Cardiff University

Problems Related to Number Theory

Sum-and-Distance Systems, Reversible Square Matrices and Divisor Functions

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This thesis is submitted in partial fulfilment of the requirement for the degree of Doctor of Philosophy

September 2017 (and in revised form April 2018)

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Summary

We say that two sets A and B, each of cardinality m, form an $m+m$ *sum-and-distance system* $\{A,B\}$ if the sum-and-distance set A^*B comprised of all the absolute values of the sums and distances $a_i \pm b_j$ contains either the consecutive odd integers $\{1,3,5,\ldots 4m^2-1\}$ or with the inclusion of the set elements themselves, the consecutive integers $\{1,2,3,\ldots,2m(m+1)\}$ (an inclusive sum-and-distance system). Sum-and-distance systems can be thought of as a discrete analogue of the union of a Minkowski sum system with a Minkowski difference system. We show that they occur naturally within a traditional reversible square matrix, where conjugation with a specific orthogonal symmetric involution, always reveals a sum-and-distance system within the block structure of the conjugated matrix. Moreover, we show that the block representation is an algebra isomorphism.

Building upon results of Ollerenshaw, and Brée, for a fixed dimension n, we establish a bijection between the set of sum-and-distance systems and the set of traditional principal reversible square matrices of size $n \times n$. Using the jth non-trivial divisor function $c_j(n)$, which counts the total number of proper ordered factorisations of the integer $n = p_1^{a_1} \ldots p_t^{a_t}$ into j parts, we prove that the total number of $n+n$ principal reversible square matrices, and so sum-and-distance systems, N_n, is given by

$$N_n = \sum_{j=1}^{\Omega(n)} \Big(c_j(n)^2 + c_{j+1}(n) c_j(n) \Big) = \sum_{j=1}^{\Omega(n)} c_j^{(0)}(n) c_j^{(1)}(n).$$

$$= \sum_{j=1}^{\Omega(n)} \left(\sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} \prod_{k=1}^{t} \binom{a_k+i-1}{i-1} \right) \left(\sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \prod_{k=1}^{t} \binom{a_k+i}{i} \right),$$

where $\Omega(n) = a_1 + a_2 + \ldots + a_t$ is the total number of prime factors (including repeats) of n.

Further relations between the divisor functions and their Dirichlet series are deduced, as well as a construction algorithm for all sum-and-distance systems of either type.

Superalgebra structures relating to the matrix symmetry properties are identified, including those for the reversible and most-perfect square matrices of those considered by Ollerenshaw and Brée. For certain symmetry types, links between the block representation constructed from a sum-and-distance system, and quadratic forms are also established.

Acknowledgements

First and foremost I would like to thank my supervisor Dr Matthew Lettington for his unwavering support and enthusiasm, without which this would not have been possible. Our meetings were productive, enjoyable and always with a lovely cup of tea. Secondly, I would like the thank my second supervisor Prof. Karl Schmidt for his invaluable comments and expertise.

To my parents for always believing in me, for their at-most support, engorgement and optimism. They have always encouraged me to pursue my dreams and not to quit. I would like to thank the rest of my family for their invaluable support those who include my Grampy and Nanny Tovey, my late Grandfather and Nanny Hill, My brother Alex, sister-in-law Vicky and the newest member little Zac who brought much happiness in the last and most stressful stages.

To my dearest friend and one of the most selfless individuals I know, Dr Waleed Ali, who has supported me both emotionally and academically for which I will always be grateful. To Dr James Evans and Martyn Deverell for their invaluable friendship, support and ability to always have a chat. To the fantastic friends during my time at Cardiff who made my time here so enjoyable those including Bro Bishop, Danny Groves, Simon Cen, Alex Mackay, Dr Xander Ramage and everyone else who has helped me.

A special thanks to Scott Morgan to all the support and encouragement.

Finally, to Misty, because I love that fluffy cat.

I acknowledge the financial support given to me by EPSRC and the school of Mathematics at Cardiff.

Notation

$\mathbb{R}$ The set of all real numbers.

$\mathbb{C}$ The set of all complex numbers.

$\mathbb{Z}$ The set of all integers.

$\mathbb{N}$ The set of all natural numbers.

$\emptyset$ The empty set.

$\mathbb{Z}_n$ The set of all integers from 1 to n, $\{1, 2, \ldots, n\}$.

$\mathbb{R}^n$ Any n-length vector of real entry values.

$\mathbb{R}^{n\times n}$ The set of all $n \times n$ matrices with real number entries.

$\mathbb{Z}^{n\times n}$ The set of all $n \times n$ matrices with integer entries.

I_n The $n \times n$ identity matrix $I_n = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$.

J_n The $n \times n$ anti-diagonal identity matrix, $J_n = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ \vdots & \iddots & & \vdots \\ 1 & \ldots & 0 & 0 \end{pmatrix}$.

E_n The $n \times n$ matrix of ones, $E_n = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}$.

$\hat{0}_n$ The $n \times n$ zero matrix, $\hat{0}_k = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}$

c_n A n-length vector consisting only of the constant c. For example, $1_n = (1, 1, \ldots, 1)^T$ and $0_n = (0, 0, \ldots, 0)^T$.

$\S_n$ is the n-length vector with alternating signed entries, 1 and -1, $\S_n = (1, -1, 1, -1, \ldots, \pm 1)^T$.

X_n The $n \times n$ X_n matrix

$$X_n = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix},$$

when $n = 2k$ is even, and

$$X_n = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0_k & J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ J_k & 0_k & -I_k \end{pmatrix}$$

when $n = 2k + 1$ is odd.

$\Omega(n)$ The sum of prime powers of $n = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$ where $p_1, p_2 \dots p_t$ are prime and $a_1, a_2 \dots a_t$ are positive integers, $\Omega(n) = a_1 + a_2 + \dots a_t$.

$\lfloor n \rfloor$ The integer part of n.

$n!$ The factorial of positive integer n, $n! = n(n-1)\dots 2 \times 1$.

$v^T u$ The dot product of two n-length vectors, $v = (v_1, v_2, \dots, v_n)$ and $u = (u_1, u_2, \dots, u_n)$, $v^T u = v_1 u_1 + v_2 u_2 + \dots + v_n u_n$.

vu^T The outer product of two vectors $v = (v_1, v_2, \dots, v_n)$ and $u = (u_1, u_2, \dots, u_n)$, giving the matrix $vu^T = (u_i v_j)_{i,j}$.

$\{v\}^\perp$ The set of orthogonal vectors to v, i.e. if $u \in \{v\}^\perp$ then $u^T v = v^T u = 0$.

$\zeta(s)$ The Riemann zeta function, defined for $s \in \mathbb{C}$, with $\Re(s) > 1$, such that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Contents

1 Introduction to Sum-and-Distance Systems

1.1 Sum-and-Distance Systems

Throughout this thesis, for two non-empty sets of integers A and B, we are interested in the *sum-set* $A+B$ and the *difference-set* $|A-B|$ defined by

$$A+B=\{a+b\,:\,a\in A,\ b\in B\},\qquad |A-B|=\{|a-b|\,:\,a\in A,\ b\in B\}.$$

Given some constant $\lambda\in\mathbb{C}$ we define

$$\lambda A=\{\lambda a\mid a\in A\},$$

and

$$A+\{\lambda\}=\{a+\lambda\mid a\in A\}.$$

More specifically we are interested in cases where the union of the sum-set and the difference-set (possibly with the set elements themselves) comprise a target set containing either consecutive integers or consecutive odd-integers with no repeats.

The combinatorial set pair formed from the union of a sum-set and difference-set (possibly with the set elements themselves) is referred to as a *sum-and-distance system*, the two types of which we now formally define.

Definition. For natural numbers m and n, let

$$A=\{a_1,\dots,a_m\},\quad \text{and}\quad B=\{b_1,\dots,b_n\},$$

be two sets of non-negative integers, respectively of m and n distinct elements, ordered in terms of increasing value, with $A\cap B=\phi$.

Then we say that the set pair A and B form an *m+n sum-and-distance system*, if and only if the corresponding *sum-and-distance set*, A^*B, defined by

$$A^*B=(A+B)\cup|A-B|=\{a_i+b_j,|a_i-b_j|\,:\,i\in\{1,\dots m\},\,j\in\{1,\dots n\}\},$$

is the set of odd integers$\{1,3,5,\dots,4mn-1\}$.

Similarly we say that A and B form an *inclusive m+n sum-and-distance system* if and only if the union of the corresponding *sum-and-distance set*, A^*B, with A and B, given by $A^*B\cup A\cup B=A\cup B\cup(A+B)\cup|A-B|$

$$=\{\ a_i,\ b_j,\ \ a_i+b_j,\ |a_i-b_j|\,:\,i\in\{1,\dots m\},\,j\in\{1,\dots n\}\},$$

is the set of consecutive integers $\{1,2,3,\dots 2mn+m+n\}$.

Example (of $m+n$ sum-and-distance sets). *When $m = 8$, $n = 2$, the set pair*

$$A = \{2, 6, 10, 14, 18, 22, 26, 30\}, \qquad B = \{31, 33\},$$

form an $8 + 2$ sum-and-distance system, because the sum-and-distance set A^*B is the set of consecutive odd integers $1, 3, 5, \ldots, 63$.

Similarly when $m = n = 4$ the set pair

$$A = \{1, 8, 9, 10\}, \qquad B = \{3, 24, 27, 30\},$$

forms an inclusive $4 + 4$ sum-and-distance system, because the inclusive sum-and-distance set, comprising the union of the sets A and B with the sum-and-distance set A^*B, is the set of consecutive integers $1, 2, 3, \ldots, 40$.

Remark. From the above definition of sum-and-distance systems (of both varieties) it can be seen that one can generalise to the d-dimensional case of the $m_1 + m_2 + \ldots + m_d$ sum-and-distance systems for some specified target set. However this more general case is a topic for future work and is not considered here.

Example (of $m+m$ sum-and-distance systems). *When $m = 3$ there are the seven $3 + 3$ (non-inclusive) sum-and-distance systems*

$$\{\{1, 3, 5\}, \{6, 18, 30\}\}, \quad \{\{1, 7, 9\}, \{2, 22, 26\}\}, \quad \{\{1, 11, 13\}, \{14, 18, 22\}\},$$

$$\{\{1, 23, 25\}, \{2, 6, 10\}\}, \quad \{\{3, 9, 15\}, \{16, 18, 20\}\}, \quad \{\{3, 21, 27\}, \{4, 6, 8\}\},$$

and $\{\{7, 9, 11\}, \{12, 18, 24\}\}$, but just the one inclusive $3 + 3$ sum-and-distance system $\{\{1, 2, 3\}, \{7, 14, 21\}\}$, where the elements of the set B are 7 times the elements of the set A. In such cases we call the set pair $\{A, B\}$ parasymmetric.

In this thesis we primarily consider the symmetric case, when both the sets A and B have cardinality m.

One of our main aims in this present work is the enumeration and construction of all such $m + m$ sum-and-distance systems.

1.2 Fundamental Properties of $m+m$ Sum-and-Distance Systems

The $m + m$ sum-and-distance systems exhibit many symmetric properties, some of which we now describe below.

LEMMA 1.1. *For a natural number m, the sum of the squares of the $2m$ elements of an $m + m$ sum-and-distance system is invariant and is given by*

$$\sum_{i=1}^{m} (a_i^2 + b_i^2) = \frac{1}{3!}(2m)((2m)^4 - 1) = \frac{m}{3}((2m)^4 - 1).$$

Similarly, the sum of the squares of the $2m$ elements of an inclusive $m + m$ sum-

and-distance system is invariant and is given by

$$\sum_{i=1}^{m}(a_i^2+b_i^2)=\frac{1}{4!}(2m+1)((2m+1)^4-1).$$

Proof. The set of $2m^2$ integers in $(A+B)\cup|A-B|$ is

$$\{a_i+b_j,\ |a_i-b_j|\ :\ i\in\{1,\dots m\},\ j\in\{1,\dots m\}\},$$

and the sum of their squares is

$$\sum_{1\le i,j\le m}\Big((a_i+b_j)^2+(a_i-b_j)^2\Big)=\sum_{1\le i,j\le m}\Big(2a_i^2+2b_j^2\Big)$$

$$=\sum_{1\le i,j\le m}2a_i^2+\sum_{1\le i,j\le m}2b_j^2=2m\sum_{i=1}^{m}a_i^2+2m\sum_{j=1}^{m}b_j^2=2m\sum_{i=1}^{m}(a_i^2+b_i^2).$$

We also have

$$(A+B)\cup|A-B|=\{1,3,5,\dots,(4m^2-1)\},$$

and we may use the result that

$$1^2+3^2+5^2+\dots+(2n-1)^2=\frac{n(4n^2-1)}{3},$$

to see that the sum of the squares is also given

$$1^2+3^2+5^2+\dots+(2m^2-1)^2=\frac{2m^2(4(2m^2)^2-1)}{3}.$$

This tells us that

$$2m\sum_{i=1}^{m}(a_i^2+b_i^2)=\frac{2m^2(16m^4-1)}{3}$$

or

$$\sum_{i=1}^{m}(a_i^2+b_i^2)=\frac{m(16m^4-1)}{3}$$

as required.

To see the second statement, from the definition of an inclusive $m+m$ sum-and-distance system we have that the set of $2m^2+2m$ integers in $A\cup B\cup(A+B)\cup|A-B|$ is

$$\{a_i,\ b_j,\ a_i+b_j,\ |a_i-b_j|\ :\ i\in\{1,\dots m\},\ j\in\{1,\dots m\}\},$$

and the sum of their squares is

$$\sum_{1\le i\le m}(a_i^2+b_i^2)+\sum_{1\le i,j\le m}\Big((a_i+b_j)^2+(a_i-b_j)^2\Big)$$

$$=\sum_{1\le i\le m}(a_i^2+b_i^2)+\sum_{1\le i,j\le m}\Big(2a_i^2+2b_j^2\Big)$$

$$= \sum_{1\le i\le m} a_i^2 + \sum_{1\le i,j\le m} 2a_i^2 + + \sum_{1\le j\le m} b_j^2 + \sum_{1\le i,j\le m} 2b_j^2$$

$$= (2m+1)\sum_{i=1}^{m} a_i^2 + (2m+1)\sum_{j=1}^{m} b_j^2 = (2m+1)\sum_{i=1}^{m}(a_i^2+b_i^2).$$

We also have

$$A \cup B \cup (A+B) \cup |A-B| = \{1,2,3,\dots,(2m^2+2m)\},$$

and we may use the result that

$$1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6},$$

to see that the sum of the squares is also given

$$1^2+2^2+3^2+\dots+(2m^2+2m)^2 = \frac{(2m^2+2m)(2m^2+2m+1)(4m^2+4m+1)}{6}$$

$$= \frac{((2m+1)^2-1)}{2}\frac{((2m+1)^2+1)}{2}\frac{(2m+1)^2}{6} = \frac{((2m+1)^4-1)}{4}\frac{(2m+1)^2}{6}.$$

This tells us that

$$(2m+1)\sum_{i=1}^{m}(a_i^2+b_i^2) = \frac{((2m+1)^2)((2m+1)^4-1)}{24},$$

or

$$\sum_{i=1}^{m}(a_i^2+b_i^2) = \frac{(2m+1)((2m+1)^4-1)}{4!}$$

as required. □

LEMMA 1.2. *For positive integer m, the two sets*

$$A = \{2k-1 | 1 \le k \le m\} \text{ and } B = \{2m(2k-1) | 1 \le k \le m\} = 2mA$$

form an $m+m$ sum-and-distance system, which we call the canonical *$m+m$ sum-and-distance system.*

Proof. The interval $[0, 4m^2]$ contains the $2m^2$ consecutive odd numbers $1, 3, \dots\ 4m^2-1$. It is the union of the m intervals

$$[0,4m],\ [4m,8m], \dots [2m(2m-2), 4m^2],$$

whose midpoints are precisely the elements of B.

Now each interval contains exactly m odd numbers greater than its midpoint and m less than its midpoint, all of which may be obtained by adding or subtracting one of the elements of set A. In this way we obtain the required odd numbers.

□

LEMMA 1.3. *For positive integer m, the two sets*

$$A = \{1, \ldots, m\}, \text{ and } B = (2m + 1)\{1, \ldots, m\} = (2m + 1)A$$

form an inclusive $m + m$ sum-and-distance system, which we call the canonical *inclusive $m + m$ sum-and-distance system.*

Proof. The set

$$A \cup B \cup (A + B) \cup |A - B| = \{ \ a_i, \ b_j, \ \ a_i + b_j, \ |a_i - b_j| \ : \ i, j \in \{1, \ldots m\}\}$$

contains exactly $2m + 2m^2$ elements. We have to show that these are the integers $1, 2, \ldots 2m + 2m^2$. The integers $1, 2 \ldots m$ are obtained directly from A. All integers in the interval $[m + 1, 2m + 2m^2]$ belong to exactly one of the subintervals

$$[2m+1-m, 2m+1+m], [2(2m+1)-m, 2(2m+1)+m], [3(2m+1)-m, 3(2m+1)+m],$$

$$\ldots, [m(2m + 1) - m, m(2m + 1) + m]$$

whose midpoints are the elements of B. To obtain any integer in the interval $[m + 1, 2m + 2m^2]$ except for the midpoints we choose the subinterval to which it belongs, take the midpoint, which belongs to B, and add or subtract the appropriate element from A. The midpoints are obtained directly from B. □

LEMMA 1.4. *Let $\{A, B\}$ be an $m + n$ non-inclusive sum-and-distance system. Then all elements of A are even and all elements of B are odd, or vice versa.*

Proof. If without loss of generality the set A contained both an even and an odd integer element then these could be combined with the same element of B to give an even and odd sum (or difference) in the sum-and-distance set. However the target set only contains odd numbers and so we obtain a contradiction. Hence the elements of A are either all odd or all even with the elements of B being of the opposite parity. □

1.3 Connection to Sum Systems

Definition (of sum systems)**.** For natural numbers m and n, let

$$A = \{a_1, \ldots, a_m\}, \quad \text{and} \quad B = \{b_1, \ldots, b_n\},$$

be two sets of integers, respectively of m and n distinct elements, ordered in terms of increasing absolute value, with $A \cap B = \phi$.

Then we say that the set pair A and B form an *m+n sum system*, if and only if the corresponding *sum set*, $A+B$, is the set of consecutive integers $\{0, 1, 2, \ldots, mn-1\}$.

There exists a close relationship between sum-and-distance systems and sum systems, as for each sum-and-distance system there is a corresponding sum system, as described in the following theorem.

THEOREM 1.5. *For each $m+m$ sum-and-distance system $\{A,B\}$, there is a corresponding $2m+2m$ sum system $\{A',B'\}$, such that $A^*B = A'+B'$.*

Similarly, for each $m+m$ inclusive sum-and-distance system $\{A,B\}$, there is a corresponding $(2m+1)+(2m+1)$ sum system $\{A',B'\}$, such that $A\cup B\cup A^*B = A'+B'$.

Proof. Let $\{A,B\}$ be a non-inclusive $m+m$ sum-and-distance system with

$$A = \{a_1,\ldots,a_m\}, \qquad B = \{b_1,\ldots,b_m\},$$

so that

$$(A+B)\cup|A-B| = \{1,3,\ldots,(2m)^2-1\}.$$

We now define the two-step map f as follows.

Step One

Consider the extended $2m+2m$ system given by

$$\{\{-A\cup A\},\{-B\cup B\}\} = \{\{-a_m,\ldots,-a_1,a_1,\ldots,a_m\},\ \{-b_m,\ldots,-b_1,b_1,\ldots,b_m\}\}.$$

The corresponding sum-set comprising all sums of pairs of elements can then be written as

$$\begin{aligned}\{-A\cup A\}+\{-B\cup B\} &= \{\pm a_j \pm b_k | a_j\in A,\ b_k\in B\}\\ &= \{\pm1,\pm3,\ldots,\pm((2m)^2-1)\}.\end{aligned}$$

Step Two

Now add a_m and b_m to the sets $\{-A\ \cup\ A\}$ and $\{-B\ \cup\ B\}$ respectively to obtain the system,

$$\{\{-A\cup A\}+a_m,\ \{-B\cup B\}+b_m\} = \{\{-a_m+a_m,\ldots,-a_1+a_m,a_1+a_m,\ldots,a_m+a_m\},$$

$$\{-b_m+b_m,\ldots,-b_1+b_m,b_1+b_m,\ldots b_m+b_m\}\}.$$

As a_m and b_m are the largest two entries in either A and B their sum must equal $(2m)^2 - 1$. Hence we have that

$$\begin{aligned}\{-A \ \cup \ A\} + a_m + \{-B \ \cup \ B\} + b_m =& \{\pm 1, \pm 3, \ldots, \pm((2m)^2 - 1)\} + a_m + b_m \\ &= \{\pm 1, \pm 3, \ldots, \pm((2m)^2 - 1)\} + (2m)^2 - 1 \\ &= \{0, 2, 4, \ldots, 2((2m)^2 - 1)\}.\end{aligned}$$

By Lemma 1.4, either the set A contains all even integers and the set B all odd integers or vice-versa. It follows that we can create the two sets of integers A' and B', by setting

$$A' = \frac{1}{2}\left(\{-A \ \cup \ A\} + a_m\right), \quad B' = \frac{1}{2}\left(\{-B \ \cup \ B\} + b_m\right)$$

we therefore have that $\{A', B'\}$ form a $2m \ + \ 2m$ sum system with

$$A' + B' = \{0, 1, 2, \ldots, 2((2m)^2 - 1)\}.$$

Similarly for an inclusive $m + m$ sum-and-distance system $\{A, B\}$ we have

$$A = \{a_1, \ldots, a_m\}, \quad B = \{b_1, \ldots, b_m\}$$

with

$$A + B \cup |A - B| \cup A \cup B = \{1, 2, \ldots, 2m^2 + 2m\}.$$

In **step one** we now include the element $\{0\}$ to obtain

$$\{\{-A \ \cup \ \{0\} \ \cup A\}, \ \{-B \cup \ \{0\} \ \cup B\}\} = \{\{-a_m, \ldots, -a_1, 0, a_1, \ldots, a_m\}$$

$$\{-b_m, \ldots, -b_1, 0, b_1, \ldots, b_m\}\},$$

so that

$$\begin{aligned}\{-A \cup \{0\} \cup A\} + \{-B \cup \{0\} \cup B\} &= \{\pm a_j \pm b_k, \ \pm a_j, \ \pm b_k, \ 0 \ |a_j \in A, \ b_k \in B\} \\ &= \{0, \pm 1, \pm 2, \ldots, \pm(2m^2 + 2m)\}.\end{aligned}$$

Step two remains unchanged with a_m and b_m added to the sets $\{-A \ \cup \{0\} \cup A\}$ and $\{-B \cup \{0\} \cup B\}$ respectively such that

$$\{\{-A, \{0\}, A\} + a_m, \{-B, \{0\}, B\} + b_m\} = \{\{-a_m + a_m, \ldots, -a_1 + a_m, a_m, a_1 + a_m, \ldots, a_m + a_m\},$$

$$\{-b_m + b_m, \ldots, -b_1 + b_m, b_m, b_1 + b_m, \ldots, b_m + b_m\}\}.$$

Here $a_m + b_m = 2m^2 + 2m$, so that $\{-A \cup \{0\} \cup A\} + a_m + \{-B \cup \{0\} \cup B\} + b_m$

$$= \{0, \pm 1, \pm 2, \ldots, \pm(2m^2 + 2m)\} + a_m + b_m$$

$$= \{0, \pm 1, \pm 2, \ldots, \pm(2m^2+2m)\} + 2m^2 + 2m = \{0, 1, 2, \ldots, 2(2m^2+2m)\}.$$

Setting

$$A' = \{-A \cup \{0\} \cup A\} + a_m, \quad B' = \{-B \cup \{0\} \cup B\}$$

we see that $\{A', B'\}$ is a $(2m+1)+(2m+1)$ sum system.

Hence with each $m+m$ sum-and-distance system $\{A, B\}$, there is a corresponding $2m+2m$ sum system $\{A', B'\}$, such that $A^*B = A' + B'$, for each $m+m$ inclusive sum-and-distance system $\{A, B\}$, there is a corresponding $(2m+1)+(2m+1)$ sum system $\{A', B'\}$, such that $A \cup B \cup A^*B = A' + B'$.

□

Example. *Consider the* $3+3$ *sum-and-distance system* $\{\{1, 7, 9\}, \{2, 22, 26\}\}$*. Applying the linear map* f *described in the proof of Theorem 1.5 we have*

$$\{\{1, 7, 9\}, \{2, 22, 26\}\} \longrightarrow \{\{-9, -7, -1, 1, 7, 9\}, \{-26, -22, -2, 2, 22, 26\}\},$$

and

$$\{\{-9, -7, -1, 1, 7, 9\}, \{-26, -22, -2, 2, 22, 26\}\},$$
$$\longrightarrow \{\{0, 2, 8, 10, 16, 18\}, \{0, 4, 24, 28, 48, 52\}\}$$

with the final step being

$$\{\{0, 2, 8, 10, 16, 18\}, \{0, 4, 24, 28, 48, 52\}\} \longrightarrow \{\{0, 1, 4, 5, 8, 9\}, \{0, 2, 12, 14, 24, 26\}\},$$

which is a $6+6$ *sum system.*

Similarly, considering the $3+3$ *inclusive sum-and-distance system and applying the above mapping described in the proof of the lemma we have*

$$\{\{1, 2, 3\}, \{7, 14, 21\}\} \longrightarrow \{\{-3, -2, -1, 0, 1, 2, 3\}, \{-21, -14, -7, 0, 7, 14, 21\}\},$$

and

$$\{\{-3, -2, -1, 0, 1, 2, 3\}, \{-21, -14, -7, 0, 7, 14, 21\}\}$$
$$\longrightarrow \{\{0, 1, 2, 3, 4, 5, 6\}, \{0, 7, 14, 21, 28, 35, 42\}\},$$

which is the canonical $7+7$ *sum system.*

Motivation

Much work concerning sum systems has been undertaken, including identifying arithmetic progressions found within the set produced by all possible sums of the two sets. One such result proven by Bourgain in [12], says that if the sets C and D have size γN and δ respectively then for N sufficiently large, there exists a fixed constant $c > 0$ such that $C + D$ contains an arithmetic progression of length at least

$$exp(c(\delta\gamma logN)^{1/3} - loglogN).$$

Other work concerning repeating sum systems, $A + A := \{a_1 + a_2 \mid a_i \in A\}$ has been extensively studied, along with repeating difference systems, $A - A := \{a_1 - a_2 \mid a_i \in A\}$ in papers [43],[51] and [45] in which the cardinalities of both sets were compared. Further in a 2005 paper by Solymosi, [53], repeating sum systems and product systems, $A, A = \{a_1.a_2 \mid a_i \in A\}$ are considered independently and their cardinalities compared.

However the important concept of constructing a sum system, or indeed a sum-and-distance system with a specified resulting sum set, or sum-and-distance set appears to have been overlooked in the literature, thus partially motivating these results. Moreover the foundations of our work were unknowingly laid by Ollerenshaw and Brée [10], who studied and enumerated all $n \times n$ reversible square matrices of even order $n = 4k$, but were not aware of the block representation which we develop in this thesis that establishes a direct link to sum-and-distance systems. In fact our enumeration argument of all $m + m$ sum-and-distance systems, involves the jth non-trivial divisor function $c_j(n)$, which counts the total number of proper ordered factorisations of the integer $n = p_1^{a_1} \dots p_t^{a_t}$ into j parts. We show that the total number of $n + n$ principal reversible square matrices (for any $n \in \mathbb{N}$), and so sum-and-distance systems, N_n, is given by

$$N_n = \sum_{j=1}^{\Omega(n)} \left(c_j(n)^2 + c_{j+1}(n) c_j(n) \right).$$

Such sums have been of interest in a different guise where one obtains asymptotic bounds, as discussed in a recent talk (1 September 2017) by the eminent number theorist Professor Christopher Hooley. Hooley disclosed that one of his PhD problems (set by his PhD supervisor Mr A. E. Ingham) was to obtain asymptotics for the sum

$$\sum_{n \leq x} d_2(n) d_3(n+a), \qquad a \neq 0,$$

where $d_j(n)$ is the jth divisor function, which counts the number of essentially different ways of writing n as the ordered product of j positive integer factors ≥ 1. Hooley solved this problem, which led to him being awarded a Corpus Christi Prize Fellowship. However the next divisor function problem set by Ingham was to obtain asymptotics for the sum

$$\sum_{n \leq x} d_3(n) d_3(n+a), \qquad a \neq 0,$$

which at the time of writing is still unsolved.

According to Professor M. N. Huxley, although there have been many attempts to bound sums of products of divisor functions, this appears to be the first instance of such a sum occurring naturally in a mathematical counting argument, and as such is of considerable interest.

1.4 Overview of Main Results and Thesis Structure

Having now introduced the sum-and-distance systems of both varieties we briefly outline the content, structure and main results contained in this thesis. In Chapter 2 we formally introduce the divisor functions $d_j(n)$ and $c_j(n)$ and some new and existing properties concerning them. In Chapter 3 we introduce reversible square matrices and their conjugated block representation. The block representation then enables to establish (Chapter 4) a bijection between the set of all $m + m$ sum-and-distance systems and the set of all $2m + 2m$ principal reversible squares, and between the set of all $m + m$ inclusive sum-and-distance systems and the set of all $(2m + 1) + (2m + 1)$ principal reversible squares; those whose top row begins with $1, 2, \ldots$, and whose individual row and column sequences are all increasing, respectively left to right and top to bottom. In Chapter 5, by reworking the block enumeration argument of K. Ollerenshaw and D. Brée in (see [10]) terms of the divisor functions $d_j(n)$ and $c_j(n)$, we enumerate the total number of $m+m$ sum-and-distance systems, N_{2m} for non-inclusive systems, and N_{2m+1} for inclusive systems, as described in Corollary 5.6.

Building on our understanding of the divisor path enumeration of Chapter 5 and the bijection between sum-and-distance systems and reversible squares of Chapter 4, we deduce in Chapter 6, an algorithmic approach to constructing all inclusive and non-inclusive sum-and-distance systems. The fact that the construction is both exhaustive and non-repetitive is establishing by a geometric lattice point argument in $\mathbb{Z}^n$. Furthermore it is shown that when $n = q^r$, with r maximal, then the number of parasymmetric $m + m$ sum-and-distance systems (those whose two ordered sets are linearly dependent) is given by $d_2(r)$, the standard divisor function.

With the enumeration and construction of all $m + m$ sum-and-distance systems established, we then revisit the Most Perfect Squares of Ollerenshaw and Brée and derive a conjugated block representation for all such squares (Chapter 7). From this block representation, in Chapter 8 we deduce a new proof for the bijection (first discovered by Ollerenshaw) between the number of *most-perfect squares* and the number of *principal reversible squares*. Moreover we go on to establish that there exists superalgebras of matrices related to these matrix symmetries and give an overview of these algebraic structures. Finally in Chapter 9 we consider the type A and B matrices studied by Brunnock, Lettington and Schmidt [14], and establish two-sided eigenvector matrices for rank $1 + 1$ constructions, from which we obtain quadratic forms.

Having outlined the thesis structure and topics we now give an overview of the main results obtained in each research direction.

Divisor Functions. The *jth divisor function*, denoted by $d_j(n)$, counts the number of essentially different ways of writing n as the ordered product of j positive integer factors ≥ 1, so that $d_1(n) = 1$ for all positive integers n. Similarly the jth *non-trivial divisor function*, denoted by $c_j(n)$, counts the number of essentially different ways of writing n as the ordered product of j positive integer factors with each factor ≥ 2. Hence $c_j(n)$ counts the total number of 'proper ordered factorisations' of n.

It is known that the Dirichlet series of these jth divisor function, $d_j(n)$, $c_j(n)$, are respectively given by

$$\zeta(s)^j = \sum_{n=1}^{\infty} \frac{d_j(n)}{n^s}, \qquad (\zeta(s) - 1)^j = \sum_{n=1}^{\infty} \frac{c_j(n)}{n^s},$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function. However the second arithmetic function $c_j(n)$ is not multiplicative and is far less well studied than its multiplicative cousin $d_j(n)$. In Chapter 2 we consider these two arithmetic functions in more detail, establishing relationships between $c_j(n)$ and $d_j(n)$.

With $c_j^{(0)}(n) = c_j(n)$, set to be the non-trivial divisor function $c_j(n)$, we define the mixed divisor function $c_j^{(r)}(n)$ such that

$$c_j^{(r)}(n) = \sum_{m|n} c_j^{(r-1)}(m),$$

and in Lemma 2.16 establish the corresponding Dirichlet series

$$\sum_{n=1}^{\infty} \frac{c_j^{(r)}(n)}{n^s} = \zeta(s)^r \left(\zeta(s) - 1\right)^j .$$

Let n have the prime factorisation $n = p_1^{a_1} \dots p_t^{a_t}$. Then with the divisor functions as defined above, we go on to show in Lemma 2.14 that

$$c_j^{(r)}(n) = \sum_{i=0}^{r} \binom{r}{i} c_{j+i}^{(0)}(n) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} d_{i+r}(n)$$
$$= \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \prod_{k=1}^{t} \binom{a_k + i + r - 1}{i + r - 1}.$$

Reversible Square Matrices. In Chapter 3 we consider the sets of $n \times n$ *reversible square matrices*, $R = (r_{i\,j})_{n \times n}$, defined to be the $n \times n$ matrices which satisfy for all i, j, i', j', the three symmetry conditions

(1) (reverse row similarity) $r_{i,j} + r_{i,n+1-j} = r_{i,j'} + r_{i,n+1-j'}$,

(2) (reverse column similarity) $r_{i,j} + r_{n+1-i,j} = r_{i',j} + r_{n+1-i',j}$,

(3) (equal cross sums property) $r_{i,j} + r_{i',j'} = r_{i,j'} + r_{i',j}$.

Building upon the work of K. Ollerenshaw and D. Brée, we extend their results to all $n \in \mathbb{N}$, rather than just the doubly-even case $n = 4k$ considered by them. We go on to show in Theorem 4.3 that with each principal reversible square matrix one can associate a unique *sum-and-distance system*, inclusive for odd-sided reversible squares and non-inclusive for even sided reversible squares. We thus obtain in Theorem 5.6 the explicit formula (not given in [10]) which says that the total number of principal reversible squares N_n is given by

$$N_n = \sum_{j=1}^{\Omega(n)} \left(c_j(n)^2 + c_{j+1}(n)c_j(n) \right) = \sum_{j=1}^{\Omega(n)} c_j^{(0)}(n)c_j^{(1)}(n).$$

Block Representations. To enable us to establish a bijection between the set of $n + n$ principal reversible squares and the set of $\lfloor n/2 \rfloor + \lfloor n/2 \rfloor$ sum-and-distance systems of the corresponding variety, we derive a rank $1 + 1$ block representation for all reversible square matrices, which is evident after conjugation with a unitary matrix operator (see Section 3.1). Let E_n be the $n \times n$ matrix with all entries equal to 1, $\hat{0}_n$ be the $n \times n$ matrix with all entries equal to 0 and 1_n the $n \times 1$ column vector with all entries equal to 1. Then in Theorem 3.13 we show that $M \in \mathbb{R}^{n \times n}$ is a reversible square if and only if it has the block representation,

even $n = 2k$

$$M = X_n \begin{pmatrix} wE_k & 1_k a^T \\ b1_k^T & \hat{0}_k \end{pmatrix} X_n$$

with $a, b \in \mathbb{R}^k$ and $w \in \mathbb{R}$,

odd $n = 2k + 1$

$$M = X_n \begin{pmatrix} \sqrt{2}wE_k & w1_k & \sqrt{2}1_k d^T \\ w1_k^T & w & d^T \\ \sqrt{2}b1_k^T & b & \hat{0}_k \end{pmatrix} X_n$$

with $a, b, c, d \in \mathbb{R}^k$ and $w \in \mathbb{R}$.

Sum-and-distance Systems, Binomial Enumeration and Prime Numbers. Utilising our block representation for reversible square matrices, we establish in Corollary 4.4 a bijection between the set of $n \times n$ principal reversible squares and the set of $\lfloor n/2 \rfloor + \lfloor n/2 \rfloor$ sum-and-distance systems of the corresponding variety. Thus in our enumeration argument for all $m + m$ sum-and-distance systems, we deduce that the total number of traditional principal reversible squares N_n, of size n, is equal to the number of $m + m$ non-inclusive sum-and-distance systems if $n = 2m$ is even, and the number of $m + m$ inclusive sum-and-distance systems if $n = 2m + 1$ is odd.

Let n have the prime factorisation $n = p_1^{a_1} \dots p_t^{a_t}$, so that n has t distinct prime factors and $\Omega(n) = a_1 \dots + a_t$ prime factors in total. From Corollary 5.6 we have that the total number of $\lfloor n/2 \rfloor + \lfloor n/2 \rfloor$ sum-and-distance systems, non-inclusive when n is even and inclusive when n is odd, is given by

$$N_n = \sum_{j=1}^{\Omega(n)} \left(c_j(n)^2 + c_{j+1}(n) c_j(n) \right),$$

and using the explicit binomial forms for $c_j^{(r)}(n)$ given in Chapter 2 we can write this as

$$N_n = \sum_{j=1}^{\Omega(n)} \left(\sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} \prod_{k=1}^{t} \binom{a_k + i - 1}{i - 1} \right) \left(\sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \prod_{k=1}^{t} \binom{a_k + i}{i} \right).$$

Theorem 5.8 then gives us the primality test that $n \geq 2$ is a prime number if and only if $N_n = 1$, so that when $n = 2$, the only $1 + 1$ sum-and-distance system is $\{\{1\}, \{2\}\}$, and when $n = 2m + 1$, the only $m + m$ inclusive sum-and-distance system is again the canonical system given by

$$A = \{1, 2, 3, \ldots m\}, \qquad B = \{n, 2n, 3n, \ldots mn\} = nA.$$

Hence with each prime number we can associate a unique sum-and-distance system.

Construction of all Sum-and-distance Systems. The construction algorithm for the sum-and-distance systems cannot be so easily stated as it requires a substantial amount of theory and notation which is given explicitly in Chapter 6. However a construction algorithm for all sum-and-distance systems is obtained with the main results stated in Theorems 6.4 and 6.6.

2 The jth Divisor Functions $d_j(n)$ and $c_j(n)$

The *jth divisor function*, denoted by $d_j(n)$, counts the number of different ways of writing n as the ordered product of j positive integer factors ≥ 1, so that 2×3 and 3×2 are counted as different factorisations of the number 6. Hence $d_1(n) = 1$ for all positive integers n, and traditionally the most studied of all the divisor functions $d_2(n)$, is denoted simply by $d(n)$.

Remark. As $d_2(n)$ counts the number of ordered pairs of divisors whose product is n, so that each pair with distinct factors is counted twice, it also counts the number of divisors of n.

Similarly the *jth non-trivial divisor function*, denoted by $c_j(n)$, counts the number of different ways of writing n as the ordered product of j positive integer factors with each factor ≥ 2. Hence $c_j(n)$ counts the total number of 'proper ordered factorisations' of n.

Example. *When $j = 2$ and $n = 12$ we have*

$$d_2(12) = |\{1,2,3,4,6,12\}| = |\{(1\times 12),(2\times 6),(3\times 4),(4\times 3),(6\times 2),(12\times 1)\}| = 6,$$

$$c_2(12) = |\{(2\times 6),(3\times 4),(4\times 3),(6\times 2)\}| = 4.$$

2.1 A Brief History of $d_2(n)$

The divisor functions belong to the class of arithmetic functions; those functions f which map the positive integers into the complex plane so that $f : \mathbb{N} \to \mathbb{C}$. By far the most famous of the divisor functions is $d_2(n)$, which arises in some classical number theoretic problems such as Dirichlet's Divisor Problem and the Gauss Circle problem. These problems have attracted the attention of some of the greatest number theorists of modern times including Littlewood [42], Hardy [26], Heath-Brown [28] and Huxley [34] [35].

In contrast there appears to have been much less study undertaken of the jth divisor function $d_j(n)$, and in the literature it is even harder to find reference to the jth non-trivial divisor function $c_j(n)$. In [55], Titchmarsh considered the problem of bounding the sum of the kth divisor functions for all positive integers up to x, denoted by $D_k(x)$, so that

$$D_k(x) = \sum_{n\leq x} d_k(n).$$

He obtained the asymptotic formula

$$D_k(x) = xP_k(\log x) + \Delta_k(x)$$

where

$$\Delta_k(x) = \mathcal{O}(x^{1-1/k}\log^{k-2} x) \qquad (k = 2, 3, \ldots)$$

and P_k is a polynomial of degree $k-1$.

Our motivation for studying these general divisor functions in more detail originates in the enumeration and construction problem concerning the number of *sum-and-distance systems* introduced in Chapter 1. However before proceeding in our study of these combinatorial objects and their number theoretic properties we first give a brief overview of the classical results concerning $d_2(n)$.

Definition (of the sum of number of divisors function)**.** Let x be a positive integer then we define $D(x)$ to be the *sum of the number of divisors* for all positive integers $n \leq x$, so that

$$D(x) = \sum_{n \leq x} d_2(n).$$

Around 200 years ago, Dirichlet gave the estimation for $D(x)$

$$D(x) = x \log x + (2\gamma - 1)x + \mathcal{O}(\sqrt{x}),$$

with γ the Euler-Mascheroni constant given by

$$\gamma = \lim_{x \to \infty} \left(-\log x + \sum_{k=1}^{x} \frac{1}{k} \right) = 0.57721\ldots,$$

and where the usual 'big $\mathcal{O}$' notation is employed (see for example [57]).

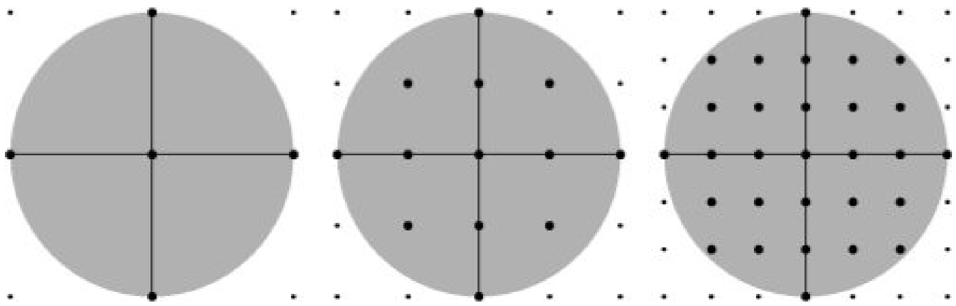
Replacing the upper bound exponent of 1/2 in the error term with θ, so that

$$D(x) = x \log x + (2\gamma - 1)x + \mathcal{O}(x^{\theta}),$$

it was shown by Hardy and Landau in 1916 [26] that θ has the lower bound $\theta \geq \frac{1}{4}$ [26].

During the last century the challenge of improving this upper bound of 1/2 has attracted some of the greatest mathematicians such as Littlewood and Walfisz 1925 [42], van der Corput 1928 [15], Vinogradov 1935 [56], Iwaniec and Mozzochi 1988 [36], Huxley 2003 [34] and most recently $\theta = \frac{517}{1648} = 0.31371\ldots$ by Bourgain and Watt 2017 [13].

Closely related to Dirichlet's divisor problem is Gauss's circle problem, whose exponent differs by only a factor of 2. Gauss's circle problem is to count the number of lattice points, denoted by $N(r)$, inside the circle of radius r, centre the origin.



Gauss showed that

$$N(r) = \pi r^2 + E(r),$$

with

$$|E(r)| \leq 2\sqrt{2}\pi r.$$

Subsequently it was shown that

$$|E(r)| \leq C r^{\theta'},$$

where $\theta' = 2\theta$, being twice that of the exponent θ found in Dirichlet's divisor problem [34].

Remark. Although we are primarily concerned with understanding the properties of the non-trivial divisor function $c_j(n)$, which occurs in our enumeration arguments, as this function is not multiplicative, it is helpful for us to understand $c_j(n)$ in terms of the multiplicative function $d_j(n)$.

2.2 Properties of $d_j(n)$

We begin by stating some well known results for the divisor function $d_2(n)$.

LEMMA 2.1. *Let p be any prime number and n any positive integer, then*

$$d_2(p^n) = n + 1.$$

LEMMA 2.2. *Let k and s_i be a positive integers, for $i = 1, \ldots, k$ with the p_i distinct prime numbers. Then*

$$d_2(p_1^{s_1} p_2^{s_2} \ldots p_k^{s_k}) = (s_1 + 1)(s_2 + 1) \ldots (s_k + 1) = \prod_{i=1}^{k} (s_i + 1).$$

For proofs of Lemmas 2.1 and 2.2 we refer the reader to [58] and [59] (pp.239) respectively.

The *jth* divisor function satisfies a sum-over-divisors recurrence relation, is multiplicative and can be expressed in terms of binomial coefficients, as described in the following lemmas.

LEMMA 2.3. *Let $j, n \in \mathbb{N}$, then the jth divisor function satisfies the sum-over-divisors recurrence relation*

$$d_j(n) = \sum_{m|n} d_{j-1}(m).$$

Proof. For a proof of this result we refer the reader to pages 334-335 of [8].

LEMMA 2.4. *The jth divisor function $d_j(n)$, is a multiplicative arithmetic function, so that for n a positive integer with n having prime factorisation $n = p_1^{s_1} p_2^{s_2} \ldots p_k^{s_k}$,* *we have that*

$$d_j(n) = d_j(p_1^{s_1} \dots p_k^{s_k}) = d_j(p_1^{s_1}) d_j(p_2^{s_2}) \dots d_j(p_k^{s_k}).$$

Proof. For a proof of this result we refer the reader to [37].

LEMMA 2.5 (*Binomial coefficient representation lemma*)**.** *Let p be a prime number and s a positive integers. Then the jth divisor function of p^s, $d_j(p^s)$, can be written as*

$$d_j(p^s) = \frac{1}{(j-1)!} \prod_{l=1}^{j-1} (s+l),$$

or equivalently as the binomial coefficient

$$d_j(p^s) = \binom{s+j-1}{j-1}.$$

COROLLARY 2.6. *Let n, j and k be positive integers with n having prime factorisation $n = p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}$. Then the jth divisor function of n, $d_j(n)$ can be written as*

$$d_j(n) = d_j(p_1^{s_1} p_2^{s_2} \dots p_k^{s_k}) = \prod_{i=1}^{k} \frac{1}{(j-1)!} \prod_{l=1}^{j-1} (s_i + l) = \left(\frac{1}{(j-1)!}\right)^k \prod_{i=1}^{k} \prod_{l=1}^{j-1} (s_i + l),$$

which has the equivalent binomial coefficient representation

$$d_j(n) = \prod_{i=1}^{k} \binom{s_i + j - 1}{j-1}.$$

Proof. For a proof of Lemma 2.5 we refer the reader to page 3 of [18], with the equivalence of the two forms then following by algebraic manipulation.

The Corollary then follows immediately by Lemma 2.4.

Remark. It is straightforward to demonstrate by counter-example that the jth divisor function is not completely multiplicative. For example, when $n = 20$ we have

$$d_2(20) = 6, \text{ and } d_2(10)d_2(2) = 4 \times 2 = 8,$$

which demonstrates that

$$d_2(20) \neq d_2(10)d_2(2).$$

Definition (of the Riemann zeta function)**.** For $s \in \mathbb{C}$, with $\Re(s) > 1$, we employ the standard notation and denote by $\zeta(s)$ the *Riemann zeta function*, so that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Definition. Let a_n, $n \in \{1, 2 \dots\}$, be an infinite sequence (possibly periodic) of complex numbers. Then the *Dirichlet series* corresponding to a_n is defined to be

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

with s a complex variable.

LEMMA 2.7 (First Dirichlet series lemma)**.** *For positive integer $j \geq 1$, and $s \in \mathbb{C}$, a complex variable, with $\Re(s) > 1$, the divisor function $d_j(n)$ has the Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{d_j(n)}{n^s} = \zeta(s)^j.$$

COROLLARY 2.8. *For $i, j \in \mathbb{N}$ and $s \in \mathbb{C}$ we have*

$$\left(\sum_{n=1}^{\infty} \frac{d_i(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{d_j(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{d_{j+i}(n)}{n^s}.$$

Proof. For this result we refer the reader to page 4 of Titchmarsh's book [55]. To see the Corollary, by the statement of the Lemma we have

$$\left(\sum_{n=1}^{\infty} \frac{d_i(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{d_j(n)}{n^s}\right) = \zeta(s)^i\zeta(s)^j = \zeta(s)^{i+j} = \sum_{n=1}^{\infty} \frac{d_{i+j}(n)}{n^s},$$

as required. □

Remark. As $d_j(1) = 1$ for all $j \in \{1, 2, 3, \ldots\}$, a common extension for the case $j = 0$ is

$$d_0(n) = \begin{cases} 1, & n = 1, \\ 0, & n > 1, \end{cases}$$

which yields

$$\sum_{n=1}^{\infty} \frac{d_0(n)}{n^s} = \frac{d_0(1)}{1^s} + \sum_{n=2}^{\infty} \frac{d_0(n)}{n^s} = \zeta(s)^0 = 1,$$

thus extending the statements of Lemma 2.7 and Corollary to the case of $j = 0$.

There are variations of the Dirichlet series associated with $d_j(n)$, such as

$$\frac{\zeta(s)^3}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d_2(n^2)}{n^s} \quad \text{and} \quad \frac{(\zeta(s))^4}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{(d_2(n))^2}{n^s}$$

(see [55] pp.5-7).

2.3 Properties of $c_j(n)$

As previously defined, we denote by $c_j(n)$, the number of ordered non-trivial factorisations $n = m_1 m_2, \cdots, m_j$, where each $m_i \geq 2$. We begin with an divisor sum recurrence for $c_j(n)$ analogous to that given for $d_j(n)$ in Lemma 2.3.

LEMMA 2.9. *Let $j, n \in \mathbb{N}$ with $j \geq 2$, then the jth non-trivial divisor function* *satisfies the recurrence relation*

$$c_j(n) = \sum_{\substack{m|n \\ m<n}} c_{j-1}(m) = \sum_{\substack{m|n \\ m\notin\{1,n\}}} c_{j-1}(m).$$

Proof. Let $n \in \mathbb{N}$ with $n = m_1 m_2 \ldots m_j$ be any ordered non-trivial factorisation into j factors. Then m_j can be any non-trivial divisor of n, so $m = \frac{n}{m_j} = m_1 \ldots m_{j-1}$ is any divisor of n, and $m_1 \ldots m_{j-1}$ is any non-trivial ordered factorisation of m.

If $m = n$, then this gives $m_j = 1$, which is impossible as each of the m_i are non-trivial factors, so $m \neq n$, and for $m < n$, there are $\sum_{m|n} c_{j-1}(m)$ distinct j-factor ordered non-trivial factorisations of n, and hence the first sum. To see the second sum we note that as the factor 1 is never counted we have $c_j(1) = 0$ for any $j \geq 1$, and hence the second sum.

□

Remark. Given that 2 is smallest prime, it follows that the smallest number with j non-trivial factors is $n = 2^j$. Hence if $n < 2^j$ then $c_j(n) = 0$.

LEMMA 2.10 (Second Dirichlet series lemma)**.** *For $s \in \mathbb{C}$, a complex variable, with $\Re(s) > 1$, the non-trivial jth divisor function has the Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{c_j(n)}{n^s} = (\zeta(s) - 1)^j .$$

Proof. For this result we refer the reader to page 7 of Titchmarsh's book [55]. □

When studying this function, the added complication that arises is that unlike $d_j(n)$, it is *not* a multiplicative arithmetic function. Hence in order to understand the less symmetric multiplicative properties of $c_j(n)$, it makes sense to try and express it in terms of its multiplicative cousin $d_j(n)$.

When $j = 2$ the non-trivial divisors for any $n > 1$ are all the divisor pairs not including 1 and n, and hence the number of non-trivial divisors is equal to the number of divisors minus 2, so that

$$c_2(n) = d_2(n) - 2 = d_2(n) - 2d_1(n) = d_2(n) - 2d_1(n) + d_0(n), \tag{2.1}$$

as $d_0(n) = 0$, for $n > 1$. However when considering the combinatorial definition of $c_2(n)$ with $n = 1$, we have $c_2(1) = 0$, so that the above becomes

$$c_2(1) = 0 = d_2(1) - 2 + 1 = d_2(1) - 2d_1(1) + d_0(1),$$

which is only true if we assume that $d_0(1) = 1$. Under this assumption we can therefore write

$$c_2(n) = d_2(n) - 2d_1(n) + d_0(n), \qquad \forall n \in \mathbb{N}.$$

Now taking $j = 3$, we have that $c_3(n)$ counts the number of ways of writing n as the product of three non-trivial divisors, and using Lemma 2.9 in conjunction with (2.1), and that for $n \geq 1$, $d_1(n) = 1$, we have

$$c_3(n) = \sum_{\substack{m|n \\ m\notin\{1,n\}}} c_2(m) = \sum_{\substack{m|n \\ m\notin\{1,n\}}} (d_2(m) - 2)$$

$$= \sum_{m|n} (d_2(m) - 2) - (d_2(n) - 2) - (d_2(1) - 2)$$

$$= \sum_{m|n} (d_2(m) - 2d_1(m)) - (d_2(n) - 2d_1(n)) - (d_1(n) - 2d_1(n))$$

$$= d_3(n) - 3d_2(n) + 3d_1(n) - d_0(n),$$

where we have again assumed that $d_0(n) = 0$, unless $n = 1$ when $d_0(1) = 1$.

The emerging binomial pattern is summarised in the following lemma.

LEMMA 2.11. *For integers $n \geq 1$, $j \geq 0$, the jth non-trivial divisor function $c_j(n)$, can be written in terms of the divisor function $d_j(n)$ such that*

$$c_j(n) = \sum_{i=0}^{j} (-1)^i \begin{pmatrix} j \\ i \end{pmatrix} d_{j-i}(n),$$

with the convention that $c_0(n) = d_0(n) = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases}$

Proof. For suitable $s \in \mathbb{C}$, by Lemma 2.10 the Dirichlet series for $c_j(n)$ is given by

$$\sum_{n=1}^{\infty} \frac{c_j(n)}{n^s} = (\zeta(s) - 1)^j = \sum_{i=0}^{j} (-1)^i \begin{pmatrix} j \\ i \end{pmatrix} \zeta^{j-i}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{i=0}^{j} (-1)^i \begin{pmatrix} j \\ i \end{pmatrix} d_{j-i}(n),$$

and by the uniqueness of Dirichlet series, [31], equating terms we obtain

$$c_j(n) = \sum_{i=0}^{j} (-1)^i \begin{pmatrix} j \\ i \end{pmatrix} d_{j-i}(n).$$

From the definitions of $c_j(n)$ and $d_j(n)$, for $n = 1$ and any $j \geq 1$ we have $c_j(1) = 0$, $d_j(1) = 1$. In consideration of the above sum with $d_0(n)$ as defined above, this gives

$$c_j(1) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} d_{j-i}(1) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} = 0,$$

$c_0(1) = d_0(1) = 1$, and for $n \geq 2$, $c_0(n) = d_0(n) = 0$, as required.

Therefore the formula is consistent for all integers $n \geq 1$, $j \geq 0$, and hence the result.

□

LEMMA 2.12. *Let p be a prime number and a a positive integer. Then*

$$c_j(p^a) = \binom{a-1}{j-1}.$$

Proof. By Lemmas 2.5 and 2.11 we can write

$$c_j(p^a) = \sum_{i=0}^{j}(-1)^i \binom{j}{i} d_{j-i}(p^a) = \sum_{i=0}^{j}(-1)^i \binom{j}{i} \binom{a+j-i-1}{j-i-1}$$

$$= \sum_{i=0}^{j}(-1)^i \binom{j}{i} \binom{a+j-i-1}{a}.$$

We now apply the binomial coefficient identity (see H. W. Gould p28, (3.49) [21])

$$\sum_{\alpha=0}^{\beta}(-1)^\alpha \binom{\beta}{\alpha}\binom{\gamma-\alpha}{\delta} = \binom{\gamma-\beta}{\delta-\beta},$$

with $\alpha = i$, $\beta = j$, $\gamma = a + j - 1$ and $\delta = a$, to obtain

$$\sum_{i=0}^{j}(-1)^i \binom{j}{i} \binom{a+j-i-1}{a} = \binom{a-1}{a-j} = \binom{a-1}{j-1}.$$

□

LEMMA 2.13. *For integers $n \geq 1$, $j \geq 0$, the jth divisor function $d_j(n)$, can be written in terms of the non-trivial divisor function $c_j(n)$ such that*

$$d_j(n) = \sum_{i=0}^{j} \binom{j}{i} c_{j-i}(n),$$

with the convention that $c_0(n) = d_0(n) = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases}$

Proof. For $n \geq 2$, $j \geq 1$. we have that

$$\sum_{i=0}^{j} \binom{j}{i} c_{j-i}(n) = \sum_{i=0}^{j-1} \binom{j}{i} c_{j-i}(n),$$

as $c_0(n) = 0$.

Now the Dirichlet series for $d_j(n)$ is given by

$$\sum_{n=2}^{\infty} \frac{d_j(n)}{n^s} = \sum_{n=1}^{\infty} \frac{d_j(n)}{n^s} - 1 = \zeta(s)^j - 1,$$

and considering the Dirichlets series of $\sum_{i=0}^{j-1} \binom{j}{i} c_{j-i}(n)$, we have

$$\sum_{n=2}^{\infty} \frac{1}{n^s} \sum_{i=0}^{j-1} \binom{j}{i} c_{j-i}(n) = \sum_{i=0}^{j-1} \binom{j}{i} \sum_{n=2}^{\infty} \frac{1}{n^s} c_{j-i}(n)$$

$$= \sum_{i=0}^{j-1} \binom{j}{i} \sum_{n=1}^{\infty} \frac{1}{n^s} c_{j-i}(n) = \sum_{i=0}^{j-1} \binom{j}{i} (\zeta(s) - 1)^{j-i}$$

$$= \sum_{i=0}^{j} \binom{j}{i} (\zeta(s) - 1)^{j-i} - 1 = (1 + (\zeta(s) - 1))^j - 1 = \zeta(s)^j - 1 = \sum_{n=2}^{\infty} \frac{d_j(n)}{n^s}$$

by the uniqueness of Dirichlet series [5](pp.127).

When $n = j = 1$ we find that $d_1(1) = 1$ and $c_1(1) + c_0(1) = 0 + 1 = 1$, so that the sum agrees, and when $j = 0$, the sum gives us $d_0(n) = c_0(n)$, as required. Hence the result holds for all integers $n \geq 1$ and $j \geq 0$. □

2.4 The Arithmetic Function $c_j^{(r)}(n)$

In analogy of the recurrence relation satisfied by the jth divisor function, we define the following sum-over-divisors recurrence for the *jth mixed divisor function* $c_j^{(r)}(n)$.

Definition (of the mixed divisor function)**.** Let n, j and r be any positive integers. Then we define the *mixed divisor function* $c_j^{(r)}(n)$ such that

$$c_j^{(0)}(n) = c_j(n), \quad c_j^{(r)}(n) = \sum_{m|n} c_j^{(r-1)}(m).$$

LEMMA 2.14. *Let n, j and r be positive integers. Then the mixed divisor function $c_j^{(r)}$ can be written in terms of the divisor function $d_j(n)$ such that*

$$c_j^{(r)}(n) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} d_{i+r}(n).$$

Proof. When $r = 0$, then by Lemma 2.11 we have

$$\sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} d_{i+0}(n) = c_j(n) = c_j^{(0)}(n),$$

where we have used $d_0(n) = c_0(n) = 0$ if $n \geq 2$ and $d_0(1) = c_0(1) = 1$.

Inductively now suppose that for $t \in \mathbb{N}$ we have that

$$c_j^{(t)}(n) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} d_{i+t}(n)$$

holds. Then

$$c_j^{(t+1)}(n) = \sum_{m|n} c_j^{(t)}(m) = \sum_{m|n} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} d_{i+t}(m)$$

$$= \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \sum_{m|n} d_{i+t}(m) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} d_{i+t+1}(n),$$

by Lemma 2.3, and so the formula holds for all positive integers r. □

Remark. Due to the symmetries of the Binomial coefficients we can also write

$$c_j^{(r)}(n) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} d_{i+r}(n) = \sum_{i=0}^{j} (-1)^{i} \binom{j}{i} d_{r+j-i}(n).$$

2.5 The Dirichlet Series of $c_j^{(r)}(n)$

We first note the well known result concerning the Dirichlet convolution

LEMMA 2.15. *The multiplication of Dirichlet series is compatible with Dirichlet convolution in the sense if $g(m) = \sum_{d|m} f(d)$, then for $F(s) = \sum_{m=1}^{\infty} \frac{f(m)}{m^s}$ and $G(s) = \sum_{m=1}^{\infty} \frac{g(m)}{m^s}$, we have*

$$G(s) = \zeta(s)F(s).$$

Proof. For a proof of this result we refer the reader to [55](pp.4-5). □

LEMMA 2.16. *The mixed divisor function $c_j^{(r)}(n)$ has the Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{c_j^{(r)}(n)}{n^s} = \zeta(s)^r (\zeta(s) - 1)^j.$$

Proof. Consider the case $r = 0$. Then by Lemma 2.10 we have

$$\sum_{n=1}^{\infty} \frac{c_j^{(0)}(n)}{n^s} = \sum_{n=1}^{\infty} \frac{c_j(n)}{n^s} = (\zeta(s) - 1)^j = \zeta(s)^0 (\zeta(s) - 1)^j.$$

Now inductively suppose that for $k \in \mathbb{N}$ we have that

$$\sum_{n=1}^{\infty} \frac{c_j^{(k)}(n)}{n^s} = \zeta(s)^k (\zeta(s) - 1)^j$$

holds. Then $c_j^{(r+1)}(n) = \sum_{m|n} c_j^{(r)}(m)$, and repeatedly using Lemma 2.15, we have that

$$\sum_{n=1}^{\infty} \frac{c_j^{(k+1)}(n)}{n^s} = \zeta(s)\zeta(s)^k(\zeta(s)-1)^j = \zeta(s)^{k+1}(\zeta(s)-1)^j,$$

as required. □

LEMMA 2.17. *Let n, j and r be positive integers. Then the mixed divisor function, $c_j^{(r)}(n)$ can be written in terms of the non-trivial divisor function such that*

$$c_j^{(r)}(n) = \sum_{i=0}^{r} \binom{r}{i} c_{j+i}(n).$$

Proof. Given that

$$c_j^{(0)}(n) = \sum_{i=0}^{0} \binom{0}{i} c_{j+i}(n) = c_j(n),$$

the identity holds for $r = 0$. Now inductively assume true for $k \in \mathbb{N}$, so that

$$c_j^{(k)}(n) = \sum_{i=0}^{k} \binom{k}{i} c_{j+i}(n).$$

Considering the sum over all divisors of n then yields

$$c_j^{(r+1)}(n) = \sum_{m|n} c_j^{(r)}(n) = \sum_{m|n} \sum_{i=0}^{k} \binom{k}{i} c_{j+i}(m) = \sum_{i=0}^{k} \binom{k}{i} \sum_{m|n} c_{j+i}(m)$$

$$= \sum_{i=0}^{k} \binom{k}{i} \left(c_{j+i}(n) + \sum_{\substack{m|n \\ m \neq n}} c_{j+i}(m) \right) = \sum_{i=0}^{k} \binom{k}{i} \left(c_{j+i}(n) + c_{j+i+1}(n) \right)$$

$$= \sum_{i=0}^{k} \binom{k}{i} c_{j+i}(n) + \sum_{i=0}^{k} \binom{k}{i} c_{j+i+1}(n) = \sum_{i=0}^{k} \binom{k}{i} c_{j+i}(n) + \sum_{i=1}^{k+1} \binom{k}{i-1} c_{j+i}(n)$$

$$= c_j(n) + \sum_{i=1}^{k+1} \left(\binom{k}{i} + \binom{k}{i-1} \right) c_{j+i}(n) = c_j(n) + \sum_{i=1}^{k+1} \binom{k+1}{i} c_{j+i}(n)$$

$$= \sum_{i=0}^{k+1} \binom{k+1}{i} c_{j+i}(n),$$

as required. □

LEMMA 2.18. *Let p be a prime number and a, r and j positive integers. Then we have*

$$c_j^{(r)}(p^a) = \binom{a+r-1}{j+r-1}.$$

COROLLARY 2.19. *We have that $c_j^{(r)}(p^a)$ is equal to the number of compositions of $a+r$ elements into $j+r$ parts, so that $c_j^{(r)}(p^a)$ is equal to the number of ways of writing the integer $a+r$ as an ordered sum of $j+r$ positive integers.*

Proof. By Lemmas 2.12, 2.17 we have that

$$c_j^{(r)}(p^a) = \sum_{i=0}^{r} \binom{r}{i} c_{j+i}(p^a) = \sum_{i=0}^{r} \binom{r}{i} \binom{a-1}{j+i-1}.$$

We now apply the binomial coefficient identity (see H. W. Gould p25, (3.20) [21])

$$\sum_{\alpha=0}^{\beta} \binom{\beta}{\alpha} \binom{\gamma}{\alpha+\delta} = \binom{\beta+\gamma}{\beta+\delta},$$

with $\alpha = i,\ \beta = r,\ \gamma = a-1$ and $\delta = j-1$, to obtain

$$c_j^{(r)}(p^a) = \sum_{i=0}^{r} \binom{r}{i} \binom{a-1}{j+i-1} = \binom{a+r-1}{j+r-1},$$

as required.

To see Corollary 2.19, it is known (see R. P. Stanley p14-15 [54]) that the number of compositions of the positive integer n into k parts is given by the binomial coefficient $\binom{n-1}{k-1}$, and the result then follows. □

3 Reversible Square Matrices

In this chapter we employ and extend results obtained by the noted mathematician, politician and educationalist Dame Kathleen Ollerenshaw (1912-2014) [46, 47, 48], whose research areas ranged from convex bodies and lattice theory to 'most-perfect magic squares'. Recently (2017), in recognition of Ollerenshaw's contribution to mathematics, Manchester University created the Dame Kathleen Ollerenshaw Research Fellowships for outstanding early career researchers.

In 1998 Ollerenshaw (at the age of 86) and Brée published the book *Most-Perfect Pandiagonal Magic Squares* [10]. In this book they enumerate a class of $n \times n$ matrices known as reversible square matrices of doubly-even order, so that $n = 4k$ for some $k \in \mathbb{N}$. By establishing a bijection between the set of all $n \times n$ reversible square matrices and the set of all $n \times n$ most-perfect pandiagonal magic square matrices (defined below) they extend this enumeration to both sets of matrices. The results contained in their book motivated and laid the foundations for much of this research.

In the following two chapters we rework the nested block construction established by Ollerenshaw and Brée, with the important distinction that we extend these results to all $n \times n$ reversible square matrices with $n \in \mathbb{N}$, not just those of the form $n = 4k$. We also employ the more powerful block representation theory for these classes of matrices, which we establish in Section 3.1. From the block structures we go on to show that the set of all $n \times n$ traditional reversible square matrices can be partitioned into equivalence classes, *for any* $n \in \mathbb{N}$. As per the method of Ollerenshaw and Brée we deduce that each equivalence class can be represented by a principal reversible square, those constructed using the integers 1 to n^2, and whose top row begins with 1 and 2, and whose individual row and column sequences are all increasing, respectively left to right and top to bottom.

From this we deduce that the entries of the first row and column of a principal reversible squares form a sum system (a point not mentioned by Ollerenshaw) and in Chapter 4 establish a bijection between the set of all $n \times n$ principal reversible square matrices, and the set of all $m + m$ sum-and-distance systems, non-inclusive when $n = 2m$ is even and inclusive when $n = 2m + 1$ is odd. In order to proceed we first need some definitions.

Definition (of vector and matrix notation)**.** Throughout the remainder of this thesis we will employ the following notational conventions.

1. A vector $v \in \mathbb{R}^n$, is defined to be an $n \times 1$ column vector, so that the corresponding row vector is denoted by v^T.

2. The two vectors 0_n and 1_n are defined to be the $n \times 1$ column vectors with every entry equal to 0 or 1, respectively.

3. The set of all vectors that are orthogonal to a given vector v is denoted by

$\{v\}^{\perp}$, so that for $u \in \{v\}^{\perp}$ we have $u^T v = v^T u = 0$.

4. The product of a column vector with a row vector is defined in the usual fashion such that

$$vu^T = v_{i,1} \otimes u_{1,j} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \otimes (u_1, u_2, \ldots, u_m) = \begin{pmatrix} v_1 u_1 & v_1 u_2 & \ldots & v_1 u_m \\ v_2 u_1 & v_2 u_2 & \ldots & v_2 u_m \\ \vdots & & & \\ v_m u_1 & v_m u_2 & & v_m u_m \end{pmatrix},$$

which is also known as the *outer product of two vectors*, or as a special case of the *tensor product*.

5. Denote by δ_{ij}, the Kronecker symbol, returning 1 if $i = j$, and 0 otherwise.

6. For $M = (m_{i,j})$, an $n_1 \times n_2$ matrix, the row index i is taken over $\mathbb{Z}_{n_1} = \mathbb{Z}/n_1\mathbb{Z}_{n_1}$, the ring of integers modulo n_1, using the residue classes $\bar{1}, \bar{2}, \ldots, \bar{n}_1$, and similarly the column index j is taken over $\mathbb{Z}_{n_2} = \mathbb{Z}/n_2\mathbb{Z}_{n_2}$, the ring of integers modulo n_2, using the residue classes $\bar{1}, \bar{2}, \ldots, \bar{n}_2$.

7. Denote by I_n, the $n \times n$ identity matrix with all diagonal entries equal to 1 and 0 elsewhere, so that $I_n = (\delta_{ij})_{i,j=1}^n$.

8. Denote by J_n, the $n \times n$ matrix with all antidiagonal entries equal to 1 and 0 elsewhere, so that $J_n = (\delta_{i,n+1-j})_{i,j=1}^n \in \mathbb{R}^{n \times n}$.

9. Denote by E_n, the $n \times n$ matrix with all entries equal to 1, so that $E_n = (1)_{i,j=1}^n \in \mathbb{R}^{n \times n}$.

10. Denote by $\hat{0}_n$, the $n \times n$ null matrix with all entries equal to 0, so that $\hat{0}_n = (0)_{i,j=1}^n$.

11. We define the $n \times n$ matrix X_n such that

$$X_n = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix},$$

when $n = 2k$ is even, and

$$X_n = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0_k & J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ J_k & 0_k & -I_k \end{pmatrix}$$

when $n = 2k + 1$ is odd.

Remark. It follows that X_n is an orthogonal symmetric involution (cf. [30] pp. 165–166) as $X_n^2 = I_n$, and $X_n^T = X_n$.

Definition (of matrix weight w)**.** Let $M = (m_{i,j})$, be an $n_1 \times n_2$ matrix with real entries. We define the weight w of M, to be the average over the $n_1 n_2$ entries of

M, so that

$$w = \frac{1}{n_1 n_2} \sum_{\substack{1 \le i \le n_1 \\ 1 \le j \le n_2}} m_{i,j}.$$

If $w = 0$ then we say that M is a *weightless matrix*, or equivalently that M has *weight zero*.

LEMMA 3.1. *Let $M = (m_{i,j})$, be an $n \times n$ matrix with weight w. Then the matrix $M^0 = M - wE_n$ has weight zero.*

Proof. By definition $M - wE_n$ has weight w_0 given by

$$w_0 = \frac{1}{n^2} \sum_{1 \le i,j \le n} (m_{i,j} - w) = \frac{1}{n^2} \left(\sum_{1 \le i,j \le n} m_{i,j} \right) - \frac{1}{n^2} \left(\sum_{1 \le i,j \le n} w \right)$$

$$= w - \frac{n^2 w}{n^2} = w - w = 0,$$

as required. □

Definition (of a traditional matrix)**.** Let $M = (m_{i,j})$, be an $n_1 \times n_2$ matrix. Then we say that M is a *traditional matrix* if its $n_1 n_2$ entries are the $n_1 n_2$ consecutive integers $1, 2, \ldots n_1 n_2$, in some order.

Additionally (by Lemma 3.1), as the matrix $M^0 = M - wE_n$ has weight $w = 0$, when M is traditional we say that M^0 is a *weightless traditional matrix.*

LEMMA 3.2 (Traditional matrix lemma)**.** *Let $M = (m_{i,j})$, be an $n_1 \times n_2$ traditional matrix, so that M contains the $n_1 n_2$ consecutive integers $1, 2, \ldots n_1 n_2$. Then M has weight*

$$w = \frac{n_1 n_2 + 1}{2},$$

and the weightless traditional matrix $M^0 = M - wE_n$, contains the $n_1 n_2$ numbers

$$\frac{1 - n_1 n_2}{2}, \frac{3 - n_1 n_2}{2}, \ldots, \frac{n_1 n_2 - 3}{2}, \frac{n_1 n_2 - 1}{2}.$$

COROLLARY 3.3. *If $n_1 = n_2 = n$ say, so that M is a traditional square matrix, then $w = \frac{n^2+1}{2}$, and when $n = 2k+1$ is odd, M^0 contains the n^2 consecutive integers*

$$-2k^2 - 2k,\ 1 - 2k^2 + 2k, \ldots, -1, 0, 1, \ldots, 2k^2 + 2k - 1,\ 2k^2 + 2k,$$

and when $n = 2k$ is even, M^0 contains the n^2 consecutive odd half-integers

$$\frac{1 - 4k^2}{2}, \frac{3 - 4k^2}{2}, \ldots, \frac{-1}{2}, \frac{1}{2}, \ldots, \frac{4k^2 - 3}{2}, \frac{4k^2 - 1}{2}.$$

Proof. From the definition of matrix weight we have that

$$w = \frac{1}{n_1 n_2} \sum_{\substack{1 \le i \le n_1 \\ 1 \le j \le n_2}} m_{i,j} = \frac{1}{n_1 n_2} \sum_{k=1}^{n_1 n_2} k,$$

as M is traditional and so contains the set of integers $1, 2, \ldots, n_1n_2$.

Applying the standard summation formula for the first n_1n_2 positive integers then yields

$$w = \frac{1}{n_1n_2}\frac{(n_1n_2)(n_1n_2+1)}{2} = \frac{n_1n_2+1}{2},$$

and subtracting this weight w from each element of the set of n^2 consecutive integers $1, 2, \ldots, n^2$, we obtain the final display of the statement of Lemma 3.2.

To see Corollary 3.3, setting $n_1 = n_2 = n$, in the above weight formula gives us $w = \frac{n^2+1}{2}$, and considering the two cases $n = 2k$ is even, and $n = 2k+1$ is odd, in the n^2 numbers

$$\frac{1-n^2}{2}, \frac{3-n^2}{2}, \ldots, \frac{n^2-3}{2}, \frac{n^2-1}{2},$$

we deduce the result. □

Definition (of most-perfect square matrices)**.** Let $n = 2k$ be a positive even integer and $w \in \mathbb{R}$ a constant. Then a square matrix $M = (m_{i,j}) \in \mathbb{R}^{n\times n}$ is defined to be a *most-perfect square* matrix if it satisfies the *type* M , *type* P and *type* S symmetry properties, defined below.

1. The matrix M is said to be *type* M if it satisfies the *most-perfect property*, where the entries of all 2×2 sub-arrays within the square matrix sum to $4w$, so that

$$m_{i,j} + m_{i,j+1} + m_{i+1,j} + m_{i+1,j+1} = 4w,$$

   for all $i, j \in \mathbb{Z}_n$, and the *alternating sum property*

$$\sum_{i,j\in\mathbb{Z}_n} (-1)^{i+j} m_{i,j} = 0.$$

2. The matrix M is said to be *type* P if it satisfies the *strong pandiagonal property*, where the pairs of entries $\frac{1}{2}n = k$ distance along any diagonal (including broken diagonals) sum to $2w$, so that

$$m_{i,j} + m_{i+\frac{1}{2}n,j+\frac{1}{2}n} = 2w, \qquad i, j \in \mathbb{Z}_n.$$

3. The matrix M is said to be *type* S if it satisfies the *constant sum property*, where the sum of the elements of each row, or column sum to nw, so that

$$\sum_{j\in\mathbb{Z}_n} m_{i,j} = \sum_{j\in\mathbb{Z}_n} m_{j,i} = nw, \qquad i \in \mathbb{Z}_n.$$

LEMMA 3.4. *Let $M = (m_{i,j}) \in \mathbb{R}^{n\times n}$ be an $n \times n$ square matrix that satisfies at least one of the three symmetry conditions (M), (P), (S). Then M has weight w.*

Proof. Each one of the three symmetry conditions are independently sufficient to imply the weight condition and hence M has weight w. □

Property (M), does not at face value presuppose even matrix dimension n in the way that property (P) does, requiring that $2 \mid n$. However if n is odd then only the null matrix $\hat{0}_n$ has this property as detailed in Lemma 3.5, below.

LEMMA 3.5. *Let $n \in \mathbb{N}$ be odd and $M \in \mathbb{R}^{n\times n}$ a matrix with property (M). Then $M = \hat{0}_n$.*

Proof. By the 2×2 array sum property, we have for each $i \in \mathbb{Z}_n$

$$m_{i,1} + m_{i+1,1} = 4w - m_{i,2} - m_{i+1,2} = m_{i,3} + m_{i+1,3} = 4w - m_{i,4} - m_{i+1,4}$$
$$= \cdots = m_{i,n} + m_{i+1,n} = 4w - m_{i,1} - m_{i+1,1},$$

so $m_{i,j} + m_{i+1,j} = 2w$ for all $i, j \in \mathbb{Z}_n$. Hence, for each $j \in \mathbb{Z}_n$,

$$m_{1,j} = 2w - m_{2,j} = m_{3,j} = \cdots = m_{n,j} = 2w - m_{1,j},$$

which implies $m_{i,j} = w$ $(i, j \in \mathbb{Z}_n)$. But then the alternating sum property requires $w = 0$. □

For readers familiar with Ollerenshaw and Brée's definition of *most-perfect squares*, the type (M) alternating property might come as a surprise as it is not mentioned in their book [10]. However whilst examining these symmetries from a deeper algebraic perspective, it was found that the alternating property was required in the definition of type M matrices, as it allowed for a type N complimentary matrix, considered later on in Chapter 8.

At a more general level, it can be deduced that if a matrix M is type P, and satisfies the most-perfect sub-array sum condition, then it also satisfies the alternating condition as detailed in Lemma 3.6, below.

LEMMA 3.6. *Let $n \in \mathbb{N}$ be even and $M \in \mathbb{R}^{n\times n}$ a matrix with property (P) and the most-perfect sub-array property, so that $m_{i,j} + m_{i,j+1} + m_{i+1,j} + m_{i+1,j+1} = 4w$, for all $i, j \in \mathbb{Z}_n$. Then M also satisfies the alternating sum condition*

$\sum_{i,j\in\mathbb{Z}_n} (-1)^{i+j} m_{i,j} = 0$.

Proof. As $n = 2k$ is even, starting in the top left-hand corner, the matrix M can be partitioned in a disjoint union of k^2, 2×2 sub-arrays. Using the pandiagonal condition, each 2×2 sub-array can be paired with its pandiagonal 2×2 sub-array partner; When k is even we get $k^2/2$ of these paired sub-arrays, and when k is odd we $(k^2 - 1)/2$ of these paired sub-arrays and a lone central sub-array.

Applying the signs of the alternating sum $(-1)^{i+j}$ to the elements of these paired sub-arrays, we find that pandiagonal pairs of entries have the same sign, two pairs positive and two pairs negative. Hence the sum of the signed entries of each pandiagonal pair of sub-arrays equals $2w + 2w - 2w - 2w = 0$. In the case that k is odd, the central 2×2 sub-array sum equals $2w - 2w = 0$. Therefore, in either case, the sum over all the 2×2 sub-arrays equates to 0, and hence M also satisfies the alternating sum condition, so is fully type M. □

Example. *Below we give an example of an* 8×8 *traditional most-perfect square.*

1	16	17	32	53	60	37	44
63	50	47	34	11	6	27	22
3	14	19	30	55	58	39	42
61	52	45	36	9	8	25	24
12	5	28	21	64	49	48	33
54	59	38	43	2	15	18	31
10	7	26	23	62	51	46	35
56	57	40	41	4	13	20	29

Here $w = \frac{65}{2}$*, so that the rows, columns and diagonals sum to* 260*; all two by two entries sum to* 130*, and the two entries* $\frac{1}{2}n$ *distance along the diagonal sum to* 65*.*

Definition (of reversible matrices)**.** A *reversible matrix* satisfies the *type* R and *type* V symmetry properties; a *weightless reversible matrix* satisfies the *type* R and *type* $\hat{\mathrm{V}}$ symmetry properties defined below.

1. A matrix $M = (m_{i,j}) \in \mathbb{R}^{n_1 \times n_2}$ is said to be *type* R if it has row and column reverse similarity, so that

$$m_{i,j} + m_{i,(n+1-j)} = m_{i,j'} + m_{i,(n+1-j')} \quad i \in \mathbb{Z}_{n_1} \quad j, j' \in \mathbb{Z}_{n_2},$$

   and

$$m_{i,j} + m_{(n+1-i),j} = m_{i',j} + m_{n+1-i',j} \quad i, i' \in \mathbb{Z}_{n_1} \quad j \in \mathbb{Z}_{n_2}.$$

2. A matrix $M = (m_{i,j}) \in \mathbb{R}^{n_1 \times n_2}$ is said to be *type* V if its rectangular subarrays have equal cross sums at their corners, so that

$$m_{i,j} + m_{i',j'} = m_{i,j'} + m_{i',j} \quad i, i' \in \mathbb{Z}_{n_1} \quad j, j' \in \mathbb{Z}_{n_2}.$$

3. A matrix $M = (m_{i,j}) \in \mathbb{R}^{n_1 \times n_2}$ is said to be *type* $\hat{\mathrm{V}}$ if it is weightless type V, so that it has the additional property that

$$\sum_{i \in \mathbb{Z}_{n_1},\, j \in \mathbb{Z}_{n_2}} m_{i,j} = 0.$$

Remark. If a matrix is type R, then the pairwise sum of entries that are diametrically opposite each other in the specified row or column is constant within that row or column.

It also follows from the Corollary to Lemma 3.2 that a traditional reversible square matrix has weight $w = \frac{n^2+1}{2}$.

