SINGULAR VECTORS FOR THE W_N ALGEBRAS

DAVID RIDOUT, STEVE SIU, AND SIMON WOOD

ABSTRACT. In this paper, we use free field realisations of the A-type principal, or Casimir, W_N algebras to derive explicit formulae for singular vectors in Fock modules. These singular vectors are constructed by applying screening operators to Fock module highest weight vectors. The action of the screening operators is then explicitly evaluated in terms of Jack symmetric functions and their skew analogues. The resulting formulae depend on sequences of pairs of integers that completely determine the Fock module as well as the Jack symmetric functions.

1. INTRODUCTION

A W-algebra is, generally speaking, a vertex operator algebra whose set of generating fields includes conformal primaries of conformal weights greater than 1. Aside from the Virasoro minimal models, such algebras were first considered by Zamolodchikov [36] shortly after the genesis of conformal field theory [6]. We refer to [10] for an overview of W-algebras and their role in conformal field theory.

Here, we focus on the so-called W_N algebras, where $N \geq 2$, which were first considered in [5]. They form a distinguished family of W-algebras that are generated by $N - 1$ fields of conformal weights $2, 3, \ldots, N$, where the generating field of conformal weight 2 is the energy momentum tensor and all the remaining generators are conformal primaries. While these algebras have received a lot of attention, there is much that remains poorly understood. Even for the case $N = 3$ there are significant difficulties, see [9] for example.

The W_N algebras were first proposed in the context of statistical mechanics, where they were linked to the continuum limits of certain $\mathbb{Z}_N$-symmetric lattice models [15]. In particular, W_3 was first introduced to describe the extended symmetry of the 3 state Potts model [36]. They enjoyed a period of intense popularity shortly thereafter as physicists explored the possibilities in their largely unsuccessful quest to classify all rational conformal field theories. More recently, under the name of minimal model holography, W_N algebras (and their supersymmetric generalisations) have been the focus of intense scrutiny because of their role in constructing the AdS/CFT duals of Vasiliev higher spin theories of gravity on AdS_3 [31], see [18] for an in depth review.

The W_N algebras are also of great mathematical interest, since they are closely related to vertex operator algebras constructed from affine Lie algebras through either cosets [4, 19] of the form

$$\frac{\widehat{\mathfrak{sl}}(N)_k \otimes \widehat{\mathfrak{sl}}(N)_1}{\widehat{\mathfrak{sl}}(N)_{k+1}} \tag{1.1}$$

or through quantum hamiltonian reductions associated with principal nilpotents of $\mathfrak{sl}(N)$ [11, 17]. Although the equivalence of the coset and principal reduction pictures is only now being established rigorously [1, 2], this has the happy consequence that even though W_N algebras are quite different, structurally, to affine vertex operator algebras, Lie-theoretic ideas can still be used to analyse them.

The purpose of this paper is to derive explicit formulae for singular vectors of the W_N algebras. Recall that the (universal) W_N algebra may be realised as a subalgebra of the rank $N - 1$ Heisenberg vertex algebra and thus, by restriction, the Fock spaces of the Heisenberg algebra become modules over the W_N algebras. It is within these Fock spaces that we derive explicit formulae for W_N singular vectors.

The derivation uses the simple, but far-reaching, fact that the universal enveloping algebra of the creation operators of a Heisenberg algebra is, as an associative algebra, isomorphic to the complexification of a ring of symmetric functions.[1] The appeal of this fact is that it allows one to express complicated singular vectors, consisting of large linear combinations

[1] We adhere to the convention that a symmetric polynomial in a countably infinite number of variables is called a symmetric function.

of products of Heisenberg creation operators, as images of symmetric functions that have a very simple form under the aforementioned isomorphism.

This identification of singular vectors with symmetric functions can be traced back to Wakimoto and Yamada [32], who discovered that Virasoro singular vectors in Fock spaces for the rank 1 Heisenberg algebra of central charge $c = 1$ can be elegantly expressed in terms of Schur functions, a much studied basis of the ring of symmetric functions. This work was later generalised to free field realisations of the Virasoro algebra at arbitrary central charge by Mimachi and Yamada [24], with Schur functions being replaced by Jack symmetric functions [20]. We recall that the Jack functions form a one-parameter family of bases of the ring of symmetric functions and that the Schur functions correspond to the parameter being set to 1.

This work relied heavily on a construction by Tsuchiya and Kanie [28, 29] of Virasoro module homomorphisms called screening operators. With these, singular vectors may be realised as images of Fock space highest-weight vectors. These screening operators are also the key ingredient in Dotsenko and Fateev's Coulomb gas formalism [14]. Similar singular vector constructions using screening operators have recently been detailed for the free field realisations of the vertex operator algebras associated with $\widehat{\mathfrak{sl}}(2)$ and the $\mathcal{N} = 1$ superconformal algebras, with singular vectors being evaluated using Jack symmetric functions, their supersymmetric generalisations and related families [7, 12, 13, 21, 26, 35].

The Jack symmetric function basis is indexed by partitions of integers and, interestingly, the singular vectors of Fock spaces for the Virasoro algebra and $\widehat{\mathfrak{sl}}(2)$ are always associated to a single Jack symmetric function indexed by a rectangular partition, that is, a partition whose parts are all equal. It is therefore natural to try and find an interpretation for the Jack symmetric functions indexed by non-rectangular partitions. One such interpretation was found by Awata, Odake, Matsuo and Shiraishi [3] while studying a connection between the Calogero-Sutherland model, a model for a system of non-relativistic particles on a circle with an inverse square potential, and the W_N algebras. In [3], it is noted that W_N singular vectors can be realised using screening operators acting on rank $N - 1$ Fock spaces. However, these singular vectors were not explicitly evaluated, but were instead projected onto a rank 1 Fock space, where they were identified as eigenstates of the hamiltonian of the Calogero-Sutherland model. As the eigenstates of the Calogero-Sutherland model may be expressed as products of a ground state wavefunction and Jack polynomials, the conclusion of [3] is then that W_N singular vectors can be used to construct arbitrary Jack polynomials.

Here, in contrast to [3], we do not project, but stay in the rank $N - 1$ Fock spaces, using symmetric function theory to find explicit formulae for W_N singular vectors. Our main result is that certain singular vectors in rank $N - 1$ Fock spaces may be identified with linear combinations of Jack symmetric functions and their skew variants, parametrised by sequences of partitions.

The motivation for this study of singular vectors is that they have proven invaluable in finding short elegant proofs of the classification theorems for modules of rank 1 vertex operator algebras [8, 25, 26, 30]. We believe that a solid understanding of higher rank singular vectors will pave the way to similar classification theorems for higher rank vertex operator algebras.

This paper is organised as follows. Sections 2 to 4 form an overview of known results and serve to fix notation. In Section 2, we review the rank r Heisenberg vertex algebra for later use. We show how energy-momentum tensors can be constructed for arbitrary choices of central charge and discuss vertex operators and their compositions. In Section 3, we give the standard free field realisation of the W_3 algebras by explicitly constructing them as subalgebras of the rank 2 Heisenberg vertex algebra. We then determine their screening operators and identify the Fock spaces on which we can consistently evaluate their action. Section 4 gives a brief overview of the ring of symmetric functions, with particular emphasis on the Jack symmetric function bases, these being the bases required to evaluate the action of screening operators.

The main results, explicit formulae for W_N singular vectors in certain Fock spaces in terms of the Jack symmetric functions, are given in Sections 5 to 7 — see Equation (5.11) for the W_3 singular vector formula and Equation (7.17) for that of W_N. In Section 5, we give a detailed derivation of these formulae for the W_3 algebras. This case already illustrates the complexity of the general computations while keeping the formulae reasonably brief. This is followed by a discussion of simple examples in Section 6 featuring low-grade singular vectors. Although we do not report the

results of investigating any really complicated examples, symbolic algebra packages incorporating symmetric functions, SAGEMATH for example, allow such investigations to be straightforwardly performed to quite high grades. In Section 7, we generalise the singular vector formulae of Section 5 to the W_N algebras for all $N \geq 3$.

2. THE RANK r HEISENBERG ALGEBRA

The theory of r free chiral bosons, also known as the rank r Heisenberg vertex algebra H_r, is a staple of conformal field theory and vertex operator algebra theory. Here, we construct the Heisenberg algebras by affinising abelian Lie algebras.

The rank r Heisenberg algebra is constructed from an r-dimensional complex vector space $\mathfrak{h}$ together with a non-degenerate symmetric bilinear form $(-,-)$. We pick a basis $\{a^1, \ldots, a^r\}$ of $\mathfrak{h}$ such that the Gram matrix of $(-,-)$ is the Cartan matrix of $\mathfrak{sl}(r+1)$:

$$\left(a^i, a^j\right) = 2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j+1}, \quad i,j = 1, \ldots, r. \tag{2.1}$$

Since $(-,-)$ is non-degenerate, it defines a vector space isomorphism $\iota : \mathfrak{h} \to \mathfrak{h}^*$ by $a \mapsto (a,-)$. The induced non-degenerate symmetric bilinear form will also be denoted by $(-,-)$. We denote the images of the basis vectors a^i by $\alpha^i = \iota(a^i)$ and the elements of the basis of $\mathfrak{h}^*$ dual to $\{a^i\}$ by ω_i. Thus, $\omega_i(a^j) = \delta_i^j$. The α^i and ω_i may therefore be identified as simple roots and fundamental weights, respectively, of $\mathfrak{sl}(r+1)$. In this picture, the basis vectors $a^i \in \mathfrak{h}$ are the simple coroots of $\mathfrak{sl}(r+1)$.

To any vector $a \in \mathfrak{h}$, one assigns a field $a(z)$ whose defining operator product expansions are

$$a(z)b(w) \sim \frac{(a,b)}{(z-w)^2}, \quad a,b \in \mathfrak{h}. \tag{2.2}$$

These fields admit Fourier expansions of the form

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad a \in \mathfrak{h}, \tag{2.3}$$

whose modes satisfy the following commutation relations:

$$\left[a_m, b_n\right] = m(a,b)\delta_{m,-n}\mathbf{1}. \tag{2.4}$$

The Heisenberg Lie algebra $\widehat{\mathfrak{h}}$ is the infinite-dimensional Lie algebra spanned by the central element $\mathbf{1}$ and the generators a_m, for all $a \in \mathfrak{h}$ and $m \in \mathbb{Z}$. We have chosen to denote the central element by $\mathbf{1}$, since we assume that it will act as the identity on any $\widehat{\mathfrak{h}}$-module. A basis of $\widehat{\mathfrak{h}}$ is then given by $\mathbf{1}$ and the a_m^i, with $i = 1, \ldots, r$ and $m \in \mathbb{Z}$.

The Heisenberg Lie algebra admits a triangular decomposition

$$\widehat{\mathfrak{h}} = \widehat{\mathfrak{h}}_- \oplus \widehat{\mathfrak{h}}_0 \oplus \widehat{\mathfrak{h}}_+, \quad \widehat{\mathfrak{h}}_{\pm} = \bigoplus_{i=1}^r \bigoplus_{m \geq 1} \mathbb{C}a_{\pm m}^i, \quad \widehat{\mathfrak{h}}_0 = \bigoplus_{i=1}^r \mathbb{C}a_0^i \oplus \mathbb{C}\mathbf{1}. \tag{2.5}$$

Verma modules over $\widehat{\mathfrak{h}}$ are commonly referred to as Fock spaces. These are induced from the one-dimensional modules $\mathbb{C}|\zeta\rangle$, $\zeta \in \mathfrak{h}^*$, over $\widehat{\mathfrak{h}}_{\geq} = \widehat{\mathfrak{h}}_0 \oplus \widehat{\mathfrak{h}}_+$ that are defined by

$$\mathbf{1}|\zeta\rangle = |\zeta\rangle, \qquad a_n|\zeta\rangle = \delta_{n,0}\zeta(a)|\zeta\rangle, \quad a \in \mathfrak{h}, \quad n \geq 0. \tag{2.6}$$

The Fock spaces

$$\mathcal{F}_\zeta = \mathrm{U}(\widehat{\mathfrak{h}}) \otimes_{\mathrm{U}(\widehat{\mathfrak{h}}_{\geq})} \mathbb{C}|\zeta\rangle \tag{2.7}$$

are well known to be simple $\widehat{\mathfrak{h}}$-modules, for all $\zeta \in \mathfrak{h}^*$.

