Lightweight Temporal Description Logics with Rigid Roles and Restricted TBoxes

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Abstract

We study temporal description logics (TDLs) based on the branching-time temporal logic CTL and the lightweight DL $\mathcal{EL}$ in the presence of rigid roles and restricted TBoxes. While TDLs designed in this way are known to be inherently nonelementary or even undecidable over general TBoxes, there is hope for a better computational behaviour over acyclic or empty TBoxes. We begin by showing that the basic DL $\mathcal{ALC}$ combined with CTL in the described way is indeed decidable, but still inherently nonelementary. As our main contribution, we identify several TDLs of elementary complexity, obtained by combining $\mathcal{EL}$ with CTL fragments that allow only restricted sets of temporal operators. We obtain upper complexity bounds ranging from PTIME to CONEXPTIME and mostly tight lower bounds. This contrasts the fact that the respective $\mathcal{ALC}$ variants are already inherently nonelementary.

1 Introduction

Classical description logics (DLs), such as those underlying the W3C standard OWL, are a successful family of knowledge representation languages. Temporal description logics (TDLs) extend classical DLs, providing built-in means to represent and reason about temporal aspects of knowledge. The importance of TDLs stems from the need of relevant applications to capture temporal and dynamic aspects of knowledge, e.g., in medical and life science ontologies, which are very large but still demand efficient reasoning, such as SNOMED CT and FMA [Bodenreider and Zhang, 2006], and the gene ontology (GO) [The Gene Ontology Consortium, 2000]. A natural task is to model *dynamic* knowledge about patient histories against *static* medical knowledge (e.g., about diseases): e.g., the temporal concept $C := \mathbf{E}\Diamond\exists\,\mathsf{requiresTransfusion}.\top$ describes a patient who may need a blood transfusion in the future, and the axiom $\mathsf{Anemic} \sqsubseteq C$ says that this applies to anemic people. In contrast, $\mathsf{Anemia} \sqsubseteq \mathsf{Disorder}$ represents static knowledge.

A notable approach to designing TDLs is to combine DLs with temporal logics commonly used in software/hardware verification such as LTL, $\text{CTL}^{(*)}$, and to provide a two-dimensional product-like semantics [Schild, 1993; Gabbay *et al.*, 2003; Lutz *et al.*, 2008]. The combination allows various design choices, e.g., we can restrict the scope of temporal operators to certain types of entities (such as concepts, roles, axioms), or declare some DL concepts or roles as rigid, meaning that their interpretation will not change over time. The need for rigid roles in TDL applications, e.g., in biomedical ontologies to accurately capture life-time relations, has been identified [Baader *et al.*, 2008]. For example, the role $\mathsf{hasBloodType}$ should be rigid since a human's blood type does not change during their lifetime.

Unfortunately, TDLs based on the Boolean-complete DL $\mathcal{ALC}$ with rigid roles cannot be effectively used since they become undecidable as soon as temporal operators are applied to concepts and a general TBox is allowed [Gabbay *et al.*, 2003; Gutiérrez-Basulto *et al.*, 2014]. This is the case even if we severely restrict the temporal operators available and use the sub-Boolean DL $\mathcal{EL}$, whose standard reasoning problems are tractable, instead of $\mathcal{ALC}$ [Artale *et al.*, 2007a; Gutiérrez-Basulto *et al.*, 2014]. In the light of these results, several efforts have been devoted to the design of decidable TDLs with rigid roles [Artale *et al.*, 2007b; 2014]; e.g., decidability can be attained by using a different lightweight DL component based on *DL-Lite*. Both the $\mathcal{EL}$ and *DL-Lite* families underlie prominent profiles of the OWL standard.

Interestingly, no research has been yet devoted to TDLs based on $\mathcal{EL}$ in the presence of restricted TBoxes, such as classical TBoxes, which consist solely of definitions of the form $A \equiv C$ with A atomic and unique, or acyclic TBoxes, which additionally forbid syntactic cycles in definitions. This is surprising since in the presence of general TBoxes TDLs based on $\mathcal{EL}$ tend to be as complex as the $\mathcal{ALC}$ variant [Artale *et al.*, 2007b; Gutiérrez-Basulto *et al.*, 2012; 2014].

These considerations lead us to investigating TDLs with rigid roles based on $\mathcal{EL}$ and the (branching-time) CTL allowing for temporal concepts and empty or acyclic TBoxes. We strongly believe that TDLs designed in this way are well-suited as temporal extensions of biomedical ontologies. After all, large parts of SNOMED CT and GO indeed are acyclic $\mathcal{EL}$-TBoxes.

Our main contributions are algorithms for standard reasoning problems and (mostly tight) complexity bounds. We begin by showing that the combination of CTL and $\mathcal{ALC}$ with empty and acyclic TBoxes is decidable. Our nonelementary upper bound is optimal even when the set of temporal operators is

Rigid roles? TBoxes	no general	yes general	yes acyclic	yes empty
$\text{CTL}_{\mathcal{ALC}}$	$=$ExpTime[1]	undecidable[2]	nonelementary, decidable [Thm. 1]	nonelem., decidable [Thm. 1]
$\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$/$\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$	$\leq$PTime[1]	nonelem./undecid.[2]	$\leq$PTime [Thm. 6]	$\leq$PTime [Thm. 6]
$\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$	$=$ExpTime[1,2]	undecidable[2]	$\geq$coNP, $\leq$coNExpTime [Thm. 2, Cor. 5]	$=$coNP [Thm. 2]
$\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond,\mathbf{A}\Box}$	$=$PSpace[1]	nonelementary[2]	$=$PSpace [Thm. 9]	$\leq$PSpace [Thm. 9]

[1] [Gutiérrez-Basulto *et al.*, 2012] [2] [Gutiérrez-Basulto *et al.*, 2014]

Table 1: Overview of previous and new complexity results. $\geq$ hardness, $\leq$ membership, $=$ completeness

restricted to $\mathbf{E}\Diamond$ ("possibly eventually") or $\mathbf{E}\bigcirc$ ("possibly next"). We then replace $\mathcal{ALC}$ with $\mathcal{EL}$ and maintain the restriction to $\mathbf{E}\Diamond, \mathbf{E}\bigcirc$ and empty TBoxes. We particularly show that the resulting TDLs are decidable in PTime with one of the two operators, and coNP-complete with both. To this aim, we employ canonical models, together with expansion vectors [Haase and Lutz, 2008] in the case with both $\mathbf{E}\Diamond, \mathbf{E}\bigcirc$. Next, we lift the PTime upper bound to the case of acyclic TBoxes, employing a completion algorithm in the style of those for $\mathcal{EL}$ and extensions, [Baader *et al.*, 2005]. Finally, we show that the combination of $\mathbf{E}\Diamond$ with $\mathbf{A}\Box$ ("always globally") and acyclic TBoxes leads to a PSpace-complete TDL, again employing a completion algorithm. An overview of existing and new results is given in Table 1, where CTL_X^Y denotes the combination of the DL X with the fragment of CTL restricted to the temporal operators Y. In particular, all the new results hold even if rigid concepts are also included.

The relatively low complexity that we obtain for $\mathcal{EL}$-based TDLs over restricted TBoxes are in sharp contrast with the undecidability and nonelementary lower bounds known for the same logics over general TBoxes [Gutiérrez-Basulto *et al.*, 2014]. With the restriction to acyclic TBoxes, we will thus identify the first computationally well-behaved TDLs with rigid roles based on $\mathcal{EL}$ and classical temporal logics.

Due to limited space, additional technical notions and proofs are in a report: http://tinyurl.com/ijcai15tdl

2 Preliminaries

We introduce $\text{CTL}_{\mathcal{ALC}}$, a TDL based on the classical DL $\mathcal{ALC}$. Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be countably infinite sets of *concept names* and *role names*, respectively. We assume that $\mathsf{N_C}$ and $\mathsf{N_R}$ are partitioned into two countably infinite sets: $\mathsf{N_C^{rig}}$ and $\mathsf{N_C^{loc}}$ of *rigid concept names* and *local concept names*, respectively; and, $\mathsf{N_R^{rig}}$ and $\mathsf{N_R^{loc}}$ of *rigid role names* and *local role names*, respectively. $\text{CTL}_{\mathcal{ALC}}$*-concepts* C are defined by the grammar

$$C := \top \mid A \mid \neg C \mid C \sqcap D \mid \exists r.C \mid \mathbf{E}\bigcirc C \mid \mathbf{E}\Box C \mid \mathbf{E}(C\mathcal{U}D)$$

where A ranges over $\mathsf{N_C}$, r over $\mathsf{N_R}$. We use standard DL abbreviations [Baader *et al.*, 2003] and temporal abbreviations $\mathbf{E}\Diamond C$, $\mathbf{A}\Box C$, $\mathbf{A}\Diamond C$ and $\mathbf{A}(C\,\mathcal{U}\,D)$ [Clarke *et al.*, 1999].

The semantics of classical DLs, such as $\mathcal{ALC}$, is given in terms of *interpretations* of the form $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$, where Δ is a non-empty set called the *domain* and $\cdot^{\mathcal{I}}$ is an *interpretation function* that maps each $A \in \mathsf{N_C}$ to a subset $A^{\mathcal{I}} \subseteq \Delta$ and each $r \in \mathsf{N_R}$ to a binary relation $r^{\mathcal{I}} \subseteq \Delta \times \Delta$. The semantics of $\text{CTL}_{\mathcal{ALC}}$ is given in terms of temporal interpretations based on infinite trees [Gutiérrez-Basulto *et al.*, 2014]: A *temporal interpretation* based on an infinite tree $T = (W, E)$ is a structure $\mathfrak{I} = (T, (\mathcal{I}_w)_{w \in W})$ such that, for each $w \in W$, $\mathcal{I}_w$ is a DL interpretation with domain Δ; and, $r^{\mathcal{I}_w} = r^{\mathcal{I}_{w'}}$ and $A^{\mathcal{I}_w} = A^{\mathcal{I}_{w'}}$ for all $r \in \mathsf{N_R^{rig}}$, $A \in \mathsf{N_C^{rig}}$ and $w, w' \in W$. We usually write $A^{\mathfrak{I},w}$ instead of $A^{\mathcal{I}_w}$. The stipulation that all worlds share the same domain is called the *constant domain assumption (CDA)*. For Boolean-complete TDLs, CDA is the most general: increasing, decreasing and varying domains can all be reduced to it [Gabbay *et al.*, 2003, Prop. 3.32]. For the sub-Boolean logics studied here, CDA is not w.l.o.g. Indeed, we identify a logic in which reasoning with increasing domains cannot be reduced to the constant domain case.