Example. *Below we give an example of an* 8×8 *traditional reversible square*

$M =$

1	2	5	6	33	34	37	38
3	4	7	8	35	36	39	40
9	10	13	14	41	42	45	46
11	12	15	16	43	44	47	48
17	18	21	22	49	50	53	54
19	20	23	24	51	52	55	56
25	26	29	30	57	58	61	62
27	28	31	32	59	60	63	64

,

so that the square satisfies the three conditions required. For example

1. *in the third row of entries,* $9, 10, 13, 14, 41, 42, 45, 46$ *the reverse similarity holds, as*
$$9 + 46 = 10 + 45 = 13 + 42 = 14 + 41 = 55;$$

2. *in the 7th column of entries,* $37, 39, 45, 47, 53, 55, 61, 63$ *the reverse similarity holds, as*
$$37 + 63 = 39 + 61 = 45 + 55 = 47 + 53 = 100,$$

3. *and in the array with corner entries* $r_{23} = 7, r_{27} = 39, r_{63} = 23, r_{67} = 55$ *the cross sums are equal as*
$$r_{23} + r_{67} = 7 + 55 = 62 = 23 + 39 = r_{27} + r_{63}.$$

Further examples of reversible square matrices are given in the appendices.

Definition (of associated matrices)**.** A matrix $M = (m_{i,j}) \in \mathbb{R}^{n_1 \times n_2}$ is said to be *associated*, or *type* A, if satisfied the associated symmetry property, which requires that
$$m_{i,j} + m_{(n+1-i),(n+1-j)} = 2w \quad \forall i \in \mathbb{Z}_{n_1}, \quad \text{and} \quad \forall j \in \mathbb{Z}_{n_2}.$$

LEMMA 3.7. *An* $n_1 \times n_2$ *associated matrix* M *has weight* w.

Proof. The sum of all the entries in M can be taken pairwise using the associated sum property of $2w$, which also implies that if M has a central cell (both n_1, n_2 being odd), then this must have the entry w. Hence the average entry of M is w, which by definition is the weight of M. □

Remark. It will be shown later in this chapter that every reversible square matrix is also an associated square matrix.

Definition. Let $n \in \mathbb{N}$. We define the following matrix symmetry type spaces.

$$\mathrm{S}_n = \{M \in \mathbb{R}^{n \times n} \mid M \text{ has property (S)}\},$$

$$\mathrm{R}_n = \{M \in \mathbb{R}^{n \times n} \mid M \text{ has property (R)}\},$$

$V_n = \{M \in \mathbb{R}^{n\times n} \mid M \text{ has property (V)}\},$

$\hat{V}_n = \{M \in \mathbb{R}^{n\times n} \mid M \text{ has property } (\hat{V})\},$

$A_n = \{M \in \mathbb{R}^{n\times n} \mid M \text{ has property (A)}\},$

For even n, we also define the symmetry type spaces

$M_n = \{M \in \mathbb{R}^{n\times n} \mid M \text{ has property (M) }\},$

$P_n = \{M \in \mathbb{R}^{n\times n} \mid M \text{ has property (P) }\}.$

Composite symmetry types are captured in the following intersections of the above spaces,

$RV_n = R_n \cap V_n$, and $R\hat{V}_n = R_n \cap \hat{V}_n$,

so that RV_n is the space of $n \times n$ reversible square matrices and $R\hat{V}_n \subset RV_n$ is the space of $n \times n$ weightless reversible square matrices. For even n we have

$MPS_n = M_n \cap P_n \cap S_n,$

so that MPS_n is the space of $n \times n$ most-perfect square matrices.

3.1 The Type R, Type V and Type $\hat{V}$ Block Representations

In this section we derive explicit block representation formulae for these matrix types after conjugation with our orthogonal symmetric involution matrix X_n. We begin with a lemma.

LEMMA 3.8. *Let $M = (m_{ij})$, $M' = (m'_{ij})$ be two $n \times n$ matrices, and J_n, I_n and E_n defined as at the beginning of this chapter. Then we have*

1. $M' = J_n M \Leftrightarrow m'_{i,j} = m_{(n+1-i),j}$, $\forall (i,j) \in \mathbb{Z}_n \times \mathbb{Z}_n$.
2. $M' = M J_n \Leftrightarrow m'_{i,j} = m_{i,(n+1-j)}$, $\forall (i,j) \in \mathbb{Z}_n \times \mathbb{Z}_n$.
3. $M' = J_n M J_n \Leftrightarrow m'_{i,j} = m_{(n+1-i),(n+1-j)}$, $\forall (i,j) \in \mathbb{Z}_n \times \mathbb{Z}_n$.
4. $J_n J_n = I_n$ *and hence* $J_n^T = J_n = J_n^{-1}$.
5. *Let M be an $n \times n$ matrix. Then when $n = 2k$ is even we can write M in the conjugated block representation form*

$$M = X_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} X_n,$$

with V, W, Y, Z, $k \times k$ matrices, and when $n = 2k+1$ is odd we can write M in the conjugated block representation form

$$M = X_n \begin{pmatrix} Y & v & V^T \\ y^T & \alpha & z^T \\ W & x & Z \end{pmatrix} X_n$$

with V, W, Y, Z, $k \times k$ matrices, $v, x, y, z \in \mathbb{R}^k$ vectors and $\alpha \in \mathbb{R}$ a real number.

6. *The weight matrix E_n satisfies*

$$E_n = X_n \begin{pmatrix} 2E_k & \hat{0}_k \\ \hat{0}_k & \hat{0}_k \end{pmatrix} X_n,$$

when $n = 2k$ is even, and

$$E_n = X_n \begin{pmatrix} 2E_k & \sqrt{2}\,1_k & \hat{0}_k \\ \sqrt{2}\,1_k^T & 1 & 0_k^T \\ \hat{0}_k & 0_k & \hat{0}_k \end{pmatrix} X_n,$$

when $n = 2k+1$ is odd.

7. *Let M be an $n \times n$ matrix, so that by the preceding result (5), we can write $M = X_n N X_n$ for some matrix $N \in \mathbb{R}^{n\times n}$. Let M have weight w (as by definition, all matrices have a weight). Then M can also be written in the form $M = X_n N^0 X_n + wE_n = M^0 + wE_n$, where $M^0 = X_n N^0 X_n$ has weight zero.*

Proof. The first four relations can be seen by noting that multiplication on the left by J_n reflects the entries of M across the central row and multiplication on the right by J_n reflects the entries of M across the central column. Hence $J_n J_n = I_n$ and conjugation with J_n rotates the entries of M around its centre.

To see the fifth relation we observe that for a given $n \times n$ matrix M we can define the matrix N such that $X_n M X_n = N$, which after conjugation yields $M = X_n N X_n$. As the even and odd block and vector representations of N allow for all possible entries in the matrix N, we have that for any matrix M we can find N with $M = X_n N X_n$.

Relation 6 follows by multiplying out the conjugated block matrix expressions for E_n, and to see the seventh relation we take $M^0 = X_n N X_n - wE_n$, using the conjugated block expressions for E_n given in the fifth identity. It then follows from the construction that M^0 has weight zero and that $N^0 = X_n M^0 X_n$, as required. □

LEMMA 3.9. *Let $M = (m_{ij}) \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$. Then $M \in \mathrm{R_n}$ if and only if*

$$(M + MJ_n)u = 0_n \quad \textit{and} \quad (M^T + M^T J_n)u = 0_n$$

for all $u \in \{1_n\}^\perp$.

These are equivalent to

$$(M^T + J_n M^T)\mathbb{R}^n \subset \mathbb{R}1_n \quad \textit{and} \quad (M + J_n M)\mathbb{R}^n \subset \mathbb{R}1_n.$$

Proof. If $(M+MJ_n)u = 0_n$ for all $u \in \{1_n\}^\perp$ then $M+MJ_n = (m_{i,j}+m_{i,n+1-j})_{i,j\in\mathbb{Z}_n}$ has constant row sum as the sum of the entries of u will be 0.

Also $(M + J_nM)\mathbb{R}^n \subset \mathbb{R}1_n$ implies that $M + J_nM = (m_{i,j} + m_{n+1-i,j})_{i,j\in\mathbb{Z}_n}$ has constant column sums, and these two statements are equivalent to M being type R. The other equivalent equations follow by considering M^T.

Conversely, if M is type R then $M+MJ_n$ and $M+J_nM$ have constant row sum and constant column sum respectively. Hence $(M + MJ_n)u = 0_n$ and $(M^T + M^TJ_n) = 0_n$, as required.

□

THEOREM 3.10. *A matrix $M \in \mathbb{R}^{n\times n}$ is an element of R_n with weight w if and only it has the block representation*

$$M = X_n \begin{pmatrix} wE_k & 1_kz^T \\ x1_k^T & Z \end{pmatrix} X_n$$

when $n = 2k$ is even, with $Z \in \mathbb{R}^{k\times k}$, $x, z \in \mathbb{R}^k$ and $w \in \mathbb{R}$,

and

$$M = X_n \begin{pmatrix} \sqrt{2}wE_k & w1_k & \sqrt{2}1_kz^T \\ w1_k^T & \frac{w}{\sqrt{2}} & z^T \\ \sqrt{2}x1_k^T & x & Z \end{pmatrix} X_n$$

when $n = 2k + 1$ is odd.

Proof. **Even case $n = 2k$.** Let M be an $n \times n$ type R matrix, so that by (5) of Lemma 3.8 we can find V, W, Y, Z, $k \times k$ matrices, such that

$$M = X_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} X_n.$$

Then by Lemma 3.9 we can write

$$(M + J_nM)\mathbb{R}^n \subset \mathbb{R}1_n,$$

so that

$$(M + J_nM)\mathbb{R}^n = \left(X_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} X_n + J_nX_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} X_n\right) \mathbb{R}^n$$

$$= X_n \left(\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} + X_nJ_nX_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix}\right) X_n\mathbb{R}^n \subset \mathbb{R}1_n.$$

Now

$$X_nJ_nX_n = \begin{pmatrix} I_k & \hat{0}_k \\ \hat{0}_k & -I_k \end{pmatrix},$$

so that

$$X_n\left(\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} + \begin{pmatrix} I_k & \hat{0}_k \\ \hat{0}_k & -I_k \end{pmatrix}\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix}\right) X_n\mathbb{R}^n \subset \mathbb{R}1_n,$$

$$X_n\left(\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} + \begin{pmatrix} I_k & \hat{0}_k \\ \hat{0}_k & -I_k \end{pmatrix}\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix}\right) X_n\mathbb{R}^n \subset \mathbb{R}X_nX_n1_n,$$

and using $X_n1_n = \sqrt{2}\begin{pmatrix} 1_k \\ 0_k \end{pmatrix}$ we obtain

$$X_n\left(\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} + \begin{pmatrix} I_k & \hat{0}_k \\ \hat{0}_k & -I_k \end{pmatrix}\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix}\right) X_n\mathbb{R}^n \subset \mathbb{R}X_n\begin{pmatrix} 1_k \\ 0_k \end{pmatrix}.$$

Moreover, as $X_n\mathbb{R}^n$ is a bijection of $\mathbb{R}^n$ we can also write

$$X_n\left(\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} + \begin{pmatrix} I_k & \hat{0}_k \\ \hat{0}_k & -I_k \end{pmatrix}\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix}\right) \mathbb{R}^n$$
$$\subset \mathbb{R}X_n\begin{pmatrix} 1_k \\ 0_k \end{pmatrix}.$$

Hence

$$2\begin{pmatrix} Y & V^T \\ \hat{0}_k & \hat{0}_k \end{pmatrix}\mathbb{R}^n = \left(\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} + \begin{pmatrix} I_k & \hat{0}_k \\ \hat{0}_k & -I_k \end{pmatrix}\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix}\right) \mathbb{R}^n$$
$$\subset \mathbb{R}\begin{pmatrix} 1_k \\ 0_k \end{pmatrix},$$

and

$$2\begin{pmatrix} Y^T & W^T \\ \hat{0}_k & \hat{0}_k \end{pmatrix}\mathbb{R}^n = \left(\begin{pmatrix} Y^T & W^T \\ V & Z^T \end{pmatrix} + \begin{pmatrix} I_k & \hat{0}_k \\ \hat{0}_k & -I_k \end{pmatrix}\begin{pmatrix} Y^T & W^T \\ V & Z^T \end{pmatrix}\right) \mathbb{R}^n$$
$$\subset \mathbb{R}\begin{pmatrix} 1_k \\ 0_k \end{pmatrix}.$$

These two expressions are equivalent to the blocks Y, Y^T, V^T and W^T having columns which are multiples of 1_n, which implies that Y is a multiple of the matrix E_k.

Conversely, if Y is a multiple of E_k, V^T and W^T have constant columns and $Z \in \mathbb{R}^{n\times n}$ we can see that the square will have reverse row and column similarity when multiplied on the left and right by the X_n block, as required.

Odd $n = 2k + 1$. Let M be an $n\times$ type R matrix. Then by relation (5) of Lemma

3.8 we can find V, W, Y, Z $k \times k$ matrices, $v, x, y, z \in \mathbb{R}^k$ vectors, and $\alpha \in \mathbb{R}$, a real number such that

$$M = X_n \begin{pmatrix} Y & v & V^T \\ y^T & \alpha & z^T \\ W & x & Z \end{pmatrix} X_n \quad \text{where we note that} \quad X_n 1_n = \begin{pmatrix} \sqrt{2} 1_k \\ 1 \\ 0 \end{pmatrix}.$$

Then in a similar fashion to the even case, we can write

$$2 \begin{pmatrix} Y & v & V^T \\ y^T & \alpha & z^T \\ \hat{0}_k & 0_k & \hat{0}_k \end{pmatrix} \mathbb{R}^n \subset \mathbb{R} \begin{pmatrix} \sqrt{2} 1_k \\ 1 \\ 0_k \end{pmatrix},$$

and

$$2 \begin{pmatrix} Y^T & y & W^T \\ v^T & \alpha & x^T \\ \hat{0}_k & 0_k & \hat{0}_k \end{pmatrix} \mathbb{R}^n \subset \mathbb{R} \begin{pmatrix} \sqrt{2} 1_k \\ 1 \\ 0_k \end{pmatrix}.$$

Again we find that these expressions are equivalent to $Y = \sqrt{2} w 1_k 1_k^T$ for some $w \in \mathbb{R}$, $v = y = w 1_k$ and $\alpha = \frac{w}{\sqrt{2}}$. Furthermore $V^T = \sqrt{2} 1_k z^T$ and $W^T = \sqrt{2} 1_k x^T$. Conversely, we can see from the block representation that

$$X_n \begin{pmatrix} \sqrt{2} w E_k & w 1_k & \sqrt{2} 1_k z^T \\ w 1_k^T & \frac{a}{\sqrt{2}} & z^T \\ \sqrt{2} x 1_k^T & x & Z \end{pmatrix} X_n$$

satisfies the type R condition, and hence we have established the equivalence of the block representations.

With the equivalence of the block representations established we now need only show that M has weight w.

Let us assume that $M = X_n N X_n$ has weight t, for some $t \in \mathbb{R}$, so that by definition the average of the sum of the entries of M is equal to t. Then by (7) of Lemma 3.8, we can write $M^0 = M - tE_n = X_n N^0 X_n$, where N^0 has weight zero.

By (6) of Lemma 3.8 we have a block representation for tE_n and so for N^0. Expanding out this block representation after conjugation with X_n we find that the elements of the vectors x and z cancel out in each row and column sum, leaving only the w and t terms. Summing over all rows, and so over all the entries of $M^0 = X_n N^0 X_n$, and equating to the zero weight, we find that $w - t = 0$, so $w = t$ and so M has weight w as required.

□

LEMMA 3.11. *A matrix $M \in \mathbb{R}^{n \times n}$ is type* V *if and only if*

$$u^T M v = 0 \quad (\textit{for all } u, v \in \{1_n\}^{\perp}).$$

A matrix $M \in \mathbb{R}^{n\times n}$ is a type $\hat{\mathrm{V}}$ if and only if

$$u^T M v = 0 \quad (\textit{for all } u, v \in \{1_n\}^{\perp})$$

$$\textit{and} \;\; 1_n^T M 1_n = 0.$$

Proof. Assuming that

$$u^T M v = 0 \quad (\text{for all } u, v \in \{1_n\}^{\perp}),$$

we define v_j to be the vector with a 1 in the jth position, -1 in the $(j+1)$-th position and 0 otherwise. Then any $u \in \{1_n\}^{\perp}$ can be written as a linear combination of v_j, and hence the set of all such v_j forms a basis for $\{1_n\}^{\perp}$. For example if $x = (-2, -2, 0, 4)^T$, then it can be written as

$$x = \begin{pmatrix} -2 \\ -2 \\ 0 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 4 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} - 4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

In a similar fashion we define the vectors u_{ij} with a 1 in the ith position, a -1 in the jth position and 0 elsewhere, so that $u_{ij} \in \{1_n\}^{\perp}$, and w.l.o.g. if $j > i$ then $u_{ij} = v_i + v_{i+1} + \ldots + v_{j-1}$, a linear combination of the v_j. Then for any $i, j, k, l \in \mathbb{Z}_n$ we have that

$$0 = u_{ik}^T M u_{jl} = \{0, \ldots, \underbrace{1}_{ith}, \ldots, \underbrace{-1}_{kth}, \ldots, 0\} \begin{pmatrix} m_{1,1} & \ldots & m_{1,n} \\ m_{2,1} & \ddots & m_{2,n} \\ \vdots & & \vdots \\ m_{n,1} & \ldots & m_{n,n} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \;\; (jth) \\ \vdots \\ -1 \;\; (lth) \\ \vdots \\ 0 \end{pmatrix}$$

$$= m_{ij} + m_{kl} - m_{il} - m_{kj}.$$

which is the required type V property

$$m_{ij} + m_{kl} = m_{il} + m_{kj},$$

and hence M is type V.

Conversely, since the type V property implies that for all $i, j, k, l \in \mathbb{Z}_n$,

$$0 = m_{i,j} + m_{k,l} - m_{i,l} - m_{k,l},$$

reversing the above argument yields

$$0 = m_{i,j} + m_{i+1,j+1} - m_{i,j+1} - m_{i+1,j} = v_i^T M v_j = (a_i v_i^T) M (b_j v_j)$$

for all $a_i, b_j \in \mathbb{R}$, $i, j \in \mathbb{Z}_n$. Hence

$$\begin{aligned} 0 =& (a_1 v_1^T) M (b_1 v_1) + (a_2 v_2^T) M (b_1 v_1) + \ldots + (a_i v_1^T) M (b_j v_j) + \ldots + (a_n v_n^T) M (b_n v_n) \\ =& (a_1 v_1^T + a_2 v_2^T + \ldots + a_n v_n^T) M (b_1 v_1 + b_2 v_2 + \ldots b_n v_n), \end{aligned}$$

with $(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n)$ and $(b_1 v_1 + b_2 v_2 + \ldots b_n v_n)$ arbitrary vectors in $\{1_n\}^\perp$.

To see the second property we observe that $1_n^T M 1_n = \sum_{i,j=1}^n m_{i,j} = 0$ and so M is type $\hat{\mathrm{V}}$, as required. □

THEOREM 3.12. *A matrix $M \in \mathbb{R}^{n \times n}$ is type* V *if and only if it has the block representation*

when $n = 2k$ is even

$$M = X_n \begin{pmatrix} Y & 1_k a^T \\ b 1_k^T & \hat{0}_k \end{pmatrix} X_n,$$

where $a, b \in \mathbb{R}^k$, $Y \in \mathbb{R}^{k \times k}$, and with Y type V,

and when $n = 2k+1$ is odd

$$M = X_n \begin{pmatrix} \sqrt{2}(a 1_k^T + 1_k c^T) - 2\alpha E_k & a & \sqrt{2} 1_k d^T \\ c^T & \alpha & d^T \\ \sqrt{2} b 1_k^T & b & \hat{0}_k \end{pmatrix} X_n,$$

with $a, b, c, d \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$.

A matrix $M \in \mathbb{R}^{n \times n}$ is of type $\hat{\mathrm{V}}$ if and only it has the block representation

when $n = 2k$ is even

$$M = X_n \begin{pmatrix} Y & 1_k a^T \\ b 1_k^T & \hat{0}_k \end{pmatrix} X_n,$$

where $a, b \in \mathbb{R}^k$, $Y \in \mathbb{R}^{k \times k}$, and with Y type $\hat{\mathrm{V}}$,

and when $n = 2k+1$ is odd

$$M = X_n \begin{pmatrix} \sqrt{2}(a 1_k^T + 1_k c^T) - \frac{2\sqrt{2}}{2k-1}(1_k^T(a+c)) E_k & a & \sqrt{2} 1_k d^T \\ c^T & \frac{\sqrt{2}}{2k-1} 1_k^T (a+c) & d^T \\ \sqrt{2} b 1_k^T & b & \hat{0}_k \end{pmatrix} X_n,$$

with $a, b, c, d \in \mathbb{R}^k$.

Proof. **Even case $n = 2k$.** Let M be type $\hat{\text{V}}$, so that by (5) of Lemma 3.8 we can write

$$M = X_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} X_n,$$

where $Y, V, W, Z \in \mathbb{R}^{k\times k}$. Then by Lemma 3.11 we have

$$0 = u^T X_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} X_n v,$$

for all $u, v \in \{1_n\}^\perp$.

Now define $u = (u_1, \ldots, u_n)^T = (\hat{u}_1, \hat{u}_2)$ and $v = (v_1, \ldots, v_n)^T = (\hat{v}, \hat{v}_2)$ such that $\hat{u_1} = (u_1, u_2, \ldots, u_k)^T$, $\hat{v_1} = (v_1, v_2, \ldots, v_k)^T$, $\hat{u_2} = (u_{k+1}, \ldots, u_n)^T$ and $\hat{v_2} = (v_{k+1}, \ldots, v_n)^T$. Then

$$\begin{aligned}0 = &\begin{pmatrix} \hat{u_1} \\ \hat{u_2} \end{pmatrix}^T X_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} X_n \begin{pmatrix} \hat{v_1} \\ \hat{v_2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \hat{u_1} + \hat{u_2} J_k \\ \hat{u_1} J_k - \hat{u_2} \end{pmatrix}^T \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} \begin{pmatrix} \hat{v_1} + J_k \hat{v_2} \\ J_k \hat{v_1} - \hat{v_2} \end{pmatrix} \\ = &\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix},\end{aligned}$$

where $\xi_1 = \hat{u_1} + \hat{u_2} J_k$, $\eta_1 = \hat{u_1} J_k - \hat{u_2}$, $\xi_2 = \hat{v_1} + J_k \hat{v_2}$ and $\eta_2 = J_k \hat{v_1} - \hat{v_2}$.

Given that $u_1 + u_2 + \ldots + u_n = 0$ and $v_1 + v_2 + \ldots + v_n = 0$, and that the entries in ξ_1 and ξ_2 also sum to zero, we deduce that $\xi_1, \xi_2 \in \{1_k\}^\perp$ and $\eta_1, \eta_2 \in \mathbb{R}^k$.

Multiplying out the final matrix display we obtain

$$\begin{aligned}0 = &\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} \\ = &\xi_1^T Y \xi_2 + \xi_1^T V^T \eta_2 + \eta_1^T W \xi_2 + \eta_1 Z \eta_2.\end{aligned}$$

Setting $\xi_1 = \xi_2 = 0_k$ yields

$$0 = \underbrace{0_k^T Y 0_k}_{=0} + \underbrace{0_k^T V^T \eta_2}_{=0} + \underbrace{\eta_1 W 0_k}_{=0} + \eta_1 Z \eta_2,$$

which implies that $Z = \hat{0}_k$.

Setting either or both of η_1 and η_2 to be the zero vector 0_k, we obtain the three equalities

$$0 = \xi_1^T Y \xi_2 + \xi_1^T V^T \eta_2, \quad 0 = \xi_1^T Y \xi_2 + \eta_1^T W \xi_2 \text{ and } 0 = \xi_1^T Y \xi_2.$$

Given that $0 = \xi_1^T V^T \eta_2$ and $0 = \eta_1^T W \xi_2$, as $\eta_1, \eta_2 \in \mathbb{R}^k$ and $\xi_1, \xi_2 \in \{1_k\}^\perp$ it follows that

$$0_k = V\xi_1 \quad \text{and} \quad 0_k = W\xi_2.$$

Let V_j^T and W_i^T be the row vectors of the matrices V and W respectively for $i, j \in \{1, 2, \ldots, n\}$, so that $V_j^T \xi_1 = W_i^T \xi_2 = 0$ for all i, j. Then $V_j \in \{\xi_1\}^\perp$ and $W_i \in \{\xi_2\}^\perp$, and as $\xi_1, \xi_2 \in \{1_k\}^\perp$ then $V_j \in \{1_k\}^{\perp^\perp} = \alpha 1_k$ and $W_i \in \{1_k\}^{\perp^\perp} = \beta 1_k$ with $\alpha, \beta \in \mathbb{R}$.

Hence $0 = \xi_1^T Y \xi_2$, with $\xi_1, \xi_2 \in \{1_k\}^\perp$ so that Y is type V by Lemma 3.11 and concluding we have the given type V block representation.

In consideration of the second condition of Lemma 3.11 for a type $\hat{V}$ matrix, which says that $0 = 1_n^T M 1_n$, we have that

$$0 = 1_n^T X_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} X_n 1_n = \frac{1}{2} \begin{pmatrix} 1_k + 1_k J_k \\ 1_k J_k - 1_k \end{pmatrix} \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} \begin{pmatrix} 1_k + J_k 1_k \\ J_k 1_k - 1_k \end{pmatrix}$$

$$= 2 \begin{pmatrix} 1_k \\ 0_k \end{pmatrix}^T \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} \begin{pmatrix} 1_k \\ 0_k \end{pmatrix} = 1_k^T Y 1_k$$

and hence Y is type $\hat{\text{V}}$, as required.

Conversely, expanding out the block representation

$$M = X_n \begin{pmatrix} Y & 1_k a^T \\ b^T 1_k & \hat{0}_k \end{pmatrix} X_n$$

with $a, b \in \mathbb{R}^k$, $Y \in \mathbb{R}^{k \times k}$ and type $\hat{V}$ we obtain,

$$M = \frac{1}{2} \begin{pmatrix} Y + J_k b 1_k^T + 1_k a^T J_k & Y J_k + J_k b 1_k^T J_k \\ J_k Y - b 1_k^T + J_k 1_k a^T J_k & J_k Y J_k - b 1_k^T J_k - J_k 1_k a^T \end{pmatrix}$$

and so by Lemma 3.11, under the assumption $0 = 1_n^T M 1_n$, we have that

$$\begin{aligned} 1_n^T M 1_n = {} & \underbrace{1_k^T Y 1_k}_{=0} + \underbrace{1_k^T J_k Y 1_k}_{=0} + \underbrace{1_k^T Y J_k 1_k}_{=0} + \underbrace{1_k^T J_k Y J_k 1_k}_{=0} \\ & + 1_k^T J_k (b 1_k^T) 1_k - 1_k^T b 1_k^T 1_k + 1_k^T J_k b 1_k^T J_k 1_k - 1_k^T b 1_k^T J_k 1_k \\ & + 1_k^T 1_k a^T J_k 1_k + 1_k^T J_k 1_k a^T J_k 1_k - 1_k^T 1_k a^T 1_k - 1_k^T J_k 1_k a^T J_k 1_k \\ = {} & 0, \end{aligned}$$

due to symmetries in the constant rows and columns of $b 1_k^T$ and $1_k a^T$ respectively.

Similarly for any vectors $u, v \in \{1_n\}^\perp$, where $\xi_1, \xi_2 \in \{1_k\}^\perp$, and $\eta_1, \eta_2 \in \mathbb{R}^k$, we have

$$u^T X_n \begin{pmatrix} Y & 1_k a^T \\ b^T 1_k & \hat{0}_k \end{pmatrix} X_n v = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}^T \begin{pmatrix} Y & 1_k a^T \\ b^T 1_k & \hat{0}_k \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}$$

$$= \xi_1^T Y \xi_2 + \xi_1^T 1_k a^T \eta_2 + \eta_1^T b 1_k^T \xi_2 + \eta_1^T \hat{0}_k \eta_2.$$

The sum of the entries of ξ_1 and ξ_2 is 0, and hence we have

$$\underbrace{\xi_1^T Y \xi_2}_{=0} + \xi_1^T 1_k a^T \eta_2 + \eta_1^T b 1_k^T \xi_2 \underbrace{\xi_1^T 1_k a^T}_{=0} \eta_2 + \eta_1^T \underbrace{b 1_k^T \xi_2}_{=0} = 0,$$

and it follows that the block representation structure ensures the type $\hat{\mathrm{V}}_n$ symmetry.

Odd case $n = 2k+1$. We proceed as in the even case with the assumption that $M \in \hat{\mathrm{V}}_n$. Setting $N = X_n M X_n$, and by (5) of Lemma 3.8 noting that any odd-sided square matrix N can be written in the form

$$N = \begin{pmatrix} Y & v & V^T \\ y^T & \alpha & z^T \\ W & x & Z \end{pmatrix} X_n \quad \text{we can write} \quad M = X_n \begin{pmatrix} Y & v & V^T \\ y^T & \alpha & z^T \\ W & x & Z \end{pmatrix} X_n,$$

for some $V, W, Y, Z \in \mathbb{R}^{k\times k}$, $x, y, v, z \in \mathbb{R}^k$, and $\alpha \in \mathbb{R}$.

By Lemma 3.11 we have

$$0 = u^T X_n \begin{pmatrix} Y & v & V^T \\ y^T & \alpha & z^T \\ W & x & Z \end{pmatrix} X_n v,$$

for all $u, v \in \{1_n\}^\perp$.

Now

$$u^T X_n = \frac{1}{\sqrt{2}} (\hat{u}_1, u_{k+1}, \hat{u}_2) \begin{pmatrix} I_k & 0_k & J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ J_k & 0_k & -I_k \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 \\ \sqrt{2} u_{k+1} \\ \eta_1 \end{pmatrix}^T = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 \\ -\sqrt{2} 1_k^T \xi_1 \\ \eta_1 \end{pmatrix}^T,$$

where $\hat{u}_1 = (u_1, u_2, \ldots, u_k)^T$ and $\hat{u}_2 = (u_{k+2}, u_{k+3}, \ldots, u_n)^T$,and again by Lemma 3.11, $\sum_{i=1}^k (\hat{u}_{1i} + \hat{u}_{2i}) + u_{k+1} = 0$ so $u_{k+1} = -\sum_{i=1}^k (\hat{u}_{1i} + \hat{u}_{2i}) = -1_k^T \xi_1$, and similarly

$$X_n v = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0_k & J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ J_k & 0_k & -I_k \end{pmatrix} \begin{pmatrix} \hat{\mathrm{V}}_1 \\ v_{k+1} \\ \hat{\mathrm{V}}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_2 \\ -\sqrt{2} 1_k^T \xi_2 \\ \eta_2 \end{pmatrix}.$$

Hence we can write,

$$
\begin{aligned}
0 =& \frac{1}{2}\begin{pmatrix} \xi_1 \\ -\sqrt{2}1_k^T\xi_1 \\ \eta_1 \end{pmatrix}^T \begin{pmatrix} Y & v & V^T \\ y^T & \alpha & z^T \\ W & x & Z \end{pmatrix} \begin{pmatrix} \xi_2 \\ -\sqrt{2}1_k^T\xi_2 \\ \eta_2 \end{pmatrix} \\
=& \xi_1^T Y\xi_2 - \sqrt{2}(\xi_1^T v)(1_k^T\xi_2) + \xi_1 V^T\eta_2 - \sqrt{2}(\xi_1^T 1_k)(y^t\xi_2) + 2(\xi_1^T 1_k)(1_k^T\xi_2)\alpha \\
&- \sqrt{2}(\xi_1^T 1_k)(z^T\eta_2) + \eta_1^T W\xi_2 - \sqrt{2}(\eta_1^T x)(1_k^T\xi_2) + \eta_1^T Z\eta_2.
\end{aligned}
$$

Setting $\xi_1 = \xi_2 = 0_k \in \mathbb{R}^k$, yields

$$
\begin{aligned}
0 =& \underbrace{\xi_1^T Y\xi_2}_{=0} - \underbrace{\sqrt{2}(\xi_1^T v)(1_k^T\xi_2)}_{=0} + \underbrace{\xi_1 V^T\eta_2}_{=0} - \underbrace{\sqrt{2}(\xi_1^T 1_k)(y^t\xi_2)}_{=0} + \underbrace{2(\xi_1^T 1_k)(1_k^T\xi_2)\alpha}_{=0} \\
&- \underbrace{\sqrt{2}(\xi_1^T 1_k)(z^T\eta_2)}_{=0} + \underbrace{\eta_1^T W\xi_2}_{=0} - \underbrace{\sqrt{2}(\eta_1^T x)(1_k^T\xi_2)}_{=0} + \eta_1^T Z\eta_2 \\
=& \eta_1^T Z\eta_2,
\end{aligned}
$$

from which we deduce that $Z = \hat{0}_k$.

Taking $\xi_1 = \xi_2 = 0_k \in \mathbb{R}^k$ gives us

$$
0 = \eta_1^T W\xi_2 - \sqrt{2}(\xi_1^T x)(1_k^T\xi_2),
$$

so that

$$
W = \sqrt{2}x1_k^T,
$$

and

$$
0 = \xi_1^T V^T\eta_2 - \sqrt{2}(\xi_1^T 1_k)(z^T\eta_2),
$$

so that

$$
V^T = \sqrt{2}1_k z^T,
$$

thus establishing W and V. It follows that

$$
\begin{aligned}
0 =& \xi_1^T Y\xi_2 - \sqrt{2}(\xi_1^T v)(1_k^T\xi_2) + \underbrace{\xi_1 V^T}_{=0_k}\eta_2 - \sqrt{2}(\xi_1^T 1_k)(y^T\xi_2) + 2(\xi_1^T 1_k)(1_k^T\xi_2)\alpha \\
&- \sqrt{2}\underbrace{\xi_1^T V^T}_{=0_k}\eta_2 + \eta_1^T\underbrace{W\xi_2}_{=0_k} - \sqrt{2}\eta_1^T\underbrace{W\xi_2}_{=0_k} = \xi_1^T Y\xi_2 - \sqrt{2}(\xi_1^T v)(1_k^T\xi_2) - \sqrt{2}(\xi_1^T 1_k)(y^T\xi_2) \\
=& \xi_1^T\left(Y - \sqrt{2}(v1_k^T) - \sqrt{2}(1_k y^T) + 2\alpha 1_k 1_k^T\right)\xi_2,
\end{aligned}
$$

and so

$$
\hat{0}_k = Y - \sqrt{2}(v1_k^T) - \sqrt{2}(1_k y^t) + 2\alpha E_k
$$

$$
Y = \sqrt{2}(v1_k^T + 1_k y^T) - 2\alpha E_k.
$$

From the second condition in 3.11, we also have

$$0 = 1_n^T M 1_n = 1_n^T X_n \begin{pmatrix} Y & v & V^T \\ y^T & \alpha & z^T \\ W & x & Z \end{pmatrix} X_n 1_n.$$

Using the representations for $1_n^T X_n$ and $X_n 1_n$,

$$\frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0_k & J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ J_k & 0_k & -I_k \end{pmatrix} = \begin{pmatrix} \sqrt{2}1_k \\ \sqrt{2} \\ 0_k \end{pmatrix}^T,$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0_k & J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ J_k & 0_k & -I_k \end{pmatrix} \begin{pmatrix} 1_k \\ 1 \\ 1_k \end{pmatrix} = \begin{pmatrix} \sqrt{2}1_k \\ \sqrt{2} \\ 0_k \end{pmatrix}$$

we can write

$$\begin{aligned} 0 &= \begin{pmatrix} \sqrt{2}1_k \\ \sqrt{2} \\ 0_k \end{pmatrix}^T \begin{pmatrix} Y & v & V^T \\ y^T & \alpha & z^T \\ W & x & Z \end{pmatrix} \begin{pmatrix} \sqrt{2}1_k \\ \sqrt{2} \\ 0_k \end{pmatrix} \\ &= 21_k^T Y 1_k + \sqrt{2}1_k^T v + \sqrt{2} y^T 1_k + \alpha. \end{aligned}$$

Substituting for $Y = \sqrt{2}(v1_k^T + 1_k y^T) - 2\alpha E_k$, then gives us

$$\begin{aligned} 0 =& 21_k^T \left(\sqrt{2}(v1_k^T + 1_k y^T) - 2\alpha E_k\right) 1_k + \sqrt{2}1_k^T v + \sqrt{2} y^T 1_k + \alpha \\ =& 2\sqrt{2}1_k^T (v1_k^T) 1_k + 2\sqrt{2}1_k^T (1_k y^T) 1_k - 4\alpha 1_k^T E_k 1_k + \sqrt{2}1_k^T v + \sqrt{2} y^T 1_k + \alpha \end{aligned}$$

so that

$$\alpha(41_k^T E_k 1_k - 1) = 2\sqrt{2}1_k^T (v1_k^T) 1_k + 2\sqrt{2}1_k^T (1_k y^T) 1_k + \sqrt{2}1_k^T v + \sqrt{2} y^T 1_k$$

$$\begin{aligned} \alpha(4k^2 - 1) &= 2\sqrt{2}1_k^T v k + 2\sqrt{2} k y^T 1_k + \sqrt{2}1_k^T v + \sqrt{2} y^T 1_k \\ &= (2\sqrt{2}k + \sqrt{2})1_k^T v + (2\sqrt{2}k + \sqrt{2}) y^T 1_k = \sqrt{2}(2k+1)\left(1_k^T v + y^T 1_k\right) \end{aligned}$$

which implies that

$$\alpha \frac{4k^2 - 1}{2k + 1} = \sqrt{2}(1_k^T v + y^T 1_k),$$

and hence

$$\alpha = \frac{\sqrt{2}}{2k - 1}(1_k^T v + y^T 1_k).$$

The converse argument then follows similarly to the even case.

□

Combining the previous theorems for type R and type V matrix block representations we obtain the following block representation theorem for a reversible square matrix M, so that M has the composite symmetry $M \in \mathrm{R}_n \cap \mathrm{V}_n = \mathrm{RV}_n$.

THEOREM 3.13. *Let $M \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$. Then M is a reversible square matrix with $M \in \mathrm{RV}_n$, if and only if M has the block representation*

when $n = 2k$ is even

$$M = X_n \begin{pmatrix} wE_k & 1_k a^T \\ b1_k^T & \hat{0}_k \end{pmatrix} X_n$$

with $a, b \in \mathbb{R}^k$, $w \in \mathbb{R}$,

and when $n = 2k+1$ is odd

$$M = X_n \begin{pmatrix} \sqrt{2}wE_k & w1_k & \sqrt{2}1_k d^T \\ w1_k^T & w & d^T \\ \sqrt{2}b1_k^T & b & \hat{0}_k \end{pmatrix} X_n$$

with $a, b, c, d \in \mathbb{R}^k$ and $w \in \mathbb{R}$.

Proof. For even $n = 2k$, considering the block representation of types R and V simultaneously yields

$$M = X_n \begin{pmatrix} wE_k & 1y^T \\ x1_k^T & Z \end{pmatrix} X_n = X_n \begin{pmatrix} Y & 1_k a^T \\ b1_k^T & \hat{0}_k \end{pmatrix} X_n = X_n \begin{pmatrix} wE_k & 1_k a^T \\ b1_k^T & \hat{0}_k \end{pmatrix} X_n,$$

and similarly for odd $n = 2k+1$ we have

$$M = X_n \begin{pmatrix} \sqrt{2}wE_k & w1_k & \sqrt{2}1_k z^T \\ w1_k^T & \frac{a}{\sqrt{2}} & z^T \\ \sqrt{2}x1_k^T & x & Z \end{pmatrix} X_n = X_n \begin{pmatrix} \sqrt{2}(a1_k^T + 1_k c^T) - 2\alpha E_k & a & \sqrt{2}1_k d^T \\ c^T & \alpha & d^T \\ \sqrt{2}b1_k^T & b & \hat{0}_k \end{pmatrix} X_n$$

$$= X_n \begin{pmatrix} \sqrt{2}wE_k & w1_k & \sqrt{2}1_k d^T \\ w1_k^T & w & d^T \\ \sqrt{2}b1_k^T & b & \hat{0}_k \end{pmatrix} X_n.$$

□

In Lemma 3.14 we use the block representation for reversible square matrices, to demonstrate that every reversible square matrix is also type A, so an associated square matrix (defined earlier in this chapter).

LEMMA 3.14. *Let $M = (m_{i,j}) \in \mathbb{R}^{n\times n}$ be an $n \times n$ reversible square matrix with weight w, so that $M \in \mathrm{RV}_n$, is both type R_n and type V_n. Then M is also an associated square matrix with weight w, so that $M \in \mathrm{A}_n$ is type A, with $m_{i,j} + m_{n+1-i,n+1-j} = 2w$, for all $i, j \in \mathbb{Z}_n$, and so $\mathrm{RV}_n \subset \mathrm{A}_n$.*

Proof. For $n = 2k$ is even we expand out the conjugated block representation of Theorem 3.13 to obtain

$$M = \frac{1}{2}\begin{pmatrix} 1_k a^T J_k + J_k b 1_k^t & -1_k a^T + J_k b 1_k \\ 1_k a^T J_k - b 1_k^T & -1_k a^T - b 1_k^T \end{pmatrix} + wE_n$$

$$= \frac{1}{2}\left(\begin{pmatrix} J_k b 1_k^T & J_k b 1_k \\ -b 1_k & -b 1_k^T \end{pmatrix} + \begin{pmatrix} 1_k a^T J_k & -1_k a^T \\ 1_k a^T J_k & -1_k a^T \end{pmatrix}\right) + wE_n$$

Written in this form it can be seen that the elements of the vectors a, b in all associated sums cancel, so that $m_{i,j} + m_{n+1-i,n+1-j} = 2w$.

Similarly with an odd sided reversible square we have

$$M = \begin{pmatrix} 1_k a^T J_k + J_k b 1_k^T & J_k b & -1_k a^T + J_k b 1_K^T \\ a^T J_k & 0 & -a^T \\ 1_k a^T J_k - b 1_k^T & -b & -1_k a^T - b 1_k^T \end{pmatrix} + wE_n,$$

and again we find that the elements of the vectors a, b in all associated sums cancel, so that $m_{i,j} + m_{n+1-i,n+1-j} = 2w$. Hence $\mathrm{RV}_n \subset \mathrm{A}_n$, as required. □

Example. *A traditional reversible square matrix of order 4 with weight* $w = \frac{17}{2}$ *and associated sum* $2w = 17$.

1	2	5	6
3	4	7	8
9	10	13	14
11	12	15	16

3.2 Legitimate Transforms and Reversible Square Equivalence Classes

We begin by defining the seven legitimate transforms for a reversible square $R = (r_{ij})$, as considered in [10] (see §2.4 pp 25-28) by Ollerenshaw and Brée.

Definition (of Legitimate Transforms)**.** Listed below are the seven legitimate transforms.

(i) Reflection across the horizontal central axis:

$$r'_{ij} = r_{(n+1-i)j} \text{ for all } i, j \in \mathbb{Z}_n;$$

(ii) Reflection across the vertical central axis:

$$r'_{ij} = r_{i(n+1-j)} \text{ for all } i, j \in \mathbb{Z}_n;$$

(iii) Reflection across either principal diagonal axis:

$$\begin{aligned} r'_{ij} =& r_{ji} \quad i,j \in \mathbb{Z}_n \quad \text{or} \\ r'_{ij} =& r_{(n+1-i)(n+1-j)} \quad \text{for all } i,j \in \mathbb{Z}_n; \end{aligned}$$

(iv) Swapping associated pairs of rows across the horizontal central axis:

$$\begin{aligned} r'_{i'j} =& r_{(n+1-i')j}, \\ r'_{(n+1-i')j} =& r_{i'j} \quad \text{for all } j \in \mathbb{Z}_n \text{ and fixed } i' \in \mathbb{Z}_n, \\ r'_{ij} =& r_{ij} \quad \text{for } i,j \in \mathbb{Z}_n, \text{ and } i \notin \{i', n+1-i'\}, \end{aligned}$$

(v) Swapping associated pairs of columns across the vertical central axis:

$$\begin{aligned} r'_{ij'} =& r_{i(n+1-j')}, \\ r'_{i(n+1-j')} =& r_{ij'} \quad \text{for all } i \in \mathbb{Z}_n \text{ and fixed } j' \in \mathbb{Z}_n, \\ r'_{ij} =& r_{ij} \quad \text{for } i,j \in \mathbb{Z}_n, \text{ and } j \notin \{j', n+1-j'\}, \end{aligned}$$

(vi) Simultaneously swapping rows i', i'' in one half of the matrix and swapping the associated two rows in the other half:

$$\begin{aligned} r'_{i'j} &= r_{i''j}, \\ r'_{i''j} &= r_{i'j}, \\ r'_{(n+1-i')j} &= r_{(n+1-i'')j}, \\ r'_{(n+1-i'')j} &= r_{(n+1-i')j} \quad \text{for all } \ i', i'', j \in \mathbb{Z}_n, \end{aligned}$$

$$r'_{ij} = r_{ij} \quad \text{for } i,j \in \mathbb{Z}_n \text{ and } i \notin \{i', i'', n+1-i', n+1-i''\}.$$

(vii) Simultaneously swapping columns j', j'' in one half of the matrix and swapping the associated two columns in the other half:

$$\begin{aligned} r'_{ij'} &= r_{ij''}, \\ r'_{ij''} &= r_{ij'}, \\ r'_{i(n+1-j')} &= r_{i(n+1-j'')}, \\ r'_{i(n+1-j'')} &= r_{i(n+1-j')} \quad \text{for all } \ j', j'', i \in \mathbb{Z}_n, \end{aligned}$$

$$r'_{ij} = r_{ij} \quad \text{for } i,j \in \mathbb{Z}_n \text{ and } \ j \notin \{j', j'', n+1-j', n+1-j''\}.$$

Definition (of reversible square equivalence classes)**.** Let $R = (r_{ij})$ and $R' = (r'_{ij})$ be two $n \times n$ reversible square matrices. Then we say that R and R' are *equivalent* if and only if there exists a finite combination of legitimate transforms which transform R to R'. If R and R' are equivalent then we write $R \sim R'$, and say that the two reversible squares lie in the same *equivalence class.*

The first three transforms lead to a reversible square $R' = (r'_{ij})$ that is considered to be the same reversible square (the dihedral symmetries), whereas the latter four transforms map the reversible square $R = (r_{ij})$ to an *essentially different* reversible square $R' = (r'_{ij})$ within the same equivalence class.

A *reversible square equivalence class* is defined to be the set of all $n \times n$ reversible squares that can be transformed into one another by any combination of legitimate transforms.

LEMMA 3.15. *The definition of two $n \times n$ reversible square matrices being equivalent if they are connected by a sequence of legitimate transforms, is in fact a true equivalence relation, satisfying the transitive, symmetric and reflexive properties required.*

Proof. Let $\sim$ mean that two $n \times n$ reversible squares are in the same equivalence class, so that there exists a sequence of legitimate transforms which takes one to the other. Then this relation is

- reflexive as the identity transform can be obtained by a sequence of legitimate transform, and so $M \sim M$,
- symmetric as the sequence of legitimate transforms which take M to N has an inverse legitimate transform which takes N to M, and so $M \sim N \Leftrightarrow N \sim M$,
- transitive, for if $L \sim M$ and $M \sim N$, then one can get from L to N via a sequence of legitimate transforms and hence L and N are not essentially different and $L \sim N$.

Hence equivalence, as defined here, is in fact a true equivalence relation.

□

In [10] Ollerenshaw and Brée's approach to counting all essentially different $n \times n$ reversible square matrices was to calculate the number of equivalence classes N_n multiplied by the size of each equivalence class M_n. A key point in their argument was that each equivalence class had the same number of M_n elements, and being represented by a principal element, called a *principal reversible square*, which we now define.

Definition (of Principal Reversible Squares)**.** An $n \times n$ square matrix $M = (m_{i,j})$, is said to be a *principal reversible square matrix* if and only if each of the row entries read from left to right, and column entries read from top to bottom form increasing integer sequences; the top left matrix entries satisfy $m_{1,1} = 1$ and $m_{1,2} = 2$ and all three conditions for a traditional reversible square matrix hold, where as previously defined the term *traditional* implies that the entries consist of the integers $1, 2, 3, \ldots, n^2$.

LEMMA 3.16. *Each equivalence class of $n \times n$ traditional reversible square matrices is represented by a unique $n \times n$ principal reversible square, so that distinct principal reversible squares lie in distinct equivalence classes.*

Proof. We sketch the proof given by Ollerenshaw and Brée in [10] (see § 2.4 pp 29-32).

Firstly we see that the numbers 1 and 2 must lie in the same row or column, otherwise there is no other pair of integers that can satisfy the equal cross-sum property, with sum $0 + 1 = 1$. Next we note that every traditional reversible square can be manipulated via the row and column swaps (i) and (ii), and the transpositions (iii), so that the first row begins $1, 2$, and that each row sequence (read left to right) and column sequence (read top to bottom) is an increasing sequence of integers. Hence the legitimate transforms enable every traditional reversible square to be transformed into a principal reversible square.

Next we need to show that there does not exist a combination of legitimate transforms that will transform one principal reversible square to another principal reversible square. As all row and column entries of a principal reversible square are in ascending order, it follows that applying any legitimate transform breaks this condition within one of the rows or columns. Hence each equivalence class of $n \times n$ traditional reversible squares is represented by a unique $n \times n$ principal reversible square, as required. □

The fundamental link between sum-systems and principal reversible squares is established in the following lemma.

LEMMA 3.17. *Given a principal reversible square $M \in \mathbb{R}^{n \times n}$ then the first row and column of the matrix $M - E_n$ form a sum-system.*

Proof. The third symmetry requirement for a square matrix to be a reversible square says that the cross sum of the corners of any subarray within the square is equal, so that

$$m_{ij} + m_{i'j'} = m_{ij'} + m_{i'j}.$$

As the first entry of $M - E_n$ is 0 then we have

$$(m_{11} - 1) + (m_{ij} - 1) = (m_{1j} - 1) + (m_{i1} - 1)$$

$$m_{ij} - 1 = m_{1j} - 1 + m_{i1} - 1$$

$$m_{ij} + 1 = m_{1j} + m_{ij},$$

and as M is traditional, it follows that $M - E_n$ contain the integers 0 to $n^2 - 1$. Therefore the first row and column of M must form an $n + n$ sum-system, as required. □

3.3 The Equivalence Class Cardinality

THEOREM 3.18. *Each reversible square equivalence class has cardinality M_n, where*

$$M_n = 2^{n-2}\left(\left\lfloor \frac{1}{2}n \right\rfloor !\right)^2 .$$

Proof. We need to count the number of equivalent reversible squares that arise from all possible combinations of legitimate transforms applied to an original $n \times n$ traditional reversible square matrix. We give the proof for even n. The proof for odd n then follows with the insertion of the floor function brackets.

Let M be our $n \times n$ principal reversible square with

$$M = (a_1, a_2, \ldots, a_{\frac{n}{2}}, b_{\frac{n}{2}}, \ldots, b_2, b_1),$$

so that a_i and b_i are the column vectors of M.

The number of equivalent reversible squares corresponds to the number of column and row swaps one can carry out. The structure of a reversible square is that column b_i must be in the mirrored position across the vertical axis from a_i for all i. The legitimate transforms tell us that we can:

1. Swap any number of a_i's with b_i's.
2. Swap any a_i and a_j provided we swap b_i and b_j for all $i, j \in \mathbb{Z}_k \quad (n = 2k)$.

Hence considering the left hand half of M whose columns are $(a_1, a_2, \ldots, a_{\frac{n}{2}})$, we have $\frac{n}{2}!$ choices for the columns. With the columns chosen we now count the number of swaps across the vertical axis which is given by the sum over zero column swaps, one column swap, two column swaps up to $\frac{n}{2}$ column swaps. This equates to

$$\binom{\frac{n}{2}}{0} + \binom{\frac{n}{2}}{1} + \binom{\frac{n}{2}}{2} + \ldots + \binom{\frac{n}{2}}{\frac{n}{2}} = \sum_{i=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{i} = 2^{\frac{n}{2}}.$$

Multiplying these together then gives $(\frac{n}{2})!2^{\frac{n}{2}}$ equivalent but different $n \times n$ reversible square matrices, and squaring this number then includes all possible row transforms.

Finally we note that in terms of excluding the dihedral symmetries, although the legitimate transforms do not allow for rotations of the original matrix, they do afford reflections across the horizontal, vertical and diagonal axes. Hence we divide through by 4, to deduce that each reversible square equivalence class has cardinality M_n with

$$M_n = 2^{n-2}\left(\left\lfloor \frac{1}{2}n \right\rfloor !\right)^2 .$$

Here the floor function incorporates the odd sided argument in which the middle row and column is fixed. □

An alternative proof can be obtained via the block representation of a legitimate transform given in Lemma 4.2.

Example. *For* $n = 4$*, there are three principal reversible squares, and so equivalence classes, each with 16 elements. We give all 16 elements for the simplest principal reversible square below.*

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

1	2	3	4
9	10	11	12
5	6	7	8
13	14	15	16

5	6	7	8
1	2	3	4
13	14	15	16
9	10	11	12

5	6	7	8
13	14	15	16
1	2	3	4
9	10	11	12

1	3	2	4
5	7	6	8
9	11	10	12
13	15	14	16

1	3	2	4
9	11	10	12
5	7	6	8
13	15	14	16

5	7	6	8
1	3	2	4
13	15	14	16
9	11	10	12

5	7	6	8
13	15	14	16
1	3	2	4
9	11	10	12

2	1	4	3
6	5	8	7
10	9	12	11
14	13	16	15

2	1	4	3
10	9	12	11
6	5	8	7
14	13	16	15

6	5	8	7
2	1	4	3
14	13	16	15
10	9	12	11

6	5	8	7
14	13	16	15
2	1	4	3
10	9	12	11

2	4	1	3
6	8	5	7
10	12	9	11
14	16	13	15

2	4	1	3
10	12	9	11
6	8	5	7
14	16	13	15

6	8	5	7
2	4	1	3
14	16	13	15
10	12	9	11

6	8	5	7
2	4	1	3
14	16	13	15
10	12	9	11

4 Sum-and-Distance to Reversible Square Bijection

In this chapter we establish that the number of principal reversible squares of size n is equal to the number of $m+m$ inclusive sum-and-distance systems if $n = 2m+1$ is odd and equal to the number of $m+m$ non inclusive sum-and-distance systems if $n = 2m$.

To begin we recall our definitions for two specific types of $m+m$ sum-and-distance systems. Note: we change the notation of the entries from a_i to $\hat{a}_i$ and b_i to $\hat{b}_i$ to distinguish between the entries of the vectors a, b in the block representation of a reversible square.

Definition (of $m+m$ sum-and-distance systems)**.** For natural number m, let

$$A = \{\hat{a}_1, \ldots, \hat{a}_m\}, \quad \text{and} \quad B = \{\hat{b}_1, \ldots, \hat{b}_m\},$$

be two sets of positive integers of m distinct elements, ordered in terms of increasing value, with $A \cap B = \phi$.

Then we say that the set pair A and B form a $m+m$ *(non-inclusive) sum-and-distance system*, if and only if the corresponding *sum-and-distance set*, A^*B, defined by

$$A^*B = (A+B) \cup |A-B| = \{\hat{a}_j + \hat{b}_k, |\hat{a}_j - \hat{b}_k| \,:\, j,k \in \{1, \ldots m\}\},$$

contains the odd integers $1, 3, 5, \ldots, 4m^2 - 1$.

Similarly we say that A and B form an *inclusive m+m sum-and-distance system* if and only if the union of the corresponding *sum-and-distance set*, A^*B, with A and B, given by $A^*B \cup A \cup B = A \cup B \cup (A+B) \cup |A-B|$

$$A^*B = \{\ \hat{a}_j,\ \hat{b}_k,\ \ \hat{a}_j + \hat{b}_k,\ |\hat{a}_j - \hat{b}_k| \,:\, j,k \in \{1, \ldots m\}\},$$

contains the consecutive integers $1, 2, 3, \ldots 2m^2 + 2m$.

4.1 Traditional Reversible Squares and Sum-and-Distance Systems

THEOREM 4.1. *A square matrix $M \in \mathbb{R}^{n\times n}$, is a traditional reversible square if and only if the set of absolute values in the vectors $a, b \in \mathbb{R}^k$ in the block representation in Theorem 3.13 comprise a sum-and-distance system, non-inclusive if $n = 2k$ is even, and inclusive if $n = 2k+1$ is odd.*

Proof. Let M be a traditional $n \times n$ reversible square with side length $n = 2k$, so that M contains the integers 1 to n^2. Subtracting the weight multiplied by E_n to obtain the weightless reversible square $M^0 = M - \frac{n^2+1}{2}E_n$, we have that M^0

contains the set of

$$\{1, 2, \ldots, n^2\} - \frac{(n^2+1)}{2} = \frac{1}{2}\Big\{-(n^2-1), -(n^2-3), \ldots, -1, 1, \ldots, (n^2-1)\Big\}.$$

From the block representation of a weightless reversible square we have that

$$M^0 = X_n \begin{pmatrix} \hat{0}_k & 1_k a^T \\ b1_k^T & \hat{0}_k \end{pmatrix} X_n = \frac{1}{2}\begin{pmatrix} J_k b 1_k + 1_k a^T J_k & J_k b 1_k J_k - 1_k a^T J \\ -b1_k + J_k 1_k a^T J_k & -b 1_k J_k - J_k 1_k a^T \end{pmatrix},$$

for some $a, b \in \mathbb{R}^k$ such that $a = (a_1, \ldots, a_k)^T,\ b = (b_1, \ldots, b_k)^T$, then comparing set entries yields

$$\begin{aligned}&\frac{1}{2}\Big\{-(n^2-1), -(n^2-3), \ldots, -1, 1, \ldots, n^2-3, n^2-1\Big\} \\ =&\frac{1}{2}\left\{b_j + a_i, b_j - a_i, -b_j + a_i, -b_j - a_i \ \mid\ i, j \in \mathbb{Z}_k\right\}.\end{aligned}$$

We can then write

$$\{1, 3, \ldots, n^2-1\} = \{|a_i| + |b_j|,\ ||a_i| - |b_j|| \ \mid\ i, j \in \mathbb{Z}_k\},$$

so that the set of absolute values of the entries of a and b form a non-inclusive sum-and-distance system.

Conversely, let $A = \{\hat{a}_1, \ldots \hat{a}_k\},\ B = \{\hat{b}_1, \ldots, \hat{b}_k\}$ form a non-inclusive sum-and-distance system, and define the k-length vectors a and b to contain every entry of A and B respectively in any permutation and with either positive or negative choice of sign. Then the block representation of the matrix

$$M^0 = X_n \begin{pmatrix} \hat{0}_k & 1_k a^T \\ b1_k^T & \hat{0}_k \end{pmatrix} X_n = \frac{1}{2}\begin{pmatrix} J_k b 1_k + 1_k a^T J_k & J_k b 1_k J_k - 1_k a^T J \\ -b1_k + J_k 1_k a^T J_k & -b 1_k J_k - J_k 1_k a^T \end{pmatrix}.$$

has the set of entries

$$\begin{aligned}&\frac{1}{2}\Big\{\hat{b}_j + \hat{a}_i, \hat{b}_j - \hat{a}_i, -\hat{b}_j + \hat{a}_i, -\hat{b}_j - \hat{a}_i \ \mid\ \hat{a}_i \in A,\ \hat{b}_j \in B\Big\} \\ =&\frac{1}{2}\Big\{-(n^2-1), -(n^2-3), \ldots, -1, 1, \ldots, n^2-3, n^2-1\Big\}\end{aligned}$$

in some order and hence $M = M^0 + \frac{n^2+1}{2}E_n$ is an $n \times n$ traditional reversible square.

For odd $n = 2k+1$, proceeding as before we remove the weight and expand the block representation to obtain

$$M^0 = X_n \begin{pmatrix} \hat{0}_k & 0_k & 2(1_k a^T) \\ 0_k^T & 0 & \sqrt{2}a^T \\ 2b1_k^T & \sqrt{2}b & \hat{0}_k \end{pmatrix} X_n = \begin{pmatrix} 1_k a^T J_k + J_k b 1_k^T & J_k b & -1_k a^T + J_k b 1_k^T \\ a^T j_k & 0 & -a^T \\ 1_k a^T j_k - b 1_k^T & -b & -1_k a^T - b1_k^T \end{pmatrix},$$

for $a, b \in \mathbb{R}^k$ such that $a = (a_1, \ldots, a_k)^T$, $b = (b_1, \ldots, b_k)^T$, from which we deduce that the four corner $k \times k$ blocks contain all entries $\pm a_i \pm b_j$ for $i, j \in \mathbb{Z}_k$, with the additional $4k$ entries $\pm a_i$ and $\pm b_j$ along the central row and column with the 0 element in the central cell. It follows that values in vectors a and b must satisfy the following,

$$\{|a_i| + |b_j|,\ ||a_i| - |b_j||,\ |a_i|,\ |b_j||\ i, j \in \mathbb{Z}_k\} = \left\{1, 2 \ldots, \frac{n^2-1}{2}\right\},$$

if M is a reversible square. Therefore the set of absolute values of the entries of a and b form an inclusive sum-and-distance system.

Conversely, let $A = \{\hat{a}_1, \ldots \hat{a}_k\}$, $B = \{\hat{b}_1, \ldots, \hat{b}_k\}$ form an inclusive sum-and-distance system, then define k-length vectors a and b to contain every entry of A and B respectively in any permutation and can be either positive or negative. Let M^0 be the resulting weightless reversible square matrix and by the expanding its block representation from above, it follows that $M = M^0 + \frac{n^2+1}{2} E_n$ is an $n \times n$ traditional reversible square, as it contains the integers

$$\{1, 2, \ldots n^2\},$$

and exhibits the three symmetry properties required.

□

LEMMA 4.2. *Let $M^0 \in \mathbb{R}^{n \times n}$ be a traditional weightless reversible square with block representation vectors a and b. Then applying any combination of the seven legitimate transforms defined previously to the matrix M^0 results in new block representation vectors c and d, whose absolute values are the same as the absolute values of a and b, i.e. the ordered sets formed from the absolute values of the two vectors are invariant under the legitimate transform actions.*

It follows that the sum-and-distance system formed from the absolute values of the block representation vectors is invariant under the action of the seven legitimate transforms.

Proof. We now describe the seven legitimate transforms in terms of matrix algebra, for both the even and the odd order cases, from which it is apparent that the legitimate transforms do not change the absolute values within the block representation vectors.

(i) Reflection across the horizontal central axis.

$$J_n X_n \begin{pmatrix} \hat{0}_k & 1_k a^T \\ b 1_k^T & \hat{0}_k \end{pmatrix} X_n = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & -J_k \\ J_k & I_k \end{pmatrix} \begin{pmatrix} \hat{0}_k & 1_k a^T \\ b 1_k^T & \hat{0}_k \end{pmatrix} X_n = X_n \begin{pmatrix} \hat{0}_k & 1_k a^T \\ -b 1_k^T & \hat{0}_k \end{pmatrix} X_n,$$

and

$$J_n X_n \begin{pmatrix} \hat{0}_k & 0_k & 2(1_k a^T) \\ 0_k^T & 0 & \sqrt{2}a^T \\ 2b1_k^T & \sqrt{2}b & \hat{0}_k \end{pmatrix} X_n = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0_k & -J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ J_k & 0_k & I_k \end{pmatrix} \begin{pmatrix} \hat{0}_k & 0_k & 2(1_k a^T) \\ 0_k^T & 0 & \sqrt{2}a^T \\ 2b1_k^T & \sqrt{2}b & \hat{0}_k \end{pmatrix} X_n$$

$$= X_n \begin{pmatrix} \hat{0}_k & 0_k & 2(1_k a^T) \\ 0_k^T & 0 & \sqrt{2}a^T \\ -2b1_k^T & \sqrt{2}b & \hat{0}_k \end{pmatrix} X_n.$$

(ii) Reflection across the vertical central axis.

$$X_n \begin{pmatrix} \hat{0}_k & 1_k a^T \\ b1_k^T & \hat{0}_k \end{pmatrix} X_n J_n = X_n \begin{pmatrix} \hat{0}_k & 1_k a^T \\ b1_k^T & \hat{0}_k \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & J_k \\ -J_k & I_k \end{pmatrix} = X_n \begin{pmatrix} \hat{0}_k & -1_k a^T \\ b1_k^T & \hat{0}_k \end{pmatrix} X_n,$$

and

$$X_n \begin{pmatrix} \hat{0}_k & 0_k & 2(1_k a^T) \\ 0_k^T & 0 & \sqrt{2}a^T \\ 2b1_k^T & \sqrt{2}b & \hat{0}_k \end{pmatrix} X_n J_n = X_n \begin{pmatrix} \hat{0}_k & 0_k & 2(1_k a^T) \\ 0_k^T & 0 & \sqrt{2}a^T \\ 2b1_k^T & \sqrt{2}b & \hat{0}_k \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0_k & J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ -J_k & 0_k & I_k \end{pmatrix}$$

$$= X_n \begin{pmatrix} \hat{0}_k & 0_k & -2(1_k a^T) \\ 0_k^T & 0 & -\sqrt{2}a^T \\ 2b1_k^T & \sqrt{2}b & \hat{0}_k \end{pmatrix} X_n.$$

(iii) Transposition.

$$\left(X_n \begin{pmatrix} \hat{0}_k & 1_k a^T \\ b1_k^T & \hat{0}_k \end{pmatrix} X_n \right)^T = X_n^T \begin{pmatrix} \hat{0}_k & 1_k a^T \\ b1_k^T & \hat{0}_k \end{pmatrix}^T X_n^T = X_n \begin{pmatrix} \hat{0}_k & 1_k b^T \\ a1_k^T & \hat{0}_k \end{pmatrix} X_n,$$

and

$$\left(X_n \begin{pmatrix} \hat{0}_k & 0_k & 2(1_k a^T) \\ 0_k^T & 0 & \sqrt{2}a^T \\ 2b1_k^T & \sqrt{2}b & \hat{0}_k \end{pmatrix} X_n \right)^T = X_n^T \left(\begin{pmatrix} \hat{0}_k & 0_k & 2(1_k a^T) \\ 0_k^T & 0 & \sqrt{2}a^T \\ 2b1_k^T & \sqrt{2}b & \hat{0}_k \end{pmatrix} \right)^T X_n^T$$

$$= X_n \begin{pmatrix} \hat{0}_k & 0_k & 2(1_k b^T) \\ 0_k^T & 0 & \sqrt{2}b^T \\ 2a1_k^T & \sqrt{2}a & \hat{0}_k \end{pmatrix} X_n.$$

In consideration of the remaining four legitimate transforms we use the block representation expansions

$$M = X_n \begin{pmatrix} \hat{0}_k & 1_k a^T \\ b 1_k^T & \hat{0}_k \end{pmatrix} X_n = \frac{1}{2}\left(\begin{pmatrix} J_k b 1_k^T & J_k b 1_k^T \\ -b 1_k^T & -b 1_k^T \end{pmatrix} + \begin{pmatrix} 1_k a^T J_k & -1_k a^T \\ 1_k a^T J_k & 1_k a^T \end{pmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{pmatrix} b_k & \dots & b_k & b_k & \dots & b_k \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ b_1 & \dots & b_1 & b_1 & \dots & b_1 \\ -b_1 & \dots & -b_1 & -b_1 & \dots & -b_1 \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ -b_k & \dots & -b_k & -b_k & \dots & -b_k \end{pmatrix} + \begin{pmatrix} a_k & \dots & a_1 & -a_1 & \dots & -a_k \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ a_k & \dots & a_1 & -a_1 & \dots & -a_k \\ a_k & \dots & a_1 & -a_1 & \dots & -a_k \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ a_k & \dots & a_1 & -a_1 & \dots & -a_k \end{pmatrix}\right),$$

and

$$M = X_n \begin{pmatrix} \hat{0}_k & 0_k & 2 1_k a^T \\ 0_k^T & 0 & \sqrt{2} a^T \\ 2 b 1_k^T & \sqrt{2} b & \hat{0}_k \end{pmatrix} X_n = \begin{pmatrix} J_k b 1_k^T & J_k b & b 1_k^T J_k \\ 0_k^T & 0 & 0_k^T \\ -b 1_k^T & -b & -b 1_k^T \end{pmatrix} + \begin{pmatrix} 1_k a^T J_k & 0_k & -J_k 1_k a^T \\ a^T J_k & 0 & -a^T \\ J_k 1_k a^T J_k & 0_k & -J_k 1_k a^T \end{pmatrix}$$

$$= \begin{pmatrix} b_k & \dots & b_k & b_k & b_k & \dots & b_k \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ b_1 & \dots & b_1 & b_1 & b_1 & \dots & b_1 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ -b_1 & \dots & -b_1 & -b_1 & -b_1 & \dots & -b_1 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_k & \dots & -b_k & -b_k & -b_k & \dots & -b_k \end{pmatrix} + \begin{pmatrix} a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \\ a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \\ a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \end{pmatrix}.$$

(iv) Swapping associated pairs of rows s and $n+1-s$ across the horizontal central axis.

$$M = X_n \begin{pmatrix} \hat{0}_k & 1_k a^T \\ b(s) 1_k^T & \hat{0}_k \end{pmatrix} X_n$$

$$= \frac{1}{2}\left(\begin{pmatrix} b_k & \dots & b_k & b_k & \dots & b_k \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ -b_{k+1-s} & \dots & -b_{k+1-s} & -b_{k+1-s} & \dots & -b_{k+1-s} \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ b_1 & \dots & b_1 & b_1 & \dots & b_1 \\ -b_1 & \dots & -b_1 & -b_1 & \dots & -b_1 \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ b_{k+1-s} & \dots & b_{k+1-s} & b_{k+1-s} & \dots & b_{k+1-s} \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ -b_k & \dots & -b_k & -b_k & \dots & -b_k \end{pmatrix} + \begin{pmatrix} a_k & \dots & a_1 & -a_1 & \dots & -a_k \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ a_k & \dots & a_1 & -a_1 & \dots & -a_k \\ a_k & \dots & a_1 & -a_1 & \dots & -a_k \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ a_k & \dots & a_1 & -a_1 & \dots & -a_k \end{pmatrix}\right),$$

where $b(s) = (b_1, \dots, -b_{k+1-s}, \dots, b_k)^T = \hat{I}(k+1-s)b$, and

$$\hat{I}(s)_{i,j} = \begin{cases} 0 & i \neq j, \\ 1 & i = j,\ i \neq s,\ j \neq s, \\ -1 & i = j = s, \end{cases}$$

and

$$M = X_n \begin{pmatrix} \hat{0}_k & 0_k & 21_k a^T \\ 0_k^T & 0 & \sqrt{2}a^T \\ 2b(s)1_k^T & \sqrt{2}b(s) & \hat{0}_k \end{pmatrix} X_n = \begin{pmatrix} a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \\ a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \\ a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \end{pmatrix}$$

$$+ \begin{pmatrix} b_k & \dots & b_k & b_k & b_k & \dots & b_k \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ -b_{k+1-s} & \dots & -b_{k+1-s} & -b_{k+1-s} & -b_{k+1-s} & \dots & -b_{k+1-s} \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_1 & \dots & b_1 & b_1 & b_1 & \dots & b_1 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ -b_1 & \dots & -b_1 & -b_1 & -b_1 & \dots & -b_1 \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots & \\ b_{k+1-s} & \dots & b_{k+1-s} & b_{k+1-s} & b_{k+1-s} & \dots & b_{k+1-s} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ -b_k & \dots & -b_k & -b_k & -b_k & \dots & -b_k \end{pmatrix}$$

where $b(s) = (b_1, \dots, -b_{k+1-s}, \dots, b_k)^T = \hat{I}(k+1-s)b$.

(v) Swapping associated pairs of columns across the vertical central axis.

$$M = X_n \begin{pmatrix} \hat{0}_k & 1_k a(s)^T \\ b1_k^T & \hat{0}_k \end{pmatrix} X_n = \frac{1}{2} \left(\begin{pmatrix} b_k & \dots & b_k & b_k & \dots & b_k \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ b_1 & \dots & b_1 & b_1 & \dots & b_1 \\ -b_1 & \dots & -b_1 & -b_1 & \dots & -b_1 \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ -b_k & \dots & -b_k & -b_k & \dots & -b_k \end{pmatrix} \right.$$

$$+\begin{pmatrix} a_k & \dots & -a_{k+1-s} & \dots & a_1 & -a_1 & \dots & a_{k+1-s} & \dots & -a_k \\ \vdots & \ddots & \vdots & \iddots & \vdots & \vdots & \iddots & \vdots & \ddots & \vdots \\ a_k & \dots & -a_{k+1-s} & \dots & a_1 & -a_1 & \dots & a_{k+1-s} & \dots & -a_k \\ a_k & \dots & -a_{k+1-s} & \dots & a_1 & -a_1 & \dots & a_{k+1-s} & \dots & -a_k \\ \vdots & \iddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \iddots & \vdots \\ a_k & \dots & -a_{k+1-s} & \dots & a_1 & -a_1 & \dots & a_{k+1-s} & \dots & -a_k \end{pmatrix}\Bigg)\Bigg),$$

where $a(s) = \hat{I}(k+1-s)(a_1, \dots, -a_{k+1-s}, \dots, a_n)^T$,

and

$$M = X_n \begin{pmatrix} \hat{0}_k & 0_k & 21_k a(s)^T \\ 0_k^T & 0 & \sqrt{2}a(s)^T \\ 2b1_k^T & \sqrt{2}b & \hat{0}_k \end{pmatrix} X_n = \begin{pmatrix} b_k & \dots & b_k & b_k & b_k & \dots & b_k \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ b_1 & \dots & b_1 & b_1 & b_1 & \dots & b_1 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ -b_1 & \dots & -b_1 & -b_1 & -b_1 & \dots & -b_1 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_k & \dots & -b_k & -b_k & -b_k & \dots & -b_k \end{pmatrix}$$

$$+\begin{pmatrix} a_k & \dots & -a_{k+1-s} & \dots & a_1 & 0 & -a_1 & \dots & a_{k+1-s} & \dots & -a_k \\ \vdots & \ddots & \vdots & \iddots & \vdots & \vdots & \vdots & \iddots & \vdots & \ddots & \vdots \\ a_k & \dots & -a_{k+1-s} & \dots & a_1 & 0 & -a_1 & \dots & a_{k+1-s} & \dots & -a_k \\ a_k & \dots & -a_{k+1-s} & \dots & a_1 & 0 & -a_1 & \dots & a_{k+1-s} & \dots & -a_k \\ a_k & \dots & -a_{k+1-s} & \dots & a_1 & 0 & -a_1 & \dots & a_{k+1-s} & \dots & -a_k \\ \vdots & \iddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \iddots & \vdots \\ a_k & \dots & -a_{k+1-s} & \dots & a_1 & 0 & -a_1 & \dots & a_{k+1-s} & \dots & -a_k \end{pmatrix}$$

where $a(s) = (a_1, \dots, -a_{k+1-s}, \dots, a_n)^T = \hat{I}(n+1-s)$.

(vi) Simultaneously swapping two rows in one half of the matrix and swapping the associated two rows s and t in the other half:

$$M = X_n \begin{pmatrix} \hat{0}_k & 1_k a^T \\ \tilde{b}(s,t)1_k^T & \hat{0}_k \end{pmatrix} X_n = \begin{pmatrix} a_k & \dots & a_1 & -a_1 & \dots & -a_k \\ \vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\ a_k & \dots & a_1 & -a_1 & \dots & -a_k \\ a_k & \dots & a_1 & -a_1 & \dots & -a_k \\ \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ a_k & \dots & a_1 & -a_1 & \dots & -a_k \end{pmatrix}\Bigg)\Bigg)$$

$$
+\frac{1}{2}\left(\begin{pmatrix}
b_k & \dots & b_k & b_k & \dots & b_k \\
\vdots & \ddots & \vdots & \vdots & & \iddots \\
b_{k+1-t} & \dots & b_{k+1-t} & b_{k+1-t} & \dots & b_{k+1-t} \\
\vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\
b_{k+1-s} & \dots & b_{k+1-s} & b_{k+1-s} & \dots & b_{k+1-s} \\
\vdots & \ddots & \vdots & \vdots & & \iddots \\
b_1 & \dots & b_1 & b_1 & \dots & b_1 \\
-b_1 & \dots & -b_1 & -b_1 & \dots & -b_1 \\
\vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\
-b_{k+1-s} & \dots & -b_{k+1-s} & -b_{k+1-s} & \dots & -b_{k+1-s} \\
\vdots & \ddots & \vdots & \vdots & & \iddots \\
-b_{k+1-t} & \dots & -b_{k+1-t} & -b_{k+1-t} & \dots & -b_{k+1-t} \\
\vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\
-b_k & \dots & -b_k & -b_k & \dots & -b_k
\end{pmatrix}\right.,
$$

where $t > s$ and

$\tilde{b}(s,t) = (b_1, \dots, b_{k-t}, b_{k+1-s}, b_{k+2-t}, \dots, b_{k-s}, b_{k+1-t}, b_{k+2-s} \dots, b_k)^T = \tilde{I}(s,t)b$,

with $\tilde{I}(s,t) = \begin{cases} 0 & i \neq j,\ i = j = s,\ i = j = t \\ 1 & i = j,\ i = t\ j = s,\ i = s\ j = t, \end{cases}$

and

$$
M = X_n \begin{pmatrix} \hat{0}_k & 0_k & 2(1_k a^T) \\ 0_k^T & 0 & \sqrt{2}a^T \\ 2\tilde{b}(s,t)1_k^T & \sqrt{2}\tilde{b}(s,t) & \hat{0}_k \end{pmatrix} X_n = \begin{pmatrix}
a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \\
\vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\
a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \\
a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \\
a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k \\
\vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_k & \dots & a_1 & 0 & -a_1 & \dots & -a_k
\end{pmatrix}
$$

$$
+\begin{pmatrix}
b_k & \dots & b_k & b_k & b_k & \dots & b_k \\
\vdots & \ddots & \vdots & \vdots & \vdots & & \iddots \\
b_{k+1-t} & \dots & b_{k+1-t} & b_{k+1-t} & b_{k+1-t} & \dots & b_{k+1-t} \\
\vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
b_{k+1-s} & \dots & b_{k+1-s} & b_{k+1-s} & b_{k+1-s} & \dots & b_{k+1-s} \\
\vdots & \ddots & \vdots & \vdots & \vdots & & \iddots \\
b_1 & \dots & b_1 & b_1 & b_1 & \dots & b_1 \\
0 & \dots & 0 & 0 & 0 & \dots & 0 \\
-b_1 & \dots & -b_1 & -b_1 & -b_1 & \dots & -b_1 \\
\vdots & \iddots & \vdots & \vdots & \ddots & \vdots & \\
-b_{k+1-s} & \dots & -b_{k+1-s} & -b_{k+1-s} & -b_{k+1-s} & \dots & -b_{k+1-s} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \\
-b_{k+1-t} & \dots & -b_{k+1-t} & -b_{k+1-t} & -b_{k+1-t} & \dots & -b_{k+1-t} \\
\vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-b_k & \dots & -b_k & -b_k & -b_k & \dots & -b_k
\end{pmatrix},
$$

where $t > s$ and $\tilde{b}(s,t) = (b_1, \dots, b_{k-t}, b_{k+1-s}, b_{k+2-t}, \dots, b_{k-s}, b_{k+1-t}, b_{k+2-s} \dots, b_k)^T = \tilde{I}(s,t)b$.

(vii) Simultaneously swapping two columns in one half of the matrix and swapping the associated two columns s and t in the other half:

$$
M = X_n \begin{pmatrix} \hat{0}_k & 1_k \tilde{a}(s,t)^T \\ b 1_k^T & \hat{0}_k \end{pmatrix} X_n = \frac{1}{2} \left(\begin{pmatrix}
b_k & \dots & b_k & b_k & \dots & b_k \\
\vdots & \ddots & \vdots & \vdots & \iddots & \vdots \\
b_1 & \dots & b_1 & b_1 & \dots & b_1 \\
-b_1 & \dots & -b_1 & -b_1 & \dots & -b_1 \\
\vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\
-b_k & \dots & -b_k & -b_k & \dots & -b_k
\end{pmatrix} \right.
$$

$$
\left. + \begin{pmatrix}
a_k & \dots & a_{k+1-t} & \dots & a_{k+1-s} & \dots & a_1 & -a_1 & \dots & -a_{k+1-s} & \dots & -a_{k+1-t} & \dots & -a_k \\
\vdots & \ddots & \vdots & \iddots & \vdots & \ddots & \vdots & \vdots & \iddots & \vdots & \ddots & \vdots & \iddots & \vdots \\
a_k & \dots & a_{k+1-t} & \dots & a_{k+1-s} & \dots & a_1 & -a_1 & \dots & -a_{k+1-s} & \dots & -a_{k+1-t} & \dots & -a_k \\
a_k & \dots & a_{k+1-t} & \dots & a_{k+1-s} & \dots & a_1 & -a_1 & \dots & -a_{k+1-s} & \dots & -a_{k+1-t} & \dots & -a_k \\
\vdots & \iddots & \vdots & \ddots & \vdots & \iddots & \vdots & \vdots & \ddots & \vdots & \iddots & \vdots & \ddots & \vdots \\
a_k & \dots & a_{k+1-t} & \dots & a_{k+1-s} & \dots & a_1 & -a_1 & \dots & -a_{k+1-s} & \dots & -a_{k+1-t} & \dots & -a_k
\end{pmatrix} \right),
$$

where $t > s$ and $\tilde{a}(s,t) = (a_1, \dots, a_{k+1-t}, \dots, a_{k+1-s}, \dots, a_k)^T = \tilde{I}(s,t)a$,

and

$$M = X_n \begin{pmatrix} \hat{0}_k & 0_k & 21_k\tilde{(a)}(s,t)^T \\ 0_k^T & 0 & \sqrt{2}\tilde{(a)}(s,t)^T \\ 2b1_k^T & \sqrt{2}b & \hat{0}_k \end{pmatrix} X_n = \begin{pmatrix} b_k & \dots & b_k & b_k & b_k & \dots & b_k \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ b_1 & \dots & b_1 & b_1 & b_1 & \dots & b_1 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ -b_1 & \dots & -b_1 & -b_1 & -b_1 & \dots & -b_1 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_k & \dots & -b_k & -b_k & -b_k & \dots & -b_k \end{pmatrix}$$

$$+ \begin{pmatrix} a_k & \dots & a_{k+1-t} & \dots & a_{k+1-s} & \dots & a_1 & 0 & -a_1 & \dots & -a_{k+1-s} & \dots & -a_{k+1-t} & \dots & -a_k \\ \vdots & \ddots & \vdots & \iddots & \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots & \ddots & \vdots & \iddots & \vdots \\ a_k & \dots & a_{k+1-t} & \dots & a_{k+1-s} & \dots & a_1 & 0 & -a_1 & \dots & -a_{k+1-s} & \dots & -a_{k+1-t} & \dots & -a_k \\ a_k & \dots & a_{k+1-t} & \dots & a_{k+1-s} & \dots & a_1 & 0 & -a_1 & \dots & -a_{k+1-s} & \dots & -a_{k+1-t} & \dots & -a_k \\ a_k & \dots & a_{k+1-t} & \dots & a_{k+1-s} & \dots & a_1 & 0 & -a_1 & \dots & -a_{k+1-s} & \dots & -a_{k+1-t} & \dots & -a_k \\ \vdots & \iddots & \vdots & \ddots & \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots & \iddots & \vdots & \ddots & \vdots \\ a_k & \dots & a_{k+1-t} & \dots & a_{k+1-s} & \dots & a_1 & 0 & -a_1 & \dots & -a_{k+1-s} & \dots & -a_{k+1-t} & \dots & -a_k \end{pmatrix},$$

where $t > s$ and $\tilde{(a)}(s,t) = (a_1, \dots, a_{k+1-t}, \dots, a_{k+1-s}, \dots, a_k)^T = \tilde{I}(s,t)a$.