As a module over itself, the Heisenberg vertex algebra H_r is identified with the Fock space $\mathcal{F}_0$ and the state-field correspondence is given by

$$|0\rangle \longleftrightarrow 1, \quad b_{-n_1-1}^1 \cdots b_{-n_k-1}^k |0\rangle \longleftrightarrow \; :\frac{\partial^{n_1}}{n_1!}b^1(z) \cdots \frac{\partial^{n_k}}{n_k!}b^k(z): , \tag{2.8}$$

where $b^1, \ldots, b^k \in \mathfrak{h}$ and normal ordering is defined in the usual way.

The Heisenberg vertex algebra H_r can be endowed with the structure of a vertex operator algebra by choosing an energy-momentum tensor. This choice is not unique. For the purposes of this note, we shall restrict our attention to the

following one-parameter family of energy-momentum tensors:

$$T(z) = \sum_{i=1}^{r} \left[\frac{1}{2} :a^i(z)a^{*i}(z): + \alpha_0 \partial a^{*i}(z) \right], \quad \alpha_0 \in \mathbb{C}. \tag{2.9}$$

Here, the $a^{*i} \in \mathfrak{h}$ are dual to the coroots a^i in the sense that $\iota(a^{*i}) = \omega_i$. We note that while the quadratic summand in the above energy-momentum tensor is basis independent, the linear summand is not. The central charge corresponding to this choice of energy-momentum tensor depends on the parameter α_0:

$$c = r - r(r+1)(r+2)\alpha_0^2. \tag{2.10}$$

By definition, the coefficients of the Fourier expansion of the energy-momentum tensor satisfy the commutation relations of the Virasoro algebra. Thus, formula (2.9) realises the Virasoro generators L_n, $n \in \mathbb{Z}$, as infinite sums of products of Heisenberg generators:

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad L_n = \sum_{i=1}^{r} \left[\frac{1}{2} \sum_{m \in \mathbb{Z}} :a_m^i a_{n-m}^{*i}: - \alpha_0(n+1)a_n^{*i} \right]. \tag{2.11}$$

This identification yields an action of the Virasoro algebra on the Fock spaces $\mathcal{F}_\zeta$, $\zeta \in \mathfrak{h}^*$. In this way, any highest-weight vector $|\zeta\rangle \in \mathcal{F}_\zeta$ is also a Virasoro highest-weight vector:

$$L_n|\zeta\rangle = h_\zeta \delta_{n,0}|\zeta\rangle, \quad n \geq 0, \quad h_\zeta = \frac{1}{2}(\zeta, \zeta - 2\alpha_0\rho). \tag{2.12}$$

Here, $\rho = \sum_i \omega_i$ is the Weyl vector of $\mathfrak{sl}(r+1)$. We note that while the Fock spaces are simple as Heisenberg modules, they need not be as Virasoro modules.

The primary fields of the free boson theory are called vertex operators (not to be confused with elements of the Heisenberg vertex operator algebra). To define them, we first need to extend $\widehat{\mathfrak{h}}$ by $\mathbb{C}[\mathfrak{h}^*]$, the group algebra of $\mathfrak{h}^*$, treating $\mathfrak{h}^*$ as an abelian group under vector addition and $\mathbb{C}[\mathfrak{h}^*]$ as an abelian Lie algebra. We denote the group algebra basis element corresponding to $\eta \in \mathfrak{h}^*$ by e^η and define the commutation relations between the generators a_m and e^η by

$$[a_m, e^\eta] = \delta_{m,0}\eta(a)e^\eta, \quad a \in \mathfrak{h}, \quad \eta \in \mathfrak{h}^*, \quad m \in \mathbb{Z}. \tag{2.13}$$

It is easy to check that this extension of $\widehat{\mathfrak{h}}$ by $\mathbb{C}[\mathfrak{h}^*]$ is a semidirect sum of Lie algebras.

A standard computation now shows that e^η maps the highest-weight vector $|\zeta\rangle \in \mathcal{F}_\zeta$ to a highest-weight vector of a_0-eigenvalue $\zeta(a) + \eta(a) = (\zeta + \eta)(a)$. Following usual practice, we shall identify $e^\eta|\zeta\rangle$ with $|\zeta + \eta\rangle$. The vertex operator corresponding to $|\zeta\rangle = e^\zeta|0\rangle$ is

$$V_\zeta(z) = e^\zeta z^{a_0} \prod_{m \geq 1} \exp\left(\frac{a_{-m}}{m}z^m\right) \exp\left(-\frac{a_m}{m}z^{-m}\right), \quad \zeta = \iota(a) \in \mathfrak{h}^*. \tag{2.14}$$

These primary fields therefore define linear maps between Fock spaces:

$$V_\zeta(z) \colon \mathcal{F}_\eta \to z^{(\zeta,\eta)}\mathcal{F}_{\zeta+\eta}[\![z, z^{-1}]\!]. \tag{2.15}$$

It is easy to check from the H_r-primary operator product expansion

$$a(z)V_\zeta(w) \sim \frac{\zeta(a)V_\zeta(w)}{z - w} \tag{2.16}$$

that $a(z)$ and $V_\zeta(w)$ are mutually local for all $a \in \mathfrak{h}$ and $\zeta \in \mathfrak{h}^*$. The same is therefore true for an arbitrary field of H_r and any vertex operator, by Dong's lemma.

Finally, suppose that $\zeta_i = \iota(b^i) \in \mathfrak{h}^*$, for $i = 1, \ldots, k$. Then, a standard computation allows one to write the composition of the k vertex operators $V_{\zeta_i}(z_i)$ as

$$V_{\zeta_1}(z_1) \cdots V_{\zeta_k}(z_k) = \prod_{i=1}^{k} e^{\zeta_i} \cdot \prod_{1 \leq i < j \leq k} (z_i - z_j)^{(\zeta_i, \zeta_j)} \cdot \prod_{i=1}^{k} z_i^{b_0^i} \cdot \prod_{m \geq 1} \exp\left(\frac{1}{m}\sum_{i=1}^{k} b_{-m}^i z_i^m\right) \exp\left(-\frac{1}{m}\sum_{i=1}^{k} b_m^i z_i^{-m}\right). \tag{2.17}$$

This explicit formula will be used many times in what follows.

3. The Universal W_3 Vertex Operator Algebra

In this section, we restrict ourselves to the rank 2 Heisenberg vertex algebra H_2 and, in the vein of [16], define a family of subalgebras, each denoted by W_3, called the W_3 vertex operator algebras, or W_3 algebras for short. These algebras are parametrised by $\alpha_0 \in \mathbb{C}$ and are strongly generated by the energy-momentum tensor defined in (2.9) and an additional primary field $W(z)$ of conformal weight 3.

In the basis $\{a^1, a^2\}$ defined above, for which the Gram matrix of the inner product $(-,-)$ is the Cartan matrix of $\mathfrak{sl}(3)$, the energy-momentum tensor $T(z)$ is

$$T(z) = \frac{1}{3}{:}a^1(z)a^1(z){:} + \frac{1}{3}{:}a^1(z)a^2(z){:} + \frac{1}{3}{:}a^2(z)a^2(z){:} + \alpha_0\partial_z a^1(z) + \alpha_0\partial_z a^2(z) \tag{3.1a}$$

and the central charge is $c = 2 - 24\alpha_0^2$. The conformal primary of weight 3 is then

$$W(z) = \frac{\sqrt{\beta}}{18\sqrt{3}}\Big[2{:}\big(a^2(z) - a^1(z)\big)\big(a^1(z) + 2a^2(z)\big)\big(2a^1(z) + a^2(z)\big){:}$$
$$+ 9\alpha_0\big({:}\partial a^2(z)\big(a^1(z) + 2a^2(z)\big){:} - {:}\partial a^1(z)\big(2a^1(z) + a^2(z)\big){:}\big) + 9\alpha_0^2\big(\partial^2 a^2(z) - \partial^2 a^1(z)\big)\Big], \tag{3.1b}$$

where

$$\beta = \frac{16}{22 + 5c} = \frac{2}{4 - 15\alpha_0^2} \tag{3.2}$$

in the conventional normalisation, appropriate for $c \neq -\frac{22}{5}$ ($\alpha_0 \neq \pm\frac{2}{\sqrt{15}}$). A somewhat involved computation now determines the operator product expansion of $W(z)$ with itself to be

$$W(z)W(w) \sim \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{\frac{3}{10}\partial^2 T(w) + 2\beta\Lambda(w)}{(z-w)^2} + \frac{\frac{1}{15}\partial^3 T(w) + \beta\partial\Lambda(w)}{z-w}, \tag{3.3}$$

where $\Lambda(z) = {:}T(z)T(z){:} - \frac{3}{10}\partial^2 T(z)$. This, along with the primary nature of $W(z)$, implies the commutation relations

$$\big[L_m, W_n\big] = (2m - n)W_{m+n}, \tag{3.4a}$$

$$\big[W_m, W_n\big] = (m - n)\Big[\frac{1}{15}(m + n + 3)(m + n + 2) - \frac{1}{6}(m + 2)(n + 2)\Big]L_{m+n}$$
$$+ \beta(m - n)\Lambda_{m+n} + \frac{c}{360}m(m^2 - 1)(m^2 - 4)\delta_{m+n,0}, \tag{3.4b}$$

where $W(z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-3}$.

Since Fock spaces are modules over the Heisenberg vertex operator algebra H_2 and we have defined the W_3 algebra as a subalgebra of H_2, each Fock space is a W_3-module, by restriction. In particular, the highest-weight vector $|\zeta\rangle \in \mathcal{F}_\zeta$, $\zeta \in \mathfrak{h}^*$, is also a highest-weight vector for W_3:

$$L_n|\zeta\rangle = \delta_{n,0}h_\zeta|\zeta\rangle, \quad W_n|\zeta\rangle = \delta_{n,0}w_\zeta|\zeta\rangle, \quad n \geq 0. \tag{3.5}$$

Here, h_ζ was given in (2.12) and the W_0-eigenvalue is given by

$$w_\zeta = \sqrt{3\beta}(\zeta, \omega_2 - \omega_1)\big((\zeta, \omega_1) - \alpha_0\big)\big((\zeta, \omega_2) - \alpha_0\big). \tag{3.6}$$

Our main reason for introducing vertex operators in (2.14) is to construct linear maps between Fock spaces that commute with the action of an appropriate subalgebra of the Heisenberg vertex algebra. Here, we wish to construct maps that commute with W_3, that is, W_3-module homomorphisms. Such module homomorphisms are called *screening operators* and they are constructed from *screening fields*, these being vertex operators whose operator product expansions with the fields of W_3 are total derivatives. For this, it clearly suffices to find fields whose operator product expansions with the generating fields $T(z)$ and $W(z)$ are total derivatives.

