n^H$ for some $H > 0$ and L slowly varying at infinity. From here, one can show that as soon as (16) holds, then there is $H > 0$ such that for every $q > 0$ satisfying

$$\frac{\mathbb{E}|Y(nt)|^q}{A_n^q} \to \mathbb{E}|Z(t)|^q, \quad \forall t \geq 0, \tag{17}$$

one has that $\tau_Y(q) = Hq$. In this setting, Y usually represents some form of cumulative process, e.g. partial sum process or integrated process. Hence, when intermittency is present, (16) and (17) cannot both hold (see Grahovac et al. (2017) for details).

The following theorem establishes intermittency of certain integrated trawl process. For the Lévy seed, any infinitely divisible distribution is allowed provided it has cumulant function analytic in the neighbourhood of the origin. However, the Gaussian case is excluded. In the Gaussian case one can apply (Taqqu 1975, Lemma 5.1) and obtain limit theorems with convergence to fractional Brownian motion (see (Grahovac et al. 2017, Example 9) for the similar argument). The underlying trawl process is assumed to a trawl function regularly varying at infinity. Additionally, the trawl function is assumed to be continuously differentiable and decreasing. An example of such trawl is given in Example 2.2.

Theorem 3.1. *Let $\{X(t),\, t \in \mathbb{R}\}$ be a zero mean non-Gaussian trawl process such that the cumulant function κ_L of the Lévy seed process is analytic in the neighborhood of the origin and suppose the trawl function g is continuously differentiable, decreasing and $(-\alpha-1)$-regularly varying at infinity for some $\alpha > 0$. If τ_{X^*} is the scaling function (14) of the process $X^* = \{X^*(t),\, t \geq 0\}$ in (11), then for every $q \geq q^*$*

$$\tau_{X^*}(q) = q - \alpha,$$

where q^* is the smallest even integer greater than 2α. In particular, for $q^* \leq p < r$

$$\frac{\tau_{X^*}(p)}{p} < \frac{\tau_{X^*}(r)}{r}$$

and hence X^* is intermittent.

Proof. First, we will investigate the asymptotic behavior of $\kappa_{X^*}^{(m)}(t)$ for $m \in \mathbb{N}$ as $t \to \infty$ using (12). By the assumptions, the trawl function $g : [0, \infty) \to (0, g(0)]$ is invertible and we can rewrite (13) in the following form

$$\begin{aligned} h_A(\xi, s, t) &= \int_0^t \mathbf{1}_{[0, g(u-s)]}(\xi) \mathbf{1}_{(-\infty, u]}(s) du \\ &= \int_0^t \mathbf{1}_{(-\infty, g^{-1}(\xi)+s]}(u) \mathbf{1}_{[s, \infty)}(u) du. \end{aligned}$$

From here we conclude that $h_A(\xi, s, t) = 0$ if either $s > t$ or $\xi < 0$ or $\xi > g(0)$ or $g^{-1}(\xi) < -s$ (which is equivalent to $\xi > g(-s)$ for $s \leq 0$). Otherwise, we have for $s \leq 0$

$$h_A(\xi, s, t) = \int_0^t \mathbf{1}_{[0, g^{-1}(\xi)+s]}(u) du = \left(g^{-1}(\xi) + s\right) \wedge t$$

and for $s > 0$

$$h_A(\xi, s, t) = \int_0^t \mathbf{1}_{[s, g^{-1}(\xi)+s]}(u) = \left(\left(g^{-1}(\xi) + s\right) \wedge t\right) - s.$$

Hence, we can write

$$h_A(\xi, s, t) = \begin{cases} t, & \text{if } s \leq 0 \text{ and } 0 \leq \xi \leq g(t-s), \\ g^{-1}(\xi) + s, & \text{if } s \leq 0 \text{ and } g(t-s) < \xi \leq g(-s), \\ t - s, & \text{if } 0 < s \leq t \text{ and } 0 \leq \xi \leq g(t-s), \\ g^{-1}(\xi), & \text{if } 0 < s \leq t \text{ and } g(t-s) < \xi \leq g(0), \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

The cumulants of the integrated process (12) can now be expressed as

$$\kappa_{X^*}^{(m)}(t) = \kappa_L^{(m)} \left(I_1^{(m)}(t) + I_2^{(m)}(t) + I_3^{(m)}(t) + I_4^{(m)}(t) \right), \tag{19}$$

where

$$I_1^{(m)}(t) = \int_{-\infty}^{0} \int_{0}^{g(t-s)} t^m d\xi ds,$$
$$I_2^{(m)}(t) = \int_{-\infty}^{0} \int_{g(t-s)}^{g(-s)} (g^{-1}(\xi)+s)^m d\xi ds,$$
$$I_3^{(m)}(t) = \int_{0}^{t} \int_{0}^{g(t-s)} (t-s)^m d\xi ds,$$
$$I_4^{(m)}(t) = \int_{0}^{t} \int_{g(t-s)}^{g(0)} (g^{-1}(\xi))^m d\xi ds.$$

We assumed X has zero mean, hence $\kappa_L^{(1)} = 0$ from (6) and consequently $\kappa_{X^*}^{(1)} = 0$.