□

4.2 Enumeration of Equivalent Vectors

With the new method of expressing legitimate transforms in terms of their block representation established, we now give an alternative proof for the number of equivalent $n \times n$ reversible square matrices M_n in each equivalence class.

Proof. (second proof of Theorem 3.18)

Let $a = \{a_1, \dots, a_{\lfloor \frac{n}{2} \rfloor}\}^T$ and $b = \{b_1, \dots, b_{\lfloor \frac{n}{2} \rfloor}\}^T$ be our block representation vectors, and

$$a' = \{\pm a_1, \dots, \pm a_{\lfloor \frac{n}{2} \rfloor}\}^T \text{ and } b' = \{\pm b_1, \dots \pm b_{\lfloor \frac{n}{2} \rfloor}\}^T.$$

Then the choice of sign, along with the extra two possibilities for the placement of the vectors on either the central row or column when n is odd give us a total of 2^n possibilities, for both odd and even n. Removing the four dihedral transforms, which we class as the same square, and whose block representation is given by

$$\left\{ X_n \begin{pmatrix} \hat{0}_k & W \\ V & \hat{0}_k \end{pmatrix} X_n,\ X_n \begin{pmatrix} \hat{0}_k & -W \\ V & \hat{0}_k \end{pmatrix} X_n,\ X_n \begin{pmatrix} \hat{0}_k & W \\ -V & \hat{0}_k \end{pmatrix} X_n,\ X_n \begin{pmatrix} \hat{0}_k & -W \\ -V & \hat{0}_k \end{pmatrix} X_n \right\},$$

where if n is even $V, W \in \mathbb{R}^{n \times n}$ and if n is odd $V \in \mathbb{R}^{k+1 \times k}$ and $W \in \mathbb{R}^{k \times k+1}$ when not counted reduces to 2^{n-2} choices.

The placing of the entries in the two vectors then give a further $\left(\lfloor \frac{n}{2} \rfloor !\right)^2$ possibilities, which in total gives us $2^{n-2} \left(\lfloor \frac{n}{2} \rfloor !\right)^2$ equivalent matrices, as required. □

THEOREM 4.3. *The number of $n \times n$ reversible square equivalence classes N_n, is equal to the number of $k + k$ inclusive sum-and-distance systems if $n = 2k + 1$ is odd, and equal to the number of $k + k$ non-inclusive sum-and-distance systems if $n = 2k$ is even.*

COROLLARY 4.4. *With each principal reversible square matrix we can associate a unique $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ sum-and-distance system of the corresponding type, depending upon the parity of n.*

Proof. By Lemma 4.2 two $n \times n$ equivalent reversible square matrices have the same absolute values of the block representation vectors, and by Theorem 4.1, the set pair formed from the block representation vectors form a sum-and-distance system. Hence the number of $n \times n$ reversible square equivalence classes N_n is equal to the number of $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ non-inclusive sum-and-distance systems if n is even and $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ inclusive sum-and-distance systems if n is odd.

By Lemma 3.16, distinct $n \times n$ principal reversible squares lie in distinct equivalence classes, and so the set of all $n \times n$ principal reversible squares can be used to represent all the $n \times n$ reversible square equivalence classes, and hence the Corollary.

□

5 Principal Reversible Squares: their Construction and Enumeration

The first known recorded occurrence of a reversible square matrix was in 1897 and is attributed to Eamon McClintock [44]. McClintock established the important result that there exists a $1-1$ correspondence between reversible squares (then called McClintock squares) and most-perfect squares (defined at the beginning of Chapter 4). He managed to construct a subset of all $n \times n$ traditional reversible squares, and it was not until the work of Ollerenshaw and Brée just over one hundred years later in 1998 [10], that a general construction method was established for the set of all such squares. This then led to the enumeration of all $n \times n$ reversible squares of doubly-even order ($n = 4k$) and so of all $n \times n$ most-perfect square matrices of doubly-even order.

Ollerenshaw and Brée approached this construction problem by initially refining the definition of the symmetry properties required and coined the term *reversible squares*. They employed a lengthy and ingenious nested block structure argument (see Chapter 4 of [10]), from which they applied combinatorial methods to enumerate all $n \times n$ principal reversible squares. In what follows we outline their approach to understanding the block structure, which we restate in terms of divisor paths. The divisor path approach lends itself to more number theoretical methods which we employ in Chapter 6 to embed the full enumeration argument with the theory of divisor functions. For the interested reader we advise that this thesis be read in conjunction with Ollerenshaw's and Brée's book.

To help progress their arguments they employed a number of concepts which we now define below.

Definition (of similar sequences)**.** Two sequences $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots, b_n$ are said to be *similar sequences* when the differences between sequence terms in corresponding positions in both sequences is constant, i.e $a_i - b_i = a_j - b_j \ \forall i, j \in \mathbb{Z}_n$.

Definition (of similar blocks)**.** An $n \times m$ *block* is defined to be an $n \times m$ matrix containing a sequence of nm entries $a_1, a_2, \ldots, a_{nm}$. Two $n \times m$ blocks are said to be *similar blocks* when the entries in all corresponding rows and columns in both blocks are similar sequences.

Definition (of reverse similarity)**.** Two sequences $a_1, a_2 \ldots a_n$ and $b_1, b_2 \ldots b_n$ are said to be in *reverse similarity*, if when the second sequence is reversed, pairs of entries in the same position sum to the same constant, i.e. $a_i + b_{n+1-i} = c \ \forall\, i \in \mathbb{Z}_n$.

An equivalent definition to that given at the beginning of Chapter 4 for an $n \times n$ reversible square can now be stated.

LEMMA 5.1. *An $n \times n$ reversible square $M = (m_{ij})$ has the following symmetry properties.*

(i) *The row sequences are similar, so that*

$$m_{ij} - m_{ij'} = m_{i'j} - m_{i'j'}, \quad i, i'j, j' \in \mathbb{Z}_n.$$

(ii) *The column sequences are similar, so that*

$$m_{ij} - m_{i'j} = m_{ij'} - m_{i'j'}, \quad i, i', j, j' \in \mathbb{Z}_n.$$

(iii) *Pairs of entries which are diametrically opposite sum to* $2w$ *(if* M *is traditional then* $w = \frac{1}{2}(n^2 + 1)$*).*

Proof. Properties (i), and (ii) follow directly from the equal cross sums condition for a reversible square, $m_{ij} + m_{i'j'} = m_{ij'} + m_{i'j}, \quad i, i', j, j' \in \mathbb{Z}_n$. Property (iii), is simply the associated property (A) previously established in Lemma 3.14 for reversible square matrices. □

5.1 Smallest Corner Blocks and Divisor Paths

Definition (of smallest corner blocks)**.** In a square matrix M containing consecutive integer entries, reading left to right along the first row, we define *the smallest corner block (SCB)* to have width j_1 equal to the first break in the sequence of consecutive integers, and depth i_1 equal to the number of rows down to which reading left to right and top to bottom in this $i_1 \times j_1$ block, that the sequence of consecutive integers continues. If $j_1 = n$, then the whole square M is taken to be the SCB.
Remark. For a principal reversible square matrix we must have $j_1 \geq 2$ otherwise (as demonstrated in the proof of Lemma 3.16), the first two entries in the first row cannot be $1, 2$, so that the SCB block has width as least two.
Definition (of divisor paths)**.** Let $n,\ i_u,\ j_v \in \mathbb{N}$ for $u, v \in \mathbb{Z}_\alpha$. We say that the ordered pairing of tuples

$$\{(i_1,\ i_2, \ldots, i_{\alpha-1},\ i_\alpha),\ (j_1,\ j_2, \ldots,\ j_{\alpha-1},\ j_\alpha)\}$$

form a *divisor path set of length* α for n, if they satisfy the divisibility chains

$$i_1 |\ i_2 |\ \ldots |\ i_{\alpha-1} |\ i_\alpha = n, \quad 1 < i_1 < i_2 < \ldots\ < i_{\alpha-1} < i_\alpha = n,$$

and

$$j_1 |\ j_2 |\ \ldots |\ j_{\alpha-1} | j_\alpha | n\ , \quad 1 < j_1 < j_2 < \ldots < j_{\alpha-1} < j_\alpha \leq n.$$

Remark. In their book, Ollerenshaw and Brée defined these divisor paths as $(\alpha - 1)$-progressive paths.

The divisor path approach gives us a new way of defining the nested block construction that uniquely determines each principal reversible square, and we now use this understanding to recursively define a block structure using the divisor path sets. In the following definition we use an existing block to recursively define a new nested block, which preserves block similarity,

Recursive nested block construction. Given the divisor path set for n of length α

$$\{(i_1,\ i_2,\ldots,i_{\alpha-1},\ i_\alpha),\ (j_1,\ j_2,\ldots,\ j_{\alpha-1},\ j_\alpha)\}$$

so that

$$i_1|\ i_2|\ \ldots|\ i_{\alpha-1}|\ i_\alpha = n,\quad 1 < i_1 < i_2 < \ldots\ < i_{\alpha-1} < i_\alpha = n,$$

and

$$j_1|\ j_2|\ \ldots|\ j_{\alpha-1}|j_\alpha|n\ ,\quad 1 < j_1 < j_2 < \ldots < j_{\alpha-1} < j_\alpha \leq n,$$

we define recursively the sequence of nested block structured matrices $R_1, R_2, \ldots, R_\alpha$ such that

$$R_1 = (k_1 + j_1(k_2-1)\mid 1 \leq k_1 \leq j_1, 1 \leq k_2 \leq i_1) = \begin{pmatrix} 1 & 2 & \ldots & j_1 \\ j_1+1 & j_1+2 & \ldots & 2j_1 \\ \vdots & \ddots & & \vdots \\ j_1+i_1 & \ldots & & i_1 j_1 \end{pmatrix},$$

and thereafter by

$$R_m = \left(R_{m-1} + (i_{m-1}j_{m-1})((k_1-1) + \left((k_2-1)\left(\frac{j_m}{j_{m-1}}\right)\right)\right.$$

$$\left.\mid 1 \leq k_1 \leq \frac{j_m}{j_{m-1}},\ 1 \leq k_2 \leq \frac{i_m}{i_{m-1}}\right)$$

$$= \begin{pmatrix} R_{m-1} & R_{m-1} + (i_{m-1}j_{m-1}) & \ldots & R_{m-1} + (i_{m-1}j_{m-1})(\frac{j_m}{j_{m-1}}-1) \\ & & \ldots & \\ \vdots & \ddots & & \vdots \\ & \ldots & & R_{m-1} + (i_{m-1}j_{m-1})(\frac{j_m}{j_{m-1}} - 1 + \frac{j_m}{j_{m-1}} - 1) \end{pmatrix},$$

where if $j_\alpha \neq n$, then in the final step we tile the $n \times n$ matrix using the $n \times j_\alpha$ largest corner block n/j_α times to obtain $R_{\alpha+1}$.

LEMMA 5.2. *The above divisor path set construction produces a principal reversible square.*

Proof. The construction ensures that the first block R_1 is a principal reversible rectangular matrix and from the recurrence, also that the recurrence sequence of matrices $R_2, \ldots, R_\alpha$ are principal reversible rectangular matrices.

It follows that the resulting square matrix, R_α when $j_\alpha = n$ and $R_{\alpha+1}$ when $j_\alpha < n$, is a principal reversible square as it has 1,2, as the first two elements of the first row, has increasing entries from left to right and top to bottom and satisfies reverse row and column similarity and the equal cross sum property, as required. □

THEOREM 5.3 (Nested block theorem)**.** *Let $\Omega(n)$ the number of prime factors of n including repeats, and let $\{(i_1,\ i_2,\ldots,i_{\alpha-1},\ i_\alpha=n),\ (j_1,\ j_2,\ldots,\ j_{\alpha-1},\ j_\alpha)\}$ be a given divisor path set for n. Then by Lemma 5.2, the recursive nested block construction defines an $n\times n$ principal reversible square, and conversely every principal reversible square R, arises from exactly one divisor path set, so that every principal reversible square can be uniquely represented by the form*

$$R(\{(i_1,i_2,\ldots,i_{\alpha-1},\ n),(j_1,j_2,\ldots,\ j_{\alpha-1},j_\alpha)\}),\quad \alpha\le\Omega(n).$$

Here in the $n\times n$ square, the $i_1\times j_1$ similar blocks are nested in $i_2\times j_2$ similar blocks which are nested in $i_3\times j_3$ similar blocks, and so on, until we obtain $i_{\alpha-1}\times j_{\alpha-1}$ similar blocks nested inside $n\times j_\alpha$ blocks. As a final step these $n\times j_\alpha$ similar blocks make up the complete $n\times n$ block. For any fixed level of nested block, the sequence of blocks from either left to right or top to bottom form an increasing sequence of similar blocks (as defined previously).

Proof. The proof that the nested block structure is uniquely defined by the divisor path set and vice versa, comprises the whole of Chapter 3 in [10]. However the fundamental idea underpinning this proof is that the row and column similarity, along with the increasing row and column sequence property of all principal reversible squares, implies that the first row defines the blocks along the first row and column, which then completely determines the nesting structure as one moves from the smallest corner block to the largest corner block in M. □

Example. *When $n=12$, consider the principal reversible square M defined by the (length 2) divisor path set $R(\{(2,12),(2,4)\})$, so that $i_1=2$, $j_1=2$, $i_2=n=12$ and $j_2=4$. Then*

$M =$

1	2	5	6	49	50	53	54	97	98	101	102
3	4	7	8	51	52	55	56	99	100	103	104
9	10	13	14	57	58	61	62	105	106	109	110
11	12	15	16	59	60	63	64	107	108	111	112
17	18	21	22	65	66	69	70	113	114	117	118
19	20	23	24	67	68	71	72	115	116	119	120
25	26	29	30	73	74	77	78	121	122	125	126
27	28	31	32	75	76	79	80	123	124	127	128
33	34	37	38	81	82	85	86	129	130	133	134
35	36	39	40	83	84	87	88	131	132	135	136
41	42	45	46	89	90	93	94	137	138	141	142
43	44	47	48	91	92	95	96	139	140	143	144

,

where | represents a gap larger then 1 *in a row or column. It can be seen that the three* 12×4 *blocks are all similar, where the* 2×2 *blocks are nested in the* 12×4 *blocks which are nested in the* 12×12 *square.*

5.2 The Number of Principal Reversible Squares N_n

Prior to her work with Brée, Ollerenshaw had succeeded in counting all principal reversible squares of the form $n = 2^r p^s$ using a complicated combinatorial method in which she considered the number of paths in an r by s factor table (see Chapter 5 of [10]). After a considerable amount of simplification Ollerenshaw obtained the concise formula for the number of $n \times n$ principal reversible squares N_n, with $n \equiv 0 \pmod 4$, given by

$$N_n = \frac{1}{2}\binom{2s}{s}\sum_{i=0}^{min[r,s]}\binom{s}{i}\binom{r+s}{r-i}\binom{2r+s-i}{r+s}.$$

Subsequently Ollerenshaw and Brée obtained a formula for the total number of principal reversible squares for any doubly-even n (so $n \equiv 0 \pmod 4$) with prime factorisation $n = p_0^{\beta_0} p_1^{\beta_1} \dots p_r^{\beta_r}$, so that $\Omega(n) = \beta_0 + \beta_1 + \dots + \beta_r$, where $p_0 = 2$ and $\beta_0 \geq 2$. Extending the methods applied to the $n = 2^r p^s$, case they established the more general formula for N_n given by

$$N_n = \sum_{v=0}^{\infty} W_v(n)\left\{W_v(n) + W_{v+1}(n)\right\},$$

where

$$W_v(n) = \begin{cases} \sum_{i=0}^{v}(-1)^{v+i}\binom{v+1}{i+1}\prod_{j=0}^{r}\binom{\beta_j+i}{i}, & \text{if } 0 \leq v \leq \Omega(n) \\ 0, & \text{otherwise.} \end{cases}$$

In the remainder of this chapter we approach the enumeration problem from a different perspective, utilising the divisor path notation, and demonstrate that a general formula for N_n, valid for all $n \in \mathbb{N}$, can be stated in terms of the divisor functions introduced in Chapter 2. Hence we extend the enumeration argument to all $n \times n$ principal reversible square matrices, and by Theorem 4.3, to all $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ sum-and-distance systems of either type, depending on the parity of n.

As detailed in Theorem 5.3, the sequence of progressive factors uniquely determine a principal reversible square, and similarly given a principal reversible square we can deduce the unique sequence of progressive factors, whose permutations we now consider in more detail.

Definition (of progressive factors)**.** The principal reversible square uniquely determined by

$$R(\{(i_1, i_2, \ldots, i_{\alpha-1}, n), (j_1, j_2, \ldots, j_\alpha)\}),$$

is said to have a *length α divisor path* or equivalently in the notation of [10], $\alpha - 1$ *progressive factors.*

LEMMA 5.4. *Let $a_j(n)$ and $b_j(n)$ respectively denote the number of possible left tuples and right tuples of divisor paths sets for n of length j (so $(j-1)$-progressive factors). Then the total number of divisor path sets is given by N_n, with*

$$N_n = \sum_{j=1}^{\Omega(n)} a_j(n) b_j(n),$$

where $\Omega(n)$ is the total number of prime factors of n, including repeats.

Proof. By Theorem 5.3, for a given integer $n \in \mathbb{N}$, there is a one-to-one correspondence between distinct divisor path sets and principal reversible square matrices. It follows that the number of distinct divisor path sets is equal to the number of $n \times n$ distinct reversible square matrices. The left and right tuple divisor paths are defined only when they are of equal sizes, then the total number of possibilities is given by the sum of the products of the possibilities for the different divisor path lengths, $a_j(n)b_j(n)$, and hence the result. □

We now approach the problem of counting the number of distinct divisor paths by considering the left and right hand tuple possibilities which correspond to the gaps greater than one in the first row and column. We illustrate the row and column breaks for divisor paths of length one, two, three and four when $n = 60$.

For $R(\{(60), (j_1)\})$, i.e. 0 progressive factors or divisor path length 1, we can have that $i_1 = n = 60$, is fixed, and that j_1 is any non-trivial divisor of 60, which we can write as

$$b_1(60) = \sum_{e|60,\ e\neq 1} 1 = \sum_{e|60,\ e\neq 1} c_1(e) = c_1(60) + c_2(60) = 1 + 10 = 11,$$

by Lemma 2.9.

In total then, a divisor path of length 1 gives us $a_1(60)b_1(60) = c_1(60) \times (c_1(60) + c_2(60)) = 1 \times 11 = 11$ choices.

For $R(\{(i_1, 60), (j_1, j_2)\})$, i.e. one progressive factor or divisor path length 2, we require that $1 < i_1 | i_2 = 60$, $i_1 < i_2 = 60$ and $1 < j_1 |\ j_2 |\ 60$, $1 < j_1 < j_2 \leq 60$. Hence, for $a_2(60)$, we have to consider all the possible divisor paths of length 2 for $i_2 = n = 60$, which is given by $c_2(60) = 10$. Similarly for $b_2(60)$, we have to consider all the possible divisor paths of length 2 for $j_2 \leq 60$, which is given by

$$b_2(60) = \sum_{e|60,\ e\neq 1} c_2(e) = c_2(60) + c_3(60) = 10 + 21 = 31,$$

by Lemma 2.9.

Hence, in total, a divisor path of length 2 gives us $a_2(60)b_2(60) = c_2(60) \times (c_2(60) + c_3(60)) = 10 \times 31 = 310$ choices.

The table below then details the 10 divisor paths of length 1 in the first row, and the 21 divisor paths of length 2 in rows two to eleven.

$j_2 \mid n$	$j_1 \mid j_2 \mid 60$with $j_1 \notin \{1, j_2\}$	Number of choices for$\{j_1, j_2\}$
60	$30, 20, 15, \ldots, 4, 3, 2$	$c_2(60) = 10$
30	$15, 10, 6, 5, 3, 2$	$c_2(30) = 6$
20	$10, 5, 4, 2$	$c_2(20) = 4$
15	$5, 3$	$c_2(15) = 2$
12	$6, 4, 3, 2$	$c_2(12) = 4$
10	$5, 2$	$c_2(10) = 2$
6	$3, 2$	$c_2(6) = 2$
5	$-$	$c_2(5) = 0$
4	2	$c_2(4) = 1$
3	$-$	$c_2(3) = 0$
2	$-$	$c_2(2) = 0$
Total	$-$	$c_2(60) + c_3(60) = 10 + 21 = 31$

,

Using a similar argument (by Lemma 2.9), the divisor paths for $n = 60$ of length 3 are then given by

$$a_3(60)b_3(60) = c_3(60) \sum_{\substack{e \mid 60 \\ e \neq 1}} c_3(e) = c_3(60) + c_4(60) = 21(21 + 12) = 693,$$

and for the divisor paths for $n = 60$ of length we have

$$a_4(60)b_4(60) = c_4(60) \sum_{\substack{e \mid 60 \\ e \neq 1}} c_4(e) = c_4(60)(c_4(60) + c_5(60)) = 12(12 + 0) = 144,$$

divisor paths.

We do not need to consider divisor paths greater than 4, as $n = 60 = 2^2.3.5$ only has 4 prime factors, so that no divisor paths of length 4 or greater exist.

Putting this altogether we have that

$$N_{60} = \sum_{j=1}^{4} a_j(60)b_j(60) = \sum_{j=1}^{4} c_j(60)\left(c_j(60) + c_{j+1}(60)\right) = \sum_{j=1}^{4} c_j(60)c_j^{(1)}(60) = 1158,$$

so that there are 1158 distinct principal reversible squares of order 60.

We now formulate the above argument in generality for N_n.

THEOREM 5.5. *With $a_j(n)$ and $b_j(n)$ defined respectively to be the total number* *of left tuples and right tuples of divisor path sets for n of length j (so $(j-1)$ progressive factors), we have that*

$$a_j(n) = c_j(n), \quad \text{and} \quad b_j(n) = a_j(n) + a_{j+1}(n) = c_j(n) + c_{j+1}(n),$$

and

$$a_j(n) = \sum_{i=0}^{j} (-1)^i \begin{pmatrix} j \\ i \end{pmatrix} d_{j-i}(n), \quad \text{and} \quad b_j(n) = \sum_{i=0}^{j} (-1)^i \begin{pmatrix} j \\ i \end{pmatrix} d_{j-i+1}(n).$$

COROLLARY 5.6. *The number of $n \times n$ principal reversible squares, and so the number of $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ non-inclusive sum-and distance systems if $n = 2k$ is even and $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ inclusive sum-and distance systems if $n = 2k+1$ is odd, is given by*

$$N_n = \sum_{j=1}^{\Omega(n)} c_j(n)\left(c_j(n) + c_{j+1}(n)\right) = \sum_{j=1}^{\Omega(n)} c_j^{(0)}(n) c_j^{(1)}(n).$$

COROLLARY 5.7. *The number of $n \times n$ traditional reversible square matrices is given by*

$$\begin{aligned} M_n N_n &= 2^{n-2} \left(\left\lfloor \frac{1}{2} n \right\rfloor ! \right)^2 \sum_{j=1}^{\Omega(n)} c_j(n)\left(c_j(n) + c_{j+1}(n)\right) \\ &= 2^{n-2} \left(\left\lfloor \frac{1}{2} n \right\rfloor ! \right)^2 \sum_{j=1}^{\Omega(n)} c_j^{(0)}(n) c_j^{(1)}(n). \end{aligned}$$

Proof. (of Theorem 5.5) Let $n \in \mathbb{N}$ and let

$$\{(i_1,\ i_2, \ldots, i_{\alpha-1},\ n),\ (j_1,\ j_2, \ldots,\ j_{\alpha-1},\ j_\alpha)\}$$

be a divisor path set of length α for n, so that they satisfy the divisibility chains

$$i_1 |\ i_2 |\ \ldots |\ i_{\alpha-1} |\ i_\alpha = n, \quad 1 < i_1 < i_2 < \ldots < i_{\alpha-1} < i_\alpha = n,$$

and

$$j_1 |\ j_2 |\ \ldots |\ j_{\alpha-1} | j_\alpha | n\ , \quad 1 < j_1 < j_2 < \ldots < j_{\alpha-1} < j_\alpha \leq n.$$

For $R(\{(n),(j_1)\})$, i.e. 0 progressive factors or divisor path length 1, we can have that $i_1 = n$, is fixed, and that j_1 is any non-trivial divisor of n, which we can write as

$$b_1(n) = \sum_{e|n,\ e\neq 1} 1 = \sum_{e|n,\ e\neq 1} c_1(e) = c_1(n) + c_2(n),$$

by Lemma 2.9.

In total then, a divisor path of length 1 gives us $a_1(n)b_1(n) = c_1(n) \times (c_1(n) + c_2(n))$ choices.

For $R(\{(i_1,n),(j_1,j_2)\})$, i.e. one progressive factor or divisor path length 2, we require that $1 < i_1 | i_2 = n$, $i_1 < i_2 = n$ and $1 < j_1 |\ j_2 |\ n$, $1 < j_1 < j_2 \leq n$. Hence, for $a_2(n)$, we have to consider all the possible divisor paths of length 2 for $i_2 = n$, which is given by $c_2(n)$. Similarly for $b_2(n)$, we have to consider all the possible divisor paths of length 2 for $j_2 \leq n$, which is given by

$$b_2(n) = \sum_{e|n,\ e\neq 1} c_2(e) = c_2(n) + c_3(n),$$

by Lemma 2.9.

Hence, in total, a divisor path for n of length 2 gives us $a_2(n)b_2(n) = c_2(n) \times (c_2(n) + c_3(n))$ choices.

Using a similar argument (by Lemma 2.9), in the general case it then follows that the total number of divisor paths of length j are given by

$$a_j(n)b_j(n) = c_j(n) \sum_{\substack{e|n \\ e\neq 1}} c_j(e) = c_j(n)(c_j(n) + c_{j+1}(n)) = c_j(n)c_j^{(1)}(n),$$

by Lemma 2.17.

Therefore we have that $a_j(n) = c_j(n)$, and $b_j(n) = c_j(n) + c_{j+1}(n) = a_j(n) + a_{j+1}(n)$, where we do not need to consider divisor paths with $j > \Omega(n)$, as then $c_j(n) = 0$, so that $a_j(n) = b_j(n) = 0$.

By Lemma 2.9, it follows that

$$a_j(n) = c_j(n) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} d_{j-i}(n),$$

and for $b_j(n)$ we use the relation

$$b_j(n) = c_j(n) + c_{j+1}(n) = \sum_{e|n} c_j(e) = \sum_{e|n} a_j(e) = \sum_{e|n} \sum_{i=0}^{j} (-1)^i \binom{j}{i} d_{j-i}(e),$$

using the above expression for $a_j(e)$ in terms of $d_j(e)$. Hence we have

$$b_j(n) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} \sum_{e|n} d_{j-i}(e) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} d_{j-i+1}(e),$$

by Lemma 2.3, as required.

To see Corollary 5.6 Summing the product $a_j(n)b_j(n)$ over all possible divisor path lengths we find that the total number of principal reversible square matrices N_n of order n (and so sum-and-distance systems of the corresponding type) is given by

$$N_n = \sum_{j=1}^{\Omega(n)} a_j(n)b_j(n) = \sum_{j=1}^{\Omega(n)} c_j(n)\left(c_j(n) + c_{j+1}(n)\right) = \sum_{j=1}^{\Omega(n)} c_j(n)c_j^{(1)}(n),$$

and the result follows.

To see Corollary 5.7, by Theorem 3.18 the cardinality of each (traditional) reversible square equivalence class M_n, is given by

$$M_n = 2^{n-2}\left(\left\lfloor\frac{1}{2}n\right\rfloor!\right)^2,$$

and by Theorem 4.3 the total number of equivalence classes is given by N_n. Hence the total number of traditional $n \times n$ reversible square matrices is given by M_nN_n, which can be written as

$$M_nN_n = 2^{n-2}\left(\left\lfloor\frac{1}{2}n\right\rfloor!\right)^2 \sum_{j=1}^{\Omega(n)} c_j(n)c_j^{(1)}(n),$$

as required. □

THEOREM 5.8. *Let $n \in \mathbb{N}$ with $n \geq 2$. Then n is a prime number if and only if $N_n = 1$, so that when $n = 2$, the only $1+1$ sum-and-distance system is $\{\{1\},\{2\}\}$, and when $n = 2m+1$, the only $m+m$ inclusive sum-and-distance system is again the canonical system given by*

$$A = \{1, 2, 3, \dots m\}, \qquad B = \{n, 2n, 3n, \dots mn\} = nA.$$

Hence there is a unique $m+m$ inclusive sum-and-distance system if $2m+1$ is a prime number.

Proof. If n is a prime number, then $\Omega(n) = 1$, and the sum in Corollary 5.6 becomes

$$= \sum_{j=1}^{1} c_j(n)\left(c_j(n) + c_{j+1}(n)\right) = c_1(n)\left(c_1(n) + c_2(n)\right) = 1(1+0) = 1,$$

as $c_2(n) = 0$ when n is a prime number. Hence when n is a prime number $N_n = 1$, and there is only one sum-and-distance system. By Lemmas 1.2 and 1.3, this must be the canonical sum-and-distance system.

Conversely, as $\Omega(n) \geq 1$ for $n \geq 2$, if $N_n = 1$, then taking into account that the sum is over non-negative terms we have

$$N_n = c_1(n)\left(c_1(n) + c_2(n)\right) + \text{ terms with } j \geq 2 \ \geq 1(1 + c_2(n)).$$

Hence for $N_n = 1$ we require that $c_j(n) = 0$ for $j \geq 2$, which only happens when n is a prime number.

By Corollary 5.6 it follows that $N_n = 1$, $\Leftrightarrow$ the only $m+m$ sum-and-distance system is the canonical sum-and-distance system given by

$$A = \{1, 2, 3, \dots m\}, \qquad B = \{n, 2n, 3n, \dots mn\} = nA,$$

so that with each prime number we can associate a unique sum-and-distance system.

□

LEMMA 5.9 (alternative proof that $b_j(n) = a_j(n)+a_{j+1}(n)$)**.** *Let $a_j(n)$ and $b_j(n)$ respectively be the total number of left tuples and right tuples of divisor path sets for n of length j (so $(j-1)$ progressive factors). Then we have*

$$b_j(n) = a_j(n) + a_{j+1}(n).$$

Proof. Let A_j and B_j be defined as the Dirichlet series for the arithmetic functions a_j and b_j. Assuming the relation (given in the proof of Theorem 5.5)

$$b_j(n) = \sum_{e|n} a_j(e)$$

and $c_j(n) = a_j(n)$, by Dirichlet's convolution of Lemma 2.15 we have that

$$\begin{aligned}
B_j(s) = \sum_{n=1}^{\infty} \frac{b_j(n)}{n^s} =& \zeta(s)A_j(s) = \zeta(s)(\zeta(s)-1)^j \\
=& (\zeta(s)-1)(\zeta(s)-1)^j + (\zeta(s)-1)^j \\
=& (\zeta(s)-1)^{j+1} + (\zeta(s)-1)^j \\
=& \sum_{n=1}^{\infty} \frac{c_{j+1}(n)}{n^s} + \sum_{n=1}^{\infty} \frac{c_j(n)}{n^s} = \sum_{n=1}^{\infty} \frac{c_{j+1}(n)+c_j(n)}{n^s} \\
=& \sum_{n=1}^{\infty} \frac{a_j(n)+a_{j+1}(n)}{n^s}.
\end{aligned}$$

By the uniqueness of Dirichlet series it follows that $a_j(n) = b_j(n) + b_{j+1}(n)$. □

6 Sum-and-Distance Systems Construction Algorithm

By Theorem 5.3 there is a one-to-one correspondence between the divisor path sets for n and the $n \times n$ principal reversible square matrices, where the recursive definition of Lemma 5.2 allows us to uniquely construct a principal reversible square from a given divisor path set for n.

In this Chapter we give a construction for all sum-and-distance systems by identifying the first row and column of each principal reversible square in terms of its divisor path block structure, and so by Theorem 1.5, the sum system corresponding to the principal reversible square. We then reverse the stepwise algorithm given Theorem 1.5 to obtain the corresponding sum-and-distance system. A lattice point argument is then used to demonstrate that the construction yields all sum-and-distance systems.

We begin by recalling that an even sided reversible square can be written as,

$$M = \frac{1}{2}\begin{pmatrix} 1_k a^T J_k + J_k b 1_k^T & -1_k a^T + J_k b 1_k \\ 1_k a^T J_k - b 1_k^T & -1_k a^T - b 1_k^T \end{pmatrix} + wE_n$$

$$= \frac{1}{2}\left(\begin{pmatrix} J_k b 1_k^T & J_k b 1_k \\ -b 1_k & -b 1_k^T \end{pmatrix} + \begin{pmatrix} 1_k a^T J_k & -1_k a^T \\ 1_k a^T J_k & -1_k a^T \end{pmatrix}\right) + wE_n$$

$$= \frac{1}{2}\left(\begin{pmatrix} b_k & \dots & b_k & b_k & \dots & b_k \\ \vdots & & \vdots & \vdots & & \vdots \\ b_1 & \dots & b_1 & b_1 & \dots & b_1 \\ -b_1 & \dots & -b_1 & -b_1 & \dots & b_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ -b_k & \dots & -b_k & -b_k & \dots & -b_k \end{pmatrix} + \begin{pmatrix} a_k & \dots & a_1 & -a_1 & \dots & -a_k \\ \vdots & & \vdots & \vdots & & \vdots \\ a_k & \dots & a_1 & -a_1 & \dots & -a_k \\ a_k & \dots & a_1 & -a_1 & \dots & -a_k \\ \vdots & & \vdots & \vdots & & \vdots \\ a_k & \dots & a_1 & -a_1 & \dots & -a_k \end{pmatrix}\right) + wE_n.$$

It follows that the first row of the square has the tuple

$$\left(\frac{1}{2}(b_k + a_k) + w, \frac{1}{2}(b_k + a_{k-1}) + w, \dots, \frac{1}{2}(b_k + a_1) + w, \frac{1}{2}(b_k - a_1) + w\ ,\right.$$
$$\left.\dots, \frac{1}{2}(b_k - a_{k-1}) + w, \frac{1}{2}(b_k - a_k) + w\right),$$

and the first column has the tuple

$$\left(\frac{1}{2}(b_k + a_k) + w, \dots, \frac{1}{2}(b_1 + a_k) + w, \frac{1}{2}(-b_1 + a_k) + w, \dots, \frac{1}{2}(-b_k + a_k) + w\right).$$

By subtracting $w + b_k/2$ and $w + a_k/2$ respectively from the first row and column of M, we obtain the sets $a = \{a_1, \dots, a_k\}$ and $b = \{b_1, \dots, b_k\}$.

Similarly with an odd sided reversible square we have

$$M = \begin{pmatrix} 1_k a^T J_k + J_k b 1_k^T & J_k b & -1_k a^T + J_k b 1_K^T \\ a^T J_k & 0 & -a^T \\ 1_k a^T J_k - b 1_k^T & -b & -1_k a^T - b 1_k^T \end{pmatrix} + wE_n.$$

Then the first row has the tuple,

$$\begin{aligned} &(a^T J_k + b_k,\ b_k,\ -a^T + b_k) + w \\ &= (b_k + a_k + w, \ldots, b_k + a_1 + w, b_k + w, b_k - a_1 + w, \ldots, b_k - a_k + w) \end{aligned}$$

and the first column has the tuple,

$$\begin{aligned} &(1_k a^T J_k + J_k b 1_k^T,\ a^T J_k,\ 1_k a^T J_k) \\ &= (b_k + a_k + w, \ldots, b_1 + a_k + w, a_k + w, -b_1 + a_k + w \ldots, -b_k + a_k + w). \end{aligned}$$

Hence by Lemma 1.5 and Lemma 3.17 we have identified the sum system comprised from the entries of the first row and column of a principal reversible square. Now the weight is found to be $w = \frac{1}{2}(a_k + b_k)$ and by Theorem 4.1 the set of absolute values of a and b form the corresponding sum-and-distance system.

In the following definition we use the fact that the dimensions of the constituent blocks that comprise an $n \times n$ principal reversible square, can be ascertained from the corresponding divisor path set for n.

Definition. Let n and α be positive integers and

$$\{\hat{i}, \hat{j}\} = \{(i_1, i_2, \ldots, i_\alpha = n),\ (j_1, j_2, \ldots, j_\alpha)\}$$

be a *divisor path set* for n of length α, so that

$$i_1 | i_2 | \ldots | i_\alpha, \quad 1 < i_1 < i_2 \ldots < i_\alpha = n,$$

$$j_1 | j_2 \ldots | j_\alpha | n, \quad 1 < j_1 < j_2 < \ldots < j_\alpha \leq n.$$

Then we define a co-ordinate set that allows us to determine in which block each entry in the first row or column lies using the ordered set of lattice points $L_\alpha \in \mathbb{N}^{\alpha+1}$ and $M_\alpha \in \mathbb{N}^\alpha$ such that

$$L_\alpha = \left((k_0, k_1, \ldots, k_\alpha) \mid 1 \leq k_0 \leq j_1, \ldots, 1 \leq k_{\alpha-1} \leq \frac{j_\alpha}{j_{\alpha-1}}, 1 \leq k_\alpha \leq \frac{n}{j_\alpha} \right)$$

$$= \left((k_0, k_1, \ldots, k_\alpha) \mid 1 \leq k_0 \leq j_1, 1 \leq k_n \leq \frac{j_{n+1}}{j_n}, 1 \leq n \leq \alpha - 1,\ 1 \leq k_\alpha \leq \frac{n}{j_\alpha} \right),$$

$$M_\alpha = \left((m_0, m_1, \ldots, m_{\alpha-1}) | 1 \leq m_0 \leq i_1,\ 1 \leq m_1 \leq \frac{i_2}{i_1}, \ldots, 1 \leq m_{\alpha-1} \leq \frac{i_\alpha}{i_{\alpha-1}} = \frac{n}{i_{\alpha-1}} \right)$$

$$= \left((m_0, m_1, \ldots, m_{\alpha-1}) | \ 1 \le m_0 \le i_1, \ 1 \le m_t \le \frac{i_{t+1}}{i_t}, \ 1 \le t \le \alpha - 1 \right).$$

With the above notation established we can now more concisely describe the sum system corresponding to the first row and column entries of a principal reversible square via its divisor path set as detailed in the following lemmas.

LEMMA 6.1. *Let $R = (r_{i,j})$ be an $n \times n$ (traditional) principal reversible square, whose first row is denoted by r. Then r can be written as the ordered tuple*

$$r = (k_0 + (k_1 - 1)(i_1 j_1) + (k_2 - 1)(i_2 j_2) + \ldots + (k_\alpha - 1)(i_\alpha j_\alpha) \mid k \in L_\alpha)$$

$$= \left(k_0 + \sum_{n=1}^{\alpha} (k_n - 1)(i_n j_n) | 1 \le k_0 \le j_1, \ 1 \le k_1 \le \frac{j_2}{j_1}, \ldots, 1 \le k_{\alpha-1} \le \frac{j_\alpha}{j_{\alpha-1}}, \ 1 \le k_\alpha \le \frac{n}{j_\alpha} \right).$$

Proof. From the recursive definition of a principal reversible square R described in Chapter 5, we have that

$$R = \left(k_0 + (m_0 - 1) j_1 + (k_1 - 1)(i_1 j_1) + (m_1 - 1) \frac{j_2}{j_1} (i_1 j_1) + \ldots + (k_\alpha - 1)(i_\alpha j_\alpha) \right.$$

$$+ \underbrace{(m_\alpha - 1)}_{=0} \left(\frac{n}{j_\alpha} \right) (i_\alpha j_\alpha) \mid 1 \le k_0 \le j_1, \ 1 \le m_0 \le i_1, \ 1 \le j_1 \le \frac{j_2}{j_1}, \ 1 \le m_1 \le \frac{i_2}{i_1},$$

$$\left. \ldots, 1 \le k_\alpha \le \frac{j_{\alpha+1}}{j_\alpha} = \frac{n}{j_\alpha}, \ 1 \le m_\alpha \le \frac{i_{\alpha+1}}{i_\alpha} = \frac{n}{n} \right)$$

$$= \left(k_0 + (m_0 - 1) j_1 + \sum_{v=1}^{\alpha} \left((k_v - 1)(i_v j_v) + (m_v - 1) \left(\frac{j_{v+1}}{j_v} (i_v j_v) \right) \right) \right. \mid$$

$$\left. 1 \le k_0 \le j_1, \ 1 \le m_0 \le i_1, \ 1 \le k_t \le \frac{j_{t+1}}{j_t}, 1 \le m_t \le \frac{i_{t+1}}{i_t}, \ 1 \le t \le \alpha \right)$$

where $k = (k_0, k_1, \ldots k_\alpha)$, $m = (m_0, m_1, \ldots, m_\alpha)$ represent the column and row co-ordinates respectively. Setting the lattice co-ordinates of M_α to be $m = (m_0, m_1, \ldots, m_\alpha) = (1, 1, \ldots, 1)$ then ensures the form required for the first row r of a principal reversible square is given by

$$r = \left(k_1 + \sum_{v=1}^{\alpha} \left((1 - 1)(i_v j_v) \left(\frac{j_{v+1}}{j_v} \right) + (k_v - 1)(i_v j_v) \right) | \ 1 \le k_0 \le i_1, \ 1 \le k_t \le \frac{j_{t+1}}{j_t}, \ 1 \le t \le \alpha \right)$$

$$= \left(k_1 + \sum_{v=1}^{\alpha} (k_v - 1)(i_v j_v) | \ 1 \le k_0 \le i_1, \ 1 \le k_t \le \frac{j_{t+1}}{j_t}, \ 1 \le t \le \alpha \right).$$

□

LEMMA 6.2. *Let $R = (r_{i,j})$ be an $n \times n$ (traditional) principal reversible square, whose first column is denoted by c. Then c can be written as the ordered tuple*

$$c = \left(\ 1 + (m_0 - 1) j_1 + \sum_{k=1}^{\alpha-1} (m_k - 1) \frac{j_{k+1}}{j_k} (i_k j_k) \mid m \in M_\alpha \right)$$

$$= \Bigg(1 + (m_0 - 1)j_1 + (m_1 - 1)\ \frac{j_2}{j_1}\ (i_1 j_1) + (m_2 - 1)\ \frac{j_3}{j_2}\ (i_2 j_2) + \ldots$$

$$+(m_{\alpha-1} - 1)\left(\frac{j_\alpha}{j_{\alpha-1}}\right)(i_{\alpha-1}j_{\alpha-1}) \mid 1 \leq m_0 \leq i_1,\ 1 \leq m_1 \leq \frac{i_2}{i_1}, \ldots, 1 \leq m_\alpha \leq \frac{i_\alpha}{i_{\alpha-1}} = 1\Bigg).$$

Proof. From the recursive definition of a principal reversible square R described in Chapter 5, we have that

$$R = \Bigg(k_0 + (m_0 - 1)j_1 + (k_1 - 1)(i_1 j_1) + (m_1 - 1)\left(\frac{j_2}{j_1}\right)(i_1 j_1) + \ldots + (k_\alpha - 1)(i_\alpha j_\alpha)$$

$$+ (m_\alpha - 1)\left(\frac{n}{j_\alpha}\right)(i_\alpha j_\alpha) \mid 1 \leq k_0 \leq j_1,\ 1 \leq m_0 \leq i_1,\ 1 \leq j_1 \leq \frac{j_2}{j_1},\ 1 \leq m_1 \leq \frac{i_2}{i_1},$$

$$\ldots, 1 \leq k_\alpha \leq \frac{j_{\alpha+1}}{j_\alpha} = \frac{n}{j_\alpha},\ 1 \leq m_\alpha \leq \frac{i_{\alpha+1}}{i_\alpha} = \frac{n}{n} = 1\Bigg)$$

$$= \Bigg(k_0 + (m_0 - 1)j_1 + \sum_{v=1}^{\alpha}\left((k_v - 1)(i_v j_v) + (m_v - 1)\left(\frac{j_{v+1}}{j_v}\right)(i_v j_v)\right) \mid$$

$$1 \leq k_0 \leq j_1,\ 1 \leq m_0 \leq i_1,\ 1 \leq k_t \leq \frac{j_{t+1}}{j_t}, 1 \leq m_t \leq \frac{i_{t+1}}{i_t},\ 1 \leq t \leq \alpha\Bigg)$$

where $k = (k_0, k_1, \ldots k_\alpha)$, $m = (m_0, m_1, \ldots, m_\alpha)$ represent the column and row co-ordinates respectively. By fixing the lattice co-ordinates of L_α to be $k = (k_0, k_1, \ldots, k_\alpha) = (1, 1, \ldots, 1)$ this gives the form for the first column of a principal reversible square, such that

$$c = \left(1 + (m_0 - 1)j_1 + \sum_{v=1}^{\alpha}\left(\underbrace{(1-1)}_{=0}(i_v j_v) + (m_v - 1)\left(\frac{j_{v+1}}{j_v}\right)(i_v j_v)\right)\right.$$

$$\left. \mid 1 \leq m_0 \leq i_1,\ 1 \leq m_t \leq \frac{i_{t+1}}{i_t},\ 1 \leq t \leq \alpha\right)$$

$$= \Bigg(1 + (m_0 - 1)j_1 + \sum_{v=1}^{\alpha}\left((m_v - 1)\left(\frac{j_{v+1}}{j_v}\right)(i_v j_v)\right)$$

$$\mid 1 \leq m_0 \leq i_1,\ 1 \leq m_t \leq \frac{i_{t+1}}{i_t},\ 1 \leq t \leq \alpha\Bigg).$$

□

Definition (of lexicographical ordering)**.** The *lexicographical ordering* defines an order on a cartesian product of ordered sets, which is a total ordering when all these sets are themselves totally ordered. Formally, we define the operator $\leq$ such that $(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$ if and only if $a_1 < b_1$ or $a_1 = b_1$ and $a_2 < b_2$ or $a_1 = b_1$ and $a_2 = b_2$ and $a_3 < b_3$ or $a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$ and $a_4 < b_4$ and so until $a_1 = b_1,\ \ldots, a_{n-1} < b_{n-1}, a_n < b_n$.

Remark. The ordered lattice point sets L_α and M_α can be thought of being "reverse" lexicographically ordered.

Example. *Consider the divisor path set* $\{\hat{i}, \hat{j}\} = \{(3, 9, 36), (6, 12, 36)\}$*, then the* *corresponding ordered lattice set L_3 is given by*

$$L_3 = \{(k_0, k_1, k_2, k_3) | 1 \leq k_0 \leq 3,\ 1 \leq k_1 \leq \frac{9}{3},\ 1 \leq k_2 \leq \frac{36}{9},\ 1 \leq k_3 \leq \frac{36}{36} = 1\}$$

$$= \{(k_0, k_1, k_2, k_3) | 1 \leq k_0 \leq 3,\ 1 \leq k_1 \leq 3,\ 1 \leq k_2 \leq 4,\ 1 \leq k_3 \leq 1\}$$

$$= \{(1,1,1,1), (2,1,1,1), (3,1,1,1), (1,2,1,1), (2,2,1,1), (3,2,1,1)$$

$$(1,3,1,1), (2,2,2,1), (3,3,1,1), (1,1,2,1), (2,1,2,1), (3,1,2,1)$$

$$(1,2,2,1), (2,2,2,1), (3,2,2,1), (1,3,2,1), (2,3,2,1), (3,3,2,1)$$

$$(1,1,3,1), (2,1,3,1), (2,1,3,1), (1,2,3,1), (2,2,3,1), (3,2,3,1)$$

$$(1,3,3,1), (2,3,3,1), (3,3,3,1), (1,1,4,1), (2,1,4,1), (3,1,4,1)$$

$$(1,2,4,1), (2,2,4,1), (3,2,4,1), (1,3,4,1), (2,3,4,1), (3,3,4,1)\}.$$

Hence the lexicographical ordering of the reversed ordered lattice set is

$$\tilde{L}_3 = \{(k_3, k_2, k_1, k_0) | 1 \leq k_0 \leq 3,\ 1 \leq k_1 \leq 3,\ 1 \leq k_2 \leq 4,\ 1 \leq k_3 \leq 1\}$$

corresponding to

$$\{(1,1,1,1), (1,1,1,2), (1,1,1,3), (1,1,2,1), (1,1,2,2), (1,1,2,3)$$

$$(1,1,3,1), (1,1,3,2), (1,1,3,3), (1,2,1,1), (1,2,1,2), (1,2,1,3)$$

$$(1,2,2,1), (1,2,2,2), (1,2,2,3), (1,2,3,1), (1,2,3,2), (1,2,3,3)$$

$$(1,3,1,1), (1,3,1,2), (1,3,1,3), (1,3,2,1), (1,3,2,2), (1,3,2,3)$$

$$(1,3,3,1), (1,3,3,2), (1,3,3,3), (1,4,1,1), (1,4,1,2), (1,4,1,3)$$

$$(1,4,2,1), (1,4,2,2), (1,4,2,3), (1,4,3,1), (1,4,3,2), (1,4,3,3)\};$$

and treating the entries as numbers gives us

$$(1111, 1112, 1113, 1121, 1122, 1123, 1131, 1132, 1133, 1211, 1212, 1213,$$

$$1221, 1222, 1223, 1231, 1232, 1233, 1311, 1312, 1313, 13121, 1322, 1323$$

$$1331, 1332, 1333, 1411, 1412, 1413, 1421, 1422, 1423, 1431, 1432, 1433)$$

so that we have the inequality chain

$$1111 < 1112 < 1113 < 1121 < 1122 < 1123 < \ldots < 1431 < 1432 < 1433.$$

For the lattice set M_3 we have

$$M_3 = \{(m_0, m_1, m_2) \mid 1 \leq m_0 \leq 6,\ 1 \leq m_1 \leq \frac{12}{2} = 2,\ 1 \leq m_2 \leq \frac{36}{12} = 3\}$$

$$= \{(1,1,1),(2,1,1),(3,1,1),(4,1,1),(5,1,1),(6,1,1),(1,2,1),(2,2,1),(3,2,1)$$
$$(4,2,1),(5,2,1),(6,2,1,),(1,1,2),(2,1,2),(3,1,2),(4,1,2),(5,1,2),(6,1,2)$$
$$(1,2,2),(2,2,2),(3,2,2),(4,2,2),(5,2,2),(6,2,2),(1,1,3),(2,1,3),(3,1,3)$$
$$(4,1,3),(5,1,3),(6,1,3),(1,2,3),(2,2,3),(3,2,3),(4,2,3),(5,2,3),(6,2,3)\}.$$

so that the reversed ordering gives the set of lattice points

$$\{\ (m_2, m_1, m_0)|\ 1 \leq m_2 \leq 3,\ \ 1 \leq m_1 \leq 2,\ \ 1 \leq m_0 \leq 6\ \}$$
$$= \{(1,1,1),(1,1,2),(1,1,3),(1,1,4),(1,1,5),(1,1,6),(1,2,1),(1,2,2),(1,2,3),$$
$$(1,2,4),(1,2,5),(1,2,6),(2,1,1),(2,1,2),(2,1,3),(2,1,4),(2,1,5),(2,1,6),$$
$$(2,2,1),(2,2,2),(2,2,3),(2,2,4),(2,2,5),(2,2,6),(3,1,1),(3,1,2),(3,1,3),$$
$$(3,1,4),(3,1,5),(3,1,6),(3,2,1),(3,2,2),(3,2,3),(3,2,4),(3,2,5),(3,2,6)\},$$

which give the lexicographical sequence of ascending numbers

$$\left\{\sum_{u=0}^{2} m_u 10^u \ \mid\ m \in M_3\right\} = \{111, 112, 113, 114, 115, 116, 121, 121, 123, 124, 125, 126,$$
$$211, 212, 213, 214, 215, 216, 221, 222, 223, 224, 225, 226,$$
$$311, 312, 313, 314, 315, 316, 321, 322, 323, 324, 325, 326\}$$

6.1 Construction of (non-inclusive) Sum-and-Distance systems

From the block representations given at the beginning of this chapter it is evident that in an even sided principal reversible square the corresponding non-inclusive sum-and-distance system can be identified from the first half of the first row r' and column c' by the sets

$$r' = \left\{\frac{1}{2}(a^T J_k + b_k) + w\right\},\quad c' = \left\{\frac{1}{2}(a_k + J_k b) + w\right\},$$

so that subtracting $\frac{1}{2}b_k + w$ from r' and $\frac{1}{2}a_k + w$ from c' yields the sets a and b. In a traditional square, the weight is calculated for any n to be $w = \frac{n^2-1}{2}$, which fixes $a_k + b_k$. In general the first and last entries of the first row and column in the principal reversible square are obtained by substituting the first and last entries of the lattice sets L_α and M_α; $(1,1,\ldots,1)$, $(j_1, \frac{j_2}{j_1}, \ldots, \frac{n}{j_\alpha})$, $(1,1,\ldots,1)$, $\left(i_1, i_2, \ldots, \frac{i_\alpha}{i_{\alpha-1}} = \frac{n}{i_{\alpha-1}}\right)$ in the equations found in Lemmas 6.1 and 6.2, such that

$$r_1 = 1,\quad r_n = j_1 + \sum_{k=1}^{\alpha}\left(\frac{j_k}{j_{k-1}} - 1\right)(i_{k-1}j_{k-1}) + \left(\frac{n}{j_{\alpha-1}} - 1\right)(i_\alpha j_\alpha)\ \text{ and}$$

$$c_1 = 1, \quad c_n = (i_1 j_1) - (j_1 - 1) + \sum_{k=1}^{\alpha-1} \left(\frac{i_{k+1}}{i_k} \right) (i_k j_k) \frac{j_{k+1}}{j_k},$$

where $i_0 = j_0 = 1$. Here r_i is the ith entry in the first row and c_j is the jth entry in the first column.

Definition. Let L'_α and M'_α be the ordered set of lattice points corresponding to the first half of the first row and column, so that when substituted into the equations in Lemmas 6.1 and 6.2 respectively, they give

$$r' = \left\{ \frac{1}{2}(b_k + a) + w \right\} = \left\{ k_0 + \sum_{m=1}^{\alpha} (k_m - 1)(i_m j_m) \mid k \in L'_\alpha \right\},$$

$$c' = \left\{ \frac{1}{2}(a_k + b) + w \right\} = \left\{ 1 + (m_0 - 1) j_1 + \sum_{u=1}^{\alpha-1} (m_u - 1)(i_u j_u) \mid m \in M'_\alpha \} \right\}.$$

LEMMA 6.3. *Let $k \in \mathbb{N}$, so that $n = 2k$ is even. Given an α length divisor path set for n (where $j_0 = i_0 = 1$, $i_\alpha = j_{\alpha+1} = n$), we have*

$$\{\hat{i}, \hat{j}\} = \{(i_1, i_2, \ldots, i_\alpha = n),\ (j_1, j_2, \ldots, j_\alpha)\},$$

for some $s \leq t \leq \alpha$ and $v \leq h \leq \alpha - 1$, where

$$\frac{j_{t+1}}{j_t} \quad \text{and} \quad \frac{i_{h+1}}{i_h}$$

are all odd, while

$$\frac{j_s}{s-1} \quad \text{and} \quad \frac{i_v}{v-1}$$

are even. The corresponding half lattice sets L'_α and M'_α are then given by

$$L'_\alpha = \left\{ k \in L_\alpha \mid \sum_{i=0}^{\alpha} k_i \ 10^{\mu i} \leq \left(\sum_{i=0}^{s-2} \frac{j_{i+1}}{j_i} \ 10^{\mu i} + \frac{1}{2} \frac{j_s}{j_{s-1}} 10^{\mu(s-1)} + \sum_{t=s}^{\alpha} \frac{1}{2} \left(\frac{j_{t+1}}{j_t} + 1 \right) 10^{\mu t} \right) \right\},$$

$$M'_\alpha = \left\{ m \in M'_\alpha \mid \sum_{i=0}^{\alpha-1} m_i 10^{\eta i} \leq \sum_{h=0}^{v-2} \frac{i_{h+1}}{i_h} + \frac{1}{2} \frac{i_v}{i_{v-1}} 10^{\eta(v-1)} + \sum_{h=v}^{\alpha-1} \frac{1}{2} \left(\frac{i_{h+1}}{i_h} + 1 \right) 10^{\eta h} \right\},$$

for any μ and η such that $\mu, \eta \in \mathbb{N}$ and $k_i < 10^\mu$ and $m_i < 10^\eta$ for all $i \in \mathbb{Z}_\alpha$.

Proof. The lattice points

$$(k_0, k_1, \ldots, k_\alpha) = \left(j_1, \frac{j_2}{j_1}, \ldots, \frac{1}{2} \frac{j_s}{j_{s-1}}, \ \frac{1}{2} \left(\frac{j_{s+1}}{j_s} + 1 \right), \ \ldots, \ \frac{1}{2} \left(\frac{n}{j_\alpha} + 1 \right) \right)$$

$$(m_0, m_1, \ldots, m_\alpha) = \left(i_1, \frac{i_2}{i_1}, \ldots, \frac{1}{2} \frac{i_v}{i_{v-1}}, \ \frac{1}{2} \left(\frac{i_{v+1}}{i_v} + 1 \right), \ \ldots, \ \frac{1}{2} \left(\frac{n}{i_{\alpha-1}} + 1 \right) \right)$$

correspond to the matrix entries directly to the left of the mid point of the first row r and similarly for the first column c, as the number of final blocks in the reversible square construction could be an odd number. Hence we need to work through the dimensional blocks until there is an even number of block in both rows and columns. □

THEOREM 6.4. *Let $k \in \mathbb{N}$, so that $n = 2k$ is even. Given an α length divisor path set for n*

$$\{\hat{i}, \hat{j}\} = \{(i_1, i_2, \ldots, i_\alpha = n), (j_1, j_2, \ldots, j_\alpha)\},$$

then its corresponding sum-and-distance system is given by

$$\{A, B\} = \{\{|a_j| \mid j \in \mathbb{Z}_{n/2}\}\ , \{|b_j| \mid j \in \mathbb{Z}_{n/2}\}\},$$

where

$$\frac{1}{2}a = \{k_0 + \sum_{i=1}^{\alpha}(k_i - 1)(i_1 j_1) | \; k \in L'_\alpha\} - \frac{1}{2}\left(1 + j_1 + \sum_{i=1}^{\alpha}\left(\frac{j_{i+1}}{j_i} - 1\right)(i_1 j_1)\right),$$

$$\frac{1}{2}b = \left\{1 + (m_0 - 1)j_1 + \sum_{u=1}^{\alpha-1}(m_u - 1)\frac{j_{u+1}}{j_u}(i_u j_u) \mid m \in M'_\alpha\right\}$$
$$-\frac{1}{2}\left(2 + (i_1 - 1)j_1 + \sum_{u=1}^{\alpha-1}\left(\frac{i_{u+1}}{i_u} - 1\right)\left(\frac{i_{u+1}}{i_u}\right)(i_u j_u)\right).$$

Proof. In the expanded block representation of a reversible square, the first row contains one set from a sum-and-distance system,

$$\left\{\frac{1}{2}(b_k + a) + w, \frac{1}{2}(b_k - a) + w\right\}.$$

By Lemma 6.1 the first row is given by

$$r = \left\{\frac{1}{2}(b_k + a) + w, \frac{1}{2}(b_k - a) + w\right\} = \left\{k_0 + \sum_{m=1}^{\alpha}(k_m - 1)(i_m j_m) \mid k \in L_\alpha\right\},$$

and with L'_α defined as above, the half-row is determined by

$$r' = \left\{\frac{1}{2}(b_k + a) + w\right\} = \left\{k_0 + \sum_{m=1}^{\alpha}(k_m - 1)(i_m j_m) \mid k \in L'_\alpha\right\}.$$

By Theorem 4.1 $A = \{|a_1|, |a_2|, \ldots, |a_k|\}$ and $B = \{|b_1|, |b_2|, \ldots, |b_k|\}$ form a sum-and-distance system, so we need only determine vectors a and b.

We can determine $\frac{1}{2}a$ by finding $\frac{1}{2}(b_k - 2w)$ and subtracting it from r'

$$r' - \frac{1}{2}(b_k + 2w) = \frac{1}{2}(b_k + a) + w - \frac{1}{2}(b_k + 2w) = \frac{1}{2}a.$$

Considering the first and last entries of our first row r, we have

$$r_1 = \frac{1}{2}(b_k + a_k) + w = k_0 + \sum_{m=1}^{\alpha}(k_m - 1)(i_m j_m) = 1, \quad \text{at} \quad k = (1, 1, \ldots, 1),$$

$$\begin{aligned} r_n &= \frac{1}{2}(b_k - a_k) + w = k_0 + \sum_{m=1}^{\alpha}(k_m - 1)(i_m j_m) \\ &= j_1 + \sum_{m=1}^{\alpha}\left(\frac{j_{m+1}}{j_m}\right), \quad \text{at} \quad k = (j_1, \frac{j_2}{j_1}, \ldots, \frac{j_\alpha}{\alpha - 1}, \frac{n}{j_\alpha}), \end{aligned}$$

so that

$$\begin{aligned} \frac{1}{2}(r_1 + r_n) &= \frac{1}{2}\left(\frac{1}{2}(b_k + a_1) + w + \frac{1}{2}(b_k - a_1) + w\right) = \frac{1}{2}(b_k + 2w) \\ &= \frac{1}{2}\left(1 + j_1 + \sum_{i=1}^{\alpha}\left(\frac{j_{i+1}}{j_i} - 1\right)(i_1 j_1)\right). \end{aligned}$$

In conclusion we therefore have

$$\frac{1}{2}a = r' - \frac{1}{2}(b_k + 2w) = r' - \frac{1}{2}\left(1 + j_1 + \sum_{i=1}^{\alpha}\left(\frac{j_{i+1}}{j_i} - 1\right)(i_1 j_1)\right).$$

Similarly the first half of the first column of a principal reversible square gives the second part of the sum-and-distance system. Consider the first column, we know from the block representation that it has the form

$$\left\{\frac{1}{2}(a_k + b) + w, \ \frac{1}{2}(a_k - b) + w\right\},$$

which can be written as

$$\begin{aligned} c &= \left\{\frac{1}{2}(a_k + b) + w, \ \frac{1}{2}(a_k - b) + w\right\} \\ &= \left\{1 + (m_0 - 1)j_1 + \sum_{u=1}^{\alpha-1}(m_u - 1)\left(\frac{j_{u+1}}{j_u}\right)(i_u j_u) \mid m \in M_\alpha\right\}, \end{aligned}$$

so that the first half of the first column by definition can be written explicitly as

$$c' = \left\{\frac{1}{2}(a_k + b) + w\right\} = \left\{1 + (m_0 - 1)j_1 + \sum_{u=1}^{\alpha-1}(m_u - 1)\left(\frac{j_{u+1}}{j_u}\right)(i_u j_u) \mid m \in M'_\alpha\right\},$$

and subtracting $\frac{1}{2}a_k + w$, gives us b.

Considering the first and last entries of the first column c of our matrix, we have $\frac{1}{2}a_k + w = \frac{1}{2}\left(\frac{1}{2}(a_k + b_k) + w + \frac{1}{2}(a_k - b_k) + w\right) = \frac{1}{2}(c_1 + c_n)$, so that

$$c_1 = 1 \quad \text{at} \quad m = (1, 1, \ldots, 1)$$

and

$$c_n = 1 + (i_1 - 1)j_1 + \sum_{u=1}^{\alpha-1} \left(\frac{i_{u+1}}{i_u} - 1\right)\left(\frac{j_{u+1}}{j_u}\right)(i_u j_u),$$

$$a_k + 2w = 2 + (i_1 - 1)j_1 + \sum_{u=1}^{\alpha-1} \left(\frac{i_{u+1}}{i_u} - 1\right)\left(\frac{j_{u+1}}{j_u}\right)(i_u j_u).$$

In conclusion we therefore have

$$\frac{1}{2}b = c' - \frac{1}{2}(a_k + 2w) = c' - \frac{1}{2}\left(2 + (i_1 - 1)j_1 + \sum_{u=1}^{\alpha-1} \left(\frac{i_{u+1}}{i_u} - 1\right)\left(\frac{j_{u+1}}{j_u}\right)(i_u j_u)\right).$$

□

Example. *We give the example when $n = 36$, for the divisor path*

$$\{\hat{\imath}, \hat{\jmath}\} = \{(3, 9, 18, 36),\ (2, 6, 12, 36)\},$$

which determines the corresponding ordered lattice set

$$L_\alpha = L_4(\hat{\imath}, \hat{\jmath}) = \{(k_0, k_1, k_2, k_3, k_4)\} = \{(1,1,1,1,1),(2,1,1,1,1),(1,2,1,1,1),$$

$$(2,2,1,1,1),(1,3,1,1,1),(2,3,1,1,1),(1,1,2,1,1),(2,1,2,1,1),(1,2,2,1,1),$$

$$(2,2,2,1,1),(1,3,2,1,1),(2,3,2,1,1),(1,1,1,2,1),(2,1,1,2,1),(1,2,1,2,1),$$

$$(2,2,1,2,1),(1,3,1,2,1),(2,3,1,2,1),(1,1,2,2,1),(2,1,2,2,1),(1,2,2,2,1),$$

$$(2,2,2,2,1),(1,3,2,2,1),(2,3,2,2,1),(1,1,1,3,1),(2,1,1,3,1),(1,2,1,3,1),$$

$$(2,2,1,3,1),(1,3,1,3,1),(2,3,1,3,1),(1,1,2,3,1),(2,1,2,3,1),(1,2,2,3,1),$$

$$(2,2,2,3,1),(1,3,2,3,1),(2,3,2,3,1).$$

The reversed lexicographical ordered set is then given by

$$\left\{\sum_{i=0}^{4} 10^i k_i \ |k \in L_4\right\} = \{11111, 11112, 11121, 11122, 11131, 11132,$$

$$11211, 11212, 11221, 11222, 11231, 11232, 12111, 12112, 12121, 12122, 12131, 12132,$$

$$12211, 12212, 12221, 12222, 12231, 12232, 13111, 13112, 13121, 13122, 13131, 13132,$$

$$13211, 13212, 13221, 13222, 13231, 13232\}.$$

When $4 \le t \le 4$, $s = 4$ we have that $\frac{j_{t+1}}{j_t}$ is odd and $\frac{j_4}{j_3}$ even, and using the above theory underpinning our construction, it therefore follows that the first half set of ordered lattice points is given by

$$j_1 = 2,\ \frac{j_2}{j_1} = 3,\ \frac{j_3}{j_2} = 2,\ \frac{j_4}{j_3} = 3,\ \frac{n}{j_4} = 1$$

$$\left\{k \in L_4 \mid \sum_{i=0}^{4} 10^i \ k_i \leq \left(\sum_{i=0}^{1} 10^i \left(\frac{j_{i+1}}{j_i}\right) + 10^2 \frac{1}{2}\left(\frac{j_4}{j_3}\right) + \sum_{i=4}^{4} 10^i \frac{1}{2}\left(\frac{j_{i+1}}{j_i} + 1\right)\right)\right\}$$

$$= \left\{k \in L_4 \mid \sum_{i=0}^{4} 10^i k_i \ \leq \ 2 + 10 * 3 + 10^2 * 1 + 10^3 * 2 + 10^4 * 1\right\}$$

$$= \left\{k \in L_4 \mid \sum_{i=0}^{4} 10^i k_i \ \leq \ 12132\right\}$$

$$= \left\{k \in L_4 \mid \sum_{i=0}^{4} 10^i k_i \in \{11111, 11112, 11121, 11122, 11131, 11132, 11211, 11212, 11221, 11222, 11231, 11232, 12111, 12112, 12121, 12122, 12131, 12132\} \ \right\}$$

$$= \{(1,1,1,1,1), (2,1,1,1,1), (1,2,1,1,1)(2,2,1,1,1), (1,3,1,1,1), (2,3,1,1,1), (1,1,2,1,1), (2,1,2,1,1), (1,2,2,1,1), (2,2,2,1,1), (1,3,2,1,1), (2,3,2,1,1), (1,1,1,2,1), (2,1,1,2,1), (1,2,1,2,1), (2,2,1,2,1), (1,3,1,2,1), (2,3,1,2,1)\}$$

The coordinate entry to the left of the middle of the first row will be the $18th$ *entry which can be seen to be* 12132. *Hence the first half set of ordered lattice points have now been determined.*

Now substituting in these k_i *values into Lemma 6.1 we determine that*

$$r = \{k_0 + (k_1 - 1)(i_1 \times j_1) + (k_2 - 1)(i_2 \times j_2) + \ldots + (k_\alpha - 1)(i_\alpha \times j_\alpha) | k \in L'_\alpha\}$$

$$= \{k_0 + 6(k_1 - 1) + 53(k_2 - 1) + 216(k_3 - 1) + \underbrace{36^2(k_4 - 1)\}}_{=0} \mid k \in L'_4\}$$

$$= \{1 + 0 + 0 + 0 + 0, 2 + 0 + 0 + 0 + 0, 1 + (1)(4) + 0 + 0 + 0, 2 + (1)(4), \ldots\}$$

$$= \{1, 2, 7, 8, 13, 14, 55, 56, 61, 62, 67, 68, 217, 218, 223, 224, 229, \ 230\} = \frac{1}{2}(a + b_k) + w.$$

Similarly, to find b_k *we add the first and last values of the first row, found by substituting the first and last coordinates* $k = (1, 1, 1, 1, 1), \ k = (2, 3, 2, 3, 1)$ *into* r, *giving*

$$b_k + 2w = \frac{1}{2}(b_k + a_1) + w + \frac{1}{2}(b_k - a_1) + w = 1 + 500 = 501.$$

Lastly we subtract half of this new expression to obtain $\frac{1}{2}a$,

$$\frac{1}{2}(a + b_k) + w - \frac{1}{2}(b_k + 2w) = \frac{1}{2}a$$

$$= -\frac{1}{2}\{41, 43, 53, 55, 65, 67, 365, 367, 377, 379, 389, 391, 473, 475, 485, \ 487, 497, 499\}$$

Hence the first part of the sum-and-distance system associated with the reversible square (given below) and divisor path set $\{(2, 6, 12, 36), \ (2, 6, 12, 36)\}$ *is twice the above form and taking the absolute values of each entry, we finally have the required* *set*

$$A = \{|a_i| \mid i \in \mathbb{Z}_k\} = \{41, 43, 53, 55, 65, 67, 365, 367, 377, 379, 389, 391, 473, 475, 485, 487, 497, 499\}.$$

Similarly for the second set b which comprises the non-inclusive sum-and-distance system for the given divisor path set (described above) we obtain the set of lattice points

$$M_\alpha = M_4(\hat{i}, \hat{j}) = \{(m_0, m_1, \ldots, m_{\alpha-1}) \mid 1 \le m_0 \le i_1 = 3,\ 1 \le m_1 \le \frac{i_2}{i_1} = \frac{9}{3} = 3, 1 \le m_2 \le \frac{i_3}{i_2} = \frac{18}{9} = 2,\ 1 \le m_3 \le \frac{n}{i_3} = \frac{36}{18} = 2\}$$

$$= \{(1,1,1,1), (2,1,1,1), (3,1,1,1), (1,2,1,1), (2,2,1,1), (3,2,1,1), (1,3,1,1), (2,3,1,1), (3,3,1,1), (1,1,2,1), (2,1,2,1), (3,1,2,1), (1,2,2,1), (2,2,2,1), (3,2,2,1), (1,3,2,1), (2,3,2,1), (3,3,2,1), (1,1,1,2), (2,1,1,2), (3,1,1,2), (1,2,1,2), (2,2,1,2), (3,2,1,2) (1,3,1,2), (2,3,1,2), (3,3,1,2), (1,1,2,2), (2,1,2,2), (3,1,2,2) (1,2,2,2), (2,2,2,2), (3,2,2,2), (1,3,2,2), (2,3,2,2), (3,3,2,2)\},$$

which has the lexicographical ordering

$$\left\{\sum_{i=0}^{3} 10^i m_i \mid m \in M_4\right\} = \{1111, 1112, 1113, 1121, 1122, 1123, 1131, 1132, 1133\ 1211, 1212, 1213, 1221, 1222, 1223, 1231, 1232, 1233\ 2111, 2112, 2113, 2121, 2122, 2123, 2131, 2132, 2133, 2211, 2212, 2213, 2221, 2222, 2223, 2231, 2232, 2233\}.$$

Therefore the half lattice point set which corresponds to the first column of the reversible square is given by

$$M_4' = \left\{m \in M_4 \mid \sum_{i=0}^{3} 10^i m_i \le 1233\right\}$$

$$= \{(1,1,1,1), (2,1,1,1), (3,1,1,1), (1,2,1,1), (2,2,1,1), (3,2,1,1)\ (1,3,1,1), (2,3,1,1), (3,3,1,1), (1,1,2,1), (2,1,2,1), (3,1,2,1)\ (1,2,2,1), (2,2,2,1), (3,2,2,1), (1,3,2,1), (2,3,2,1), (3,3,2,1)\},$$

and substituting into the first column equation we obtain

$$-\frac{1}{2}(b-\frac{1}{2}a_k-w)$$

$= \{500, 504, 508, 536, 540, 544, 572, 576, 580, 716, 720, 724, 752, 756, 760, 788, 792, 796\}.$

Hence the second part of the sum-and-distance system associated with the reversible square (given below) and divisor path set $\{(2,6,12,36),\ (2,6,12,36)\}$ *is twice the above form, and taking the absolute value of each entry gives*

$$B = \{|b_i| \mid i \in \mathbb{Z}_k\} = \{500, 504, 508, 536, 540, 544, 572, 576, 580, 716, 720, 724, 752, 756, 760, 788, 792, 796\}.$$

By the theory it follows that all the sums and distances of the entries in A *and* B *give the odd positive integers* $1, 3, 5, \ldots 4k^2 - 1$*. We give the corresponding* 36×36 *principal reversible square matrix in the appendix.*

Remark. It may be that a simpler method exists to determine an expression for the middle lattice point, but this would be a point for further work.

6.2 Construction of Inclusive Sum-and-Distance Systems

The method for constructing a non-inclusive sum-and-distance system can be expanded and simplified to construct an inclusive sum-and-distance system corresponding to an odd sided principal reversible square matrix. Again we use the fact that the first row of a weightless principal square can be written as

$$\{a^T J_k + b_k,\ b_k,\ -a^T + b_k\},$$

and we can generalise the first $1_k a^T J_k + b_k$ because b_k can be found from the $\{\hat{\imath}, \hat{\jmath}\}$ divisor path set and subtracted from all entries, giving

$$\{a^T J_k,\ 0,\ -a^T\}.$$

Hence by taking the first half of the first row we will obtain the a part of the sum-and-distance set, and similarly for the first column we obtain the entries

$$\{J_k b + a_k \quad, a_k \quad, -J_k b + a_k\},$$

and by subtracting the middle value a_k we obtain

$$\{J_k b,\ 0,\ -J_k b\}.$$

It follows that by taking the first half of the first column we will then obtain the b part of the sum-and-distance set.

Our approach again uses the two lattice sets L'_α and M'_α, from which we can obtain the explicit forms for the first row and column (respectively r' and c') of the odd sided reversible square given by

$$r' = \{(b_k + a) + w\} = \left\{ k_0 + \sum_{m=1}^{\alpha} (k_m - 1)(i_m j_m) \mid k \in L'_\alpha \right\},$$

$$c' = \{(a_k + b) + w\} = \left\{ 1 + (m_0 - 1) j_1 + \sum_{u=1}^{\alpha-1} (m_u - 1) \left(\frac{j_{u+1}}{j_u} \right) (i_u j_u) \mid m \in M'_\alpha \} \right\}.$$

LEMMA 6.5. *Given an α length divisor path set for n (with $j_0 = i_0 = 1,\ j_{\alpha+1} = i_\alpha = n$)*

$$\{\hat{i}, \hat{j}\} = \{(i_1, i_2, \ldots, i_\alpha = n),\ (j_1, j_2, \ldots, j_\alpha)\},$$

the first half of lattice points for L_α and M_α are given by

$$L'_\alpha = \left\{ k \in L_\alpha \mid \sum_{u=0}^{\alpha} k_u 10^{u\mu} < \sum_{u=0}^{\alpha} \frac{1}{2} \left(\frac{j_{u+1}}{j_u} + 1 \right) 10^{u\mu} \right\},$$

$$M'_\alpha = \left\{ m \in M_\alpha \mid \sum_{u=0}^{\alpha-1} m_u 10^{u\eta} < \sum_{u=0}^{\alpha-1} \frac{1}{2} \left(\frac{i_{u+1}}{i_u} + 1 \right) 10^{u\eta} \right\},$$

for any μ and η such that $\mu, \eta \in \mathbb{N}$ and $k_i < 10^\mu$ and $m_i < 10^\eta$ for all $i \in \mathbb{Z}_\alpha$

Proof. Corresponding to the sets L'_α and M'_α we have (as in the non-inclusive case) the reversed ordered lexicographical interpretation for the lattice points, which when substituted into Lemmas 6.1 and 6.2, the explicit expressions for the first row and column are obtained, where the middle lattice points for the first row and column are given by

$$k = \left\{ \frac{1}{2} (j_1 + 1),\ \frac{1}{2} \left(\frac{j_2}{j_1} + 1 \right),\ \ldots, \frac{1}{2} \left(\frac{n}{j_\alpha} + 1 \right) \right\}$$

and

$$m = \left\{ \frac{1}{2} (i_1 + 1),\ \frac{1}{2} \left(\frac{i_2}{i_1} + 1 \right),\ \ldots, \frac{1}{2} \left(\frac{i_\alpha = n}{i_{\alpha-1}} + 1 \right) \right\}.$$

Hence, all lattice points whose reversed lexicographic ordering is smaller than the reversed lexicographic mid-point will give the first half of the row and column required. □

THEOREM 6.6. *Let $k \in \mathbb{N} \cup \{0\}$, so that $n = 2k + 1$ is odd. Given any α length divisor path set*

$$\{\hat{i}, \hat{j}\} = \{(i_1, i_2, \ldots, i_\alpha = n),\ (j_1, j_2, \ldots, j_\alpha)\},$$

then its associated inclusive sum-and-distance system is given by

$$\{A, B\} = \left\{ \ \{|a_j| \ \mid \ j \in \mathbb{Z}_{\frac{n-1}{2}}\},\ \{\ |b_j| \ \mid \ k \in \mathbb{Z}_{\frac{n-1}{2}}\} \right\},$$

with

$$a = \left\{ k_0 + \sum_{u=1}^{\alpha} (k_i - 1)(i_u j_u) \mid k \in L'_\alpha \right\} - \left(\frac{1}{2}(j_1 + 1) + \sum_{u=1}^{\alpha} \frac{1}{2} \left(\frac{j_{u+1}}{j_u} - 1 \right) (i_u j_u) \right)$$

$$b = \left\{ 1 + (m_0 - 1) j_1 + \sum_{u=1}^{\alpha-1} (m_u - 1) \left(\frac{j_{u+1}}{j_u} \right) (i_u j_u) \mid m \in M'_\alpha \right\}$$
$$- \left(1 + \frac{1}{2}(i_1 - 1) j_1 + \sum_{u=1}^{\alpha-1} \frac{1}{2} \left(\frac{i_{u+1}}{i_u} - 1 \right) (i_u j_u) \left(\frac{j_{u+1}}{j_u} \right) \right).$$

Proof. We know from the block representation expansion that the first row is given by

$$r = \{b_k + a + w, b_k + w, b_k - a + w\}$$
$$= \left\{ k_0 + \sum_{u=1}^{\alpha} (k_m - 1)(i_m j_m) \mid k \in L_\alpha \right\}.$$

Then from the definition the first half of r gives

$$r' = \{b_k + a + w\} = \left\{ k_0 + \sum_{u=1}^{\alpha} (k_u - 1)(i_u j_u) \mid k \in L'_\alpha \right\}.$$

To obtain the a vector we again find and subtract $b_k + w$, so that

$$r' - b_k - w = \{a\} = \left\{ k_0 + \sum_{u=1}^{\alpha} (k_u - 1)(i_u j_u) \mid k \in L'_\alpha \right\} - b_k - w.$$

The value in the centre of the first row is $b_k + w$, which we can determine by substituting the mid lattice point

$$\hat{k} = (\hat{k}_0, \hat{k}_1, \ldots, \hat{k}_\alpha) = \left\{ \frac{1}{2}(j_1 + 1), \frac{1}{2}\left(\frac{j_2}{j_1} + 1 \right), \ldots, \frac{1}{2}\left(\frac{n}{j_\alpha} + 1 \right) \right\}$$

into the form for the first row and obtain

$$b_k + w = \frac{1}{2}(j_1 + 1) + \sum_{u=1}^{\alpha} \frac{j_{u+1}}{j_u}(i_u j_u).$$

So finally we have that

$$a = r' - b_k - w = \left\{ k_0 + \sum_{u=1}^{\alpha} (k_u - 1)(i_u j_u) \mid k \in L'_\alpha \right\} - \left(\frac{1}{2}(j_1 + 1) + \sum_{u=1}^{\alpha} \frac{1}{2} \left(\frac{j_{u+1}}{j_u} - 1 \right) (i_u j_u) \right)$$

and by Theorem 4.1 $A = \{|a_j| \ \ j \in \mathbb{Z}_k\}$ gives one part of an inclusive sum-and-distance system.