As the vertex operator $V_\zeta(w)$ is a conformal primary of weight h_ζ, its operator product expansion with $T(z)$ will be a total derivative if and only if $h_\zeta = 1$. Unsurprisingly, the analogous computation for $W(z)$ is more involved (we used Thielemans' OPEDEFS package for MATHEMATICA. We shall not give the unpleasant details, noting instead that a necessary condition for the operator product expansion $W(z)V_\zeta(w)$ to be a total derivative is that the coefficient of $(z - w)^{-1}$ in this expansion is a total derivative. Analysing this explicitly, for general $\zeta \in \mathfrak{h}^*$, and recalling that $h_\zeta = 1$,

we conclude that this coefficient will be a total derivative if $\zeta_1 = 0$, $\zeta_2 = 0$ or if $\zeta_1 = \zeta_2$ and $\alpha_0 = 0$ (where the ζ_i denote Dynkin labels: $\zeta = \sum_i \zeta_i \omega_i$). As we are only interested in screening operators that exist for all values of α_0, it follows that there are exactly four possible weights ζ that can be used to construct screening operators: $\zeta = \alpha_\pm \alpha^1, \alpha_\pm \alpha^2$. Here, we define

$$\alpha_+ = \frac{1}{2}\left(\alpha_0 + \sqrt{\alpha_0^2 + 4}\right), \quad \alpha_- = \frac{1}{2}\left(\alpha_0 - \sqrt{\alpha_0^2 + 4}\right) \tag{3.7}$$

to be the solutions of the quadratic equations $h_{\zeta_i \alpha^i} = 1$, for $i = 1, 2$. It only remains to confirm that the full operator product expansion $W(z)V_\zeta(w)$, when ζ is one of the above weights, is indeed a total derivative:

$$W(z)V_{\alpha_\pm \alpha^i}(w) \sim -(-1)^i \sqrt{\frac{6}{4 - 15\alpha_0^2}} \partial_w \left(\frac{\alpha_0 V_{\alpha_\pm \alpha^i}(w)}{2(z-w)^2} - \frac{:a^{*1}(w)V_{\alpha_\pm \alpha^i}(w):}{z-w} \right), \quad i = 1, 2. \tag{3.8}$$

Having identified screening fields for W_3, we construct screening operators by taking residues:

$$\mathcal{S}_{\pm i} = \oint_0 V_{\alpha_\pm \alpha^i}(w)\, dw. \tag{3.9}$$

Here, the residue is indicated using a simple anticlockwise contour that encircles 0 once (we absorb the usual factor of $2\pi i$ into the definition of the contour integral). These screening operators define W_3-module homomorphisms since

$$[T(z), \mathcal{S}_{\pm i}] = -\oint_z T(z)V_{\alpha_\pm \alpha^i}(w)\, dw = 0, \quad [W(z), \mathcal{S}_{\pm i}] = -\oint_z W(z)V_{\alpha_\pm \alpha^i}(w)\, dw = 0. \tag{3.10}$$

These identities follow from the mutual locality of Heisenberg fields and vertex operators, see (2.16).

Taking the residue of a screening field $V_{\alpha_\pm \alpha^i}(z)$ is of course only well defined when it is acting on a H_2-module for which the exponents of z in the Fourier expansion of $V_{\alpha_\pm \alpha^i}(z)$ are all integers. In case the H_2-module is the Fock space $\mathcal{F}_\eta$, this is satisfied if and only if $\alpha_\pm(\alpha^i, \eta) \in \mathbb{Z}$. Fortunately, one can also construct screening operators by integrating compositions (2.17) of multiple screening fields. In particular, composing r_2 copies of $V_{\alpha_\pm \alpha^2}(w)$ with r_1 copies of $V_{\alpha_\pm \alpha^1}(z)$ and then acting on $\mathcal{F}_\eta$ gives

$$\left. V_{\alpha_\pm \alpha^1}(z_1) \cdots V_{\alpha_\pm \alpha^1}(z_{r_1}) V_{\alpha_\pm \alpha^2}(w_1) \cdots V_{\alpha_\pm \alpha^2}(w_{r_2}) \right|_{\mathcal{F}_\eta}$$

$$= \prod_{1 \le i < j \le r_1} (z_i - z_j)^{2\alpha_\pm^2} \cdot \prod_{1 \le i < j \le r_2} (w_i - w_j)^{2\alpha_\pm^2} \cdot \prod_{i=1}^{r_1} \prod_{j=1}^{r_2} (z_i - w_j)^{-\alpha_\pm^2} \cdot \prod_{i=1}^{r_1} z_i^{\alpha_\pm(\alpha^1, \eta)} \cdot \prod_{j=1}^{r_2} w_j^{\alpha_\pm(\alpha^2, \eta)}$$

$$\cdot e^{r_1 \alpha_\pm \alpha^1 + r_2 \alpha_\pm \alpha^2} \prod_{m \ge 1} \exp\left[\alpha_\pm \left(a_{-m}^1 \sum_{i=1}^{r_1} \frac{z_i^m}{m} + a_{-m}^2 \sum_{i=1}^{r_2} \frac{w_i^m}{m} \right) \right] \exp\left[-\alpha_\pm \left(a_m^1 \sum_{i=1}^{r_1} \frac{z_i^{-m}}{m} + a_m^2 \sum_{i=1}^{r_2} \frac{w_i^{-m}}{m} \right) \right]. \tag{3.11}$$

Up to a complex phase, which we suppress, the first five multivalued factors in this expression can be rewritten in the form

$$\prod_{1 \le i \ne j \le r_1} \left(1 - \frac{z_i}{z_j}\right)^{\alpha_\pm^2} \cdot \prod_{1 \le i \ne j \le r_2} \left(1 - \frac{w_i}{w_j}\right)^{\alpha_\pm^2} \cdot \prod_{i=1}^{r_1} \prod_{j=1}^{r_2} \left(1 - \frac{w_j}{z_i}\right)^{-\alpha_\pm^2}$$

$$\cdot \prod_{i=1}^{r_1} z_i^{\alpha_\pm(\alpha^1, \eta) + \alpha_\pm^2(r_1 - r_2 - 1)} \cdot \prod_{j=1}^{r_2} w_j^{\alpha_\pm(\alpha^2, \eta) + \alpha_\pm^2(r_2 - 1)}, \tag{3.12}$$

thereby isolating the non-integer exponents of the z_i and w_j in the last two factors.

Finding closed (multivariable) contours over which multivalued functions such as (3.12) can be integrated (to obtain W_3-module homomorphisms) is a highly non-trivial problem. Fortunately, Tsuchiya and Kanie solved this problem for the rank 1 Heisenberg vertex algebra [29] by constructing cycles with non-trivial homology classes over which screening operators can be integrated. These cycles, which we shall denote by $\Gamma(m; t)$ for $m \in \mathbb{Z}_{\ge 0}$ and $t \in \mathbb{C} \setminus \mathbb{Q}_{\le 0}$,[2] allow one to integrate expressions of the form

$$\int_{\Gamma(m; t)} \prod_{1 \le i \ne j \le m} \left(1 - \frac{z_i}{z_j}\right)^{1/t} \cdot f(z)\, dz_1 \cdots dz_m, \tag{3.13}$$

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[2]The range of the parameter t could in principle be extended to $\mathbb{C} \setminus \{0\}$. However, to avoid singularities in certain coefficients, this would require one to use a different normalisation of the Jack symmetric function basis presented in Section 4. Moreover, some linear independence arguments would become more complicated. For simplicity, we therefore avoid non-positive rational values of the parameter t.

where $f(z)$ is a Laurent polynomial in $z_1, \ldots, z_m$ which is invariant with respect to permuting the indices of its variables. We shall not describe the construction of these cycles in any detail. It will, however, be convenient to normalise them by requiring that

$$\int_{\Gamma(m;t)} \prod_{1 \le i \ne j \le m} \left(1 - \frac{z_i}{z_j}\right)^{1/t} \frac{dz_1 \cdots dz_m}{z_1 \cdots z_m} = 1. \tag{3.14}$$

The cycles $\Gamma(m;t)$ can be used to construct screening operators from the compositions (3.11) whenever the exponents of the z_i and w_j are integers. If this is the case, then the screening operators are defined as

$$\mathcal{S}_\pm^{[r_1,r_2]} = \int_{\Gamma(r_1;1/\alpha_\pm^2)} \int_{\Gamma(r_2;1/\alpha_\pm^2)} V_{\alpha_\pm \alpha^1}(z_1) \cdots V_{\alpha_\pm \alpha^1}(z_{r_1}) V_{\alpha_\pm \alpha^2}(w_1) \cdots V_{\alpha_\pm \alpha^2}(w_{r_2}) dz_1 \cdots dz_{r_1} dw_1 \cdots dw_{r_2}. \tag{3.15}$$

By construction, these screening operators are W_3-module homomorphisms when acting on appropriate Fock spaces.

We parametrise the Fock space weights for which the screening operators (3.15) are defined as follows:

$$\zeta_{u_1,v_1;u_2,v_2} = \left((1-u_1)\alpha_+ + (1-v_1)\alpha_-\right)\omega_1 + \left((1-u_2)\alpha_+ + (1-v_2)\alpha_-\right)\omega_2, \quad u_1,u_2,v_1,v_2 \in \mathbb{Z}. \tag{3.16}$$

Considering the exponents of the last two factors of (3.12), we conclude that the screening operators define W_3-module homomorphisms between the following Fock spaces:

$$\begin{aligned}
\mathcal{S}_+^{[r_1,r_2]} &: \mathcal{F}_{\zeta_{r_1-r_2,s_1;r_2,s_2}} \to \mathcal{F}_{\zeta_{-r_1,s_1;r_1-r_2,s_2}}, \quad r_1,r_2 \in \mathbb{Z}_{\ge 0}, \quad s_1,s_2 \in \mathbb{Z}, \\
\mathcal{S}_-^{[s_1,s_2]} &: \mathcal{F}_{\zeta_{r_1,s_1-s_2;r_2,s_2}} \to \mathcal{F}_{\zeta_{r_1,-s_1;r_2,s_1-s_2}}, \quad r_1,r_2 \in \mathbb{Z}, \quad s_1,s_2 \in \mathbb{Z}_{\ge 0}.
\end{aligned} \tag{3.17}$$

Evaluating the action of these screening operators initially appears rather daunting. However, we know from (3.11) that compositions of screening fields factorise into a product of a multivalued function and certain power series in the z_i and w_j that are *symmetric* with respect to permuting the z_i among themselves and, separately, the w_j among themselves. The theory of symmetric functions provides the tools that allow us to evaluate the action of these screening operators on certain Fock spaces. We therefore turn to a discussion of these tools.

4. THE RING OF SYMMETRIC FUNCTIONS

The purpose of this section is to review various results from the theory of symmetric functions that will be used to evaluate the action of screening operators on certain Fock spaces. The standard reference for symmetric functions and their myriad properties is Macdonald's book [23] to which we refer the reader for more details.