We now define the semantics of $\text{CTL}_{\mathcal{ALC}}$-concepts. A *path* in $T = (W, E)$ starting at a node w is an infinite sequence $\pi = w_0 w_1 w_2 \cdots$ with $w_0 = w$ and $(w_i, w_{i+1}) \in E$. We write $\pi[i]$ for w_i, and use $\mathsf{Paths}(w)$ to denote the set of all paths starting at the node w. The mapping $\cdot^{\mathfrak{I},w}$ is extended from concept names to $\text{CTL}_{\mathcal{ALC}}$-concepts as follows.

$$\begin{aligned}
\top^{\mathfrak{I},w} &= \Delta \quad (C \sqcap D)^{\mathfrak{I},w} = C^{\mathfrak{I},w} \cap D^{\mathfrak{I},w}\\
(\exists r.C)^{\mathfrak{I},w} &= \{d \in \Delta \mid \exists e \,.\, (d, e) \in r^{\mathfrak{I},w} \wedge e \in C^{\mathfrak{I},w}\}\\
(\mathbf{E}\bigcirc C)^{\mathfrak{I},w} &= \{d \mid \exists \pi \in \mathsf{Paths}(w) \,.\, d \in C^{\mathfrak{I},\pi[1]}\}\\
(\mathbf{E}\Box C)^{\mathfrak{I},w} &= \{d \mid \exists \pi \in \mathsf{Paths}(w) \,.\, \forall j \geq 0 \,.\, d \in C^{\mathfrak{I},\pi[j]}\}\\
(\mathbf{E}(C\mathcal{U}D))^{\mathfrak{I},w} &= \{d \mid \exists \pi \in \mathsf{Paths}(w) \,.\, \exists j \geq 0 \,.\, (d \in D^{\mathfrak{I},\pi[j]}\\
&\qquad \wedge (\forall 0 \leq k < j \,.\, d \in C^{\mathfrak{I},\pi[k]}))\}
\end{aligned}$$

An *acyclic* $\text{CTL}_{\mathcal{ALC}}$*-TBox* $\mathcal{T}$ is a finite set of *concept definitions (CDs)* $A \equiv D$ with $A \in \mathsf{N_C}$ and D a $\text{CTL}_{\mathcal{ALC}}$ concept, such that (1) no two CDs have the same left-hand side, and (2) there are no CDs $A_1 \equiv C_1, \ldots, A_k \equiv C_k$ in $\mathcal{T}$ such that A_{i+1} occurs in C_i for $1 \leq i \leq k$, where $A_{k+1} = A_1$.

A temporal interpretation $\mathfrak{I}$ is a *model* of a concept C if $C^{\mathfrak{I},\varepsilon} \neq \emptyset$; it is a model of an acyclic TBox $\mathcal{T}$, written $\mathfrak{I} \models \mathcal{T}$, if $A^{\mathfrak{I},w} = C^{\mathfrak{I},w}$ for all $A \equiv C$ in $\mathcal{T}$ and $w \in W$; it is a model of a *concept inclusion* $C \sqsubseteq D$, written $\mathfrak{I} \models C \sqsubseteq D$, if $C^{\mathfrak{I},w} \subseteq D^{\mathfrak{I},w}$ for all $w \in W$.

The two main reasoning tasks we consider are concept satisfiability and subsumption. A concept C is *satisfiable* relative to an acyclic TBox $\mathcal{T}$ if there is a common model of C and $\mathcal{T}$. A concept D *subsumes* a concept C relative to an acyclic TBox $\mathcal{T}$, written $\mathcal{T} \models C \sqsubseteq D$, if $\mathfrak{I} \models C \sqsubseteq D$ for all models $\mathfrak{I}$ of $\mathcal{T}$. If $\mathcal{T}$ is empty, we write $\models C \sqsubseteq D$.

3 First Observations

We start by observing that the combination of CTL and $\mathcal{ALC}$ with rigid roles relative to empty and acyclic TBoxes is decidable and inherently nonelementary. In a nutshell, we show the upper bounds using a variant of the quasimodel technique [Gabbay *et al.*, 2003, Thm. 13.6]; the lower bound follows from the fact that satisfiability for the product modal logics S4×K and K×K is inherently nonelementary [Göller *et al.*, 2015]. Indeed, the fragment of $\mathrm{CTL}_{\mathcal{ALC}}$ allowing $\mathbf{E}\Diamond$ ($\mathbf{E}\bigcirc$) as the only temporal operator is a notational variant of S4×K (K×K) [Gutiérrez-Basulto *et al.*, 2014].

Theorem 1 *Concept satisfiability relative to acyclic and empty TBoxes for* $\mathrm{CTL}_{\mathcal{ALC}}$ *with rigid roles is decidable and inherently nonelementary.*

With Theorem 1 and the third column of Table 1 in mind, we particularly set as our goal the identification of elementary (ideally tractable) TDLs. To this aim, we study combinations of (fragments of) CTL with the lightweight DL $\mathcal{EL}$. $\mathrm{CTL}_{\mathcal{EL}}$ is the fragment of $\mathrm{CTL}_{\mathcal{ALC}}$ that disallows the constructor $\neg$ (and thus the abbreviations $C \sqcup D$, $\forall r.C$, $\mathbf{A}\Box, \dots$). The standard reasoning problem for $\mathrm{CTL}_{\mathcal{EL}}$, as for $\mathcal{EL}$, is concept subsumption since each concept and TBox are trivially satisfiable. In what follows we consider various fragments of $\mathrm{CTL}_{\mathcal{EL}}$ obtained by restricting the available temporal operators. We denote the fragments by putting the allowed operators as a superscript. In this context, we view each of the operators $\mathbf{E}\Diamond$, $\mathbf{A}\Box$ as primitive instead of as an abbreviation.

In order to keep the presentation of our main results accessible, in Sections 5-6, we concentrate on the case where only rigid role names and local concept names are present. Later on, in Section 7, we explain how to deal with the general case.

4 $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$ relative to the Empty TBox

We begin by investigating the complexity of subsumption relative to the empty TBox for a TDL whose subsumption relative to general TBoxes is undecidable: $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$.

Theorem 2 *Concept subsumption relative to the empty TBox is* CONP*-complete for* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$ *with rigid roles and in* PTIME *for* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ *and* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ *with rigid roles.*

CONP-hardness is obtained by embedding $\mathcal{EL}$ plus transitive closure into $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$; the jump in complexity comes from the ability to express disjunctions, e.g., $\models \mathbf{E}\Diamond C \sqsubseteq C \sqcup \mathbf{E}\bigcirc\mathbf{E}\Diamond C$. We next explain CONP-membership; the PTIME results are a byproduct and improved later.

We proceed in two steps: first we provide a characterization of $\models C \sqsubseteq D$ where C is an $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$-concept and D an $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$-concept. Next we generalize this characterization to $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$-concepts C.

Given a $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$-concept C, the *description tree* $t_C = (V_C, L_C, E_C)$ *for* C is a labeled graph corresponding to C's syntax tree; we denote its *root* by x_C. For example, if $C = \mathbf{E}\bigcirc(\exists r.A \sqcap \exists s.B)$, then t_C is given in Figure 1, left.

For plain $\mathcal{EL}$, we have $\models C \sqsubseteq D$ if and only if there is a homomorphism from t_D to t_C, which can be tested in polynomial time [Baader *et al.*, 1999]. This criterion cannot directly be transferred to $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ because t_C does not explicitly represent all pairs of worlds and domain elements whose existence is implied by t_C, e.g., for $\models \mathbf{E}\bigcirc\exists r.A \sqsubseteq \exists r.\mathbf{E}\bigcirc A$ with r rigid,

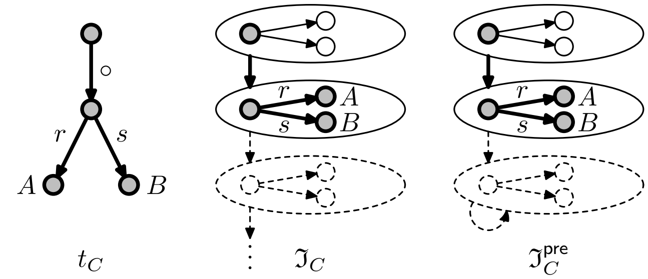

Figure 1: Description tree t_C, canonical model $\mathfrak{I}_C$, and finite representation $\mathfrak{I}_C^{\mathsf{pre}}$ for the concept $C = \mathbf{E}\bigcirc(\exists r.A \sqcap \exists s.B)$

there is no homomorphism from t_D to t_C. We overcome this problem by transforming t_C into a *canonical model* $\mathfrak{I}_C$ of C, i.e., (1) its distinguished root is an instance of C and (2) $\mathfrak{I}_C$ homomorphically embeds into every model of C. The construction of $\mathfrak{I}_C$ from t_C is straightforward: for every node with an incoming $\bigcirc$-edge (r-edge, r being a role) create a fresh world (domain element); for the root x_C create *both* a world and domain element. The temporal relation and the interpretation of r and concept names is read off E_C and L_C. To transform (W, R) into an infinite tree, we add an infinite path of fresh worlds to every world without R-successor. The canonical model for the above concept C is shown in Fig. 1, center; the infinite path of worlds is dashed.

From (1), (2), and the preservation properties of homomorphisms, we obtain the desired characterization of subsumption.

Lemma 3 *For all* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$*-concepts* C *and all* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$*-concepts* D*, we have* $\models C \sqsubseteq D$ *if and only if* $x_C \in D^{\mathfrak{I}_C,x_C}$.

Now $x_C \in D^{\mathfrak{I}_C,x_C}$ can be verified by model-checking D in world x_C and element x_C of $\mathfrak{I}_C^{\mathsf{pre}}$, which is the polynomial-sized modification of $\mathfrak{I}$ where the lastly added infinite path of worlds is replaced by a single loop, see Figure 1, right. Since $\mathfrak{I}_C$ is the unraveling of $\mathfrak{I}_C^{\mathsf{pre}}$ into the temporal dimension, both interpretations satisfy the same concepts in their roots. Theorem 2 for $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ therefore follows. The $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ part can be obtained by representing every $\mathbf{E}\Diamond$ in C by a $\bigcirc$-edge in t_C and modifying the notion of a homomorphism.