Similarly, the first column of a weightless reversible square is given by

$$c = \{a_k + b + w,\ a_k + b,\ a_k - b + w\}^T$$

$$= \left\{ 1 + (m_0 - 1)j_1 + \sum_{u=1}^{\alpha-1} (m_u - 1) \frac{j_{u+1}}{j_u} (i_u j_u) \ \mid \ m \ \in M'_\alpha \right\}.$$

As before we determine and subtract $a_k + w$ from the above expression, where similarly $a_k + w$ can be found using the mid lattice point and then substituting it into our expression for the first column c. The mid point is given by

$$\hat{m} = (\hat{m}_0, \hat{m}_1, \ldots, \ \hat{m}_{\alpha-1}) = \left(\frac{1}{2}(i_1 + 1), \frac{1}{2}\left(\frac{i_2}{i_1} + 1\right), \ldots, \ \frac{1}{2}\left(\frac{i_\alpha}{i_{\alpha-1}} + 1\right) \right)$$

and remembering that $i_\alpha = n$), we have

$$a_k + w = 1 + \frac{1}{2}(i_1 + 1) + \sum_{u=1}^{\alpha-1} \frac{1}{2} \frac{i_{u+1}}{i_u} \frac{j_{u+1}}{j_u}.$$

It follows that

$$b = c' - a_k - w = \left\{ 1 + (m_0 - 1)j_1 + \sum_{u=1}^{\alpha-1} (m_u - 1) \left(\frac{j_{u+1}}{j_u} \right) (i_u j_u) \mid m \in M'_\alpha \right\}$$

$$- \left(1 + \frac{1}{2}(i_1 - 1) + \sum_{u=1}^{\alpha-1} \frac{1}{2} \left(\frac{i_{u+1}}{i_u} - 1 \right) (i_u j_u) \left(\frac{j_{u+1}}{j_u} \right) \right),$$

and by Theorem 4.1 $B = \{|b_j| \ \mid \ j \in \mathbb{Z}_k\}$ gives the remaining part of the inclusive sum-and-distance system.

□

6.3 Geometric Interpretation

Another way we can determine uniquely a sum-and-distance system from a principal reversible square divisor path is to obtain the sets L'_α and M'_α, from sets of lattice points bounded by certain hyperplanes.

Remark. As before, we let the set of lattice points, L'_α represent the first half of entries in L_α that when substituted in yields $k_0 + (k_1 - 1)(i_1 j_1) + \ldots (k_\alpha - 1)(i_\alpha j_\alpha)$ from which we obtain the first half of the first row in the principal reversible square. So the sum-and-distance part a can be recovered as

$$a = \{k_0 + (k_1 - 1)(i_1 j_1) + \ldots (k_\alpha - 1)(i_\alpha j_\alpha) - b_k - w \mid k \in L'_\alpha, \}$$

where w is the weight of the reversible square.

Definition. Let α be a positive integer with $\hat{k}$ and $\hat{m}$ the mid lattice points which when substituted into the forms of the first row and column give the middle entry,

$$\hat{k} = (\hat{k}_0, \hat{k}_1, \ldots, \hat{k}_\alpha) = \left(\frac{1}{2}(j_1 + 1), \ \frac{1}{2}\left(\frac{j_2}{j_1} + 1\right), \ \ldots, \frac{1}{2}\left(\frac{n}{j_\alpha} + 1\right) \right)$$

$$\hat{m} = (\hat{m}_0, \hat{m}_1, \ldots, \hat{m}_{\alpha-1}) = \left(\frac{1}{2}(i_1+1), \ \frac{1}{2}\left(\frac{i_2}{i_1}+1\right), \ \ldots, \frac{1}{2}\left(\frac{i_\alpha = n}{i_{\alpha-1}}+1\right)\right).$$

LEMMA 6.7. *For a divisor path system* $\{\hat{\imath}, \hat{\jmath}\} = \{(i_1, \ldots, i_\alpha), (j_1, \ldots, j_\alpha)\}$ *we define the following sets*

$$S_0 = \left\{(k_0, k_1, \ldots, k_\alpha) \mid 1 \le k_0 \le 2\hat{k}_0 - 1, \ \ldots, \ 1 \le k_{\alpha-1} \le 2\hat{k}_{\alpha-1} - 1, \ 1 \le k_\alpha \le \hat{k}_\alpha - 1\right\},$$

for $0 < t < \alpha$

$$S_t = \{(k_0, \ldots, k_\alpha) \mid 1 \le k_0 \le 2\hat{k}_0 - 1, \ \ldots, 1 \le k_{\alpha-t-1} \le 2\hat{k}_{\alpha-t-1} - 1, \ 1 \le k_{\alpha-t} \le \hat{k}_{\alpha-t} - 1,$$

$$k_{\alpha-t+1} = \hat{k}_{\alpha-t+1}, \ \ldots, \ k_\alpha = \hat{k}_\alpha\}$$

and

$$S_\alpha = \{(k_0, k_1, \ldots, k_\alpha) \mid 1 \le k_0 \le \hat{k}_0 - 1, \ k_1 = \hat{k}_1, \ldots, \ k_\alpha = \hat{k}_\alpha\}.$$

Then

$$L'_\alpha = \bigcup_{u=0}^{\alpha} S_u.$$

Proof. Firstly, S_0 gives the first $\frac{1}{2}\left(\frac{n}{j_\alpha} - 1\right) = \hat{k}_\alpha - 1$, j_α length sequence in the first row of the reversible square excluding the centre j_α length sequence. S_1 with fixed $k_\alpha = \hat{k}_\alpha$ co-ordinate is then found in the centre j_α length sequence, and this then gives the first $\hat{k}_{\alpha-1} - 1$, $j_{\alpha-1}$ length sequence within the first row of the reversible square. Hence as the index of S increase we move towards the centre most point in the first row of the reversible square. The set of points S_t is then found in the centre $j_\alpha, j_{\alpha-1}, \ldots, j_{\alpha-t-1}$ length sequences and this gives the $\hat{k}_{\alpha-t} - 1$, $j_{\alpha-t}$ length strings.

Continuing with this method we finally arrive in the centre most j_1 length sequence in which the first $\hat{k}_0 - 1$ entries are given. It follows that the union of the previous sets will give all the points before this set, and hence the union over all of these sets will give the points that explicitly determine the first half of the first row of our reversible square.

□

LEMMA 6.8. *The points of* L'_α *can be described by all the points under a hyperplane in the full lattice, with corresponding normal vector defined by*

$$c_i = \begin{cases} 1 & \text{if } i = 0, \\ 2^{i-1} & \text{if } i > 0, \end{cases} \quad \text{and} \quad v_i = \frac{c_i}{\prod_{j=i}^{\alpha-1} \hat{k}_j}$$

where the points under the plane have a one-to-one correspondence to

$$L'_\alpha = \ \{(k_0, k_1, \ldots, k_\alpha) | (k - \hat{k}).v < 0, \ k \in L_\alpha\}.$$

Proof. Firstly we will show that any $k \in \cup_{t=0}^{\alpha} S_t$ will give $(k - \hat{k}).v < 0$. Let $k \in S_0$, then

$$k \in \Big\{(k_0, k_1, \ldots, k_\alpha) \mid 1 \le k_0 \le j_1 = 2\hat{k}_1 - 1,\ 1 \le k_1 \le \frac{j_2}{j_1} = 2\hat{k}_1 - 1, \ldots$$

$$,\ 1 \le k_{\alpha-1} \le \frac{j_\alpha}{j_{\alpha-1}} = 2\hat{k}_{\alpha-1} - 1,\ 1 \le k_\alpha \le \frac{1}{2}\left(\frac{n}{j_\alpha} - 1\right) = \hat{k}_\alpha - 1\Big\}.$$

For $0 < u \le \alpha - 1$ we have

$$\frac{2^{u-1}(k_u - \hat{k}_u)}{\prod_{j=u}^{\alpha-1} \hat{k}_j} \le \frac{2^{u-1}(2\hat{k}_u - 1 - \hat{k}_u)}{\prod_{j=u}^{\alpha-1} \hat{k}_j} = \frac{2^{u-1}(\hat{k}_u - 1)}{\prod_{j=u}^{\alpha-1} \hat{k}_j} = \frac{\hat{k}_u - 1}{\hat{k}_u} \frac{2^{u-1}}{\prod_{j=u+1}^{\alpha-1} \hat{k}_j} < 2^{u+1},$$

for $u = 0$ that

$$\frac{(k_0 - \hat{k}_0)}{\prod_{j=0}^{\alpha-1} \hat{k}_j} \le \frac{2\hat{k}_0 - 1 - \hat{k}_0}{\hat{k}_0 \prod_{j=1}^{\alpha-1} \hat{k}_j} = \frac{\hat{k}_0 - 1}{\hat{k}_0} \frac{1}{\prod_{j=1}^{\alpha-1} \hat{k}_j} < \frac{1}{\prod_{j=1}^{\alpha-1} \hat{k}_j} \le 1$$

and for $u = \alpha$ we have

$$2^{\alpha-1}(k_\alpha - \hat{k}_\alpha) \le 2^{\alpha-1}(\hat{k}_\alpha - 1 - \hat{k}_\alpha) = -2^{\alpha-1}.$$

Combining these results in $(k - \hat{k}).v$ gives us the inequality

$$(k - \hat{k}).v = \frac{k_0 - \hat{k}_0}{\prod_{j=0}^{\alpha-1} \hat{k}_j} + \frac{(k_1 - \hat{k}_1)}{\prod_{j=1}^{\alpha-1} \hat{k}_j} + \frac{2(k_2 - \hat{k}_2)}{\prod_{j=2}^{\alpha-1} \hat{k}_j} + \ldots + \frac{2^{\alpha-2}(k_{\alpha-1} - \hat{k}_{\alpha-1})}{\prod_{j=\alpha-1}^{\alpha-1} \hat{k}_j} + 2^{\alpha-1}(k_\alpha - \hat{k}_\alpha)$$

$$< 1 + 1 + 2 + \ldots + 2^{\alpha-2} - 2^{\alpha-1} = 1 + \sum_{i=0}^{\alpha-2} 2^i - 2^{\alpha-1} = 1 + 2^{\alpha-1} - 1 - 2^{\alpha-1} = 0.$$

Similarly for $1 \le t \le \alpha - 1$, then for any

$$k \in \Big\{(k_0, k_1, \ldots, k_\alpha) \mid 1 \le k_0 \le 2\hat{k}_0 - 1, \ldots,\ 1 \le k_{\alpha-t-1} \le 2\hat{k}_{\alpha-t-1},\ 1 \le k_{\alpha-t} \le \hat{k}_{\alpha-t} - 1,$$

$$k_{\alpha-t+1} = \hat{k}_{\alpha-t+1}, \ldots, k_\alpha = \hat{k}_\alpha\Big\}$$

we have that

$$(k - \hat{k}).v = \frac{k_0 - \hat{k}_0}{\prod_{j=0}^{\alpha-1} \hat{k}_j} + \frac{(k_1 - \hat{k}_1)}{\prod_{j=1}^{\alpha-1} \hat{k}_j} + \frac{2(k_2 - \hat{k}_2)}{\prod_{j=2}^{\alpha-1} \hat{k}_j} + \ldots + \frac{2^{t-\alpha-2}(k_{\alpha-t-1} - \hat{k}_{\alpha-t-1})}{\prod_{j=\alpha-t-1}^{\alpha-1} \hat{k}_j}$$

$$+ \frac{2^{t-\alpha-1}(k_{\alpha-t} - \hat{k}_{\alpha-t})}{\prod_{j=\alpha-t}^{\alpha-1} \hat{k}_j} + \ldots + \frac{2^{\alpha-2}(k_{\alpha-1} - \hat{k}_{\alpha-1})}{\prod_{j=\alpha-1}^{\alpha-1} \hat{k}_j} + 2^{\alpha-1}(k_\alpha - \hat{k}_\alpha)$$

$$= \frac{k_0 - \hat{k}_0}{\prod_{j=0}^{\alpha-1} \hat{k}_j} + \frac{(k_1 - \hat{k}_1)}{\prod_{j=1}^{\alpha-1} \hat{k}_j} + \frac{2(k_2 - \hat{k}_2)}{\prod_{j=2}^{\alpha-1} \hat{k}_j} + \ldots + \frac{2^{t-\alpha-2}(k_{\alpha-t-1} - \hat{k}_{\alpha-t-1})}{\prod_{j=\alpha-t-1}^{\alpha-1} \hat{k}_j} + \frac{2^{t-\alpha-1}(k_{\alpha-t} - \hat{k}_{\alpha-t})}{\prod_{j=\alpha-t}^{\alpha-1} \hat{k}_j}$$

$$= \frac{1}{\prod_{j=\alpha-t}^{\alpha-1} \hat{k}_j} \Bigg(\frac{k_0 - \hat{k}_0}{\prod_{j=0}^{\alpha-t-1} \hat{k}_j} + \frac{(k_1 - \hat{k}_1)}{\prod_{j=1}^{\alpha-t-1} \hat{k}_j} + \frac{2(k_2 - \hat{k}_2)}{\prod_{j=2}^{\alpha-t-1} \hat{k}_j} + \ldots + \frac{2^{t-\alpha-2}(k_{\alpha-t-1} - \hat{k}_{\alpha-t-1})}{\prod_{j=\alpha-t-1}^{\alpha-t-1}}$$

$$+ 2^{t-\alpha-1}(k_{\alpha-t} - \hat{k}_{\alpha-t})\Bigg)$$

$$< 1 + 1 + 2 + \ldots + 2^{\alpha-t-2} - 2^{\alpha-t-1} = 1 + \sum_{i=0}^{\alpha-t-2} 2^i - 2^{\alpha-t-1} = 0.$$

Lastly for

$$k \in \left\{(k_0, k_1, \ldots, k_\alpha) \mid 1 \leq k_0 \leq \hat{k}_0 - 1,\ k_1 = \hat{k}_1 \ldots,\ k_\alpha = \hat{k}_\alpha\right\}$$

$$(k - \hat{k}).v = \frac{k_0 - \hat{k}_0}{\prod_{j=0}^{\alpha-1} \hat{k}_j} < -1.$$

Therefore for any $k \in L'_\alpha$ we have $(k - \hat{k}).v < 0$.

We now show that points not in L'_α but in L_α give $(k - \hat{k}).v \geq 0$. First, consider the points not in S_α, the set of which we denote by

$$S'_\alpha = \left\{(k_0, \ldots, k_\alpha) \mid \hat{k}_0 + 1 \leq k_0 \leq 2\hat{k}_0 - 1,\ k_1 = \hat{k}_1, \ldots, k_\alpha = \hat{k}_\alpha\right\}.$$

For any points in this set we have that

$$(k - \hat{k}).v = \frac{k_0 - \hat{k}_0}{\prod_{j=0}^{\alpha-1} \hat{k}_j} > \frac{1}{\prod_{j=0}^{\alpha-1} \hat{k}_j} > 0.$$

Corresponding to the set S_t for $0 \leq t \leq \alpha - 1$ we define the *complimentary set* such that

$$S'_t = \Big\{k \mid 1 \leq k_0 \leq 2\hat{k}_0 - 1,\ \ldots, 1 \leq k_{\alpha-t-1} \leq 2\hat{k}_{\alpha-t-1} - 1,\ \hat{k}_{\alpha-t} + 1 \leq k_{\alpha-t} \leq 2\hat{k}_{\alpha-t} - 1,$$

$$k_{\alpha-t+1} = \hat{k}_{\alpha-t+1}, \ldots, k_\alpha = \hat{k}_\alpha\Big\},$$

which yields the inequality chain

$$(k-\hat{k}).v = \frac{k_0 - \hat{k}_0}{\prod_{j=0}^{\alpha-1} \hat{k}_j} + \frac{(k_1 - \hat{k}_1)}{\prod_{j=1}^{\alpha-1} \hat{k}_j} + \frac{2(k_2 - \hat{k}_2)}{\prod_{j=2}^{\alpha-1} \hat{k}_j} + \ldots + \frac{2^{t-\alpha-2}(k_{\alpha-t-1} - \hat{k}_{\alpha-t-1})}{\prod_{j=\alpha-t-1}^{\alpha-1} \hat{k}_j} + \frac{2^{t-\alpha-1}(k_{\alpha-t} - \hat{k}_{\alpha-t})}{\prod_{j=\alpha-t}^{\alpha-1} \hat{k}_j}$$

$$\geq \frac{1 - \hat{k}_0}{\prod_{j=0}^{\alpha-1} \hat{k}_j} + \frac{(1 - \hat{k}_1)}{\prod_{j=1}^{\alpha-1} \hat{k}_j} + \frac{2(1 - \hat{k}_2)}{\prod_{j=2}^{\alpha-1} \hat{k}_j} + \ldots + \frac{2^{t-\alpha-2}(1 - \hat{k}_{\alpha-t-1})}{\prod_{j=\alpha-t-1}^{\alpha-1} \hat{k}_j} + \frac{2^{t-\alpha-1}(1)}{\prod_{j=\alpha-t}^{\alpha-1} \hat{k}_j}$$

$$= \frac{1}{\prod_{j=u-t}^{\alpha-t-1}} \left(\frac{1 - \hat{k}_0}{\prod_{j=0}^{\alpha-t-1} \hat{k}_j} + \frac{(1 - \hat{k}_1)}{\prod_{j=1}^{\alpha-t-1} \hat{k}_j} + \frac{2(1 - \hat{k}_2)}{\prod_{j=2}^{\alpha-t-1} \hat{k}_j} + \ldots + \frac{2^{t-\alpha-2}(1 - \hat{k}_{\alpha-t-1})}{\prod_{j=\alpha-t-1}^{\alpha-t-1} \hat{k}_j} + 2^{t-\alpha-1} \right)$$

$$> \frac{1}{\prod_{j=u-t}^{\alpha-t-1}} (1 + 1 + 2 + \ldots + 2^{\alpha-t-2} - 2^{\alpha-t-1}) = 0.$$

Finally, for $k = \hat{k}$, we obtain $(k - \hat{k}).v = 0.v = 0$, for any $k \in L'_\alpha$ and putting this all together we have $(k - \hat{k}).v < 0$ and for all $k \in L_\alpha / L'_\alpha$ we have $(k - \hat{k}).v \geq 0$, as required.

□

LEMMA 6.9. *Given an α length divisor path set for $n \in \mathbb{N}$,*

$$\{\hat{i}, \hat{j}\} = \{\ (i_1, \dots, i_\alpha = n),\ (j_1, \dots, j_\alpha)\ \}$$

then the first half of the lattice points can be described as

$$M'_\alpha = \bigcup_{u=0}^{\alpha-1} T_u$$

where

$$T_0 = \left\{ (m_0, m_1, \dots, m_{\alpha-1}) \mid 1 \leq m_0 \leq i_1,\ 1 \leq m_1 \leq \frac{i_2}{i_1},\ \dots, 1 \leq m_{\alpha-1} \leq \frac{1}{2}\left(\frac{n}{i_{\alpha-1}} - 1\right) \right\},$$

and for $1 \leq u \leq \alpha - 2$,

$$T_u = \left\{ (m_0, m_1, \dots, m_{\alpha-1}) \mid 1 \leq m_0 \leq i_1,\ 1 \leq m_1 \leq \frac{i_2}{i_1}, \dots \right.$$

$$\left. \dots, 1 \leq m_{\alpha-1-u} \leq \frac{1}{2}\left(\frac{i_{\alpha-u}}{i_{\alpha-1-u}} - 1\right), m_{\alpha-u} = \hat{m}_{\alpha-u}, \dots, m_{\alpha-1} = \hat{m}_{\alpha-1} \right\}$$

and

$$T_{\alpha-1} = \left\{ (m_0, m_1, \dots, m_{\alpha-1}) \mid 1 \leq m_0 \leq \frac{1}{2}(i_1 - 1),\ m_1 = \hat{m}_1, \dots, m_{\alpha-1} = \hat{m}_{\alpha-1} \right\}.$$

Proof. T_0 gives the first $\frac{1}{2}\left(\frac{n}{i_{\alpha-1}} - 1\right)$, $i_{\alpha-1}$ length sets of lattice points which does not include the middle $i_{\alpha-1}$ length set. Further, T_1 gives the first $\frac{1}{2}\left(\frac{i_{\alpha-1}}{\alpha-2}\right)$, $i_{\alpha-2}$ length set of lattice points within the middle $i_{\alpha-1}$ set. This holds for all u, T_u gives the first $\frac{1}{2}\left(\frac{i_{\alpha-u}}{i_{\alpha-u-1}} - 1\right)$, $i_{\alpha-u-1}$ length sets of lattice points in the middle $i_{\alpha-u}$ set. Lastly, T_α gives the middle $\frac{1}{2}(i_1 - 1)$ values in the middle i_1 length set of lattice points. So to conclude, when we take the union of all of these points we obtain all the points in the first half of the column. □

LEMMA 6.10. *The set of lattice points M'_α can be described by all the points under a hyperplane in the full α lattice, with corresponding normal vector defined by*

$$d_i = \begin{cases} 1 & \textit{if } i = 0, \\ 2^{i-1} & \textit{if } i > 0, \end{cases} \quad \textit{and} \quad u_i = \frac{d_i}{\prod_{j=1}^{\alpha-1} \hat{m}_j}$$

then

$$M'_\alpha = \{(m_0, m_1, \dots, m_{\alpha-1}) \mid (m - \hat{m}).u < 0,\ m \in M_\alpha\}.$$

Proof. Firstly we will show that any $m \in \cup_{u=0}^{\alpha-1} T_u$ will give $(m - \hat{m}).u < 0$, so let us consider any

$$m \in \{(m_0, \dots, m_{\alpha-1}) \mid 1 \leq m_0 \leq 2\hat{m}_0 - 1, \dots, 1 \leq m_{\alpha-2} \leq 2\hat{m}_{\alpha-1} \leq \hat{m}_\alpha - 1\}.$$

So for $0 < u \leq \alpha - 2$

$$\frac{2^{u-1}(m_u - \hat{m}_u)}{\prod_{j=u}^{\alpha-2} \hat{m}_j} \leq \frac{2^{u-1}(2\hat{m}_u - 1 - \hat{m}_u)}{\prod_{j=u}^{\alpha-2} \hat{m}_j} = \frac{(\hat{m}_u - 1)}{\hat{m}_u} \frac{2^{u-1}}{\prod_{j=u+1}^{\alpha-2} \hat{m}_j} < 2^{u-1},$$

for $u = 0$

$$\frac{m + 0 - \hat{m}_0}{\prod_{j=0}^{\alpha-2} \hat{m}_j} \leq \frac{2\hat{m}_0 - 1 - \hat{m}_0}{\prod_{j=0}^{\alpha-2} \hat{m}_j} = \frac{(m_0 - \hat{m}_0)}{\hat{m}_0} \frac{1}{\prod_{j=1}^{\alpha-2} \hat{m}_j} < \frac{1}{\prod_{j=1}^{\alpha-2} \hat{m}_j} \leq 1,$$

and for $u = \alpha - 1$

$$2^{\alpha-2}(m_{\alpha-1} - \hat{m}_{\alpha-1}) \leq 2^{\alpha-2}(\hat{m}_{\alpha-1} - 1 - \hat{m}_{\alpha-1}) = 2^{\alpha-2}.$$

The we have the inequality

$$(m - \hat{m}).u = \frac{(m_0 - \hat{m}_0)}{\prod_{j=0}^{\alpha-2} \hat{m}_j} + \frac{(m_1 - \hat{m}_1)}{\prod_{j=1}^{\alpha-2} \hat{m}_j} + \frac{2(m_2 - \hat{m}_2)}{\prod_{j=2}^{\alpha-2} \hat{m}_j} + \ldots + \frac{2^{\alpha-3}(m_{\alpha-2} - \hat{m}_{\alpha-2})}{\prod_{j=\alpha-2}^{\alpha-2} \hat{m}_j}$$

$$< 1 + 1 + 2 + \ldots + 2^{\alpha-2} - 2^{\alpha-2} = 0.$$

Now consider $1 \leq t \leq \alpha - 2$ such that

$$m \in \{(m_0, \ldots, m_{\alpha-1}) \mid 1 \leq m_0 \leq 2\hat{m}_0 - 1, \ldots, 1 \leq m_{\alpha-t-2} \leq 2\hat{m}_{\alpha-t-2} - 1,$$

$$1 \leq m_{\alpha-t-1} \leq \hat{m}_{\alpha-t-1} - 1,\ m_{\alpha-t} = \hat{m}_{\alpha-t},\ \ldots, m_{\alpha-1} = \hat{m}_{\alpha-1}\}$$

$$(m - \hat{m}).u = \frac{(m_0 - \hat{m}_0)}{\prod_{j=0}^{\alpha-2} \hat{m}_j} + \frac{(m_1 - \hat{m}_1)}{\prod_{j=1}^{\alpha-2} \hat{m}_j} + \frac{2(m_2 - \hat{m}_2)}{\prod_{j=2}^{\alpha-2} \hat{m}_j} + \ldots \frac{2^{\alpha-t-3}(m_{\alpha-t-2} - \hat{m}_{\alpha-t-2})}{\prod_{j=\alpha-t-2}^{\alpha-2} \hat{m}_j}$$
$$+ \frac{2^{\alpha-t-2}(m_{\alpha-t-1} - \hat{m}_{\alpha-t-1})}{\prod_{j=\alpha-t-1}^{\alpha-2} \hat{m}_j} + \frac{2^{\alpha-t-1}(m_{\alpha-t} - \hat{m}_{\alpha-t})}{\prod_{j=\alpha-t}^{\alpha-2} \hat{m}_j} + \ldots + 2^{\alpha-2}(m_{\alpha-1} - \hat{m}_{\alpha-1})$$

$$= \frac{(m_0 - \hat{m}_0)}{\prod_{j=0}^{\alpha-2} \hat{m}_j} + \frac{(m_1 - \hat{m}_1)}{\prod_{j=1}^{\alpha-2} \hat{m}_j} + \frac{2(m_2 - \hat{m}_2)}{\prod_{j=2}^{\alpha-2} \hat{m}_j} + \ldots \frac{2^{\alpha-t-3}(m_{\alpha-t-2} - \hat{m}_{\alpha-t-2})}{\prod_{j=\alpha-t-2}^{\alpha-2} \hat{m}_j}$$
$$+ \frac{2^{\alpha-t-2}(m_{\alpha-t-1} - \hat{m}_{\alpha-t-1})}{\prod_{j=\alpha-t-1}^{\alpha-2} \hat{m}_j}$$

$$= \frac{1}{\prod_{j=\alpha-t-1}^{\alpha-2}\hat{m}_j}\left(\frac{(m_0-\hat{m}_0)}{\prod_{j=0}^{\alpha-t-2}\hat{m}_j}+\frac{(m_1-\hat{m}_1)}{\prod_{j=1}^{\alpha-t-2}\hat{m}_j}+\frac{2(m_2-\hat{m}_2)}{\prod_{j=2}^{\alpha-t-2}\hat{m}_j}+\dots\frac{2^{\alpha-t-3}(m_{\alpha-t-2}-\hat{m}_{\alpha-t-2})}{\prod_{j=\alpha-t-2}^{\alpha-t-2}\hat{m}_j}\right.$$
$$\left.+2^{\alpha-t-2}(m_{\alpha-t-1}-\hat{m}_{\alpha-t-1})\right)$$

$$< \frac{1}{\prod_{j=\alpha-t-1}^{\alpha-2}\hat{m}_j}\left(1+1+2+\dots+2^{\alpha-t-3}-2^{\alpha-t-2}\right)=0.$$

Similarly, for

$$m \in \{(m_0,\dots,m_{\alpha-1}) \mid 1 \le m_0 \le \hat{m}_0-1,\ m_1=\hat{m}_1,\ \dots,\ m_{\alpha-1}=\hat{m}_{\alpha-1}\}$$

we have that

$$(m-\hat{m}).u = \frac{\hat{m}_0-1-\hat{m}_0}{\prod_{j=0}^{\alpha-2}\hat{m}_j} < 0.$$

We again consider the complimentary set of points not in $\cup_{u=0}^{\alpha-1}T_u$ and deduce that for all those points $(m-\hat{m}).u \ge 0$. We begin with the points not in the set $T_{\alpha-1}$ denoted by

$$m \in T'_{\alpha-1} = \{(m_0,\dots,m_{\alpha-1}) \mid \hat{m}_0+1 \le m_0 \le 2\hat{m}_0-1, m_1=\hat{m}_1,\dots,m_{\alpha-1}=\hat{m}_{\alpha-1}\},$$

from which we deduce that

$$(m-\hat{m}).u = \frac{(m_0-\hat{m}_0)}{\prod_{j=0}^{\alpha-2}\hat{m}_j} \ge \frac{\hat{m}_0+1-\hat{m}_0}{\prod_{j=0}^{\alpha-2}\hat{m}_j} = \frac{1}{\prod_{j=0}^{\alpha-2}\hat{m}_j} > 0.$$

For the set T_u for $0 \le u \le \alpha-2$ we denote its complimentary set by

$$\begin{aligned} T'_u = &\{(m_0,\dots,m_{\alpha-1}) \mid 1 \le m_0 \le 2\hat{m}_0-1,\ \dots, 1 \le m_{\alpha-u-2} \le 2\hat{m}_{\alpha-u-2}-1,\\ &\hat{m}_{\alpha-u-1}+1 \le m_{\alpha-u-1} \le 2\hat{m}_{\alpha-u-1}-1,\ m_{\alpha-u}=\hat{m}_{\alpha-u},\dots,m_{\alpha-1}=\hat{m}_{\alpha-1}\ \}\end{aligned}$$

We then calculate

$$(m-\hat{m}).u = \frac{(m_0-\hat{m}_0)}{\prod_{j=0}^{\alpha-2}\hat{m}_j}+\frac{(m_1-\hat{m}_1)}{\prod_{j=1}^{\alpha-2}\hat{m}_j}+\frac{2(m_2-\hat{m}_2)}{\prod_{j=2}^{\alpha-2}\hat{m}_j}+\dots\frac{2^{\alpha-t-3}(m_{\alpha-t-2}-\hat{m}_{\alpha-t-2})}{\prod_{j=\alpha-t-2}^{\alpha-2}\hat{m}_j}$$
$$+\frac{2^{\alpha-t-2}(m_{\alpha-t-1}-\hat{m}_{\alpha-t-1})}{\prod_{j=\alpha-t-1}^{\alpha-2}\hat{m}_j}+\frac{2^{\alpha-t-1}(m_{\alpha-t}-\hat{m}_{\alpha-t})}{\prod_{j=\alpha-t}^{\alpha-2}\hat{m}_j}+\dots+2^{\alpha-2}(m_{\alpha-1}-\hat{m}_{\alpha-1})$$
$$=\frac{(m_0-\hat{m}_0)}{\prod_{j=0}^{\alpha-2}\hat{m}_j}+\frac{(m_1-\hat{m}_1)}{\prod_{j=1}^{\alpha-2}\hat{m}_j}+\frac{2(m_2-\hat{m}_2)}{\prod_{j=2}^{\alpha-2}\hat{m}_j}+\dots\frac{2^{\alpha-t-3}(m_{\alpha-t-2}-\hat{m}_{\alpha-t-2})}{\prod_{j=\alpha-t-2}^{\alpha-2}\hat{m}_j}$$
$$+\frac{2^{\alpha-t-2}(m_{\alpha-t-1}-\hat{m}_{\alpha-t-1})}{\prod_{j=\alpha-t-1}^{\alpha-2}\hat{m}_j}$$

$$= \frac{(m_0 - \hat{m}_0)}{\prod_{j=0}^{\alpha-2} \hat{m}_j} + \frac{(m_1 - \hat{m}_1)}{\prod_{j=1}^{\alpha-2} \hat{m}_j} + \frac{2(m_2 - \hat{m}_2)}{\prod_{j=2}^{\alpha-2} \hat{m}_j} + \dots \frac{2^{\alpha-t-3}(m_{\alpha-t-2} - \hat{m}_{\alpha-t-2})}{\prod_{j=\alpha-t-2}^{\alpha-2} \hat{m}_j}$$
$$+ \frac{2^{\alpha-t-2}(m_{\alpha-t-1} - \hat{m}_{\alpha-t-1})}{\prod_{j=\alpha-t-1}^{\alpha-2} \hat{m}_j}$$

$$= \frac{1}{\prod_{j=\alpha-t-1}^{\alpha-2} \hat{m}_j} \left(\frac{(m_0 - \hat{m}_0)}{\prod_{j=0}^{\alpha-t-2} \hat{m}_j} + \frac{(m_1 - \hat{m}_1)}{\prod_{j=1}^{\alpha-t-2} \hat{m}_j} + \frac{2(m_2 - \hat{m}_2)}{\prod_{j=2}^{\alpha-t-2} \hat{m}_j} + \dots \frac{2^{\alpha-t-3}(m_{\alpha-t-2} - \hat{m}_{\alpha-t-2})}{\prod_{j=\alpha-t-2}^{\alpha-t-2} \hat{m}_j} \right.$$
$$\left. +2^{\alpha-t-2}(m_{\alpha-t-1} - \hat{m}_{\alpha-t-1}) \right)$$

$$= \frac{1}{\prod_{j=\alpha-u-1}^{\alpha-2} \hat{m}_j} \left(\underbrace{\frac{(1 - \hat{m}_0)}{\hat{m}_0} \frac{1}{\prod_{j=1}^{\alpha-u-2} \hat{m}_j}}_{>-1} + \underbrace{\frac{(1 - \hat{m}_1)}{\hat{m}_1} \frac{1}{\prod_{j=2}^{\alpha-u-2}}}_{>-1} + \underbrace{\frac{(1 - \hat{m}_2)}{\hat{m}_2} \frac{2}{\prod_{j=3}^{\alpha-u-2} \hat{m}_j}}_{>-2} \right.$$

$$\left. \dots + \underbrace{2^{\alpha-u-3} \frac{1 - \hat{m}_{\alpha-u-2}}{\hat{m}_{\alpha-u-2}}}_{>-2^{\alpha-u-3}} + \underbrace{2^{\alpha-t-2}(m_{\alpha-t-1} - \hat{m}_{\alpha-t-1})}_{\geq 2^{\alpha-t-2}} \right)$$
$$< \frac{1}{\prod_{j=\alpha-u-1}^{\alpha-2} \hat{m}_j} \left(-1 - 1 - 2 - \dots - 2^{\alpha-t-3} + 2^{\alpha-t-2} \right) = \frac{1}{\prod_{j=\alpha-u-1}^{\alpha-2} \hat{m}_j}(0). = 0$$

Finally, for $m = \hat{m}$ we have $(m - \hat{m}).u = (0).u = 0$, therefore for any $m \in M_\alpha / M'_\alpha$ we have that $(m - \hat{m}).u \geq 0$, as required.

□

Definition. A parasymmetric inclusive or non-inclusive sum-and-distance system is defined to be one that satisfies,

$$\{a, b\} = \{a, \lambda a\},$$

for some $\lambda \in \mathbb{Z}$ and $|a| = |b|$.

THEOREM 6.11. *Let $n = q^\gamma$ be odd, where $\gamma \in \mathbb{N}$. Then any parasymmetric sum-and-distance system is given by Lemma 6.6 with the corresponding divisor path set*

$$\{\hat{\imath}, \hat{\jmath}\} = \{(q^w, q^{2w}, \dots,\ q^{vw}),\ (q^w, q^{2w}, \dots,\ q^{vw})\}$$

such that $w|\gamma$, $vw = \gamma$ and $\lambda = j_1 = i_1$.

Proof. To show that a sum-and-distance is parasymmetric if and only if its divisor path has the following structure

$$\{\hat{\imath}, \hat{\jmath}\} = \{(q^w, q^{2w}, \dots,\ q^{vw}),\ (q^w, q^{2w}, \dots,\ q^{vw})\}$$

we must show that if $\lambda a = b$ then it will indeed have the above structure. Given that within the first row and column the parts b and a can be found, i.e.

$$\lambda\tilde{a} = \lambda(r - b_k - w) = \lambda\{Ja,\ 0,\ -a\} = c - a_k - w = \{Jb,\ 0,\ -b\} = \lambda\tilde{b},$$

we have that when the corresponding lattice points are substituted into the explicit formula for the first row and column, the same values will be obtained.

For the midpoints $k = (\hat{k_0}, \hat{k}_1, \ldots, \hat{k}_\alpha)$ and $m = (\hat{m}_0, \hat{m}_1, \ldots, \hat{m}_{\alpha-1})$ we know from the preceding results that both the values will be equal to 0, and similarly for the point to the immediate left of the midpoint, which will have lattice co-ordinates

$$k = (\hat{k}_0 - 1, \hat{k}_1, \ldots, \hat{k}_\alpha),\ \text{and}\ m = (\hat{m}_0 - 1, \hat{m}_1, \ldots, \hat{m}_{\alpha-1}),$$

which when substituted in to the equations for the first row and column give

$$\lambda\left((k_0 - 1) - \frac{1}{2}(j_1 - 1) + \sum_{u=1}^{\alpha}(i_u j_u)\left((k_u - 1) - \frac{1}{2}\left(\frac{j_{u+1}}{j_u} - 1\right)\right)\right)$$

$$= \left(j_1((m_0 - 1) - \frac{1}{2}(i_1 - 1)) + \sum_{u=1}^{\alpha-1}(i_u j_u)\left(\frac{j_{u+1}}{j_u}\right)\left((m_u - 1) - \frac{1}{2}\left(\frac{i_{u+1}}{i_u} - 1\right)\right)\right)$$

Rearranging and simplifying we obtain,

$$\begin{aligned}
\lambda\left((k_0 - \hat{k}) + \sum_{u=1}^{\alpha}(i_u j_u)\left(k_u - \hat{k}_u\right)\right) &= \left(j_1(m_0 - \hat{m}_0) + \sum_{u=1}^{\alpha-1}(i_u j_u)\left(\frac{j_{u+1}}{j_u}\right)(m_u - \hat{m}_u)\right) \\
\lambda\left((\hat{k} - 1 - \hat{k}) + \sum_{u=1}^{\alpha}(i_u j_u)\left(\hat{k}_u - \hat{k}_u\right)\right) &= \left(j_1(\hat{m}_0 - 1 - \hat{m}_0) + \sum_{u=1}^{\alpha-1}(i_u j_u)\left(\frac{j_{u+1}}{j_u}\right)(\hat{m}_u - \hat{m}_u)\right) \\
\lambda\left((\hat{k} - 1 - \hat{k})\right) &= (j_1(\hat{m}_0 - 1 - \hat{m}_0)) \\
\lambda &= j_1.
\end{aligned}$$

Hence we have the equality

$$j_1\tilde{a} = j_1(k_0 - \hat{k}_0) + \sum_{u=1}^{\alpha} j_1(i_u j_u)(k_u - \hat{k}_u) = j_1(m_0 - \hat{m}) + \sum_{u=1}^{\alpha-1}\frac{j_{u+1}}{u}(i_u j_u)(m_u - \hat{m}_u) = \tilde{b}.$$

For the base case we will show that $i_1 = j_1$ and $\frac{j_2}{j_1} = j_1$, so w.l.o.g. let us assume that $i_1 > j_1$ so that $i_1 \geq j_1 + 2$. We can then say that the middle j_1 sequence in the first row is smaller then the middle i_1 sequence in the first column. It follows that the $\hat{k}_0 th$ entry from the middle of the i_1 sequence is at the same positioned entry in $j_1\tilde{a} = \tilde{b}$ as the last entry of the first j_1 sequence to the left of the middle j_1 sequence, due to $i_1 > j_1$. Hence the lattice co-ordinates

$$k = (j_1,\ \hat{k}_1 - 1,\ \hat{k}_2,\ \ldots,\ \hat{k}_\alpha),\quad \text{and}\quad m = (\hat{m}_0 - \hat{k}_0,\ \hat{m}_1,\ \ldots,\ \hat{m}_{\alpha-1})$$

give the same entry, and

$$j_1\tilde{a} = j_1(j_1 - \hat{k}_0) + j_1(i_1 j_1)(\hat{k}_1 - 1 - \hat{k}_1) + \sum_{u=2}^{\alpha} j_1(i_u j_u)(\hat{k}_u - \hat{k}_u)$$

$$= \; j_1(\hat{m}_0 - \hat{k}_0 - \hat{m}_0) + \sum_{u=1}^{\alpha-1} \frac{j_{u+1}}{j_u}(i_u j_u)(\hat{m}_u - \hat{m}_u) = \tilde{b}.$$

Hence

$$j_1(j_1 - \hat{k}_0) - j_1(i_1 j_1) = -j_1 \hat{k}_0,$$

and therefore

$$i_1 = 1,$$

which is impossible as by definition $i_1 \neq 1$ so that $i_1 \geq 3$.

Similarly, say $j_1 > i_1$, then $j_1 \geq i_1 + 2$ the middle j_1 sequence in $j_1\tilde{a}$ is longer than the middle i_1 sequence in $\tilde{b}$, so the $\hat{m}_1$ entry from the middle point will be in the middle j_1 sequence. Now the same positioned entry is the last entry in the next i_1 length sequence to the left of the middle i_1 length sequence in $\tilde{b}$, and so the lattice co-ordinates

$$k = (\hat{k}_0 - \hat{m}_0,\ \hat{k}_1, \ldots, \hat{k}_\alpha), \quad \text{and} \quad m = (i_1,\ \hat{m}_1 - 1,\ \hat{m}_2,\ \ldots,\ \hat{m}_{\alpha-1})$$

give the same entry

$$j_1\tilde{a} = j_1(\hat{k}_0 - \hat{m}_0 - \hat{k}_0) + \sum_{u=1}^{\alpha} j_1(i_u j_u)(\hat{k}_u - \hat{k}_u)$$

$$= j_1(i_1 - \hat{m}_0) + \frac{j_2}{j_1}(i_1 j_1)(\hat{m}_1 - 1 - \hat{m}_1) + \sum_{u=2}^{\alpha-1} = \tilde{b},$$

which rearranging gives,

$$-j_1\hat{m}_0 = j_1 i_1 j_1 \hat{m}_0 - \frac{j_2}{j_1}(i_1 j_1)$$

$$j_1 = j_2.$$

However, by definition $j_2 > j_1$ so from this contradiction we conclude that $i_1 = j_1$.

As $i_1 = j_1$ the entries at

$$k = (\hat{k}_0,\ \hat{k}_1 - 1,\ \hat{k}_2,\ \ldots,\ \hat{k}_\alpha) \;\; \text{and} \;\; m = (\hat{m}_0,\ \hat{m}_1 - 1,\ \ldots,\ \hat{m}_{\alpha-1})$$

in $j_1\tilde{a} = \tilde{b}$ are equal regardless of the size of i_2 or j_2 as both are at least 3. We then have the equality

$$j_1(\hat{k}_0-\hat{k}_0)+j_1(i_1j_1)(\hat{k}_1-1-\hat{k}_1)+\sum_{u=2}^{\alpha}j_1(i_uj_u)(\hat{k}_u-\hat{k}_u)$$

$$=j_1(\hat{m}_0-\hat{m}_0)+\frac{j_2}{j_1}(i_1j_1)(\hat{m}_1-1-\hat{m}_1)+\sum_{u=2}\frac{j_{u+1}}{j_u}(\hat{m}_u-\hat{m}_u)$$

which by rearranging gives

$$\begin{aligned}-j_1(i_1j_1)=&-\frac{j_2}{j_1}(i_1j_1)\\ j_1=&\frac{j_2}{j_1}.\end{aligned}$$

Inductively assuming that for all $u\leq w$, $1\leq w\leq\alpha$ that $i_u=j_u$ and $\frac{j_{u+1}}{j_u}$ then we need only show that $j_{w+1}=i_{w+1}$ and $\frac{j_{w+2}}{j_{w+1}}=j_1$.

If $i_{w+1}>j_{w+1}$, then $i_{w+1}\geq j_{w+2}$, and similarly for the argument when $i_1>j_1$, from which we deduce that the two lattice co-ordinates

$$k=(j_1,\ \frac{j_2}{j_1},\ \ldots,\ \frac{j_w}{j_{w-1}},\frac{j_{w+1}}{j_w},\ \hat{k}_{w+1}-1,\ \hat{k}_{w+2},\ldots,\ \hat{k}_\alpha),\quad\text{and}$$

$$m=(i_1,\ \frac{i_2}{i_1},\ \ldots,\ \frac{i_w}{i_{w+1}},\ \hat{m}_w-\hat{k}_w,\ \hat{m}_{w+1},w\hat{+}2,\ldots,\ \hat{m}_{\alpha-1})$$

give the same entries in $j_1\tilde{a}=\tilde{b}$, with the resulting equality

$$j_1(j_1-\hat{k}_0)+\sum_{u=1}^{w-1}j_1(i_uj_u)\left(\frac{j_{u+1}}{j_u}-\hat{k}_u\right)+j_1(i_wj_w)\left(\frac{j_{w+1}}{j_w}-\hat{k}_w\right)+j_1(i_{w+1}j_{w+1})(\hat{k}_{w+1}-1-\hat{k}_{w+1})$$

$$+\underbrace{\sum_{u=w+2}^{\alpha}j_1(i_uj_u)(\hat{k}_u-\hat{k}_u)}_{=0}$$

$$=j_1(i_1-\hat{m}_1)+\sum_{u=1}^{w+1}\frac{j_{u+1}}{j_u}(i_uj_u)\left(\frac{i_{u+1}}{i_u}-\hat{m}_u\right)+\frac{j_{w+1}}{j_w}(i_wj_w)(\hat{m}_w-\hat{k}_w-\hat{m}_w)$$

$$+\underbrace{\sum_{u=w+2}^{\alpha-1}\frac{j_{u+1}}{j_u}(i_uj_u)(\hat{m}_u-\hat{m}_u)}_{=0}.$$

So as $i_u=j_u$ and $\frac{j_{u+1}}{j_u}$ for $u\leq w$ then

$$j_1(j_1-\hat{k}_0)+\sum_{u=1}^{w-1}j_1(i_uj_u)\left(\frac{j_{u+1}}{j_u}-\hat{k}_u\right)=j_1(i_1-\hat{m}_1)+\sum_{u=1}^{w+1}\frac{j_{u+1}}{j_u}(i_uj_u)\left(\frac{i_{u+1}}{i_u}-\hat{m}_u\right)$$

and so

$$j_1(i_w j_w)\left(\frac{j_{w+1}}{j_w} - \hat{k}_w\right) - j_1(i_{w+1}j_{w+1}) = -\frac{j_{w+1}}{j_w}(i_w j_w)\hat{k}_w.$$

From our assumption $i_w = j_w$ and $\frac{j_{w+1}}{j_w} = j_1$ we thus obtain

$$j_1^2 j_w^2 - j_1 j_w^2 \hat{k}_w - j_1(i_{w+1}j_{w+1}) = -j_1 j_w^2 \hat{k}_w,$$

and rearranging we have

$$\begin{aligned} j_1 j_w^2 &= j_1 i_{w+1} j_{w+1} \\ \frac{j_{w+1}}{j_w} j_w^2 &= i_{w+1} j_{w+1} \\ i_w = j_w &= i_{w+1} \end{aligned}$$

which is impossible as $i_w < i_{w+1}$.

Similarly for $j_{w+1} > i_{w+1}$ we have that $j_{w+1} \geq i_{w+1} + 2$ from which we deduce that for the lattice co-ordinates

$$k = (j_1,\ \frac{j_2}{j_1},\ \ldots,\ \frac{j_w}{j_{w-1}},\ \hat{k}_w - \hat{m}_w,\ \hat{k}_{w+1},\ \ldots,\ \hat{k}_{\alpha-1},\ \hat{k}_\alpha) \text{ and}$$

$$m = (i_1, \frac{i_2}{i_1},\ \ldots,\ \frac{i_w}{i_{w-1}},\ \frac{i_{w+1}}{i_w},\ \hat{m}_k - 1,\ \hat{m}_{w+2},\ \ldots,\ \hat{m}_{\alpha-1})$$

which gives the equality

$$j_1(j_1 - \hat{k}_0) + \sum_{u=1}^{w-1} j_1(i_u j_u)\left(\frac{j_{u+1}}{j_u} - \hat{k}_u\right) + j_1(i_w j_w)(\hat{k}_w - \hat{m}_w - \hat{k}_w)$$

$$+ \underbrace{\sum_{u=w+1}^{\alpha} j_1(i_u j_u)(\hat{k}_u - \hat{k}_u)}_{=0}$$

$$= j_1(i_1 - \hat{m}_0) + \sum_{u=1}^{w-1} \frac{j_{u+1}}{j_u}\left(\frac{i_{u+1}}{i_u} - \hat{m}_u\right) + \frac{j_{w+1}}{j_w}(i_w j_w)\left(\frac{i_{w+1}}{i_w} - \hat{m}_w\right)$$

$$+\frac{j_{w+2}}{j_{w+1}}(i_{w+1}j_{w+1})(\hat{m}_{w+1} - 1 - \hat{m}_{w+1}) + \underbrace{\sum_{u=w+2}^{\alpha-1} \frac{j_{u+1}}{j_u}(i_u j_u)(\hat{m}_u - \hat{m}_u)}_{=0}.$$

With $i_w = j_w$ and $\frac{j_{w+1}}{j_w}$ this becomes

$$j_1^2 j_w^2 - j_1 j_w^2 \hat{k}_w - j_1(i_{w+1}j_{w+1}) = -j_1 j_w^2 \hat{k}_w$$

which rearranges to

$$j_1^2 j_w^2 = j_1 i_{w+1} j_{w+1}$$
$$\frac{j_{w+1}}{j_w} j_w^2 = i_{w+1} j_{w+1}$$
$$i_w = j_w = i_{w+1},$$

which again is impossible as $j_{w+1} < j_{w+2}$. Therefore to conclude we must have $i_{w+1} = j_{w+1}$.

As a final step we need to show that $\frac{j_{w+2}}{j_{w+1}} = j_1$, similarly to when we deduced that $\frac{j_2}{j_1} = j_1$. Now as $i_u = j_u$ for all $u \leq w+1$ then we have that the lattice co-ordinates

$$k = (\hat{k}_0,\ \hat{k}_1,\ \ldots, \hat{k}_w,\ \hat{k}_{w+1} - 1,\ \hat{k}_{w+2}, \ldots, \hat{k}_\alpha)$$

$$m = (\hat{m}_0,\ \hat{m}_1,\ \ldots, \hat{m}_w,\ \hat{m}_{w+1} - 1,\ \hat{m}_{w+2}, \ldots, \hat{m}_{\alpha-1})$$

give the same entry in $j_1 \tilde{a} = \tilde{b}$, which when substituted and simplified gives

$$-j_1(i_{w+1} j_{w+1}) = -\frac{j_{w+2}}{j_{w+1}}(i_{w+1} j_{w+1})$$

$$j_1 = \frac{j_{w+2}}{j_{w+1}}.$$

Therefore for all $1 \leq u \leq \alpha - 1$ (where it can be deduced that $j_\alpha = i_\alpha = n$ as the sum for $\tilde{b}$ can be extended to include $i_\alpha = n$) we have that

$$i_u = j_u \quad \text{and} \quad \frac{j_{u+1}}{j_u} = j_1.$$

To conclude if $\frac{j_{u+1}}{j_u} = j_1$ for all u then $j_2 = j_1^2$, $j_3 = j_1^3, \ldots,\ j_u = j_1^u$, and eventually $j_1^\alpha = n$, so that $j_1 = \lambda$.

Conversely, say $\{\hat{i}, \hat{j}\} = \{(q^w, q^{2w}, \ldots, q^{vw} = n)(q^w, q^{2w}, \ldots, q^{vw} = n)\}$. Then for any $0 \leq u \leq \alpha - 1$, we have $\frac{j_{u+1}}{j_u} = q^w = j_1$. Also the lattice sets for the first row and column are equal so that $M_\alpha = L_\alpha = \{(m_0, m_1, \ldots, m_{\alpha-1}, 1) = (k_0, k_1, \ldots, k_{\alpha-1}, 1) | 1 \leq m_0, k_0 \leq q^w,\ 1 \leq m_1, k_1 \leq q^w, \ldots, 1 \leq m_{\alpha-1}, k_{\alpha-1} \leq$

$q^w, 1 \leq k_\alpha \leq 1\}$. Hence we can write the equality

$$\begin{aligned}
\lambda a = \lambda \left((k_0 - \hat{k}) + \sum_{u=1}^{\alpha} (i_u j_u) \left(k_u - \hat{k}_u \right) \right) &= j_1 \left((k_0 - \hat{k}) + \sum_{u=1}^{\alpha} (i_u j_u) \left(k_u - \hat{k}_u \right) \right) \\
&= q^w \left((k_0 - \hat{k}) + \sum_{u=1}^{\alpha} (i_u j_u) \left(k_u - \hat{k}_u \right) \right) \\
&= \left(q^w (k_0 - \hat{k}) + \sum_{u=1}^{\alpha} (i_u j_u) q^w \left(k_u - \hat{k}_u \right) \right) \\
&= \left(q^w (k_0 - \hat{k}) + \sum_{u=1}^{\alpha} (i_u j_u) \frac{j_{u+1}}{j_u} \left(k_u - \hat{k}_u \right) \right) \\
&= \left(q^w (m_0 - \hat{m}) + \sum_{u=1}^{\alpha} (i_u j_u) \frac{j_{u+1}}{j_u} \left(m_u - \hat{m}_u \right) \right) \\
&= \left(j_1 (m_0 - \hat{m}) + \sum_{u=1}^{\alpha} (i_u j_u) \frac{j_{u+1}}{j_u} \left(m_u - \hat{m}_u \right) \right) = b,
\end{aligned}$$

as required.

□

Example. *Given from the divisor path set* $(\hat{\imath}, \hat{\jmath}) = \{(6, 36), (6, 36)\} = \{(6, 6^2), (6, 6^2)\}$ *we obtain the corresponding sum-and-distance system,*

$$a = \{31, 33, 35, 37, 39, 41, 103, 105, 107, 109, 111, 113, 175, 177, 179, 181, 183, 185\},$$

$$b = \{186, 198, 210, 222, 234, 246, 618, 630, 642, 654, 666, 678, 1050, 1062, 1074, 1086, 1098, 1110\}$$

so that here $b = 6a$.

7 Most-Perfect Squares and Reversible Squares

To date, there does not exist a simple method to directly construct and enumerate most-perfect squares for a given side length n. However as briefly stated in Chapter 3, there exists a deep connection between most-perfect and reversible squares and in their book [10], Ollerenshaw and Brée develop a bijective mapping between these two types of matrices, which will be described in this chapter.

However using our results in Chapters 3-5, for the enumeration and construction of reversible square matrices, in conjunction with such a bijective mapping, would similarly lead to the enumeration and construction of all most-perfect square matrices.

To this end we now derive a block-representation for all most-perfect square matrices, establishing a bijection with the block representation of reversible square matrices, and hence a new bijective mapping between reversible and most-perfect squares. When the most-perfect square is traditional, so containing the first n^2 consecutive positive integers, we again establish a direct link with sum-and-distance systems.

To begin, let us describe Ollerenshaw and Brée's bijective mapping.

7.1 The Most Perfect to Reversible Transform

We recall from Chapter 3 the definition of an $n \times n$ most-perfect square matrix $M = (m_{i,j})$, where $n = 2k$ is even, which say a *most-perfect square* is a square matrix whose entries satisfy the three symmetry conditions (M), (P), (S), which respectively say that any 2×2 array entries within the square sum to $4w$ and the alternating sum of all the entries of M is zero; any pair of entries $\frac{1}{2}n$ distant along any diagonal sum to $2w$, and that the sum of the entries in any row or column is nw, for some $w \in \mathbb{R}$.

By Lemma 3.4 any of these three conditions imply that M has weight w, and additionally if M is *traditional*, so that it contains the set of consecutive integers $1, 2, \ldots, n$, then by Corollary 3.3, M has weight $w = \frac{n^2+1}{2}$.

Definition (of transformation T_1)**.** Let $n = 2k$ be even, and let $R = (r_{i,j}) \in \mathbb{R}^{n \times n}$ be an even sided reversible squre matrix. We set T_1 to be the transformation of the matrix R, defined stepwise such that

1. First we define an intermediate matrix $Z = (z)_{i,j}$ such that

$$z_{i,j} = \begin{cases} r_{i,j}, & \text{if } i, j \leq n; \\ r_{i,3k+1-j}, & \text{if } i \leq k,\ j > k; \\ r_{3k+1-i,j}, & \text{if } i > k,\ j \leq k; \\ r_{3k+1-i,3k+1-j}, & \text{if } i, j > k \end{cases}$$

This is essentially the block matrix transform, $Z = \begin{pmatrix} R_1 & R_2 J_k \\ J_k R_4 & J_k R_3 J_k \end{pmatrix}$, where $R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}$ and J_k the anti-diagonal identity matrix.

2. A transform is applied to the indices of the entries of the resulting square which correspond to the entries in the intermediate matrix, such that

$$\begin{pmatrix} i'-1 \\ j'-1 \end{pmatrix} = T \begin{pmatrix} i-1 \\ j-1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ n & n+1 \end{pmatrix} \begin{pmatrix} i-1 \\ j-1 \end{pmatrix}.$$

   So the entries $m_{i,j}$ are in the position (i', j') described above and given by a new matrix $Q = q_{i',j'} = z_{i,j}$[10].

An alternative description of the transform T_1 is given by

- Reverse the right-hand half of each row.
- Reverse the bottom half of each column.
- Apply the indices transform

$$\begin{pmatrix} i'-1 \\ j'-1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{n}{2} \\ \frac{n}{2} & \frac{n}{2}+1 \end{pmatrix} \begin{pmatrix} i-1 \\ j-1 \end{pmatrix},$$

  where i', j' are the new indices of the transformed square.

Remark. We note that a traditional square in Ollerenshaw and Brée's book [10] is defined to contain the consecutive integers $0, 1, \ldots, n^2 - 1$.

Example. *Consider a 4×4 principal reversible square and let us reverse the right hand half of each row (i.e. swapping the end 2 columns) and then reverse the bottom half of each column (i.e. swapping the bottom 2 rows), to obtain.*

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & 3 \\ 5 & 6 & 8 & 7 \\ 9 & 10 & 12 & 11 \\ 13 & 14 & 16 & 15 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & 3 \\ 5 & 6 & 8 & 7 \\ 13 & 14 & 16 & 15 \\ 9 & 10 & 12 & 11 \end{pmatrix}.$$

Lastly, applying the indices transform gives,

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 5 & 6 & 8 & 7 \\ 13 & 14 & 16 & 15 \\ 9 & 10 & 12 & 11 \end{pmatrix} \to \begin{pmatrix} 1 & 15 & 4 & 14 \\ 8 & 10 & 5 & 11 \\ 13 & 3 & 16 & 2 \\ 12 & 6 & 9 & 7 \end{pmatrix}.$$

which can be seen is a most-perfect square.

A detailed proof that this transform is a bijection is given in [10] (see Chapter 3 pp. 33-38) where they show that this transformation leads to a unique most-perfect square.

LEMMA 7.1. *There exists no most-perfect magic squares of order* $n \equiv 1, 2, 3 \pmod 4$.

Proof. When $n \equiv 1, 3 \pmod 4$ the type P property (strongly pandiagonal) fails as $2 \nmid n$. When $n \equiv 2 \pmod 4$ we refer the reader to a proof given by C. Planck (see [49] pp.308-309, 1919). □

7.2 The Most-Perfect Square Block Representation

We recall from Chapter 3 the definition of a most-perfect square matrix whose definition we now restate for clarity.

Definition. Let $n = 2k$ be a positive even integer and $w \in \mathbb{R}$ a constant. Then a square matrix $M = (m_{i,j}) \in \mathbb{R}^{n\times n}$ is defined to be a *most-perfect square* matrix if it satisfies the *type* M , *type* P and *type* S symmetry properties, defined below.

1. The matrix M is said to be *type* M if it satisfies the *most-perfect property*, where the entries of all 2×2 sub-arrays within the square matrix sum to $4w$, so that
$$m_{i,j} + m_{i,j+1} + m_{i+1,j} + m_{i+1,j+1} = 4w,$$
for all $i, j \in \mathbb{Z}_n$, and the *alternating sum property*
$$\sum_{i,j\in\mathbb{Z}_n} (-1)^{i+j} m_{i,j} = 0.$$

2. The matrix M is said to be *type* P if it satisfies the *strong pandiagonal property*, where the pairs of entries $\frac{1}{2}n = k$ distance along any diagonal (including broken diagonals) sum to $2w$, so that
$$m_{i,j} + m_{i+\frac{1}{2}n, j+\frac{1}{2}n} = 2w, \qquad i, j \in \mathbb{Z}_n.$$

3. The matrix M is said to be *type* S if it satisfies the *constant sum property*, where the sum of the elements of each row, or column sum to nw, so that
$$\sum_{j\in\mathbb{Z}_n} m_{i,j} = \sum_{j\in\mathbb{Z}_n} m_{j,i} = nw, \qquad i \in \mathbb{Z}_n.$$

By Lemma 3.4, we know that any one of the three symmetry conditions (M), (P), (S), imply that a given matrix has weight w, so that an $n \times n$ most-perfect square matrix M has weight w, and by Lemma 3.1, that the matrix $M - wE_n$ has weight zero.

Additionally from Lemma 3.6 we have that if a matrix is type (P) and also satisfies the 2×2 sub-array condition then this implies that the matrix also satisfies the alternating sum condition and so is type (M).

Definition (of the set of weightless type M squares M_n^0)**.** We define M_n^0 to be the set of all weightless $n \times n$ type M square matrices, so that for any $M = (m_{i,j}) \in \mathrm{M}_n^0$, we have that M has weight $w = 0$, the alternating sum property, and all two-by-two sub-arrays sum to zero, so that

$$m_{i,j} + m_{i+1,j} + m_{i,j+1} + m_{i+1,j+1} = 0.$$

Definition (of the vector $\S_n$)**.** We define $\S_n$ to be the column vector containing alternately signed ones, such that $\S_n = (1, -1, 1, -1, 1 \ldots, \pm 1)^T \in \mathbb{R}^n$, where the final entry is -1 if n is even and $+1$ if n is odd.

LEMMA 7.2. *Let $k \in \mathbb{N}$, and $n = 2k$ be even. Then the square matrix $M = (m_{i,j}) \in \mathbb{R}^{n \times n}$ is weightless type* M*, and so $M \in \mathrm{M}_n^0$ if and only if*

$$u^T M v = 0 \quad (u, v \in \{\S_n\}^\perp),$$

and

$$\S_n^T M \S_n = 0.$$

Proof. Given that n is even and $u \in \{\S_n\}^\perp$ then

$$u_1 - u_2 + u_2 \ldots - u_n = 0,$$

so we have the following equality

$$u^T M v = \left(\sum_{i=1}^n u_i m_{i,1}, \sum_{i=1}^n u_i m_{i,2}, \ldots, \sum_{i=1}^n u_i m_{i,n} \right) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{i,j}^n u_i v_j m_{i,j}.$$

Considering that as M is of type M^0 it satisfies

$$m_{i,j} + m_{i+1,j} + m_{i+1,j+1} + m_{i,j+1} = 0$$

for all $i, j \in \mathbb{Z}_n$. Taking the column vectors

$$U_i = (0, \ldots, 1, 1, 0, \ldots, 0)^T, \quad U_j = (0, \ldots, \ldots, 1, 1, 0, \ldots, 0)^T$$

where U_i has a 1 in entries i and $i+1$ and U_j has a 1 in entries j and $j+1$ we see these are in $\{\S_n\}^\perp$.

Then we compute

$$U_i M V_j = (m_{i,j} + m_{i+1,k}) + (m_{i+1,j} + m_{i+1,j+1}) = 0.$$

Further, it can be shown that the set of vectors, $U_i \in \{\S_n\}^\perp$ for all $i \in \{1, 2, \dots n-1\}$ form a basis for the set $\{\S_n\}^\perp$. So given any u and v in $\{\S_n\}^\perp$ we have

$$u^T M v = (\alpha_1 U_1 + \alpha_2 U_2 + \dots \alpha_{n-1} U_{n-1})^T M (\beta_1 U_1 + \beta_2 U_2 + \dots + \beta_{n-1} U_{n-1})$$

$$\sum_{i,j}^{n-1} U_i^T M U_j^T = \sum_{i,j}^{n-1} \alpha_i \beta_j \underbrace{((m_{i,j} + m_{i+1,k}) + (m_{i+1,j} + m_{i+1,j+1}))}_{=0} = 0.$$

For the second condition,

$$\S_n^T M \S_n = (-1)^{i+j} m_{i,j} = 0$$

by definition.

Conversely, considering that U_i for $i \in \{1, 2, \dots, n-1\}$ forms a basis for $\{\S_n\}^\perp$ taking all combinations of i and j in $\{1, 2 \dots n-1\}$ we obtain

$$0 = U_i M U_j = m_{i,j} + m_{i+1,j} + m_{i+1,j+1} + m_{i,j+1}$$

for all $i, j \in \{1, 2, \dots n\}$.

Similarly with

$$0 = \S_n^T M \S_n = \sum_{i,j \in \mathbb{Z}_n} (-1)^{i+j} m_{i,j}.$$

□

THEOREM 7.3. *Let $k \in \mathbb{N}$, and $n = 2k$ be even. Then $M \in \mathbb{R}^{n \times n}$, is a weightless type* M *matrix if and only if it has block representation*

$$M = X_n \begin{pmatrix} \hat{0}_k & a\S_k^T \\ \S_k b^T & Z \end{pmatrix} X_n,$$

with $Z \in M_K^0$ and $a, b \in \mathbb{R}^k$.

Proof. Given that if M is type M^0 by Lemma 7.2 we can write

$$0 = u^T X_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} X_n v$$

where $u, v \in \{\S_n\}^\perp$ and $Y, V, W, Z \in \mathbb{R}^{k \times k}$.

Consider now $u^T X_n$ and $X_n v$,

$$u^T X_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix}^T \begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{u}_1^T + \tilde{u}_2^T J_k \\ \tilde{u}_1 J_k - \tilde{u}_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}.$$

Similarly,

$$X_n v = \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}$$

where $\xi_1, \xi_2 \in \mathbb{R}^k$ and $\eta_1, \eta_2 \in \{\S_n\}^\perp$ as $\eta_1 = \tilde{u}_1 - J_k \tilde{u}_2$ and $\eta_2 = J_k \tilde{u}_1 - \tilde{v}_2$.

Expanding the equation

$$0 = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}^T \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}$$

then gives

$$0 = \xi_1^T Y \xi_2 + \xi_1^T V^T \eta_2 + \eta_1^T W \xi_2 + \eta_1^T Z \eta_2.$$

Firstly, if $\eta_1 = \eta_2 = 0_k \in \{\S_n\}^\perp$ then we have

$$0 = \xi_1^T Y \xi_2 + \xi_1^T \underbrace{V^T \eta_2}_{=0} + \underbrace{\eta_1^T W}_{=0} \xi_2 + \underbrace{\eta_1^T Z \eta_2}_{=0} = \xi_1^T Y \xi_2,$$

which shows $Y = \hat{0}_k$ as $\xi_1, \xi_2 \in \mathbb{R}^k$.

Next consider $\eta_1 = \eta_2 = 0_k$ we obtain

$$0 = \underbrace{\xi_1^T V^T}_{=0} \eta_2 + \eta_1^T \underbrace{W \xi_2}_{=0} + \eta_1^T Z \eta_2$$

$$0 = \eta_1^T Z \eta_2,$$

which implies that Z satisfies the first condition of a type M.

Now separately considering $\eta_1 = 0_k$ and $\eta_2 = 0_k$ then we have the two equalities

$$0 = \xi_1^T V^T \eta_2 + \eta_1^T W \xi_2 + \eta_1^T Z \eta_2$$

$$0 = \xi_1^T V^T \eta_2$$

$$0 = \underbrace{\xi_1^T V^T}_{=0} \eta_2 + \eta_1^T W \xi_2 + \underbrace{\eta_1^T Z}_{=0} \eta_2$$

$$0 = \eta_1^T W \xi_2,$$

from which we deduce that

$$\eta_1^T W = 0_k \quad \text{and} \quad \eta_2^T V = 0_k,$$

as $\eta_1, \eta_2 \in \{\S_n\}^\perp$. Labelling the columns of W and V as column vectors W_i and V_i respectively for $i \in \{1, 2, \ldots k\}$ then it must hold that $\eta_1^T W_i = 0$ and $\eta_2^T V_i = 0$ for all i. Hence $V_i \{\eta_1\}^\perp \in \{\S_n\}^{\perp\perp} = \mathbb{R}\S_k$ and similarly for $W_i \in \{\S_n\}^{\perp\perp} = \mathbb{R}\S_k$. In consideration of the second condition of Lemma 7.2 we have

$$
\begin{aligned}
0 =& \S_n^T M \S_n = \begin{pmatrix} \S_k \\ \S_k \end{pmatrix}^T \begin{pmatrix} \hat{0}_k & V^T \\ W & Z \end{pmatrix} \begin{pmatrix} \S_k \\ \S_k \end{pmatrix} \\
=& \S_k \underbrace{V^T \S_k}_{=0_k} + \underbrace{\S_k^T W}_{=0_k} \S_k + \S_k^T Z \S_k = \S_k^T Z \S_k
\end{aligned}
$$

which demonstrates that the blocks satisfy all of the type M symmetry properties. Conversely, say $Z \in M_k^0$, $a, b \in \mathbb{R}^k$ and $u, v\{\S_k\}^\perp$ then by Lemma 7.2 we have

$$
\begin{aligned}
u^T X_n \begin{pmatrix} \hat{0}_k & a\S_k^T \\ \S_k b^T & Z \end{pmatrix} X_n v = \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}^T \begin{pmatrix} \hat{0}_k & a\S_k^T \\ \S_k b^T & Z \end{pmatrix} \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} \\
= \underbrace{\xi_1^T \S_k b^T}_{=0_k^T} \xi_2 + \xi_1^T \underbrace{a \S_k \eta_2}_{=0_k} + \underbrace{\eta_1^T Z \eta_2}_{=0} = 0.
\end{aligned}
$$

Lastly, consider

$$
\begin{aligned}
\S_n^T M \S_k = \begin{pmatrix} \S_k \\ \S_k \end{pmatrix}^T \begin{pmatrix} \hat{0}_k & a\S_k^T \\ \S_k b^T & Z \end{pmatrix} \begin{pmatrix} \S_k \\ \S_k \end{pmatrix} \\
= \S_k^T \S_k b^T \S_k + \S_k^T a \S_k^T \S_k + \S_k^T Z \S_k = 0
\end{aligned}
$$

by Lemma 7.2 as required.

□

Definition (of the set of weightless type P squares P_n^0)**.** Let $n \in \mathbb{N}$ be even. We define P_n^0 to be the set of all weightless $n \times n$ type P square matrices, so that for any $M = (m_{i,j}) \in \mathrm{P}_n^0$, we have that M has weight $w = 0$ and

$$
m_{i,j} + m_{i+\frac{1}{2}n, j+\frac{1}{2}n} = 0, \qquad i, j \in \mathbb{Z}_n.
$$

LEMMA 7.4. *A square matrix $M \in \mathbb{R}^{2k \times 2k}$ is a weightless pandiagonal type* P *matrix if and only if*

$$
M = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}.
$$

Proof. By the definition of a weightless pandiagonal square we have the condition, $m_{i,j} + m_{i+k,j+k} = 0$ for $i, j \in \mathbb{Z}_k$ therefore we must have $m_{i,j} = -m_{i+k,j+k}$ and so we have the structure given. Conversely, given the structure above it satisfies the definition of a pandiagonal square. □

Definition (of the set of weightless type S squares S_n^0)**.** Let $n \in \mathbb{N}$. We define S_n^0 to be the set of all weightless $n \times n$ type S square matrices, so that for any

$M = (m_{i,j}) \in \mathrm{S}_n^0$, we have that M has weight $w = 0$, and

$$\sum_{j\in\mathbb{Z}_n} m_{i,j} = \sum_{j\in\mathbb{Z}_n} m_{j,i} = 0, \qquad i \in \mathbb{Z}_n.$$

Remark. We recall from Lemma 3.1 that for $M = (m_{i,j})$, an $n \times n$ matrix with weight w, we can write $M = M_0 + wE_m$, where M_0 has weight zero.

LEMMA 7.5. *Let $M \in \mathbb{R}^{n\times n}$. Then the square matrix M is type* S *if and only if*

$$1_n^T M u = 0 \ \textit{ and } \ u^T M 1_n = 0 \ \ (u \in \{1_n\}^\perp).$$

Proof. Since $0 = u^T M 1_n$ for all $u \in \{1_n\}^\perp$ then $M1_n \in \{1_n\}^{\perp\perp} = \mathbb{R}1_n$. Similarly we have $0 = 1_n^T M u = (u^T M^T 1_n)^T$ then $M^T 1_n \in \{1_n\}^{\perp\perp}$

So say $M1_n = \lambda 1_n$ and $M^T 1_n = \lambda' 1_n$ for $\lambda, \lambda' \in \mathbb{R}$ then we have

$$\lambda 1_n^T 1_n = 1_n^T(\lambda 1_n) = 1_n^T M 1_n = (1_n^T M)1_n = (M^T 1_n)^T 1_n = (\lambda' 1_n)^T 1_n = \lambda' 1_n^T 1_n$$

and as $1_n 1_n^T \neq 0$ as $1_n \neq 0_n$ we have $\lambda = \lambda'$. So to conclude we have $M1_n = M^T 1_n = \lambda 1_n$ where we can set $\lambda = nw$.

The converse then follows directly, as say $M1_n = M^T 1_n = \lambda 1_n$, then

$$u^T M 1_n = \lambda u^T 1_n = 0,$$

$$1_n^T M u = \lambda 1_n^T u = 0$$

if $u \in \{1_n\}^\perp$. □

THEOREM 7.6. *Let $n \in \mathbb{N}$, and $M = (m_{i,j}) \in \mathbb{R}^{n\times n}$. Then M is type* S *if and only if it can be written as the following block representation,*

even $n = 2k$**:**

$$M = X_n \begin{pmatrix} Y + 2wE_k & V^T \\ W & Z \end{pmatrix} X_n,$$

with Y a semimagic square with weight 0, matrices V,W have row sum 0 and Z that can be any $k \times k$ square matrix.

odd $2k + 1$**:**

$$M = X_n \begin{pmatrix} Y + 2wE_k & \sqrt{2}(w1_k - Y1_k) & V^T \\ \sqrt{2}(w1_k - Y^T 1_k)^T & w + 21_k^T Y 1_k & -\sqrt{2}(V1_k)^T \\ W & -\sqrt{2}W1_k & Z \end{pmatrix} X_n$$

with arbitrary $V, W, Y, Z \in \mathbb{R}^{k\times k}$ and $w \in \mathbb{R}$.

Proof. Even case $n = 2k$: If $M \in \mathbb{R}^{n\times n}$ is a type S square with weight w, then we can write

$$X_n \begin{pmatrix} Y + 2wE_k & V^T \\ W & Z \end{pmatrix} X_n = X_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} X_n + X_n \begin{pmatrix} 2wE_n & 0 \\ 0 & 0 \end{pmatrix} X_n$$
$$= X_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} X_n + wE_n,$$

and as wE_n is type S square we need only show that $X_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} X_n$ is a weightless type S square.

Considering the expansion,

$$\frac{1}{\sqrt{2}} \begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix} \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} Y + J_kW & V^T + J_kZ \\ J_kY - W & J_kV^T - Z \end{pmatrix} \begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} Y + J_kW + V^TJ_k + J_kZJ_k & YJ_k + J_k - V^T - J_kZ \\ J_kY - W + J_kV^TJ_k - ZJ_k & J_kTJ_k - WJ_k - J_kV^T + Z \end{pmatrix}$$
$$= \frac{1}{2} \left(\begin{pmatrix} Y & YJ_k \\ J_kY & J_kYJ_k \end{pmatrix} + \begin{pmatrix} J_kW & J_jWJ_k \\ -W & -WJ_k \end{pmatrix} + \begin{pmatrix} V^TJ_k & -V^T \\ J_kV^TJ_k & -J_kV^T \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} J_kZJ_k & -J_kZ \\ -ZJ_k & Z \end{pmatrix} \right),$$

as the sum of any type S square is also a type S square we need only demonstrate that each matrix is type S as the sum is also type S.

We will now use the properties of multiplying a block matrix by the antidiagonal identity matrix J_n.

If Y is a weightless type S square then $\begin{pmatrix} Y & YJ_k \\ J_kY & J_kYJ_k \end{pmatrix}$ has row and column 0, therefore is type S. As J_kW & W and J_kWJ_k & WJ_k have the same column entries the sum of the columns in the matrix $\begin{pmatrix} J_kW & J_kWJ_k \\ -W & -WJ_k \end{pmatrix}$. The rows have sum zero as the matrix W has row sum zero, therefore the matrix is type S. Similarly, V^TJ_k & V^T and $J_kV^TJ_k$ & J_kV^T have the same row entries therefore the matrix $\begin{pmatrix} V^TJ_k & -V^T \\ J_kV^TJ_k & -J_kV^T \end{pmatrix}$ has row sum zero. The columns have sum zero as the matrix V^T has column sum zero, therefore the matrix is type S. Lastly,

J_kZJ_k & J_kZ and ZJ_k & Z have the same row entries and J_kZJ_k & ZJ_k and J_kZ & Z have the same column entries. So the matrix $\begin{pmatrix} J_kZJ_k & -J_kZ \\ -ZJ_k & Z \end{pmatrix}$ has row and column zero and so is type S. Therefore the sum of the these four type S square is also type S.

Now if M is a type S square with weight w, it can be written as

$$M = X_n \begin{pmatrix} A + 2wE_k & B \\ C & D \end{pmatrix} X_n = X_n \begin{pmatrix} A & B \\ C & D \end{pmatrix} X_n + wE_n$$

by Lemma 3.8, the row and column sum of $X_n \begin{pmatrix} A & B \\ C & D \end{pmatrix} X_n$ equals zero.

Expanding this block representation we have

$$M = \frac{1}{2} \begin{pmatrix} A + J_kC + BJ_k + J_kDJ_k & AJ_k + J_kCJ_k - B - J_kD \\ J_kA - C + J_kBJ_k - J_kDJ_k & J_kA - CJ_k - J_kB + D \end{pmatrix},$$

and by assuming these properties we will prove A, B, C, D also have the given symmetry properties.

Considering the *column* sums, of the expanded matrix, we have

$$\begin{pmatrix} A + J_kC + BJ_k + J_kDJ_k \\ J_kA - C + J_kBJ_k - J_kDJ_k \end{pmatrix}, \quad \begin{pmatrix} AJ_k + J_kCJ_k - B - J_kD \\ J_kA - CJ_k - J_kB + D \end{pmatrix}.$$

As the entries in the variations of D (J_kDJ_k and J_kDJ_k or J_kD and D) have the same column entries with some permutation give sum 0, similarly the variations of C give column sum 0. Hence the column sum depends only on the matrix

$$\begin{pmatrix} A + BJ_k & AJ_k - B \\ J_kA + J_kBJ_k & J_kAJ_k - J_kB \end{pmatrix}.$$

Considering now the *row* sums of the expanded matrix, we have

$$\begin{pmatrix} A + J_kC + BJ_k + J_kDJ_k & AJ_k + J_kCJ_k - B - J_kD \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} J_kA - C + J_kBJ_k - J_kDJ_k & J_kA - CJ_k - J_kB + D \end{pmatrix}.$$

Similarly, due to symmetries in the variations in D and B, their row sums are 0. Hence we need only look at rows

$$\begin{pmatrix} A + J_kC & AJ_k + J_kCJ_k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} J_kA - C & J_kA - CJ_k \end{pmatrix}.$$

With these results we can now establish a contradiction in that if A were not semimagic and-or B^T and C did not have row sum 0 then the square would not have column and row sum 0 and so would not be a semimagic square. Say the row and column sum for A was non-zero then B^T and C would have non zero column sum. Similarly if B^T and-or C have non-zero row sum then the row and-or column sum will be non-zero, as required.

Odd case $n = 2k + 1$: We assume that M has the given block representation, by taking any row and column sum as in the even case

$$M = X_n \begin{pmatrix} Y + 2wE_k & \sqrt{2}(w1_k - Y1_k) & V^T \\ \sqrt{2}(w1_k - Y^T 1_k)^T & w + 21_k^T Y1_k & -\sqrt{2}(V1_k)^T \\ W & -\sqrt{2}W1_k & Z \end{pmatrix} X_n.$$

Expanding the block structure then gives us

$$= \frac{1}{2} \begin{pmatrix} Y + 2wE_k + V^T J_k + J_k W + J_k & 2(w1_k - Y1_k) - 2J_k W1_k & (Y + 2wE_k)J_k - V^T + J_k W J_k - J_k Z \\ 2(w1_k - Y^T 1_k)^T - 2(V1_k)^T J_k & 2(w + 21_k^T Y1_k) & 2(w1_k - Y^T 1_k)^T J_k + 2(V1_k)^T \\ J_k Y + J_k 2wE_k + J_k V^T J_k - W - ZJ_k & 2J_k(w1_k - Y1_k) + 2W1_k & J_k(Y + 2wE_k)J_k - J_k V^T - WJ_k + Z \end{pmatrix},$$

and by taking any row and column sums we can show that they have the same sum nw, and therefore we have that M is a type S square matrix.