Let Λ_n denote the ring of symmetric polynomials in the n variables $z_1, \ldots, z_n$. This is the subring of $\mathbb{C}[z_1, \ldots, z_n]$ that consists of the polynomials that are invariant with respect to permuting the indices of the z_i. It admits numerous interesting generators such as the power sums

$$p_k = \sum_{i=1}^n z_i^k, \quad k \ge 1. \tag{4.1}$$

For $1 \le k \le n$, the p_k are algebraically independent and freely generate Λ_n, that is,

$$\Lambda_n = \mathbb{C}[p_1, \cdots, p_n]. \tag{4.2}$$

We can therefore use partitions $\lambda = [\lambda_1, \lambda_2, \ldots]$, whose parts λ_i are bounded by n, to define

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}. \tag{4.3}$$

These power sums, labelled by partitions whose parts do not exceed n, thus form a basis of Λ_n:

$$\Lambda_n = \bigoplus_{\lambda, \lambda_1 \le n} \mathbb{C} p_\lambda. \tag{4.4}$$

Another family of symmetric polynomials is given by the monomial symmetric polynomials

$$m_\lambda = \sum_\sigma z_1^{\lambda_{\sigma(1)}} \cdots z_n^{\lambda_{\sigma(n)}}, \tag{4.5}$$

where σ runs over all distinct permutations of the partition λ. In this case, λ is not constrained by a bound on its individual parts, but by their number $\ell(\lambda)$ (the length of λ) which is at most n. Note that each monomial summand of m_λ has coefficient 1. For example,

$$\mathsf{m}_{[2,2]}(z_1, z_2) = z_1^2 z_2^2, \quad \mathsf{m}_{[2,2]}(z_1, z_2, z_3) = z_1^2 z_2^2 + z_1^2 z_3^2 + z_2^2 z_3^2. \tag{4.6}$$

The monomial symmetric polynomials also form a basis of Λ_n:

$$\Lambda_n = \bigoplus_{\lambda, \ell(\lambda) \leq n} \mathbb{C}\mathsf{m}_\lambda. \tag{4.7}$$

The respective restrictions on parts and lengths of partitions in the definitions of these symmetric polynomials can be avoided by taking a formal limit to infinitely many variables. The resulting ring Λ is called the ring of symmetric functions and, unsurprisingly, its elements are called symmetric functions. The ring Λ_n of symmetric polynomials in n variables can then be easily recovered from Λ by setting all but the first n variables to 0. This amounts to a projection

$$\pi_n : \Lambda \to \Lambda_n, \quad f(x_1, x_2, \ldots) \mapsto f(x_1, \ldots, x_n, 0, 0, \ldots). \tag{4.8}$$

In Λ, the power sums p_k are algebraically independent for all $k \geq 1$ and they freely generate Λ, that is,

$$\Lambda = \mathbb{C}[\mathsf{p}_1, \mathsf{p}_2, \ldots]. \tag{4.9}$$

Similarly, the restrictions on the sizes of the parts and the lengths of the partitions labelling power sums and monomial symmetric functions, respectively, no longer apply. Both classes of symmetric functions give bases of Λ:

$$\Lambda = \bigoplus_\lambda \mathbb{C}\mathsf{p}_\lambda = \bigoplus_\lambda \mathbb{C}\mathsf{m}_\lambda. \tag{4.10}$$

We note that $\pi_n(\mathsf{m}_\lambda) = 0$ if and only if $\ell(\lambda) > n$, but that no such truncations exist for the power sums p_k: their images under π_n are all non-zero.

There exists another family of bases of Λ and Λ_n labelled by partitions, called the Jack symmetric functions and Jack symmetric polynomials (or just Jack functions or polynomials for short), respectively. These are defined using the dominance partial ordering of partitions: if λ and μ are both partitions of the same non-negative integer, then we write $\lambda \geq \mu$ (and say that λ dominates μ) if

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i, \tag{4.11}$$

for all $i \geq 1$.

For each $t \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$ (the non-positive rationals are excluded to avoid certain normalisation problems), the Jack functions P_λ^t are uniquely defined by the following two properties:

(1) For any partition λ, P_λ^t admits an upper triangular decomposition of the form

$$\mathsf{P}_\lambda^t = \mathsf{m}_\lambda + \sum_{\lambda > \mu} v_{\lambda, \mu}(t) \mathsf{m}_\mu, \qquad v_{\lambda, \mu}(t) \in \mathbb{C}. \tag{4.12}$$

(2) The Jack functions form an orthogonal basis of Λ with respect to the inner product defined by

$$\langle \mathsf{p}_\lambda, \mathsf{p}_\mu \rangle^t = t^{\ell(\lambda)} \delta_{\lambda\mu} \prod_{i \geq 1} i^{m_i} m_i!, \tag{4.13}$$

where m_i denotes of number of parts of λ equal to i.

For each $n \geq 1$, the Jack polynomials in Λ_n may be defined as the images of the corresponding Jack functions in Λ under the projection π_n. As with monomial symmetric polynomials, we have $\pi_n(\mathsf{P}_\lambda^t) = 0$ if and only if $\ell(\lambda) > n$. For $\ell(\lambda) \leq n$, the Jack polynomials

$$\mathsf{P}_\lambda^t(z_1, \ldots, z_n) = \pi_n(\mathsf{P}_\lambda^t) \tag{4.14}$$

are linearly independent and form a basis of Λ_n. For the application to follow, we mention the following important examples in Λ_n called the *rectangular* Jack polynomials. In these, the partition has the form $\lambda = [m^n]$ in which all n parts

are equal to m. Rectangular Jack polynomials have a very simple form:

$$\mathsf{P}^t_{[m^n]}(z_1,\ldots,z_n) = \mathsf{m}_{[m^n]}(z_1,\ldots,z_n) = \prod_{i=1}^n z_i^m. \tag{4.15}$$

This follows because all partitions of mn that are strictly dominated by $[m^n]$ have length greater than n. They also have extremely simple products with other Jacks. For $\ell(\lambda) \leq n$, denote by $\lambda + [m^n]$ the partition with parts $\lambda_i + m$. Then,

$$\mathsf{P}^t_{[m^n]}(z_1,\ldots,z_n)\mathsf{P}^t_\lambda(z_1,\ldots,z_n) = \mathsf{P}^t_{\lambda+[m^n]}(z_1,\ldots,z_n). \tag{4.16}$$

We emphasise that rectangular Jack polynomials are independent of the parameter t.

The Jack functions and polynomials satisfy many properties that shall be essential for what follows. We list some of them here for convenience.

(1) We denote by Q^t_λ the elements of the basis dual to the P^t_λ with respect to the inner product (4.13). Since the Jack functions form an orthogonal basis, P^t_λ is proportional to Q^t_λ:

$$\mathsf{Q}^t_\lambda = b^t_\lambda \mathsf{P}^t_\lambda, \quad b^t_\lambda = \frac{1}{\left\langle \mathsf{P}^t_\lambda, \mathsf{P}^t_\lambda \right\rangle^t}. \tag{4.17}$$

The proportionality constant b^t_λ is given explicitly by

$$b^t_\lambda = \prod_{s\in\lambda} \frac{a(s)t+l(s)+1}{(a(s)+1)t+l(s)}, \tag{4.18}$$

where $a(s)$ and $l(s)$ denote the arm and leg lengths, respectively, of the box s in the Young diagram of λ.

(2) The Jack functions and their duals admit a kind of generating function called the *Cauchy kernel*:

$$\prod_{i,j}(1-y_iz_j)^{-1/t} = \prod_{m\geq 1}\exp\left(\frac{1}{t}\frac{\mathsf{p}_m(y)\mathsf{p}_m(z)}{m}\right) = \sum_\lambda \mathsf{P}^t_\lambda(y)\mathsf{Q}^t_\lambda(z). \tag{4.19}$$

In this identity, the two alphabets $\{y_i\}$ and $\{z_j\}$ may be finite or infinite.

(3) Given partitions λ and μ, the skew Jack functions $\mathsf{P}^t_{\lambda/\mu}$ and $\mathsf{Q}^t_{\lambda/\mu}$ are defined to be the unique symmetric functions satisfying

$$\left\langle \mathsf{P}^t_{\lambda/\mu}, \mathsf{Q}^t_\nu \right\rangle^t = \left\langle \mathsf{P}^t_\lambda, \mathsf{Q}^t_\mu\mathsf{Q}^t_\nu \right\rangle^t \quad\text{and}\quad \left\langle \mathsf{Q}^t_{\lambda/\mu}, \mathsf{P}^t_\nu \right\rangle^t = \left\langle \mathsf{Q}^t_\lambda, \mathsf{P}^t_\mu\mathsf{P}^t_\nu \right\rangle^t \tag{4.20}$$

for all partitions ν. Let us write $\mu \subseteq \lambda$ if the Young diagram of μ is contained in that of λ. Then, $\mathsf{P}^t_{\lambda/\mu} = \mathsf{Q}^t_{\lambda/\mu} = 0$ unless $\mu \subseteq \lambda$. Finally, the ordinary and dual skew Jack functions are proportional:

$$\mathsf{Q}^t_{\lambda/\mu} = \frac{b^t_\lambda}{b^t_\mu}\mathsf{P}^t_{\lambda/\mu}. \tag{4.21}$$

(4) Consider an alphabet $z = (z_1, z_2, \ldots)$, partitioned into two subsets $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$. Any symmetric function in z may obviously be decomposed into symmetric functions in x and y. For Jack functions, this decomposition is

$$\mathsf{P}^t_\lambda(z) = \mathsf{P}^t_\lambda(x\cup y) = \sum_\nu \mathsf{P}^t_\nu(x)\mathsf{P}^t_{\lambda/\nu}(y), \quad \mathsf{Q}^t_\lambda(z) = \mathsf{Q}^t_\lambda(x\cup y) = \sum_\nu \mathsf{Q}^t_\nu(x)\mathsf{Q}^t_{\lambda/\nu}(y). \tag{4.22}$$

Both sums may clearly be restricted to partitions satisfying $\nu \subseteq \lambda$.

(5) The Jack polynomials $\mathsf{P}^t_\lambda(z_1,\ldots,z_n)$ are orthogonal with respect to the inner product

$$\langle f,g\rangle^t_n = \int_{\Gamma(n;t)} G^t_n(x)f(x)\overline{g(x)}\frac{\mathrm{d}x_1\cdots\mathrm{d}x_n}{x_1\cdots x_n}, \tag{4.23}$$

where $\Gamma(n;t)$ is the cycle normalised in (3.14), $\overline{g(x_1,x_2,\ldots)} = g(x_1^{-1},x_2^{-1},\ldots)$ and

$$G^t_n(x) = \prod_{1\leq i\neq j\leq n}\left(1-\frac{x_i}{x_j}\right)^{1/t} \tag{4.24}$$

is called the integrating kernel. With respect to this integral inner product, the Jack polynomials satisfy

$$\left\langle \mathsf{P}_\lambda^t(x), \mathsf{Q}_\mu^t(x) \right\rangle_n^t = \delta_{\lambda,\mu} b_\lambda^t(n), \quad b_\lambda^t(n) = \prod_{s\in\lambda} \frac{n+a'(s)t-l'(s)}{n+(a'(s)+1)t-l'(s)-1}, \tag{4.25}$$

where $a'(s)$ and $l'(s)$ denote the arm and leg colengths, respectively, of the box s in the Young diagram of λ.

5. Explicit evaluation of W_3 singular vectors

With this symmetric function technology under our belts, we now turn to the computation of singular vectors in Fock spaces, the idea being to realise them as images of highest-weight vectors under a W_3-module homomorphism (screening operator). For definiteness, we shall choose the screening operator $\mathcal{S}_+^{[r_1,r_2]}$ defined in (3.15) that was constructed from r_1 copies of $\mathsf{V}_{\alpha_+\alpha_1}$ and r_2 copies of $\mathsf{V}_{\alpha_+\alpha_2}$. The computation for $\mathcal{S}_-^{[r_1,r_2]}$ is exactly the same and will be omitted. By (3.17), $\mathcal{S}_+^{[r_1,r_2]}$ has a well defined action on the Fock space $\mathcal{F}_\eta$, where $\eta = \zeta_{r_1-r_2,s_1;r_2,s_2}$, sending it into $\mathcal{F}_\theta$, where $\theta = \zeta_{-r_1,s_1;r_1-r_2,s_2}$.