For $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$, we use expansion vectors introduced by Haase and Lutz [2008], applied to the temporal dimension. Let C be a $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$-concept with n occurrences of $\mathbf{E}\Diamond$. An *expansion vector for* C is an n-tuple $U = (u_1, \dots, u_n)$ of natural numbers $u_i \in \mathbb{N}$ (including 0). Intuitively, U fixes a specific number of temporal steps taken for each $\mathbf{E}\Diamond$ in C when constructing t_C and $\mathfrak{I}_C$. More precisely, we use $C[U]$ to denote the $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$-concept obtained from C by replacing the i-th occurrence of $\mathbf{E}\Diamond$ with $(\mathbf{E}\bigcirc)^{u_i}$, that is, a sequence of u_i $\mathbf{E}\bigcirc$-operators. For example, if $C = \mathbf{E}\Diamond\exists r.\mathbf{E}\Diamond(A \sqcap \mathbf{E}\bigcirc B)$ and $U = (2, 0)$, then $C[U] = \mathbf{E}\bigcirc\mathbf{E}\bigcirc\exists r.(A \sqcap \mathbf{E}\bigcirc B)$.

Let $\mathbb{U}_C^m$ be the set of all expansion vectors $(u_1, \dots, u_n)$ with $0 \leqslant u_i \leqslant m$, for all $i = 1, \dots, n$. We denote with $\mathsf{td}(D)$ the nesting depth of temporal operators in D. We use expansion vectors with entries bounded by $\mathsf{td}(D)$ to reduce $\not\models C \sqsubseteq D$ for $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$ to the case where C is from $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$.

Lemma 4 *For all* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$*-concepts* C, D*, we have* $\models C \sqsubseteq D$ *if and only if* $\models C[\overline{U}] \sqsubseteq D$ *for all* $\overline{U} \in \mathbb{U}_C^{\mathsf{td}(D)+1}$*.*

Together with Lemma 3, this yields the desired polynomial-time guess-and-check procedure for deciding $\models C \sqsubseteq D$.

5 $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ and $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ relative to Acyclic TBoxes

The results of Theorem 2 transfer to acyclic TBoxes with an exponential blowup due to unfolding [Nebel, 1990], that is:

Corollary 5 *Concept subsumption relative to acyclic* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$*-TBoxes with rigid roles is in* coNExpTime*.*

For the subfragments $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ and $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$, we can even show polynomial complexity as in the empty TBox case.

Theorem 6 *Concept subsumption relative to acyclic* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$*- and* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$*-TBoxes with rigid roles is in* PTime*.*

We first concentrate on the $\mathbf{E}\Diamond$ case and explain below how to deal with the $\mathbf{E}\bigcirc$ one. We focus w.l.o.g. on subsumption between concept *names* and assume that the input TBox is in normal form (NF), i.e., each axiom is of the shape $A \equiv A_1 \sqcap A_2$, $A \equiv \mathbf{E}\Diamond A_1$, or $A \equiv \exists r.A_1$, where $A_i \in \mathsf{N_C} \cup \{\top\}$ and $r \in \mathsf{N_R}$. As usual, a subsumption-equivalent TBox in NF can be computed in polynomial time [Baader, 2003]. We use CN and ROL to denote the sets of concept names and roles occurring in $\mathcal{T}$.

To prove a PTime upper bound, we devise a completion algorithm in the style of those known for $\mathcal{EL}$ and (two-dimensional) extensions, cf. [Baader *et al.*, 2005; Gutiérrez-Basulto *et al.*, 2011], which build an abstract representation of the 'minimal' model of the input TBox $\mathcal{T}$ (in the sense of Horn logic). The main difficulty is that different occurrences of the same concept name in the TBox cannot all be treated uniformly (as it is the case for, say, $\mathcal{EL}$), due to the two-dimensional semantics. Instead, we have to carefully choose witnesses for $\mathbf{E}\Diamond A$ and $\exists r.A$, respectively. Our algorithm constructs a graph $G = (W, E, Q, R)$ based on a set W, a binary relation E, a mapping Q that associates with each $A \in \mathsf{CN}$ and each $w \in W$ a subset $Q(A, w) \subseteq \mathsf{CN}$, and a mapping R that associates with each rigid role $r \in \mathsf{ROL}$ a relation $R(r) \subseteq \mathsf{CN} \times W \times \mathsf{CN} \times W$. For brevity, we write $(A, w) \xrightarrow{r} (B, w')$ instead of $(A, w, B, w') \in R(r)$ and denote with E^* the reflexive, transitive closure of E.

The algorithm for deciding subsumption initializes G by setting $R(r) = \emptyset$ for all $r \in \mathsf{ROL}$ and for all $A \in \mathsf{CN}$:

$$\begin{aligned} W &= \mathsf{CN} \times \mathsf{CN} \cup \{\mathbf{E}\Diamond A \mid A \in \mathsf{CN}\}; \\ E &= \{(\mathbf{E}\Diamond A, AA), (AB, A\top) \mid A, B \in \mathsf{CN}\}; \\ Q(A, w) &= \{\top, B\}, \text{ if } w = AB; \{\top\}, \text{ otherwise.} \end{aligned}$$

Intuitively, the unraveling of (W, E) is the temporal tree underlying the canonical model and the mappings Q and R contain condensed information on how to interpret concepts and roles, respectively. More specifically, the data stored in $Q(A, \cdot)$ describes the temporal evolution of an instance of A. For example, $Q(A, AA)$ collects all concept names B such that $\mathcal{T} \models A \sqsubseteq B$; likewise, $Q(A, \mathbf{E}\Diamond A)$ captures everything that follows from $\mathbf{E}\Diamond A$. Finally, $Q(A, AB)$ contains concept names that are implied by B given that B appears in the temporal evolution of an instance of A, i.e., $B' \in Q(A, AB)$ implies $\mathcal{T} \models A \sqcap \mathbf{E}\Diamond B \sqsubseteq \mathbf{E}\Diamond(B \sqcap B')$.

F1 If $B \in Q(A, AA')$ & $B \equiv \mathbf{E}\Diamond B' \in \mathcal{T}$, add (AA', AB') to E

F2 If $B \in Q(A, w)$ and $B \equiv \exists r.B' \in \mathcal{T}$,
set $(A, w) \xrightarrow{r} (B', B'B')$

F3 If $B \in Q(A, w)$ & $B \equiv A_1 \sqcap A_2 \in \mathcal{T}$, add A_1, A_2 to $Q(A, w)$

C1 If $(BB, w) \in E$ and $(A, w') \xrightarrow{r} (B, BB)$, add (w', w) to E

C2 If $(A, w) \xrightarrow{r} (B, BB)$, then
a) $(A, w') \xrightarrow{r} (B, \mathbf{E}\Diamond B)$ for all $w' \neq w$ with $(w', w) \in E^*$
b) $(A, w') \xrightarrow{r} (B, w')$ for all w' with $(w', w) \notin E^*$

B1 If $B \in Q(A, w)$, $(w', w) \in E^*$, and $A' \equiv \mathbf{E}\Diamond B \in \mathcal{T}$,
add A' to $Q(A, w')$

B2 If $A \in Q(B, w)$, $(A', w') \xrightarrow{r} (B, w)$, and $A'' \equiv \exists r.A \in \mathcal{T}$
add A'' to $Q(A', w')$

B3 If $A_1, A_2 \in Q(B, w)$ & $A \equiv A_1 \sqcap A_2 \in \mathcal{T}$
add A to $Q(B, w)$

Figure 2: Completion rules

After initialization, the algorithm extends G by applying the completion rules depicted in Figure 2 in three phases. In the first phase – also called Forward-phase, since definitions $A \equiv C \in \mathcal{T}$ are read as $A \sqsubseteq C$ – rules **F1**-**F3** are exhaustively applied in order to generate a fusion-like representation by adding witness-worlds and witness-existentials as demanded. Most notably, rule **F2** introduces a pointer to the structure representing the temporal evolution of an instance of B'.

Subsequently, G is extended to conform with the constant domain assumption and reflect rigidity of roles by exhaustively applying rules **C1** and **C2**. For example, one can read **C2** as '*if two points are connected via* r *in some world, then they should be connected in all worlds*.' Note that $Q(B, \mathbf{E}\Diamond B)$ is used as a representative for the entire "past" of B in part **a)**.

In the final phase, Backward-completion rules **B1**-**B3** are exhaustively applied in order to respect the 'backwards'-direction of definitions, i.e., definitions $A \equiv C \in \mathcal{T}$ are read as $A \sqsupseteq C$. This separation into a Forward and Backward phase is sanctioned by acyclicity of the TBox. In fact, one run through each phase is enough; note that no new tuples are added to E or R in the Backward-phase.

The following lemma shows correctness of our algorithm.

Lemma 7 *Let* $\mathcal{T}$ *be an acyclic* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$*-TBox in normal form. Then for all* $A, B \in \mathsf{CN}$*, we have* $\mathcal{T} \models A \sqsubseteq B$ *iff, after exhaustive rule application,* $B \in Q(A, AA)$*.*

For proving "$\Leftarrow$", we show that (a certain unraveling of) G "embeds" into every model of A and $\mathcal{T}$. For this purpose, we need to adapt the notion of a homomorphism to temporal interpretations and rigid roles. For the reverse direction, we construct from G a model $\mathfrak{I}$ of $\mathcal{T}$ such that $d \in A^{\mathfrak{I},w} \setminus B^{\mathfrak{I},w}$ for some d, w. It is not hard to see that the algorithm runs in polynomial time: The size of the data structures W, E, and R is clearly polynomial and the mapping $Q(\cdot,\cdot)$ is extended in every rule application, so the algorithm stops after polynomially many steps.

Finally, we sketch two modifications of the algorithm such that it works for $\mathbf{E}\bigcirc$ instead of $\mathbf{E}\Diamond$. First, we have to use a non-transitive version of **B1**. Second, and a bit more subtly, we have to replace $\mathbf{E}\Diamond A \in W$ with $\mathbf{E}\bigcirc^k A$, $1 \leq k \leq |\mathcal{T}|$ to capture what is implied by $\mathbf{E}\bigcirc^k A$; more precisely, $B' \in Q(A, \mathbf{E}\bigcirc^k A)$ implies $\mathcal{T} \models \mathbf{E}\bigcirc^k A \sqsubseteq B'$, where $\mathbf{E}\bigcirc^k$ denotes $\mathbf{E}\bigcirc \cdots \mathbf{E}\bigcirc$ k times.