Now say we have an odd sided type S square $M \in \mathbb{R}^{n \times n}$ with weight w such that for some $A, B, C, D \in \mathbb{R}^{k \times k}$, $a, b, c, d \in \mathbb{R}^k$ and $e \in \mathbb{R}$ this can be written as,

$$\begin{aligned} M &= \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0_k & J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ J_k & 0_k & -I_k \end{pmatrix} \begin{pmatrix} A & a & B^T \\ c^T & e & b^T \\ C & d & D \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0_k & J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ J_k & 0_k & -I_k \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} A + J_k C & I_k a + J_k d & B^T + J_k D \\ \sqrt{2}c^T & \sqrt{2}e & \sqrt{2}b^T \\ J_k A - C & J_k a - I_k d & J_k B^T - D \end{pmatrix} \begin{pmatrix} I_k & 0_k & J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ J_k & 0_k & -I_k \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} A + J_k C + B^T J_k + J_k D J_k & \sqrt{2}I_k a + \sqrt{2}J_k b & AJ_k + J_k C J_k - B^T - J_k D \\ \sqrt{2}c^T I_k + \sqrt{2}b^T J_k & 2e & \sqrt{2}c^T J_k - \sqrt{2}b^T \\ J_k A - C + J_k B^T J_k - DJ_k & \sqrt{2}J_k a - \sqrt{2}I_k d & J_k A J_k - CJ_k - J_k B^T + D \\ . & & \end{pmatrix} \end{aligned}$$

As the square is type S then the sum of any row or column is nw, so if we consider all columns and rows separately we can deduce the correct conditions of the separate blocks.

The first three columns are given by the first $n \times k$ block, the n length vector and the end $n \times k$ block,

$$\begin{pmatrix} A + J_kC + B^T + J_kDJ_k \\ \sqrt{2}c^T + \sqrt{2}b^TJ_k \\ J_kA - C + J_kB^TJ_k - DJ_k \end{pmatrix}, \quad \begin{pmatrix} \sqrt{2}a + \sqrt{2}J_kd \\ 2e \\ \sqrt{2}J_ka - \sqrt{2}d \end{pmatrix}, \quad \begin{pmatrix} AJ_k + J_kCJ_k - B^T - J_kD \\ \sqrt{2}c^TJ_k - \sqrt{2}b^T \\ J_kAJ_k - CJ_k - J_kB^T + D \end{pmatrix}.$$

Then taking sum of the entries in any of these columns by Lemma 3.8 gives the equalities. For the $n \times k$ blocks

$$\begin{aligned}
2nw &= \sum_{i=1}^{k} A_{i,j} + \sum_{i=1}^{k}(J_kC)_{i,j} + \sum_{i=1}^{k}(B^TJ_k)_{i,j} + \sum_{i=1}^{k}(J_kDJ_k)_{i,j} + \sqrt{2}c_j + \sqrt{2}(b^TJ_k)_j \\
&+ \sum_{i=1}^{k}(J_kA)_{i,j} - \sum_{i=1}^{k} C_{i,j} - \sum_{i=1}^{k}(DJ_k)_{i,j} + \sum_{i=1}^{k}(J_kB^TJ_k)_{i,j} \\
&= \sum_{i=1}^{k} A_{i,j} + \sum_{i=1}^{k} C_{k+1-i,j} + \sum_{i=1}^{k} B^T_{i,k+1-j} + \sum_{i=1}^{k} D_{k+1-i,k+1-j} + \sqrt{2}c_j + \sqrt{2}b^T_{k+1-j} \\
&+ \sum_{i=1}^{k} A_{k+1-i,j} - \sum_{i=1}^{k} C_{i,j} + \sum_{i=1}^{k} B^T_{k+1-i,k+1-j} - \sum_{i=1}^{k} D_{i,k+1-j} \\
&= 2\sum_{i=1}^{k} A_{i,j} + 2\sum_{i=1}^{k} B^T_{k+1-i,k+1-j} + \sqrt{2}c_j + \sqrt{2}b^T_{k+1-j}
\end{aligned}$$

where j denotes the fixed column and we have used the properties of the summed entries,

$$\begin{aligned}
\sum_{i=1}^{k} A_{i,j} &= \sum_{i=1}^{k} A_{k+1-i,j} \\
\sum_{i=1}^{k} C_{k+1-i,j} &= \sum_{i=1}^{k} C_{i,j} \\
\sum_{i=1}^{k} B^T_{i,k+1-j} &= \sum_{i=1}^{k} B^T_{k+1-i,k+1-j} \\
\sum_{i=1}^{k} D_{k+1-i,k+1-j} &= \sum_{i=1}^{k} D_{i,k+1-j}.
\end{aligned}$$

The middle column sum yields

$$\begin{aligned}
2nw &= \sqrt{2}\sum_{i=1}^{k} a_i + \sqrt{2}\sum_{i=1}^{k} d_{k+1-i} + 2e + \sqrt{2}\sum_{i=1}^{k} a_{k+1-i} - \sqrt{2}\sum_{i=1}^{k} d_i \\
&= 2\sqrt{2}\sum_{i=1}^{k} a_i + 2e.
\end{aligned}$$

Similarly to the first $n \times k$ column block, the last $n \times k$ yields

$$2nw = 2\sum_{i=1}^{k} A_{k+1-i,k+1-j} - 2\sum_{i=1}^{k} B_{i,j}^T + \sqrt{2}c_{k+1-j} - \sqrt{2}b_j.$$

Likewise to the column blocks, we consider the three row blocks individually

$$(A + J_kC + B^T J_k + J_kDJ_k,\ \sqrt{2}a + \sqrt{2}J_kd,\quad A + J_kCJ_k - B^T - J_kD),$$

$$(\sqrt{2}c^T + \sqrt{2}b^T J_k,\quad 2e,\quad \sqrt{2}c^T J_k - \sqrt{2}b^T),$$

$$(J_kA - C + J_kB^T J_k - DJ_k,\quad \sqrt{2}J_ka - \sqrt{2}d,\quad J_kAJ_k - CJ_k - J_kB^T + D),$$

By fixing any row we obtain the three equations

$$\begin{aligned}
2nw &= 2\sum_{j=1}^{k} A_{i,j} + 2\sum_{j=1}^{k} C_{k+1-i,k+1-j} + \sqrt{2}a_i + \sqrt{2}a_i + \sqrt{2}d_{k+1-i} \\
2nw &= 2\sqrt{2}\sum_{j=1}^{k} c_i + 2e \\
2nw &= 2\sum_{j=1}^{k} A_{k+1-i.k+1-j} - 2\sum_{j=1}^{k} C_{i,j} + \sqrt{2}a_{k+1-i} - \sqrt{2}d_i
\end{aligned}$$

We note that as none of the six equations contain any variation of the matrix D, it follows that D must be any arbitrary matrix in $\mathbb{R}^{k\times k}$.

Continuing, the six equations allow us to deduce the correct conditions,

$$2nw = 2\sum_{i=1}^{k} A_{i,j} + 2\sum_{i=1}^{k} B^T_{k+1-i,k+1-j} + \sqrt{2}c_j + \sqrt{2}b^T_{k+1-j} \tag{7.1}$$

$$2wn = 2\sqrt{2}\sum_{i=1}^{k} a_i + 2e \tag{7.2}$$

$$2nw = 2\sum_{i=1}^{k} A_{k+1-i,k+1-j} - 2\sum_{i=1}^{k} B^T_{i,j} + \sqrt{2}c_{k+1-j} - \sqrt{2}b_j \tag{7.3}$$

$$2nw = 2\sum_{j=1}^{k} A_{i,j} + 2\sum_{j=1}^{k} C_{k+1-i,k+1-j} + \sqrt{2}a_i + \sqrt{2}d_{k+1-i} \tag{7.4}$$

$$2nw = 2\sqrt{2}\sum_{j=1}^{k} c_i + 2e \tag{7.5}$$

$$2nw = 2\sum_{j=1}^{k} A_{k+1-i.k+1-j} - 2\sum_{j=1}^{k} C_{i,j} + \sqrt{2}a_{k+1-i} - \sqrt{2}d_i. \tag{7.6}$$

We can equate (7.1) and (7.3) for the columns j and $k+1-j$ respectively, to obtain

$$\begin{aligned} &2\sum_{i=1}^{k} A_{i,j} + 2\sum_{i=1}^{k} B^T_{k+1-i,k+1-j} + \sqrt{2}c_j + \sqrt{2}b^T_{k+1-j} \\ &= 2\sum_{i=1}^{k} A_{k+1-i,k+1-j} - 2\sum_{i=1}^{k} B^T_{i,j} + \sqrt{2}c_{k+1-j} - \sqrt{2}b_j \end{aligned}$$

and as $\sum_{i=1}^{k} A_{i,j} = \sum_{i=1}^{k} A_{k+1-i,j}$ and $\sum_{i=1}^{k} B^T_{k+1-i,k+1-j} = \sum_{i=1}^{k} B^T_{i,k+1-j}$,

$$\begin{aligned} 2\sum_{i=1}^{k} B^T_{k+1-i,k+1-j} + \sqrt{2}b_{k+1-j} &= -2B^T_{k+1-i,k+1-j} - \sqrt{2}b_{k+1-j} \\ -\sqrt{2}\sum_{i=1}^{k} B^T_{k+1-i,k+1-j} &= b_{k+1-j} \\ -\sqrt{2}\sum_{i=1}^{k} B^T_{i,j} &= b_j, \end{aligned}$$

which for all $j \in \mathbb{Z}_k$ we can write in the vector form

$$-\sqrt{2}(B1_k)^T = b^T.$$

Similarly, equating (4) and (6) for the rows i and $k+1-i$ respectively we obtain

$$2\sum_{j=1}^{k} A_{i,j} + 2\sum_{j=1}^{k} C_{k+1-i,k+1-j} + \sqrt{2}a_i + \sqrt{2}d_{k+1-i}$$

$$= 2\sum_{j=1}^{k} A_{i,k+1-j} - 2\sum_{j=1}^{k} C_{k+1-i,k+1-j} + \sqrt{2}a_i - \sqrt{2}d_{k+1-i}$$

$$4\sum_{j=1}^{k} C_{k+1-i,k+1-j} = -2\sqrt{2}d_{k+1-i}$$

$$-\sqrt{2}\sum_{j=1}^{k} C_{i,j} = d_i$$

which for all $i \in \mathbb{Z}_k$ we can write as a vector form

$$-\sqrt{2}(C1_k) = d.$$

Further, adding equations (4) and (6) for i and $k+1-i$ respectively we obtain

$$4nw = 2\sum_{j=1}^{k} A_{i,j} + 2\sum_{j=1}^{k} C_{k+1-i,k+1-j} + \sqrt{2}d_{k+1-i} + \sqrt{2}a_i$$
$$+ 2\sum_{j=1}^{k} A_{i,k+1-j} - \sum_{j=1}^{k} C_{k+1-i,j} + \sqrt{2}a_i - \sqrt{2}d_{k+1-i}$$

which reduces to give

$$a_i = \sqrt{2}\left(nw - \sum_{j=1}^{k} A_{i,j}\right).$$

Similarly, adding equations (1) and (3) for i and $k+1-i$ respectively and simplifying gives,

$$c_j = \sqrt{2}\left(nw - \sum_{i=1}^{k} A_{i,j}\right).$$

Finally, any $A \in \mathbb{R}^{k\times k}$ can be written in the form

$$A = A_0 + 2wE_k$$

for some $A_0 \in \mathbb{R}^{k\times k}$. Rewriting as

$$\begin{aligned}
c_j =&\sqrt{2}\left(nw - \sum_{i=1}^{k} A_{i,j}\right) = \sqrt{2}\left(nw - \left(\sum_{i=1}^{k}(A_0)_{i,j} + \sum_{i=1}^{k} 2wE_k\right)\right) \\
=&\sqrt{2}\left(nw - \sum_{i=1}^{k}(A_0)_{i,j} - 2wk\right) \\
=&\sqrt{2}\left((2k+1)w - \sum_{i=1}^{k}(A_0)_{i,j} - 2kw\right) \\
=&\sqrt{2}\left(w - \sum_{i=1}^{k}(A_0)_{i,j}\right),
\end{aligned}$$

this holds for all $j \in \mathbb{Z}_k$ so it can be written as the vector form

$$c = \sqrt{2}(w1_k - (A_0)^T 1_k).$$

Similarly, for any i we have

$$\begin{aligned}
a_i &= \sqrt{2}\left(nw - \sum_{j=1}^{k} A_{i,j}\right) \\
&= \sqrt{2}\left(nw - \sum_{j=1}^{k}(A_0)_{i,j} - 2w\sum_{j=1}^{k} E_{i,j}\right) \\
&= \sqrt{2}\left(nw - 2kw - \sum_{j=1}^{k}(A_0)_{i,j}\right) \\
&= \sqrt{2}\left(w - \sum_{j=1}^{k}(A_0)_{i,j}\right).
\end{aligned}$$

Then for all $i \in \mathbb{Z}_k$ we again have the vector form

$$a = \sqrt{2}(w1_k - A^0 1_k).$$

Lastly, given that

$$2nw = 2\sqrt{2}\sum_{i=1}^{k} a_i + 2e$$

then for any $i \in \mathbb{Z}_k$ we have

$$a_i = \sqrt{2}\left(nw - \sum_{j=1}^{k} A_{i,j}\right)$$

and so

$$
\begin{aligned}
e &= nw - 2\sum_{i=1}^{k}\left(nw - \sum_{j=1} A_{i,j}\right) \\
&= nw - 2\sum_{i=1}^{k} nw + 2\sum_{i,j}\left(A^0_{i,j} + 2wE_{i,j}\right) \\
&= nw - 2knw + 2\sum_{i,j} A^0_{i,j} + 4wk^2 \\
&= w + 2\sum_{i,j} A^0_{i,j} = w + 21_k^T A^0 1_k.
\end{aligned}
$$

Hence we conclude that if M is an odd sided type S (semimagic) square then it has the block representation stated.

□

In the following theorem we consider the block representation of a weightless most-perfect square M of side length $n = 2k$

THEOREM 7.7. *Let $k \in \mathbb{N}$, so that $n = 2k$ is even, $\S_k = (1, -1, 1, -1, \ldots, \pm 1)^T$, the column vector of length n defined earlier in this chapter, and $M = (m_{i,j}) \in \mathbb{R}^{n\times n}$ a weightless type* M *square, so that $w = 0$. Then M has the block representation*

$$
M = X_n \begin{pmatrix} \hat{0}_k & \gamma \S_k^T \\ \S_k \delta^T & Z \end{pmatrix} X_n,
$$

where $\gamma, \delta \in \mathbb{R}^n$ with $J_k\gamma = (-1)^{k-1}\gamma$, $J_k\delta = (-1)^{k-1}\delta$ and $Z \in \mathbb{R}^{n\times n}$ is both type M *and type* A*, so that M satisfies the 2×2 array sum, alternating sum and the associated pair sum property ($m_{i,j} + m_{n+1-i,+n+1-j} = 0$) conditions.*

Proof. A most-perfect square is of type M, P and S, so combining the block representation of Theorems 7.3 and 7.6 we find that

$$
M = X_n \begin{pmatrix} \hat{0}_k & a\S_k^T \\ \S_k b^T & Z \end{pmatrix} X_n
$$

where $a^T 1_k$, $b^T 1_k$ (as $a^T \S_k$ and $b^T \S_k$ has row sum zero) and $Z \in M_n^0$. Using the type P block structure given in Lemma 7.4 we see that,

$$
X_n \begin{pmatrix} A & B \\ -B & -A \end{pmatrix} X_n = \frac{1}{2}\begin{pmatrix} A + BJ_k - J_kB - J_kAJ_k & A - J_kB - J_kBJ_k + J_kA \\ J_kA + J_kBJ_k + B + AJ_k & J_kAJ_k - J_kB + BJ_k - A \end{pmatrix}.
$$

Considering the bottom left and top right blocks we obtain the equalities,

$$
J_k(J_kA \pm B \pm J_kBJ_k + AJ_k)J_k = AJ_k \pm J_kBJ_k \pm B + J_kA
$$

and therefore $\S_k a^T = J_k \S_k a^T J_k = \mp \S_k a^T J_k$ and $\S_k b^T = J_k \S_k b^T J_k = \mp \S_k b^T J_k$. Hence we deduce that $a = \pm J_k a$ and $b = \mp J_k b$. Lastly, considering the bottom right block we have

$$J_k Z J_k = \frac{1}{2} J_k (J_k A J_k - J_k B + B J_k - A) J_k = \frac{1}{2}(A - B J_k + J_k B - J_k A J_k) = -Z.$$

Now if a matrix $Z = (z_{i,j}) \in \mathbb{R}^{k \times k}$ satisfies $J_k Z J_k + Z = \hat{0}_n$, then this means that $z_{i,j} + z_{n+1-i,+n+1-j} = 0,\ i, j \in \mathbb{Z}_k$, and it follows that $Z \in A_n^0$ is type A with weight zero.

Conversely, let

$$M = \begin{pmatrix} \hat{0}_k & a\S_k^T \\ \S_k b^T & Z \end{pmatrix}$$

where a, b, Z have the properties in the statement of the theorem. Then by Theorems 7.3 and 7.6 $M \in M_n \cap S_n$ and expanding this block representation gives

$$M = \frac{1}{2} \begin{pmatrix} a\S_k^T J_k + J_k \S_k b^T + J_k Z J_k & -a\S_k^T + J_k \S_k b^T J_k - J_k Z \\ J_k a \S_k^T J_k - \S_k b^T - Z J_k & -J_k a \S_k^T - \S_k b^T J_k + Z \end{pmatrix}$$

which by Lemma 7.4 has the type P (pandiagonal) structure. Therefore we have that the matrix M is of type M, P and S, so $M \in \mathrm{MPS}_n$ and M is a most-perfect square matrix. □

THEOREM 7.8. *Let $k \in \mathbb{N}$, so that $n = 2k$ is even. Then the square matrix $M = (m_{i,j}) \in \mathbb{R}^{n \times n}$ is a weightless most-perfect square if and only if*

$$M = \gamma \S_n^T + \S_n \delta^T$$

where

(i) for even k $\quad \gamma = \begin{pmatrix} \tilde{\gamma} \\ -\tilde{\gamma} \end{pmatrix}, \quad \delta = \begin{pmatrix} \tilde{\delta} \\ -\tilde{\delta} \end{pmatrix}, \quad \tilde{\gamma}, \tilde{\delta} \in \mathbb{R}^k$

(ii) for odd k $\quad \gamma = \begin{pmatrix} \tilde{\gamma} \\ \tilde{\gamma} \end{pmatrix}, \quad \delta = \begin{pmatrix} \tilde{\delta} \\ \tilde{\delta} \end{pmatrix}, \quad \tilde{\gamma}, \tilde{\delta} \in \{1_k\}^\perp.$

Proof. If M is a most-perfect square then $M \in M_n^0 \cap P_n^0 \cap S_n^0$ and by Lemma 7.2 $\frac{1}{n} M \S_n \in \{\S_n\}^\perp$ (Let us set this as $\gamma = \frac{1}{n} M \S_n$).

Let $v \in \{\S_n\}^\perp$ then by Lemma 7.2 $u^T(Mv) = 0$ for $u \in \{\S_n\}^\perp$ and $Mv \in \{\S_n\}^{\perp\perp} = c\S_n$ for $c \in \mathbb{R}$. Here $c \in \mathbb{R}$ can be represented by a linear form $c = f(v)$ for $v \in \{\S_n\}^\perp$ such that $Mv = f(v)\S_n$. Then by the Riesz Representation Theorem there is a vector $\delta \in \{\S_n\}^\perp$ such that $f(v) = \delta^T v$ so that $Mv = \S_n \delta^T v$.

Hence any $x \in \mathbb{R}^n$ can be written in the form $x = \alpha \S_n + v$ for $\alpha \in \mathbb{R}$ and $v \in \{\S_n\}^\perp$.

$$Mx = M(\alpha \S_n + v) = \alpha M \S_n + Mv = \alpha M \S_n + \S_n \delta^T v = \alpha n \gamma + \S_n \delta^T v$$

$$= \alpha n\gamma + \S_n\delta^T v + \underbrace{\alpha\S_n\delta^T\S_n}_{=0} + \underbrace{\gamma\S_n^T v}_{=0} = (\gamma\S_n^T + \S_n\delta^T)(\alpha\S_n + v) = (\gamma\S_n^T + \S_n\delta^T)x,$$

showing that M has the desired form.

Furthermore, let us write $\gamma = \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$ and $\delta = \begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}$ with $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}^k$. Then we have

$$M = \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\begin{pmatrix}\S_k\\ \pm\S_k\end{pmatrix}^T + \begin{pmatrix}\S_k\\ \pm\S_k\end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}^T = \begin{pmatrix}\gamma_1\S_k^T + \S_k\delta_1 & \pm\gamma_1 + \S_k\delta_2^T\\ \gamma_2\S_k^T \pm \S_k\delta_1^T & \pm\gamma_2\S_k^T \pm \S_k\delta_2^T\end{pmatrix}$$

Using the structure of a type P matrix given in Theorem 7.4, we have

$$(\gamma_1 \pm \gamma_2)\S_k^T + \S_k(\delta_1 \pm \delta_2)^T = \hat{0}_k.$$

As $\gamma = \frac{1}{n}M\S_n \in \{\S_n\}^\perp$ by Lemma 7.2 then

$$0 = \S_n^T\gamma = \begin{pmatrix}\S_k\\ \pm\S_k\end{pmatrix}^T\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} = \S_k^T(\gamma_1 \pm \gamma_2)$$

and it follows that

$$0_k = \S_n^T(\gamma_1 \pm \gamma_2)\S_k + \S_k^T\S_k(\delta_1 = \pm\delta_2) = 0_k + k(\delta_1 \pm \delta_2)$$

implying that $\delta_1 \mp \delta_2$. Similarly, applying these calculations to the vector γ we deduce that $\gamma_1 = \mp\gamma_2$.

By Lemmas 7.5 and 7.2 we have that $\S_n^T 1_n = 0$, and that $\gamma, \delta \in \{1_n\}^\perp$, so for even k the conditions are satisfied. For odd k we require the further condition that $\gamma_1, \delta_1 \in \{1_k\}^\perp$ (are orthogonal to 1_k). The converse then follows by Lemmas 7.2, 7.5 and 7.4. □

THEOREM 7.9. *Let $k \in \mathbb{N}$, so that $n = 2k$ is even and $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_k)^T$, $\tilde{\delta} = (\tilde{\delta}_1, \tilde{\delta}_2, \ldots, \tilde{\delta}_k)^T$, be two k-dimensional column vectors. Then the square matrix $M = (m_{i,j}) \in \mathbb{R}^{n\times n}$ is a (weightless) traditional most-perfect square if and only if the sets of absolute values of the vectors $\tilde{\gamma}$, $\tilde{\delta} \in \mathbb{R}^k$ in the representation*

$$2M = \gamma\S_n^T + \S_n\delta^T,$$

with $\gamma = \begin{pmatrix}\tilde{\gamma}\\ \mp\tilde{\gamma}\end{pmatrix}$, $\delta = \begin{pmatrix}\tilde{\delta}\\ \mp\tilde{\delta}\end{pmatrix}$ (and if k is odd orthogonal to 1_k) form a sum-and-distance system, so that $\{\tilde{\gamma}', \tilde{\delta}'\}$ is a sum-and-distance system with

$$\tilde{\gamma}' = \{|\tilde{\gamma}_1|, |\tilde{\gamma}_2|, \ldots, |\tilde{\gamma}_k|\}, \quad \tilde{\delta}' = \{|\tilde{\delta}_1|, |\tilde{\delta}_2|, \ldots, |\tilde{\delta}_k|\}.$$

Proof. We recall that a weightless traditional square has the set if entries

$$\frac{1}{2}\{-(n^2-1),\ -(n^2-3),\ldots,-1,1,\ldots,n^2-3,\ n^2-1\}.$$

Considering the structure of a most-perfect square. Then by Theorem 7.8 (for all n) we can write

$$2M=\begin{pmatrix}\tilde{\gamma}\\ \mp\tilde{\gamma}\end{pmatrix}\begin{pmatrix}\S_k\\ \pm\S_k\end{pmatrix}^T+\begin{pmatrix}\S_k\\ \pm\S_k\end{pmatrix}\begin{pmatrix}\tilde{\delta}\\ \mp\tilde{\delta}\end{pmatrix}=\begin{pmatrix}\tilde{\gamma}\S_k^T+\S_k\tilde{\delta}^T & \tilde{\gamma}\S_k\mp\S_k\tilde{\delta}^T\\ \mp\tilde{\gamma}\S_k^T\pm\S_k\tilde{\delta}^T & -\tilde{\gamma}\S_k^T-\S_k\tilde{\delta}\end{pmatrix}.$$

For even k, the set of all entries in these quadrants give

$$\{-(n^2-1),\ -(n^2-3)\ldots,-1,1,\ldots,n^2-3,\ n^2-1\}$$

$$=\{(-1)^j\tilde{\gamma}_i+(-1)^i\tilde{\delta},\ (-1)^{j+1}\tilde{\gamma}_i+(-1)^i\tilde{\delta},\ (-1)^{j+1}\tilde{\gamma}_i+(-1)^{i+1}\tilde{\delta},\ (-1)^j\tilde{\gamma}_i+(-1)^{i+1}\tilde{\delta}\mid i,j\in\mathbb{Z}_k\}$$

$$=\{\pm(-1)^j\tilde{\gamma}_i\pm(-1)^i\tilde{\delta}_j\ \mid\ i,j\in\mathbb{Z}_k\}=\{\pm\tilde{\gamma}_i\pm\tilde{\delta}\ \mid\ i,j\in\mathbb{Z}_k\}$$

which by definition implies that the sets of absolute values form a sum-and-distance system.

For odd k we have the representation

$$2M=\begin{pmatrix}\tilde{\gamma}\S_k^T+\S_k\tilde{\delta}^T & -\tilde{\gamma}\S_k+\S_k\tilde{\delta}^T\\ \tilde{\gamma}\S_k^T-\S_k\tilde{\delta}^T & -\tilde{\gamma}\S_k^T-\S_k\tilde{\delta}\end{pmatrix}$$

which similarly gives the same set as above and so again the sets of absolute values form a sum-and-distance system.

Conversely, it can be shown by a simple calculation,

$$\{(-1)^j\tilde{\gamma}_i+(-1)^i\tilde{\delta},\ (-1)^{j+1}\tilde{\gamma}_i+(-1)^i\tilde{\delta},\ (-1)^{j+1}\tilde{\gamma}_i+(-1)^{i+1}\tilde{\delta},\ (-1)^j\tilde{\gamma}_i+(-1)^{i+1}\tilde{\delta}\mid i,j\in\mathbb{Z}_k\}$$

$$=\{-(n^2-1),\ -(n^2-3),\ldots,-1,1,n^2-3,\ \ldots,n^2-1\}$$

and therefore the square matrix M is a weightless traditional most-perfect square, as required. □

It follows from Theorems 4.1 and 7.9 that a sum-and-distance system can be found from the vectors used in the representations of each type of square. We will now use this connection to describe a new bijective mapping between the two sets of $n\times n$ traditional most-perfect squares and $n\times n$ traditional reversible squares.

THEOREM 7.10. *Let $k\in\mathbb{N}$, so that $n=2k$ is even. Then there exists a (new) one-to-one mapping between the set of all $n\times n$ traditional reversible square matrices and the set of all $n\times n$ traditional most-perfect square matrices.*

Proof. Let $a,b\in\mathbb{R}^k$ such that the sets of their absolute values form a sum-and-distance system, and let M_1 and M_2 be $n\times n$ matrices constructed from the two vectors a, b, according to Theorems 4.1 and 7.9. Then M_1 and M_2 are respectively an $n \times n$ (weightless) traditional reversible square matrix and an $n \times n$ (weightless) most-perfect square matrix. Conversely, by Theorems 3.13 and 7.8, writing

$$M_1 = X_n \begin{pmatrix} \hat{0}_k & 1_k a^T \\ b 1_k^T & \hat{0}_k \end{pmatrix} X_n, \quad 2M_2 = \tilde{a}^T \S_n + \S_n^T \tilde{b}$$

where $\tilde{a} = \begin{pmatrix} a \\ \mp a \end{pmatrix}$, $\tilde{b} = \begin{pmatrix} b \\ \mp b \end{pmatrix}$ and $a, b \in \mathbb{R}^k$. It follows that any vectors a, b that give a traditional reversible square in its block representation M_1 will also give a most-perfect square when substituted into the representation of M_2. Hence, each distinct pair of vectors uniquely define both an $n \times n$ reversible square matrix and an $n \times n$ most perfect square matrix. Hence there exists a bijection between the two sets of $n \times n$ reversible and most perfect square matrices. □

Remark. Combining our understanding of the construction of sum-and-distance systems (detailed in Chapter 6) with the legitimate transform vector changes (detailed in Chapter 4) means that we are now able to construct all $n \times n$ traditional most-perfect squares for any given even side length $n = 2k$.

8 Algebras of Matrices with Symmetric Properties

In this section we consider our families of $n \times n$ matrices, defined by their symmetry properties, in terms of $\mathbb{Z}_2$-graded algebras (also known as *superalgebras*), which have a decomposition into an 'even' subalgebra and an 'odd' complementary part which is a bimodule over the 'even' subalgebra and squares into it.

We use the convention of calling a direct sum $\Xi \oplus \mathrm{H}$, where Ξ, H are vector subspaces of $\mathbb{R}^{n\times n}$, a $\mathbb{Z}_2$-graded algebra if the first direct summand Ξ is the 'even' subalgebra and the second direct summand H is the 'odd' complement, i.e. if

$$\Xi\Xi \subset \Xi, \qquad \mathrm{H}\Xi \subset \mathrm{H}, \qquad \Xi\mathrm{H} \subset \mathrm{H}, \qquad \mathrm{HH} \subset \Xi.$$

The following statements follow immediately from this definition.

LEMMA 8.1. *Let $n \in \mathbb{N}$.*

(a) If $\Xi \oplus \mathrm{H} \in \mathbb{R}^{n\times n}$ is a $\mathbb{Z}_2$-graded algebra and $\Gamma \subset \mathbb{R}^{n\times n}$ is a matrix algebra, then $(\Xi \cap \Gamma) \oplus (\mathrm{H} \cap \Gamma)$ is a $\mathbb{Z}_2$-graded algebra.

(b) If $\Xi \oplus \mathrm{H}, \Xi' \oplus \mathrm{H}' \subset \mathbb{R}^{n\times n}$ are $\mathbb{Z}_2$-graded algebras, then $(\Xi \cap \Xi') \oplus (\mathrm{H} \cap \mathrm{H}')$ is a $\mathbb{Z}_2$-graded algebra.

So far we have focused on the matrix symmetry type spaces (R_n, V_n, $\hat{V}_n$, M_n, P_n and S_n), for the matrix symmetry types (R, V, $\hat{\mathrm{V}}$, M, P and S).

To complement the M_n subspace of $\mathbb{R}^{n\times n}$, in the following definition we introduce the matrix symmetry type N, and respective vector space $N_n \subset \mathbb{R}^{n\times n}$.

Definition. Let $n \in \mathbb{N}$. Then we define a matrix $M = (m_{i,j}) \in \mathbb{R}^{n\times n}$ to be *type* N, if it has the consecutive row and column alternating sum property

$$\sum_{i\in\mathbb{Z}_n} (-1)^i (m_{i,j} + m_{i,j+1}) = \sum_{i\in\mathbb{Z}_n} (-1)^i (m_{j,i} + m_{j+1,i}) = 0 \quad (j \in \mathbb{Z}_n).$$

We denote by N_n, the space of all $n \times n$ type N matrices, so that

$$\mathrm{N}_n = \{M \in \mathbb{R}^{n\times n} \mid M \text{ has property (N).}\}$$

LEMMA 8.2. *Let $n = 2k$ be even, then a square matrix $M \in \mathbb{R}^{n\times n}$ is a type N if and only if it satisfies,*

$$\S_n^T M u = u^T M \S_n = 0.$$

Proof. Let the vector v_k contain 1 in entries k and $k+1$ and 0 elsewhere. The vectors v_k for $k \in \{1, 2 \ldots, n-1\}$ form a basis for the set of vectors $u \in \{\S_n\}^{\perp}$, so given any $u = \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_{n-1} v_{n-1}$ and we may write

$$0 = u^T M\S_n = (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{n-1} v_{n-1})^T M\S_n$$
$$= \alpha_1 v_1^T M\S_n + \alpha_2 v_2^T M\S_n + \ldots + \alpha_{n-1} v_{n-1}^T M\S_n,$$

where $\alpha_k \in \mathbb{R}$ for all $k \in \{1, 2, \ldots n-1\}$. Now say that all $\alpha_k = 0$ for all but $j \in \{1, 2, \ldots n-1\}$ we then have

$$0 = \alpha_j v_j^T M\S_n = \alpha_j \sum_{i\in\mathbb{Z}_n} (-1)^{i-1}(m_{j,i} + m_{j+1,i}) = \sum_{i\in\mathbb{Z}_n} (-1)^i (m_{j,i} + m_{j+1,i}),$$

similarly for

$$0 = \alpha_j \S_n^T M v_j = \sum_{i\in\mathbb{Z}_n} (-1)^i (m_{i,j} + m_{i,j+1}).$$

Conversely, if M is type N then for all of the basis vectors v_j we have

$$v_j^T M\S_n = -\sum_{i\in\mathbb{Z}_n} (-1)^i (m_{j,i} + m_{j+1,i}) = 0.$$

Similarly,

$$\S_n^T M v_j = -\sum_{i\in\mathbb{Z}_n} (m_{i,j} + m_{i,j+1}) = 0$$

as required.

□

As one of the central results of this chapter, we shall obtain the following symmetry superalgebras.

THEOREM 8.3. *Let $n \in \mathbb{N}$.*

(a) The following are $\mathbb{Z}_2$-graded algebras,

$$\mathbb{R}^{n\times n} = S_n \oplus V_n, \quad \textit{if } n \textit{ is even, then also} \quad \mathbb{R}^{n\times n} = N_n \oplus M_n.$$

(b) The space of type R *square matrices R_n, is a subalgebra of $\mathbb{R}^{n\times n}$, so that $R_n R_n \subset R_n \subset \mathbb{R}^{n\times n}$.*

To help us prove these results we first need to establish some algebraic results.

LEMMA 8.4. *The conjugation between matrices and X_n is an algebra isomorphism and so has the following properties;*

1.
$$X_n(\alpha M + N)X_n = \alpha X_n M X_n + X_n N X_n \quad (\alpha \in \mathbb{R},\ M, N \in \mathbb{R}^{n\times n}),$$

2.

$$X_n(MN)X_n = X_nMX_nX_nNX_n \quad (M, N \in \mathbb{R}^{n\times n}),$$

3.

$$X_n(X_nM_nX_n)X_n = M_n.$$

Further, the transpose conjugation of X_n and M is the conjugation of the transpose of M,

$$(X_nMX_n)^T = X_nM^TX_n \quad (m \in \mathbb{R}^{n\times n}).$$

Proof. These properties follow directly from the properties of X_n such that,

$$X_nX_n = I_n \text{ and } X_n = X_n^T.$$

□

LEMMA 8.5. *Let $n \in \mathbb{N}$ and $y \in \mathbb{R}^n/\{0_n\}$. Then $M \in \mathbb{R}^{n\times n}$ satisfies*

$$y^TMu = u^TMy = 0 \qquad (u \in \{y\}^\perp)$$

(where $\{y\}^\perp$ is the orthogonal set of vectors to y) if and only if there is some $\lambda \in \mathbb{R}$ such that $My = M^Ty = \lambda y$.

Proof. Since $0 = u^TMy$ for all $u \in \{y\}^\perp$ then $My \in \{y\}^{\perp\perp} = \mathbb{R}y$. Similarly we have $0 = y^TMu = (u^TM^Ty)^T$ then $M^Ty \in \{y\}^{\perp\perp}$

So say $My = \lambda y$ and $M^Ty = \lambda' y$ for $\lambda, \lambda' \in \mathbb{R}$ then we have

$$\lambda y^Ty = y^T(\lambda y) = y^TMy = (y^TM)y = (M^Ty)^Ty = (\lambda' y)^Ty = \lambda' y^Ty$$

and as $yy^T \neq 0$ as $y \neq 0_n$ we have $\lambda = \lambda'$. So to conclude we have $My = M^Ty = \lambda y$.

The converse falls out directly, as say $My = M^Ty = \lambda y$, then

$$u^TMy = \lambda u^Ty = 0$$

and

$$y^TMu = \lambda y^Tu = 0$$

if $u \in \{y\}^\perp$. □

Definition. A matrix $P \in \mathbb{R}^{n\times m}$ is a *projector* matrix if and only if it is symmetrical and idempotent ($P = PP^T = P^2$).
Remark. We use the following square projector matrix $P = (y^Ty)^{-1}yy^T$ for some $y \in \mathbb{R}^n$,

$$P^2 = (y^Ty)^{-1}yy^T(y^Ty)^{-1}yy^T = (y^Ty)^{-1}(y^Ty)^{-1}yy^Tyy^T = (y^Ty)^{-1}(y^Ty)^{-1}y(y^Ty)y^T = P,$$

$$P^T = ((y^Ty)^{-1}yy^T)^T = (yy^T)^T(y^Ty)^{-1} = (y^Ty)^{-1}yy^T = P.$$

These square matrices can then be generalised by the following two defined sets which depend on a choice of vector $y \in \mathbb{R}^n$.

Definition (of $W_{y,n}$)**.** Let any $n \in \mathbb{N}$, $y \in \mathbb{R}^n/\{0_n\}$ and $M \in \mathbb{R}^{n\times n}$. Then we denote by $W_{y,n}$, the set of all $n \times n$ matrices M that satisfy

$$y^T Mu = 0 \ \text{ and } \ u^T My = 0 \ \ (u \in \{y\}^\perp).$$

Definition (of $X_{y,n}$)**.** Let any $n \in \mathbb{N}$, $y \in \mathbb{R}^n/\{0_n\}$ and $M \in \mathbb{R}^{n\times n}$. Then we denote by $X_{y,n}$, the set of all $n \times n$ matrices M that satisfy

$$y^T My = 0 \ \text{ and } \ u^T Mv = 0 \ \ (u, v \in \{y\}^\perp).$$

Example. *The set of all type S_n matrices can be defined as $W_{1_n,n}$, similarly the set of all type V_n matrices is denoted by $X_{1_n,n}$.*

THEOREM 8.6. *Let $y \in \mathbb{R}^n/\{0_n\}$, then the sets $W_{y,n}$ and $X_{y,n}$ span $\mathbb{R}^{n\times n}$, so that,*

$$W_{y,n} \oplus X_{y,n} = \mathbb{R}^{n\times n}.$$

Proof. Let $M \in \mathbb{R}^{n\times n}$. We now show that M can be decomposed into the sum of two matrices, one from $W_{y,n}$, and the other from $X_{y,n}$. Consider the projector matrix $P = (y^Ty)^{-1}yy^T$, and write

$$M = (PMP + (I - P)M(I - P)) + (PM(I - P) + (I - P)MP),$$

where we set

$$M_1 = PMP + (I - P)M(I - P) \ \text{ and } \ M_2 = PM(I - P) + (I - P)MP.$$

Now let $u \in \{y\}^\perp$, so that

$$u^T M_1 y = u^T PMPy + u^T(I - P)M(I - P)y$$

$$= \underbrace{(Pu)^T}_{=0_n^T} MPy + ((I - P)u)^T M \underbrace{(I - P)y}_{=0_n} = 0,$$

$$y^T M_1 u = y^T PMPu + y^T(I - P)M(I - P)u$$

$$= (Py)^T M \underbrace{Pu}_{=0_n} + \underbrace{((I - P)y)^T}_{=0_n^T} M(I - P)u = 0$$

so $M_1 \in W_{y,n}$.

Similarly,

$$y^T M_2 y = (Pu)^T M \underbrace{(I - P)y}_{=0_n} + \underbrace{((I - P)y)^T}_{=0_n^T} MPy,$$

$$u^T M_2 v = \underbrace{(Pu)^T}_{=0_n^T} M(I-P)v + ((I-P)u)^T M \underbrace{Pv}_{=0_n} = 0$$

so $M_2 \in X_{y,n}$. □

THEOREM 8.7. *We have that $W_{y,n} \oplus X_{y,n} = \mathbb{R}^{n\times n}$, form a superalgebra, with $X_{y,n}$ the odd part, that squares into $W_{y,n}$ the even subalgebra, so that*

$$W_{y,n}W_{y,n} \subset W_{y,n},\ X_{y,n}X_{y,n} \subset W_{y,n},\ W_{y,n}X_{y,n} \subset X_{y,n},\ W_{y,n}X_{y,n} \subset X_{y,n}.$$

Proof. Let P the projector matrix $P = (y^T y)^{-1} y y^T$. Then for $W_1, W_2 \in W_{y,n}$ and $u \in \{y\}^\perp$

$$y^T W_1 W_2 u = y^T W_1 \underbrace{P W_2 u}_{0_n} + \underbrace{y^T W_1 (I_n - P)}_{0_n} W_2 u = 0,$$

$$u^T W_1 W_2 y = \underbrace{u^T W_1 P}_{=0_n} W_2 y^T + u^T W_1 \underbrace{(I_n - P) W_2 y}_{0_n} = 0$$

so that $W_1 W_2 \in W_{y,n}$. Similarly for $X_1, X_2 \in X_{y,n}$ and $u \in \{u\}^\perp$. Hence

$$y^T X_1 X_2 u = y^T X_1 \underbrace{P X_2 u}_{=0_n} + \underbrace{y^T X_1 (I_n - P)}_{0_n} X_2 u = 0,$$

$$u^T X_1 X_2 y = \underbrace{u^T X_1 P}_{0_n} X_2 y + u^T X_1 \underbrace{(I_n - P) X_2 y}_{0_n} = 0,$$

and it follows that $X_1 X_2 \in W_{y,n}$.

Now let $W_1 \in W_{y,n}$, $X_1 \in X_{y,n}$ and $u, v \in \{y\}^\perp$ Then

$$y^T W_1 X_1 y = y^T W_1 \underbrace{P X_1 y}_{0_n} + \underbrace{y^T W_1 (I_n - P)}_{0_n} X_1 y = 0,$$

$$u^T W_1 X_1 v = u^T W_1 \underbrace{P X_1 v}_{=0_n} + u^T W_1 \underbrace{(I_n - P) X_1 v}_{=0_n} = 0$$

and it follows that $W_1 X_1 \in X_{y,n}$.

Lastly, we take $W_1 \in W_{y,n}$, $X_1 \in X_{y,n}$ and $u, v \in \{y\}^\perp$,

$$y^T X_1 W_1 y = \underbrace{y^T X_1 P}_{0_n} W_2 y + y^T X_1 \underbrace{(I_n - P) W_1 y}_{=0_n} = 0,$$

$$u^T X_1 W_1 v = u^T X_1 \underbrace{P W_1 v}_{0_n} + \underbrace{u^T X_1 (I_n - P)}_{0_n} W_1 v = 0$$

so that $X_1 W_1 \in X_{y,n}$. Therefore, $W_{y,n} \oplus X_{y,n} = \mathbb{R}^{n\times n}$, form a superalgebra, with $X_{y,n}$ the odd part, and $W_{y,n}$ the even subalgebra as required. □

8.1 The Superlagebras $S_n \oplus V_n$ and $N_n \oplus M_n$

THEOREM 8.8. *The two sets of all $n \times n$ type S and type V matrices, respectively S_n and V_n, form a superalgebra such that*

$$S_n \oplus V_n = W_{1_n,n} \oplus X_{1_n,n} = \mathbb{R}^{n\times n},$$

with

$$S_nS_n \subset S_n, \quad V_nV_n \subset S_n, \quad S_nV_n \subset V_n, \quad V_nS_n \subset V_n,$$

so that V_n is the odd part, which squares into S_n, the even subalgebra.

Proof. By Lemmas 3.11 and 7.5 the set of type S and type V matrices can respectively be written in the form $W_{y,n}$ and $X_{y,n}$, with $y = 1_n$. By Theorem 8.7 it then follows that $S_n \oplus V_n = W_{1_n,n} \oplus X_{1_n,n} = \mathbb{R}^{n\times n}$, form a superalgebra, with V_n the odd part, and S_n the even subalgebra, as required. □

THEOREM 8.9. *Let $k \in \mathbb{N}$, so that $n = 2k$ is even. Then the two sets of all $n \times n$ type N and type M matrices, respectively N_n and M_n, form a superalgebra such that*

$$N_n \oplus M_n = W_{\S_n,n} \oplus X_{\S_n,n} = \mathbb{R}^{n\times n},$$

with

$$N_nN_n \subset N_n, \quad M_nM_n \subset N_n, \quad M_nN_n \subset M_n, \quad N_nM_n \subset M_n,$$

so that M_n is the odd part, which squares into N_n, the even subalgebra.

Proof. By Lemmas 7.2 and 8.2 the set of type N and type M matrices can respectively be written in the form $W_{y,n}$ and $X_{y,n}$, with $y = \S_n$. By Theorem 8.7 it then follows that $N_n \oplus M_n = W_{\S_n,n} \oplus X_{\S_n,n} = \mathbb{R}^{n\times n}$, form a superalgebra, with M_n the odd part, and N_n the even subalgebra, as required. □

8.2 The Type R Matrix Algebra R_n

THEOREM 8.10. *The set of all $n \times n$ type R matrices R_n is a subalgebra of the matrix algebra $\mathbb{R}^{n\times n}$, so that for $n \in \mathbb{N}$, we have $R_n \subset \mathbb{R}^{n\times n}$, and $R_nR_n \subset R_n$.*

Proof. Let $M_1, M_2 \in R_n$, be two type R $n \times n$ matrices. Then

$$M_1M_2 = X_nN_1X_nX_nN_2X_n = X_nN_1N_2X_n,$$

so we need only show that N_1N_2 has the same block representation found in Theorem 3.10.

For even $n = 2k$ let $\gamma_1, \gamma_2 \in \mathbb{R}$, $x_1, x_2, z_1, z_2 \in \mathbb{R}^k$ and $Z_1, Z_2 \in \mathbb{R}^{k\times k}$, so that

$$M_1M_2 = X_n \begin{pmatrix} \gamma_1 E_k & 1_k z_1^T \\ x_1 1_k^T & Z_1 \end{pmatrix} \begin{pmatrix} \gamma_2 E_k & 1_k z_2^T \\ x_2 1_k^T & Z_2 \end{pmatrix} X_n$$

$$= X_n \begin{pmatrix} \gamma_1 E_k \gamma_2 E_k + 1_k z_1^T x_2 1_k^T & \gamma_1 E_k 1_k z_2^T + 1_k z_1^T Z_2 \\ x_1 1_k^T \gamma_2 E_k + Z_1 x_2 1_k^T & x_1 1_k^T z_2^T + Z_1 Z_2 \end{pmatrix} X_n$$

$$= X_n \begin{pmatrix} (k\gamma_1\gamma_2 + z_1^T x_2) E_k & 1_k(k\gamma_1 z_2^T + z_1^T Z_2) \\ (k\gamma_2 x_1 + Z_1 x_2) 1_k^T & k x_1 z_2^T + Z_1 Z_2 \end{pmatrix} X_n.$$

Similarly, for odd $n = 2k+1$, we have

$$X_n \begin{pmatrix} \sqrt{2}\gamma_1 E_k & \gamma_1 1_k & \sqrt{2} 1_k z_1^T \\ \gamma_1 1_k^T & \frac{\gamma_1}{\sqrt{2}} & z_1^T \\ \sqrt{2} x_1 1_k^T & x_1 & Z_1 \end{pmatrix} \begin{pmatrix} \sqrt{2}\gamma_2 E_k & \gamma_2 1_k & \sqrt{2} 1_k z_2^T \\ \gamma_2 1_k^T & \frac{\gamma_2}{\sqrt{2}} & z_2^T \\ \sqrt{2} x_2 1_k^T & x_2 & Z_2 \end{pmatrix} X_n$$

$$= X_n \begin{pmatrix} (n\gamma_1\gamma_2 + 2z_1^T x_2) E_k & \frac{1}{\sqrt{2}}(n\gamma_1\gamma_2 + 2z_1^T x_2) & 1_k(n\gamma_1 z_2^T + \sqrt{2} z_1 Z_2) \\ \frac{1}{\sqrt{2}}(n\gamma_1\gamma_1 + 2z_1^T x_2) 1_k^T & \frac{1}{2}(n\gamma_1\gamma_2 + 2z_1^T x_2) & n\gamma_1 z_2^T + \sqrt{2} z_1 Z_2 \\ (n\gamma_2 x_1 + \sqrt{2} z_1 x_2) 1_k^T & \frac{1}{\sqrt{2}}(n\gamma_2 x_1 + \sqrt{2} z_1 x_2) & n x_1 z_2^T + Z_1 Z_2 \end{pmatrix} X_n.$$

From the block representations of the space R_n, it then follows that R_n is a subalgebra of the matrix algebra $\mathbb{R}^{n \times n}$, as required. □

9 Quadratic Forms from Type A Matrices

In this chapter we consider matrices with the type A symmetry property, deducing their block representation, eigenvalue characteristics and the construction of a two-sided eigenvector matrix, from which we establish a connection to quadratic forms.

9.1 The Type A Block Representation

We begin by restating the type A condition.

Definition. A matrix $M = (m_{i,j}) \in \mathbb{R}^{n_1 \times n_2}$ is said to be *associated*, or *type* A, if satisfied the associated symmetry condition

$$m_{i,j} + m_{(n+1-i),(n+1-j)} = 2w \quad \forall i \in \mathbb{Z}_{n_1}, \quad \text{and} \quad \forall j \in \mathbb{Z}_{n_2},$$

so that by the definition of matrix weight (stated in Chapter 3), M has weight w.

By Lemma 3.1 it follows that if $n_1 = n_2 = n$ say, so that M is a type A square matrix, then $M - wE_n$ is a type A matrix with weight zero.

Definition (of weightless type A squares A_n^0)**.** Let $n \in \mathbb{N}$. We define A_n^0 to be the set of all weightless $n \times n$ type A square matrices, so that for any $M = (m_{i,j}) \in \mathrm{A}_n^0$, we have that M has weight $w = 0$ and

$$m_{i,j} + m_{(n+1-i),(n+1-j)} = 0, \qquad i, j \in \mathbb{Z}_n.$$

LEMMA 9.1. *Let $n \in \mathbb{N}$, and let M be an $n \times n$ weightless type A matrix $M \in \mathbb{R}^{n \times n}$. Then M has the block representation*

$$M = \begin{pmatrix} R & S \\ -J_k S J_k & -J_k R J_k \end{pmatrix}$$

when $n = 2k$ is even, and

$$M = \begin{pmatrix} S & s & R \\ -rJ_k & 0 & r \\ -J_k R J_k & -J_k s & -J_k S J_k \end{pmatrix}$$

when $n = 2k+1$ is odd, where $S, R \in \mathbb{R}^{k \times k}$ and $s, r \in \mathbb{R}^k$.

Proof. It follows from the type A property that any associated pair of cells must sum to 0, so that $m_{i,j} + m_{n+1-i,n+1-j} = 0$ for any $i, j \in \mathbb{Z}_n$, so we can write $m_{i,j} = -m_{n+1-i,n+1-j}$ and hence M has the block structure stated. □

Remark. It follows that if $M \in A_n^0$ then $J_n M J_n = -M$.

THEOREM 9.2 (Weightless type A block representation)**.** *A matrix $M \in \mathbb{R}^{n\times n}$ is an element of A_n^0 if an only if*

$$M = X_n \begin{pmatrix} \hat{0} & D \\ C & \hat{0} \end{pmatrix} X_n,$$

where $C, D \in \mathbb{R}^{k\times k}$ and both the top left and bottom right $\hat{0} = 0_{k\times k}$, when $n = 2k$ is even, and where $C \in \mathbb{R}^{k\times(k+1)}$ and the top left $\hat{0}$ has the same size, $D \in \mathbb{R}^{(k+1)\times k}$, when $n = 2k+1$ is odd.

Proof. Let $M \in A_n^0$. Then by Lemma 9.1 we can write

$$M = \begin{pmatrix} R & S \\ -J_kSJ_k & -J_kRJ_k \end{pmatrix}$$

for some $S, R \in \mathbb{R}^{k\times k}$. Multiplying the left and right hand side by X_n and expanding the block representation then yields

$$X_nMX_n = X_n \begin{pmatrix} R & S \\ -J_kSJ_k & -J_kRJ_k \end{pmatrix} X_n$$

$$= \frac{1}{2}\begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix}\begin{pmatrix} R & S \\ -J_kSJ_k & -J_kRJ_k \end{pmatrix}\begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} R - SJ_k & S - RJ_k \\ J_k + J_kR + J_kSJ_k & J_kS + J_kRJ_k \end{pmatrix}\begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \hat{0}_k & 2RJ_k - 2S \\ 2J_kR + J_kSJ_k & \hat{0}_k \end{pmatrix} = \begin{pmatrix} \hat{0} & RJ_k - S \\ J_kR + J_kSJ_k & \hat{0}_k \end{pmatrix},$$

and setting $D = RJ_k - S$, $C = J_kR + J_kSJ_k$ we then have the desired block representation.

Conversely, let us take the block structure and expand

$$X_n \begin{pmatrix} \hat{0}_k & D \\ C & \hat{0}_k \end{pmatrix} X_n = \frac{1}{2}\begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix}\begin{pmatrix} \hat{0}_k & D \\ C & \hat{0}_k \end{pmatrix}\begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} J_kC & D \\ -C & J_kD \end{pmatrix}\begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix} = \frac{1}{2}\begin{pmatrix} J_kC + DJ_k & J_kCJ_k - D \\ -C + J_kDJ_k & -CJ_k - J_kD \end{pmatrix}.$$

Then we have

$$J_kC + DJ_k = -J_k(-CJ_k - J_kDJ_k)J_k$$

$$J_kCJ_k - D = -J_k(-C + J_kDJ_k)J_k$$

and by Lemma 9.1 the structure of this square is that of a weightless type A.

When $n = 2k + 1$ is odd, we begin with

$$\begin{pmatrix} A & v & B \\ -w^T J_k & 0 & w^T \\ -J_k B J_k & -J_k v & -J_k A J_k \end{pmatrix}$$

with $A, B \in \mathbb{R}^{k\times k}$ and $w, v \in \mathbb{R}^k$. Then multiplying the left and right hand side by X_n and expanding the block representation we have

$$X_n M X_n = X_n \begin{pmatrix} A & v & B \\ -w^T J_k & 0 & w^T \\ -J_k B J_k & -J_k v & -J_k A J_k \end{pmatrix} X_n$$

$$= \frac{1}{2} \begin{pmatrix} I_k & 0_k & J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ J_k & 0_k & -I_k \end{pmatrix} \begin{pmatrix} A & v & B \\ -w^T J_k & 0 & w^T \\ -J_k B J_k & -J_k v & -J_k A J_k \end{pmatrix} \begin{pmatrix} I_k & 0_k & J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ J_k & 0_k & -I_k \end{pmatrix}$$

$$= \begin{pmatrix} A - BJ_k & 0_k & B - AJ_k \\ -\sqrt{2}wJ_k & 0 & \sqrt{2}w^T \\ J_k A + J_k B J_k & 2J_k v & J_k B + J_k A J_k \end{pmatrix} \begin{pmatrix} I_k & 0_k & J_k \\ 0_k^T & \sqrt{2} & 0_k^T \\ J_k & 0_k & -I_k \end{pmatrix}$$

$$= \begin{pmatrix} A - BJ_k & 0_k & AJ_k - B \\ \sqrt{2}w^T J_k & 0 & \sqrt{2}w^T \\ J_k A + J_k B J_k & J_k v & J_k A J_k + J_k B \end{pmatrix}$$

where

$$A - BJ_k = J_k(J_k A J_k + J_k B)J_k,$$

$$AJ_k - B = J_k(J_k A + J_k B J_k)J_k,$$

which is of the desired form.

The converse follows similarly as in the even-dimensional case.

□

9.2 Eigenvalues of the square of a type $A \cap S$ Square

THEOREM 9.3. *Let $n = 2k$, $V, W \in \mathbb{R}^{k\times k}$ and*

$$M_0 = X_n \begin{pmatrix} \hat{0}_k & V^T \\ W & \hat{0}_k \end{pmatrix} X_n$$

be a weightless associated magic square (i.e, type A and S) and for $w \in \mathbb{R}\backslash\{0\}$, let $M_w := M_0 + wE_n$. Then M_w^2 has the same non-zero eigenvalues (including multiplicities) as M_0^2, with $w-$independent eigenvectors, and additional eigenvalue

n^2w^2. *Moreover,*

$$rkM_w^2 = rkM_0^2 + I_n.$$

Proof. As M_0 is weightless and semimagic we have $M_0E_n = E_nM_0 = \hat{0}_n$, then

$$\begin{aligned} M_w^2 = (M_0 + wE_n)^2 &= M_0^2 + w(M_0E_n + E_nM_0) + w^2E_n^2 \\ &= M_0^2 + nw^2E_n. \end{aligned}$$

Let us assume that we have a non-zero eigenvalue $\lambda \neq 0$ of the block representation of $M_0^2 = (X_nN_0X_n)^2$, (which as X_n is has the same eigenvalues as $(X_n(X_nM_0X_n)X_n)^2 = M_0^2$), so there is a vector

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^n \backslash \{0_n\},$$

such that

$$\begin{pmatrix} V^TW & \hat{0}_k \\ \hat{0}_k & WV^T \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Then $V^TWv_1 = \lambda v_1$ so $v_1 \in range(V^T)$ and so we can write $1_k^Tv_1 = 0$ (i.e. we can say that v_1 is some linear combination of the columns of V^T). Following this, $E_kv_1 = 0_k$, and considering the block representation of M_w^2 we have

$$\begin{pmatrix} V^TW + 4kw^2E_k & \hat{0}_k \\ \hat{0}_k & WV^T \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 + 0_k \\ \lambda v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

which shows that λ is also an eigenvalue of M_w^2. Moreover, using the fact that $W1_k = 0_k$ we can see that

$$\begin{pmatrix} V^TW + 4kw^2E_k & \hat{0}_k \\ \hat{0}_k & WV^T \end{pmatrix} \begin{pmatrix} 1_k \\ 0_k \end{pmatrix} = \begin{pmatrix} V^TW1_k + 4k^2w^2E_k1_k \\ 0_k \end{pmatrix} = 4k^2w^2 \begin{pmatrix} 1_k \\ 0_k \end{pmatrix},$$

so we have that $4k^2w^2$ is an additional eigenvalue of M_w^2.

Conversely, if λ is an eigenvalue of M_w^2, so we have

$$\begin{pmatrix} V^TW + 4kw^2E_k & \hat{0}_k \\ \hat{0}_k & WV^T \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Hence we can write $v_1 = \alpha 1_k + z$ where $z \in \mathbb{R}^k$ and $z^T1_k = 0$ (i.e. the entries sum to 0) and expanding and simplifying gives

$$(V^TW + 4kw^2E_k)(\alpha 1_k + z) = \alpha \underbrace{V^TW1_k}_{0_k} + V^TWz + 4kw^2\alpha E_k1_k + \underbrace{4kw^2E_kz}_{0_k}$$

$$= V^TWz + 4k^2w^2\alpha 1_k = \lambda(\alpha 1_k + z).$$

Now V^TWz is orthogonal to 1_k as $z^T1_k = 0$ and we have

$$V^TWz = \lambda z, \quad \lambda\alpha = 4k^2w^2\alpha.$$

Concluding it follows that $\lambda = 4k^2w^2$ unless $\alpha = 0$ and λ is an eigenvalue of M_0^2 unless $z = w_2 = 1_k$. □

Definition (of a parasymmetric matrix)**.** We call an associated magic square (type S and A) matrix M *parasymmetric* if its square M^2 is symmetric. In terms of the block representation, parasymmetry can be characterised by the two blocks of M both having rank 1.

LEMMA 9.4. *Let $V, W \in \mathbb{R}^{k\times k}$ be rank 1 matrices with row sum 0, and let M have the usual type A and S block representation so that*

$$M = X_n \begin{pmatrix} \hat{0}_k & V^T \\ W & \hat{0}_k \end{pmatrix} X_n.$$

If $M^2 \neq \hat{0}_n^2$, then M is parasymmetric if and only if W is a multiple of V.

Proof. We can write $V = uv^T$, $W = xy^T$ with non-null vectors $u, v, x, y \in \mathbb{R}^n\ \{0_k\}$ so that $V^T = vu^T$ and $W^T = yx^T$. Now if V^TW is symmetric then

$$(u^Tx)vy^T = V^TW = W^TV = (x^Tu)yv^T,$$

so we either have $u^Tx = 0$ or $vy^T = yv^T$. Now by taking the latter case and multiplying by y on the right we obtain

$$(vy^T)y = (yv^T)y$$

$$v(yy^T) = y(v^Ty)$$

and setting $yy^T = \lambda_1$ and $v^Ty = \lambda_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ we see that v and y are linearly dependent. Similarly if WV^T is symmetric, the either $v^Ty = 0$ or u and v are linearly dependent.

Now if $ux^T = 0$ the we also have $yv^T = 0$ as if not then u and x would be linearly dependent which they cannot be simultaneously as orthogonal ($u^Tx = 0$) and both u, v are non-zero vectors. Hence we obtain

$$M_0^2 = X_n \begin{pmatrix} V^TW & \hat{0}_k \\ \hat{0}_k & WV^T \end{pmatrix} X_n = X_n \begin{pmatrix} (u^Tx)vy^T & \hat{0}_k \\ \hat{0}_k & (y^Tv)xu^T \end{pmatrix} X_n$$

$$= X_n \begin{pmatrix} \hat{0}_k & \hat{0}_k \\ \hat{0}_k & \hat{0}_k \end{pmatrix} X_n = \hat{0}_n.$$

So we can conclude that if u and x, and v and y are linearly dependent then

$$x = \lambda_1 u, \qquad y = \lambda_2 v$$

for $\lambda_1 \lambda_2 \in \mathbb{R}$. So we have $V = vu^T$ and $W = yx^T = \lambda_2 v(\lambda_1 u)^T = \lambda_1\lambda_2 vu^T = \lambda V$, as required.

The converse can be found to be true trivially by substitution,

$$(M_0^2)^T = \left(X_n \begin{pmatrix} \lambda V^T V & \hat{0}_k \\ \hat{0} & \lambda V V^T \end{pmatrix} X_n \right)^T$$

$$= X_n \begin{pmatrix} (\lambda V^T V)^T & \hat{0}_k \\ \hat{0} & (\lambda V V^T)^T \end{pmatrix} X_n = X_n \begin{pmatrix} \lambda V^T V & \hat{0}_k \\ \hat{0} & \lambda V V^T \end{pmatrix} X_n = M_0^2.$$

□

THEOREM 9.5. *Let $M \in \mathbb{R}^{n\times n}$ be a weightless associated magic square (type S and A). Then when $n = 2k$ is even we have*

$$M = X_n \begin{pmatrix} \hat{0}_k & vu^T \\ xy^T & \hat{0}_k \end{pmatrix} X_n$$

with rank 1 block components $V = uv^T$, $W = xy^T$, and M^2 has eigenvalues 0 and $(u^T x)(y^T v) = Tr(V^T W)$. We note that if V, W have integer entries, then the eigenvalues of M^2 are integers.

Proof. For $N \in \mathbb{R}^{n\times n}$ and as X_n is unitary the eigenvalues of $M = X_n N X_n$ and N are always identical. Now consider the block representation of M_0^2 such that

$$N = X_n M_0^2 X_n = \begin{pmatrix} V^T W & \hat{0}_k \\ \hat{0}_k & W V^T \end{pmatrix} = \begin{pmatrix} (u^T x) v y^T & \hat{0}_k \\ \hat{0}_k & (y^T v) x u^T \end{pmatrix}.$$

By the laws of determinants of block representation we have (see [6])

$$0 = \det(N - \lambda I_n) = \det\left(\begin{pmatrix} V^T W - \lambda I_k & \hat{0}_k \\ \hat{0}_k & W V^T - \lambda I_k \end{pmatrix} \right)$$

$$= \det(V^T W - \lambda I_k)\det(W V^T - \lambda I_k).$$

Hence the eigenvalues of N (and so M_0^2) are the eigenvalues of both $V^T W$ and $W V^T$.

Now when $u^T x = y^T v = 0$ then $V^T W = W V^T = \hat{0}_k$, so the matrix only has the zero eigenvalue.

For $u^T = 0$ and $y^T v \neq 0$ we have the k multiplicities of the eigenvalue 0 from

$$V^T W = (u^T x) v y^T = \hat{0}_k$$

and as WV^T is non-zero and rank 1 it has at least $k-1$ multiplicities of the 0 eigenvalues. Now the potential non-zero eigenvalue of WV^T can be found by considering the product

$$WV^T x = (y^T v) x u^T x = (y^T v)(u^T x) x.$$

Therefore $(y^T v)(u^T x)$ is an eigenvalue with x its corresponding eigenvector. Similarly for $u^T x \neq 0$ and $y^T v = 0$ then the only non-zero eigenvalue is $(y^T v)(u^T x)$ and has $n-1$ zero eigenvalues. Finally for $u^T x \neq 0$ and $y^T v \neq 0$ we have two potential non-zero eigenvalues and $n-2$ multiplicities of the zero eigenvalue. □

9.3 Eigenvector Matrices of the Square of a Type $A \cap S$ Square

If $(u^T x)(y^T v) \neq 0$ then it can be shown simply that the linear independent (right) eigenvectors of M^2 are

$$\sqrt{2} X_n \begin{pmatrix} v \\ 0_k \end{pmatrix} = \begin{pmatrix} v \\ J_k v \end{pmatrix}, \quad -\sqrt{2} X_n \begin{pmatrix} 0_k \\ x \end{pmatrix} = \begin{pmatrix} -J_k x \\ x \end{pmatrix},$$

so that if x, v have integer entries then so do these eigenvectors. Note also that the left eigenvectors (i.e. eigenvectors for $(M^2)^T$) for M^2 are given by

$$\begin{pmatrix} y \\ J_n y \end{pmatrix}, \quad \begin{pmatrix} -J_k u \\ u \end{pmatrix},$$

where the right and left eigenvectors are orthogonal.
Remark. In the parasymmetric case $y = v$ and $x = \alpha u$ for $u, v \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$, the matrix M^2 always has a non-zero eigenvalue $\alpha(u^T u)(v^T v)$.

In the following subsection we will show a construction method that yields a two-sided eigenvector matrix for M^2 where M is the rank $1+1$ associated magic square defined above. A two-sided eigenvector matrix is one in which the columns are the right eigenvectors of M^2 and the rows are the left eigenvectors of M^2 . To begin let us consider two right eigenvectors of M^2 which correspond to the non-zero eigenvalue $\lambda = (u^T x)(y^T v)$. Placing vectors side by side to form a $2k \times 2$ matrix we have

$$P_1 = \begin{pmatrix} B_1 \\ A_1 \\ C_1 \end{pmatrix} = \begin{pmatrix} v & -J_k x \\ J_k v & x \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} v_k & -x_1 \\ v_k & x_1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} v_1 & -x_k \\ v_2 & -x_{k-1} \\ \vdots & \vdots \\ v_{k-1} & -x_2 \end{pmatrix}, \text{ and } C_1 = \begin{pmatrix} v_{k-1} & -x_1 \\ v_{k-2} & -x_2 \\ \vdots & \vdots \\ v_1 & x_k \end{pmatrix},$$

with $A_1 \in \mathbb{R}^{2\times 2}$ and $B_1, C_1 \in \mathbb{R}^{k-1\times 2}$. Now defining $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ we obtain the identity $C_1 = J_{k-1}B_1\sigma$, and with the assumption that $v_k, x_1 \neq 0$ we have that the matrix A_1 is non-singular. We note that the construction can be generalised to any non-singular A_1 composed from two rows of P_1 (as one can always find two linearly independent rows within P_1 due to the columns being linearly independent). However, for simplicity we will base our construction method on the two central rows.

Let

$$\tilde{P}_1 = -P_1 A_1^{-1} = \begin{pmatrix} -B_1 A_1^{-1} \\ -I_2 \\ -C_1 A_1^{-1} \end{pmatrix},$$

and similarly starting from the matrix of left eigenvectors from above, we define

$$P_2 = \begin{pmatrix} B_2 \\ A_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} y & -J_k u \\ J_n y & u \end{pmatrix}.$$

With the assumption that $y_k, u_1 \neq 0$ so that $A_2 = \begin{pmatrix} y_k & -u_1 \\ J_k y_k & u_1 \end{pmatrix}$ is non-singular , enables us to define

$$\tilde{P}_2 = -P_2 A_2^{-1} = \begin{pmatrix} -B_2 A_2^{-1} \\ -I_2 \\ -C_2 A_2^{-1} \end{pmatrix}.$$

From our construction it is clear that the columns of $\tilde{P}_1$ and $\tilde{P}_2$ will also be linearly independent eigenvectors for eigenvalue λ and matrices M^2 and $(M^2)^T$ respectively. For matrices A_1 and A_2 we obtain the symmetries,

$$J_2 A_j \sigma = A_j \quad \text{and} \quad \sigma A_j^{-1} J_2 = A_j^{-1} \quad \text{for } j \in \{1, 2\}.$$

Hence it follows that

$$J_{2k}\tilde{P}_j J_2 = \begin{pmatrix} -J_{k-1}C_j A_j^{-1} \\ -J_2 \\ -J_{k-1}B_j A_j^{-1} \end{pmatrix} J_2 = \begin{pmatrix} -B_j \sigma A_j^{-1} J_2 \\ -I_2 \\ -C_j \sigma A_j^{-1} J_2 \end{pmatrix} = \tilde{P}_j \quad (j \in \{1, 2\}),$$

and we deduce that swapping the columns of $\tilde{P}_j$ equates to reversing the order of the column elements.

Definition. Let $A_1, A_2, \ldots, A_m$ be $n \times a_i$ matrices for $a_i, n \in \mathbb{N}$, $i \in \{1, 2, \ldots, m\}$ and $a_1 + a_2 + \ldots + a_m = n$. Then we say that

$$(A_1 | A_2 | \ldots | A_m |)$$

is an $n \times n$ matrix constructed from the sub-matrices $A_1, A_2, \ldots, A_m$.

LEMMA 9.6. *The matrices $\tilde{P}_1$ and $\tilde{P}_2$ satisfy the identities*

$$(\hat{0}_{2k,k-1} \; | \tilde{P}_1 \; | \hat{0}_{2k,k-1}) M = -M,$$

$$(\hat{0}_{2k,k-1} \; | \tilde{P}_2 \; | \hat{0}_{2k,k-1}) M^T = -M^T,$$

where $M \in \mathbb{R}^{n \times n}$ is of type $A \cap S$ and $n = 2k$ is a positive integer.

Proof. We have

$$\begin{aligned}
(\hat{0}_{2k,k-1} \; | \tilde{P}_1 \; | \hat{0}_{2k,k-1}) M &= -P_1 (\hat{0}_{2,k-1} \; | A_1^{-1} \; | \hat{0}_{2,k-1}) X_n \begin{pmatrix} \hat{0}_k & vu^T \\ xy^T & \hat{0}_k \end{pmatrix} X_n \\
&= -\frac{1}{2} P_1 (\hat{0}_{2,k-1} \mid A_1^{-1} X_2 | \hat{0}_{2,k-1}) \begin{pmatrix} \hat{0}_k & vu^T \\ xy^T & \hat{0}_k \end{pmatrix} \begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix} \\
&= -\frac{1}{2} P_1 \left(A_1^{-1} \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} y^T \mid A_1^{-1} \begin{pmatrix} v_k \\ v_k \end{pmatrix} u^T \right) \begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix} \\
&= -\frac{1}{2} \begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix} \begin{pmatrix} v & 0_k \\ 0_k & -x \end{pmatrix} \begin{pmatrix} 0_k^T & u^T \\ -y^T & 0_k^T \end{pmatrix} \begin{pmatrix} I_k & J_k \\ J_k & -I_k \end{pmatrix} \\
&= -X_n \begin{pmatrix} \hat{0}_{k,k} & vu^T \\ xy^T & \hat{0}_{k,k} \end{pmatrix} = M,
\end{aligned}$$

as required. The identity $(\hat{0}_{2k,k-1} \; | \tilde{P}_2 \; | \hat{0}_{2k,k-1}) M^T = -M^T$ can be similarly obtained. □

The following theorem gives a two-sided eigenvector matrix construction for M^2.

THEOREM 9.7. *Given the vectors $u, v, x, y \in \mathbb{R}^k$ such that we have $\lambda = (u^T x)(y^T v) \neq 0$ and $u_1, v_k, x_1, y_k \neq 0$. Let M be an $2k \times 2k$ associated magic square, and let $\tilde{P}_1$ and $\tilde{P}_2$ be as defined above. Then the two-sided eigenvector matrix for M^2 is given by*

$$P = I_{2k} + (\hat{0}_{2k,k-1} \mid \tilde{P}_1 \; | \hat{0}_{2k,k-1}) + \begin{pmatrix} \hat{0}_{k-1,2k} \\ \tilde{P}_2 \\ \hat{0}_{k-1,2k} \end{pmatrix},$$

so that P satisfies the relations

$$M^2 P = P diag(0_{k-1}, \; \lambda 1_2, \; 0_{k-1}) \quad and \quad P M^2 = diag(0_{k-1}, \lambda 1_2, 0_{k-1}) P.$$

Furthermore, the inverse matrix of P exists and is given by

$$P^{-1} = diag(1_{k-1}, 0_2, 1_{k-1}) - \frac{M^2}{\lambda}.$$

Proof. As the columns of $\tilde{P}_1$ are eigenvectors of M^2 and by Lemma 10.6, we can write

$$M^2 P = M^2 \left(I_{2k} + (\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1}) + \begin{pmatrix} \hat{0}_{k-1,2k} \\ \tilde{P}_2 \\ \hat{0}_{k-1,2k} \end{pmatrix} \right)$$

$$= M^2 + M^2 (\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1}) + M \underbrace{\left(M \begin{pmatrix} \hat{0}_{k-1,2k} \\ \tilde{P}_2^T \\ \hat{0}_{k-1,2k} \end{pmatrix} \right)}_{=-M}$$

$$= M^2 + \lambda(\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1}) + M(-M) = \lambda(\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1}),$$

and as $(\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1})$ are right eigenvectors for M^2, similarly for $PM^2 = \begin{pmatrix} \hat{0}_{k-1,2k} \\ \tilde{P}_2^T \\ \hat{0}_{k-1,2k} \end{pmatrix} M^2$.

On the other hand, the centre 2×2 matrix is the negative identity matrix $-I_2$, so we have

$$P\, diag(0_{k-1}, \lambda 1_2, 0_{k-1}) = diag(0_{k-1}, \lambda 1_2, 0_{k-1}) \left(I_{2k} + (\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1}) + \begin{pmatrix} \hat{0}_{k-1,2k} \\ \tilde{P}_2^T \\ \hat{0}_{k-1,2k} \end{pmatrix} \right)$$

$$= diag(0_{k-1,\lambda 1_2, 0_{k-1}}) + (\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1}) + \begin{pmatrix} \hat{0}_{k-1,2k} \\ \tilde{P}_2^T \\ \hat{0}_{k-1,2k} \end{pmatrix} diag(0_{k-1}, \lambda 1_2, 0_{k-1})$$

$$diag(0_{k-1}, \lambda 1_2, 0_{k-1}) + \lambda(\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1}) - diag(0_{k-1}, \lambda 1_2, 0_{k-1})$$
$$= \lambda(\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1}).$$

Again the calculations are the same for PM^2, where $diag(0_{k-1} | \lambda 1_2 | 0_{k-1}) P = \lambda \begin{pmatrix} \hat{0}_{k-1,2k} \\ \tilde{P}_2^T \\ \hat{0}_{k-1,2k} \end{pmatrix}$.

Lastly, consider the inverse expression,

$$P^{-1} = diag(1_{k-1}, 0_2, 1_{k-1}) - \frac{M^2}{\lambda}.$$

By multiplying the right hand side of this expression by P we obtain

$$(diag(1_{k-1}, 0_2, 1_{k-1}) - \frac{M^2}{\lambda})P = diag(1_{k-1}, 0_2, 1_{k-1})P - \frac{M^2}{\lambda}P$$

$$= diag(1_{k-1}, 0_2, 1_{k-1})P - Pdiag(0_{k-1}, 1_2, 0_{k-1})$$

$$= diag(1_{k-1}, 0_2, 1_{k-1}) \left(I_{2k} + (\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1}) + \begin{pmatrix} \hat{0}_{k-1,2k} \\ \tilde{P}_2^T \\ \hat{0}_{k-1,2k} \end{pmatrix} \right)$$

$$- \left(I_{2k} + (\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1}) + \begin{pmatrix} \hat{0}_{k-1,2k} \\ \tilde{P}_2^T \\ \hat{0}_{k-1,2k} \end{pmatrix} \right) diag(0_{k-1}, 1_2, 0_{k-1})$$

$$= diag(1_{k-1}, 0_2, 1_{k-1}) + diag(1_{k-1}, 0_2, 1_{k-1})(\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1})$$

$$+diag(1_{k-1}, 0_2, 1_{k-1}) \begin{pmatrix} \hat{0}_{k-1,2k} \\ \tilde{P}_2^T \\ \hat{0}_{k-1,2k} \end{pmatrix} - diag(0_{k-1}, 1_2, 0_{k-1})$$

$$-(\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1})diag(0_{k-1}, 1_2, 0_{k-1}) - \begin{pmatrix} \hat{0}_{k-1,2k} \\ \tilde{P}_2^T \\ \hat{0}_{k-1,2k} \end{pmatrix} diag(0_{k-1}, 1_2, 0_{k-1})$$

$$= diag(1_{k-1}, 0_2, 1_{k-1}) + diag(1_{k-1}, 0_2, 1_{k-1})(\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1})$$

$$+diag(1_{k-1}, 0_2, 1_{k-1}) - diag(1_{k-1}, 0_2, 1_{k-1}) - (\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1})diag(1_{k-1}, 0_2, 1_{k-1})$$

$$= diag(1_{k-1}, 0_2, 1_{k-1}) + ((\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1}) + diag(0_{k-1}, 1_2, 0_{k-1}))$$

$$-diag(0_{k-1}, 1_2, 0_{k-1}) - (\hat{0}_{2k,k-1} \mid \tilde{P}_1 \mid \hat{0}_{2k,k-1}) + diag(0_{k-1}, 1_2, 0_{k-1})$$

$$= diag(1_{2k}) = I_{2k}.$$

The opposite product follows similarly. □

Remark. Considering the parasymmetric case where $y = v$ and $x = \alpha u$ for $\alpha \in \mathbb{R}$, then the above construction yields

$$P_1 = \begin{pmatrix} y & -J_n \alpha u \\ -J_n y & \alpha u \end{pmatrix} = P_2 \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}.$$

Here $\begin{pmatrix} -J_k u \\ u \end{pmatrix}$ will be an eigenvector of M^2 as well as $\begin{pmatrix} -J_k \alpha u \\ \alpha u \end{pmatrix}$, so we take $P_1 = P_2$, which ensures that

$$P = I_{2k} + (\hat{0}_{k-1}|\ \tilde{P}_1\ |\hat{0}_{k-1}) + (\hat{0}_{k-1}\ |\ \tilde{P}_2\ |\ \hat{0}_{k-1})^T$$

is a symmetric matrix.

9.4 Quadratic Forms Constructed from the Square of a Type $A \cap S$ Matrix

We now show that there is a connection between the block representation vectors and certain types of quadratic forms. We start by observing the block representation of a type S and A (associated) square,

$$M = X_n \begin{pmatrix} 2wE_k & V^T \\ W & \hat{0}_k \end{pmatrix} X_n$$

which can be decomposed into 3 parts such that

$$\begin{aligned} \begin{pmatrix} 2wE_k & V^T \\ W & \hat{0}_k \end{pmatrix} &= 2w \begin{pmatrix} E_k & \hat{0}_k \\ \hat{0}_k & \hat{0}_k \end{pmatrix} + \begin{pmatrix} \hat{0}_k & V^T \\ \hat{0}_k & \hat{0}_k \end{pmatrix} + \begin{pmatrix} \hat{0}_k & \hat{0}_k \\ W & \hat{0}_k \end{pmatrix} \\ &= 2we + a + b, \end{aligned}$$

with $e = \begin{pmatrix} E_k & \hat{0}_k \\ \hat{0}_k & \hat{0}_k \end{pmatrix}$, $a = \begin{pmatrix} \hat{0}_k & V^T \\ \hat{0}_k & \hat{0}_k \end{pmatrix}$ and $b = \begin{pmatrix} \hat{0}_k & \hat{0}_k \\ W & \hat{0}_k \end{pmatrix}$. These individual blocks have the properties,

$$a^2 = b^2 = eb = ea = be = \hat{0}_n,\ ae = ne,\ ab = \begin{pmatrix} V^TW & \hat{0}_k \\ \hat{0}_k & \hat{0}_k \end{pmatrix} \text{ and } ba = \begin{pmatrix} \hat{0}_k & \hat{0}_k \\ \hat{0}_k & WV^T \end{pmatrix},$$

which can be deduced from the rows of the block matrices V and W summing to 0.

This decomposition corresponds to the splitting of the square such that

$$\begin{aligned} X_n(2we + a + b)X_n &= ewX_neX_n + X_naX_n + X_nbX_n \\ &= wE_{2k} + \frac{1}{2} \begin{pmatrix} V^TJ_k & -V^T \\ J_kV^TJ_k & -J_kV^T \end{pmatrix} + \frac{1}{2} \begin{pmatrix} J_kW & J_kWJ_k \\ -W & -WJ_k \end{pmatrix} \\ &= wE_{2k} + A + B, \end{aligned}$$

with $A = X_naX_n = \frac{1}{2} \begin{pmatrix} V^TJ_k & -V^T \\ J_kV^TJ_k & -J_kV^T \end{pmatrix}$, $B = X_nbX_n = \frac{1}{2} \begin{pmatrix} J_kW & J_kWJ_k \\ -W & -WJ_k \end{pmatrix}$.