We can now explicitly evaluate the action of the screening operator $\mathcal{S}_+^{[r_1,r_2]}$ on the highest-weight vector $|\eta\rangle \in \mathcal{F}_\eta$. Using (3.11), (3.12) and (3.16), this action is

$$\begin{aligned}
\mathcal{S}_+^{[r_1,r_2]}|\eta\rangle &= \int_\Delta \mathsf{V}_{\alpha_+\alpha^1}(z_1^1)\cdots\mathsf{V}_{\alpha_+\alpha^1}(z_{r_1}^1)\mathsf{V}_{\alpha_+\alpha^2}(z_1^2)\cdots\mathsf{V}_{\alpha_+\alpha^2}(z_{r_2}^2)|\eta\rangle \prod_{k=1}^2\prod_{i=1}^{r_k} dz_i^k \\
&= \int_\Delta \prod_{k=1}^2 \prod_{1\le i\ne j\le r_k} \left(1-\frac{z_i^k}{z_j^k}\right)^{\alpha_+^2} \cdot \prod_{i=1}^{r_1}\prod_{j=1}^{r_2}\left(1-\frac{z_j^2}{z_i^1}\right)^{-\alpha_+^2} \cdot \prod_{k=1}^2\prod_{i=1}^{r_k}(z_i^k)^{\alpha_+(\alpha^k,\eta)+\alpha_+^2(r_k-1)+1} \\
&\qquad \cdot \prod_{i=1}^{r_1}(z_i^1)^{-\alpha_+^2 r_2} \cdot \prod_{k=1}^2\prod_{m\ge 1}\exp\left(\frac{\alpha_+ a_{-m}^k}{m}\sum_{i=1}^{r_k}(z_i^k)^m\right)\cdot|\theta\rangle \prod_{k=1}^2\prod_{i=1}^{r_k}\frac{dz_i^k}{z_i^k} \\
&= \int_\Delta \prod_{k=1}^2 \prod_{1\le i\ne j\le r_k} \left(1-\frac{z_i^k}{z_j^k}\right)^{\alpha_+^2} \cdot \prod_{i=1}^{r_1}\prod_{j=1}^{r_2}\left(1-\frac{z_j^2}{z_i^1}\right)^{-\alpha_+^2} \\
&\qquad \cdot \prod_{k=1}^2\prod_{i=1}^{r_k}(z_i^k)^{s_k} \cdot \prod_{k=1}^2\prod_{m\ge 1}\exp\left(\frac{\alpha_+ a_{-m}^k \mathsf{p}_m(z^k)}{m}\right)\cdot|\theta\rangle \prod_{k=1}^2\prod_{i=1}^{r_k}\frac{dz_i^k}{z_i^k}.
\end{aligned} \tag{5.1}$$

Here, the integrals are over the product cycle $\Delta = \Gamma(r_1;\alpha_+^{-2})\times\Gamma(r_2;\alpha_+^{-2})$, see Section 3.

To proceed, we note that the tensor product $\Lambda\otimes_\mathbb{C}\Lambda$ is isomorphic to $\mathrm{U}(\widehat{\mathfrak{h}}_-) = \mathbb{C}[a_{-m}^k : k=1,2, m\in\mathbb{Z}_{>0}]$ as an algebra, by (4.9). Concretely, let y_i^1 and y_i^2 denote the variables for the two factors of $\Lambda\otimes_\mathbb{C}\Lambda$ and consider the isomorphism

$$\rho_+\colon \Lambda\otimes_\mathbb{C}\Lambda \longrightarrow \mathrm{U}(\widehat{\mathfrak{h}}_-), \quad \mathsf{p}_m(y^k)\longmapsto \frac{1}{\alpha_+}a_{-m}^k, \quad k=1,2, m\in\mathbb{Z}_{>0}. \tag{5.2}$$

Then, we may write

$$\prod_{m\ge 1}\exp\left(\frac{\alpha_+ a_{-m}^k \mathsf{p}_m(z^k)}{m}\right) = \rho_+\left(\prod_{m\ge 1}\exp\left(\alpha_+^2\frac{\mathsf{p}_m(y^k)\mathsf{p}_m(z^k)}{m}\right)\right) = \rho_+\left(\prod_{i\ge 1}\prod_{j=1}^{r_k}(1-y_i^k z_j^k)^{-\alpha_+^2}\right), \tag{5.3}$$

recognising the Cauchy kernel (4.19) with parameter $t=\alpha_+^{-2}$.

For $k=1$, we expand this Cauchy kernel in terms of Jacks and their duals as in (4.19):

$$\prod_{m\ge 1}\exp\left(\frac{\alpha_+ a_{-m}^1 \mathsf{p}_m(z^1)}{m}\right) = \rho_+\left(\sum_\lambda \mathsf{P}_\lambda^t(y^1)\mathsf{Q}_\lambda^t(z^1)\right). \tag{5.4}$$

For $k=2$, we first combine the Cauchy kernel with that appearing in the second factor of the integrand of (5.1):

$$\rho_+\left(\prod_{i\ge 1}\prod_{j=1}^{r_2}(1-y_i^2 z_j^2)^{-\alpha_+^2}\right)\prod_{i=1}^{r_1}\prod_{j=1}^{r_2}\left(1-(z_i^1)^{-1}z_j^2\right)^{-\alpha_+^2} = \rho_+\left(\sum_\mu \mathsf{P}_\mu^t\left(y^2\cup(z^1)^{-1}\right)\mathsf{Q}_\mu^t(z^2)\right). \tag{5.5}$$

Here, we have noted that the product is a Cauchy kernel in the alphabets $\{y_i^2\}\cup\{(z_\ell^1)^{-1}\}$ and $\{z_j^2\}$. This may be further simplified using skew-Jacks as in (4.22):

$$\mathsf{P}_\mu^t\left(y^2\cup(z^1)^{-1}\right) = \sum_\nu \overline{\mathsf{P}_\nu^t(z^1)}\mathsf{P}_{\mu/\nu}^t(y^2). \tag{5.6}$$

We recall that the skew-Jack $\mathsf{P}^t_{\mu/\nu}$ is 0 unless $\nu \subseteq \mu$.

Next, note that we also have the integrating kernels

$$\prod_{k=1}^{2} \prod_{1 \le i \ne j \le r_k} \left(1 - \frac{z_i^k}{z_j^k}\right)^{\alpha_+^2} = G^t_{r_1}(z^1) G^t_{r_2}(z^2) \tag{5.7}$$

of the symmetric polynomial inner product (4.23). Finally, the product $\prod_{k=1}^{2} \prod_{i=1}^{r_k} (z_i^k)^{s_k}$ is a product of rectangular Jack polynomials. However, here we have to be careful with the signs of the s_k. Indeed, (2.12) and (3.16) show that the conformal weights of the highest-weight vectors $|\eta\rangle$ and $|\theta\rangle$ differ by

$$h_\eta - h_\theta = -r_1 s_1 - r_2 s_2. \tag{5.8}$$

This must be non-negative if the screening operator $\mathcal{S}_+^{[r_1, r_2]}$ is to map $|\eta\rangle$ to a singular descendant of $|\theta\rangle$. We shall therefore assume from here on that $s_1, s_2 \in \mathbb{Z}_{\le 0}$. Thus,

$$\prod_{k=1}^{2} \prod_{i=1}^{r_k} (z_i^k)^{s_k} = \overline{\mathsf{P}^t_{[-s_1^{r_1}]}(z^1)}\, \overline{\mathsf{P}^t_{[-s_2^{r_2}]}(z^2)}, \tag{5.9}$$

using (4.15).

Putting all this back in (5.1), the integrand factorises and we get

$$\mathcal{S}_+^{[r_1,r_2]}|\eta\rangle = \sum_{\lambda,\mu,\nu} \rho_+\left(\mathsf{P}^t_\lambda(y^1)\right) \int_{\Gamma(r_1;t)} G^t_{r_1}(z^1)\overline{\mathsf{P}^t_{[-s_1^{r_1}]}(z^1)}\,\overline{\mathsf{P}^t_\nu(z^1)} \mathsf{Q}^t_\lambda(z^1) \prod_{i=1}^{r_1} \frac{\mathrm{d}z_i^1}{z_i^1}$$

$$\cdot \rho_+\left(\mathsf{P}^t_{\mu/\nu}(y^2)\right) \int_{\Gamma(r_2;t)} G^t_{r_2}(z^2)\overline{\mathsf{P}^t_{[-s_2^{r_2}]}(z^2)}\mathsf{Q}^t_\mu(z^2) \prod_{i=1}^{r_2} \frac{\mathrm{d}z_i^2}{z_i^2} \cdot |\theta\rangle$$

$$= \sum_{\lambda,\mu,\nu} \left\langle \mathsf{Q}^t_\lambda, \mathsf{P}^t_{\nu+[-s_1^{r_1}]}\right\rangle^t_{r_1} \left\langle \mathsf{Q}^t_\mu, \mathsf{P}^t_{[-s_2^{r_2}]}\right\rangle^t_{r_2} \rho_+\left(\mathsf{P}^t_\lambda(y^1)\mathsf{P}^t_{\mu/\nu}(y^2)\right)|\theta\rangle$$

$$= \sum_{\substack{\nu \subseteq [-s_2^{r_2}] \\ \ell(\nu) \le r_1}} b^t_{\nu+[-s_1^{r_1}]}(r_1) b^t_{[-s_2^{r_2}]}(r_2) \rho_+\left(\mathsf{P}^t_{\nu+[-s_1^{r_1}]}(y^1)\mathsf{P}^t_{[-s_2^{r_2}]/\nu}(y^2)\right)|\theta\rangle, \tag{5.10}$$

by using (4.16), (4.23) and (4.25). As the second factor is independent of ν (and non-zero), it may be absorbed into the normalisation of the singular vector. Our final result is therefore

$$\mathcal{S}_+^{[r_1,r_2]}|\eta\rangle = \sum_{\substack{\nu \subseteq [-s_2^{r_2}] \\ \ell(\nu) \le r_1}} b^t_{\nu+[-s_1^{r_1}]}(r_1) \rho_+\left(\mathsf{P}^t_{\nu+[-s_1^{r_1}]}(y^1)\mathsf{P}^t_{[-s_2^{r_2}]/\nu}(y^2)\right)|\theta\rangle. \tag{5.11}$$

This form is now easily implemented in computer algebra packages.

The right-hand side of (5.11) is easily seen to be manifestly non-zero by noting that the total degree, with respect to the a_{-m}^2, of the summand corresponding to the empty partition $\nu = [\,]$ is maximal and that all other summands have strictly lesser degrees. Since $b^t_{[-s_1^{r_1}]}(r_1)$, $\mathsf{P}^t_{[-s_1^{r_1}]}(y^1)$ and $\mathsf{P}^t_{[-s_2^{r_2}]}(y^2)$ are all non-zero, this summand is therefore linearly independent of all others. The conclusion is that (5.11) defines a singular vector for every $r_1, r_2 \in \mathbb{Z}_{\ge 0}$, $s_1, s_2 \in \mathbb{Z}_{\le 0}$ and $t \in \mathbb{C} \setminus \mathbb{Q}_{\le 0}$.

6. EXAMPLES

We now illustrate the W_3 singular vector formula (5.11) with three examples.

6.1. Example 1.
For our first example, we compute a singular vector for the case when $t = \frac{4}{5}$, so that

$$\alpha_+ = \frac{\sqrt{5}}{2}, \quad \alpha_- = -\frac{2}{\sqrt{5}}, \quad \alpha_0 = \frac{1}{2\sqrt{5}}, \quad c = \frac{4}{5}. \tag{6.1}$$

This central charge corresponds to that of the 3-state Potts model, described by the W_3 minimal model $\mathsf{W}_3(4,5)$ (the parameters here are the numerator and denominator of t in reduced form). Take $r_1 = r_2 = -s_1 = -s_2 = 1$ for simplicity.