We next show that there is a jump in the complexity if increasing domains are considered instead of constant ones. Intuitively, this can be explained by the fact that increasing domains allow rigid roles to mimic the behaviour of the $\mathbf{A}\Box$-operator. In the next section, we show that the addition of $\mathbf{A}\Box$ to $\{\mathbf{E}\Diamond\}$ indeed leads to PSPACE hardness.

Theorem 8 *Concept subsumption relative to acyclic* $\mathrm{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$*- and* $\mathrm{CTL}^{\mathbf{E}\Diamond}_{\mathcal{EL}}$*-TBoxes with rigid roles and increasing domains is* PSPACE*-hard.*

6 $\mathrm{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\Box}_{\mathcal{EL}}$ relative to Acyclic TBoxes

We now add $\mathbf{A}\Box$ and observe that this leads to an increase in complexity to polynomial space over acyclic TBoxes.

Theorem 9 *Concept subsumption relative to acyclic* $\mathrm{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\Box}_{\mathcal{EL}}$*-TBoxes with rigid roles is* PSPACE*-complete.*

The lower bound is obtained via a reduction from QBF validity. For the upper bound, we again consider w.l.o.g. subsumption between concept *names* and assume that the acyclic TBox is in normal form, i.e., axioms are of the shape $A \equiv A_1 \sqcap A_2$, $A \equiv \mathbf{E}\Diamond A_1$, $A \equiv \mathbf{A}\Box A_1$, or $A \equiv \exists r.A_1$, where $A_i \in \mathsf{N_C} \cup \{\top\}$ and $r \in \mathsf{N_R}$. We also restrict ourselves again to only rigid roles. CN and ROL are used as before.

In contrast to the previous section, we cannot maintain the entire minimal model in memory since the added operator $\mathbf{A}\Box$ can be used to enforce models of exponential size. Instead, we will compute all concepts implied by the input concept A (the left-hand side of the subsumption to be checked) by iteratively visiting relevant parts of the minimal model. Our main tool for doing so are *traces*.

Definition 1 *A* trace *is a tuple* (σ, E, R) *where* σ *is a sequence* $(d_0, w_0) \cdots (d_n, w_n)$ *such that for all* $0 \leq i < n$ *one of the following is true:*

- $d_i = d_{i+1}$ *and* $(w_i, w_{i+1}) \in E$*;*
- $w_i = w_{i+1}$ *and* $(d_i, d_{i+1}) \in R(r)$ *for some* $r \in \mathsf{ROL}$.

Intuitively, traces represent paths through temporal interpretations, which in each step follow either the temporal relation (first item of Definition 1) or a DL relation r (second item of Definition 1); so, in a pair (d, w), d can be thought of as a domain element and w as a world.

Our algorithm, whose basic structure is depicted in Algorithm 1, enumerates on input A and $\mathcal{T}$, in a systematic tableau-like way, all traces that *must* appear in every model of A and $\mathcal{T}$. It is important to note that in the context of Algorithm 1 a trace is used as the basis for inducing a richer structure that conforms with the constant domain assumption and captures rigidity; see Example 1 below. The algorithm also maintains an additional mapping Q that labels each point (d, w) of the trace (and all the induced points) with a set $Q(d, w) \subseteq \mathsf{CN}$. The set $Q(d, w)$ captures all concept names that are satisfied in the minimal model at points represented by (d, w).

Algorithm 1: Subsumption in $\mathrm{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\Box}_{\mathcal{EL}}$

Input: Acyclic TBox $\mathcal{T}$, concept names A, B
Output: `true` if $\mathcal{T} \models A \sqsubseteq B$, `false` otherwise

1 $\sigma := (d_0, w_0)$; $Q(d_0, w_0) := \{A, \top\}$;
2 $E := \emptyset$; $R(r) := \emptyset$ for all $r \in \mathsf{ROL}$;
3 `expand`(σ, E, R);
4 **return** `true` if $B \in Q(d_0, w_0)$, `false` otherwise;
5 **procedure** `expand` (σ, E, R) **:**
6   `complete` (σ, E, R, Q);
7   **if** (σ, Q) *is periodic at* (i, j) **then**
8   add (w_{j-1}, w_i) to E;
9   `truncate`;
10   `complete` (σ, E, R, Q);
11   **return**;
12     $(d, w) :=$ last element of σ;
13   **foreach** $A \in Q(d, w)$ *with* $A \equiv \exists r.B \in \mathcal{T}$ **do**
14         $Q(d', w) = \{B, \top\}$ for a fresh d';
15   add (d, d') to $R(r)$;
16   `expand` $(\sigma \cdot (d', w), E, R)$;
17   **foreach** $A \in Q(d, w)$ *with* $A \equiv \mathbf{E}\Diamond B \in \mathcal{T}$ **do**
18         $Q(d, w') = \{B, \top\}$ for a fresh w';
19   add (w, w') to E;
20   `expand` $(\sigma \cdot (d, w'), E, R)$;

The basics of Algorithm 1 are the following. In Lines 1 and 2, it creates a trace consisting of a single point representing A and initializes the necessary data structures. In Line 3, the systematic expansion is set off. When that is finished, the algorithm just returns whether or not B (the right-hand of the subsumption) has been added during the expansion. As for the `expand` procedure:

- in Line 6 and 10, the algorithm updates the mapping Q;
- Line 7 contains some termination condition; and finally,
- the loops in Lines 13 and 17 enumerate all $\exists r.B$ and $\mathbf{E}\Diamond B$ that appear in the set $Q(d, w)$ of the last element of the trace and expand the trace to witness these concepts.

This basic description of the algorithm leaves open several points: *(i)* the precise behavior of the subroutine `complete`, *(ii)* when a trace is *periodic*, and *(iii)* what happens inside the `truncate` command in Line 9. Let us start with describing the subroutine `complete`. It uses additional mappings $Q_{\mathsf{cert}}(d) \subseteq \mathsf{CN}$ and $Q_{\mathbf{A}\Box}(d, w) \subseteq \mathsf{CN}$, which intuitively contain all the concept names that d satisfies *certainly*, i.e., in all worlds, and starting from world w, respectively. It proceeds in two steps:

1. Initialize undefined $Q(d, w)$ and $Q_{\mathsf{cert}}(d)$ with $\{\top\}$, and undefined $Q_{\mathbf{A}\Box}(d, w)$ with $Q_{\mathsf{cert}}(d)$; and
2. apply rules **R1**-**R12** in Figure 3 to $Q(\cdot)$, $Q_{\mathsf{cert}}(\cdot)$ and $Q_{\mathbf{A}\Box}(\cdot)$.

The number of rules is indeed scarily high; however, they can be divided into four digestible groups: **R1** and **R2** are used

R1 If $A \equiv A_1 \sqcap A_2 \in \mathcal{T}$ and $A \in Q_*(\cdot)$, add A_1, A_2 to $Q_*(\cdot)$
R2 If $A \equiv A_1 \sqcap A_2 \in \mathcal{T}$ and $A_1, A_2 \in Q_*(\cdot)$, add A to $Q_*(\cdot)$

R3 If $(d, d') \in R(r)$, $B \in Q(d', w)$, $A \equiv \exists r.B \in \mathcal{T}$, add A to $Q(d, w)$
R4 If $B \in Q(d, w)$, $(w', w) \in E^*$, $A \equiv \mathbf{E}\Diamond B \in \mathcal{T}$, add A to $Q(d, w')$
R5 If $B \in Q(d, w)$, $(w, w') \in E^*$, $B \equiv \mathbf{A}\Box A \in \mathcal{T}$, add B, A to $Q(d, w')$

R6 If $(d, d') \in R(r)$, $B \in Q_{\mathsf{cert}}(d')$, $A \equiv \exists r.B \in \mathcal{T}$, add A to $Q_{\mathsf{cert}}(d)$
R7 If $B \in Q_{\mathsf{cert}}(d)$, $A \equiv \mathbf{A}\Box B \in \mathcal{T}$, add A to $Q_{\mathsf{cert}}(d)$
R8 If $B \in Q_{\mathsf{cert}}(d)$, add B to $Q(d, w)$ for all w

R9 If $B \in Q_{\mathbf{A}\Box}(d, w)$, $A \equiv \mathbf{A}\Box B \in \mathcal{T}$, add A to $Q(d, w)$
R10 If $A \in Q(d, w)$, $A \equiv \mathbf{A}\Box B \in \mathcal{T}$, add A, B to $Q_{\mathbf{A}\Box}(d, w)$
R11 If $(d, d') \in R(r)$, $B \in Q_{\mathbf{A}\Box}(d', w)$, $A \equiv \exists r.B \in \mathcal{T}$, add A to $Q_{\mathbf{A}\Box}(d, w)$
R12 If $A \in Q_{\mathbf{A}\Box}(d, w)$, $A \equiv \mathbf{E}\Diamond B \in \mathcal{T}$, w' added due to $A \in Q(d, w)$ in Line 18, $B' \in Q(d, w')$, $A' \equiv \mathbf{E}\Diamond B' \in \mathcal{T}$, add A' to $Q_{\mathbf{A}\Box}(d, w)$

Figure 3: Saturation rules, where in **R1** and **R2** the set $Q_*(\cdot)$ ranges over all $Q(d, w)$, $Q_{\mathsf{cert}}(d)$, and $Q_{\mathbf{A}\Box}(d, w)$.

to ensure that all sets Q_* are closed under conjunction; **R3**-**R5** are used to complete $Q(\cdot)$. Note that **R1**-**R4** are already known from the algorithm of the previous section. Furthermore, **R6**-**R8** are used to deal with $Q_{\mathsf{cert}}(\cdot)$; and **R9**-**R12** to update $Q_{\mathbf{A}\Box}(\cdot)$. As an example of the interplay between the different mappings take **R9**: If B is certain for d starting in world w and $A \equiv \mathbf{A}\Box B$, then we also know that d satisfies A in w; and **R11** for the interplay between temporal operators and rigid roles: indeed, for r rigid, $\models \exists r.\mathbf{A}\Box B \sqsubseteq \mathbf{A}\Box\exists r.B$.