Now consider the square of M which gives the type B square,

$$M^2 = X_n \begin{pmatrix} 2wE_k & V^T \\ W & \hat{0}_k \end{pmatrix} X_nX_n \begin{pmatrix} 2wE_k & V^T \\ W & \hat{0}_k \end{pmatrix} X_n$$

$$= X_n \begin{pmatrix} 2wE_k & V^T \\ W & \hat{0}_k \end{pmatrix} \begin{pmatrix} 2wE_k & V^T \\ W & \hat{0}_k \end{pmatrix} X_n = X_n \begin{pmatrix} 4kw^2E_k + V^TW & \hat{0}_k \\ \hat{0}_k & WV^T \end{pmatrix} X_n.$$

For $\xi, \eta \in \mathbb{R}^k$ we can associate with

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}^T X_n M^2 X_n \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}^T \begin{pmatrix} 4kw^2E_k + V^TW & \hat{0}_k \\ \hat{0}_k & WV^T \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$= 4kw^2\xi^T E_k \xi + \xi^T V^T W \xi + \eta^T W V^T \eta$$

For a given $\eta, \xi \in \mathbb{R}^k$ the weight term is independent of u and v.

Considering the weightless parasymmetric case, such that $w = 0$, $y = v$, $x = \alpha u$ we find that

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}^T \begin{pmatrix} V^TW & \hat{0}_k \\ \hat{0}_k & WV^T \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}^T \begin{pmatrix} \alpha(u^Tu)vv^T & \hat{0}_k \\ \hat{0}_k & \alpha(v^Tv)uu^T \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$= \alpha(u^Tu)(v^T\xi)^2 + \alpha(v^Tv)(u^T\eta)^2,$$

and setting

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = X_{2k} Y = \frac{1}{\sqrt{2}} \begin{pmatrix} Y_1 + J_k Y_2 \\ J_k Y_1 - Y_2 \end{pmatrix}$$

with $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \in \mathbb{R}^{2k}$, we obtain the quadratic form

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix}^T X_{2k} M^2 X_{2k} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = Y^T X_{2k} X_{2k} M^2 X_{2k} X_{2k} = Y^T M^2 Y$$

$$= \frac{\alpha}{2}(u^Tu) \left(\begin{pmatrix} v \\ J_k v \end{pmatrix}^T Y \right)^2 + \frac{\alpha}{2}(v^Tv) \left(\begin{pmatrix} J_k u \\ -u \end{pmatrix}^T Y \right)^2.$$

Introducing the variables

$$z_1 = \begin{pmatrix} v \\ J_k v \end{pmatrix}^T Y \quad z_2 = \begin{pmatrix} J_k u \\ -u \end{pmatrix}^T Y$$

then yields the reduced quadratic form

$$q_1(z_1, z_2) = \frac{\alpha}{2}((u^Tu)z_1^2 + (v^Tv)z_2^2).$$

For any two vectors that make up the rank $1+1$ block or type S and S (i.e. not necessarily parasymmetric) we can similarly obtain corresponding quadratic form from the decomposition of M^2, which we now describe.

To begin with we consider any (right) eigenvector matrix $P \in \mathbb{R}^{2k \times 2k}$ which contains the two non-zero eigenvector columns of M^2 corresponding to the eigenvalue λ, with the other columns $b_3, \ldots, b_{2k}$ being the eigenvectors with eigenvalue 0.

In the consideration of the transpose,

$$Y = P\gamma = \sum_{j=1}^{2k} \gamma_j b_j$$

(with γ_j is the jth entry of the vector $\gamma \in \mathbb{R}^{2k}$) we then obtain the quadratic form

$$q(\gamma) = Y^T M^2 Y = \lambda(\gamma_1 b_1 + \gamma_2 b_2) \left(\gamma_1 b_1 + \gamma_2 b_2 + \sum_{j=3}^{2k} \gamma_j b_j^T \right)$$

$$= \lambda \left(b_1^T b_1 \gamma_1^2 + 2 b_1^T b_1 \gamma_1 \gamma_2 + b_2^T b_2 \gamma_2^2 + \sum_{j=3}^{2k} \left(b_j^T b_1 \gamma_j \gamma_1 + b_j^T b_2 \gamma_j \gamma_2 \right) \right).$$

For the parasymmetric case where M^2 is symmetric the quadratic form simplifies to the binary form

$$q_2(\gamma_1, \gamma_2) = Y^T M^2 Y = \lambda(b_1^T b_1 \gamma_1^2 + 2 b_1^T b_2 \gamma_1 \gamma_2 + b_2^T b_2 \gamma_2^2)$$

as $b_3, \ldots, b_{2k}$ are orthogonal to b_1, b_2.

Also, if b_1, b_2 are mutually orthogonal (i.e. not transformed eigenvectors from the constructed matrix P from above) then we have the simplified form

$$q_3(\gamma_1, \gamma_2) = Y^T M^2 Y = \lambda(b_1^T b_1 \gamma_1^2 + b_2^T b_2 \gamma_2^2).$$

Examples of these quadratic form constructions are given in the Appendices.

10 Further Work

Having now completed the present work, it is apparent that the subject of sum systems and sum-and-distance systems has many possible extensions such as considering $m_1 + \ldots + m_d$ sum-and-distance systems for a given integer n. Accompanying questions include whether there exist similar enumeration arguments and constructions for such larger systems. However at this present juncture we will reflect on the progress made in this thesis and leave these musings for another day.

Appendix 1: Examples

Example. *of representation Lemma 3.9*

Let $u = \begin{pmatrix} 2 \\ -3 \\ -1 \\ 2 \end{pmatrix}$ *and a type R square matrix* $M = \begin{pmatrix} 13 & 8 & 4 & -1 \\ 10 & 3 & 5 & -2 \\ 0 & 5 & -1 & 4 \\ -3 & 0 & 0 & 3 \end{pmatrix}$ *then it holds that*

$$(M + J_4 M)u = \left(\begin{pmatrix} 13 & 8 & 4 & -1 \\ 10 & 3 & 5 & -2 \\ 0 & 5 & -1 & 4 \\ -3 & 0 & 0 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 4 & 8 & 13 \\ -2 & 5 & 3 & 10 \\ 4 & -1 & 5 & 0 \\ 3 & 0 & 0 & -3 \end{pmatrix} \right) \begin{pmatrix} 2 \\ -3 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 12 & 12 & 12 \\ 8 & 8 & 8 & 8 \\ 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$(M^T + J_4 M^T)u = \left(\begin{pmatrix} 13 & 10 & 0 & -3 \\ 8 & 3 & 5 & 0 \\ 4 & 5 & -1 & 0 \\ -1 & -2 & 4 & 3 \end{pmatrix} + \begin{pmatrix} -3 & 0 & 10 & 13 \\ 0 & 5 & 3 & 8 \\ 0 & -1 & 5 & 4 \\ 3 & 4 & -2 & -1 \end{pmatrix} \right) u$$

$$= \begin{pmatrix} 10 & 10 & 10 & 10 \\ 8 & 8 & 8 & 8 \\ 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Example. *of block representation Lemma 3.10*

Let $y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $x = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $1_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $Z = \begin{pmatrix} 3 & 7 \\ 1 & 2 \end{pmatrix}$ *and* $w = 3$ *then*

$$X_n \begin{pmatrix} wE_2 & y1_k^T \\ 1_k x^T & Z \end{pmatrix} X_n = X_n \begin{pmatrix} 3 & 3 & 1 & 2 \\ 3 & 3 & 1 & 2 \\ 1 & 1 & -2 & 4 \\ 3 & 3 & 1 & 5 \end{pmatrix} X_n = \frac{1}{2} \begin{pmatrix} 13 & 8 & 4 & -1 \\ 10 & 3 & 5 & -2 \\ 0 & 5 & -1 & 4 \\ -3 & 0 & 0 & 3 \end{pmatrix}$$

is a type R *square matrix.*

Example. *of representation Lemma 3.11*

Let $u = \begin{pmatrix} 2 \\ -3 \\ -1 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} 4 \\ -1 \\ -2 \\ -1 \end{pmatrix}$ *and a type* V *square matrix* $M = \begin{pmatrix} 7 & 6 & 4 & 1 \\ 6 & 5 & 3 & 0 \\ 4 & 3 & 1 & -2 \\ 3 & 2 & 0 & -3 \end{pmatrix}$

then

$$\begin{pmatrix} 2 \\ -3 \\ -1 \\ 2 \end{pmatrix}^T \begin{pmatrix} 6 & 2 & 0 & 0 \\ 5 & 1 & -1 & -1 \\ 3 & -1 & -3 & -3 \\ 2 & -2 & -4 & -4 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ -2 \\ -1 \end{pmatrix} = 0$$

and

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 6 & 2 & 0 & 0 \\ 5 & 1 & -1 & -1 \\ 3 & -1 & -3 & -3 \\ 2 & -2 & -4 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0.$$

Example. *of block representation Lemma 3.12*

Let $a = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *and type* V *matrix* $Y = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$ *then*

$$X_n \begin{pmatrix} Y & 1_2 a^T \\ b1_2^T & \hat{0}_2 \end{pmatrix} X_n = X_n \begin{pmatrix} 2 & 3 & 1 & 3 \\ 2 & 3 & 1 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix} X_n = \frac{1}{2} \begin{pmatrix} 7 & 6 & 4 & 1 \\ 6 & 5 & 3 & 0 \\ 4 & 3 & 1 & -2 \\ 3 & 2 & 0 & -3 \end{pmatrix}$$

is a type V *matrix.*

Example. *of representation Lemma 3.11*

Let $u = \begin{pmatrix} 2 \\ -3 \\ -1 \\ 2 \end{pmatrix}$, $v = \begin{pmatrix} 4 \\ -1 \\ -2 \\ -1 \end{pmatrix}$ *and a type* $\hat{V}$ *square matrix* $M = \begin{pmatrix} 6 & 2 & 0 & 0 \\ 5 & 1 & -1 & -1 \\ 3 & -1 & -3 & -3 \\ 2 & -2 & -4 & -4 \end{pmatrix}$

then

$$\begin{pmatrix}2\\-3\\-1\\2\end{pmatrix}^T\begin{pmatrix}7&6&4&1\\6&5&3&0\\4&3&1&-2\\3&2&0&-3\end{pmatrix}\begin{pmatrix}4\\-1\\-2\\-1\end{pmatrix}=0$$

and

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}^T\begin{pmatrix}7&6&4&1\\6&5&3&0\\4&3&1&-2\\3&2&0&-3\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=0$$

Example. *of block representation 3.12*

Let $a=\begin{pmatrix}1\\3\end{pmatrix}$, $b=\begin{pmatrix}1\\2\end{pmatrix}$ *and type* $\hat{V}$ *matrix* $Y=\begin{pmatrix}1&-1\\1&-1\end{pmatrix}$ *then*

$$X_n\begin{pmatrix}Y & 1_2\ a^T\\ b\ 1_2^T & \hat{0}_2\end{pmatrix}X_n=X_n\begin{pmatrix}1&-1&1&3\\1&-1&1&3\\1&1&0&0\\2&2&0&0\end{pmatrix}X_n=\frac{1}{2}\begin{pmatrix}6&2&0&0\\5&1&-1&-1\\3&-1&-3&-3\\2&-2&-4&-4\end{pmatrix}.$$

Example. *of representation Lemma 7.2*

Let $u=\begin{pmatrix}2\\1\\3\\4\end{pmatrix}$, $v=\begin{pmatrix}-1\\0\\-1\\-2\end{pmatrix}$, $\S_4=\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}$ *and* $M=\begin{pmatrix}-1&-1&5&-1\\5&-3&-1&-3\\1&-3&7&-3\\5&-3&-1&-3\end{pmatrix}$ *then*

$$\begin{pmatrix}2\\1\\3\\4\end{pmatrix}^T\begin{pmatrix}-1&-1&5&-1\\5&-3&-1&-3\\1&-3&7&-3\\5&-3&-1&-3\end{pmatrix}\begin{pmatrix}-1\\0\\-1\\-2\end{pmatrix}=0$$

and

$$\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}\begin{pmatrix}-1&-1&5&-1\\5&-3&-1&-3\\1&-3&7&-3\\5&-3&-1&-3\end{pmatrix}\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}=0.$$

Example. *of block representation Lemma 7.3*

Let $a = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ *and weightless type* $\hat{M}$ *matrix* $Z = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}$ *then*

$$X_n \begin{pmatrix} \hat{0}_k & a\ \S_2^T \\ \S_k\ b^T & Z \end{pmatrix} X_n = X_n \begin{pmatrix} 0 & 0 & -2 & 2 \\ 0 & 0 & -3 & 3 \\ 1 & -2 & 2 & 1 \\ -1 & 2 & -1 & -2 \end{pmatrix} X_n = \begin{pmatrix} -1 & -1 & 5 & -1 \\ 5 & -3 & -1 & -3 \\ 1 & -3 & 7 & -3 \\ 5 & -3 & -1 & -3 \end{pmatrix}$$

Example. *of representation Lemma 7.5*

Let $u = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} -1 \\ 4 \\ -3 \\ 0 \end{pmatrix}$, $1_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ *and* $M = \begin{pmatrix} 6 & 2 & 0 & -8 \\ -1 & 3 & -3 & 1 \\ -7 & 3 & 1 & 3 \\ 2 & -8 & 2 & 4 \end{pmatrix}$

then

$$\begin{pmatrix} -2 \\ -1 \\ 3 \\ 0 \end{pmatrix}^T \begin{pmatrix} 6 & 2 & 0 & -8 \\ -1 & 3 & -3 & 1 \\ -7 & 3 & 1 & 3 \\ 2 & -8 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

and

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 6 & 2 & 0 & -8 \\ -1 & 3 & -3 & 1 \\ -7 & 3 & 1 & 3 \\ 2 & -8 & 2 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \\ -3 \\ 0 \end{pmatrix} = 0$$

Example. *of block representation Lemma 7.6*

Let $Y = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $V = \begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}$, $W = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$ *and* $Z = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ *then*

$$X_n \begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix} X_n = X_n \begin{pmatrix} 1 & -1 & -2 & 3 \\ -1 & 1 & 2 & -3 \\ 1 & -1 & 1 & 2 \\ -2 & 2 & 3 & 4 \end{pmatrix} X_n = \begin{pmatrix} 6 & 2 & 0 & -8 \\ -1 & 3 & -3 & 1 \\ -7 & 3 & 1 & 3 \\ 2 & -8 & 2 & 4 \end{pmatrix}.$$

Example. *Quadratic Forms*

Here we take the fundamental parasymmetric 2+2 *sum-and-distance system* $\{u, 4u\}$, *so that*

$$u = \{-1,\ 3\}, \quad x = 4u = \{-4, 12\} \quad \textit{and} \quad v = y = \{-1, 1\},$$

then

$$M = X_4 \begin{pmatrix} \hat{0}_2 & vu^T \\ xy^T & \hat{0}_2 \end{pmatrix} X_4 = \frac{1}{2} \begin{pmatrix} 15 & -13 & -11 & 9 \\ -7 & 5 & 3 & -1 \\ 1 & -1 & -5 & 7 \\ -9 & 11 & 13 & -15 \end{pmatrix},$$

and

$$X_4 M^2 X_4 = 4 \begin{pmatrix} 10 & -10 & 0 & 0 \\ -10 & 10 & 0 & 0 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & -6 & 18 \end{pmatrix}.$$

We deduce that the reduced quadratic form is given by

$$q_1(z_1, z_2) = \frac{\alpha}{2} \left((u^T u) z_1^2 + (v^T v) z_2^2 \right)$$

$$= \frac{4}{2} \left((u^T u) z_1^2 + (v^T v) z_2^2 \right) = 2((1+9)z_1^2 + (1+1)z_2^2)$$

$$= 4(5z_1^2 + z_2^2).$$

Example. *Here we take the fundamental $4+4$ parasymmetric sum-and-distance system $\{u, 8u\}$, so that*

$$u = \{-1, 3, 5, -7\}, \quad x = 8u = \{-8, 24, 40, -56\} \quad \textit{and} \quad v = y = \{-1, 1, 1, -1\}.$$

Then we have

$$M = X_8 \begin{pmatrix} \hat{0}_4 & vu^T \\ xy^T & \hat{0}_4 \end{pmatrix} X_8 = \frac{1}{2} \begin{pmatrix} 63 & -61 & -59 & 57 & 55 & -53 & -51 & 49 \\ -47 & 45 & 43 & -41 & -39 & 37 & 35 & -33 \\ -31 & 29 & 27 & -25 & -23 & 21 & 19 & -17 \\ 15 & -13 & -11 & 9 & 7 & -5 & -3 & 1 \\ -1 & 3 & 5 & -7 & -9 & 11 & 13 & -15 \\ 17 & -19 & -21 & 23 & 25 & -27 & -29 & 31 \\ 33 & -35 & -37 & 39 & 41 & -43 & -45 & 47 \\ -49 & 51 & 53 & -55 & -57 & 59 & 61 & -63 \end{pmatrix},$$

and

$$X_8 M^2 X_8 = \begin{pmatrix} 84 & -84 & -84 & 84 & 0 & 0 & 0 & 0 \\ -84 & 84 & 84 & -84 & 0 & 0 & 0 & 0 \\ -84 & 84 & 84 & -84 & 0 & 0 & 0 & 0 \\ 84 & -84 & -84 & 84 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -12 & -20 & 28 \\ 0 & 0 & 0 & 0 & -12 & 36 & 60 & -84 \\ 0 & 0 & 0 & 0 & -20 & 60 & 100 & -140 \\ 0 & 0 & 0 & 0 & 28 & -84 & -140 & 196 \end{pmatrix}.$$

In this instance we deduce that the reduced quadratic form is given by

$$q_1(z_1, z_2) = \frac{\alpha}{2}\left((u^T u)z_1^2 + (v^T v)z_2^2\right)$$

$$= 4((1 + 9 + 25 + 49)z_1^2 + 4z_2^2) = 4(84z_1^2 + 4z_2^2) = 16(21z_1^2 + z_2^2).$$

Example. *Here we take the* $4{+}4$ *parasymmetric sum-and-distance systems* $\{u, 2u\}$ *with*

$$u = \{11, -13, -19, 21\}, \quad x = 2u = \{22, -26, -38, 42\} \quad \text{and} \quad v = y = \{-1, 1, 1, -1\}$$

and

$$M = X_8 \begin{pmatrix} \hat{0}_4 & vu^T \\ xy^T & \hat{0}_4 \end{pmatrix} X_8 = \frac{1}{2} \begin{pmatrix} -63 & 61 & 55 & -53 & -31 & 29 & 23 & -21 \\ 59 & -57 & -51 & 49 & 27 & -25 & -19 & 17 \\ 47 & -45 & -39 & 37 & 15 & -13 & -7 & 5 \\ -43 & 41 & 35 & -33 & -11 & 9 & 3 & -1 \\ 1 & -3 & -9 & 11 & 33 & -35 & -41 & 43 \\ -5 & 7 & 13 & -15 & -37 & 39 & 45 & -47 \\ -17 & 19 & 25 & -27 & -49 & 51 & 57 & -59 \\ 21 & -23 & -29 & 31 & 53 & -55 & -61 & 63 \end{pmatrix}.$$

Then M^2 *has the block representation*

$$M^2 = 8X_8 \begin{pmatrix} 273 & -273 & -273 & 273 & 0 & 0 & 0 & 0 \\ -273 & 273 & 273 & -273 & 0 & 0 & 0 & 0 \\ -273 & 273 & 273 & -273 & 0 & 0 & 0 & 0 \\ 273 & -273 & -273 & 273 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 121 & -143 & -209 & 231 \\ 0 & 0 & 0 & 0 & -143 & 169 & 247 & -273 \\ 0 & 0 & 0 & 0 & -209 & 247 & 361 & -399 \\ 0 & 0 & 0 & 0 & 231 & -273 & -399 & 411 \end{pmatrix} X_8.$$

In this instance the formula given by q_1*, gives the reduced quadratic form*

$$q_1(z_1, z_2) = 1092z_1^2 + 4z_2^2.$$

The matrix M^2 *has the two eigenvectors*

$$b_1 = \begin{pmatrix} v \\ J_k v \end{pmatrix}, \quad b_2 = \begin{pmatrix} -J_k u \\ u \end{pmatrix}$$

(as we divided b_2 *by* α*) for eigenvalue* $\lambda = \alpha(u^T u)(v^T v) = 8736$*, so* $b_1^T b_2 = 2u^T u = 2184$*, and these vectors are orthogonal. We can then write the quadratic form such that*

$$q_3(\gamma_1, \gamma_2) = 8736(8\gamma_1^2 + 2184\gamma_2^2) = 17472(4\gamma_1^2 + 1092\gamma_2^2).$$

Applying our two-sided eigenvector matrix construction from theorem 9.7, we obtain the rational symmetric (left and right) eigenvector matrix

$$P = \frac{1}{11}\begin{pmatrix} 11 & 0 & 0 & -16 & 5 & 0 & 0 & 0 \\ 0 & 11 & 0 & 15 & -4 & 0 & 0 & 0 \\ 0 & 0 & 11 & 12 & -1 & 0 & 0 & 0 \\ -16 & 15 & 12 & -11 & 0 & -1 & -4 & 5 \\ 5 & -4 & -1 & 0 & -11 & 12 & 15 & -16 \\ 0 & 0 & 0 & -1 & 12 & 11 & 0 & 0 \\ 0 & 0 & 0 & -4 & 15 & 0 & 11 & 0 \\ 0 & 0 & 0 & 5 & -16 & 0 & 0 & 11 \end{pmatrix}.$$

Taking the middle columns of this matrix for the (non-orthogonal) eigenvectors $b_1, b_2 = J_8 b_1$, *we now have* $b_1^T b_1 = b_2^T b_2 = \frac{788}{121}$, $b_1^T b_2 = -\frac{304}{121}$ *and the above formula gives*

$$q_2(\gamma_1,\gamma_2) = 8736\left(\frac{788}{121}\gamma_1^2 - \frac{608}{121}\gamma_1\gamma_2 + \frac{788}{121}\gamma_2^2\right) = \frac{34944}{232}(197\gamma_1^2 - 152\gamma_1\gamma_2) + 197\gamma_2^2.$$

The inverse for the eigenvector matrix can similarly be obtained from the formula

$$P^{-1} = diag(1_3, 0_2, 1_3) - \frac{M}{\lambda} = \frac{8}{\lambda}\begin{pmatrix} 735 & 336 & 273 & -252 & -21 & 0 & -63 & 84 \\ 336 & 775 & -260 & 241 & 32 & -13 & 44 & -63 \\ 273 & -260 & 871 & 208 & 65 & -52 & -13 & 0 \\ -252 & 241 & 208 & -197 & -76 & 65 & 32 & -21 \\ -21 & 32 & 65 & -76 & -179 & 208 & 241 & -252 \\ 0 & -13 & -52 & 65 & 208 & 871 & -260 & 273 \\ -63 & 44 & -13 & 32 & 241 & -260 & 775 & 336 \\ 84 & -63 & 0 & -21 & 252 & 273 & 336 & 735 \end{pmatrix},$$

so that $P^{-1}M^2P = PM^2P^{-1} = diag(0_3, \lambda, \lambda, 0_3)$.

Although the three quadratic forms q_1, q_2 *and* q_3 *are different they are in fact equivalent to each other as a transformation matrix exists with determinant* 1 *for the co-ordinate transformation.*

Example. *For our last example, we consider* $\{u, x\}$ *non-parasymmetric sum-and-distance system given by,*

$$u = \{10, -14, -18, 22\}, \quad x = \{23, -25, -39, 41\} \quad and \quad v = y = \{-1, 1, 1, -1\},$$

so that

$$M = X_8 \begin{pmatrix} \hat{0}_4 & vu^T \\ xy^T & \hat{0}_4 \end{pmatrix} X_8 = \begin{pmatrix} -63 & 59 & 55 & -51 & -31 & 27 & 23 & -19 \\ 61 & -57 & -53 & 49 & 29 & -25 & -21 & 17 \\ 47 & -43 & -39 & 35 & 15 & -11 & -7 & 3 \\ -45 & 41 & 37 & -33 & -13 & 9 & 5 & -1 \\ 1 & -5 & 9 & 13 & 33 & -37 & -41 & 45 \\ -3 & 7 & 11 & -15 & -35 & 39 & 43 & -47 \\ -17 & 21 & 25 & -29 & -49 & 53 & 57 & -61 \\ 19 & -23 & -27 & 31 & 51 & -55 & -59 & 63 \end{pmatrix}.$$

Then M^2 has the non-symmetric block representation

$$M^2 = 8X_8 \begin{pmatrix} 273 & -273 & -273 & 273 & 0 & 0 & 0 & 0 \\ -273 & 273 & 273 & -273 & 0 & 0 & 0 & 0 \\ -273 & 273 & 273 & -273 & 0 & 0 & 0 & 0 \\ 273 & -273 & -273 & 273 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 115 & -161 & -207 & 253 \\ 0 & 0 & 0 & 0 & -125 & 175 & 225 & -275 \\ 0 & 0 & 0 & 0 & -195 & 273 & 351 & -429 \\ 0 & 0 & 0 & 0 & 205 & -287 & -369 & 451 \end{pmatrix} X_8.$$

Here the non-zero eigenvalue is given by $\lambda = (u^T x)(y^T v) = 8736$, and the (right and left) eigenvector matrix P has the form

$$P = \frac{1}{115} \begin{pmatrix} 115 & 0 & 0-160 & 45 & 0 & 0 & 0 & \\ 0 & 115 & 0 & 155 & -40 & 0 & 0 & 0 \\ 0 & 0 & 115 & 120 & -5 & 0 & 0 & 0 \\ -184 & 161 & 138 & -115 & 0 & -23 & -46 & 69 \\ 69 & -46 & -23 & 0 & -115 & 138 & 161 & -184 \\ 0 & 0 & 0 & -5 & 120 & 115 & 0 & 0 \\ 0 & 0 & 0 & -40 & 155 & 0 & 115 & 0 \\ 0 & 0 & 0 & 45 & 160 & 0 & 0 & 115 \end{pmatrix}.$$

Now ordering the columns of P as $P = (b_3, b_4, b_5, b_1, b_2, b_8, b_7, b_6)$ to define eigenvectors $b_1, b_2, \ldots, b_8$, we see that $b_2 = J_8 b_1$ and $b_{j+3} = J_8 b_j$ ($j \in \{3, 5, 6\}$), and calculate further $b_1^T b_1 = b_2^T b_2 = \frac{80900}{115^2}, -\frac{28000}{115^2}$ as well as $b_3^T b_1 = -b_3^T b_2 = -b_6^T b_1 = b_6^T b_2 = \frac{2760}{115^2}$, $b_4^T b_1 = b_4^T b_2 = -b_7^T b_1 = b_7^T b_2 = -\frac{690}{115^2}$ and $b_5^T b_1 = b_5^T b_2 = -b_8^T b_1 = b_8^T b_2 = -\frac{2070}{115^2}$. Hence the quadratic form associated with this sum-and-distance system is given by

$$\sum_{j=3} (b_j^T b_1 \gamma_j \gamma_1 + b_j^T b_2 \gamma_j \gamma_2) = \sum_{j=3}^{5} (b_j^T b_1 (\gamma_j - \gamma_{j+3}) \gamma_1 + b_j^T b_2 (\gamma_j - \gamma_{j+3}) \gamma_2)$$

$$= \sum_{j=3} (b_j^T b_1)(\gamma_j - \gamma_{j+3})(\gamma_1 - \gamma_2),$$

for our example

$$\sum_{j=3}^{5} (b_j^T b_1)(\gamma_j - \gamma_{j+3}) = \frac{6}{115}(4(\gamma_3 - \gamma_6) - (\gamma_4 - \gamma_7) - 3(\gamma_5 - \gamma_8)),$$

writing the linear combination $4(\gamma_3 - \gamma_6) - (\gamma_4 - \gamma_7) - 3(\gamma_5 - \gamma_8)$ *as one new variable* γ_0, *we obtain the ternary quadratic form*

$$q(\gamma_1, \gamma_2, \gamma_0) = 8736 \left(\frac{4}{529}(809\gamma_1^2 - 560\gamma_1\gamma_2 + 809\gamma_2^2) + \frac{6}{115}\gamma_0(\gamma_1 - \gamma_2) \right).$$

Appendix 2

Divisor Paths, Non-Inclusive Sum-and-Distance Systems and Even Sided Reversible Squares The following tables list all even sided reversible squares for $n \in \{2,4,6,8,10,12\}$, there corresponding divisor path sets, block representations and sum-and-distance systems.

Table 1: n=2

	Divisor Paths	Reversible Square	Block Representation	Sum-and-Distance System
1	$\{(2),\ (2)\}$	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	$\begin{pmatrix} 5 & -1 \\ -2 & 0 \end{pmatrix}$	$\{\{1\},\ \{2\}\}$

Table 2: n=4

	Divisor Paths	Reversible Square	Block Representation	Sum-and-Distance System
1	$\{(4),\ (4)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$	$\begin{pmatrix} 17 & 17 & -1 & -3 \\ 17 & 17 & -1 & -3 \\ -4 & -4 & 0 & 0 \\ -12 & -12 & 0 & 0 \end{pmatrix}$	$\{\{1,3\},\ \{4,12\}\} **$
2	$\{(4),\ (2)\}$	$\begin{pmatrix} 1 & 2 & 9 & 10 \\ 3 & 4 & 11 & 12 \\ 5 & 6 & 13 & 14 \\ 7 & 8 & 15 & 16 \end{pmatrix}$	$\begin{pmatrix} 17 & 17 & -7 & -9 \\ 17 & 17 & -7 & -9 \\ -2 & -2 & 0 & 0 \\ -6 & -6 & 0 & 0 \end{pmatrix}$	$\{\{2,6\},\ \{7,9\}\}$
3	$\{(2,4),\ (2,4)\}$	$\begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ 9 & 10 & 13 & 14 \\ 11 & 12 & 15 & 16 \end{pmatrix}$	$\begin{pmatrix} 17 & 17 & -3 & -5 \\ 17 & 17 & -3 & -5 \\ -6 & -6 & 0 & 0 \\ -10 & -10 & 0 & 0 \end{pmatrix}$	$\{\{3,5\},\ \{6,10\}\}$

Table 3: n=6

	Divisor Paths	Reversible Square	Block Representation	Sum-and-Distance System
1	$\{(6),\ (6)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 \\ 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 \\ 31 & 32 & 33 & 34 & 35 & 36 \end{pmatrix}$	$\begin{pmatrix} 37 & 37 & 37 & -1 & -3 & -5 \\ 37 & 37 & 37 & -1 & -3 & -5 \\ 37 & 37 & 37 & -1 & -3 & -5 \\ -6 & -6 & -6 & 0 & 0 & 0 \\ -18 & -18 & -18 & 0 & 0 & 0 \\ -30 & -30 & -30 & 0 & 0 & 0 \end{pmatrix}$	$\{\{1,3,5\},\ \{6,18,30\}\}$
2	$\{(6),(3)\}$	$\begin{pmatrix} 1 & 2 & 3 & 19 & 20 & 21 \\ 4 & 5 & 6 & 22 & 23 & 24 \\ 7 & 8 & 9 & 25 & 26 & 27 \\ 10 & 11 & 12 & 28 & 29 & 30 \\ 13 & 14 & 15 & 31 & 32 & 33 \\ 16 & 17 & 18 & 34 & 35 & 36 \end{pmatrix}$	$\begin{pmatrix} 37 & 37 & 37 & -16 & -18 & -20 \\ 37 & 37 & 37 & -16 & -18 & -20 \\ 37 & 37 & 37 & -16 & -18 & -20 \\ -3 & -3 & -3 & 0 & 0 & 0 \\ -9 & -9 & -9 & 0 & 0 & 0 \\ -15 & -15 & -15 & 0 & 0 & 0 \end{pmatrix}$	$\{\{16,18,20\},\ \{3,9,15\}\}$
3	$\{(6),(2)\}$	$\begin{pmatrix} 1 & 2 & 13 & 14 & 25 & 26 \\ 3 & 4 & 15 & 16 & 27 & 28 \\ 5 & 6 & 17 & 18 & 29 & 30 \\ 7 & 8 & 19 & 20 & 31 & 32 \\ 9 & 10 & 21 & 22 & 33 & 34 \\ 11 & 12 & 23 & 24 & 35 & 36 \end{pmatrix}$	$\begin{pmatrix} 37 & 37 & 37 & -1 & -23 & -25 \\ 37 & 37 & 37 & -1 & -23 & -25 \\ 37 & 37 & 37 & -1 & -23 & -25 \\ -2 & -2 & -2 & 0 & 0 & 0 \\ -6 & -6 & -6 & 0 & 0 & 0 \\ -10 & -10 & -10 & 0 & 0 & 0 \end{pmatrix}$	$\{\{1,23,25\},\ \{2,6,10\}\}$
4	$\{(2,6),\ (2,6)\}$	$\begin{pmatrix} 1 & 2 & 5 & 6 & 9 & 10 \\ 3 & 4 & 7 & 8 & 11 & 12 \\ 13 & 14 & 17 & 18 & 21 & 22 \\ 15 & 16 & 19 & 20 & 23 & 24 \\ 25 & 26 & 29 & 30 & 33 & 34 \\ 27 & 28 & 31 & 32 & 35 & 36 \end{pmatrix}$	$\begin{pmatrix} 37 & 37 & 37 & -1 & -7 & -9 \\ 37 & 37 & 37 & -1 & -7 & -9 \\ 37 & 37 & 37 & -1 & -7 & -9 \\ -2 & -2 & -2 & 0 & 0 & 0 \\ -22 & -22 & -22 & 0 & 0 & 0 \\ -26 & -26 & -26 & 0 & 0 & 0 \end{pmatrix}$	$\{\{1,7,9\},\{2,22,26\}\}$

	Divisor Paths	Reversible Square	Block Representation	Sum-and-Distance System
5	$\{(2,6),\ (3,6)\}$	$\begin{pmatrix} 1 & 2 & 3 & 7 & 8 & 9 \\ 4 & 5 & 6 & 10 & 11 & 12 \\ 13 & 14 & 15 & 19 & 20 & 21 \\ 16 & 17 & 18 & 22 & 23 & 24 \\ 25 & 26 & 27 & 31 & 32 & 33 \\ 28 & 29 & 30 & 34 & 35 & 36 \end{pmatrix}$	$\begin{pmatrix} 37 & 37 & 37 & -4 & -6 & -8 \\ 37 & 37 & 37 & -4 & -6 & -8 \\ 37 & 37 & 37 & -4 & -6 & -8 \\ -3 & -3 & -3 & 0 & 0 & 0 \\ -21 & -21 & -21 & 0 & 0 & 0 \\ -27 & -27 & -27 & 0 & 0 & 0 \end{pmatrix}$	$\{\{4,6,8\},\ \{3,21,27\}\}$
6	$\{(3,6),\ (2,6)\}$	$\begin{pmatrix} 1 & 2 & 7 & 8 & 13 & 14 \\ 3 & 4 & 9 & 10 & 15 & 16 \\ 5 & 6 & 11 & 12 & 17 & 18 \\ 19 & 20 & 25 & 26 & 31 & 32 \\ 21 & 22 & 27 & 28 & 33 & 34 \\ 23 & 24 & 29 & 30 & 35 & 36 \end{pmatrix}$	$\begin{pmatrix} 37 & 37 & 37 & -1 & -11 & -13 \\ 37 & 37 & 37 & -1 & -11 & -13 \\ 37 & 37 & 37 & -1 & -11 & -13 \\ -14 & -14 & -14 & 0 & 0 & 0 \\ -18 & -18 & -18 & 0 & 0 & 0 \\ -22 & -22 & -22 & 0 & 0 & 0 \end{pmatrix}$	$\{\{1,11,13\},\ \{14,18,22\}\}$
7	$\{(3,6),\ (3,6)\}$	$\begin{pmatrix} 1 & 2 & 3 & 10 & 11 & 12 \\ 4 & 5 & 6 & 13 & 14 & 15 \\ 7 & 8 & 9 & 16 & 17 & 18 \\ 19 & 20 & 21 & 28 & 29 & 30 \\ 22 & 23 & 24 & 31 & 32 & 33 \\ 25 & 26 & 27 & 34 & 35 & 36 \end{pmatrix}$	$\begin{pmatrix} 37 & 37 & 37 & -7 & -9 & -11 \\ 37 & 37 & 37 & -7 & -9 & -11 \\ 37 & 37 & 37 & -7 & -9 & -11 \\ -12 & -12 & -12 & 0 & 0 & 0 \\ -18 & -18 & -18 & 0 & 0 & 0 \\ -24 & -24 & -24 & 0 & 0 & 0 \end{pmatrix}$	$\{\{7,9,11\},\ \{12,18,24\}\}$

Table 4: n=8

	Divisor Paths	Reversible Square	Block Representation	Sum-and-Distance System
1	$\{(8),\ (8)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 \\ 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 \\ 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 \end{pmatrix}$	$\begin{pmatrix} 65 & 65 & 65 & 65 & -1 & -3 & -5 & -7 \\ 65 & 65 & 65 & 65 & -1 & -3 & -5 & -7 \\ 65 & 65 & 65 & 65 & -1 & -3 & -5 & -7 \\ 65 & 65 & 65 & 65 & -1 & -3 & -5 & -7 \\ -8 & -8 & -8 & -8 & 0 & 0 & 0 & 0 \\ -24 & -24 & -24 & -24 & 0 & 0 & 0 & 0 \\ -40 & -40 & -40 & -40 & 0 & 0 & 0 & 0 \\ -56 & -56 & -56 & -56 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{1,3,5,7\},\ \{8,24,40,56\}\}$
2	$\{(8),\ (2)\}$	$\begin{pmatrix} 1 & 2 & 17 & 18 & 33 & 34 & 49 & 50 \\ 3 & 4 & 19 & 20 & 35 & 36 & 51 & 52 \\ 5 & 6 & 21 & 22 & 37 & 38 & 53 & 54 \\ 7 & 8 & 23 & 24 & 39 & 40 & 55 & 56 \\ 9 & 10 & 25 & 26 & 41 & 42 & 57 & 58 \\ 11 & 12 & 27 & 28 & 43 & 44 & 59 & 60 \\ 13 & 14 & 29 & 30 & 45 & 46 & 61 & 62 \\ 15 & 16 & 31 & 32 & 47 & 48 & 63 & 64 \end{pmatrix}$	$\begin{pmatrix} 65 & 65 & 65 & 65 & -15 & -17 & -47 & -49 \\ 65 & 65 & 65 & 65 & -15 & -17 & -47 & -49 \\ 65 & 65 & 65 & 65 & -15 & -17 & -47 & -49 \\ 65 & 65 & 65 & 65 & -15 & -17 & -47 & -49 \\ -2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 \\ -6 & -6 & -6 & -6 & 0 & 0 & 0 & 0 \\ -10 & -10 & -10 & -10 & 0 & 0 & 0 & 0 \\ -14 & -14 & -14 & -14 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{15,17,47,49\},\ \{2,6,10,14\}\}$
3	$\{(8),\ (4)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 33 & 34 & 35 & 36 \\ 5 & 6 & 7 & 8 & 37 & 38 & 39 & 40 \\ 9 & 10 & 11 & 12 & 41 & 42 & 43 & 44 \\ 13 & 14 & 15 & 16 & 45 & 46 & 47 & 48 \\ 17 & 18 & 19 & 20 & 49 & 50 & 51 & 52 \\ 21 & 22 & 23 & 24 & 53 & 54 & 55 & 56 \\ 25 & 26 & 27 & 28 & 57 & 58 & 59 & 60 \\ 29 & 30 & 31 & 32 & 61 & 62 & 63 & 64 \end{pmatrix}$	$\begin{pmatrix} 65 & 65 & 65 & 65 & -29 & -31 & -33 & -35 \\ 65 & 65 & 65 & 65 & -29 & -31 & -33 & -35 \\ 65 & 65 & 65 & 65 & -29 & -31 & -33 & -35 \\ 65 & 65 & 65 & 65 & -29 & -31 & -33 & -35 \\ -4 & -4 & -4 & -4 & 0 & 0 & 0 & 0 \\ -12 & -12 & -12 & -12 & 0 & 0 & 0 & 0 \\ -20 & -20 & -20 & -20 & 0 & 0 & 0 & 0 \\ -28 & -28 & -28 & -28 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{29,31,33,35\},\ \{4,12,20,28\}\}$
4	$\{(2,8),\ (2,8)\}$	$\begin{pmatrix} 1 & 2 & 5 & 6 & 9 & 10 & 13 & 14 \\ 3 & 4 & 7 & 8 & 11 & 12 & 15 & 16 \\ 17 & 18 & 21 & 22 & 25 & 26 & 29 & 30 \\ 19 & 20 & 23 & 24 & 27 & 28 & 31 & 32 \\ 33 & 34 & 37 & 38 & 41 & 42 & 45 & 46 \\ 35 & 36 & 39 & 40 & 43 & 44 & 47 & 48 \\ 49 & 50 & 53 & 54 & 57 & 58 & 61 & 62 \\ 51 & 52 & 55 & 56 & 59 & 60 & 63 & 64 \end{pmatrix}$	$\begin{pmatrix} 65 & 65 & 65 & 65 & -3 & -5 & -11 & -13 \\ 65 & 65 & 65 & 65 & -3 & -5 & -11 & -13 \\ 65 & 65 & 65 & 65 & -3 & -5 & -11 & -13 \\ 65 & 65 & 65 & 65 & -3 & -5 & -11 & -13 \\ -14 & -14 & -14 & -14 & 0 & 0 & 0 & 0 \\ -18 & -18 & -18 & -18 & 0 & 0 & 0 & 0 \\ -46 & -46 & -46 & -46 & 0 & 0 & 0 & 0 \\ -50 & -50 & -50 & -50 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{3,5,11,13\},\ \{14,18,46,50\}\}$

	Divisor Paths	Reversible Square	Block Representation	Sum-and-Distance System
5	$\{(2,8),\ (4,8)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 & 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 & 25 & 26 & 27 & 28 \\ 21 & 22 & 23 & 24 & 29 & 30 & 31 & 32 \\ 33 & 34 & 35 & 36 & 41 & 42 & 43 & 44 \\ 37 & 38 & 39 & 40 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 57 & 58 & 59 & 60 \\ 53 & 54 & 55 & 56 & 61 & 62 & 63 & 64 \end{pmatrix}$	$\begin{pmatrix} 65 & 65 & 65 & 65 & -5 & -7 & -9 & -11 \\ 65 & 65 & 65 & 65 & -5 & -7 & -9 & -11 \\ 65 & 65 & 65 & 65 & -5 & -7 & -9 & -11 \\ 65 & 65 & 65 & 65 & -5 & -7 & -9 & -11 \\ -12 & -12 & -12 & -12 & 0 & 0 & 0 & 0 \\ -20 & -20 & -20 & -20 & 0 & 0 & 0 & 0 \\ -44 & -44 & -44 & -44 & 0 & 0 & 0 & 0 \\ -52 & -52 & -52 & -52 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{5,7,9,11\},\ \{12,20,44,52\}\}$
6	$\{(8),\ (2,4)\}$	$\begin{pmatrix} 1 & 2 & 5 & 6 & 33 & 34 & 37 & 38 \\ 3 & 4 & 7 & 8 & 35 & 36 & 39 & 40 \\ 9 & 10 & 13 & 14 & 41 & 42 & 45 & 46 \\ 11 & 12 & 15 & 16 & 43 & 44 & 47 & 48 \\ 17 & 18 & 21 & 22 & 49 & 50 & 53 & 54 \\ 19 & 20 & 23 & 24 & 51 & 52 & 55 & 56 \\ 25 & 26 & 29 & 30 & 57 & 58 & 61 & 62 \\ 27 & 28 & 31 & 32 & 59 & 60 & 63 & 64 \end{pmatrix}$	$\begin{pmatrix} 65 & 65 & 65 & 65 & -27 & -29 & -35 & -37 \\ 65 & 65 & 65 & 65 & -27 & -29 & -35 & -37 \\ 65 & 65 & 65 & 65 & -27 & -29 & -35 & -37 \\ 65 & 65 & 65 & 65 & -27 & -29 & -35 & -37 \\ -6 & -6 & -6 & -6 & 0 & 0 & 0 & 0 \\ -10 & -10 & -10 & -10 & 0 & 0 & 0 & 0 \\ -22 & -22 & -22 & -22 & 0 & 0 & 0 & 0 \\ -26 & -26 & -26 & -26 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{27,29,35,37\},\ \{6,10,22,26\}\}$
7	$\{(4,8),\ (2,8)\}$	$\begin{pmatrix} 1 & 2 & 9 & 10 & 17 & 18 & 25 & 26 \\ 3 & 4 & 11 & 12 & 19 & 20 & 27 & 28 \\ 5 & 6 & 13 & 14 & 21 & 22 & 29 & 30 \\ 7 & 8 & 15 & 16 & 23 & 24 & 31 & 32 \\ 33 & 34 & 41 & 42 & 49 & 50 & 57 & 58 \\ 35 & 36 & 43 & 44 & 51 & 52 & 59 & 60 \\ 37 & 38 & 45 & 46 & 53 & 54 & 61 & 62 \\ 39 & 40 & 47 & 48 & 55 & 56 & 63 & 64 \end{pmatrix}$	$\begin{pmatrix} 65 & 65 & 65 & 65 & -7 & -9 & -23 & -25 \\ 65 & 65 & 65 & 65 & -7 & -9 & -23 & -25 \\ 65 & 65 & 65 & 65 & -7 & -9 & -23 & -25 \\ 65 & 65 & 65 & 65 & -7 & -9 & -23 & -25 \\ -26 & -26 & -26 & -26 & 0 & 0 & 0 & 0 \\ -30 & -30 & -30 & -30 & 0 & 0 & 0 & 0 \\ -34 & -34 & -34 & -34 & 0 & 0 & 0 & 0 \\ -38 & -38 & -38 & -38 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{7,9,23,25\},\ \{26,30,34,38\}\}$
8	$\{(4,8),\ (4,8)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 17 & 18 & 19 & 20 \\ 5 & 6 & 7 & 8 & 21 & 22 & 23 & 24 \\ 9 & 10 & 11 & 12 & 25 & 26 & 27 & 28 \\ 13 & 14 & 15 & 16 & 29 & 30 & 31 & 32 \\ 33 & 34 & 35 & 36 & 49 & 50 & 51 & 52 \\ 37 & 38 & 39 & 40 & 53 & 54 & 55 & 56 \\ 41 & 42 & 43 & 44 & 57 & 58 & 59 & 60 \\ 45 & 46 & 47 & 48 & 61 & 62 & 63 & 64 \end{pmatrix}$	$\begin{pmatrix} 65 & 65 & 65 & 65 & -13 & -15 & -17 & -19 \\ 65 & 65 & 65 & 65 & -13 & -15 & -17 & -19 \\ 65 & 65 & 65 & 65 & -13 & -15 & -17 & -19 \\ 65 & 65 & 65 & 65 & -13 & -15 & -17 & -19 \\ -20 & -20 & -20 & -20 & 0 & 0 & 0 & 0 \\ -28 & -28 & -28 & -28 & 0 & 0 & 0 & 0 \\ -36 & -36 & -36 & -36 & 0 & 0 & 0 & 0 \\ -44 & -44 & -44 & -44 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{13,15,17,19\},\ \{20,28,36,44\}\}$

	Divisor Paths	Reversible Square	Block Representation	Sum-and-Distance System
5	$\{(4,8),\ (2,4)\}$	$\begin{pmatrix} 1 & 2 & 9 & 10 & 33 & 34 & 41 & 42 \\ 3 & 4 & 11 & 12 & 35 & 36 & 43 & 44 \\ 5 & 6 & 13 & 14 & 37 & 38 & 45 & 46 \\ 7 & 8 & 15 & 16 & 39 & 40 & 47 & 48 \\ 17 & 18 & 25 & 26 & 49 & 50 & 57 & 58 \\ 19 & 20 & 27 & 28 & 51 & 52 & 59 & 60 \\ 21 & 22 & 29 & 30 & 53 & 54 & 61 & 62 \\ 23 & 24 & 31 & 32 & 55 & 56 & 63 & 64 \end{pmatrix}$	$\begin{pmatrix} 65 & 65 & 65 & 65 & -23 & -25 & -39 & -41 \\ 65 & 65 & 65 & 65 & -23 & -25 & -39 & -41 \\ 65 & 65 & 65 & 65 & -23 & -25 & -39 & -41 \\ 65 & 65 & 65 & 65 & -23 & -25 & -39 & -41 \\ -10 & -10 & -10 & -10 & 0 & 0 & 0 & 0 \\ -14 & -14 & -14 & -14 & 0 & 0 & 0 & 0 \\ -18 & -18 & -18 & -18 & 0 & 0 & 0 & 0 \\ -22 & -22 & -22 & -22 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{23,25,39,41\},\ \{10,14,18,22\}\}$
6	$\{(2,4,8),\ (2,4,8)\}$	$\begin{pmatrix} 1 & 2 & 5 & 6 & 17 & 18 & 21 & 22 \\ 3 & 4 & 7 & 8 & 19 & 20 & 23 & 24 \\ 9 & 10 & 13 & 14 & 25 & 26 & 29 & 30 \\ 11 & 12 & 15 & 16 & 27 & 28 & 31 & 32 \\ 33 & 34 & 37 & 38 & 49 & 50 & 53 & 54 \\ 35 & 36 & 39 & 40 & 51 & 52 & 55 & 56 \\ 41 & 42 & 45 & 46 & 57 & 58 & 61 & 62 \\ 43 & 44 & 47 & 48 & 59 & 60 & 63 & 64 \end{pmatrix}$	$\begin{pmatrix} 65 & 65 & 65 & 65 & -11 & -13 & -19 & -21 \\ 65 & 65 & 65 & 65 & -11 & -13 & -19 & -21 \\ 65 & 65 & 65 & 65 & -11 & -13 & -19 & -21 \\ 65 & 65 & 65 & 65 & -11 & -13 & -19 & -21 \\ -22 & -22 & -22 & -22 & 0 & 0 & 0 & 0 \\ -26 & -26 & -26 & -26 & 0 & 0 & 0 & 0 \\ -38 & -38 & -38 & -38 & 0 & 0 & 0 & 0 \\ -42 & -42 & -42 & -42 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{11,13,19,21\},\ \{22,26,38,42\}\}$

Table 5: n=10

	Divisor Paths	Reversible Square	Block Representation	Sum-and-Distance System
1	$\{(10),(10)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 \\ 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 \\ 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 & 49 & 50 \\ 51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 & 59 & 60 \\ 61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 & 69 & 70 \\ 71 & 72 & 73 & 74 & 75 & 76 & 77 & 78 & 79 & 80 \\ 81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 & 89 & 90 \\ 91 & 92 & 93 & 94 & 95 & 96 & 97 & 98 & 99 & 100 \end{pmatrix}$	$\begin{pmatrix} 101 & 101 & 101 & 101 & 101 & -1 & -3 & -5 & -7 & -9 \\ 101 & 101 & 101 & 101 & 101 & -1 & -3 & -5 & -7 & -9 \\ 101 & 101 & 101 & 101 & 101 & -1 & -3 & -5 & -7 & -9 \\ 101 & 101 & 101 & 101 & 101 & -1 & -3 & -5 & -7 & -9 \\ 101 & 101 & 101 & 101 & 101 & -1 & -3 & -5 & -7 & -9 \\ -10 & -10 & -10 & -10 & -10 & 0 & 0 & 0 & 0 & 0 \\ -30 & -30 & -30 & -30 & -30 & 0 & 0 & 0 & 0 & 0 \\ -50 & -50 & -50 & -50 & -50 & 0 & 0 & 0 & 0 & 0 \\ -70 & -70 & -70 & -70 & -70 & 0 & 0 & 0 & 0 & 0 \\ -90 & -90 & -90 & -90 & -90 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{1,3,5,7,9\},\ \{10,30,50,70,90\}\}$
2	$\{(10),(2)\}$	$\begin{pmatrix} 1 & 2 & 21 & 22 & 41 & 42 & 61 & 62 & 81 & 82 \\ 3 & 4 & 23 & 24 & 43 & 44 & 63 & 64 & 83 & 84 \\ 5 & 6 & 25 & 26 & 45 & 46 & 65 & 66 & 85 & 86 \\ 7 & 8 & 27 & 28 & 47 & 48 & 67 & 68 & 87 & 88 \\ 9 & 10 & 29 & 30 & 49 & 50 & 69 & 70 & 89 & 90 \\ 11 & 12 & 31 & 32 & 51 & 52 & 71 & 72 & 91 & 92 \\ 13 & 14 & 33 & 34 & 53 & 54 & 73 & 74 & 93 & 94 \\ 15 & 16 & 35 & 36 & 55 & 56 & 75 & 76 & 95 & 96 \\ 17 & 18 & 37 & 38 & 57 & 58 & 77 & 78 & 97 & 98 \\ 19 & 20 & 39 & 40 & 59 & 60 & 79 & 80 & 99 & 100 \end{pmatrix}$	$\begin{pmatrix} 101 & 101 & 101 & 101 & 101 & -1 & -39 & -41 & -79 & -81 \\ 101 & 101 & 101 & 101 & 101 & -1 & -39 & -41 & -79 & -81 \\ 101 & 101 & 101 & 101 & 101 & -1 & -39 & -41 & -79 & -81 \\ 101 & 101 & 101 & 101 & 101 & -1 & -39 & -41 & -79 & -81 \\ 101 & 101 & 101 & 101 & 101 & -1 & -39 & -41 & -79 & -81 \\ -2 & -2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ -6 & -6 & -6 & -6 & -6 & 0 & 0 & 0 & 0 & 0 \\ -10 & -10 & -10 & -10 & -10 & 0 & 0 & 0 & 0 & 0 \\ -14 & -14 & -14 & -14 & -14 & 0 & 0 & 0 & 0 & 0 \\ -18 & -18 & -18 & -18 & -18 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{1,39,41,79,81\},\ \{2,6,10,14,18\}\}$
3	$\{(10),(5)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 51 & 52 & 53 & 54 & 55 \\ 6 & 7 & 8 & 9 & 10 & 56 & 57 & 58 & 59 & 60 \\ 11 & 12 & 13 & 14 & 15 & 61 & 62 & 63 & 64 & 65 \\ 16 & 17 & 18 & 19 & 20 & 66 & 67 & 68 & 69 & 70 \\ 21 & 22 & 23 & 24 & 25 & 71 & 72 & 73 & 74 & 75 \\ 26 & 27 & 28 & 29 & 30 & 76 & 77 & 78 & 79 & 80 \\ 31 & 32 & 33 & 34 & 35 & 81 & 82 & 83 & 84 & 85 \\ 36 & 37 & 38 & 39 & 40 & 86 & 87 & 88 & 89 & 90 \\ 41 & 42 & 43 & 44 & 45 & 91 & 92 & 93 & 94 & 95 \\ 46 & 47 & 48 & 49 & 50 & 96 & 97 & 98 & 99 & 100 \end{pmatrix}$	$\begin{pmatrix} 101 & 101 & 101 & 101 & 101 & -46 & -48 & -50 & -52 & -54 \\ 101 & 101 & 101 & 101 & 101 & -46 & -48 & -50 & -52 & -54 \\ 101 & 101 & 101 & 101 & 101 & -46 & -48 & -50 & -52 & -54 \\ 101 & 101 & 101 & 101 & 101 & -46 & -48 & -50 & -52 & -54 \\ 101 & 101 & 101 & 101 & 101 & -46 & -48 & -50 & -52 & -54 \\ -5 & -5 & -5 & -5 & -5 & 0 & 0 & 0 & 0 & 0 \\ -15 & -15 & -15 & -15 & -15 & 0 & 0 & 0 & 0 & 0 \\ -25 & -25 & -25 & -25 & -25 & 0 & 0 & 0 & 0 & 0 \\ -35 & -35 & -35 & -35 & -35 & 0 & 0 & 0 & 0 & 0 \\ -45 & -45 & -45 & -45 & -45 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{46,48,50,52,54\},\ \{5,15,25,35,45\}\}$
4	$\{(2,10),(2,10)\}$	$\begin{pmatrix} 1 & 2 & 5 & 6 & 9 & 10 & 13 & 14 & 17 & 18 \\ 3 & 4 & 7 & 8 & 11 & 12 & 15 & 16 & 19 & 20 \\ 21 & 22 & 25 & 26 & 29 & 30 & 33 & 34 & 37 & 38 \\ 23 & 24 & 27 & 28 & 31 & 32 & 35 & 36 & 39 & 40 \\ 41 & 42 & 45 & 46 & 49 & 50 & 53 & 54 & 57 & 58 \\ 43 & 44 & 47 & 48 & 51 & 52 & 55 & 56 & 59 & 60 \\ 61 & 62 & 65 & 66 & 69 & 70 & 73 & 74 & 77 & 78 \\ 63 & 64 & 67 & 68 & 71 & 72 & 75 & 76 & 79 & 80 \\ 81 & 82 & 85 & 86 & 89 & 90 & 93 & 94 & 97 & 98 \\ 83 & 84 & 87 & 88 & 91 & 92 & 95 & 96 & 99 & 100 \end{pmatrix}$	$\begin{pmatrix} 101 & 101 & 101 & 101 & 101 & -1 & -7 & -9 & -15 & -17 \\ 101 & 101 & 101 & 101 & 101 & -1 & -7 & -9 & -15 & -17 \\ 101 & 101 & 101 & 101 & 101 & -1 & -7 & -9 & -15 & -17 \\ 101 & 101 & 101 & 101 & 101 & -1 & -7 & -9 & -15 & -17 \\ 101 & 101 & 101 & 101 & 101 & -1 & -7 & -9 & -15 & -17 \\ -2 & -2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ -38 & -38 & -38 & -38 & -38 & 0 & 0 & 0 & 0 & 0 \\ -42 & -42 & -42 & -42 & -42 & 0 & 0 & 0 & 0 & 0 \\ -78 & -78 & -78 & -78 & -78 & 0 & 0 & 0 & 0 & 0 \\ -82 & -82 & -82 & -82 & -82 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{1,7,9,15,17\},\ \{2,38,42,78,82\}\}$

	Divisor Paths	Reversible Square	Block Representation	Sum-and-Distance System
5	$\{(2,10),(5,10)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 11 & 12 & 13 & 14 & 15 \\ 6 & 7 & 8 & 9 & 10 & 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 & 31 & 32 & 33 & 34 & 35 \\ 26 & 27 & 28 & 29 & 30 & 36 & 37 & 38 & 39 & 40 \\ 41 & 42 & 43 & 44 & 45 & 51 & 52 & 53 & 54 & 55 \\ 46 & 47 & 48 & 49 & 50 & 56 & 57 & 58 & 59 & 60 \\ 61 & 62 & 63 & 64 & 65 & 71 & 72 & 73 & 74 & 75 \\ 66 & 67 & 68 & 69 & 70 & 76 & 77 & 78 & 79 & 80 \\ 81 & 82 & 83 & 84 & 85 & 91 & 92 & 93 & 94 & 95 \\ 86 & 87 & 88 & 89 & 90 & 96 & 97 & 98 & 99 & 100 \end{pmatrix}$	$\begin{pmatrix} 101 & 101 & 101 & 101 & 101 & -6 & -8 & -10 & -12 & -14 \\ 101 & 101 & 101 & 101 & 101 & -6 & -8 & -10 & -12 & -14 \\ 101 & 101 & 101 & 101 & 101 & -6 & -8 & -10 & -12 & -14 \\ 101 & 101 & 101 & 101 & 101 & -6 & -8 & -10 & -12 & -14 \\ 101 & 101 & 101 & 101 & 101 & -6 & -8 & -10 & -12 & -14 \\ -5 & -5 & -5 & -5 & -5 & 0 & 0 & 0 & 0 & 0 \\ -35 & -35 & -35 & -35 & -35 & 0 & 0 & 0 & 0 & 0 \\ -45 & -45 & -45 & -45 & -45 & 0 & 0 & 0 & 0 & 0 \\ -75 & -75 & -75 & -75 & -75 & 0 & 0 & 0 & 0 & 0 \\ -85 & -85 & -85 & -85 & -85 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{6,8,10,12,14\},\ \{5,35,45,75,85\}\}$
6	$\{(5,10),(2,10)\}$	$\begin{pmatrix} 1 & 2 & 11 & 12 & 21 & 22 & 31 & 32 & 41 & 42 \\ 3 & 4 & 13 & 14 & 23 & 24 & 33 & 34 & 43 & 44 \\ 5 & 6 & 15 & 16 & 25 & 26 & 35 & 36 & 45 & 46 \\ 7 & 8 & 17 & 18 & 27 & 28 & 37 & 38 & 47 & 48 \\ 9 & 10 & 19 & 20 & 29 & 30 & 39 & 40 & 49 & 50 \\ 51 & 52 & 61 & 62 & 71 & 72 & 81 & 82 & 91 & 92 \\ 53 & 54 & 63 & 64 & 73 & 74 & 83 & 84 & 93 & 94 \\ 55 & 56 & 65 & 66 & 75 & 76 & 85 & 86 & 95 & 96 \\ 57 & 58 & 67 & 68 & 77 & 78 & 87 & 88 & 97 & 98 \\ 59 & 60 & 69 & 70 & 79 & 80 & 89 & 90 & 99 & 100 \end{pmatrix}$	$\begin{pmatrix} 101 & 101 & 101 & 101 & 101 & -1 & -19 & -21 & -39 & -41 \\ 101 & 101 & 101 & 101 & 101 & -1 & -19 & -21 & -39 & -41 \\ 101 & 101 & 101 & 101 & 101 & -1 & -19 & -21 & -39 & -41 \\ 101 & 101 & 101 & 101 & 101 & -1 & -19 & -21 & -39 & -41 \\ 101 & 101 & 101 & 101 & 101 & -1 & -19 & -21 & -39 & -41 \\ -42 & -42 & -42 & -42 & -42 & 0 & 0 & 0 & 0 & 0 \\ -46 & -46 & -46 & -46 & -46 & 0 & 0 & 0 & 0 & 0 \\ -50 & -50 & -50 & -50 & -50 & 0 & 0 & 0 & 0 & 0 \\ -54 & -54 & -54 & -54 & -54 & 0 & 0 & 0 & 0 & 0 \\ -58 & -58 & -58 & -58 & -58 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{1,19,21,39,41\},\ \{42,46,50,54,58\}\}$
7	$\{(5,10),(5,10)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 26 & 27 & 28 & 29 & 30 \\ 6 & 7 & 8 & 9 & 10 & 31 & 32 & 33 & 34 & 35 \\ 11 & 12 & 13 & 14 & 15 & 36 & 37 & 38 & 39 & 40 \\ 16 & 17 & 18 & 19 & 20 & 41 & 42 & 43 & 44 & 45 \\ 21 & 22 & 23 & 24 & 25 & 46 & 47 & 48 & 49 & 50 \\ 51 & 52 & 53 & 54 & 55 & 76 & 77 & 78 & 79 & 80 \\ 56 & 57 & 58 & 59 & 60 & 81 & 82 & 83 & 84 & 85 \\ 61 & 62 & 63 & 64 & 65 & 86 & 87 & 88 & 89 & 90 \\ 66 & 67 & 68 & 69 & 70 & 91 & 92 & 93 & 94 & 95 \\ 71 & 72 & 73 & 74 & 75 & 96 & 97 & 98 & 99 & 100 \end{pmatrix}$	$\begin{pmatrix} 101 & 101 & 101 & 101 & 101 & -21 & -23 & -25 & -27 & -29 \\ 101 & 101 & 101 & 101 & 101 & -21 & -23 & -25 & -27 & -29 \\ 101 & 101 & 101 & 101 & 101 & -21 & -23 & -25 & -27 & -29 \\ 101 & 101 & 101 & 101 & 101 & -21 & -23 & -25 & -27 & -29 \\ 101 & 101 & 101 & 101 & 101 & -21 & -23 & -25 & -27 & -29 \\ -30 & -30 & -30 & -30 & -30 & 0 & 0 & 0 & 0 & 0 \\ -40 & -40 & -40 & -40 & -40 & 0 & 0 & 0 & 0 & 0 \\ -50 & -50 & -50 & -50 & -50 & 0 & 0 & 0 & 0 & 0 \\ -60 & -60 & -60 & -60 & -60 & 0 & 0 & 0 & 0 & 0 \\ -70 & -70 & -70 & -70 & -70 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{21,23,25,27,29\},\ \{30,40,50,60,70\}\}$

Table 6: n=12

	Divisor Paths	Reversible Square	Block Representation
1	$\{(12),\ (12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 \\ 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 & 59 & 60 \\ 61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 & 69 & 70 & 71 & 72 \\ 73 & 74 & 75 & 76 & 77 & 78 & 79 & 80 & 81 & 82 & 83 & 84 \\ 85 & 86 & 87 & 88 & 89 & 90 & 91 & 92 & 93 & 94 & 95 & 96 \\ 97 & 98 & 99 & 100 & 101 & 102 & 103 & 104 & 105 & 106 & 107 & 108 \\ 109 & 110 & 111 & 112 & 113 & 114 & 115 & 116 & 117 & 118 & 119 & 120 \\ 121 & 122 & 123 & 124 & 125 & 126 & 127 & 128 & 129 & 130 & 131 & 132 \\ 133 & 134 & 135 & 136 & 137 & 138 & 139 & 140 & 141 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -5 & -7 & -9 & -11 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -5 & -7 & -9 & -11 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -5 & -7 & -9 & -11 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -5 & -7 & -9 & -11 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -5 & -7 & -9 & -11 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -5 & -7 & -9 & -11 \\ -12 & -12 & -12 & -12 & -12 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ -36 & -36 & -36 & -36 & -36 & -36 & 0 & 0 & 0 & 0 & 0 & 0 \\ -60 & -60 & -60 & -60 & -60 & -60 & 0 & 0 & 0 & 0 & 0 & 0 \\ -84 & -84 & -84 & -84 & -84 & -84 & 0 & 0 & 0 & 0 & 0 & 0 \\ -108 & -108 & -108 & -108 & -108 & -108 & 0 & 0 & 0 & 0 & 0 & 0 \\ -132 & -132 & -132 & -132 & -132 & -132 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
2	$\{(12),\ (2)\}$	$\begin{pmatrix} 1 & 2 & 25 & 26 & 49 & 50 & 73 & 74 & 97 & 98 & 121 & 122 \\ 3 & 4 & 27 & 28 & 51 & 52 & 75 & 76 & 99 & 100 & 123 & 124 \\ 5 & 6 & 29 & 30 & 53 & 54 & 77 & 78 & 101 & 102 & 125 & 126 \\ 7 & 8 & 31 & 32 & 55 & 56 & 79 & 80 & 103 & 104 & 127 & 128 \\ 9 & 10 & 33 & 34 & 57 & 58 & 81 & 82 & 105 & 106 & 129 & 130 \\ 11 & 12 & 35 & 36 & 59 & 60 & 83 & 84 & 107 & 108 & 131 & 132 \\ 13 & 14 & 37 & 38 & 61 & 62 & 85 & 86 & 109 & 110 & 133 & 134 \\ 15 & 16 & 39 & 40 & 63 & 64 & 87 & 88 & 111 & 112 & 135 & 136 \\ 17 & 18 & 41 & 42 & 65 & 66 & 89 & 90 & 113 & 114 & 137 & 138 \\ 19 & 20 & 43 & 44 & 67 & 68 & 91 & 92 & 115 & 116 & 139 & 140 \\ 21 & 22 & 45 & 46 & 69 & 70 & 93 & 94 & 117 & 118 & 141 & 142 \\ 23 & 24 & 47 & 48 & 71 & 72 & 95 & 96 & 119 & 120 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -23 & -25 & -71 & -73 & -119 & -121 \\ 145 & 145 & 145 & 145 & 145 & 145 & -23 & -25 & -71 & -73 & -119 & -121 \\ 145 & 145 & 145 & 145 & 145 & 145 & -23 & -25 & -71 & -73 & -119 & -121 \\ 145 & 145 & 145 & 145 & 145 & 145 & -23 & -25 & -71 & -73 & -119 & -121 \\ 145 & 145 & 145 & 145 & 145 & 145 & -23 & -25 & -71 & -73 & -119 & -121 \\ 145 & 145 & 145 & 145 & 145 & 145 & -23 & -25 & -71 & -73 & -119 & -121 \\ -2 & -2 & -2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -6 & -6 & -6 & -6 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & -10 & -10 & -10 & -10 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \\ -14 & -14 & -14 & -14 & -14 & -14 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & -18 & -18 & -18 & -18 & -18 & 0 & 0 & 0 & 0 & 0 & 0 \\ -22 & -22 & -22 & -22 & -22 & -22 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
3	$\{(12),\ (3)\}$	$\begin{pmatrix} 1 & 2 & 3 & 37 & 38 & 39 & 73 & 74 & 75 & 109 & 110 & 111 \\ 4 & 5 & 6 & 40 & 41 & 42 & 76 & 77 & 78 & 112 & 113 & 114 \\ 7 & 8 & 9 & 43 & 44 & 45 & 79 & 80 & 81 & 115 & 116 & 117 \\ 10 & 11 & 12 & 46 & 47 & 48 & 82 & 83 & 84 & 118 & 119 & 120 \\ 13 & 14 & 15 & 49 & 50 & 51 & 85 & 86 & 87 & 121 & 122 & 123 \\ 16 & 17 & 18 & 52 & 53 & 54 & 88 & 89 & 90 & 124 & 125 & 126 \\ 19 & 20 & 21 & 55 & 56 & 57 & 91 & 92 & 93 & 127 & 128 & 129 \\ 22 & 23 & 24 & 58 & 59 & 60 & 94 & 95 & 96 & 130 & 131 & 132 \\ 25 & 26 & 27 & 61 & 62 & 63 & 97 & 98 & 99 & 133 & 134 & 135 \\ 28 & 29 & 30 & 64 & 65 & 66 & 100 & 101 & 102 & 136 & 137 & 138 \\ 31 & 32 & 33 & 67 & 68 & 69 & 103 & 104 & 105 & 139 & 140 & 141 \\ 34 & 35 & 36 & 70 & 71 & 72 & 106 & 107 & 108 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -34 & -36 & -38 & -106 & -108 & -110 \\ 145 & 145 & 145 & 145 & 145 & 145 & -34 & -36 & -38 & -106 & -108 & -110 \\ 145 & 145 & 145 & 145 & 145 & 145 & -34 & -36 & -38 & -106 & -108 & -110 \\ 145 & 145 & 145 & 145 & 145 & 145 & -34 & -36 & -38 & -106 & -108 & -110 \\ 145 & 145 & 145 & 145 & 145 & 145 & -34 & -36 & -38 & -106 & -108 & -110 \\ 145 & 145 & 145 & 145 & 145 & 145 & -34 & -36 & -38 & -106 & -108 & -110 \\ -3 & -3 & -3 & -3 & -3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & -9 & -9 & -9 & -9 & -9 & 0 & 0 & 0 & 0 & 0 & 0 \\ -15 & -15 & -15 & -15 & -15 & -15 & 0 & 0 & 0 & 0 & 0 & 0 \\ -21 & -21 & -21 & -21 & -21 & -21 & 0 & 0 & 0 & 0 & 0 & 0 \\ -27 & -27 & -27 & -27 & -27 & -27 & 0 & 0 & 0 & 0 & 0 & 0 \\ -33 & -33 & -33 & -33 & -33 & -33 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
4	$\{(12),\ (4)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 49 & 50 & 51 & 52 & 97 & 98 & 99 & 100 \\ 5 & 6 & 7 & 8 & 53 & 54 & 55 & 56 & 101 & 102 & 103 & 104 \\ 9 & 10 & 11 & 12 & 57 & 58 & 59 & 60 & 105 & 106 & 107 & 108 \\ 13 & 14 & 15 & 16 & 61 & 62 & 63 & 64 & 109 & 110 & 111 & 112 \\ 17 & 18 & 19 & 20 & 65 & 66 & 67 & 68 & 113 & 114 & 115 & 116 \\ 21 & 22 & 23 & 24 & 69 & 70 & 71 & 72 & 117 & 118 & 119 & 120 \\ 25 & 26 & 27 & 28 & 73 & 74 & 75 & 76 & 121 & 122 & 123 & 124 \\ 29 & 30 & 31 & 32 & 77 & 78 & 79 & 80 & 125 & 126 & 127 & 128 \\ 33 & 34 & 35 & 36 & 81 & 82 & 83 & 84 & 129 & 130 & 131 & 132 \\ 37 & 38 & 39 & 40 & 85 & 86 & 87 & 88 & 133 & 134 & 135 & 136 \\ 41 & 42 & 43 & 44 & 89 & 90 & 91 & 92 & 137 & 138 & 139 & 140 \\ 45 & 46 & 47 & 48 & 93 & 94 & 95 & 96 & 141 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -93 & -95 & -97 & -99 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -93 & -95 & -97 & -99 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -93 & -95 & -97 & -99 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -93 & -95 & -97 & -99 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -93 & -95 & -97 & -99 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -93 & -95 & -97 & -99 \\ -4 & -4 & -4 & -4 & -4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -12 & -12 & -12 & -12 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ -20 & -20 & -20 & -20 & -20 & -20 & 0 & 0 & 0 & 0 & 0 & 0 \\ -28 & -28 & -28 & -28 & -28 & -28 & 0 & 0 & 0 & 0 & 0 & 0 \\ -36 & -36 & -36 & -36 & -36 & -36 & 0 & 0 & 0 & 0 & 0 & 0 \\ -44 & -44 & -44 & -44 & -44 & -44 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

	Divisor Paths	Reversible Square	Block Representation
5	$\{(12),\ (6)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 73 & 74 & 75 & 76 & 77 & 78 \\ 7 & 8 & 9 & 10 & 11 & 12 & 79 & 80 & 81 & 82 & 83 & 84 \\ 13 & 14 & 15 & 16 & 17 & 18 & 85 & 86 & 87 & 88 & 89 & 90 \\ 19 & 20 & 21 & 22 & 23 & 24 & 91 & 92 & 93 & 94 & 95 & 96 \\ 25 & 26 & 27 & 28 & 29 & 30 & 97 & 98 & 99 & 100 & 101 & 102 \\ 31 & 32 & 33 & 34 & 35 & 36 & 103 & 104 & 105 & 106 & 107 & 108 \\ 37 & 38 & 39 & 40 & 41 & 42 & 109 & 110 & 111 & 112 & 113 & 114 \\ 43 & 44 & 45 & 46 & 47 & 48 & 115 & 116 & 117 & 118 & 119 & 120 \\ 49 & 50 & 51 & 52 & 53 & 54 & 121 & 122 & 123 & 124 & 125 & 126 \\ 55 & 56 & 57 & 58 & 59 & 60 & 127 & 128 & 129 & 130 & 131 & 132 \\ 61 & 62 & 63 & 64 & 65 & 66 & 133 & 134 & 135 & 136 & 137 & 138 \\ 67 & 68 & 69 & 70 & 71 & 72 & 139 & 140 & 141 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -67 & -69 & -71 & -73 & -75 & -77 \\ 145 & 145 & 145 & 145 & 145 & 145 & -67 & -69 & -71 & -73 & -75 & -77 \\ 145 & 145 & 145 & 145 & 145 & 145 & -67 & -69 & -71 & -73 & -75 & -77 \\ 145 & 145 & 145 & 145 & 145 & 145 & -67 & -69 & -71 & -73 & -75 & -77 \\ 145 & 145 & 145 & 145 & 145 & 145 & -67 & -69 & -71 & -73 & -75 & -77 \\ 145 & 145 & 145 & 145 & 145 & 145 & -67 & -69 & -71 & -73 & -75 & -77 \\ -6 & -6 & -6 & -6 & -6 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & -18 & -18 & -18 & -18 & -18 & 0 & 0 & 0 & 0 & 0 & 0 \\ -30 & -30 & -30 & -30 & -30 & -30 & 0 & 0 & 0 & 0 & 0 & 0 \\ -42 & -42 & -42 & -42 & -42 & -42 & 0 & 0 & 0 & 0 & 0 & 0 \\ -54 & -54 & -54 & -54 & -54 & -54 & 0 & 0 & 0 & 0 & 0 & 0 \\ -66 & -66 & -66 & -66 & -66 & -66 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
6	$\{(2,12),\ (2,12)\}$	$\begin{pmatrix} 1 & 2 & 5 & 6 & 9 & 10 & 13 & 14 & 17 & 18 & 21 & 22 \\ 3 & 4 & 7 & 8 & 11 & 12 & 15 & 16 & 19 & 20 & 23 & 24 \\ 25 & 26 & 29 & 30 & 33 & 34 & 37 & 38 & 41 & 42 & 45 & 46 \\ 27 & 28 & 31 & 32 & 35 & 36 & 39 & 40 & 43 & 44 & 47 & 48 \\ 49 & 50 & 53 & 54 & 57 & 58 & 61 & 62 & 65 & 66 & 69 & 70 \\ 51 & 52 & 55 & 56 & 59 & 60 & 63 & 64 & 67 & 68 & 71 & 72 \\ 73 & 74 & 77 & 78 & 81 & 82 & 85 & 86 & 89 & 90 & 93 & 94 \\ 75 & 76 & 79 & 80 & 83 & 84 & 87 & 88 & 91 & 92 & 95 & 96 \\ 97 & 98 & 101 & 102 & 105 & 106 & 109 & 110 & 113 & 114 & 117 & 118 \\ 99 & 100 & 103 & 104 & 107 & 108 & 111 & 112 & 115 & 116 & 119 & 120 \\ 121 & 122 & 125 & 126 & 129 & 130 & 133 & 134 & 137 & 138 & 141 & 142 \\ 123 & 124 & 127 & 128 & 131 & 132 & 135 & 136 & 139 & 140 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -11 & -13 & -19 & -21 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -11 & -13 & -19 & -21 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -11 & -13 & -19 & -21 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -11 & -13 & -19 & -21 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -11 & -13 & -19 & -21 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -11 & -13 & -19 & -21 \\ -22 & -22 & -22 & -22 & -22 & -22 & 0 & 0 & 0 & 0 & 0 & 0 \\ -26 & -26 & -26 & -26 & -26 & -26 & 0 & 0 & 0 & 0 & 0 & 0 \\ -70 & -70 & -70 & -70 & -70 & -70 & 0 & 0 & 0 & 0 & 0 & 0 \\ -74 & -74 & -74 & -74 & -74 & -74 & 0 & 0 & 0 & 0 & 0 & 0 \\ -118 & -118 & -118 & -118 & -118 & -118 & 0 & 0 & 0 & 0 & 0 & 0 \\ -122 & -122 & -122 & -122 & -122 & -122 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
7	$\{(2,12),\ (3,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 7 & 8 & 9 & 13 & 14 & 15 & 19 & 20 & 21 \\ 4 & 5 & 6 & 10 & 11 & 12 & 16 & 17 & 18 & 22 & 23 & 24 \\ 25 & 26 & 27 & 31 & 32 & 33 & 37 & 38 & 39 & 43 & 44 & 45 \\ 28 & 29 & 30 & 34 & 35 & 36 & 40 & 41 & 42 & 46 & 47 & 48 \\ 49 & 50 & 51 & 55 & 56 & 57 & 61 & 62 & 63 & 67 & 68 & 69 \\ 52 & 53 & 54 & 58 & 59 & 60 & 64 & 65 & 66 & 70 & 71 & 72 \\ 73 & 74 & 75 & 79 & 80 & 81 & 85 & 86 & 87 & 91 & 92 & 93 \\ 76 & 77 & 78 & 82 & 83 & 84 & 88 & 89 & 90 & 94 & 95 & 96 \\ 97 & 98 & 99 & 103 & 104 & 105 & 109 & 110 & 111 & 115 & 116 & 117 \\ 100 & 101 & 102 & 106 & 107 & 108 & 112 & 113 & 114 & 118 & 119 & 120 \\ 121 & 122 & 123 & 127 & 128 & 129 & 133 & 134 & 135 & 139 & 140 & 141 \\ 124 & 125 & 126 & 130 & 131 & 132 & 136 & 137 & 138 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -4 & -6 & -8 & -16 & -18 & -20 \\ 145 & 145 & 145 & 145 & 145 & 145 & -4 & -6 & -8 & -16 & -18 & -20 \\ 145 & 145 & 145 & 145 & 145 & 145 & -4 & -6 & -8 & -16 & -18 & -20 \\ 145 & 145 & 145 & 145 & 145 & 145 & -4 & -6 & -8 & -16 & -18 & -20 \\ 145 & 145 & 145 & 145 & 145 & 145 & -4 & -6 & -8 & -16 & -18 & -20 \\ 145 & 145 & 145 & 145 & 145 & 145 & -4 & -6 & -8 & -16 & -18 & -20 \\ -21 & -21 & -21 & -21 & -21 & -21 & 0 & 0 & 0 & 0 & 0 & 0 \\ -27 & -27 & -27 & -27 & -27 & -27 & 0 & 0 & 0 & 0 & 0 & 0 \\ -69 & -69 & -69 & -69 & -69 & -69 & 0 & 0 & 0 & 0 & 0 & 0 \\ -75 & -75 & -75 & -75 & -75 & -75 & 0 & 0 & 0 & 0 & 0 & 0 \\ -117 & -117 & -117 & -117 & -117 & -117 & 0 & 0 & 0 & 0 & 0 & 0 \\ -123 & -123 & -123 & -123 & -123 & -123 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
8	$\{(2,12),\ (4,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 9 & 10 & 11 & 12 & 17 & 18 & 19 & 20 \\ 5 & 6 & 7 & 8 & 13 & 14 & 15 & 16 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 33 & 34 & 35 & 36 & 41 & 42 & 43 & 44 \\ 29 & 30 & 31 & 32 & 37 & 38 & 39 & 40 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 57 & 58 & 59 & 60 & 65 & 66 & 67 & 68 \\ 53 & 54 & 55 & 56 & 61 & 62 & 63 & 64 & 69 & 70 & 71 & 72 \\ 73 & 74 & 75 & 76 & 81 & 82 & 83 & 84 & 89 & 90 & 91 & 92 \\ 77 & 78 & 79 & 80 & 85 & 86 & 87 & 88 & 93 & 94 & 95 & 96 \\ 97 & 98 & 99 & 100 & 105 & 106 & 107 & 108 & 113 & 114 & 115 & 116 \\ 101 & 102 & 103 & 104 & 109 & 110 & 111 & 112 & 117 & 118 & 119 & 120 \\ 121 & 122 & 123 & 124 & 129 & 130 & 131 & 132 & 137 & 138 & 139 & 140 \\ 125 & 126 & 127 & 128 & 133 & 134 & 135 & 136 & 141 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -13 & -15 & -17 & -19 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -13 & -15 & -17 & -19 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -13 & -15 & -17 & -19 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -13 & -15 & -17 & -19 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -13 & -15 & -17 & -19 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -13 & -15 & -17 & -19 \\ -20 & -20 & -20 & -20 & -20 & -20 & 0 & 0 & 0 & 0 & 0 & 0 \\ -28 & -28 & -28 & -28 & -28 & -28 & 0 & 0 & 0 & 0 & 0 & 0 \\ -68 & -68 & -68 & -68 & -68 & -68 & 0 & 0 & 0 & 0 & 0 & 0 \\ -76 & -76 & -76 & -76 & -76 & -76 & 0 & 0 & 0 & 0 & 0 & 0 \\ -116 & -116 & -116 & -116 & -116 & -116 & 0 & 0 & 0 & 0 & 0 & 0 \\ -124 & -124 & -124 & -124 & -124 & -124 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