Then, the map $\mathcal{S}_+^{[1,1]}$ sends $\mathcal{F}_\eta$, where $\eta = \zeta_{0,-1;1,-1}$, into $\mathcal{F}_\theta$, where $\theta = \zeta_{-1,-1;0,-1}$. We note that

$$h_\eta = \frac{13}{6}, \quad h_\theta = \frac{1}{6}, \quad w_\eta = \frac{187}{9\sqrt{390}}, \quad w_\theta = -\frac{7}{9\sqrt{390}}, \tag{6.2}$$

by (2.12) and (3.6). The conformal weight h_θ is not one of those associated with the 3-state Potts model. Nevertheless, the Fock space $\mathcal{F}_\theta$ has a singular vector at grade 2 in accordance with (5.8). Equation (5.11) writes it in the form

$$\mathcal{S}_+^{[1,1]}|\eta\rangle = \sum_{\nu \subseteq [1]} b_{\nu+[1]}^{4/5}(1) \rho_+ \left(\mathsf{P}_{\nu+[1]}^{4/5}(y^1) \mathsf{P}_{[1]/\nu}^{4/5}(y^2) \right) |\theta\rangle. \tag{6.3}$$

There are only two partitions ν to consider. Using (4.25), (5.2) and SAGEMATH to write Jacks and skew-Jacks in terms of power sums, we have

$$\nu = [0]: \quad b_{[0]+[1]}^{4/5}(1) = \frac{5}{4}, \quad \rho_+\left(\mathsf{P}_{[0]+[1]}^{4/5}(y^1) \right) = \rho_+\left(\mathsf{p}_{[1]}(y^1) \right) = \frac{2}{\sqrt{5}} a_{-1}^1,$$

$$\rho_+\left(\mathsf{P}_{[1]/[0]}^{4/5}(y^2) \right) = \rho_+\left(\mathsf{p}_{[1]}(y^2) \right) = \frac{2}{\sqrt{5}} a_{-1}^2.$$

$$\nu = [1]: \quad b_{[1]+[1]}^{4/5}(1) = \frac{5}{4}\frac{9}{8}, \quad \rho_+\left(\mathsf{P}_{[1]+[1]}^{4/5}(y^1) \right) = \rho_+\left(\frac{5}{9}\mathsf{p}_{[1,1]}(y^1) + \frac{4}{9}\mathsf{p}_{[2]}(y^1) \right) \tag{6.4}$$

$$= \frac{4}{5}\frac{5}{9} a_{-1}^1 a_{-1}^1 + \frac{2}{\sqrt{5}}\frac{4}{9} a_{-2}^1,$$

$$\rho_+\left(\mathsf{P}_{[1]/[1]}^{4/5}(y^1) \right) = \rho_+\left(\mathsf{p}_{[0]}(y^1) \right) = 1.$$

The singular vector is therefore explicitly identified as

$$\mathcal{S}_+^{[1,1]}|\eta\rangle = \left(a_{-1}^1 a_{-1}^2 + \frac{5}{8} a_{-1}^1 a_{-1}^1 + \frac{\sqrt{5}}{4} a_{-2}^1 \right) |\theta\rangle. \tag{6.5}$$

Consider the W_3 Verma module $\mathcal{V}_\vartheta$ whose highest-weight vector $|\vartheta\rangle$ has L_0- and W_0-eigenvalue h_θ and w_θ, as given in (6.2). By direct calculation, $\mathcal{V}_\vartheta$ has a singular vector $|\chi\rangle$, unique up to normalisation, at grade 2:

$$|\chi\rangle = \left(\frac{390}{119} W_{-1}W_{-1} - \frac{\sqrt{390}}{17} W_{-2} + \frac{10\sqrt{390}}{119} L_{-1}W_{-1} + L_{-1}L_{-1} \right) |\vartheta\rangle. \tag{6.6}$$

The free field realisation $f: \mathsf{W}_3 \hookrightarrow \mathsf{H}_2$ defined by (3.1) induces a W_3-module homomorphism

$$f_\vartheta: \mathcal{V}_\vartheta \longrightarrow \mathcal{F}_\theta, \quad f_\vartheta(U|\vartheta\rangle) = f(U)|\theta\rangle. \tag{6.7}$$

Here, U is an arbitrary element of the W_3 mode algebra, this being the (unital) associative algebra generated by the L_m and W_n subject to (3.4). Explicit calculation now verifies that the image of the singular vector $|\chi\rangle$ under f_ϑ is, of course, that constructed in (6.5):

$$f_\vartheta(|\chi\rangle) = \left(\frac{5}{4} a_{-1}^1 a_{-1}^2 + \frac{25}{32} a_{-1}^1 a_{-1}^1 + \frac{5\sqrt{5}}{16} a_{-2}^1 \right) |\theta\rangle = \frac{5}{4}\mathcal{S}_+^{[1,1]}|\eta\rangle. \tag{6.8}$$

6.2. **Example 2.** For our second example, we compute a grade three singular vector for general central charges. Let $r_1 = 2$ and $r_2 = -s_1 = -s_2 = 1$, so that $\eta = \zeta_{1,-1;1,-1}$ and $\theta = \zeta_{-2,-1;1,-1}$. In order to evaluate the singular vector formula (5.11), we need to compute

$$b_{[1,1]}^t(2) = \frac{2}{t+1}\frac{1}{t}, \quad b_{[2,1]}^t(2) = \frac{2}{t+1}\frac{1}{t}\frac{t+2}{2t+1},$$

$$\mathsf{P}_{[1,1]}^t = \frac{1}{2}\mathsf{p}_{[1,1]} - \frac{1}{2}\mathsf{p}_{[2]}, \quad \mathsf{P}_{[1]/[0]}^t = \mathsf{p}_{[1]}, \tag{6.9}$$

$$\mathsf{P}_{[2,1]}^t = \frac{1}{t+2}\mathsf{p}_{[1,1,1]} + \frac{t-1}{t+2}\mathsf{p}_{[2,1]} - \frac{t}{t+2}\mathsf{p}_{[3]}, \quad \mathsf{P}_{[1]/[1]}^t = 1,$$

again using (4.25), (5.2) and SAGEMATH. The singular vector is thus

$$\mathcal{S}_+^{[2,1]}|\eta\rangle = \left[b_{[1,1]}^t(2)\rho_+\left(\mathsf{P}_{[1,1]}^t(y^1)\mathsf{P}_{[1]/[0]}^t(y^2) \right) + b_{[2,1]}^t(2)\rho_+\left(\mathsf{P}_{[2,1]}^t(y^1)\mathsf{P}_{[1]/[1]}^t(y^2) \right) \right] |\theta\rangle$$

$$= \left[\frac{1}{t(t+1)} \left(\frac{1}{\alpha_+^2} a_{-1}^1 a_{-1}^1 - \frac{1}{\alpha_+} a_{-2}^1 \right) \frac{1}{\alpha_+} a_{-1}^2 \right.$$

$$\left. + \frac{2}{t(t+1)(2t+1)} \left(\frac{1}{\alpha_+^3} a_{-1}^1 a_{-1}^1 a_{-1}^1 + \frac{t-1}{\alpha_+^2} a_{-2}^1 a_{-1}^1 - \frac{t}{\alpha_+} a_{-3}^1 \right) \right] |\theta\rangle$$

$$= \left[\frac{2/\alpha_+}{(t+1)(2t+1)} a_{-1}^1 a_{-1}^1 a_{-1}^1 + \frac{1/\alpha_+}{t+1} a_{-1}^1 a_{-1}^1 a_{-1}^2 \right.$$

$$\left. + \frac{2(t-1)}{(t+1)(2t+1)} a_{-2}^1 a_{-1}^1 - \frac{1}{t+1} a_{-2}^1 a_{-1}^2 - \frac{1/\alpha_+}{(t+1)(2t+1)} a_{-3}^1 \right] |\theta\rangle. \tag{6.10}$$

We note that the result is manifestly well defined and non-zero for all α_+ such that $t = \alpha_+^{-2} \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$, as expected. This region includes all central charges less than 98.

6.3. **Example 3.** Our final example concerns singular vectors for quite arbitrary central charges (including all $c < 98$). This time, we fix $\theta = 0$ and use (5.11) to construct singular vectors in the Fock space $\mathcal{F}_0$.

First, we note that $\theta = \zeta_{-r_1,s_1;r_1-r_2,s_2} = 0$ may be solved for r_1 and s_1:

$$r_1 = -1 + (-s_1 + 1)t, \quad r_2 = -2 + (-s_1 - s_2 + 2)t. \tag{6.11}$$

Since $r_1, r_2, s_1, s_2 \in \mathbb{Z}$, we will only find singular vectors when $t \in \mathbb{Q}_{>0}$. Writing $t = \frac{u}{v}$, where u and v are coprime integers, it follows that

$$r_1 = mu - 1, \quad -s_1 = mv - 1, \quad r_2 = nu - 2, \quad -s_2 = (n-m)v - 1, \tag{6.12}$$

for some $m, n \in \mathbb{Z}$. Given that $r_1, r_2 \in \mathbb{Z}_{\geq 0}$ and $s_1, s_2 \in \mathbb{Z}_{<0}$, we conclude that m, n and $n - m$ must be positive integers. We thereby obtain, for each fixed $t \in \mathbb{Q}_{>0}$, an infinite sequence of singular vectors, generically indexed by integers $n > m > 0$, of the form $\mathcal{S}_+^{[mu-1,nu-2]} |\zeta_{(m-n)u+1,-mv+1;nu-2,-(n-m)v+1}\rangle$. Among these, the singular vector of lowest grade corresponds, assuming that $u > 1$, to $(m,n) = (1,2)$. Moreover, the grade of $\mathcal{S}_+^{[u-1,2(u-1)]} |\zeta_{-(u-1),-(v-1);2(u-1),-(v-1)}\rangle$ is $3(u-1)(v-1)$, by (5.8).

It is not clear if these singular vectors of the Fock space $\mathcal{F}_0$ correspond, in the sense of Example 6.1 to singular vectors in the W$_3$ vacuum Verma module $\mathcal{V}_0$ or not. However, there are five other Fock spaces $\mathcal{F}_\zeta$ whose highest-weight vectors $|\zeta\rangle$ have $h_\zeta = w_\zeta = 0$. This follows from the easily verified fact that both h_ζ and w_ζ are left invariant by the following shifted action of the Weyl group S$_3$:

$$\sigma \cdot \zeta = \sigma(\zeta - \alpha_0 \rho) + \alpha_0 \rho, \quad \sigma \in \mathsf{S}_3. \tag{6.13}$$

Each of these five other Fock spaces has an infinite sequence of singular vectors given by (5.11) and it is interesting to ask whether these also correspond to singular vectors in $\mathcal{V}_0$ or not. We shall not investigate this question here. We only note the following observation: $\mathcal{F}_{2\alpha_0\rho}$ has such a singular vector at grade 3 and it corresponds to just one of the *two* linearly independent grade 3 singular vectors of $\mathcal{V}_0$. Which one is obtained depends on the branch of the square root of β chosen in (3.1).