Example 1 *Let $\mathcal{T} = \{A \equiv \mathbf{E}\Diamond A_1,\ A_1 \equiv \exists r.B,\ B \equiv \mathbf{E}\Diamond A_1\}$ be the input TBox; and $\mathcal{T} \models A \sqsubseteq A_1$ is to be checked. Figure 4 (left) shows the trace initiated at (d_0, w_0) with $Q(d_0, w_0) = \{\top, A\}$, and further expanded in Lines 13 and 17. The trace, as mentioned above, induces a richer structure, reflecting rigid roles and the constant domain assumption; see Fig. 4 (center). This richer structure is then `completed` to properly enrich the types $Q(d, w)$ of each element. In particular, during completion, further concept names are added to the corresponding types (Fig. 4, right). One can now easily see that $\mathcal{T} \models A \sqsubseteq A_1$ indeed holds. Furthermore, note that*

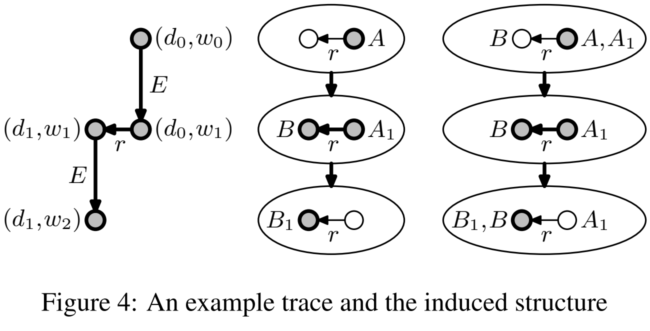

Figure 4: An example trace and the induced structure

$\mathcal{T} \not\models A \sqsubseteq A_1$, if r is local or increasing domains are assumed. This is the case since, in both cases, the r-connection is not necessarily present in the 'root world'.

For the termination condition in Line 7, we take the following definition of periodicity.

Definition 2 *A trace (σ, E, R) together with a mapping Q is called* periodic at (i, j) *if $\sigma = (d_0, w_0) \cdots (d_n, w_n)$, $i < j$, $d_i = d_j = d_n$, and $Q(d_i, w_i) = Q(d_j, w_j)$.*

This means that during the evolution of element $d = d_i = d_j$, we find two different worlds w_i, w_j such that d has the same type in w_i and w_j. We can stop expanding worlds appearing after w_j since their behavior is already captured by the successors of w_i. If a trace periodic at (i, j) is found, we add an edge (w_{j-1}, w_i) to E reflecting the periodic behavior, see Line 8. Then, in `truncate`, the trace is shortened to $(d_0, w_0) \cdots (d_{j-1}, w_{j-1})$ and the relations E and $R(r)$, $r \in \mathsf{ROL}$, and the mappings $Q, Q_{\mathsf{cert}}, Q_{\mathbf{A}\Box}$ are restricted to those d and w that appear in the shortened trace.

Lemma 10 *On every input $\mathcal{T}$, A, B, Algorithm 1 terminates and returns `true` iff $\mathcal{T} \models A \sqsubseteq B$.*

For termination, consider a trace with suffix $(d, w_1) \cdots (d, w_n)$ and let additionally $A_1, \ldots, A_n$ be the concept names such that $\mathbf{E}\Diamond A_i$ lead to w_i, see Line 17 of Algorithm 1. It is not difficult to show that if $A_i = A_j$ for $i < j$, then $Q(d, w_i) \subseteq Q(d, w_j)$ after application of `complete`. Since $Q(d, w) \subseteq \mathsf{CN}$, there are no infinite (strictly) increasing sequences. Hence, the expansion in Lines 17ff. will not indefinitely be applied. Also, the expansion in Lines 13ff. stops due to acyclicity of the TBox. Together, this guarantees termination.

Correctness is shown similar to Lemma 7. For "⇒", we show that every trace together with the labeling so far computed in Q can be embedded into every model of A and $\mathcal{T}$. For "⇐", we present a model of $\mathcal{T}$ witnessing $\mathcal{T} \not\models A \sqsubseteq B$.

To finish the proof of Theorem 9, it remains to note that the termination argument indeed yields a polynomial bound on the length of the traces encountered during the run of Algorithm 1.

7 Local Roles and Rigid Concepts

One can easily extend the above algorithms so as to deal with local roles. In fact, e.g., in Section 5 only **B4** in Figure 5 needs to be added to the BACKWARD-rules in Figure 3. Note that **F2** is only applied to rigid roles and **C2** is therefore not applied to local ones. Clearly, the algorithm in Section 6 can be extended with a similar rule.

Recall that rigid concepts are concepts whose interpretation does not change over time. In the first example from Section 1, the concept $\mathsf{Disorder}$ should be rigid because we consider medical knowledge as static. In contrast, $\mathsf{PatientWithDisorder}$ should be local because a disease history has a begin and end.

In the presence of general TBoxes, rigid concepts can be simulated by rigid roles: replace each rigid concept name A with $\exists r_A.\top$, where r_A is a fresh rigid role. Unfortunately, this simulation does not work in the context of acyclic TBoxes since the result of replacing A with $\exists r_A.\top$ in a CD $A \equiv D$ is not a CD anymore. Nevertheless, our algorithms can be extended, without increasing the complexity, to consider rigid

B4 If $A \in Q(B, w)$, $A \equiv \exists r.A'$, $B' \in Q(A', A'A')$ and $B'' \equiv \exists r.B' \in \mathcal{T}$, add B'' to $Q(B, w)$
RC If $B \in Q(A, w)$, $B \in \mathsf{CN}_{\mathsf{rig}}$, add B to $Q(A, w')$, $\forall w' \in W$
R13 If $B \in Q(d, w)$ or $B \in Q_{\mathbf{A}\Box}(d, w)$ & $B \in \mathsf{CN}_{\mathsf{rig}}$, then add B to $Q_{\mathsf{cert}}(d)$

Figure 5: Rules for Local Roles and Rigid Concepts

concepts: e.g., the algorithm in Section 5 can be extended by adding **RC** above to the FORWARD *and* BACKWARD rules – $\mathsf{CN}_{\mathsf{rig}}$ denotes the set of rigid concepts occurring in the input TBox. Note that the intermediate phase remains the same, i.e., rules **C1** and **C2** are neither extended nor modified.

Rigid concepts can analogously be included in Section 6 by adding **R13** to the rules in Figure 3. Recall that, intuitively, $Q_{\mathsf{cert}}(d)$ contains the concepts that hold for d in any world.

Finally, note that in the empty TBox case rigid roles can indeed simulate rigid concepts, as described above.

8 Conclusions and Future Work

In this paper we have initiated the investigation of TDLs based on $\mathcal{EL}$ allowing for rigid roles and restricted TBoxes. We indeed achieved our main goal: we identified fragments of the combination of CTL and $\mathcal{EL}$ that have elementary, some even polynomial, complexity.

One important conclusion is that the use of acyclic TBoxes, instead of general ones, allows to design TDLs based on $\mathcal{EL}$ with dramatically better complexity than the $\mathcal{ALC}$ variant; e.g., for the fragment allowing only $\mathbf{E}\bigcirc$ the complexity drops from nonelementary to PTIME. As an important byproduct, the studied fragments of $\text{CTL}_{\mathcal{EL}}$ can be seen as *positive fragments* of product modal logics with elementary complexity, e.g., implication for the positive fragment of $\mathsf{K}\times\mathsf{K}$ is in PTIME.

As a next step, we plan to look at more expressive fragments of $\text{CTL}_{\mathcal{EL}}$ or at classical (cyclic) TBoxes, e.g., consider *non-convex* fragments, such as $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$, with (a)cyclic TBoxes. We plan to incorporate temporal roles, too. It is also worth exploring how restricting TBoxes can help tame other TDLs with bad computational behavior over general TBoxes, such as TDLs based on LTL or the μ-calculus. We believe that the LTL case is technically easier than ours since it does not have the extra '$\frac{1}{2}$-dimension' introduced by branching.

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Appendix

A Additional Preliminaries

- A *tree* is a directed graph $T = (W, E)$ where $W \subseteq (\mathbb{N} \setminus \{0\})^*$ is a prefix-closed non-empty set of *nodes* and $E = \{(w, wc) \mid wc \in W, w \in \mathbb{N}^*, c \in \mathbb{N}\}$ a set of *edges*; we generally assume that $wc \in W$ and $c' < c$ implies $wc' \in W$ and that E is a total relation. The node $\varepsilon \in W$ is the *root* of T. For brevity and since E can be reconstructed from W, we will usually identify T with W.
- If the constant domain assumption is made, we sometimes write $\mathfrak{I} = (\Delta, T, (\mathcal{I}_w)_{w\in W})$, instead of $\mathfrak{I} = (T, (\mathcal{I}_w)_{w\in W})$, to denote a temporal interpretation.

B Proofs for the CTL$_{\mathcal{ALC}}$

Theorem 1 *Concept satisfiability relative to acyclic and empty TBoxes for* CTL$_{\mathcal{ALC}}$ *is decidable and inherently nonelementary.*

Proof. The proof of the upper bound for the empty TBox case follows a two-step strategy similar to that for LTL$_{\mathcal{ALC}}$ [Gabbay *et al.*, 2003, Thm. 13.6]. Let C be the CTL$_{\mathcal{ALC}}$ concept whose satisfiability is to be decided. First, we define quasimodels, which are abstractions of temporal interpretations, and we show that satisfiability of C is characterized by the existence of a quasimodel for C. Second, we express the latter as a monadic second-order formula. We can thus infer decidability from the fact that the monadic second-order theory of countably branching trees is decidable [Hodkinson *et al.*, 2002]. Our proof requires a careful treatment of the definition of the quasimodel and of the reduction to monadic second-order logic to conform with the branching structure of time.

The case of acyclic TBoxes can be reduced to the empty-TBox case using standard unfolding [Nebel, 1990].

The nonelementary lower bound follows from the fact that satisfiability for the product modal logics $\mathsf{S4}\times\mathsf{K}$ and $\mathsf{K}\times\mathsf{K}$ is inherently nonelementary [Göller *et al.*, 2015]. Indeed, the fragment of CTL$_{\mathcal{ALC}}$ allowing $\mathbf{E}\Diamond$ ($\mathbf{E}\bigcirc$) as the only temporal operator is a notational variant of $\mathsf{S4}\times\mathsf{K}$ ($\mathsf{K}\times\mathsf{K}$) [Gutiérrez-Basulto *et al.*, 2014].