	Divisor Paths	Reversible Square	Block Representation
9	$\{(2,12),\ (6,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 13 & 14 & 15 & 16 & 17 & 18 \\ 7 & 8 & 9 & 10 & 11 & 12 & 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 & 37 & 38 & 39 & 40 & 41 & 42 \\ 31 & 32 & 33 & 34 & 35 & 36 & 43 & 44 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 53 & 54 & 61 & 62 & 63 & 64 & 65 & 66 \\ 55 & 56 & 57 & 58 & 59 & 60 & 67 & 68 & 69 & 70 & 71 & 72 \\ 73 & 74 & 75 & 76 & 77 & 78 & 85 & 86 & 87 & 88 & 89 & 90 \\ 79 & 80 & 81 & 82 & 83 & 84 & 91 & 92 & 93 & 94 & 95 & 96 \\ 97 & 98 & 99 & 100 & 101 & 102 & 109 & 110 & 111 & 112 & 113 & 114 \\ 103 & 104 & 105 & 106 & 107 & 108 & 115 & 116 & 117 & 118 & 119 & 120 \\ 121 & 122 & 123 & 124 & 125 & 126 & 133 & 134 & 135 & 136 & 137 & 138 \\ 127 & 128 & 129 & 130 & 131 & 132 & 139 & 140 & 141 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -11 & -13 & -15 & -17 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -11 & -13 & -15 & -17 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -11 & -13 & -15 & -17 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -11 & -13 & -15 & -17 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -11 & -13 & -15 & -17 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -11 & -13 & -15 & -17 \\ -18 & -18 & -18 & -18 & -18 & -18 & 0 & 0 & 0 & 0 & 0 & 0 \\ -30 & -30 & -30 & -30 & -30 & -30 & 0 & 0 & 0 & 0 & 0 & 0 \\ -66 & -66 & -66 & -66 & -66 & -66 & 0 & 0 & 0 & 0 & 0 & 0 \\ -78 & -78 & -78 & -78 & -78 & -78 & 0 & 0 & 0 & 0 & 0 & 0 \\ -114 & -114 & -114 & -114 & -114 & -114 & 0 & 0 & 0 & 0 & 0 & 0 \\ -126 & -126 & -126 & -126 & -126 & -126 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
10	$\{(2,12),\ (2,4)\}$	$\begin{pmatrix} 1 & 2 & 5 & 6 & 49 & 50 & 53 & 54 & 97 & 98 & 101 & 102 \\ 3 & 4 & 7 & 8 & 51 & 52 & 55 & 56 & 99 & 100 & 103 & 104 \\ 9 & 10 & 13 & 14 & 57 & 58 & 61 & 62 & 105 & 106 & 109 & 110 \\ 11 & 12 & 15 & 16 & 59 & 60 & 63 & 64 & 107 & 108 & 111 & 112 \\ 17 & 18 & 21 & 22 & 65 & 66 & 69 & 70 & 113 & 114 & 117 & 118 \\ 19 & 20 & 23 & 24 & 67 & 68 & 71 & 72 & 115 & 116 & 119 & 120 \\ 25 & 26 & 29 & 30 & 73 & 74 & 77 & 78 & 121 & 122 & 125 & 126 \\ 27 & 28 & 31 & 32 & 75 & 76 & 79 & 80 & 123 & 124 & 127 & 128 \\ 33 & 34 & 37 & 38 & 81 & 82 & 85 & 86 & 129 & 130 & 133 & 134 \\ 35 & 36 & 39 & 40 & 83 & 84 & 87 & 88 & 131 & 132 & 135 & 136 \\ 41 & 42 & 45 & 46 & 89 & 90 & 93 & 94 & 137 & 138 & 141 & 142 \\ 43 & 44 & 47 & 48 & 91 & 92 & 95 & 96 & 139 & 140 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -91 & -93 & -99 & -101 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -91 & -93 & -99 & -101 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -91 & -93 & -99 & -101 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -91 & -93 & -99 & -101 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -91 & -93 & -99 & -101 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -91 & -93 & -99 & -101 \\ -6 & -6 & -6 & -6 & -6 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & -10 & -10 & -10 & -10 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \\ -22 & -22 & -22 & -22 & -22 & -22 & 0 & 0 & 0 & 0 & 0 & 0 \\ -26 & -26 & -26 & -26 & -26 & -26 & 0 & 0 & 0 & 0 & 0 & 0 \\ -38 & -38 & -38 & -38 & -38 & -38 & 0 & 0 & 0 & 0 & 0 & 0 \\ -42 & -42 & -42 & -42 & -42 & -42 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
11	$\{(2,12),\ (2,6)\}$	$\begin{pmatrix} 1 & 2 & 5 & 6 & 9 & 10 & 73 & 74 & 77 & 78 & 81 & 82 \\ 3 & 4 & 7 & 8 & 11 & 12 & 75 & 76 & 79 & 80 & 83 & 84 \\ 13 & 14 & 17 & 18 & 21 & 22 & 85 & 86 & 89 & 90 & 93 & 94 \\ 15 & 16 & 19 & 20 & 23 & 24 & 87 & 88 & 91 & 92 & 95 & 96 \\ 25 & 26 & 29 & 30 & 33 & 34 & 97 & 98 & 101 & 102 & 105 & 106 \\ 27 & 28 & 31 & 32 & 35 & 36 & 99 & 100 & 103 & 104 & 107 & 108 \\ 37 & 38 & 41 & 42 & 45 & 46 & 109 & 110 & 113 & 114 & 117 & 118 \\ 39 & 40 & 43 & 44 & 47 & 48 & 111 & 112 & 115 & 116 & 119 & 120 \\ 49 & 50 & 53 & 54 & 57 & 58 & 121 & 122 & 125 & 126 & 129 & 130 \\ 51 & 52 & 55 & 56 & 59 & 60 & 123 & 124 & 127 & 128 & 131 & 132 \\ 61 & 62 & 65 & 66 & 69 & 70 & 133 & 134 & 137 & 138 & 141 & 142 \\ 63 & 64 & 67 & 68 & 71 & 72 & 135 & 136 & 139 & 140 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -63 & -65 & -71 & -73 & -79 & -81 \\ 145 & 145 & 145 & 145 & 145 & 145 & -63 & -65 & -71 & -73 & -79 & -81 \\ 145 & 145 & 145 & 145 & 145 & 145 & -63 & -65 & -71 & -73 & -79 & -81 \\ 145 & 145 & 145 & 145 & 145 & 145 & -63 & -65 & -71 & -73 & -79 & -81 \\ 145 & 145 & 145 & 145 & 145 & 145 & -63 & -65 & -71 & -73 & -79 & -81 \\ 145 & 145 & 145 & 145 & 145 & 145 & -63 & -65 & -71 & -73 & -79 & -81 \\ -10 & -10 & -10 & -10 & -10 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \\ -14 & -14 & -14 & -14 & -14 & -14 & 0 & 0 & 0 & 0 & 0 & 0 \\ -34 & -34 & -34 & -34 & -34 & -34 & 0 & 0 & 0 & 0 & 0 & 0 \\ -38 & -38 & -38 & -38 & -38 & -38 & 0 & 0 & 0 & 0 & 0 & 0 \\ -58 & -58 & -58 & -58 & -58 & -58 & 0 & 0 & 0 & 0 & 0 & 0 \\ -62 & -62 & -62 & -62 & -62 & -62 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
12	$\{(3,12),\ (3,6)\}$	$\begin{pmatrix} 1 & 2 & 3 & 7 & 8 & 9 & 73 & 74 & 75 & 79 & 80 & 81 \\ 4 & 5 & 6 & 10 & 11 & 12 & 76 & 77 & 78 & 82 & 83 & 84 \\ 13 & 14 & 15 & 19 & 20 & 21 & 85 & 86 & 87 & 91 & 92 & 93 \\ 16 & 17 & 18 & 22 & 23 & 24 & 88 & 89 & 90 & 94 & 95 & 96 \\ 25 & 26 & 27 & 31 & 32 & 33 & 97 & 98 & 99 & 103 & 104 & 105 \\ 28 & 29 & 30 & 34 & 35 & 36 & 100 & 101 & 102 & 106 & 107 & 108 \\ 37 & 38 & 39 & 43 & 44 & 45 & 109 & 110 & 111 & 115 & 116 & 117 \\ 40 & 41 & 42 & 46 & 47 & 48 & 112 & 113 & 114 & 118 & 119 & 120 \\ 49 & 50 & 51 & 55 & 56 & 57 & 121 & 122 & 123 & 127 & 128 & 129 \\ 52 & 53 & 54 & 58 & 59 & 60 & 124 & 125 & 126 & 130 & 131 & 132 \\ 61 & 62 & 63 & 67 & 68 & 69 & 133 & 134 & 135 & 139 & 140 & 141 \\ 64 & 65 & 66 & 70 & 71 & 72 & 136 & 137 & 138 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -64 & -66 & -68 & -76 & -78 & -80 \\ 145 & 145 & 145 & 145 & 145 & 145 & -64 & -66 & -68 & -76 & -78 & -80 \\ 145 & 145 & 145 & 145 & 145 & 145 & -64 & -66 & -68 & -76 & -78 & -80 \\ 145 & 145 & 145 & 145 & 145 & 145 & -64 & -66 & -68 & -76 & -78 & -80 \\ 145 & 145 & 145 & 145 & 145 & 145 & -64 & -66 & -68 & -76 & -78 & -80 \\ 145 & 145 & 145 & 145 & 145 & 145 & -64 & -66 & -68 & -76 & -78 & -80 \\ -9 & -9 & -9 & -9 & -9 & -9 & 0 & 0 & 0 & 0 & 0 & 0 \\ -15 & -15 & -15 & -15 & -15 & -15 & 0 & 0 & 0 & 0 & 0 & 0 \\ -33 & -33 & -33 & -33 & -33 & -33 & 0 & 0 & 0 & 0 & 0 & 0 \\ -39 & -39 & -39 & -39 & -39 & -39 & 0 & 0 & 0 & 0 & 0 & 0 \\ -57 & -57 & -57 & -57 & -57 & -57 & 0 & 0 & 0 & 0 & 0 & 0 \\ -63 & -63 & -63 & -63 & -63 & -63 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

	Divisor Paths	Reversible Square	Block Representation
13	$\{(3,12),\ (2,12)\}$	$\begin{pmatrix} 1 & 2 & 7 & 8 & 13 & 14 & 19 & 20 & 25 & 26 & 31 & 32 \\ 3 & 4 & 9 & 10 & 15 & 16 & 21 & 22 & 27 & 28 & 33 & 34 \\ 5 & 6 & 11 & 12 & 17 & 18 & 23 & 24 & 29 & 30 & 35 & 36 \\ 37 & 38 & 43 & 44 & 49 & 50 & 55 & 56 & 61 & 62 & 67 & 68 \\ 39 & 40 & 45 & 46 & 51 & 52 & 57 & 58 & 63 & 64 & 69 & 70 \\ 41 & 42 & 47 & 48 & 53 & 54 & 59 & 60 & 65 & 66 & 71 & 72 \\ 73 & 74 & 79 & 80 & 85 & 86 & 91 & 92 & 97 & 98 & 103 & 104 \\ 75 & 76 & 81 & 82 & 87 & 88 & 93 & 94 & 99 & 100 & 105 & 106 \\ 77 & 78 & 83 & 84 & 89 & 90 & 95 & 96 & 101 & 102 & 107 & 108 \\ 109 & 110 & 115 & 116 & 121 & 122 & 127 & 128 & 133 & 134 & 139 & 140 \\ 111 & 112 & 117 & 118 & 123 & 124 & 129 & 130 & 135 & 136 & 141 & 142 \\ 113 & 114 & 119 & 120 & 125 & 126 & 131 & 132 & 137 & 138 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -17 & -19 & -29 & -31 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -17 & -19 & -29 & -31 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -17 & -19 & -29 & -31 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -17 & -19 & -29 & -31 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -17 & -19 & -29 & -31 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -17 & -19 & -29 & -31 \\ -32 & -32 & -32 & -32 & -32 & -32 & 0 & 0 & 0 & 0 & 0 & 0 \\ -36 & -36 & -36 & -36 & -36 & -36 & 0 & 0 & 0 & 0 & 0 & 0 \\ -40 & -40 & -40 & -40 & -40 & -40 & 0 & 0 & 0 & 0 & 0 & 0 \\ -104 & -104 & -104 & -104 & -104 & -104 & 0 & 0 & 0 & 0 & 0 & 0 \\ -108 & -108 & -108 & -108 & -108 & -108 & 0 & 0 & 0 & 0 & 0 & 0 \\ -112 & -112 & -112 & -112 & -112 & -112 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
14	$\{(3,12),\ (3,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 10 & 11 & 12 & 19 & 20 & 21 & 28 & 29 & 30 \\ 4 & 5 & 6 & 13 & 14 & 15 & 22 & 23 & 24 & 31 & 32 & 33 \\ 7 & 8 & 9 & 16 & 17 & 18 & 25 & 26 & 27 & 34 & 35 & 36 \\ 37 & 38 & 39 & 46 & 47 & 48 & 55 & 56 & 57 & 64 & 65 & 66 \\ 40 & 41 & 42 & 49 & 50 & 51 & 58 & 59 & 60 & 67 & 68 & 69 \\ 43 & 44 & 45 & 52 & 53 & 54 & 61 & 62 & 63 & 70 & 71 & 72 \\ 73 & 74 & 75 & 82 & 83 & 84 & 91 & 92 & 93 & 100 & 101 & 102 \\ 76 & 77 & 78 & 85 & 86 & 87 & 94 & 95 & 96 & 103 & 104 & 105 \\ 79 & 80 & 81 & 88 & 89 & 90 & 97 & 98 & 99 & 106 & 107 & 108 \\ 109 & 110 & 111 & 118 & 119 & 120 & 127 & 128 & 129 & 136 & 137 & 138 \\ 112 & 113 & 114 & 121 & 122 & 123 & 130 & 131 & 132 & 139 & 140 & 141 \\ 115 & 116 & 117 & 124 & 125 & 126 & 133 & 134 & 135 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -11 & -25 & -27 & -29 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -11 & -25 & -27 & -29 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -11 & -25 & -27 & -29 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -11 & -25 & -27 & -29 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -11 & -25 & -27 & -29 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -11 & -25 & -27 & -29 \\ -30 & -30 & -30 & -30 & -30 & -30 & 0 & 0 & 0 & 0 & 0 & 0 \\ -36 & -36 & -36 & -36 & -36 & -36 & 0 & 0 & 0 & 0 & 0 & 0 \\ -42 & -42 & -42 & -42 & -42 & -42 & 0 & 0 & 0 & 0 & 0 & 0 \\ -102 & -102 & -102 & -102 & -102 & -102 & 0 & 0 & 0 & 0 & 0 & 0 \\ -108 & -108 & -108 & -108 & -108 & -108 & 0 & 0 & 0 & 0 & 0 & 0 \\ -114 & -114 & -114 & -114 & -114 & -114 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
15	$\{(3,12),\ (4,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 13 & 14 & 15 & 16 & 25 & 26 & 27 & 28 \\ 5 & 6 & 7 & 8 & 17 & 18 & 19 & 20 & 29 & 30 & 31 & 32 \\ 9 & 10 & 11 & 12 & 21 & 22 & 23 & 24 & 33 & 34 & 35 & 36 \\ 37 & 38 & 39 & 40 & 49 & 50 & 51 & 52 & 61 & 62 & 63 & 64 \\ 41 & 42 & 43 & 44 & 53 & 54 & 55 & 56 & 65 & 66 & 67 & 68 \\ 45 & 46 & 47 & 48 & 57 & 58 & 59 & 60 & 69 & 70 & 71 & 72 \\ 73 & 74 & 75 & 76 & 85 & 86 & 87 & 88 & 97 & 98 & 99 & 100 \\ 77 & 78 & 79 & 80 & 89 & 90 & 91 & 92 & 101 & 102 & 103 & 104 \\ 81 & 82 & 83 & 84 & 93 & 94 & 95 & 96 & 105 & 106 & 107 & 108 \\ 109 & 110 & 111 & 112 & 121 & 122 & 123 & 124 & 133 & 134 & 135 & 136 \\ 113 & 114 & 115 & 116 & 125 & 126 & 127 & 128 & 137 & 138 & 139 & 140 \\ 117 & 118 & 119 & 120 & 129 & 130 & 131 & 132 & 141 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -21 & -23 & -25 & -27 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -21 & -23 & -25 & -27 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -21 & -23 & -25 & -27 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -21 & -23 & -25 & -27 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -21 & -23 & -25 & -27 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -21 & -23 & -25 & -27 \\ -28 & -28 & -28 & -28 & -28 & -28 & 0 & 0 & 0 & 0 & 0 & 0 \\ -36 & -36 & -36 & -36 & -36 & -36 & 0 & 0 & 0 & 0 & 0 & 0 \\ -44 & -44 & -44 & -44 & -44 & -44 & 0 & 0 & 0 & 0 & 0 & 0 \\ -100 & -100 & -100 & -100 & -100 & -100 & 0 & 0 & 0 & 0 & 0 & 0 \\ -108 & -108 & -108 & -108 & -108 & -108 & 0 & 0 & 0 & 0 & 0 & 0 \\ -116 & -116 & -116 & -116 & -116 & -116 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
16	$\{(3,12),\ (6,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 19 & 20 & 21 & 22 & 23 & 24 \\ 7 & 8 & 9 & 10 & 11 & 12 & 25 & 26 & 27 & 28 & 29 & 30 \\ 13 & 14 & 15 & 16 & 17 & 18 & 31 & 32 & 33 & 34 & 35 & 36 \\ 37 & 38 & 39 & 40 & 41 & 42 & 55 & 56 & 57 & 58 & 59 & 60 \\ 43 & 44 & 45 & 46 & 47 & 48 & 61 & 62 & 63 & 64 & 65 & 66 \\ 49 & 50 & 51 & 52 & 53 & 54 & 67 & 68 & 69 & 70 & 71 & 72 \\ 73 & 74 & 75 & 76 & 77 & 78 & 91 & 92 & 93 & 94 & 95 & 96 \\ 79 & 80 & 81 & 82 & 83 & 84 & 97 & 98 & 99 & 100 & 101 & 102 \\ 85 & 86 & 87 & 88 & 89 & 90 & 103 & 104 & 105 & 106 & 107 & 108 \\ 109 & 110 & 111 & 112 & 113 & 114 & 127 & 128 & 129 & 130 & 131 & 132 \\ 115 & 116 & 117 & 118 & 119 & 120 & 133 & 134 & 135 & 136 & 137 & 138 \\ 121 & 122 & 123 & 124 & 125 & 126 & 139 & 140 & 141 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -13 & -15 & -17 & -19 & -21 & -23 \\ 145 & 145 & 145 & 145 & 145 & 145 & -13 & -15 & -17 & -19 & -21 & -23 \\ 145 & 145 & 145 & 145 & 145 & 145 & -13 & -15 & -17 & -19 & -21 & -23 \\ 145 & 145 & 145 & 145 & 145 & 145 & -13 & -15 & -17 & -19 & -21 & -23 \\ 145 & 145 & 145 & 145 & 145 & 145 & -13 & -15 & -17 & -19 & -21 & -23 \\ 145 & 145 & 145 & 145 & 145 & 145 & -13 & -15 & -17 & -19 & -21 & -23 \\ -24 & -24 & -24 & -24 & -24 & -24 & 0 & 0 & 0 & 0 & 0 & 0 \\ -36 & -36 & -36 & -36 & -36 & -36 & 0 & 0 & 0 & 0 & 0 & 0 \\ -48 & -48 & -48 & -48 & -48 & -48 & 0 & 0 & 0 & 0 & 0 & 0 \\ -96 & -96 & -96 & -96 & -96 & -96 & 0 & 0 & 0 & 0 & 0 & 0 \\ -108 & -108 & -108 & -108 & -108 & -108 & 0 & 0 & 0 & 0 & 0 & 0 \\ -120 & -120 & -120 & -120 & -120 & -120 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

	Divisor Paths	Reversible Square	Block Representation
17	$\{(3,12),\ (2,4)\}$	$\begin{pmatrix} 1 & 2 & 7 & 8 & 49 & 50 & 55 & 56 & 97 & 98 & 103 & 104 \\ 3 & 4 & 9 & 10 & 51 & 52 & 57 & 58 & 99 & 100 & 105 & 106 \\ 5 & 6 & 11 & 12 & 53 & 54 & 59 & 60 & 101 & 102 & 107 & 108 \\ 13 & 14 & 19 & 20 & 61 & 62 & 67 & 68 & 109 & 110 & 115 & 116 \\ 15 & 16 & 21 & 22 & 63 & 64 & 69 & 70 & 111 & 112 & 117 & 118 \\ 17 & 18 & 23 & 24 & 65 & 66 & 71 & 72 & 113 & 114 & 119 & 120 \\ 25 & 26 & 31 & 32 & 73 & 74 & 79 & 80 & 121 & 122 & 127 & 128 \\ 27 & 28 & 33 & 34 & 75 & 76 & 81 & 82 & 123 & 124 & 129 & 130 \\ 29 & 30 & 35 & 36 & 77 & 78 & 83 & 84 & 125 & 126 & 131 & 132 \\ 37 & 38 & 43 & 44 & 85 & 86 & 91 & 92 & 133 & 134 & 139 & 140 \\ 39 & 40 & 45 & 46 & 87 & 88 & 93 & 94 & 135 & 136 & 141 & 142 \\ 41 & 42 & 47 & 48 & 89 & 90 & 95 & 96 & 137 & 138 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -89 & -91 & -101 & -103 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -89 & -91 & -101 & -103 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -89 & -91 & -101 & -103 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -89 & -91 & -101 & -103 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -89 & -91 & -101 & -103 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -89 & -91 & -101 & -103 \\ -8 & -8 & -8 & -8 & -8 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -12 & -12 & -12 & -12 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ -16 & -16 & -16 & -16 & -16 & -16 & 0 & 0 & 0 & 0 & 0 & 0 \\ -32 & -32 & -32 & -32 & -32 & -32 & 0 & 0 & 0 & 0 & 0 & 0 \\ -36 & -36 & -36 & -36 & -36 & -36 & 0 & 0 & 0 & 0 & 0 & 0 \\ -40 & -40 & -40 & -40 & -40 & -40 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
18	$\{(3,12),\ (2,6)\}$	$\begin{pmatrix} 1 & 2 & 7 & 8 & 13 & 14 & 73 & 74 & 79 & 80 & 85 & 86 \\ 3 & 4 & 9 & 10 & 15 & 16 & 75 & 76 & 81 & 82 & 87 & 88 \\ 5 & 6 & 11 & 12 & 17 & 18 & 77 & 78 & 83 & 84 & 89 & 90 \\ 19 & 20 & 25 & 26 & 31 & 32 & 91 & 92 & 97 & 98 & 103 & 104 \\ 21 & 22 & 27 & 28 & 33 & 34 & 93 & 94 & 99 & 100 & 105 & 106 \\ 23 & 24 & 29 & 30 & 35 & 36 & 95 & 96 & 101 & 102 & 107 & 108 \\ 37 & 38 & 43 & 44 & 49 & 50 & 109 & 110 & 115 & 116 & 121 & 122 \\ 39 & 40 & 45 & 46 & 51 & 52 & 111 & 112 & 117 & 118 & 123 & 124 \\ 41 & 42 & 47 & 48 & 53 & 54 & 113 & 114 & 119 & 120 & 125 & 126 \\ 55 & 56 & 61 & 62 & 67 & 68 & 127 & 128 & 133 & 134 & 139 & 140 \\ 57 & 58 & 63 & 64 & 69 & 70 & 129 & 130 & 135 & 136 & 141 & 142 \\ 59 & 60 & 65 & 66 & 71 & 72 & 131 & 132 & 137 & 138 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -59 & -61 & -71 & -73 & -83 & -85 \\ 145 & 145 & 145 & 145 & 145 & 145 & -59 & -61 & -71 & -73 & -83 & -85 \\ 145 & 145 & 145 & 145 & 145 & 145 & -59 & -61 & -71 & -73 & -83 & -85 \\ 145 & 145 & 145 & 145 & 145 & 145 & -59 & -61 & -71 & -73 & -83 & -85 \\ 145 & 145 & 145 & 145 & 145 & 145 & -59 & -61 & -71 & -73 & -83 & -85 \\ 145 & 145 & 145 & 145 & 145 & 145 & -59 & -61 & -71 & -73 & -83 & -85 \\ -14 & -14 & -14 & -14 & -14 & -14 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & -18 & -18 & -18 & -18 & -18 & 0 & 0 & 0 & 0 & 0 & 0 \\ -22 & -22 & -22 & -22 & -22 & -22 & 0 & 0 & 0 & 0 & 0 & 0 \\ -50 & -50 & -50 & -50 & -50 & -50 & 0 & 0 & 0 & 0 & 0 & 0 \\ -54 & -54 & -54 & -54 & -54 & -54 & 0 & 0 & 0 & 0 & 0 & 0 \\ -58 & -58 & -58 & -58 & -58 & -58 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
19	$\{(3,12),\ (3,6)\}$	$\begin{pmatrix} 1 & 2 & 3 & 10 & 11 & 12 & 73 & 74 & 75 & 82 & 83 & 84 \\ 4 & 5 & 6 & 13 & 14 & 15 & 76 & 77 & 78 & 85 & 86 & 87 \\ 7 & 8 & 9 & 16 & 17 & 18 & 79 & 80 & 81 & 88 & 89 & 90 \\ 19 & 20 & 21 & 28 & 29 & 30 & 91 & 92 & 93 & 100 & 101 & 102 \\ 22 & 23 & 24 & 31 & 32 & 33 & 94 & 95 & 96 & 103 & 104 & 105 \\ 25 & 26 & 27 & 34 & 35 & 36 & 97 & 98 & 99 & 106 & 107 & 108 \\ 37 & 38 & 39 & 46 & 47 & 48 & 109 & 110 & 111 & 118 & 119 & 120 \\ 40 & 41 & 42 & 49 & 50 & 51 & 112 & 113 & 114 & 121 & 122 & 123 \\ 43 & 44 & 45 & 52 & 53 & 54 & 115 & 116 & 117 & 124 & 125 & 126 \\ 55 & 56 & 57 & 64 & 65 & 66 & 127 & 128 & 129 & 136 & 137 & 138 \\ 58 & 59 & 60 & 67 & 68 & 69 & 130 & 131 & 132 & 139 & 140 & 141 \\ 61 & 62 & 63 & 70 & 71 & 72 & 133 & 134 & 135 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -61 & -63 & -65 & -79 & -81 & -83 \\ 145 & 145 & 145 & 145 & 145 & 145 & -61 & -63 & -65 & -79 & -81 & -83 \\ 145 & 145 & 145 & 145 & 145 & 145 & -61 & -63 & -65 & -79 & -81 & -83 \\ 145 & 145 & 145 & 145 & 145 & 145 & -61 & -63 & -65 & -79 & -81 & -83 \\ 145 & 145 & 145 & 145 & 145 & 145 & -61 & -63 & -65 & -79 & -81 & -83 \\ 145 & 145 & 145 & 145 & 145 & 145 & -61 & -63 & -65 & -79 & -81 & -83 \\ -12 & -12 & -12 & -12 & -12 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & -18 & -18 & -18 & -18 & -18 & 0 & 0 & 0 & 0 & 0 & 0 \\ -24 & -24 & -24 & -24 & -24 & -24 & 0 & 0 & 0 & 0 & 0 & 0 \\ -48 & -48 & -48 & -48 & -48 & -48 & 0 & 0 & 0 & 0 & 0 & 0 \\ -54 & -54 & -54 & -54 & -54 & -54 & 0 & 0 & 0 & 0 & 0 & 0 \\ -60 & -60 & -60 & -60 & -60 & -60 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
20	$\{(4,12),\ (2,12)\}$	$\begin{pmatrix} 1 & 2 & 9 & 10 & 17 & 18 & 25 & 26 & 33 & 34 & 41 & 42 \\ 3 & 4 & 11 & 12 & 19 & 20 & 27 & 28 & 35 & 36 & 43 & 44 \\ 5 & 6 & 13 & 14 & 21 & 22 & 29 & 30 & 37 & 38 & 45 & 46 \\ 7 & 8 & 15 & 16 & 23 & 24 & 31 & 32 & 39 & 40 & 47 & 48 \\ 49 & 50 & 57 & 58 & 65 & 66 & 73 & 74 & 81 & 82 & 89 & 90 \\ 51 & 52 & 59 & 60 & 67 & 68 & 75 & 76 & 83 & 84 & 91 & 92 \\ 53 & 54 & 61 & 62 & 69 & 70 & 77 & 78 & 85 & 86 & 93 & 94 \\ 55 & 56 & 63 & 64 & 71 & 72 & 79 & 80 & 87 & 88 & 95 & 96 \\ 97 & 98 & 105 & 106 & 113 & 114 & 121 & 122 & 129 & 130 & 137 & 138 \\ 99 & 100 & 107 & 108 & 115 & 116 & 123 & 124 & 131 & 132 & 139 & 140 \\ 101 & 102 & 109 & 110 & 117 & 118 & 125 & 126 & 133 & 134 & 141 & 142 \\ 103 & 104 & 111 & 112 & 119 & 120 & 127 & 128 & 135 & 136 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -23 & -25 & -39 & -41 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -23 & -25 & -39 & -41 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -23 & -25 & -39 & -41 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -23 & -25 & -39 & -41 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -23 & -25 & -39 & -41 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -23 & -25 & -39 & -41 \\ -2 & -2 & -2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -6 & -6 & -6 & -6 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -90 & -90 & -90 & -90 & -90 & -90 & 0 & 0 & 0 & 0 & 0 & 0 \\ -94 & -94 & -94 & -94 & -94 & -94 & 0 & 0 & 0 & 0 & 0 & 0 \\ -98 & -98 & -98 & -98 & -98 & -98 & 0 & 0 & 0 & 0 & 0 & 0 \\ -102 & -102 & -102 & -102 & -102 & -102 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

	Divisor Paths	Reversible Square	Block Representation
21	$\{(4,12),\ (2,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 13 & 14 & 15 & 25 & 26 & 27 & 37 & 38 & 39 \\ 4 & 5 & 6 & 16 & 17 & 18 & 28 & 29 & 30 & 40 & 41 & 42 \\ 7 & 8 & 9 & 19 & 20 & 21 & 31 & 32 & 33 & 43 & 44 & 45 \\ 10 & 11 & 12 & 22 & 23 & 24 & 34 & 35 & 36 & 46 & 47 & 48 \\ 49 & 50 & 51 & 61 & 62 & 63 & 73 & 74 & 75 & 85 & 86 & 87 \\ 52 & 53 & 54 & 64 & 65 & 66 & 76 & 77 & 78 & 88 & 89 & 90 \\ 55 & 56 & 57 & 67 & 68 & 69 & 79 & 80 & 81 & 91 & 92 & 93 \\ 58 & 59 & 60 & 70 & 71 & 72 & 82 & 83 & 84 & 94 & 95 & 96 \\ 97 & 98 & 99 & 109 & 110 & 111 & 121 & 122 & 123 & 133 & 134 & 135 \\ 100 & 101 & 102 & 112 & 113 & 114 & 124 & 125 & 126 & 136 & 137 & 138 \\ 103 & 104 & 105 & 115 & 116 & 117 & 127 & 128 & 129 & 139 & 140 & 141 \\ 106 & 107 & 108 & 118 & 119 & 120 & 130 & 131 & 132 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -10 & -12 & -14 & -34 & -36 & -38 \\ 145 & 145 & 145 & 145 & 145 & 145 & -10 & -12 & -14 & -34 & -36 & -38 \\ 145 & 145 & 145 & 145 & 145 & 145 & -10 & -12 & -14 & -34 & -36 & -38 \\ 145 & 145 & 145 & 145 & 145 & 145 & -10 & -12 & -14 & -34 & -36 & -38 \\ 145 & 145 & 145 & 145 & 145 & 145 & -10 & -12 & -14 & -34 & -36 & -38 \\ 145 & 145 & 145 & 145 & 145 & 145 & -10 & -12 & -14 & -34 & -36 & -38 \\ -3 & -3 & -3 & -3 & -3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & -9 & -9 & -9 & -9 & -9 & 0 & 0 & 0 & 0 & 0 & 0 \\ -87 & -87 & -87 & -87 & -87 & -87 & 0 & 0 & 0 & 0 & 0 & 0 \\ -93 & -93 & -93 & -93 & -93 & -93 & 0 & 0 & 0 & 0 & 0 & 0 \\ -99 & -99 & -99 & -99 & -99 & -99 & 0 & 0 & 0 & 0 & 0 & 0 \\ -105 & -105 & -105 & -105 & -105 & -105 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
22	$\{(4,12),\ (3,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 17 & 18 & 19 & 20 & 33 & 34 & 35 & 36 \\ 5 & 6 & 7 & 8 & 21 & 22 & 23 & 24 & 37 & 38 & 39 & 40 \\ 9 & 10 & 11 & 12 & 25 & 26 & 27 & 28 & 41 & 42 & 43 & 44 \\ 13 & 14 & 15 & 16 & 29 & 30 & 31 & 32 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 65 & 66 & 67 & 68 & 81 & 82 & 83 & 84 \\ 53 & 54 & 55 & 56 & 69 & 70 & 71 & 72 & 85 & 86 & 87 & 88 \\ 57 & 58 & 59 & 60 & 73 & 74 & 75 & 76 & 89 & 90 & 91 & 92 \\ 61 & 62 & 63 & 64 & 77 & 78 & 79 & 80 & 93 & 94 & 95 & 96 \\ 97 & 98 & 99 & 100 & 113 & 114 & 115 & 116 & 129 & 130 & 131 & 132 \\ 101 & 102 & 103 & 104 & 117 & 118 & 119 & 120 & 133 & 134 & 135 & 136 \\ 105 & 106 & 107 & 108 & 121 & 122 & 123 & 124 & 137 & 138 & 139 & 140 \\ 109 & 110 & 111 & 112 & 125 & 126 & 127 & 128 & 141 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -29 & -31 & -33 & -35 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -29 & -31 & -33 & -35 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -29 & -31 & -33 & -35 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -29 & -31 & -33 & -35 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -29 & -31 & -33 & -35 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -29 & -31 & -33 & -35 \\ -4 & -4 & -4 & -4 & -4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -12 & -12 & -12 & -12 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ -84 & -84 & -84 & -84 & -84 & -84 & 0 & 0 & 0 & 0 & 0 & 0 \\ -92 & -92 & -92 & -92 & -92 & -92 & 0 & 0 & 0 & 0 & 0 & 0 \\ -100 & -100 & -100 & -100 & -100 & -100 & 0 & 0 & 0 & 0 & 0 & 0 \\ -108 & -108 & -108 & -108 & -108 & -108 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
23	$\{(4,12),\ (4,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 25 & 26 & 27 & 28 & 29 & 30 \\ 7 & 8 & 9 & 10 & 11 & 12 & 31 & 32 & 33 & 34 & 35 & 36 \\ 13 & 14 & 15 & 16 & 17 & 18 & 37 & 38 & 39 & 40 & 41 & 42 \\ 19 & 20 & 21 & 22 & 23 & 24 & 43 & 44 & 45 & 46 & 47 & 48 \\ 49 & 50 & 51 & 52 & 53 & 54 & 73 & 74 & 75 & 76 & 77 & 78 \\ 55 & 56 & 57 & 58 & 59 & 60 & 79 & 80 & 81 & 82 & 83 & 84 \\ 61 & 62 & 63 & 64 & 65 & 66 & 85 & 86 & 87 & 88 & 89 & 90 \\ 67 & 68 & 69 & 70 & 71 & 72 & 91 & 92 & 93 & 94 & 95 & 96 \\ 97 & 98 & 99 & 100 & 101 & 102 & 121 & 122 & 123 & 124 & 125 & 126 \\ 103 & 104 & 105 & 106 & 107 & 108 & 127 & 128 & 129 & 130 & 131 & 132 \\ 109 & 110 & 111 & 112 & 113 & 114 & 133 & 134 & 135 & 136 & 137 & 138 \\ 115 & 116 & 117 & 118 & 119 & 120 & 139 & 140 & 141 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -19 & -21 & -23 & -25 & -27 & -29 \\ 145 & 145 & 145 & 145 & 145 & 145 & -19 & -21 & -23 & -25 & -27 & -29 \\ 145 & 145 & 145 & 145 & 145 & 145 & -19 & -21 & -23 & -25 & -27 & -29 \\ 145 & 145 & 145 & 145 & 145 & 145 & -19 & -21 & -23 & -25 & -27 & -29 \\ 145 & 145 & 145 & 145 & 145 & 145 & -19 & -21 & -23 & -25 & -27 & -29 \\ 145 & 145 & 145 & 145 & 145 & 145 & -19 & -21 & -23 & -25 & -27 & -29 \\ -6 & -6 & -6 & -6 & -6 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & -18 & -18 & -18 & -18 & -18 & 0 & 0 & 0 & 0 & 0 & 0 \\ -78 & -78 & -78 & -78 & -78 & -78 & 0 & 0 & 0 & 0 & 0 & 0 \\ -90 & -90 & -90 & -90 & -90 & -90 & 0 & 0 & 0 & 0 & 0 & 0 \\ -102 & -102 & -102 & -102 & -102 & -102 & 0 & 0 & 0 & 0 & 0 & 0 \\ -114 & -114 & -114 & -114 & -114 & -114 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
24	$\{(4,12),\ (6,12)\}$	$\begin{pmatrix} 1 & 2 & 9 & 10 & 49 & 50 & 57 & 58 & 97 & 98 & 105 & 106 \\ 3 & 4 & 11 & 12 & 51 & 52 & 59 & 60 & 99 & 100 & 107 & 108 \\ 5 & 6 & 13 & 14 & 53 & 54 & 61 & 62 & 101 & 102 & 109 & 110 \\ 7 & 8 & 15 & 16 & 55 & 56 & 63 & 64 & 103 & 104 & 111 & 112 \\ 17 & 18 & 25 & 26 & 65 & 66 & 73 & 74 & 113 & 114 & 121 & 122 \\ 19 & 20 & 27 & 28 & 67 & 68 & 75 & 76 & 115 & 116 & 123 & 124 \\ 21 & 22 & 29 & 30 & 69 & 70 & 77 & 78 & 117 & 118 & 125 & 126 \\ 23 & 24 & 31 & 32 & 71 & 72 & 79 & 80 & 119 & 120 & 127 & 128 \\ 33 & 34 & 41 & 42 & 81 & 82 & 89 & 90 & 129 & 130 & 137 & 138 \\ 35 & 36 & 43 & 44 & 83 & 84 & 91 & 92 & 131 & 132 & 139 & 140 \\ 37 & 38 & 45 & 46 & 85 & 86 & 93 & 94 & 133 & 134 & 141 & 142 \\ 39 & 40 & 47 & 48 & 87 & 88 & 95 & 96 & 135 & 136 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -87 & -89 & -103 & -105 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -87 & -89 & -103 & -105 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -87 & -89 & -103 & -105 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -87 & -89 & -103 & -105 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -87 & -89 & -103 & -105 \\ 145 & 145 & 145 & 145 & 145 & 145 & -7 & -9 & -87 & -89 & -103 & -105 \\ -2 & -2 & -2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -6 & -6 & -6 & -6 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -26 & -26 & -26 & -26 & -26 & -26 & 0 & 0 & 0 & 0 & 0 & 0 \\ -30 & -30 & -30 & -30 & -30 & -30 & 0 & 0 & 0 & 0 & 0 & 0 \\ -34 & -34 & -34 & -34 & -34 & -34 & 0 & 0 & 0 & 0 & 0 & 0 \\ -38 & -38 & -38 & -38 & -38 & -38 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

	Divisor Paths	Reversible Square	Block Representation
25	$\{(4,12),\ (2,6)\}$	$\begin{pmatrix} 1 & 2 & 9 & 10 & 17 & 18 & 73 & 74 & 81 & 82 & 89 & 90 \\ 3 & 4 & 11 & 12 & 19 & 20 & 75 & 76 & 83 & 84 & 91 & 92 \\ 5 & 6 & 13 & 14 & 21 & 22 & 77 & 78 & 85 & 86 & 93 & 94 \\ 7 & 8 & 15 & 16 & 23 & 24 & 79 & 80 & 87 & 88 & 95 & 96 \\ 25 & 26 & 33 & 34 & 41 & 42 & 97 & 98 & 105 & 106 & 113 & 114 \\ 27 & 28 & 35 & 36 & 43 & 44 & 99 & 100 & 107 & 108 & 115 & 116 \\ 29 & 30 & 37 & 38 & 45 & 46 & 101 & 102 & 109 & 110 & 117 & 118 \\ 31 & 32 & 39 & 40 & 47 & 48 & 103 & 104 & 111 & 112 & 119 & 120 \\ 49 & 50 & 57 & 58 & 65 & 66 & 121 & 122 & 129 & 130 & 137 & 138 \\ 51 & 52 & 59 & 60 & 67 & 68 & 123 & 124 & 131 & 132 & 139 & 140 \\ 53 & 54 & 61 & 62 & 69 & 70 & 125 & 126 & 133 & 134 & 141 & 142 \\ 55 & 56 & 63 & 64 & 71 & 72 & 127 & 128 & 135 & 136 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -55 & -57 & -71 & -73 & -87 & -89 \\ 145 & 145 & 145 & 145 & 145 & 145 & -55 & -57 & -71 & -73 & -87 & -89 \\ 145 & 145 & 145 & 145 & 145 & 145 & -55 & -57 & -71 & -73 & -87 & -89 \\ 145 & 145 & 145 & 145 & 145 & 145 & -55 & -57 & -71 & -73 & -87 & -89 \\ 145 & 145 & 145 & 145 & 145 & 145 & -55 & -57 & -71 & -73 & -87 & -89 \\ 145 & 145 & 145 & 145 & 145 & 145 & -55 & -57 & -71 & -73 & -87 & -89 \\ -2 & -2 & -2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -6 & -6 & -6 & -6 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -42 & -42 & -42 & -42 & -42 & -42 & 0 & 0 & 0 & 0 & 0 & 0 \\ -46 & -46 & -46 & -46 & -46 & -46 & 0 & 0 & 0 & 0 & 0 & 0 \\ -50 & -50 & -50 & -50 & -50 & -50 & 0 & 0 & 0 & 0 & 0 & 0 \\ -54 & -54 & -54 & -54 & -54 & -54 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
26	$\{(4,12),\ (3,6)\}$	$\begin{pmatrix} 1 & 2 & 3 & 13 & 14 & 15 & 73 & 74 & 75 & 85 & 86 & 87 \\ 4 & 5 & 6 & 16 & 17 & 18 & 76 & 77 & 78 & 88 & 89 & 90 \\ 7 & 8 & 9 & 19 & 20 & 21 & 79 & 80 & 81 & 91 & 92 & 93 \\ 10 & 11 & 12 & 22 & 23 & 24 & 82 & 83 & 84 & 94 & 95 & 96 \\ 25 & 26 & 27 & 37 & 38 & 39 & 97 & 98 & 99 & 109 & 110 & 111 \\ 28 & 29 & 30 & 40 & 41 & 42 & 100 & 101 & 102 & 112 & 113 & 114 \\ 31 & 32 & 33 & 43 & 44 & 45 & 103 & 104 & 105 & 115 & 116 & 117 \\ 34 & 35 & 36 & 46 & 47 & 48 & 106 & 107 & 108 & 118 & 119 & 120 \\ 49 & 50 & 51 & 61 & 62 & 63 & 121 & 122 & 123 & 133 & 134 & 135 \\ 52 & 53 & 54 & 64 & 65 & 66 & 124 & 125 & 126 & 136 & 137 & 138 \\ 55 & 56 & 57 & 67 & 68 & 69 & 127 & 128 & 129 & 139 & 140 & 141 \\ 58 & 59 & 60 & 70 & 71 & 72 & 130 & 131 & 132 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -58 & -60 & -62 & -82 & -84 & -86 \\ 145 & 145 & 145 & 145 & 145 & 145 & -58 & -60 & -62 & -82 & -84 & -86 \\ 145 & 145 & 145 & 145 & 145 & 145 & -58 & -60 & -62 & -82 & -84 & -86 \\ 145 & 145 & 145 & 145 & 145 & 145 & -58 & -60 & -62 & -82 & -84 & -86 \\ 145 & 145 & 145 & 145 & 145 & 145 & -58 & -60 & -62 & -82 & -84 & -86 \\ 145 & 145 & 145 & 145 & 145 & 145 & -58 & -60 & -62 & -82 & -84 & -86 \\ -3 & -3 & -3 & -3 & -3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & -9 & -9 & -9 & -9 & -9 & 0 & 0 & 0 & 0 & 0 & 0 \\ -39 & -39 & -39 & -39 & -39 & -39 & 0 & 0 & 0 & 0 & 0 & 0 \\ -45 & -45 & -45 & -45 & -45 & -45 & 0 & 0 & 0 & 0 & 0 & 0 \\ -51 & -51 & -51 & -51 & -51 & -51 & 0 & 0 & 0 & 0 & 0 & 0 \\ -57 & -57 & -57 & -57 & -57 & -57 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
27	$\{(6,12),\ (2,12)\}$	$\begin{pmatrix} 1 & 2 & 13 & 14 & 25 & 26 & 37 & 38 & 49 & 50 & 61 & 62 \\ 3 & 4 & 15 & 16 & 27 & 28 & 39 & 40 & 51 & 52 & 63 & 64 \\ 5 & 6 & 17 & 18 & 29 & 30 & 41 & 42 & 53 & 54 & 65 & 66 \\ 7 & 8 & 19 & 20 & 31 & 32 & 43 & 44 & 55 & 56 & 67 & 68 \\ 9 & 10 & 21 & 22 & 33 & 34 & 45 & 46 & 57 & 58 & 69 & 70 \\ 11 & 12 & 23 & 24 & 35 & 36 & 47 & 48 & 59 & 60 & 71 & 72 \\ 73 & 74 & 85 & 86 & 97 & 98 & 109 & 110 & 121 & 122 & 133 & 134 \\ 75 & 76 & 87 & 88 & 99 & 100 & 111 & 112 & 123 & 124 & 135 & 136 \\ 77 & 78 & 89 & 90 & 101 & 102 & 113 & 114 & 125 & 126 & 137 & 138 \\ 79 & 80 & 91 & 92 & 103 & 104 & 115 & 116 & 127 & 128 & 139 & 140 \\ 81 & 82 & 93 & 94 & 105 & 106 & 117 & 118 & 129 & 130 & 141 & 142 \\ 83 & 84 & 95 & 96 & 107 & 108 & 119 & 120 & 131 & 132 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -11 & -13 & -35 & -37 & -59 & -61 \\ 145 & 145 & 145 & 145 & 145 & 145 & -11 & -13 & -35 & -37 & -59 & -61 \\ 145 & 145 & 145 & 145 & 145 & 145 & -11 & -13 & -35 & -37 & -59 & -61 \\ 145 & 145 & 145 & 145 & 145 & 145 & -11 & -13 & -35 & -37 & -59 & -61 \\ 145 & 145 & 145 & 145 & 145 & 145 & -11 & -13 & -35 & -37 & -59 & -61 \\ 145 & 145 & 145 & 145 & 145 & 145 & -11 & -13 & -35 & -37 & -59 & -61 \\ -62 & -62 & -62 & -62 & -62 & -62 & 0 & 0 & 0 & 0 & 0 & 0 \\ -66 & -66 & -66 & -66 & -66 & -66 & 0 & 0 & 0 & 0 & 0 & 0 \\ -70 & -70 & -70 & -70 & -70 & -70 & 0 & 0 & 0 & 0 & 0 & 0 \\ -74 & -74 & -74 & -74 & -74 & -74 & 0 & 0 & 0 & 0 & 0 & 0 \\ -78 & -78 & -78 & -78 & -78 & -78 & 0 & 0 & 0 & 0 & 0 & 0 \\ -82 & -82 & -82 & -82 & -82 & -82 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
28	$\{6,12),\ (3,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 19 & 20 & 21 & 37 & 38 & 39 & 55 & 56 & 57 \\ 4 & 5 & 6 & 22 & 23 & 24 & 40 & 41 & 42 & 58 & 59 & 60 \\ 7 & 8 & 9 & 25 & 26 & 27 & 43 & 44 & 45 & 61 & 62 & 63 \\ 10 & 11 & 12 & 28 & 29 & 30 & 46 & 47 & 48 & 64 & 65 & 66 \\ 13 & 14 & 15 & 31 & 32 & 33 & 49 & 50 & 51 & 67 & 68 & 69 \\ 16 & 17 & 18 & 34 & 35 & 36 & 52 & 53 & 54 & 70 & 71 & 72 \\ 73 & 74 & 75 & 91 & 92 & 93 & 109 & 110 & 111 & 127 & 128 & 129 \\ 76 & 77 & 78 & 94 & 95 & 96 & 112 & 113 & 114 & 130 & 131 & 132 \\ 79 & 80 & 81 & 97 & 98 & 99 & 115 & 116 & 117 & 133 & 134 & 135 \\ 82 & 83 & 84 & 100 & 101 & 102 & 118 & 119 & 120 & 136 & 137 & 138 \\ 85 & 86 & 87 & 103 & 104 & 105 & 121 & 122 & 123 & 139 & 140 & 141 \\ 88 & 89 & 90 & 106 & 107 & 108 & 124 & 125 & 126 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -16 & -18 & -20 & -52 & -54 & -56 \\ 145 & 145 & 145 & 145 & 145 & 145 & -16 & -18 & -20 & -52 & -54 & -56 \\ 145 & 145 & 145 & 145 & 145 & 145 & -16 & -18 & -20 & -52 & -54 & -56 \\ 145 & 145 & 145 & 145 & 145 & 145 & -16 & -18 & -20 & -52 & -54 & -56 \\ 145 & 145 & 145 & 145 & 145 & 145 & -16 & -18 & -20 & -52 & -54 & -56 \\ 145 & 145 & 145 & 145 & 145 & 145 & -16 & -18 & -20 & -52 & -54 & -56 \\ -57 & -57 & -57 & -57 & -57 & -57 & 0 & 0 & 0 & 0 & 0 & 0 \\ -63 & -63 & -63 & -63 & -63 & -63 & 0 & 0 & 0 & 0 & 0 & 0 \\ -69 & -69 & -69 & -69 & -69 & -69 & 0 & 0 & 0 & 0 & 0 & 0 \\ -75 & -75 & -75 & -75 & -75 & -75 & 0 & 0 & 0 & 0 & 0 & 0 \\ -81 & -81 & -81 & -81 & -81 & -81 & 0 & 0 & 0 & 0 & 0 & 0 \\ -87 & -87 & -87 & -87 & -87 & -87 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

	Divisor Paths	Reversible Square	Block Representation
29	$\{(6,12),\ (4,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 25 & 26 & 27 & 28 & 49 & 50 & 51 & 52 \\ 5 & 6 & 7 & 8 & 29 & 30 & 31 & 32 & 53 & 54 & 55 & 56 \\ 9 & 10 & 11 & 12 & 33 & 34 & 35 & 36 & 57 & 58 & 59 & 60 \\ 13 & 14 & 15 & 16 & 37 & 38 & 39 & 40 & 61 & 62 & 63 & 64 \\ 17 & 18 & 19 & 20 & 41 & 42 & 43 & 44 & 65 & 66 & 67 & 68 \\ 21 & 22 & 23 & 24 & 45 & 46 & 47 & 48 & 69 & 70 & 71 & 72 \\ 73 & 74 & 75 & 76 & 97 & 98 & 99 & 100 & 121 & 122 & 123 & 124 \\ 77 & 78 & 79 & 80 & 101 & 102 & 103 & 104 & 125 & 126 & 127 & 128 \\ 81 & 82 & 83 & 84 & 105 & 106 & 107 & 108 & 129 & 130 & 131 & 132 \\ 85 & 86 & 87 & 88 & 109 & 110 & 111 & 112 & 133 & 134 & 135 & 136 \\ 89 & 90 & 91 & 92 & 113 & 114 & 115 & 116 & 137 & 138 & 139 & 140 \\ 93 & 94 & 95 & 96 & 117 & 118 & 119 & 120 & 141 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -45 & -47 & -49 & -51 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -45 & -47 & -49 & -51 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -45 & -47 & -49 & -51 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -45 & -47 & -49 & -51 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -45 & -47 & -49 & -51 \\ 145 & 145 & 145 & 145 & 145 & 145 & -1 & -3 & -45 & -47 & -49 & -51 \\ -52 & -52 & -52 & -52 & -52 & -52 & 0 & 0 & 0 & 0 & 0 & 0 \\ -60 & -60 & -60 & -60 & -60 & -60 & 0 & 0 & 0 & 0 & 0 & 0 \\ -68 & -68 & -68 & -68 & -68 & -68 & 0 & 0 & 0 & 0 & 0 & 0 \\ -76 & -76 & -76 & -76 & -76 & -76 & 0 & 0 & 0 & 0 & 0 & 0 \\ -84 & -84 & -84 & -84 & -84 & -84 & 0 & 0 & 0 & 0 & 0 & 0 \\ -92 & -92 & -92 & -92 & -92 & -92 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
30	$\{(6,12),\ (6,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 37 & 38 & 39 & 40 & 41 & 42 \\ 7 & 8 & 9 & 10 & 11 & 12 & 43 & 44 & 45 & 46 & 47 & 48 \\ 13 & 14 & 15 & 16 & 17 & 18 & 49 & 50 & 51 & 52 & 53 & 54 \\ 19 & 20 & 21 & 22 & 23 & 24 & 55 & 56 & 57 & 58 & 59 & 60 \\ 25 & 26 & 27 & 28 & 29 & 30 & 61 & 62 & 63 & 64 & 65 & 66 \\ 31 & 32 & 33 & 34 & 35 & 36 & 67 & 68 & 69 & 70 & 71 & 72 \\ 73 & 74 & 75 & 76 & 77 & 78 & 109 & 110 & 111 & 112 & 113 & 114 \\ 79 & 80 & 81 & 82 & 83 & 84 & 115 & 116 & 117 & 118 & 119 & 120 \\ 85 & 86 & 87 & 88 & 89 & 90 & 121 & 122 & 123 & 124 & 125 & 126 \\ 91 & 92 & 93 & 94 & 95 & 96 & 127 & 128 & 129 & 130 & 131 & 132 \\ 97 & 98 & 99 & 100 & 101 & 102 & 133 & 134 & 135 & 136 & 137 & 138 \\ 103 & 104 & 105 & 106 & 107 & 108 & 139 & 140 & 141 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -31 & -33 & -35 & -37 & -39 & -41 \\ 145 & 145 & 145 & 145 & 145 & 145 & -31 & -33 & -35 & -37 & -39 & -41 \\ 145 & 145 & 145 & 145 & 145 & 145 & -31 & -33 & -35 & -37 & -39 & -41 \\ 145 & 145 & 145 & 145 & 145 & 145 & -31 & -33 & -35 & -37 & -39 & -41 \\ 145 & 145 & 145 & 145 & 145 & 145 & -31 & -33 & -35 & -37 & -39 & -41 \\ 145 & 145 & 145 & 145 & 145 & 145 & -31 & -33 & -35 & -37 & -39 & -41 \\ -42 & -42 & -42 & -42 & -42 & -42 & 0 & 0 & 0 & 0 & 0 & 0 \\ -54 & -54 & -54 & -54 & -54 & -54 & 0 & 0 & 0 & 0 & 0 & 0 \\ -66 & -66 & -66 & -66 & -66 & -66 & 0 & 0 & 0 & 0 & 0 & 0 \\ -78 & -78 & -78 & -78 & -78 & -78 & 0 & 0 & 0 & 0 & 0 & 0 \\ -90 & -90 & -90 & -90 & -90 & -90 & 0 & 0 & 0 & 0 & 0 & 0 \\ -102 & -102 & -102 & -102 & -102 & -102 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
31	$\{(6,12),\ (2,4)\}$	$\begin{pmatrix} 1 & 2 & 13 & 14 & 49 & 50 & 61 & 62 & 97 & 98 & 109 & 110 \\ 3 & 4 & 15 & 16 & 51 & 52 & 63 & 64 & 99 & 100 & 111 & 112 \\ 5 & 6 & 17 & 18 & 53 & 54 & 65 & 66 & 101 & 102 & 113 & 114 \\ 7 & 8 & 19 & 20 & 55 & 56 & 67 & 68 & 103 & 104 & 115 & 116 \\ 9 & 10 & 21 & 22 & 57 & 58 & 69 & 70 & 105 & 106 & 117 & 118 \\ 11 & 12 & 23 & 24 & 59 & 60 & 71 & 72 & 107 & 108 & 119 & 120 \\ 25 & 26 & 37 & 38 & 73 & 74 & 85 & 86 & 121 & 122 & 133 & 134 \\ 27 & 28 & 39 & 40 & 75 & 76 & 87 & 88 & 123 & 124 & 135 & 136 \\ 29 & 30 & 41 & 42 & 77 & 78 & 89 & 90 & 125 & 126 & 137 & 138 \\ 31 & 32 & 43 & 44 & 79 & 80 & 91 & 92 & 127 & 128 & 139 & 140 \\ 33 & 34 & 45 & 46 & 81 & 82 & 93 & 94 & 129 & 130 & 141 & 142 \\ 35 & 36 & 47 & 48 & 83 & 84 & 95 & 96 & 131 & 132 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -11 & -13 & -83 & -85 & -107 & -109 \\ 145 & 145 & 145 & 145 & 145 & 145 & -11 & -13 & -83 & -85 & -107 & -109 \\ 145 & 145 & 145 & 145 & 145 & 145 & -11 & -13 & -83 & -85 & -107 & -109 \\ 145 & 145 & 145 & 145 & 145 & 145 & -11 & -13 & -83 & -85 & -107 & -109 \\ 145 & 145 & 145 & 145 & 145 & 145 & -11 & -13 & -83 & -85 & -107 & -109 \\ 145 & 145 & 145 & 145 & 145 & 145 & -11 & -13 & -83 & -85 & -107 & -109 \\ -14 & -14 & -14 & -14 & -14 & -14 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & -18 & -18 & -18 & -18 & -18 & 0 & 0 & 0 & 0 & 0 & 0 \\ -22 & -22 & -22 & -22 & -22 & -22 & 0 & 0 & 0 & 0 & 0 & 0 \\ -26 & -26 & -26 & -26 & -26 & -26 & 0 & 0 & 0 & 0 & 0 & 0 \\ -30 & -30 & -30 & -30 & -30 & -30 & 0 & 0 & 0 & 0 & 0 & 0 \\ -34 & -34 & -34 & -34 & -34 & -34 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
32	$\{6,12),\ (2,6)\}$	$\begin{pmatrix} 1 & 2 & 13 & 14 & 25 & 26 & 73 & 74 & 85 & 86 & 97 & 98 \\ 3 & 4 & 15 & 16 & 27 & 28 & 75 & 76 & 87 & 88 & 99 & 100 \\ 5 & 6 & 17 & 18 & 29 & 30 & 77 & 78 & 89 & 90 & 101 & 102 \\ 7 & 8 & 19 & 20 & 31 & 32 & 79 & 80 & 91 & 92 & 103 & 104 \\ 9 & 10 & 21 & 22 & 33 & 34 & 81 & 82 & 93 & 94 & 105 & 106 \\ 11 & 12 & 23 & 24 & 35 & 36 & 83 & 84 & 95 & 96 & 107 & 108 \\ 37 & 38 & 49 & 50 & 61 & 62 & 109 & 110 & 121 & 122 & 133 & 134 \\ 39 & 40 & 51 & 52 & 63 & 64 & 111 & 112 & 123 & 124 & 135 & 136 \\ 41 & 42 & 53 & 54 & 65 & 66 & 113 & 114 & 125 & 126 & 137 & 138 \\ 43 & 44 & 55 & 56 & 67 & 68 & 115 & 116 & 127 & 128 & 139 & 140 \\ 45 & 46 & 57 & 58 & 69 & 70 & 117 & 118 & 129 & 130 & 141 & 142 \\ 47 & 48 & 59 & 60 & 71 & 72 & 119 & 120 & 131 & 132 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -47 & -49 & -71 & -73 & -95 & -97 \\ 145 & 145 & 145 & 145 & 145 & 145 & -47 & -49 & -71 & -73 & -95 & -97 \\ 145 & 145 & 145 & 145 & 145 & 145 & -47 & -49 & -71 & -73 & -95 & -97 \\ 145 & 145 & 145 & 145 & 145 & 145 & -47 & -49 & -71 & -73 & -95 & -97 \\ 145 & 145 & 145 & 145 & 145 & 145 & -47 & -49 & -71 & -73 & -95 & -97 \\ 145 & 145 & 145 & 145 & 145 & 145 & -47 & -49 & -71 & -73 & -95 & -97 \\ -26 & -26 & -26 & -26 & -26 & -26 & 0 & 0 & 0 & 0 & 0 & 0 \\ -30 & -30 & -30 & -30 & -30 & -30 & 0 & 0 & 0 & 0 & 0 & 0 \\ -34 & -34 & -34 & -34 & -34 & -34 & 0 & 0 & 0 & 0 & 0 & 0 \\ -38 & -38 & -38 & -38 & -38 & -38 & 0 & 0 & 0 & 0 & 0 & 0 \\ -42 & -42 & -42 & -42 & -42 & -42 & 0 & 0 & 0 & 0 & 0 & 0 \\ -46 & -46 & -46 & -46 & -46 & -46 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

	Divisor Paths	Reversible Square	Block Representation
33	$\{(6,12),\ (3,6)\}$	$\begin{pmatrix} 1 & 2 & 3 & 19 & 20 & 21 & 73 & 74 & 75 & 91 & 92 & 93 \\ 4 & 5 & 6 & 22 & 23 & 24 & 76 & 77 & 78 & 94 & 95 & 96 \\ 7 & 8 & 9 & 25 & 26 & 27 & 79 & 80 & 81 & 97 & 98 & 99 \\ 10 & 11 & 12 & 28 & 29 & 30 & 82 & 83 & 84 & 100 & 101 & 102 \\ 13 & 14 & 15 & 31 & 32 & 33 & 85 & 86 & 87 & 103 & 104 & 105 \\ 16 & 17 & 18 & 34 & 35 & 36 & 88 & 89 & 90 & 106 & 107 & 108 \\ 37 & 38 & 39 & 55 & 56 & 57 & 109 & 110 & 111 & 127 & 128 & 129 \\ 40 & 41 & 42 & 58 & 59 & 60 & 112 & 113 & 114 & 130 & 131 & 132 \\ 43 & 44 & 45 & 61 & 62 & 63 & 115 & 116 & 117 & 133 & 134 & 135 \\ 46 & 47 & 48 & 64 & 65 & 66 & 118 & 119 & 120 & 136 & 137 & 138 \\ 49 & 50 & 51 & 67 & 68 & 69 & 121 & 122 & 123 & 139 & 140 & 141 \\ 52 & 53 & 54 & 70 & 71 & 72 & 124 & 125 & 126 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -52 & -54 & -56 & -88 & -90 & -92 \\ 145 & 145 & 145 & 145 & 145 & 145 & -52 & -54 & -56 & -88 & -90 & -92 \\ 145 & 145 & 145 & 145 & 145 & 145 & -52 & -54 & -56 & -88 & -90 & -92 \\ 145 & 145 & 145 & 145 & 145 & 145 & -52 & -54 & -56 & -88 & -90 & -92 \\ 145 & 145 & 145 & 145 & 145 & 145 & -52 & -54 & -56 & -88 & -90 & -92 \\ 145 & 145 & 145 & 145 & 145 & 145 & -52 & -54 & -56 & -88 & -90 & -92 \\ -21 & -21 & -21 & -21 & -21 & -21 & 0 & 0 & 0 & 0 & 0 & 0 \\ -27 & -27 & -27 & -27 & -27 & -27 & 0 & 0 & 0 & 0 & 0 & 0 \\ -33 & -33 & -33 & -33 & -33 & -33 & 0 & 0 & 0 & 0 & 0 & 0 \\ -39 & -39 & -39 & -39 & -39 & -39 & 0 & 0 & 0 & 0 & 0 & 0 \\ -45 & -45 & -45 & -45 & -45 & -45 & 0 & 0 & 0 & 0 & 0 & 0 \\ -51 & -51 & -51 & -51 & -51 & -51 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
34	$\{(2,4,12),\ (2,4,12)\}$	$\begin{pmatrix} 1 & 2 & 5 & 6 & 17 & 18 & 21 & 22 & 33 & 34 & 37 & 38 \\ 3 & 4 & 7 & 8 & 19 & 20 & 23 & 24 & 35 & 36 & 39 & 40 \\ 9 & 10 & 13 & 14 & 25 & 26 & 29 & 30 & 41 & 42 & 45 & 46 \\ 11 & 12 & 15 & 16 & 27 & 28 & 31 & 32 & 43 & 44 & 47 & 48 \\ 49 & 50 & 53 & 54 & 65 & 66 & 69 & 70 & 81 & 82 & 85 & 86 \\ 51 & 52 & 55 & 56 & 67 & 68 & 71 & 72 & 83 & 84 & 87 & 88 \\ 57 & 58 & 61 & 62 & 73 & 74 & 77 & 78 & 89 & 90 & 93 & 94 \\ 59 & 60 & 63 & 64 & 75 & 76 & 79 & 80 & 91 & 92 & 95 & 96 \\ 97 & 98 & 101 & 102 & 113 & 114 & 117 & 118 & 129 & 130 & 133 & 134 \\ 99 & 100 & 103 & 104 & 115 & 116 & 119 & 120 & 131 & 132 & 135 & 136 \\ 105 & 106 & 109 & 110 & 121 & 122 & 125 & 126 & 137 & 138 & 141 & 142 \\ 107 & 108 & 111 & 112 & 123 & 124 & 127 & 128 & 139 & 140 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -27 & -29 & -35 & -37 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -27 & -29 & -35 & -37 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -27 & -29 & -35 & -37 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -27 & -29 & -35 & -37 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -27 & -29 & -35 & -37 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -27 & -29 & -35 & -37 \\ -6 & -6 & -6 & -6 & -6 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & -10 & -10 & -10 & -10 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \\ -86 & -86 & -86 & -86 & -86 & -86 & 0 & 0 & 0 & 0 & 0 & 0 \\ -90 & -90 & -90 & -90 & -90 & -90 & 0 & 0 & 0 & 0 & 0 & 0 \\ -102 & -102 & -102 & -102 & -102 & -102 & 0 & 0 & 0 & 0 & 0 & 0 \\ -106 & -106 & -106 & -106 & -106 & -106 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
35	$\{(2,4,12),\ (2,6,12)\}$	$\begin{pmatrix} 1 & 2 & 5 & 6 & 9 & 10 & 25 & 26 & 29 & 30 & 33 & 34 \\ 3 & 4 & 7 & 8 & 11 & 12 & 27 & 28 & 31 & 32 & 35 & 36 \\ 13 & 14 & 17 & 18 & 21 & 22 & 37 & 38 & 41 & 42 & 45 & 46 \\ 15 & 16 & 19 & 20 & 23 & 24 & 39 & 40 & 43 & 44 & 47 & 48 \\ 49 & 50 & 53 & 54 & 57 & 58 & 73 & 74 & 77 & 78 & 81 & 82 \\ 51 & 52 & 55 & 56 & 59 & 60 & 75 & 76 & 79 & 80 & 83 & 84 \\ 61 & 62 & 65 & 66 & 69 & 70 & 85 & 86 & 89 & 90 & 93 & 94 \\ 63 & 64 & 67 & 68 & 71 & 72 & 87 & 88 & 91 & 92 & 95 & 96 \\ 97 & 98 & 101 & 102 & 105 & 106 & 121 & 122 & 125 & 126 & 129 & 130 \\ 99 & 100 & 103 & 104 & 107 & 108 & 123 & 124 & 127 & 128 & 131 & 132 \\ 109 & 110 & 113 & 114 & 117 & 118 & 133 & 134 & 137 & 138 & 141 & 142 \\ 111 & 112 & 115 & 116 & 119 & 120 & 135 & 136 & 139 & 140 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -15 & -17 & -23 & -25 & -31 & -33 \\ 145 & 145 & 145 & 145 & 145 & 145 & -15 & -17 & -23 & -25 & -31 & -33 \\ 145 & 145 & 145 & 145 & 145 & 145 & -15 & -17 & -23 & -25 & -31 & -33 \\ 145 & 145 & 145 & 145 & 145 & 145 & -15 & -17 & -23 & -25 & -31 & -33 \\ 145 & 145 & 145 & 145 & 145 & 145 & -15 & -17 & -23 & -25 & -31 & -33 \\ 145 & 145 & 145 & 145 & 145 & 145 & -15 & -17 & -23 & -25 & -31 & -33 \\ -10 & -10 & -10 & -10 & -10 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \\ -14 & -14 & -14 & -14 & -14 & -14 & 0 & 0 & 0 & 0 & 0 & 0 \\ -82 & -82 & -82 & -82 & -82 & -82 & 0 & 0 & 0 & 0 & 0 & 0 \\ -86 & -86 & -86 & -86 & -86 & -86 & 0 & 0 & 0 & 0 & 0 & 0 \\ -106 & -106 & -106 & -106 & -106 & -106 & 0 & 0 & 0 & 0 & 0 & 0 \\ -110 & -110 & -110 & -110 & -110 & -110 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
36	$\{2,4,12),\ (3,6,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 7 & 8 & 9 & 25 & 26 & 27 & 31 & 32 & 33 \\ 4 & 5 & 6 & 10 & 11 & 12 & 28 & 29 & 30 & 34 & 35 & 36 \\ 13 & 14 & 15 & 19 & 20 & 21 & 37 & 38 & 39 & 43 & 44 & 45 \\ 16 & 17 & 18 & 22 & 23 & 24 & 40 & 41 & 42 & 46 & 47 & 48 \\ 49 & 50 & 51 & 55 & 56 & 57 & 73 & 74 & 75 & 79 & 80 & 81 \\ 52 & 53 & 54 & 58 & 59 & 60 & 76 & 77 & 78 & 82 & 83 & 84 \\ 61 & 62 & 63 & 67 & 68 & 69 & 85 & 86 & 87 & 91 & 92 & 93 \\ 64 & 65 & 66 & 70 & 71 & 72 & 88 & 89 & 90 & 94 & 95 & 96 \\ 97 & 98 & 99 & 103 & 104 & 105 & 121 & 122 & 123 & 127 & 128 & 129 \\ 100 & 101 & 102 & 106 & 107 & 108 & 124 & 125 & 126 & 130 & 131 & 132 \\ 109 & 110 & 111 & 115 & 116 & 117 & 133 & 134 & 135 & 139 & 140 & 141 \\ 112 & 113 & 114 & 118 & 119 & 120 & 136 & 137 & 138 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -16 & -18 & -20 & -28 & -30 & -32 \\ 145 & 145 & 145 & 145 & 145 & 145 & -16 & -18 & -20 & -28 & -30 & -32 \\ 145 & 145 & 145 & 145 & 145 & 145 & -16 & -18 & -20 & -28 & -30 & -32 \\ 145 & 145 & 145 & 145 & 145 & 145 & -16 & -18 & -20 & -28 & -30 & -32 \\ 145 & 145 & 145 & 145 & 145 & 145 & -16 & -18 & -20 & -28 & -30 & -32 \\ 145 & 145 & 145 & 145 & 145 & 145 & -16 & -18 & -20 & -28 & -30 & -32 \\ -9 & -9 & -9 & -9 & -9 & -9 & 0 & 0 & 0 & 0 & 0 & 0 \\ -15 & -15 & -15 & -15 & -15 & -15 & 0 & 0 & 0 & 0 & 0 & 0 \\ -81 & -81 & -81 & -81 & -81 & -81 & 0 & 0 & 0 & 0 & 0 & 0 \\ -87 & -87 & -87 & -87 & -87 & -87 & 0 & 0 & 0 & 0 & 0 & 0 \\ -105 & -105 & -105 & -105 & -105 & -105 & 0 & 0 & 0 & 0 & 0 & 0 \\ -111 & -111 & -111 & -111 & -111 & -111 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

	Divisor Paths	Reversible Square	Block Representation
37	$\{(2,6,12),\ (2,4,12)\}$	$\begin{pmatrix} 1 & 2 & 5 & 6 & 25 & 26 & 29 & 30 & 49 & 50 & 53 & 54 \\ 3 & 4 & 7 & 8 & 27 & 28 & 31 & 32 & 51 & 52 & 55 & 56 \\ 9 & 10 & 13 & 14 & 33 & 34 & 37 & 38 & 57 & 58 & 61 & 62 \\ 11 & 12 & 15 & 16 & 35 & 36 & 39 & 40 & 59 & 60 & 63 & 64 \\ 17 & 18 & 21 & 22 & 41 & 42 & 45 & 46 & 65 & 66 & 69 & 70 \\ 19 & 20 & 23 & 24 & 43 & 44 & 47 & 48 & 67 & 68 & 71 & 72 \\ 73 & 74 & 77 & 78 & 97 & 98 & 101 & 102 & 121 & 122 & 125 & 126 \\ 75 & 76 & 79 & 80 & 99 & 100 & 103 & 104 & 123 & 124 & 127 & 128 \\ 81 & 82 & 85 & 86 & 105 & 106 & 109 & 110 & 129 & 130 & 133 & 134 \\ 83 & 84 & 87 & 88 & 107 & 108 & 111 & 112 & 131 & 132 & 135 & 136 \\ 89 & 90 & 93 & 94 & 113 & 114 & 117 & 118 & 137 & 138 & 141 & 142 \\ 91 & 92 & 95 & 96 & 115 & 116 & 119 & 120 & 139 & 140 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -43 & -45 & -51 & -53 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -43 & -45 & -51 & -53 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -43 & -45 & -51 & -53 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -43 & -45 & -51 & -53 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -43 & -45 & -51 & -53 \\ 145 & 145 & 145 & 145 & 145 & 145 & -3 & -5 & -43 & -45 & -51 & -53 \\ -54 & -54 & -54 & -54 & -54 & -54 & 0 & 0 & 0 & 0 & 0 & 0 \\ -58 & -58 & -58 & -58 & -58 & -58 & 0 & 0 & 0 & 0 & 0 & 0 \\ -70 & -70 & -70 & -70 & -70 & -70 & 0 & 0 & 0 & 0 & 0 & 0 \\ -74 & -74 & -74 & -74 & -74 & -74 & 0 & 0 & 0 & 0 & 0 & 0 \\ -86 & -86 & -86 & -86 & -86 & -86 & 0 & 0 & 0 & 0 & 0 & 0 \\ -90 & -90 & -90 & -90 & -90 & -90 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
38	$\{(2,6,12),\ (2,6,12)\}$	$\begin{pmatrix} 1 & 2 & 5 & 6 & 9 & 10 & 37 & 38 & 41 & 42 & 45 & 46 \\ 3 & 4 & 7 & 8 & 11 & 12 & 39 & 40 & 43 & 44 & 47 & 48 \\ 13 & 14 & 17 & 18 & 21 & 22 & 49 & 50 & 53 & 54 & 57 & 58 \\ 15 & 16 & 19 & 20 & 23 & 24 & 51 & 52 & 55 & 56 & 59 & 60 \\ 25 & 26 & 29 & 30 & 33 & 34 & 61 & 62 & 65 & 66 & 69 & 70 \\ 27 & 28 & 31 & 32 & 35 & 36 & 63 & 64 & 67 & 68 & 71 & 72 \\ 73 & 74 & 77 & 78 & 81 & 82 & 109 & 110 & 113 & 114 & 117 & 118 \\ 75 & 76 & 79 & 80 & 83 & 84 & 111 & 112 & 115 & 116 & 119 & 120 \\ 85 & 86 & 89 & 90 & 93 & 94 & 121 & 122 & 125 & 126 & 129 & 130 \\ 87 & 88 & 91 & 92 & 95 & 96 & 123 & 124 & 127 & 128 & 131 & 132 \\ 97 & 98 & 101 & 102 & 105 & 106 & 133 & 134 & 137 & 138 & 141 & 142 \\ 99 & 100 & 103 & 104 & 107 & 108 & 135 & 136 & 139 & 140 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -27 & -29 & -35 & -37 & -43 & -45 \\ 145 & 145 & 145 & 145 & 145 & 145 & -27 & -29 & -35 & -37 & -43 & -45 \\ 145 & 145 & 145 & 145 & 145 & 145 & -27 & -29 & -35 & -37 & -43 & -45 \\ 145 & 145 & 145 & 145 & 145 & 145 & -27 & -29 & -35 & -37 & -43 & -45 \\ 145 & 145 & 145 & 145 & 145 & 145 & -27 & -29 & -35 & -37 & -43 & -45 \\ 145 & 145 & 145 & 145 & 145 & 145 & -27 & -29 & -35 & -37 & -43 & -45 \\ -46 & -46 & -46 & -46 & -46 & -46 & 0 & 0 & 0 & 0 & 0 & 0 \\ -50 & -50 & -50 & -50 & -50 & -50 & 0 & 0 & 0 & 0 & 0 & 0 \\ -70 & -70 & -70 & -70 & -70 & -70 & 0 & 0 & 0 & 0 & 0 & 0 \\ -74 & -74 & -74 & -74 & -74 & -74 & 0 & 0 & 0 & 0 & 0 & 0 \\ -94 & -94 & -94 & -94 & -94 & -94 & 0 & 0 & 0 & 0 & 0 & 0 \\ -98 & -98 & -98 & -98 & -98 & -98 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
39	$\{(2,6,12),\ (3,6,12)\}$	$\begin{pmatrix} 1 & 2 & 3 & 7 & 8 & 9 & 37 & 38 & 39 & 43 & 44 & 45 \\ 4 & 5 & 6 & 10 & 11 & 12 & 40 & 41 & 42 & 46 & 47 & 48 \\ 13 & 14 & 15 & 19 & 20 & 21 & 49 & 50 & 51 & 55 & 56 & 57 \\ 16 & 17 & 18 & 22 & 23 & 24 & 52 & 53 & 54 & 58 & 59 & 60 \\ 25 & 26 & 27 & 31 & 32 & 33 & 61 & 62 & 63 & 67 & 68 & 69 \\ 28 & 29 & 30 & 34 & 35 & 36 & 64 & 65 & 66 & 70 & 71 & 72 \\ 73 & 74 & 75 & 79 & 80 & 81 & 109 & 110 & 111 & 115 & 116 & 117 \\ 76 & 77 & 78 & 82 & 83 & 84 & 112 & 113 & 114 & 118 & 119 & 120 \\ 85 & 86 & 87 & 91 & 92 & 93 & 121 & 122 & 123 & 127 & 128 & 129 \\ 88 & 89 & 90 & 94 & 95 & 96 & 124 & 125 & 126 & 130 & 131 & 132 \\ 97 & 98 & 99 & 103 & 104 & 105 & 133 & 134 & 135 & 139 & 140 & 141 \\ 100 & 101 & 102 & 106 & 107 & 108 & 136 & 137 & 138 & 142 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -28 & -30 & -32 & -40 & -42 & -44 \\ 145 & 145 & 145 & 145 & 145 & 145 & -28 & -30 & -32 & -40 & -42 & -44 \\ 145 & 145 & 145 & 145 & 145 & 145 & -28 & -30 & -32 & -40 & -42 & -44 \\ 145 & 145 & 145 & 145 & 145 & 145 & -28 & -30 & -32 & -40 & -42 & -44 \\ 145 & 145 & 145 & 145 & 145 & 145 & -28 & -30 & -32 & -40 & -42 & -44 \\ 145 & 145 & 145 & 145 & 145 & 145 & -28 & -30 & -32 & -40 & -42 & -44 \\ -45 & -45 & -45 & -45 & -45 & -45 & 0 & 0 & 0 & 0 & 0 & 0 \\ -51 & -51 & -51 & -51 & -51 & -51 & 0 & 0 & 0 & 0 & 0 & 0 \\ -69 & -69 & -69 & -69 & -69 & -69 & 0 & 0 & 0 & 0 & 0 & 0 \\ -75 & -75 & -75 & -75 & -75 & -75 & 0 & 0 & 0 & 0 & 0 & 0 \\ -93 & -93 & -93 & -93 & -93 & -93 & 0 & 0 & 0 & 0 & 0 & 0 \\ -99 & -99 & -99 & -99 & -99 & -99 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
40	$\{3,6,12),\ (2,4,12)\}$	$\begin{pmatrix} 1 & 2 & 7 & 8 & 25 & 26 & 31 & 32 & 49 & 50 & 55 & 56 \\ 3 & 4 & 9 & 10 & 27 & 28 & 33 & 34 & 51 & 52 & 57 & 58 \\ 5 & 6 & 11 & 12 & 29 & 30 & 35 & 36 & 53 & 54 & 59 & 60 \\ 13 & 14 & 19 & 20 & 37 & 38 & 43 & 44 & 61 & 62 & 67 & 68 \\ 15 & 16 & 21 & 22 & 39 & 40 & 45 & 46 & 63 & 64 & 69 & 70 \\ 17 & 18 & 23 & 24 & 41 & 42 & 47 & 48 & 65 & 66 & 71 & 72 \\ 73 & 74 & 79 & 80 & 97 & 98 & 103 & 104 & 121 & 122 & 127 & 128 \\ 75 & 76 & 81 & 82 & 99 & 100 & 105 & 106 & 123 & 124 & 129 & 130 \\ 77 & 78 & 83 & 84 & 101 & 102 & 107 & 108 & 125 & 126 & 131 & 132 \\ 85 & 86 & 91 & 92 & 109 & 110 & 115 & 116 & 133 & 134 & 139 & 140 \\ 87 & 88 & 93 & 94 & 111 & 112 & 117 & 118 & 135 & 136 & 141 & 142 \\ 89 & 90 & 95 & 96 & 113 & 114 & 119 & 120 & 137 & 138 & 143 & 144 \end{pmatrix}$	$\begin{pmatrix} 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -41 & -43 & -53 & -55 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -41 & -43 & -53 & -55 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -41 & -43 & -53 & -55 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -41 & -43 & -53 & -55 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -41 & -43 & -53 & -55 \\ 145 & 145 & 145 & 145 & 145 & 145 & -5 & -7 & -41 & -43 & -53 & -55 \\ -56 & -56 & -56 & -56 & -56 & -56 & 0 & 0 & 0 & 0 & 0 & 0 \\ -60 & -60 & -60 & -60 & -60 & -60 & 0 & 0 & 0 & 0 & 0 & 0 \\ -64 & -64 & -64 & -64 & -64 & -64 & 0 & 0 & 0 & 0 & 0 & 0 \\ -80 & -80 & -80 & -80 & -80 & -80 & 0 & 0 & 0 & 0 & 0 & 0 \\ -84 & -84 & -84 & -84 & -84 & -84 & 0 & 0 & 0 & 0 & 0 & 0 \\ -88 & -88 & -88 & -88 & -88 & -88 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

#	Divisor Paths	Reversible Square	Block Representation
41	$\{(3,6,12),\ (2,6,12)\}$	(see below)	(see below)
42	$\{(3,6,12),\ (3,6,12)\}$	(see below)	(see below)

41 — $\{(3,6,12),\ (2,6,12)\}$

Reversible Square

1	2	5	6	9	10	25	26	29	30	33	34
3	4	7	8	11	12	27	28	31	32	35	36
13	14	17	18	21	22	37	38	41	42	45	46
15	16	19	20	23	24	39	40	43	44	47	48
49	50	53	54	57	58	73	74	77	78	81	82
51	52	55	56	59	60	75	76	79	80	83	84
61	62	65	66	69	70	85	86	89	90	93	94
63	64	67	68	71	72	87	88	91	92	95	96
97	98	101	102	105	106	121	122	125	126	129	130
99	100	103	104	107	108	123	124	127	128	131	132
109	110	113	114	117	118	133	134	137	138	141	142
111	112	115	116	119	120	135	136	139	140	143	144

Block Representation

145	145	145	145	145	145	−23	−25	−35	−37	−47	−49
145	145	145	145	145	145	−23	−25	−35	−37	−47	−49
145	145	145	145	145	145	−23	−25	−35	−37	−47	−49
145	145	145	145	145	145	−23	−25	−35	−37	−47	−49
145	145	145	145	145	145	−23	−25	−35	−37	−47	−49
145	145	145	145	145	145	−23	−25	−35	−37	−47	−49
−50	−50	−50	−50	−50	−50	0	0	0	0	0	0
−54	−54	−54	−54	−54	−54	0	0	0	0	0	0
−58	−58	−58	−58	−58	−58	0	0	0	0	0	0
−86	−86	−86	−86	−86	−86	0	0	0	0	0	0
−90	−90	−90	−90	−90	−90	0	0	0	0	0	0
−94	−94	−94	−94	−94	−94	0	0	0	0	0	0

42 — $\{(3,6,12),\ (3,6,12)\}$

Reversible Square

1	2	3	10	11	12	37	38	39	46	47	48
4	5	6	13	14	15	40	41	42	49	50	51
7	8	9	16	17	18	43	44	45	52	53	54
19	20	21	28	29	30	55	56	57	64	65	66
22	23	24	31	32	33	58	59	60	67	68	69
25	26	27	34	35	36	61	62	63	70	71	72
73	74	75	82	83	84	109	110	111	118	119	120
76	77	78	85	86	87	112	113	114	121	122	123
79	80	81	88	89	90	115	116	117	124	125	126
91	92	93	100	101	102	127	128	129	136	137	138
94	95	96	103	104	105	130	131	132	139	140	141
97	98	99	106	107	108	133	134	135	142	143	144

Block Representation

145	145	145	145	145	145	−25	−27	−29	−43	−45	−47
145	145	145	145	145	145	−25	−27	−29	−43	−45	−47
145	145	145	145	145	145	−25	−27	−29	−43	−45	−47
145	145	145	145	145	145	−25	−27	−29	−43	−45	−47
145	145	145	145	145	145	−25	−27	−29	−43	−45	−47
145	145	145	145	145	145	−25	−27	−29	−43	−45	−47
−48	−48	−48	−48	−48	−48	0	0	0	0	0	0
−54	−54	−54	−54	−54	−54	0	0	0	0	0	0
−60	−60	−60	−60	−60	−60	0	0	0	0	0	0
−84	−84	−84	−84	−84	−84	0	0	0	0	0	0
−90	−90	−90	−90	−90	−90	0	0	0	0	0	0
−96	−96	−96	−96	−96	−96	0	0	0	0	0	0

The 42 Sum-and-Distance Systems when $n = 12$

$$SD_1 = \{\{1,3,5,7,9,11\},\ \{12,36,60,84,108,132\}\}$$
$$SD_2 = \{\{23,25,71,73,119,121\},\ \{2,6,10,14,18,22\}\}$$
$$SD_3 = \{\{34,36,38,106,108,110\},\ \{3,9,15,21,27,33\}\}$$
$$SD_4 = \{\{1,3,93,95,97,99\},\ \{4,12,20,28,36,44\}\}$$
$$SD_5 = \{\{67,69,71,73,75,77\},\ \{6,18,30,42,45,66\}\}$$
$$SD_6 = \{\{3,5,11,13,19,21\},\ \{22,26,70,74,118,122\}\}$$
$$SD_7 = \{\{4,6,8,16,18,20\},\ \{21,27,69,75,117,123\}\}$$
$$SD_8 = \{\{1,3,13,15,17,19\},\ \{20,28,68,76,116,124\}\}$$
$$SD_9 = \{\{7,9,11,13,15,17\},\ \{18,30,66,78,114,126\}\}$$
$$SD_{10} = \{\{3,5,91,93,99,101\},\ \{6,10,22,26,28,42\}\}$$
$$SD_{11} = \{\{63,65,71,73,79,81\},\ \{10,14,34,38,58,62\}\}$$
$$SD_{12} = \{\{64,66,68,76,78,80\},\ \{9,15,33,39,57,63\}\}$$
$$SD_{13} = \{\{5,7,17,19,29,31\},\ \{32,36,40,104,108,112\}\}$$
$$SD_{14} = \{\{7,9,11,25,27,29\},\ \{30,36,42,102,108,114\}\}$$
$$SD_{15} = \{\{1,3,21,23,25,27\},\ \{28,36,44,100,108,116\}\}$$
$$SD_{16} = \{\{13,15,17,19,21,23\},\ \{24,36,48,96,108,120\}\}$$
$$SD_{17} = \{\{5,7,89,91,101,103\},\ \{8,12,16,32,36,40\}\}$$
$$SD_{18} = \{\{59,61,71,73,83,85\},\ \{14,18,22,50,54,58\}\}$$
$$SD_{19} = \{\{61,63,65,79,81,83\},\ \{12,18,24,48,54,60\}\}$$
$$SD_{20} = \{\{7,9,23,25,39,41\},\ \{2,6,90,94,98,102\}\}$$
$$SD_{21} = \{\{10,12,14,34,36,38\},\ \{3,9,87,93,99,105\}\}$$
$$SD_{22} = \{\{1,3,29,31,33,35\},\ \{4,12,84,92,100,108\}\}$$
$$SD_{23} = \{\{19,21,23,25,27,29\},\ \{6,18,78,90,102,114\}\}$$
$$SD_{24} = \{\{7,9,87,89,103,105\},\ \{2,6,26,30,34,38\}\}$$
$$SD_{25} = \{\{55,57,71,73,87,89\},\ \{2,6,42,46,50,54\}\}$$
$$SD_{26} = \{\{58,60,62,82,84,86\},\ \{3,9,39,45,51,57\}\}$$
$$SD_{27} = \{\{11,13,35,37,59,61\},\ \{62,66,70,74,78,82\}\}$$
$$SD_{28} = \{\{16,18,20,52,54,56\},\ \{57,63,69,75,81,87\}\}$$
$$SD_{29} = \{\{1,3,45,47,49,51\},\ \{52,60,68,76,84,92\}\}$$
$$SD_{30} = \{\{31,33,35,37,39,41\},\ \{42,54,66,78,90,102\}\}$$
$$SD_{31} = \{\{11,13,83,85,107,109\},\ \{14,18,22,26,30,34\}\}$$
$$SD_{32} = \{\{47,49,71,73,95,97\},\ \{26,30,34,38,42,46\}\}$$
$$SD_{33} = \{\{52,54,56,88,90,82\},\ \{21,27,33,39,45,51\}\}$$
$$SD_{34} = \{\{3,5,27,29,35,37\},\ \{6,10,86,90,102,106\}\}$$
$$SD_{35} = \{\{15,17,23,25,31,33\},\ \{10,14,82,86,106,110\}\}$$
$$SD_{36} = \{\{16,18,20,28,30,32\},\ \{9,15,81,87,105,111\}\}$$
$$SD_{37} = \{\{3,5,43,45,51,53\},\ \{54,58,70,74,86,90\}\}$$
$$SD_{38} = \{\{27,29,35,37,43,45\},\ \{46,50,70,74,94,98\}\}$$
$$SD_{39} = \{\{28,30,32,40,42,44\},\ \{45,51,69,75,93,99\}\}$$
$$SD_{40} = \{\{5,7,41,43,53,55\},\ \{56,60,64,80,84,88\}\}$$
$$SD_{41} = \{\{23,25,35,37,47,49\},\ \{50,54,58,86,90,94\}\}$$
$$SD_{42} = \{\{25,27,29,43,45,47\},\ \{48,54,60,84,90,96\}\}$$

Appendix 3: Divisor Paths, Inclusive Sum-and-Distance Systems and Odd Sided Reversible Squares

The following tables list all even sided reversible squares for $n \in \{3,5,7,9,11,13,15\}$, there corresponding divisor path sets, block representations and inclusive sum-and-distance systems.