We conclude by remarking that the question of whether the Fock space singular vectors constructed here exhaust the singular vectors of $\mathcal{V}_0$ is much easier to answer. They do not. We cannot obtain the two linearly independent singular vectors at grade 1 using (5.11) (for $c \neq 2$; when $c = 2$, W_0 acts non-diagonalisably). Nor can we obtain, when $t = \frac{u}{v} \in \mathbb{Q}_{>0}$, the grade $(u-2)(v-2)$ singular vector whose image is non-zero in the universal W$_3$ vacuum module. This singular vector can be constructed formally using screening operators, but we do not know how to actually evaluate the integral in this case. What is needed is a certain $\mathfrak{sl}(3)$ analogue of the theory of Jack functions, something which does not appear to have yet been developed (see [27, 33, 34] for work in this direction). We hope to return to this important point in the future.

7. EXPLICIT EVALUATION OF W$_N$ SINGULAR VECTORS

In the previous section, we computed explicit formulae for W$_3$ singular vectors in Fock spaces. In this section we generalise the results of Section 5 and derive explicit formulae for W$_N$ singular vectors in Fock spaces. Continuing the pattern of ranks 1 and 2, the rank $N - 1$ Heisenberg vertex operator algebra, with choice of energy-momentum tensor

(2.9), has $2(N-1)$ screening operators, $V_\zeta(w)$ for $\zeta = \alpha_\pm \alpha^i$, where $\alpha_\pm$ was defined in (3.7) and the α^i are the simple roots of $\mathfrak{sl}(N)$.

The W_N vertex operator algebra W_N is usually described as being generated by the Virasoro field $T(z)$ and $N-2$ Virasoro primary fields $W^3(z), \ldots, W^N(z)$ of conformal weights $3, \ldots, N$, respectively. Unfortunately, explicit formulae for these primaries, for example in terms of Heisenberg fields, rapidly increase in complexity as N increases and there are no known closed formulae for general N. Fortunately our computations do not require explicit expressions for the $W^k(z)$, only the fact that they commute with the screening operators.

We therefore turn to the definition of the W_N vertex operator algebra [22] in terms of a generating function called the quantum Miura transform. This constructs a different set of generators of W_N that are not conformal primaries in general, but which are easily verified to commute with screening operators. We denote these new generating fields by $U^2(z) = T(z)$, $U^3(z), \ldots, U^N(z)$ and their generating function by

$$R_N(z) = -\sum_{k=0}^{N} U_k(z)(\alpha_0 \partial)^{N-k} = \; :(\alpha_0 \partial_z - \varepsilon^1(z)) \cdots (\alpha_0 \partial_z - \varepsilon^N(z)): , \tag{7.1}$$

where the ε^i are the weights of the defining representation of $\mathfrak{sl}(N)$ so that $\varepsilon^1 + \cdots + \varepsilon^N = 0$ and $\alpha^i = \varepsilon^i - \varepsilon^{i+1}$, for $i = 1, \ldots, N-1$.

With the W_N algebra now defined explicitly as the algebra generated by the $U^i(z)$, $i = 2, \ldots N$, we construct screening fields in a manner similar to W_3. As mentioned above, the vertex operators $V_\zeta(w)$ with Heisenberg weights $\zeta = \alpha_\pm \alpha_1, \ldots, \alpha_\pm \alpha_{N-1}$ are screening fields, because their operator product expansions with $R_N(z)$ are total derivatives:

$$R_N(z) V_{\alpha_\pm \alpha_i}(w) \sim \partial_w \left(\frac{:R_N^i(w) V_{\alpha_\pm \alpha_i}(w):}{z-w} \right). \tag{7.2}$$

Here, $R_N^i(z)$ is defined as the product in (7.1), but without the factors involving ε^i and ε^{i+1}.

As in the rank 2 case, the residues of the screening fields, when defined, commute with the W_N algebra, because their operator product expansions with the generating U^i fields are total derivatives, and therefore define module homomorphisms. Also as in the rank 2 case, one can compose screening fields and integrate them over suitable contours to construct yet more module homomorphisms. Note that it is sufficient to only compose screening fields whose weights are all rescalings of simple $\mathfrak{sl}(N)$ roots by either α_+ or α_-. This is because the two screening operators corresponding to the residues of $V_{\alpha_+ \alpha^i}(w)$ and $V_{\alpha_- \alpha^j}(w)$ commute and can thus be considered independently. We shall therefore only present calculations involving the $V_{\alpha_+ \alpha^i}(w)$; those involving the $V_{\alpha_- \alpha^j}(w)$ work in exactly the same way.

We therefore compose r_1 copies of $V_{\alpha_+ \alpha^1}(z^1)$ with r_2 copies of $V_{\alpha_+ \alpha^2}(z^2)$ and so on, evaluating this composition on a Fock space of weight η, to obtain

$$\prod_{i=1}^{r_1} V_{\alpha_+ \alpha^1}(z_i^1) \cdots \prod_{i=1}^{r_{N-1}} V_{\alpha_+ \alpha^{N-1}}(z_i^{N-1}) \Big|_{\mathcal{F}_\eta}$$

$$= \prod_{k=1}^{N-1} \prod_{1 \le i < j \le r_k} (z_i^k - z_j^k)^{2\alpha_+^2} \cdot \prod_{k=1}^{N-2} \prod_{i=1}^{r_k} \prod_{j=1}^{r_{k+1}} (z_i^k - z_j^{k+1})^{-\alpha_+^2} \cdot \prod_{k=1}^{N-1} \prod_{i=1}^{r_k} (z_i^k)^{\alpha_+(\alpha^k, \eta)}$$

$$\cdot \prod_{k=1}^{N-1} e^{r_k \alpha_+ \alpha^k} \cdot \prod_{k=1}^{N-1} \prod_{m \ge 1} \exp\left(\frac{\alpha_+ a_{-m}^k}{m} \sum_{i=1}^{r_k} (z_i^k)^m \right) \exp\left(-\frac{\alpha_+ a_m^k}{m} \sum_{i=1}^{r_k} (z_i^k)^{-m} \right)$$

$$= \prod_{k=1}^{N-1} \prod_{1 \le i \ne j \le r_k} \left(1 - \frac{z_i^k}{z_j^k} \right)^{\alpha_+^2} \cdot \prod_{k=2}^{N-1} \prod_{i=1}^{r_k} \prod_{j=1}^{r_{k+1}} \left(1 - \frac{z_j^k}{z_i^{k-1}} \right)^{-\alpha_+^2} \cdot \prod_{k=1}^{N-1} \prod_{i=1}^{r_k} (z_i^k)^{\alpha_+^2(r_k - r_{k+1} - 1) + \alpha_+(\alpha^k, \eta)}$$

$$\cdot \prod_{k=1}^{N-1} e^{r_k \alpha_+ \alpha^k} \cdot \prod_{k=1}^{N-1} \prod_{m \ge 1} \exp\left(\frac{\alpha_+ a_{-m}^k}{m} \sum_{i=1}^{r_k} (z_i^k)^m \right) \exp\left(-\frac{\alpha_+ a_m^k}{m} \sum_{i=1}^{r_k} (z_i^k)^{-m} \right), \tag{7.3}$$

where we define $r_N = 0$. In analogy to the reasoning presented for the W_3 algebra in Section 3, one can construct a W_N-module homomorphism by choosing an appropriate contour. Integrating over the contours of Tsuchiya and Kanie [29] is well defined whenever

$$\alpha_+^2 (r_k - r_{k+1} - 1) + \alpha_+ \left(\alpha^k, \eta \right) \in \mathbb{Z}, \quad \text{for all } k = 1, \ldots, N-1. \tag{7.4}$$

To parametrise the weights satisfying these constraints, we define

$$\zeta_{\mathbf{u},\mathbf{v}} = \sum_{i=1}^{N-1}\big((1-u_i)\alpha_+ + (1-v_i)\alpha_-\big)\omega_i, \quad \mathbf{u}=(u_1,\ldots,u_{N-1}),\ \mathbf{v}=(v_1,\ldots,v_{N-1})\in\mathbb{Z}^{N-1}, \tag{7.5}$$

and define screening operators

$$\mathcal{S}_+^{[\mathbf{r}]} = \int_{\Gamma(r_1;1/\alpha_+^2)}\cdots\int_{\Gamma(r_{N-1};1/\alpha_+^2)} \prod_{i=1}^{r_1}V_{\alpha_\pm\alpha^1}(z_i^1)\cdots\prod_{i=1}^{r_{N-1}}V_{\alpha_+\alpha^{N-1}}(z_i^{N-1})\cdot\prod_{k=1}^{N-1}\prod_{i=1}^{r_k}dz_i^k, \quad \mathbf{r}\in\mathbb{Z}_{\geq 0}^{N-1}. \tag{7.6}$$

These, in turn, induce W_N-module homomorphisms

$$\mathcal{S}_+^{[\mathbf{r}]}\colon \mathcal{F}_{\eta_{\mathbf{r},\mathbf{s}}^+}\to\mathcal{F}_{\theta_{\mathbf{r},\mathbf{s}}^+}, \quad \mathbf{r}\in\mathbb{Z}_{\geq 0}^{N-1},\ \mathbf{s}\in\mathbb{Z}^{N-1}, \tag{7.7}$$

where $\eta_{\mathbf{r},\mathbf{s}}^+ = \zeta_{(r_1-r_2,\ldots,r_{N-2}-r_{N-1},r_{N-1}),\mathbf{s}}$ and $\theta_{\mathbf{r},\mathbf{s}}^+ = \zeta_{(-r_1,r_1-r_2,\ldots,r_{N-2}-r_{N-1}),\mathbf{s}}$. Similar screening operators $\mathcal{S}_-^{[\mathbf{s}]}$ are obtained by swapping the roles of α_+ and α_-, as well as $\mathbf{r}$ and $\mathbf{s}$, in this development.

If we apply the screening operator $\mathcal{S}_+^{[\mathbf{r}]}$ to the H_{N-1} highest-weight vector $|\eta_{\mathbf{r},\mathbf{s}}^+\rangle$, we get

$$\begin{aligned}
\mathcal{S}_+^{[\mathbf{r}]}|\eta_{\mathbf{r},\mathbf{s}}^+\rangle &= \int_{\Gamma(r_1;1/\alpha_+^2)}\cdots\int_{\Gamma(r_{N-1};1/\alpha_+^2)}\prod_{i=1}^{r_1}V_{\alpha_+\alpha^1}(z_i^1)\cdots\prod_{i=1}^{r_{N-1}}V_{\alpha_+\alpha^{N-1}}(z_i^{N-1})\cdot|\eta_{\mathbf{r},\mathbf{s}}^+\rangle\prod_{k=1}^{N-1}\prod_{i=1}^{r_k}dz_i^k\\
&= \int_{\Gamma(r_1;1/\alpha_+^2)}\cdots\int_{\Gamma(r_{N-1};1/\alpha_+^2)}\prod_{k=1}^{N-1}\prod_{1\leq i\neq j\leq r_k}\left(1-\frac{z_i^k}{z_j^k}\right)^{\alpha_+^2}\cdot\prod_{k=2}^{N-1}\prod_{i=1}^{r_{k-1}}\prod_{j=1}^{r_k}\left(1-\frac{z_j^k}{z_i^{k-1}}\right)^{-\alpha_+^2}\\
&\quad\cdot\prod_{k=1}^{N-1}\prod_{i=1}^{r_k}(z_i^k)^{\alpha_+(\alpha^k,\eta_{\mathbf{r},\mathbf{s}}^+)+\alpha_+^2(r_k-1)+1}\cdot\prod_{k=2}^{N-1}\prod_{i=1}^{r_k}(z_i^1)^{-\alpha_+^2 r_k}\\
&\quad\cdot\prod_{k=1}^{N-1}\prod_{m\geq 1}\exp\left(\frac{\alpha_+a_{-m}^k}{m}\sum_{i=1}^{r_k}(z_i^k)^m\right)\cdot|\theta_{\mathbf{r},\mathbf{s}}^+\rangle\prod_{k=1}^{N-1}\prod_{i=1}^{r_k}\frac{dz_i^k}{z_i^k}\\
&= \int_{\Gamma(r_1;1/\alpha_+^2)}\cdots\int_{\Gamma(r_{N-1};1/\alpha_+^2)}\prod_{k=1}^{N-1}\prod_{1\leq i\neq j\leq r_k}\left(1-\frac{z_i^k}{z_j^k}\right)^{\alpha_+^2}\cdot\prod_{k=1}^{N-2}\prod_{i=1}^{r_{k-1}}\prod_{j=1}^{r_k}\left(1-\frac{z_j^k}{z_i^{k-1}}\right)^{-\alpha_+^2}\\
&\quad\cdot\prod_{k=1}^{N-1}\prod_{i=1}^{r_k}(z_i^k)^{s_k}\cdot\prod_{k=1}^{N-1}\prod_{m\geq 1}\exp\left(\frac{\alpha_+a_{-m}^k\mathsf{p}_m(z^k)}{m}\right)\cdot|\theta_{\mathbf{r},\mathbf{s}}^+\rangle\prod_{k=1}^{N-1}\prod_{i=1}^{r_k}\frac{dz_i^k}{z_i^k}.
\end{aligned} \tag{7.8}$$