It is worth noting that a similar technique has been used to show decidability of the so-called *monodic* fragment of first-order branching temporal logic [Hodkinson *et al.*, 2002, Theorem 8], which is closely related to CTL$_{\mathcal{ALC}}$. However, the expressivity of this logic is orthogonal to our CTL$_{\mathcal{ALC}}$ since it does not allow rigid roles, but temporal operators can be applied to TBoxes.

We next proceed with the first step. Let us fix a CTL$_{\mathcal{ALC}}$ concept C. We use $\mathsf{cl}(C)$ to denote the set of concepts that occur in C, closed under subconcepts and single negation. We moreover use $\mathsf{rd}(C)$ to denote the *role depth* of C, that is, the maximal nesting depth of existential restrictions in C. The *depth* of a tree T is the length of the longest path of T. The *co-depth of* $w \in T$ *(*$\mathsf{cd}_T(w)$*)* is the distance from the root ε to w. The *depth of* $w \in T$ it is the depth of the subtree of T rooted at w. Let Σ be a finite alphabet. A Σ-*labeled tree* is a pair (T, τ) where T is a tree and $\tau : T \to \Sigma$ assigns a letter from Σ to each world. We sometimes identify (T, τ) with τ.

A *type for* C is a set $t \subseteq \mathsf{cl}(C)$ such that $D \sqcap E \in t$ iff $D \in t$ and $E \in t$, for all $D \sqcap E \in \mathsf{cl}(C)$, and $\neg D \in t$ iff $D \notin t$, for every $D \in \mathsf{cl}(C)$. We denote by $\mathsf{tp}(C)$ the set of all types for C. In the following, we restrict ourselves to a single role r, which is rigid. All arguments work in the presence of arbitrarily many roles, including local roles, but the technical notation required for writing them down becomes more complex.

We next introduce the structure representing a DL interpretation in a given world.

Definition 3 *A* quasistate for C *is finite* $\mathsf{tp}(C)$*-labeled tree* (T, τ) *such that*

1. *for all* $w \in T$ *and* $\exists r.D \in \mathsf{cl}(C)$, $\exists r.D \in \tau(w)$ *iff* $\exists w' \in T$ *such that* $w' \in \mathsf{children}(w)$ *and* $D \in \tau(w')$*;*
2. *for all* w, w_1, w_2*, if* $w_1, w_2 \in \mathsf{children}(w)$ *and* $w_1 \neq w_2$*, the subtrees generated by* w_1 *and* w_2 *are not isomorphic;*
3. (T, τ) *is of depth* $\leq \mathsf{rd}(C)$.

Note that there are at most $2^{\mathsf{cl}(C)}$ types, and therefore the number of pairwise non-isomorphic quasistates of depth 0 is at most the number of types. We define $n_k(C)$ inductively as follows:

$$n_0(C) = 2^{\mathsf{cl}(C)}, \qquad n_{k+1}(C) = 2^{\mathsf{cl}(C)} \times 2^{n_k(C)}$$

It is clear that $n_k(C)$ bounds the number of non-isomorphic quasistates for C of depth k.

We now introduce the structure used to reconstruct a CTL$_{\mathcal{ALC}}$-interpretation of C. Let T be a total tree. A *basic structure of depth* m for C is a pair $(T, \mathbf{q})$, where $\mathbf{q}$ is a function associating with each $w \in T$ a quasistate $\mathbf{q}(w) = (T_w, \tau_w)$ for C such that the depth of each T_w is m. Now we introduce run-functions which are used to recover the temporal relation between types at different quasistates.

Definition 4 *A* k-run through $(T, \mathbf{q})$ *is a function* ρ *such that for each* $w \in T$ *it assigns a node* $\rho(w) \in T_w$ *of co-depth* k*. Given a set of runs* $\mathfrak{R}$*, we denote by* $\mathfrak{R}_k$ *the set of all* k*-runs in* $\mathfrak{R}$*. We moreover say that a run is* proper *if the following hold:*

- *for every* $\mathbf{E}\bigcirc D \in \mathsf{cl}(C)$ *and every* $w \in T$*, we have* $\mathbf{E}\bigcirc D \in \tau_w(\rho(w))$ *iff* $D \in \tau_{\pi[1]}(\rho(\pi[1]))$ *for some* $\pi \in \mathsf{Paths(w)}$;
- *for every* $\mathbf{E}\Box D \in \mathsf{cl}(C)$ *and every* $w \in T$*, we have* $\mathbf{E}\Box D \in \tau_w(\rho(w))$ *iff* $\forall j \geq 0. D \in \tau_{\pi[j]}(\rho(\pi[j]))$ *for some* $\pi \in \mathsf{Paths(w)}$;
- *for every* $\mathbf{E}(D_1 \mathcal{U} D_2) \in \mathsf{cl}(C)$ *and every* $w \in T$*, we have* $\mathbf{E}(D_1 \mathcal{U} D_2) \in \tau_w(\rho(w))$ *iff* $\exists j \geq 0.(D_2 \in \tau_{\pi[j]}(\rho(\pi[j])) \wedge (\forall\, 0 \leq k < j. C \in \tau_{\pi[k]}(\rho(\pi[k]))))$ *for some* $\pi \in \mathsf{Paths}(w)$.

We now have the required ingredients to define an abstraction of a CTL$_{\mathcal{ALC}}$ model.

Definition 5 *A quadruple* $\mathfrak{M} = \langle T, \mathbf{q}, \mathfrak{R}, \lhd\rangle$ *is a* quasimodel for C *if* $(T, \mathbf{q})$ *is a basic structure for* C *of depth* $m \leq \mathsf{rd}(C)$*,* $\mathfrak{R}$ *is a set of proper runs through* $(T, \mathbf{q})$*, and* $\lhd$ *is a binary relation on* $\mathfrak{R}$ *such that*

1. *$C \in \tau_\varepsilon(\rho_0(\varepsilon))$, where $\rho_0 \in \mathfrak{R}_0$ and $\rho_0(\varepsilon) = \varepsilon$;*
2. *for all $\rho, \rho' \in \mathfrak{R}$, if $\rho \lhd \rho'$ then $\rho'(w) \in \mathsf{children}(\rho(w))$ for all $w \in T$;*
3. *for all $k < m, \rho \in \mathfrak{R}_k, w \in T$ and $x \in T_w$, if $x \in \mathsf{children}(\rho(w))$ then there is a $\rho' \in \mathfrak{R}_{k+1}$ such that $\rho'(w) = x$ and $\rho \lhd \rho'$.*

The following result can be proved as the analogous lemma for $\mathsf{K}_{\mathcal{ALC}}$ [Wolter and Zakharyaschev, 1999, Theorem 14].

Lemma 11 *A CTL$_{\mathcal{ALC}}$ concept C is satisfiable iff there is a quasimodel for C.*

Proof. "$\Leftarrow$:" Let $\mathfrak{M} = \langle T, \mathbf{q}, \mathfrak{R}, \lhd\rangle$ be a quasimodel for C. We construct a model $\mathfrak{I} = (T, \Delta, \{\mathcal{I}_w\}_{w\in W})$ of C, where T is defined as in $\mathfrak{M}$ and $\Delta = \mathfrak{R}$. It remains to define the interpretation of concept names and role names:

$$A^{\mathfrak{I},w} = \{\rho \in \Delta \mid A \in \rho(w)\};$$
$$r^{\mathfrak{I},w} = \{(\rho, \rho') \in \Delta\times\Delta \mid \rho' \lhd \rho\}.$$

One can check by structural induction that the following claim holds.

Claim. For every $D \in \mathsf{cl}(C)$, $\rho \in \Delta$, $w \in W$, we have that

$$C^{\mathfrak{I},w} \text{ iff } C \in \rho(w).$$

Therefore, by the previous claim and Condition 1 of Definition 5, $\rho_0 \in C^{\mathfrak{I},\varepsilon}$.

"$\Rightarrow$:" Let $\mathfrak{I} = (\Delta, T, (\mathcal{I}_w)_{w\in W})$ be a model of C. We construct a quasimodel $\mathfrak{M}$ for C. We begin by defining the set of types for C induced by $\mathfrak{I}$: For every $d \in \Delta$ and $w \in W$, we set:

$$\mathsf{tp}(d, w) = \{C \in \mathsf{cl}(C) \mid d \in C^{\mathfrak{I},w}\}.$$

It is well-known that every satisfiable CTL$_{\mathcal{ALC}}$ concept C is satisfiable in a model where for every $w \in W$, $\mathcal{I}_w$ is an intransitive tree of depth $\leq \mathsf{rd}(C) = m$. From now on we assume this is the case for $\mathfrak{I}$.

Now, we have to define a quasistate (T_w, τ_w) for all $w \in W$. To this aim, one could choose Δ to be the set of nodes associated to T_w, and the 'r-connections' to be the order on the nodes. However, note that Δ might be infinite and, by Condition 3 of Definition 3, a quasistate is finite. Hence we need to make Δ finite without violating Conditions 1 and 2 of Definition 3. Fix a $w \in W$ and define a binary relation $\sim_w$ on Δ as follows:

- if $d, e \in \Delta$ are of depth 0 (that is, d, e are at the leaves-level), then
$$d \sim_w e \text{ iff } \mathsf{tp}(d, w) = \mathsf{tp}(e, w);$$
- for $d, e \in \Delta$ at depth $0 < k \leq \mathsf{rd}(C)$ $d \sim_w e$ iff
  - $\mathsf{tp}(d, w) = \mathsf{tp}(e, w)$;
  - $\forall d' \in \Delta\ \big((d, d') \in r^{\mathfrak{I},w} \rightarrow \exists e' \in \Delta\ ((e, e') \in r^{\mathfrak{I},w} \wedge e \sim_w e')\big)$;
  - $\forall e' \in \Delta\ \big((e, e') \in r^{\mathfrak{I},w} \rightarrow \exists d' \in \Delta\ ((d, d') \in r^{\mathfrak{I},w} \wedge d \sim_w d')\big)$.