1

Table 7: n=3

	Divisor Paths	Reversible Square	Block Representation	Sum-and-Distance System
1	$\{(3),\ (3)\}$	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$	$\begin{pmatrix} 10 & 5\sqrt{2} & -2 \\ 5\sqrt{2} & 5 & -\sqrt{2} \\ -6 & -3\sqrt{2} & 0 \end{pmatrix}$	$\{\{1\},\ \{3\}\}$

Table 8: n=5

	Divisor Paths	Reversible Square	Block Representation	S & D System*
1	$\{(5),\ (5)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{pmatrix}$	$\begin{pmatrix} 26 & 26 & 13\sqrt{2} & -2 & -4 \\ 26 & 26 & 13\sqrt{2} & -2 & -4 \\ 13\sqrt{2} & 13\sqrt{2} & 13 & -\sqrt{2} & -2\sqrt{2} \\ -10 & -10 & -5\sqrt{2} & 0 & 0 \\ -20 & -20 & -10\sqrt{2} & 0 & 0 \end{pmatrix}$	$\{\{1,2\},\ \{5,10\}\}$

[1]* Here $S\&D$ stands for sum-and-distance and D. P. for divisor path.

Table 9: n=7

	Divisor Paths	Reversible Square	Block Representation	S & D System*
1	$\{(7),\ (7)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ 22 & 23 & 24 & 25 & 26 & 27 & 28 \\ 29 & 30 & 31 & 32 & 33 & 34 & 35 \\ 36 & 37 & 38 & 39 & 40 & 41 & 42 \\ 43 & 44 & 45 & 46 & 47 & 48 & 49 \end{pmatrix}$	$\begin{pmatrix} 50 & 50 & 50 & 25\sqrt{2} & -2 & -4 & -6 \\ 50 & 50 & 50 & 25\sqrt{2} & -2 & -4 & -6 \\ 50 & 50 & 50 & 25\sqrt{2} & -2 & -4 & -6 \\ 25\sqrt{2} & 25\sqrt{2} & 25\sqrt{2} & 25 & -\sqrt{2} & -2\sqrt{2} & -3\sqrt{2} \\ -14 & -14 & -14 & -7\sqrt{2} & 0 & 0 & 0 \\ -28 & -28 & -28 & -14\sqrt{2} & 0 & 0 & 0 \\ -42 & -42 & -42 & -21\sqrt{2} & 0 & 0 & 0 \end{pmatrix}$	$\{\{1,2,3\},\ \{7,14,21\}\}$

Table 10: n=9

	Divisor Paths	Reversible Square	Block Representation	S & D System*
1	$\{(9),\ (9)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 \\ 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 & 45 \\ 46 & 47 & 48 & 49 & 50 & 51 & 52 & 53 & 54 \\ 55 & 56 & 57 & 58 & 59 & 60 & 61 & 62 & 63 \\ 64 & 65 & 66 & 67 & 68 & 69 & 70 & 71 & 72 \\ 73 & 74 & 75 & 76 & 77 & 78 & 79 & 80 & 81 \end{pmatrix}$	$\begin{pmatrix} 82 & 82 & 82 & 82 & 41\sqrt{2} & -2 & -4 & -6 & -8 \\ 82 & 82 & 82 & 82 & 41\sqrt{2} & -2 & -4 & -6 & -8 \\ 82 & 82 & 82 & 82 & 41\sqrt{2} & -2 & -4 & -6 & -8 \\ 82 & 82 & 82 & 82 & 41\sqrt{2} & -2 & -4 & -6 & -8 \\ 41\sqrt{2} & 41\sqrt{2} & 41\sqrt{2} & 41\sqrt{2} & 41 & -\sqrt{2} & -2\sqrt{2} & -3\sqrt{2} & -4\sqrt{2} \\ -18 & -18 & -18 & -18 & -9\sqrt{2} & 0 & 0 & 0 & 0 \\ -36 & -36 & -36 & -36 & -18\sqrt{2} & 0 & 0 & 0 & 0 \\ -54 & -54 & -54 & -54 & -27\sqrt{2} & 0 & 0 & 0 & 0 \\ -72 & -72 & -72 & -72 & -36\sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{1,2,3,4\},\ \{9,18,27,36\}\}$
2	$\{(9),\ (3)\}$	$\begin{pmatrix} 1 & 2 & 3 & 28 & 29 & 30 & 55 & 56 & 57 \\ 4 & 5 & 6 & 31 & 32 & 33 & 58 & 59 & 60 \\ 7 & 8 & 9 & 34 & 35 & 36 & 61 & 62 & 63 \\ 10 & 11 & 12 & 37 & 38 & 39 & 64 & 65 & 66 \\ 13 & 14 & 15 & 40 & 41 & 42 & 67 & 68 & 69 \\ 16 & 17 & 18 & 43 & 44 & 45 & 70 & 71 & 72 \\ 19 & 20 & 21 & 46 & 47 & 48 & 73 & 74 & 75 \\ 22 & 23 & 24 & 49 & 50 & 51 & 76 & 77 & 78 \\ 25 & 26 & 27 & 52 & 53 & 54 & 79 & 80 & 81 \end{pmatrix}$	$\begin{pmatrix} 82 & 82 & 82 & 82 & 41\sqrt{2} & -2 & -52 & -54 & -56 \\ 82 & 82 & 82 & 82 & 41\sqrt{2} & -2 & -52 & -54 & -56 \\ 82 & 82 & 82 & 82 & 41\sqrt{2} & -2 & -52 & -54 & -56 \\ 82 & 82 & 82 & 82 & 41\sqrt{2} & -2 & -52 & -54 & -56 \\ 41\sqrt{2} & 41\sqrt{2} & 41\sqrt{2} & 41\sqrt{2} & 41 & -\sqrt{2} & -26\sqrt{2} & -27\sqrt{2} & -28\sqrt{2} \\ -6 & -6 & -6 & -6 & -3\sqrt{2} & 0 & 0 & 0 & 0 \\ -12 & -12 & -12 & -12 & -6\sqrt{2} & 0 & 0 & 0 & 0 \\ -18 & -18 & -18 & -18 & -9\sqrt{2} & 0 & 0 & 0 & 0 \\ -24 & -24 & -24 & -24 & -12\sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{1,26,27,28\},\ \{3,6,9,12\}\}$

	Divisor Paths	Reversible Square	Block Representation	S & D System*
3	{(5), (5)}	$\begin{pmatrix} 1 & 2 & 3 & 10 & 11 & 12 & 19 & 20 & 21 \\ 4 & 5 & 6 & 13 & 14 & 15 & 22 & 23 & 24 \\ 7 & 8 & 9 & 16 & 17 & 18 & 25 & 26 & 27 \\ 28 & 29 & 30 & 37 & 38 & 39 & 46 & 47 & 48 \\ 31 & 32 & 33 & 40 & 41 & 42 & 49 & 50 & 51 \\ 34 & 35 & 36 & 43 & 44 & 45 & 52 & 53 & 54 \\ 55 & 56 & 57 & 64 & 65 & 66 & 73 & 74 & 75 \\ 58 & 59 & 60 & 67 & 68 & 69 & 76 & 77 & 78 \\ 61 & 62 & 63 & 70 & 71 & 72 & 79 & 80 & 81 \end{pmatrix}$	$\begin{pmatrix} 82 & 82 & 82 & 82 & 41\sqrt{2} & -2 & -16 & -18 & -20 \\ 82 & 82 & 82 & 82 & 41\sqrt{2} & -2 & -16 & -18 & -20 \\ 82 & 82 & 82 & 82 & 41\sqrt{2} & -2 & -16 & -18 & -20 \\ 82 & 82 & 82 & 82 & 41\sqrt{2} & -2 & -16 & -18 & -20 \\ 41\sqrt{2} & 41\sqrt{2} & 41\sqrt{2} & 41\sqrt{2} & 41 & -\sqrt{2} & -8\sqrt{2} & -9\sqrt{2} & -10\sqrt{2} \\ -6 & -6 & -6 & -6 & -3\sqrt{2} & 0 & 0 & 0 & 0 \\ -48 & -48 & -48 & -48 & -24\sqrt{2} & 0 & 0 & 0 & 0 \\ -54 & -54 & -54 & -54 & -27\sqrt{2} & 0 & 0 & 0 & 0 \\ -60 & -60 & -60 & -60 & -30\sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\{1, 8, 9, 10\},\ \{3, 24, 27, 30\}\}$

Table 11: n=11

	Divisor Paths	Reversible Square	Block Representation
1	{(11), (11)}	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 \\ 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 \\ 45 & 46 & 47 & 48 & 49 & 50 & 51 & 52 & 53 & 54 & 55 \\ 56 & 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 & 65 & 66 \\ 67 & 68 & 69 & 70 & 71 & 72 & 73 & 74 & 75 & 76 & 77 \\ 78 & 79 & 80 & 81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 \\ 89 & 90 & 91 & 92 & 93 & 94 & 95 & 96 & 97 & 98 & 99 \\ 100 & 101 & 102 & 103 & 104 & 105 & 106 & 107 & 108 & 109 & 110 \\ 111 & 112 & 113 & 114 & 115 & 116 & 117 & 118 & 119 & 120 & 121 \end{pmatrix}$	$\begin{pmatrix} 122 & 122 & 122 & 122 & 122 & 61\sqrt{2} & -2 & -4 & -6 & -8 & -10 \\ 122 & 122 & 122 & 122 & 122 & 61\sqrt{2} & -2 & -4 & -6 & -8 & -10 \\ 122 & 122 & 122 & 122 & 122 & 61\sqrt{2} & -2 & -4 & -6 & -8 & -10 \\ 122 & 122 & 122 & 122 & 122 & 61\sqrt{2} & -2 & -4 & -6 & -8 & -10 \\ 122 & 122 & 122 & 122 & 122 & 61\sqrt{2} & -2 & -4 & -6 & -8 & -10 \\ 61\sqrt{2} & 61\sqrt{2} & 61\sqrt{2} & 61\sqrt{2} & 61\sqrt{2} & 61 & -\sqrt{2} & -2\sqrt{2} & -3\sqrt{2} & -4\sqrt{2} & -5\sqrt{2} \\ -22 & -22 & -22 & -22 & -22 & -11\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ -44 & -44 & -44 & -44 & -44 & -22\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ -66 & -66 & -66 & -66 & -66 & -33\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ -88 & -88 & -88 & -88 & -88 & -44\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ -110 & -110 & -110 & -110 & -110 & -55\sqrt{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
		S & D System	$\{\{1, 2, 3, 4, 5\},\ \{11, 22, 33, 44, 55\}\ \}$

Table 12: n=13

	D. P.	Reversible Square	Block Representation
1	$\{(13),\ (13)\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \\ 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 \\ 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 & 49 & 50 & 51 & 52 \\ 53 & 54 & 55 & 56 & 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 & 65 \\ 66 & 67 & 68 & 69 & 70 & 71 & 72 & 73 & 74 & 75 & 76 & 77 & 78 \\ 79 & 80 & 81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 & 89 & 90 & 91 \\ 92 & 93 & 94 & 95 & 96 & 97 & 98 & 99 & 100 & 101 & 102 & 103 & 104 \\ 105 & 106 & 107 & 108 & 109 & 110 & 111 & 112 & 113 & 114 & 115 & 116 & 117 \\ 118 & 119 & 120 & 121 & 122 & 123 & 124 & 125 & 126 & 127 & 128 & 129 & 130 \\ 131 & 132 & 133 & 134 & 135 & 136 & 137 & 138 & 139 & 140 & 141 & 142 & 143 \\ 144 & 145 & 146 & 147 & 148 & 149 & 150 & 151 & 152 & 153 & 154 & 155 & 156 \\ 157 & 158 & 159 & 160 & 161 & 162 & 163 & 164 & 165 & 166 & 167 & 168 & 169 \end{pmatrix}$	$\begin{pmatrix} 170 & 170 & 170 & 170 & 170 & 170 & 85\sqrt{2} & -2 & -4 & -6 & -8 & -10 & -12 \\ 170 & 170 & 170 & 170 & 170 & 170 & 85\sqrt{2} & -2 & -4 & -6 & -8 & -10 & -12 \\ 170 & 170 & 170 & 170 & 170 & 170 & 85\sqrt{2} & -2 & -4 & -6 & -8 & -10 & -12 \\ 170 & 170 & 170 & 170 & 170 & 170 & 85\sqrt{2} & -2 & -4 & -6 & -8 & -10 & -12 \\ 170 & 170 & 170 & 170 & 170 & 170 & 85\sqrt{2} & -2 & -4 & -6 & -8 & -10 & -12 \\ 170 & 170 & 170 & 170 & 170 & 170 & 85\sqrt{2} & -2 & -4 & -6 & -8 & -10 & -12 \\ 85\sqrt{2} & 85\sqrt{2} & 85\sqrt{2} & 85\sqrt{2} & 85\sqrt{2} & 85\sqrt{2} & 85 & -\sqrt{2} & -2\sqrt{2} & -3\sqrt{2} & -4\sqrt{2} & -5\sqrt{2} & -6\sqrt{2} \\ -26 & -26 & -26 & -26 & -26 & -26 & -13\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -52 & -52 & -52 & -52 & -52 & -52 & -26\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -78 & -78 & -78 & -78 & -78 & -78 & -39\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -104 & -104 & -104 & -104 & -104 & -104 & -52\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -130 & -130 & -130 & -130 & -130 & -130 & -65\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -156 & -156 & -156 & -156 & -156 & -156 & -78\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
		S & D System	$\{\{1,2,3,4,5,6\},\ \{13,26,39,52,65,78\}\ \}$

Table 13: n=15

	D. P.	Reversible Square	Block Representation
1	{(15), (15}	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 \\ 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 & 45 \\ 46 & 47 & 48 & 49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 & 59 & 60 \\ 61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 & 69 & 70 & 71 & 72 & 73 & 74 & 75 \\ 76 & 77 & 78 & 79 & 80 & 81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 & 89 & 90 \\ 91 & 92 & 93 & 94 & 95 & 96 & 97 & 98 & 99 & 100 & 101 & 102 & 103 & 104 & 105 \\ 106 & 107 & 108 & 109 & 110 & 111 & 112 & 113 & 114 & 115 & 116 & 117 & 118 & 119 & 120 \\ 121 & 122 & 123 & 124 & 125 & 126 & 127 & 128 & 129 & 130 & 131 & 132 & 133 & 134 & 135 \\ 136 & 137 & 138 & 139 & 140 & 141 & 142 & 143 & 144 & 145 & 146 & 147 & 148 & 149 & 150 \\ 151 & 152 & 153 & 154 & 155 & 156 & 157 & 158 & 159 & 160 & 161 & 162 & 163 & 164 & 165 \\ 166 & 167 & 168 & 169 & 170 & 171 & 172 & 173 & 174 & 175 & 176 & 177 & 178 & 179 & 180 \\ 181 & 182 & 183 & 184 & 185 & 186 & 187 & 188 & 189 & 190 & 191 & 192 & 193 & 194 & 195 \\ 196 & 197 & 198 & 199 & 200 & 201 & 202 & 203 & 204 & 205 & 206 & 207 & 208 & 209 & 210 \\ 211 & 212 & 213 & 214 & 215 & 216 & 217 & 218 & 219 & 220 & 221 & 222 & 223 & 224 & 225 \end{pmatrix}$	$\begin{pmatrix} 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -6 & -8 & -10 & -12 & -14 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -6 & -8 & -10 & -12 & -14 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -6 & -8 & -10 & -12 & -14 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -6 & -8 & -10 & -12 & -14 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -6 & -8 & -10 & -12 & -14 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -6 & -8 & -10 & -12 & -14 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -6 & -8 & -10 & -12 & -14 \\ 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113 & -\sqrt{2} & -2\sqrt{2} & -3\sqrt{2} & -4\sqrt{2} & -5\sqrt{2} & -6\sqrt{2} & -7\sqrt{2} \\ -30 & -30 & -30 & -30 & -30 & -30 & -30 & -15\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -60 & -60 & -60 & -60 & -60 & -60 & -60 & -30\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -90 & -90 & -90 & -90 & -90 & -90 & -90 & -45\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -120 & -120 & -120 & -120 & -120 & -120 & -120 & -60\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -150 & -150 & -150 & -150 & -150 & -150 & -150 & -75\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -180 & -180 & -180 & -180 & -180 & -180 & -180 & -90\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -210 & -210 & -210 & -210 & -210 & -210 & -210 & -105\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
		S & D System	$\{\{1, 2, 3, 4, 5, 6, 7\},\ \{15, 30, 45, 60, 75, 90, 105\}\ \}$

	D. P.	Reversible Square	Block Representation
2	{(15), (3}	$\begin{pmatrix} 1 & 2 & 3 & 46 & 47 & 48 & 91 & 92 & 93 & 136 & 137 & 138 & 181 & 182 & 183 \\ 4 & 5 & 6 & 49 & 50 & 51 & 94 & 95 & 96 & 139 & 140 & 141 & 184 & 185 & 186 \\ 7 & 8 & 9 & 52 & 53 & 54 & 97 & 98 & 99 & 142 & 143 & 144 & 187 & 188 & 189 \\ 10 & 11 & 12 & 55 & 56 & 57 & 100 & 101 & 102 & 145 & 146 & 147 & 190 & 191 & 192 \\ 13 & 14 & 15 & 58 & 59 & 60 & 103 & 104 & 105 & 148 & 149 & 150 & 193 & 194 & 195 \\ 16 & 17 & 18 & 61 & 62 & 63 & 106 & 107 & 108 & 151 & 152 & 153 & 196 & 197 & 198 \\ 19 & 20 & 21 & 64 & 65 & 66 & 109 & 110 & 111 & 154 & 155 & 156 & 199 & 200 & 201 \\ 22 & 23 & 24 & 67 & 68 & 69 & 112 & 113 & 114 & 157 & 158 & 159 & 202 & 203 & 204 \\ 25 & 26 & 27 & 70 & 71 & 72 & 115 & 116 & 117 & 160 & 161 & 162 & 205 & 206 & 207 \\ 28 & 29 & 30 & 73 & 74 & 75 & 118 & 119 & 120 & 163 & 164 & 165 & 208 & 209 & 210 \\ 31 & 32 & 33 & 76 & 77 & 78 & 121 & 122 & 123 & 166 & 167 & 168 & 211 & 212 & 213 \\ 34 & 35 & 36 & 79 & 80 & 81 & 124 & 125 & 126 & 169 & 170 & 171 & 214 & 215 & 216 \\ 37 & 38 & 39 & 82 & 83 & 84 & 127 & 128 & 129 & 172 & 173 & 174 & 217 & 218 & 219 \\ 40 & 41 & 42 & 85 & 86 & 87 & 130 & 131 & 132 & 175 & 176 & 177 & 220 & 221 & 222 \\ 43 & 44 & 45 & 88 & 89 & 90 & 133 & 134 & 135 & 178 & 179 & 180 & 223 & 224 & 225 \end{pmatrix}$	$\begin{pmatrix} 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -88 & -90 & -92 & -178 & -180 & -182 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -88 & -90 & -92 & -178 & -180 & -182 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -88 & -90 & -92 & -178 & -180 & -182 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -88 & -90 & -92 & -178 & -180 & -182 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -88 & -90 & -92 & -178 & -180 & -182 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -88 & -90 & -92 & -178 & -180 & -182 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -88 & -90 & -92 & -178 & -180 & -182 \\ 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113 & -\sqrt{2} & -44\sqrt{2} & -45\sqrt{2} & -46\sqrt{2} & -89\sqrt{2} & -90\sqrt{2} & -91\sqrt{2} \\ -6 & -6 & -6 & -6 & -6 & -6 & -6 & -3\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -12 & -12 & -12 & -12 & -12 & -12 & -6\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & -18 & -18 & -18 & -18 & -18 & -18 & -9\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -24 & -24 & -24 & -24 & -24 & -24 & -24 & -12\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -30 & -30 & -30 & -30 & -30 & -30 & -30 & -15\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -36 & -36 & -36 & -36 & -36 & -36 & -36 & -18\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -42 & -42 & -42 & -42 & -42 & -42 & -42 & -21\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
		S & D System	$\{\{1, 44, 45, 46, 89, 90, 91\},\ \{3, 6, 9, 12, 15, 18, 21\}\ \}$

	D. P.	Reversible Square	Block Representation
3	{(15), (5)}	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 76 & 77 & 78 & 79 & 80 & 151 & 152 & 153 & 154 & 155 \\ 6 & 7 & 8 & 9 & 10 & 81 & 82 & 83 & 84 & 85 & 156 & 157 & 158 & 159 & 160 \\ 11 & 12 & 13 & 14 & 15 & 86 & 87 & 88 & 89 & 90 & 161 & 162 & 163 & 164 & 165 \\ 16 & 17 & 18 & 19 & 20 & 91 & 92 & 93 & 94 & 95 & 166 & 167 & 168 & 169 & 170 \\ 21 & 22 & 23 & 24 & 25 & 96 & 97 & 98 & 99 & 100 & 171 & 172 & 173 & 174 & 175 \\ 26 & 27 & 28 & 29 & 30 & 101 & 102 & 103 & 104 & 105 & 176 & 177 & 178 & 179 & 180 \\ 31 & 32 & 33 & 34 & 35 & 106 & 107 & 108 & 109 & 110 & 181 & 182 & 183 & 184 & 185 \\ 36 & 37 & 38 & 39 & 40 & 111 & 112 & 113 & 114 & 115 & 186 & 187 & 188 & 189 & 190 \\ 41 & 42 & 43 & 44 & 45 & 116 & 117 & 118 & 119 & 120 & 191 & 192 & 193 & 194 & 195 \\ 46 & 47 & 48 & 49 & 50 & 121 & 122 & 123 & 124 & 125 & 196 & 197 & 198 & 199 & 200 \\ 51 & 52 & 53 & 54 & 55 & 126 & 127 & 128 & 129 & 130 & 201 & 202 & 203 & 204 & 205 \\ 56 & 57 & 58 & 59 & 60 & 131 & 132 & 133 & 134 & 135 & 206 & 207 & 208 & 209 & 210 \\ 61 & 62 & 63 & 64 & 65 & 136 & 137 & 138 & 139 & 140 & 211 & 212 & 213 & 214 & 215 \\ 66 & 67 & 68 & 69 & 70 & 141 & 142 & 143 & 144 & 145 & 216 & 217 & 218 & 219 & 220 \\ 71 & 72 & 73 & 74 & 75 & 146 & 147 & 148 & 149 & 150 & 221 & 222 & 223 & 224 & 225 \end{pmatrix}$	$\begin{pmatrix} 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -146 & -148 & -150 & -152 & -154 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -146 & -148 & -150 & -152 & -154 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -146 & -148 & -150 & -152 & -154 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -146 & -148 & -150 & -152 & -154 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -146 & -148 & -150 & -152 & -154 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -146 & -148 & -150 & -152 & -154 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -146 & -148 & -150 & -152 & -154 \\ 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113 & -\sqrt{2} & -2\sqrt{2} & -73\sqrt{2} & -74\sqrt{2} & -75\sqrt{2} & -76\sqrt{2} & -77\sqrt{2} \\ -10 & -10 & -10 & -10 & -10 & -10 & -10 & -5\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -20 & -20 & -20 & -20 & -20 & -20 & -20 & -10\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -30 & -30 & -30 & -30 & -30 & -30 & -30 & -15\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -40 & -40 & -40 & -40 & -40 & -40 & -40 & -20\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -50 & -50 & -50 & -50 & -50 & -50 & -50 & -25\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -60 & -60 & -60 & -60 & -60 & -60 & -60 & -30\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -70 & -70 & -70 & -70 & -70 & -70 & -70 & -35\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
		S & D System	$\{\{1, 2, 73, 74, 75, 76, 77\},\ \{5, 10, 15, 20, 25, 30, 35\}\ \}$

	D. P.	Reversible Square	Block Representation
4	$\{(3,15),\ (3,15\}$	$\begin{pmatrix} 1 & 2 & 3 & 10 & 11 & 12 & 19 & 20 & 21 & 28 & 29 & 30 & 37 & 38 & 39 \\ 4 & 5 & 6 & 13 & 14 & 15 & 22 & 23 & 24 & 31 & 32 & 33 & 40 & 41 & 42 \\ 7 & 8 & 9 & 16 & 17 & 18 & 25 & 26 & 27 & 34 & 35 & 36 & 43 & 44 & 45 \\ 46 & 47 & 48 & 55 & 56 & 57 & 64 & 65 & 66 & 73 & 74 & 75 & 82 & 83 & 84 \\ 49 & 50 & 51 & 58 & 59 & 60 & 67 & 68 & 69 & 76 & 77 & 78 & 85 & 86 & 87 \\ 52 & 53 & 54 & 61 & 62 & 63 & 70 & 71 & 72 & 79 & 80 & 81 & 88 & 89 & 90 \\ 91 & 92 & 93 & 100 & 101 & 102 & 109 & 110 & 111 & 118 & 119 & 120 & 127 & 128 & 129 \\ 94 & 95 & 96 & 103 & 104 & 105 & 112 & 113 & 114 & 121 & 122 & 123 & 130 & 131 & 132 \\ 97 & 98 & 99 & 106 & 107 & 108 & 115 & 116 & 117 & 124 & 125 & 126 & 133 & 134 & 135 \\ 136 & 137 & 138 & 145 & 146 & 147 & 154 & 155 & 156 & 163 & 164 & 165 & 172 & 173 & 174 \\ 139 & 140 & 141 & 148 & 149 & 150 & 157 & 158 & 159 & 166 & 167 & 168 & 175 & 176 & 177 \\ 142 & 143 & 144 & 151 & 152 & 153 & 160 & 161 & 162 & 169 & 170 & 171 & 178 & 179 & 180 \\ 181 & 182 & 183 & 190 & 191 & 192 & 199 & 200 & 201 & 208 & 209 & 210 & 217 & 218 & 219 \\ 184 & 185 & 186 & 193 & 194 & 195 & 202 & 203 & 204 & 211 & 212 & 213 & 220 & 221 & 222 \\ 187 & 188 & 189 & 196 & 197 & 198 & 205 & 206 & 207 & 214 & 215 & 216 & 223 & 224 & 225 \end{pmatrix}$	$\begin{pmatrix} 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -16 & -18 & -20 & -34 & -36 & -38 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -16 & -18 & -20 & -34 & -36 & -38 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -16 & -18 & -20 & -34 & -36 & -38 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -16 & -18 & -20 & -34 & -36 & -38 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -16 & -18 & -20 & -34 & -36 & -38 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -16 & -18 & -20 & -34 & -36 & -38 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -16 & -18 & -20 & -34 & -36 & -38 \\ 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113 & -\sqrt{2} & -8\sqrt{2} & -9\sqrt{2} & -10\sqrt{2} & -17\sqrt{2} & -18\sqrt{2} & -19\sqrt{2} \\ -6 & -6 & -6 & -6 & -6 & -6 & -6 & -3\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -84 & -84 & -84 & -84 & -84 & -84 & -84 & -42\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -90 & -90 & -90 & -90 & -90 & -90 & -90 & -45\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -96 & -96 & -96 & -96 & -96 & -96 & -96 & -48\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -174 & -174 & -174 & -174 & -174 & -174 & -174 & -87\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -180 & -180 & -180 & -180 & -180 & -180 & -180 & -90\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -186 & -186 & -186 & -186 & -186 & -186 & -186 & -93\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
		S & D System	$\{\{1,8,9,10,17,18,19\},\ \{42,45,48,87,90,93\}\ \}$

	D. P.	Reversible Square	Block Representation
5	$\{(3,15),\ (5,15\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 16 & 17 & 18 & 19 & 20 & 31 & 32 & 33 & 34 & 35 \\ 6 & 7 & 8 & 9 & 10 & 21 & 22 & 23 & 24 & 25 & 36 & 37 & 38 & 39 & 40 \\ 11 & 12 & 13 & 14 & 15 & 26 & 27 & 28 & 29 & 30 & 41 & 42 & 43 & 44 & 45 \\ 46 & 47 & 48 & 49 & 50 & 61 & 62 & 63 & 64 & 65 & 76 & 77 & 78 & 79 & 80 \\ 51 & 52 & 53 & 54 & 55 & 66 & 67 & 68 & 69 & 70 & 81 & 82 & 83 & 84 & 85 \\ 56 & 57 & 58 & 59 & 60 & 71 & 72 & 73 & 74 & 75 & 86 & 87 & 88 & 89 & 90 \\ 91 & 92 & 93 & 94 & 95 & 106 & 107 & 108 & 109 & 110 & 121 & 122 & 123 & 124 & 125 \\ 96 & 97 & 98 & 99 & 100 & 111 & 112 & 113 & 114 & 115 & 126 & 127 & 128 & 129 & 130 \\ 101 & 102 & 103 & 104 & 105 & 116 & 117 & 118 & 119 & 120 & 131 & 132 & 133 & 134 & 135 \\ 136 & 137 & 138 & 139 & 140 & 151 & 152 & 153 & 154 & 155 & 166 & 167 & 168 & 169 & 170 \\ 141 & 142 & 143 & 144 & 145 & 156 & 157 & 158 & 159 & 160 & 171 & 172 & 173 & 174 & 175 \\ 146 & 147 & 148 & 149 & 150 & 161 & 162 & 163 & 164 & 165 & 176 & 177 & 178 & 179 & 180 \\ 181 & 182 & 183 & 184 & 185 & 196 & 197 & 198 & 199 & 200 & 211 & 212 & 213 & 214 & 215 \\ 186 & 187 & 188 & 189 & 190 & 201 & 202 & 203 & 204 & 205 & 216 & 217 & 218 & 219 & 220 \\ 191 & 192 & 193 & 194 & 195 & 206 & 207 & 208 & 209 & 210 & 221 & 222 & 223 & 224 & 225 \end{pmatrix}$	$\begin{pmatrix} 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -26 & -28 & -30 & -32 & -34 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -26 & -28 & -30 & -32 & -34 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -26 & -28 & -30 & -32 & -34 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -26 & -28 & -30 & -32 & -34 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -26 & -28 & -30 & -32 & -34 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -26 & -28 & -30 & -32 & -34 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -26 & -28 & -30 & -32 & -34 \\ 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113 & -\sqrt{2} & -2\sqrt{2} & -13\sqrt{2} & -14\sqrt{2} & -15\sqrt{2} & -16\sqrt{2} & -17\sqrt{2} \\ -10 & -10 & -10 & -10 & -10 & -10 & -10 & -5\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -80 & -80 & -80 & -80 & -80 & -80 & -80 & -40\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -90 & -90 & -90 & -90 & -90 & -90 & -90 & -45\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -100 & -100 & -100 & -100 & -100 & -100 & -100 & -50\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -170 & -170 & -170 & -170 & -170 & -170 & -170 & -85\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -180 & -180 & -180 & -180 & -180 & -180 & -180 & -90\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -190 & -190 & -190 & -190 & -190 & -190 & -190 & -95\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
		S & D System	$\{\{1, 2, 13, 14, 15, 16, 17\},\ \{40, 45, 50, 85, 90, 95\}\ \}$

	D. P.	Reversible Square	Block Representation
6	$\{(5,15),\ (3,15\}$	$\begin{pmatrix} 1 & 2 & 3 & 16 & 17 & 18 & 31 & 32 & 33 & 46 & 47 & 48 & 61 & 62 & 63 \\ 4 & 5 & 6 & 19 & 20 & 21 & 34 & 35 & 36 & 49 & 50 & 51 & 64 & 65 & 66 \\ 7 & 8 & 9 & 22 & 23 & 24 & 37 & 38 & 39 & 52 & 53 & 54 & 67 & 68 & 69 \\ 10 & 11 & 12 & 25 & 26 & 27 & 40 & 41 & 42 & 55 & 56 & 57 & 70 & 71 & 72 \\ 13 & 14 & 15 & 28 & 29 & 30 & 43 & 44 & 45 & 58 & 59 & 60 & 73 & 74 & 75 \\ 76 & 77 & 78 & 91 & 92 & 93 & 106 & 107 & 108 & 121 & 122 & 123 & 136 & 137 & 138 \\ 79 & 80 & 81 & 94 & 95 & 96 & 109 & 110 & 111 & 124 & 125 & 126 & 139 & 140 & 141 \\ 82 & 83 & 84 & 97 & 98 & 99 & 112 & 113 & 114 & 127 & 128 & 129 & 142 & 143 & 144 \\ 85 & 86 & 87 & 100 & 101 & 102 & 115 & 116 & 117 & 130 & 131 & 132 & 145 & 146 & 147 \\ 88 & 89 & 90 & 103 & 104 & 105 & 118 & 119 & 120 & 133 & 134 & 135 & 148 & 149 & 150 \\ 151 & 152 & 153 & 166 & 167 & 168 & 181 & 182 & 183 & 196 & 197 & 198 & 211 & 212 & 213 \\ 154 & 155 & 156 & 169 & 170 & 171 & 184 & 185 & 186 & 199 & 200 & 201 & 214 & 215 & 216 \\ 157 & 158 & 159 & 172 & 173 & 174 & 187 & 188 & 189 & 202 & 203 & 204 & 217 & 218 & 219 \\ 160 & 161 & 162 & 175 & 176 & 177 & 190 & 191 & 192 & 205 & 206 & 207 & 220 & 221 & 222 \\ 163 & 164 & 165 & 178 & 179 & 180 & 193 & 194 & 195 & 208 & 209 & 210 & 223 & 224 & 225 \end{pmatrix}$	$\begin{pmatrix} 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -28 & -30 & -32 & -58 & -60 & -62 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -28 & -30 & -32 & -58 & -60 & -62 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -28 & -30 & -32 & -58 & -60 & -62 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -28 & -30 & -32 & -58 & -60 & -62 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -28 & -30 & -32 & -58 & -60 & -62 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -28 & -30 & -32 & -58 & -60 & -62 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -28 & -30 & -32 & -58 & -60 & -62 \\ 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113 & -\sqrt{2} & -14\sqrt{2} & -15\sqrt{2} & -16\sqrt{2} & -29\sqrt{2} & -30\sqrt{2} & -31\sqrt{2} \\ -6 & -6 & -6 & -6 & -6 & -6 & -6 & -3\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -12 & -12 & -12 & -12 & -12 & -12 & -6\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -138 & -138 & -138 & -138 & -138 & -138 & -138 & -69\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -144 & -144 & -144 & -144 & -144 & -144 & -144 & -72\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -150 & -150 & -150 & -150 & -150 & -150 & -150 & -75\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -156 & -156 & -156 & -156 & -156 & -156 & -156 & -78\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -162 & -162 & -162 & -162 & -162 & -162 & -162 & -81\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
		S & D System	$\{\{1,14,15,16,29,30,31\},\ \{3,6,69,72,75,78,81\}\ \}$

	D. P.	Reversible Square	Block Representation
7	$\{(5,15),\ (5,15\}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 26 & 27 & 28 & 29 & 30 & 51 & 52 & 53 & 54 & 55 \\ 6 & 7 & 8 & 9 & 10 & 31 & 32 & 33 & 34 & 35 & 56 & 57 & 58 & 59 & 60 \\ 11 & 12 & 13 & 14 & 15 & 36 & 37 & 38 & 39 & 40 & 61 & 62 & 63 & 64 & 65 \\ 16 & 17 & 18 & 19 & 20 & 41 & 42 & 43 & 44 & 45 & 66 & 67 & 68 & 69 & 70 \\ 21 & 22 & 23 & 24 & 25 & 46 & 47 & 48 & 49 & 50 & 71 & 72 & 73 & 74 & 75 \\ 76 & 77 & 78 & 79 & 80 & 101 & 102 & 103 & 104 & 105 & 126 & 127 & 128 & 129 & 130 \\ 81 & 82 & 83 & 84 & 85 & 106 & 107 & 108 & 109 & 110 & 131 & 132 & 133 & 134 & 135 \\ 86 & 87 & 88 & 89 & 90 & 111 & 112 & 113 & 114 & 115 & 136 & 137 & 138 & 139 & 140 \\ 91 & 92 & 93 & 94 & 95 & 116 & 117 & 118 & 119 & 120 & 141 & 142 & 143 & 144 & 145 \\ 96 & 97 & 98 & 99 & 100 & 121 & 122 & 123 & 124 & 125 & 146 & 147 & 148 & 149 & 150 \\ 151 & 152 & 153 & 154 & 155 & 176 & 177 & 178 & 179 & 180 & 201 & 202 & 203 & 204 & 205 \\ 156 & 157 & 158 & 159 & 160 & 181 & 182 & 183 & 184 & 185 & 206 & 207 & 208 & 209 & 210 \\ 161 & 162 & 163 & 164 & 165 & 186 & 187 & 188 & 189 & 190 & 211 & 212 & 213 & 214 & 215 \\ 166 & 167 & 168 & 169 & 170 & 191 & 192 & 193 & 194 & 195 & 216 & 217 & 218 & 219 & 220 \\ 171 & 172 & 173 & 174 & 175 & 196 & 197 & 198 & 199 & 200 & 221 & 222 & 223 & 224 & 225 \end{pmatrix}$	$\begin{pmatrix} 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -46 & -48 & -50 & -52 & -54 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -46 & -48 & -50 & -52 & -54 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -46 & -48 & -50 & -52 & -54 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -46 & -48 & -50 & -52 & -54 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -46 & -48 & -50 & -52 & -54 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -46 & -48 & -50 & -52 & -54 \\ 226 & 226 & 226 & 226 & 226 & 226 & 226 & 113\sqrt{2} & -2 & -4 & -46 & -48 & -50 & -52 & -54 \\ 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113\sqrt{2} & 113 & -\sqrt{2} & -2\sqrt{2} & -23\sqrt{2} & -24\sqrt{2} & -25\sqrt{2} & -26\sqrt{2} & -27\sqrt{2} \\ -10 & -10 & -10 & -10 & -10 & -10 & -10 & -5\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -20 & -20 & -20 & -20 & -20 & -20 & -20 & -10\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -130 & -130 & -130 & -130 & -130 & -130 & -130 & -65\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -140 & -140 & -140 & -140 & -140 & -140 & -140 & -70\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -150 & -150 & -150 & -150 & -150 & -150 & -150 & -75\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -160 & -160 & -160 & -160 & -160 & -160 & -160 & -80\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -170 & -170 & -170 & -170 & -170 & -170 & -170 & -85\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
		S & D System	$\{\{1, 2, 23, 24, 25, 26, 27\},\ \{5, 10, 65, 70, 75, 80, 85\}\ \}$

Appendix 4 : Prime Factorisation and the Number of Sum-and-Distance Systems, N_n

The following table lists the number of sum-and-distance systems, N_n, for the first 200 values of n.

n	1	2	3	4	5	6	7	8	9	10
N_n	0	1	1	3	1	7	1	10	3	7
n	11	12	13	14	15	16	17	18	19	20
N_n	1	42	1	7	7	35	1	42	1	42
n	21	22	23	24	25	26	27	28	29	30
N_n	7	7	1	230	3	7	10	42	1	115
n	31	32	33	34	35	36	37	38	39	40
N_n	1	126	7	7	7	393	1	7	7	230
n	41	42	43	44	45	46	47	48	49	50
N_n	1	115	1	42	42	7	1	1190	3	42
n	51	52	53	54	55	56	57	58	59	60
N_n	7	42	1	230	7	230	7	7	1	1158
n	61	62	63	64	65	66	67	68	69	70
N_n	1	7	42	462	7	115	1	42	7	115
n	71	72	73	74	75	76	77	78	79	80
N_n	1	3030	1	7	42	42	7	115	1	1190
n	81	82	83	84	85	86	87	88	89	90
N_n	35	7	1	1158	7	7	7	230	1	1158
n	91	92	93	94	95	96	97	98	99	100
N_n	7	42	7	7	7	5922	1	42	42	393
n	101	102	103	104	105	106	107	108	109	110
N_n	1	115	1	230	115	7	1	3030	1	115
n	111	112	113	114	115	116	117	118	119	120
N_n	7	1190	1	115	7	42	42	7	7	9350
n	121	122	123	124	125	126	127	128	129	130
N_n	3	7	7	42	10	1158	1	1716	7	115
n	131	122	133	134	135	136	137	138	139	140
N_n	1	1158	7	7	230	230	1	115	1	1158
n	141	142	143	144	145	146	147	148	149	150
N_n	7	7	7	20790	7	7	42	42	1	1158
n	151	152	153	154	155	156	157	158	159	160
N_n	1	230	42	115	7	1158	1	7	7	5922
n	161	162	163	164	165	166	167	168	169	170
N_n	7	1190	1	42	115	7	1	9350	3	115
n	171	172	173	174	175	176	177	178	179	180
N_n	42	42	1	115	42	1190	7	7	1	16782
n	181	182	183	184	185	186	187	188	189	190
N_n	1	115	7	230	7	115	7	42	230	115
n	191	192	193	194	195	196	197	198	199	200
N_n	1	28644	1	7	115	393	1	1158	1	3030

$$
\begin{pmatrix}
1 & 2 & 7 & 8 & 13 & 14 & 55 & 56 & 61 & 62 & 67 & 68 & 217 & 218 & 223 & 224 & 229 & 230 & 271 & 272 & 277 & 278 & 283 & 284 & 433 & 434 & 439 & 440 & 445 & 446 & 487 & 488 & 493 & 494 & 499 & 500 \\
3 & 4 & 9 & 10 & 15 & 16 & 57 & 58 & 63 & 64 & 69 & 70 & 219 & 220 & 225 & 226 & 231 & 232 & 273 & 274 & 279 & 280 & 285 & 286 & 435 & 436 & 441 & 442 & 447 & 448 & 489 & 490 & 495 & 496 & 501 & 502 \\
5 & 6 & 11 & 12 & 17 & 18 & 59 & 60 & 65 & 66 & 71 & 72 & 221 & 222 & 227 & 228 & 233 & 234 & 275 & 276 & 281 & 282 & 287 & 288 & 437 & 438 & 443 & 444 & 449 & 450 & 491 & 492 & 497 & 498 & 503 & 504 \\
19 & 20 & 25 & 26 & 31 & 32 & 73 & 74 & 79 & 80 & 85 & 86 & 235 & 236 & 241 & 242 & 247 & 248 & 289 & 290 & 295 & 296 & 301 & 302 & 451 & 452 & 457 & 458 & 463 & 464 & 505 & 506 & 511 & 512 & 517 & 518 \\
21 & 22 & 27 & 28 & 33 & 34 & 75 & 76 & 81 & 82 & 87 & 88 & 237 & 238 & 243 & 244 & 249 & 250 & 291 & 292 & 297 & 298 & 303 & 304 & 453 & 454 & 459 & 460 & 465 & 466 & 507 & 508 & 513 & 514 & 519 & 520 \\
23 & 24 & 29 & 30 & 35 & 36 & 77 & 78 & 83 & 84 & 89 & 90 & 239 & 240 & 245 & 246 & 251 & 252 & 293 & 294 & 299 & 300 & 305 & 306 & 455 & 456 & 461 & 462 & 467 & 468 & 509 & 510 & 515 & 516 & 521 & 522 \\
37 & 38 & 43 & 44 & 49 & 50 & 91 & 92 & 97 & 98 & 103 & 104 & 253 & 254 & 259 & 260 & 265 & 266 & 307 & 308 & 313 & 314 & 319 & 320 & 469 & 470 & 475 & 476 & 481 & 482 & 523 & 524 & 529 & 530 & 535 & 536 \\
39 & 40 & 45 & 46 & 51 & 52 & 93 & 94 & 99 & 100 & 105 & 106 & 255 & 256 & 261 & 262 & 267 & 268 & 309 & 310 & 315 & 316 & 321 & 322 & 471 & 472 & 477 & 478 & 483 & 484 & 525 & 526 & 531 & 532 & 537 & 538 \\
41 & 42 & 47 & 48 & 53 & 54 & 95 & 96 & 101 & 102 & 107 & 108 & 257 & 258 & 263 & 264 & 269 & 270 & 311 & 312 & 317 & 318 & 323 & 324 & 473 & 474 & 479 & 480 & 485 & 486 & 527 & 528 & 533 & 534 & 539 & 540 \\
109 & 110 & 115 & 116 & 121 & 122 & 163 & 164 & 169 & 170 & 175 & 176 & 325 & 326 & 331 & 332 & 337 & 338 & 379 & 380 & 385 & 386 & 391 & 392 & 541 & 542 & 547 & 548 & 553 & 554 & 595 & 596 & 601 & 602 & 607 & 608 \\
111 & 112 & 117 & 118 & 123 & 124 & 165 & 166 & 171 & 172 & 177 & 178 & 327 & 328 & 333 & 334 & 339 & 340 & 381 & 382 & 387 & 388 & 393 & 394 & 543 & 544 & 549 & 550 & 555 & 556 & 597 & 598 & 603 & 604 & 609 & 610 \\
113 & 114 & 119 & 120 & 125 & 126 & 167 & 168 & 173 & 174 & 179 & 180 & 329 & 330 & 335 & 336 & 341 & 342 & 383 & 384 & 389 & 390 & 395 & 396 & 545 & 546 & 551 & 552 & 557 & 558 & 599 & 600 & 605 & 606 & 611 & 612 \\
127 & 128 & 133 & 134 & 139 & 140 & 181 & 182 & 187 & 188 & 193 & 194 & 343 & 344 & 349 & 350 & 355 & 356 & 397 & 398 & 403 & 404 & 409 & 410 & 559 & 560 & 565 & 566 & 571 & 572 & 613 & 614 & 619 & 620 & 625 & 626 \\
129 & 130 & 135 & 136 & 141 & 142 & 183 & 184 & 189 & 190 & 195 & 196 & 345 & 346 & 351 & 352 & 357 & 358 & 399 & 400 & 405 & 406 & 411 & 412 & 561 & 562 & 567 & 568 & 573 & 574 & 615 & 616 & 621 & 622 & 627 & 628 \\
131 & 132 & 137 & 138 & 143 & 144 & 185 & 186 & 191 & 192 & 197 & 198 & 347 & 348 & 353 & 354 & 359 & 360 & 401 & 402 & 407 & 408 & 413 & 414 & 563 & 564 & 569 & 570 & 575 & 576 & 617 & 618 & 623 & 624 & 629 & 630 \\
145 & 146 & 151 & 152 & 157 & 158 & 199 & 200 & 205 & 206 & 211 & 212 & 361 & 362 & 367 & 368 & 373 & 374 & 415 & 416 & 421 & 422 & 427 & 428 & 577 & 578 & 583 & 584 & 589 & 590 & 631 & 632 & 637 & 638 & 643 & 644 \\
147 & 148 & 153 & 154 & 159 & 160 & 201 & 202 & 207 & 208 & 213 & 214 & 363 & 364 & 369 & 370 & 375 & 376 & 417 & 418 & 423 & 424 & 429 & 430 & 579 & 580 & 585 & 586 & 591 & 592 & 633 & 634 & 639 & 640 & 645 & 646 \\
149 & 150 & 155 & 156 & 161 & 162 & 203 & 204 & 209 & 210 & 215 & 216 & 365 & 366 & 371 & 372 & 377 & 378 & 419 & 420 & 425 & 426 & 431 & 432 & 581 & 582 & 587 & 588 & 593 & 594 & 635 & 636 & 641 & 642 & 647 & 648 \\
649 & 650 & 655 & 656 & 661 & 662 & 703 & 704 & 709 & 710 & 715 & 716 & 865 & 866 & 871 & 872 & 877 & 878 & 919 & 920 & 925 & 926 & 931 & 932 & 1081 & 1082 & 1087 & 1088 & 1093 & 1094 & 1135 & 1136 & 1141 & 1142 & 1147 & 1148 \\
651 & 652 & 657 & 658 & 663 & 664 & 705 & 706 & 711 & 712 & 717 & 718 & 867 & 868 & 873 & 874 & 879 & 880 & 921 & 922 & 927 & 928 & 933 & 934 & 1083 & 1084 & 1089 & 1090 & 1095 & 1096 & 1137 & 1138 & 1143 & 1144 & 1149 & 1150 \\
653 & 654 & 659 & 660 & 665 & 666 & 707 & 708 & 713 & 714 & 719 & 720 & 869 & 870 & 875 & 876 & 881 & 882 & 923 & 924 & 929 & 930 & 935 & 936 & 1085 & 1086 & 1091 & 1092 & 1097 & 1098 & 1139 & 1140 & 1145 & 1146 & 1151 & 1152 \\
667 & 668 & 673 & 674 & 679 & 680 & 721 & 722 & 727 & 728 & 733 & 734 & 883 & 884 & 889 & 890 & 895 & 896 & 937 & 938 & 943 & 944 & 949 & 950 & 1099 & 1100 & 1105 & 1106 & 1111 & 1112 & 1153 & 1154 & 1159 & 1160 & 1165 & 1166 \\
669 & 670 & 675 & 676 & 681 & 682 & 723 & 724 & 729 & 730 & 735 & 736 & 885 & 886 & 891 & 892 & 897 & 898 & 939 & 940 & 945 & 946 & 951 & 952 & 1101 & 1102 & 1107 & 1108 & 1113 & 1114 & 1155 & 1156 & 1161 & 1162 & 1167 & 1168 \\
671 & 672 & 677 & 678 & 683 & 684 & 725 & 726 & 731 & 732 & 737 & 738 & 887 & 888 & 893 & 894 & 899 & 900 & 941 & 942 & 947 & 948 & 953 & 954 & 1103 & 1104 & 1109 & 1110 & 1115 & 1116 & 1157 & 1158 & 1163 & 1164 & 1169 & 1170 \\
685 & 686 & 691 & 692 & 697 & 698 & 739 & 740 & 745 & 746 & 751 & 752 & 901 & 902 & 907 & 908 & 913 & 914 & 955 & 956 & 961 & 962 & 967 & 968 & 1117 & 1118 & 1123 & 1124 & 1129 & 1130 & 1171 & 1172 & 1177 & 1178 & 1183 & 1184 \\
687 & 688 & 693 & 694 & 699 & 700 & 741 & 742 & 747 & 748 & 753 & 754 & 903 & 904 & 909 & 910 & 915 & 916 & 957 & 958 & 963 & 964 & 969 & 970 & 1119 & 1120 & 1125 & 1126 & 1131 & 1132 & 1173 & 1174 & 1179 & 1180 & 1185 & 1186 \\
689 & 690 & 695 & 696 & 701 & 702 & 743 & 744 & 749 & 750 & 755 & 756 & 905 & 906 & 911 & 912 & 917 & 918 & 959 & 960 & 965 & 966 & 971 & 972 & 1121 & 1122 & 1127 & 1128 & 1133 & 1134 & 1175 & 1176 & 1181 & 1182 & 1187 & 1188 \\
757 & 758 & 763 & 764 & 769 & 770 & 811 & 812 & 817 & 818 & 823 & 824 & 973 & 974 & 979 & 980 & 985 & 986 & 1027 & 1028 & 1033 & 1034 & 1039 & 1040 & 1189 & 1190 & 1195 & 1196 & 1201 & 1202 & 1243 & 1244 & 1249 & 1250 & 1255 & 1256 \\
759 & 760 & 765 & 766 & 771 & 772 & 813 & 814 & 819 & 820 & 825 & 826 & 975 & 976 & 981 & 982 & 987 & 988 & 1029 & 1030 & 1035 & 1036 & 1041 & 1042 & 1191 & 1192 & 1197 & 1198 & 1203 & 1204 & 1245 & 1246 & 1251 & 1252 & 1257 & 1258 \\
761 & 762 & 767 & 768 & 773 & 774 & 815 & 816 & 821 & 822 & 827 & 828 & 977 & 978 & 983 & 984 & 989 & 990 & 1031 & 1032 & 1037 & 1038 & 1043 & 1044 & 1193 & 1194 & 1199 & 1200 & 1205 & 1206 & 1247 & 1248 & 1253 & 1254 & 1259 & 1260 \\
775 & 776 & 781 & 782 & 787 & 788 & 829 & 830 & 835 & 836 & 841 & 842 & 991 & 992 & 997 & 998 & 1003 & 1004 & 1045 & 1046 & 1051 & 1052 & 1057 & 1058 & 1207 & 1208 & 1213 & 1214 & 1219 & 1220 & 1261 & 1262 & 1267 & 1268 & 1273 & 1274 \\
777 & 778 & 783 & 784 & 789 & 790 & 831 & 832 & 837 & 838 & 843 & 844 & 993 & 994 & 999 & 1000 & 1005 & 1006 & 1047 & 1048 & 1053 & 1054 & 1059 & 1060 & 1209 & 1210 & 1215 & 1216 & 1221 & 1222 & 1263 & 1264 & 1269 & 1270 & 1275 & 1276 \\
779 & 780 & 785 & 786 & 791 & 792 & 833 & 834 & 839 & 840 & 845 & 846 & 995 & 996 & 1001 & 1002 & 1007 & 1008 & 1049 & 1050 & 1055 & 1056 & 1061 & 1062 & 1211 & 1212 & 1217 & 1218 & 1223 & 1224 & 1265 & 1266 & 1271 & 1272 & 1277 & 1278 \\
793 & 794 & 799 & 800 & 805 & 806 & 847 & 848 & 853 & 854 & 859 & 860 & 1009 & 1010 & 1015 & 1016 & 1021 & 1022 & 1063 & 1064 & 1069 & 1070 & 1075 & 1076 & 1225 & 1226 & 1231 & 1232 & 1237 & 1238 & 1279 & 1280 & 1285 & 1286 & 1291 & 1292 \\
795 & 796 & 801 & 802 & 807 & 808 & 849 & 850 & 855 & 856 & 861 & 862 & 1011 & 1012 & 1017 & 1018 & 1023 & 1024 & 1065 & 1066 & 1071 & 1072 & 1077 & 1078 & 1227 & 1228 & 1233 & 1234 & 1239 & 1240 & 1281 & 1282 & 1287 & 1288 & 1293 & 1294 \\
797 & 798 & 803 & 804 & 809 & 810 & 851 & 852 & 857 & 858 & 863 & 864 & 1013 & 1014 & 1019 & 1020 & 1025 & 1026 & 1067 & 1068 & 1073 & 1074 & 1079 & 1080 & 1229 & 1230 & 1235 & 1236 & 1241 & 1242 & 1283 & 1284 & 1289 & 1290 & 1295 & 1296
\end{pmatrix}
$$

The following table lists the number of number of sum-and-distance systems N_n for n in terms of the prime factorisation of n.

Table 14: $n = p_1^{a_1}\ p_2^0\ p_3^0\ p_4^0\ p_5^0$ (one prime)

a_1	a_2	a_3	a_4	a_5	$\Omega(n)$	N_n
1	0	0	0	0	1	1
2	0	0	0	0	2	3
3	0	0	0	0	3	10
4	0	0	0	0	4	35
5	0	0	0	0	5	126
6	0	0	0	0	6	462

Table 15: $n = p_1^{a_1}\ p_2^{a_2}\ p_3^0\ p_4^0\ p_5^0$ (two primes)

a_1	a_2	a_3	a_4	a_5	$\Omega(n)$	N_n
1	1	0	0	0	2	7
1	2	0	0	0	3	42
1	3	0	0	0	4	230
1	4	0	0	0	5	1190
1	5	0	0	0	6	5922
2	2	0	0	0	4	393
2	3	0	0	0	5	3030
2	4	0	0	0	6	20790
2	5	0	0	0	7	131796
3	3	0	0	0	6	30670
3	4	0	0	0	7	264740
3	5	0	0	0	8	2050020
4	4	0	0	0	8	2781065
4	5	0	0	0	9	25586694
5	5	0	0	0	10	271679058

Table 16: $n = p_1^{a_1}\ p_2^{a_2}\ p_3^{a_3}\ p_4^0\ p_5^0$ (three primes)

a_1	a_2	a_3	a_4	a_5	$\Omega(n)$	N_n
1	1	1	0	0	3	115
1	1	2	0	0	4	1158
1	1	3	0	0	5	9350
1	1	4	0	0	6	66290
1	1	5	0	0	7	430794
1	1	6	0	0	8	2628780
1	2	2	0	0	5	16782
1	2	3	0	0	6	180990
1	2	4	0	0	7	1636740
1	2	5	0	0	8	13141044
1	2	6	0	0	9	96687612
1	3	3	0	0	7	2474030
1	3	4	0	0	8	27413540
1	3	5	0	0	9	262999044
1	3	6	0	0	10	2243103996
1	4	4	0	0	9	361969790
1	4	5	0	0	10	4001024034
1	4	6	0	0	11	37210138644
1	5	5	0	0	11	48035790810
2	2	2	0	0	6	334833
2	2	3	0	0	7	4676670
2	2	4	0	0	8	52682490
2	2	5	0	0	9	512075340
2	3	3	0	0	7	80988270

Table 17: $n = p_1^{a_1}\ p_2^{a_2}\ p_3^{a_3}\ p_4^{a_4}\ p_5^{0}$ (four primes)

a_1	a_2	a_3	a_4	a_5	$\Omega(n)$	N_n
1	1	1	1	0	4	3451
1	1	1	2	0	5	52422
1	1	1	3	0	6	583670
1	1	1	4	0	7	5404490
1	1	1	5	0	8	44200170
1	1	2	2	0	6	1083318
1	1	2	3	0	7	15509070
1	1	2	4	0	8	178011540
1	1	2	5	0	9	1 758 179 556
1	1	2	6	0	10	15 558 091 164
1	2	3	3	0	9	11 521 530 270
1	2	3	4	0	10	190 441 098 540

Table 18: $n = p_1^{a_1}\ p_2^{a_2}\ p_3^{a_3}\ p_4^{a_4}\ p_5^{a_5}$ (four primes)

a_1	a_2	a_3	a_4	a_5	$\Omega(n)$	N_n
1	1	1	1	1	5	164731
1	1	1	1	2	6	3 518 358
1	1	1	1	3	7	51 569 510
1	1	1	1	4	8	602 678 090
1	1	1	1	5	9	6 038 986 842
1	1	1	2	2	7	98 090 142
1	1	1	2	3	8	1 795 856 670
1	1	1	2	3	9	25 445 723 940

Appendix 5: Mathematica Scripts for Inclusive Sum-and-Distance Systems

The following contain Mathematica scripts that produce inclusive sum-and-distance systems for any n with length $\alpha = 1, 2$ or 3 divisor path sets.

```
(* For all k_i < 10 and alpha = 2 *)

(*Defining the alpha=2 length divisor path set*)

In[10]:= S1 = {{i1, n}, {j1, j2}}

Out[10]= {{i1, n}, {j1, j2}}

(*Defining the set of L_{2} lattice points associated with divisor path set "set1"*)

In[13]:= L1[S_] := Flatten[Table[{k0, k1, k2},
    {k0, 1, S[[2]][[1]]},
    {k1, 1, S[[2]][[2]] / S[[2]][[1]]},
    {k2, 1, S[[1]][[2]] / S[[2]][[2]]}], 2]

(*Defining the set of L'_{2} lattice
 points associated with divisor path set "set1"*)

In[98]:= L1a[S_] := Flatten[Table[
        If[
            k0 + k1 * 10 + k2 * 10^2 < (1 / 2) * ((S[[2]][[1]] + 1)
                        + (S[[2]][[2]] / S[[2]][[1]] + 1) * 10
                        + (S[[1]][[2]] / S[[2]][[2]] + 1) * 10^2),
            {k0, k1, k2},
            Nothing],
    {k0, 1, S[[2]][[1]]},
    {k1, 1, S[[2]][[2]] / S[[2]][[1]]},
    {k2, 1, S[[1]][[2]] / S[[2]][[2]]} ], 2]

In[99]:= (*The formula that gives the left set of the sum and
 distance system associated with divisor path set "set1"*)
    SDL1[set1_] := Sort[-1 * (Table[
                L1a[set1][[k]][[1]]
                  + (L1a[set1][[k]][[2]] - 1) (set1[[1]][[1]] * set1[[2]][[1]])
                  + (L1a[set1][[k]][[3]] - 1) (set1[[1]][[2]] * set1[[2]][[2]]),
            {k, 1, Length[L1a[set1]]}] - 1 / 2 (1 + set1[[2]][[1]]
          + (set1[[2]][[2]] / set1[[2]][[1]] - 1) (set1[[1]][[1]] * set1[[2]][[1]])
             + (set1[[1]][[2]] / set1[[2]][[2]] - 1) (set1[[1]][[2]] * set1[[2]][[2]])))]

(*Defining the set of M_{2} lattice points associated with divisor path set "set1"*)

In[100]:= M1[S_] := Table[{m0, m1},
   {m0, 1, S[[1]][[1]]},
   {m1, 1, S[[1]][[2]] / S[[1]][[1]]}]

(*Defining the set of M'_{2} lattice
 points associated with divisor path set "set1"*)
```

In[101]:=
```
M1a[S_] := Flatten[
  Table[If[m0 + m1 * 10 < (1 / 2) * ((S[[1]][[1]] + 1) + (S[[1]][[2]] / S[[1]][[1]] + 1) * 10),
    {m0, m1}, Nothing],
   {m0, 1, S[[1]][[1]]},
   {m1, 1, S[[1]][[2]] / S[[1]][[1]]}], 1]

(*The formula that gives the left set of the sum and
 distance system associated with divisor path set "set1"*)
```

In[107]:=
```
SDR1[set1_] :=
 Sort[-1 * (Table[1 + (M1a[set1][[m]][[1]] - 1) set1[[2]][[1]] + (M1a[set1][[m]][[2]] - 1)
       (set1[[2]][[2]] / set1[[2]][[1]]) (set1[[1]][[1]] * set1[[2]][[1]])
     , {m, 1, Length[M1a[set1]]}] - ((1 + (1 / 2) (set1[[1]][[1]] - 1) set1[[2]][[1]]) +
      (1 / 2) (set1[[1]][[2]] / set1[[1]][[1]] - 1)
       (set1[[1]][[1]] * set1[[2]][[1]]) (set1[[2]][[2]] / set1[[2]][[1]]))))]

(*Function to check if SDR2 and SDL2 form a sum and distance system*)
```

In[124]:=
```
PMU1[set1_] := Union[
  Sort[
   Flatten[
    Table[
       {Abs[SDR1[set1][[k]] - SDL1[set1][[m]]],
        SDR1[set1][[k]] + SDL1[set1][[m]],
        SDR1[set1][[k]],
        SDL1[set1][[m]]},
     {k, 1, Length[SDR1[set1]]},
     {m, 1, Length[SDL1[set1]]}]]]]
```

In[112]:=
```
(*Example*)
TestSet = {{3, 15}, {5, 15}}
```

Out[112]= {{3, 15}, {5, 15}}

In[114]:= `L1[TestSet]`

Out[114]= {{1, 1, 1}, {1, 2, 1}, {1, 3, 1}, {2, 1, 1}, {2, 2, 1}, {2, 3, 1}, {3, 1, 1}, {3, 2, 1},
{3, 3, 1}, {4, 1, 1}, {4, 2, 1}, {4, 3, 1}, {5, 1, 1}, {5, 2, 1}, {5, 3, 1}}

In[115]:= `L1a[TestSet]`

Out[115]= {{1, 1, 1}, {1, 2, 1}, {2, 1, 1}, {2, 2, 1}, {3, 1, 1}, {4, 1, 1}, {5, 1, 1}}

In[116]:= `M1[TestSet]`

Out[116]= {{{1, 1}, {1, 2}, {1, 3}, {1, 4}, {1, 5}},
{{2, 1}, {2, 2}, {2, 3}, {2, 4}, {2, 5}}, {{3, 1}, {3, 2}, {3, 3}, {3, 4}, {3, 5}}}

In[117]:= `M1a[TestSet]`

Out[117]= {{1, 1}, {1, 2}, {1, 3}, {2, 1}, {2, 2}, {3, 1}, {3, 2}}

```
In[119]:= SDL1[TestSet]
Out[119]= {1, 2, 13, 14, 15, 16, 17}

In[118]:= SDR1[TestSet]
Out[118]= {5, 40, 45, 50, 85, 90, 95}

In[125]:= PMU1[{{3, 15}, {5, 15}}]
Out[125]= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26,
 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49,
 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71,
 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93,
 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112}
```

```
(* For all k_i < 10^2 and alpha = 3 *)

(*Defining the alpha=3 length divisor path set*)

In[1]:= S2 = {{i1, i2, n}, {j1, j2, j3}}

Out[1]= {{i1, i2, n}, {j1, j2, j3}}

(*Defining the set of L_{3} lattice points associated with divisor path set "set1"*)

In[8]:= L2[S_] := Flatten[Table[{k0, k1, k2, k3},
    {k0, 1, S[[2]][[1]]},
    {k1, 1, S[[2]][[2]] / S[[2]][[1]]},
    {k2, 1, S[[2]][[3]] / S[[2]][[2]]},
    {k3, 1, S[[1]][[3]] / S[[2]][[3]]}], 3]

(*Defining the set of L'_{3} lattice
 points associated with divisor path set "set1"*)

In[38]:= L2a[S_] := Flatten[Table[
        If[k0 + k1 * 10^2 + k2 * 10^4 + k3 * 10^6
                           < (1 / 2) * ((S[[2]][[1]] + 1)
                          + (S[[2]][[2]] / S[[2]][[1]] + 1) * 10^2
                          + (S[[2]][[3]] / S[[2]][[2]] + 1) * 10^4
                          + (S[[1]][[3]] / S[[2]][[3]] + 1) * 10^6),
             {k0, k1, k2, k3},
             Nothing],
    {k0, 1, S[[2]][[1]]},
    {k1, 1, S[[2]][[2]] / S[[2]][[1]]},
    {k2, 1, S[[2]][[3]] / S[[2]][[2]]} ,
    {k3, 1, S[[1]][[3]] / S[[2]][[3]]}], 3]

(*The formula that gives the left set of the sum and
 distance system associated with divisor path set "set1"*)
SDL2[set1_] := Sort[-1 * (Table[
              L2a[set1][[k]][[1]]
               + (L2a[set1][[k]][[2]] - 1) (set1[[1]][[1]] * set1[[2]][[1]])
               + (L2a[set1][[k]][[3]] - 1) (set1[[1]][[2]] * set1[[2]][[2]])
               + (L2a[set1][[k]][[4]] - 1) (set1[[1]][[3]] * set1[[2]][[3]]),
       {k, 1, Length[L2a[set1]]}]
      - 1 / 2 (1 + set1[[2]][[1]]
             + (set1[[2]][[2]] / set1[[2]][[1]] - 1) (set1[[1]][[1]] * set1[[2]][[1]])
             + (set1[[2]][[3]] / set1[[2]][[2]] - 1) (set1[[1]][[2]] * set1[[2]][[2]])
             + (set1[[1]][[3]] / set1[[2]][[3]] - 1) (set1[[1]][[3]] * set1[[2]][[3]]))
        )]

(*Defining the set of M_{3} lattice points associated with divisor path set "set1"*)
```

In[49]:=
```
M2[S_] := Flatten[Table[{m0, m1, m2},
   {m0, 1, S[[1]][[1]]},
   {m1, 1, S[[1]][[2]] / S[[1]][[1]]},
   {m2, 1, S[[1]][[3]] / S[[1]][[2]]}], 2]

(*Defining the set of M'_{3} lattice
 points associated with divisor path set "set1"*)
```

In[54]:=
```
M2a[S_] := Flatten[Table[If[m0  + m1 * 10^2 + m2 * 10^4
      < (1 / 2) * ((S[[1]][[1]] + 1)
          + (S[[1]][[2]] / S[[1]][[1]]  + 1) * 10^2
          + (S[[1]][[3]] / S[[1]][[2]] + 1) * 10^4), {m0, m1, m2}, Nothing],
   {m0, 1, S[[1]][[1]]},
   {m1, 1, S[[1]][[2]] / S[[1]][[1]]},
   {m2, 1, S[[1]][[3]] / S[[1]][[2]]}], 2]

(*The formula that gives the left set of the sum and
 distance system associated with divisor path set "set1"*)

SDR2[set1_] := Sort[-1 (Table[1 + (M2a[set1][[m]][[1]] - 1) set1[[2]][[1]]
         + (M2a[set1][[m]][[2]] - 1)
          (set1[[2]][[2]] / set1[[2]][[1]]) (set1[[1]][[1]] * set1[[2]][[1]])
         + (M2a[set1][[m]][[3]] - 1) (set1[[2]][[3]] / set1[[2]][[2]])
          (set1[[1]][[2]] * set1[[2]][[2]])
       , {m, 1, Length[M2a[set1]]}] - ((1 + (1 / 2) (set1[[1]][[1]] - 1) set1[[2]][[1]])
        + (1 / 2) (set1[[1]][[2]] / set1[[1]][[1]] - 1)
          (set1[[1]][[1]] * set1[[2]][[1]]) (set1[[2]][[2]] / set1[[2]][[1]])
        + (1 / 2) (set1[[1]][[3]] / set1[[1]][[2]] - 1) (set1[[1]][[2]] * set1[[2]][[2]])
          (set1[[2]][[3]] / set1[[2]][[2]]))))]
(*Function to check if SDR2 and SDL2 form a sum and distance system*)
```

In[122]:=
```
PMU2[set1_] := Union[
  Sort[
   Flatten[
    Table[
        {Abs[SDR2[set1][[k]] - SDL2[set1][[m]]],
         SDR2[set1][[k]] + SDL2[set1][[m]],
         SDR2[set1][[k]],
         SDL2[set1][[m]]},
      {k, 1, Length[SDR2[set1]]},
      {m, 1, Length[SDL2[set1]]}]]]]
```

In[147]:=
```
(*Example*)
TestSet = {{3, 9, 27}, {3, 9, 27}}
```

Out[147]=
```
{{3, 9, 27}, {3, 9, 27}}
```

```
In[148]:= L2[TestSet]

Out[148]= {{1, 1, 1, 1}, {1, 1, 2, 1}, {1, 1, 3, 1}, {1, 2, 1, 1}, {1, 2, 2, 1},
  {1, 2, 3, 1}, {1, 3, 1, 1}, {1, 3, 2, 1}, {1, 3, 3, 1}, {2, 1, 1, 1},
  {2, 1, 2, 1}, {2, 1, 3, 1}, {2, 2, 1, 1}, {2, 2, 2, 1}, {2, 2, 3, 1},
  {2, 3, 1, 1}, {2, 3, 2, 1}, {2, 3, 3, 1}, {3, 1, 1, 1}, {3, 1, 2, 1}, {3, 1, 3, 1},
  {3, 2, 1, 1}, {3, 2, 2, 1}, {3, 2, 3, 1}, {3, 3, 1, 1}, {3, 3, 2, 1}, {3, 3, 3, 1}}

In[149]:= L2a[TestSet]

Out[149]= {{1, 1, 1, 1}, {1, 1, 2, 1}, {1, 2, 1, 1}, {1, 2, 2, 1},
  {1, 3, 1, 1}, {2, 1, 1, 1}, {2, 1, 2, 1}, {2, 2, 1, 1},
  {2, 3, 1, 1}, {3, 1, 1, 1}, {3, 1, 2, 1}, {3, 2, 1, 1}, {3, 3, 1, 1}}

In[150]:= M2[TestSet]

Out[150]= {{1, 1, 1}, {1, 1, 2}, {1, 1, 3}, {1, 2, 1}, {1, 2, 2}, {1, 2, 3},
  {1, 3, 1}, {1, 3, 2}, {1, 3, 3}, {2, 1, 1}, {2, 1, 2}, {2, 1, 3}, {2, 2, 1},
  {2, 2, 2}, {2, 2, 3}, {2, 3, 1}, {2, 3, 2}, {2, 3, 3}, {3, 1, 1}, {3, 1, 2},
  {3, 1, 3}, {3, 2, 1}, {3, 2, 2}, {3, 2, 3}, {3, 3, 1}, {3, 3, 2}, {3, 3, 3}}

In[151]:= M2a[TestSet]

Out[151]= {{1, 1, 1}, {1, 1, 2}, {1, 2, 1}, {1, 2, 2}, {1, 3, 1}, {2, 1, 1},
  {2, 1, 2}, {2, 2, 1}, {2, 3, 1}, {3, 1, 1}, {3, 1, 2}, {3, 2, 1}, {3, 3, 1}}

In[152]:= SDL2[TestSet]

Out[152]= {1, 8, 9, 10, 71, 72, 73, 80, 81, 82, 89, 90, 91}

In[153]:= SDR2[TestSet]

Out[153]= {3, 24, 27, 30, 213, 216, 219, 240, 243, 246, 267, 270, 273}
```

```
In[154]:= PMU2[TestSet]

Out[154]= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25,
  26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47,
  48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69,
  70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90,
  91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109,
  110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126,
  127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143,
  144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160,
  161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177,
  178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194,
  195, 196, 197, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211,
  212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228,
  229, 230, 231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243, 244, 245,
  246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 256, 257, 258, 259, 260, 261, 262,
  263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279,
  280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296,
  297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309, 310, 311, 312, 313,
  314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330,
  331, 332, 333, 334, 335, 336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347,
  348, 349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361, 362, 363, 364}
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