Case $m > \alpha+1$. We now take $m > \alpha+1$ and consider each integral one by one. Since g is $(-\alpha-1)$-regularly varying, it can be written in the form $g(x) = L(x)x^{-\alpha-1}$ with L slowly varying at infinity. For $I_1^{(m)}(t)$ using change of variable and Karamata's theorem (Bingham et al. 1989, Proposition 1.5.10.) we get

$$I_1^{(m)}(t) = t^m \int_{-\infty}^{0} g(t-s)ds = t^m \int_{t}^{\infty} g(u)du \sim \frac{1}{\alpha} L(t)t^{m-\alpha}, \qquad \text{as } t \to \infty. \tag{20}$$

For the second integral, since g is assumed to be continuously differentiable, we have by the change of variable $u = g^{-1}(\xi) + s$ and Fubini's theorem

$$I_2^{(m)}(t) = \int_{-\infty}^{0} \int_{t}^{0} u^m g'(u-s)duds = \int_{t}^{0} u^m \int_{-\infty}^{0} g'(u-s)dsdu = \int_{0}^{t} u^m g(u)du.$$

Now from (Bingham et al. 1989, Proposition 1.5.11. (i)) it follows that

$$I_2^{(m)}(t) \sim \frac{1}{m-\alpha} L(t)t^{m-\alpha}, \qquad \text{as } t \to \infty.$$

Similarly, for $I_3^{(m)}(t)$ we obtain

$$I_3^{(m)}(t) = \int_{0}^{t} (t-s)^m g(t-s)ds = \int_{0}^{t} u^m g(u)du \sim \frac{1}{m-\alpha} L(t)t^{m-\alpha}, \qquad \text{as } t \to \infty.$$

Finally, for $I_4^{(m)}(t)$ by the change of variable $u = g^{-1}(\xi)$, Fubini's theorem and integration by parts it follows that

$$
\begin{aligned}
I_4^{(m)}(t) &= \int_0^t \int_{t-s}^0 u^m g'(u) du ds \\
&= -\int_0^t u^m g'(u) \int_0^{t-u} ds du \\
&= \int_0^t u^{m+1} g'(u) du - t \int_0^t u^m g'(u) du \\
&= t^{m+1} g(t) - (m+1) \int_0^t u^m g(u) du - t^{m+1} g(t) + tm \int_0^t u^{m-1} g(u) du \\
&= tm \int_0^t u^{m-1} g(u) du - (m+1) \int_0^t u^m g(u) du. \qquad (21)
\end{aligned}
$$

Since we have assumed $m > \alpha + 1$ and (Bingham et al. 1989, Proposition 1.5.11. (i)) can be applied to get as $t \to \infty$

$$
I_4^{(m)}(t) \sim \frac{m}{m-\alpha-1} L(t) t^{m-\alpha} - \frac{m+1}{m-\alpha} L(t) t^{m-\alpha} = \frac{\alpha+1}{(m-\alpha-1)(m-\alpha)} L(t) t^{m-\alpha}.
$$

We now conclude from (19) that for every $m > \alpha + 1$ such that $\kappa_L^{(m)} \neq 0$ there exists a slowly varying function L_m such that $\kappa_{X^*}^{(m)}(t) \sim L_m(t) t^{m-\alpha}$.

Case $m < \alpha + 1$. In this case we will only need an upper bound on $\kappa_{X^*}^{(m)}(t)$. The equation (20) remains valid anyway and shows that $I_1^{(m)}(t) \leq C_1 t$ for t large enough. Next, since $u^m g(u) = u^{m-\alpha-1} L(u)$ is bounded at infinity, we have

$$
\left| I_2^{(m)}(t) \right| = \left| I_3^{(m)}(t) \right| = \int_0^t u^m g(u) du \leq C_2 t.
$$

Similarly, we can take $0 < \varepsilon < \alpha + 1 - m$ and u large enough so that $u^{m-1} g(u) = u^{m-\alpha-2} L(u) \leq C_3 u^{-1-\varepsilon}$. Hence, we have from (21)

$$
\left| I_4^{(m)}(t) \right| \leq tm \int_0^t u^{-1-\varepsilon} du + (m+1) C_2 t \leq C_4 t.
$$

We conclude from (19) that $\left| \kappa_{X^*}^{(m)}(t) \right| \leq Ct$ for $m < \alpha + 1$.

Case $m = \alpha + 1$. Note that this is possible only if α is an integer. If the slowly varying function L is bounded, everything remains the same as in prof of the previous case. Otherwise, for arbitrary $\varepsilon > 0$, we can take u large enough so that $L(u) \leq u^{\varepsilon}$. Now one can proceed as in the previous case to obtain that $\left| \kappa_{X^*}^{(m)}(t) \right| \leq C t^{1+\varepsilon}$ for $m = \alpha + 1$.

Having established these results now, we can relate cumulants to moments as in the proof of (Grahovac et al. 2017, Theorem 7) and show that for some slowly varying function $\widetilde{L}$

$$
E|X^*(t)|^m \sim \widetilde{L}(t) t^{m-\alpha}
$$

and consequently $\tau_{X^*}(m) = m - \alpha$, for any even integer m greater than 2α. As in (Grahovac et al. 2017, Lemma 3), the convexity of τ_{X^*} is then used to extend the validity of $\tau_{X^*}(q) = q - \alpha$ to any real $q \geq q^*$. $\square$

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