We denote by $[d]_w$ the equivalence class of d in w. We now define for all $w \in W$.

$$V_w := \{[d]_w \mid d \in \Delta\};$$
$$E_w := \{([d]_w, [e]_w) \mid \exists e' \in [e]_w \wedge (d, e') \in r^{\mathfrak{I},w}\};$$
$$\tau_w([d]_w) := \mathsf{tp}(d, w).$$

Note that, however, $((V_w, E_w), \tau_w)$ might not be a tree. Nevertheless, we can unravel $((V_w, E_w), \tau_w)$ to obtain a tree, as follows. A *role path in* w is a sequence $[d_0]_w \cdots [d_k]_w$ such that for each $0 \leq i < k$, $([d_i]_w, [d_{i+1}]_w) \in E_w$. We denote by V^*_w the set of all role paths in w and define $\mathsf{tail}([d_0]_w \cdots [d_k]_w) = [d_k]_w$ and $E^*_w = \{(\sigma, \sigma') \in V^*_w\times V^*_w \mid (\mathsf{tail}(\sigma), \mathsf{tail}(\sigma')) \in E_w\}$. Finally, let $\tau^*_w(\sigma) = \tau_w(\mathsf{tail}(\sigma))$. Set now

$$T_w = ((V^*_w, E^*_w), \tau^*_w).$$

It is not hard to see that for any $w \in W$, $T_w = ((V^*_w, E^*_w), \tau^*_w)$ satisfies Conditions 1 and 2 from Definition 3. Now, take $\mathbf{q}(w) = ((V^*_w, E^*_w), \tau^*_w)$ for each $w \in W$ and get a basic structure $(T, \mathbf{q})$. It remains to define the runs through $(T, \mathbf{q})$ as follows. For each $k \leq m$ and each sequence $d_0, \ldots, d_k$ such that $(d_i, d_{i+1}) \in r^{\mathfrak{I},w}$ take the map:

$$\rho : w \rightarrow ([d_0]_w, \ldots, [d_k]_w)$$

ρ is thus a k-run through $(T, \mathbf{q})$. Finally, let $\mathfrak{R}$ be the set of such runs.

With this definitions at hand, it is routine to see that $\mathfrak{R}$ is a set of proper runs, and moreover that they satisfy Conditions 2 and 3 of Definition 5. Therefore, $\langle T, \mathbf{q}, \mathfrak{R}, \lhd\rangle$ is a quasimodel. ❑

We now proceed with the *second step* of the proof. We translate into monadic second order the statement 'there is a quasimodel for C'. We fix an arbitrary $m \leq \mathsf{rd}(C)$, and denote by $\mathsf{qs}_m(C)$ the set of all quasistates for C of depth m.

We introduce a unary predicate variable $P_\mathbf{q}$ for each $\mathbf{q} \in \mathsf{qs}_m(C)$ and a unary predicate variable R^k_D for each $D \in \mathsf{cl}(C)$ and $k \leq m$. Now, given a type t for C and a $k \leq m$, let

$$\chi_t(R^k(x)) = \bigwedge_{D\in t} R^k_D(x) \wedge \bigwedge_{D\in\mathsf{cl}(C)\setminus t} \neg R^k_D(x)$$

Intuitively, it says that the type t at point x of co-depth k is defined using

$$R^k(x) = \langle R^k_D(x) \mid D \in \mathsf{cl}(D)\rangle.$$

We next proceed to capture that R^k defines a proper k-run through a 'path' of quasistates using $P = \langle P_\mathbf{q} \mid \mathbf{q} \in \mathsf{qs}_m(C)\rangle$ as follows. For each $k \leq m$, $\mathsf{run}^0(P, R^k)$ denotes the conjunction of the following formulas:

$$\forall x\ \textstyle\bigwedge_{\mathbf{q}\in\mathsf{qs}_m(C)} \big(P_\mathbf{q}(x) \rightarrow \bigvee_{\substack{w\in T_\mathbf{q} \\ \mathsf{cd}_\mathbf{q}(w)=k}} \chi_{\tau_\mathbf{q}(w)}(R^k(x))\big)$$
$$\forall x \textstyle\bigwedge_{\mathbf{E}\mathcal{C}\in\mathsf{cl}(C)} \big(R^k_{\mathbf{E}\mathcal{C}}(x) \leftrightarrow \exists\pi(\beta(\pi, x) \wedge \gamma(\mathcal{C}, \pi, x))\big)$$

where $\beta(\pi, x)$ denotes the MSO formula saying that π –π a set variable– is a path containing x, and $\gamma(\mathcal{C}, \pi, x)$ is defined as follows:

$\gamma(\bigcirc D, \pi, x) = R^k_D(\pi[1])$
$\gamma(\Box D, \pi, x) = \forall j \geq 0. R^k_D(\pi[j])$
$\gamma(D_1 \mathcal{U} D_2, \pi, x) =$
$$\exists j \geq 0. \big(R^k_{D_2}(\pi[j]) \wedge \forall 0 \leq l \leq j. R^k_{D_1}(\pi[l])\big)$$

In the previous definition, we slightly abuse notation in the sense that we see π as in $\mathsf{Paths}(x)$.

We next ensure that Condition 3 of Definition 5 is satisfied, by defining, by 'backwards' induction on k (following [Gabbay *et al.*, 2003, Theorem 13.6]), the formula $\mathsf{run}(P, R^k)$.

- If $k = m$, then $\mathsf{run}(P, R^m) = \mathsf{run}^0(P, R^m)$.
- For the inductive step, suppose that for $k \leq m$, $\mathsf{run}(P, R^k)$ is defined. We then define $\mathsf{run}(P, R^{k-1})$ as follows:

$$\mathsf{run}^0(P, R^{k-1}) \wedge \forall x \bigwedge_{\mathbf{q} \in \mathsf{qs}_m(C)} \bigwedge_{\substack{w \in T_\mathbf{q} \\ \mathsf{cd}_\mathbf{q}(w) = k-1}} \Big[P_\mathbf{q}(x) \wedge \chi_{\tau_\mathbf{q}(w)}(R^{k-1}(x)) \rightarrow \bigwedge_{\substack{w' \in T_\mathbf{q} \\ w' \in \mathsf{children}(w)}} \exists_{D \in \mathsf{cl}(C)} R^k_D \Big(\mathsf{run}(P, R^k) \wedge \chi_{\tau_\mathbf{q}(w')}(R^k(x)) \wedge \forall z \bigwedge_{\mathbf{q}' \in \mathsf{qs}_m(C)} \bigwedge_{\substack{v \in T_{\mathbf{q}'} \\ \mathsf{cd}_{\mathbf{q}'}(v) = k-1}} \big(P_{\mathbf{q}'}(z) \wedge \chi_{\tau_{\mathbf{q}'}(v)}(R^{k-1}(z)) \rightarrow \bigvee_{\substack{v' \in T_{\mathbf{q}'} \\ v' \in \mathsf{children}(v)}} (R^k(z))\big)\Big)\Big]$$

To finish the translation, we define the MSO sentence qm^m_C as follows:

$$\mathsf{qm}^m_C = \exists_{\mathbf{q} \in \mathsf{qs}_m(C)} P_\mathbf{q} \Big[\forall x \bigvee_{\mathbf{q} \in \mathsf{qs}_m(C)} \big(P_\mathbf{q}(x) \wedge \bigwedge_{\substack{\mathbf{q}' \in \mathsf{qs}_m(C) \\ \mathbf{q} \neq \mathbf{q}'}} \neg P_{\mathbf{q}'}(x)\big) \wedge \bigvee_{\substack{\mathbf{q}', \varepsilon \in T_{\mathbf{q}'} \\ \mathsf{cd}_{\mathbf{q}'(\varepsilon)=0} \\ C \in \tau_{\mathbf{q}'}(\varepsilon)}} \exists x \Big(P_{\mathbf{q}'}(x) \wedge \exists_{D \in \mathsf{cl}(C)} R^0_D \big(\mathsf{run}(P, R^0) \wedge \chi_{\tau_{\mathbf{q}'}(\varepsilon)}(R^0(x))\big)\Big)\Big]$$

Evaluated in a (time) tree T, as discussed in [Gabbay *et al.*, 2003, Theorem 13.6], the first line of qm^m_C states that the sets $P_\mathbf{q} \subseteq W$ make a partition on W. We can then obtain a quasimodel $\mathfrak{M} = \langle T, \mathbf{q}, \mathfrak{R}, \lhd \rangle$ for C by defining a mapping $\mathbf{q} : W \rightarrow \mathsf{qs}_m(C)$ as

$$\mathbf{q}(w) = \mathbf{q} \text{ iff } w \in P_\mathbf{q}$$

and a relation $\lhd$ on the runs as follows: $r \lhd r'$ iff r is defined by R^{k-1} and r' is defined by R^k for some $k \leq m$.

The second line of qm^m_C states conditions of Definition 5 are satisfied by the definitions of $\lhd$ and $\mathsf{run}(P, R_k)$. With this at hand, it is not hard to see that the following holds.

Lemma 12 $T \models \mathsf{qm}^m_C$ *iff there exists a quasimodel for* C.

Finally, to deduce decidability we use the fact that the monadic-second order theory of countably branching trees is decidable [Hodkinson *et al.*, 2002]. ❑

C Proofs and additional details for Section 4

C.1 Additional notation

The *depth* of a $\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$-concept C, denoted $\mathsf{d}(C)$, is the combined nesting depth of $\exists r$ and $\mathbf{E}\bigcirc$ operators in C:

$$\mathsf{d}(A) = 0, \quad A \in \mathsf{N_C}$$
$$\mathsf{d}(C_1 \sqcap C_2) = \max\{\mathsf{d}(C_1), \mathsf{d}(C_2)\}$$
$$\mathsf{d}(\exists r.D) = \mathsf{d}(\mathbf{E}\bigcirc D) = \mathsf{d}(\mathbf{E}\Diamond D) = \mathsf{d}(D) + 1$$

Thus, a depth-0 concept has the form $C = A_1 \sqcap \cdots \sqcap A_n$, where $A_i \in \mathsf{N_C}$, and a concept of depth $d \geqslant 1$ has the form

$$C = \bigsqcap_{i=1}^{n} A_i \sqcap \bigsqcap_{i=1}^{m} \exists r_i.D_i \sqcap \bigsqcap_{i=m+1}^{\ell} \mathbf{E}\bigcirc D_i \sqcap \bigsqcap_{i=\ell+1}^{k} \mathbf{E}\Diamond D_i, \quad (1)$$

where all D_i have depth at most $d - 1$.