As in the W_3 case, we can evaluate these integral formulae for singular vectors in terms of symmetric functions. Recall that the tensor product of $N-1$ copies of the ring of symmetric functions $\Lambda^{\otimes N-1}$ is isomorphic to

$$U(\widehat{\mathfrak{h}}_-) = \mathbb{C}[a_{-m}^k \ : \ k=1,\ldots,N-1,\ m\in\mathbb{Z}_{>0}] \tag{7.9}$$

as an algebra, by (4.9). Distinguishing the alphabets of the tensor factors by superscripts, so that the alphabet in the i-th tensor factor is denoted by y^i, we define the following algebra isomorphism generalising that of (5.2):

$$\rho_+\colon \Lambda^{\otimes N-1}\to U(\widehat{\mathfrak{h}}_-^{\otimes N-1}), \quad \mathsf{p}_n(y^k)\mapsto\frac{1}{\alpha_+}a_{-n}^k. \tag{7.10}$$

This isomorphism allows us to write

$$\prod_{m\geq 1}\exp\left(\frac{\alpha_+a_{-m}^k\mathsf{p}_m(z^k)}{m}\right) = \rho_+\left(\prod_{m\geq 1}\exp\left(\alpha_+^2\frac{\mathsf{p}_m(y^k)\mathsf{p}_m(z^k)}{m}\right)\right) = \rho_+\left(\prod_{i\geq 1}\prod_{j=1}^{r_k}(1-y_i^k z_j^k)^{-\alpha_+^2}\right), \tag{7.11}$$

for $k=1,\ldots,N-1$. We now identify, with $t=\alpha_+^{-2}$,

$$\prod_{k=1}^{N-1}\prod_{1\leq i\neq j\leq r_k}\left(1-\frac{z_i^k}{z_j^k}\right)^{\alpha_+^2} = \prod_{k=1}^{N-1}G_{r_k}^t(z^k) \tag{7.12}$$

as the product of the integrating kernels for the variables z_k. For $k=1$, as in (5.4), we write

$$\prod_{m\geq 1}\exp\left(\frac{\alpha_+a_{-m}^1\mathsf{p}_m(z^1)}{m}\right) = \rho_+\left(\sum_{\mu_1}\mathsf{P}_{\mu_1}^t(y^1)\mathsf{Q}_{\mu_1}^t(z^1)\right). \tag{7.13}$$

For $k = 2, \ldots, N-1$, similar to (5.5), we have instead

$$\prod_{m \geq 1} \exp\left(\frac{\alpha_+ a_{-m}^k \mathsf{p}_m(z^k)}{m}\right) \cdot \prod_{i=1}^{r_{k-1}} \prod_{j=1}^{r_k} \left(1 - \frac{z_j^k}{z_i^{k-1}}\right)^{-\alpha_+^2} = \rho_+\left(\sum_{\mu_k} \mathsf{P}_{\mu_k}^t(y^k \cup (z^{k-1})^{-1}) \mathsf{Q}_{\mu_k}^t(z^k)\right)$$

$$= \rho_+\left(\sum_{\mu_k, \nu_k} \overline{\mathsf{P}_{\nu_k}^t(z^{k-1})} \mathsf{P}_{\mu_k/\nu_k}^t(y^k) \mathsf{Q}_{\mu_k}^t(z^k)\right). \tag{7.14}$$

Putting everything together, we have

$$\mathcal{S}_+^{[\mathbf{r}]}|\eta_{\mathbf{r},\mathbf{s}}^+\rangle = \sum_{\substack{\mu_1,\mu_2,\ldots,\mu_{N-1} \\ \nu_2,\ldots,\nu_{N-1}}} \rho_+(\mathsf{P}_{\mu_1}^t(y^1)) \int_{\Gamma(r_1;t)} G_{r_1}^t(z^1) \overline{\mathsf{P}_{[-s_1^{r_1}]}^t(z^1)} \overline{\mathsf{P}_{\nu_2}^t(z^1)} \mathsf{Q}_{\mu_1}^t(z^1) \prod_{i=1}^{r_1} \frac{\mathrm{d}z_i^1}{z_i^1}$$

$$\cdot \rho_+\left(\mathsf{P}_{\mu_2/\nu_2}^t(y^2)\right) \int_{\Gamma(r_2;t)} G_{r_2}^t(z^2) \overline{\mathsf{P}_{[-s_2^{r_2}]}^t(z^2)} \overline{\mathsf{P}_{\nu_3}^t(z^2)} \mathsf{Q}_{\mu_2}^t(z^2) \prod_{i=1}^{r_2} \frac{\mathrm{d}z_i^2}{z_i^2}$$

$$\vdots$$

$$\cdot \rho_+\left(\mathsf{P}_{\mu_{N-2}/\nu_{N-2}}^t(y^{N-2})\right) \int_{\Gamma(r_{N-2};t)} G_{r_{N-2}}^t(z^{N-2}) \overline{\mathsf{P}_{[-s_{N-2}^{r_{N-2}}]}^t(z^{N-2})} \overline{\mathsf{P}_{\nu_{N-1}}^t(z^{N-2})} \mathsf{Q}_{\mu_{N-2}}^t(z^{N-2}) \prod_{i=1}^{r_{N-2}} \frac{\mathrm{d}z_i^{N-2}}{z_i^{N-2}}$$

$$\cdot \rho_+\left(\mathsf{P}_{\mu_{N-1}/\nu_{N-1}}^t(y^{N-1})\right) \int_{\Gamma(r_{N-1};t)} G_{r_{N-1}}^t(z^{N-1}) \overline{\mathsf{P}_{[-s_{N-1}^{r_{N-1}}]}^t(z^{N-1})} \mathsf{Q}_{\mu_{N-1}}^t(z^{N-1}) \prod_{i=1}^{r_{N-1}} \frac{\mathrm{d}z_i^{N-1}}{z_i^{N-1}} \cdot |\theta_{\mathbf{r},\mathbf{s}}^+\rangle$$

$$= \sum_{\substack{\mu_1,\mu_2,\ldots,\mu_{N-1} \\ \nu_2,\ldots,\nu_{N-1}}} \prod_{k=1}^{N-2} \left\langle \mathsf{Q}_{\mu_k}^t, \mathsf{P}_{\nu_{k+1}+[-s_k^{r_k}]}^t \right\rangle_{r_k}^t \cdot \left\langle \mathsf{Q}_{\mu_{N-1}}^t, \mathsf{P}_{[-s_{N-1}^{r_{N-1}}]}^t \right\rangle_{r_{N-1}}^t \rho_+(\mathsf{P}_{\mu_1}^t(y^1)) \prod_{k=2}^{N-1} \rho_+\left(\mathsf{P}_{\mu_k/\nu_k}^t(y^k)\right) \cdot |\theta_{\mathbf{r},\mathbf{s}}^+\rangle$$

$$= \sum_{\nu_2,\ldots,\nu_{N-1}} \left(\prod_{k=1}^{N-2} b_{\nu_{k+1}+[-s_k^{r_k}]}^t(r_k) \cdot b_{[-s_{N-1}^{r_{N-1}}]}^t(r_{N-1})\right)$$

$$\cdot \rho_+\left(\mathsf{P}_{\nu_2+[-s_1^{r_1}]}^t(y^1)\right) \prod_{k=2}^{N-2} \rho_+\left(\mathsf{P}_{(\nu_{k+1}+[-s_k^{r_k}])/\nu_k}^t(y^k)\right) \cdot \rho_+\left(\mathsf{P}_{[-s_{N-1}^{r_{N-1}}]/\nu_{N-1}}^t(y^{N-1})\right) |\theta_{\mathbf{r},\mathbf{s}}^+\rangle. \tag{7.15}$$

As before, the factor $b_{[-s_{N-1}^{r_{N-1}}]}^t(r_{N-1})$ does not depend on the summation indices $\nu_2,\ldots,\nu_{N-1}$, appears in every summand, and is non-zero, so it can be suppressed. Moreover, the skew-Jack polynomials vanish unless the summation indices $\nu_2,\ldots,\nu_{N-1}$ satisfy the relations

$$\nu_k \subseteq \nu_{k+1} + [-s_k^{r_k}], \quad k = 2,\ldots,N-2, \quad \nu_{N-1} \subseteq [-s_{N-1}^{r_{N-1}}]. \tag{7.16}$$

Thus, the singular vector $\mathcal{S}_+^{[\mathbf{r}]}|\eta_{\mathbf{r},\mathbf{s}}^+\rangle \in \mathcal{F}_{\theta_{\mathbf{r},\mathbf{s}}^+}$ is proportional to

$$\sum_{\nu_2,\ldots,\nu_{N-1}} \left(\prod_{k=1}^{N-2} b_{\nu_{k+1}+[-s_k^{r_k}]}^t(r_k)\right) \rho_+\left(\mathsf{P}_{\nu_2+[-s_1^{r_1}]}^t(y^1)\right)$$

$$\cdot \prod_{k=2}^{N-2} \rho_+\left(\mathsf{P}_{(\nu_{k+1}+[-s_k^{r_k}])/\nu_k}^t(y^k)\right) \cdot \rho_+\left(\mathsf{P}_{[-s_{N-1}^{r_{N-1}}]/\nu_{N-1}}^t(y^{N-1})\right) |\theta_{\mathbf{r},\mathbf{s}}^+\rangle. \tag{7.17}$$

This is our final formula for W_n singular vectors generalising the $n = 3$ case in (5.11). As before, considering the summand with $\nu_2 = \cdots = \nu_{N-1} = [\,]$ shows that the right-hand side is non-zero for every $\mathbf{r} \in \mathbb{Z}_{\geq 0}^{N-1}$, $\mathbf{s} \in \mathbb{Z}_{\leq 0}^{N-1}$ and $t \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$. This singular vector formula also has the nice property of being comparatively easy to evaluate using computer algebra packages such as SAGEMATH.

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(David Ridout) School of Mathematics and Statistics, University of Melbourne, Parkville, Australia, 3010.
E-mail address: `david.ridout@unimelb.edu.au`

(Steve Siu) School of Mathematics and Statistics, University of Melbourne, Parkville, Australia, 3010.
E-mail address: `ssiu1@student.unimelb.edu.au`

(Simon Wood) School of Mathematics, Cardiff University, Cardiff, United Kingdom, CF24 4AG.
E-mail address: `woodsi@cardiff.ac.uk`