C.2 Description graphs

We use description graphs to represent concepts and interpretations alike.

A *description graph* is a labelled directed graph $G = (V, L, E)$, where V is a non-empty set of *nodes*, L is a map from V to $2^{\mathsf{N_C}}$, and E is a set of *edges*, i.e., triples $(v, \bullet, v')$ where $\bullet \in \mathsf{N_R} \cup \{\bigcirc\}$. We write

$v \xrightarrow{\bullet}_G v'$ for $(v, \bullet, v') \in E$,

$\xrightarrow{\mathsf{N_R}}_G$ for the union of all $\xrightarrow{r}_G$, $r \in \mathsf{N_R}$, and

$\xrightarrow{\bullet}^*_G$ for the reflexive and transitive closure of $\xrightarrow{\bullet}_G$.

Given a temporal interpretation $\mathfrak{I} = (\Delta, T, (\mathcal{I}_w)_{w \in W})$ with $T = (W, R)$, the *description graph* $G_\mathfrak{I} = (V_\mathfrak{I}, L_\mathfrak{I}, E_\mathfrak{I})$ *associated with* $\mathfrak{I}$ is defined as follows.

$$V_\mathfrak{I} = \{\langle w, x \rangle \mid w \in W, x \in \Delta\}$$
$$L_\mathfrak{I}(\langle w, x \rangle) = \{A \mid x \in A^{\mathfrak{I}, w}\}$$
$$\langle w, x \rangle \xrightarrow{r}_G \langle w', x' \rangle \Leftrightarrow w = w' \text{ and } (x, x') \in r^{\mathfrak{I}, w}$$
$$\langle w, x \rangle \xrightarrow{\bigcirc}_G \langle w', x' \rangle \Leftrightarrow (w, w') \in R \text{ and } x = x'$$

Given a $\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$-concept C, the *description tree* $t_C = (V_C, L_C, E_C)$ *associated with* C and its *root* $x_C \in V_C$ are straightforwardly obtained from the tree representation of C and its root node. They are defined recursively over the depth of C as follows.

- If $C = A_1 \sqcap \cdots \sqcap A_n$, then $V_C = \{x_C\}$, $L_C(x_C) = \{A_1, \ldots, A_n\}$, and $E_C = \emptyset$.
- If C is of the form (1), then

$$V_C = \biguplus_{i=1}^{\ell} V_{D_i} \uplus \{x_C\}$$

$$L_C(v) = \begin{cases} L_{D_i}(v) & \text{if } v \in V_{D_i} \\ \{A_1, \ldots, A_n\} & \text{if } v = x_C \end{cases}$$

$$E_C = \bigcup_{i=1}^{\ell} E_{D_i} \cup \bigcup_{i=1}^{m} \{(x_C, r_i, x_{D_i})\} \cup \bigcup_{i=m+1}^{\ell} \{(x_C, \bigcirc, x_{D_i})\} \cup \bigcup_{i=\ell+1}^{k} \{(x_C, \Diamond, x_{D_i})\}$$

C.3 Temporary restriction to rigid roles

From here on, we are restricting ourselves to the case $\mathsf{N}_\mathsf{C}^{\mathsf{loc}} = \emptyset$, i.e., no local roles are allowed. The only reason is that the technical notation would become more complex if we had to consider both rigid and local roles. Later in this section, we will explain the modifications necessary to incorporate local roles.

C.4 Homomorphisms and the fragments $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ and $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$

Given description graphs $G_i = (V_i, L_i, E_i)$, $i = 1, 2$, a *homomorphism from G_1 to G_2* is a map $h : V_1 \to V_2$ satisfying

1. $L_1(v) \subseteq L_2(h(v))$ for all $v \in V_1$.
2. $v \xrightarrow{\bullet}_{G_1} v'$ implies $h(v) \xrightarrow{\bullet}_{G_2} h(v')$, for $\bullet \in \mathsf{N_R} \cup \{\bigcirc\}$.
3. $v \xrightarrow{\Diamond}_{G_1} v'$ implies $h(v) \xrightarrow{\bigcirc}{}^{*}_{G_2} h(v')$.

For our purposes, it suffices to consider as ranges of h only description graphs G_2 associated with temporal interpretations, and those do not have $\Diamond$-edges. Therefore our definition does not need to include, e.g., $h(v) \xrightarrow{\Diamond}_{G_2} h(v')$.

Since homomorphisms operate on pairs of worlds and domain elements, we will ease notation by writing, from now on, $\langle w, x\rangle \in C^{\mathfrak{I}}$ instead of $x \in C^{\mathfrak{I},w}$. As expected, homomorphisms preserve instanceship:

Lemma 13 *Let $\mathfrak{I}_1, \mathfrak{I}_2$ be temporal interpretations and h a homomorphism from $G_{\mathfrak{I}_1}$ to $G_{\mathfrak{I}_2}$. Then, for all $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$-concepts C, $\langle w, x\rangle \in C^{\mathfrak{I}_1}$ implies $h(\langle w, x\rangle) \in C^{\mathfrak{I}_2}$.*

Proof. Via straightforward induction on the structure of C.

$C = A$. Follows from condition 1 of homomorphisms and the construction of $L_{\mathfrak{I}_1}/L_{\mathfrak{I}_2}$.

$C = C_1 \sqcap C_2$. Via a simple argument involving the induction hypothesis.

$C = \mathbf{E}\bigcirc D$. If $\langle w, x\rangle \in (\mathbf{E}\bigcirc D)^{\mathfrak{I}_1}$, then there is some $v \in \Delta^{\mathfrak{I}_1}$ with $(w, v) \in R^{\mathfrak{I}_1}$ and $\langle v, x\rangle \in D^{\mathfrak{I}_1}$. Let $\langle w', x'\rangle = h(\langle w, x\rangle)$ and $\langle v', x''\rangle = h(\langle v, x\rangle)$. By condition 2 of homomorphisms and the construction of $E_{\mathfrak{I}_1}/E_{\mathfrak{I}_2}$, we have that $(w', v') \in R^{\mathfrak{I}_2}$ and $x' = x''$. Furthermore, by induction hypothesis we get $\langle v', x'\rangle \in D^{\mathfrak{I}_2}$. Hence $\langle w', x'\rangle \in (\mathbf{E}\bigcirc D)^{\mathfrak{I}_2}$.

$C = \mathbf{E}\Diamond D$. Via the same argument as in the previous case, with a modification to the second-last step: instead of $(w', v') \in R^{\mathfrak{I}_2}$ we conclude that either $w' = v'$ or w', v' are connected via an $R^{\mathfrak{I}_2}$-chain of arbitrary length. As before, the induction hypothesis yields $\langle v', x'\rangle \in D^{\mathfrak{I}_2}$, and we get $\langle w', x'\rangle \in (\mathbf{E}\Diamond D)^{\mathfrak{I}_2}$.

$C = \exists r.D$. Analogous to the case $C = \mathbf{E}\bigcirc D$, swapping the temporal and DL "dimensions".

❑

Lemma 14 *For every $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$-concept C, temporal interpretation $\mathfrak{I}$, $w_0 \in W$ and $x_0 \in \Delta$, we have $\langle w_0, x_0\rangle \in C^{\mathfrak{I}}$ if and only if there is a homomorphism h from t_C to $G_{\mathfrak{I}}$ with $h(x_C) = \langle w_0, x_0\rangle$.*

Proof. "$\Rightarrow$". Via induction on the depth of C. If $C = A_1 \sqcap \cdots \sqcap A_n$, then h with $h(x_C) = \langle w_0, x_0\rangle$ is the required homomorphism.

If C has the form (1), then $k = \ell$ because C is from $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$. Let $\mathfrak{I} = (\Delta, T, (\mathcal{I}_w)_{w\in W})$ with $T = (W, R)$. Since $\langle w_0, x_0\rangle \in C^{\mathfrak{I}}$, there are

- $x_1, \ldots, x_m \in \Delta$ with $(x_0, x_i) \in r_i^{\mathfrak{I},w_0}$ and $\langle w_0, x_i\rangle \in D_i^{\mathfrak{I}}$ and
- $w_{m+1}, \ldots, w_\ell \in W$ with $(w_0, w_i) \in R$ and $\langle w_i, x_0\rangle \in D_i^{\mathfrak{I}}$.

By induction hypothesis, there are homomorphisms h_i from t_{D_i} to $G_{\mathfrak{I}}$ with $h_i(x_{D_i}) = \langle w_0, x_i\rangle$ if $i \leqslant m$ and $h_i(x_{D_i}) = \langle w_i, x_0\rangle$ if $i > m$. It is immediate from the construction of t_C that h with

$$h(y) = \begin{cases} \langle w_0, x_0\rangle & \text{if } y = x_C \\ h_i(y) & \text{if } y \in V_{D_i} \end{cases}$$

is the required homomorphism.

"$\Leftarrow$". Again via induction on the depth of C. If $C = A_1 \sqcap \cdots \sqcap A_n$, then $h(x_C) = \langle w_0, x_0\rangle$ together with Property 1 of being a homomorphism guarantees that $\langle w_0, x_0\rangle \in A_i^{\mathfrak{I}}$ for every A_i.

If C has the form (1), then $k = \ell$ as above. Let $\mathfrak{I} = (\Delta, T, (\mathcal{I}_w)_{w\in W})$ with $T = (W, R)$. As in the base case, we get $\langle w_0, x_0\rangle \in A_i^{\mathfrak{I}}$ for every A_i. It remains to show

(i) $\langle w_0, x_0\rangle \in (\exists r_i.D_i)^{\mathfrak{I}}$ for every $i \leqslant m$ and

(ii) $\langle w_0, x_0\rangle \in (\mathbf{E}\bigcirc D_i)^{\mathfrak{I}}$ for every $i = m+1, \ldots, \ell$.

which we obtain as follows. Due to the construction of t_C we have

- $x_C \xrightarrow{r_i}_{t_C} x_{D_i}$ for every $i \leqslant m$ and
- $x_C \xrightarrow{\bigcirc}_{t_C} x_{D_i}$ for every $i = m+1, \ldots, \ell$.

Let $h(x_{D_i}) = \langle w_i, x_i\rangle$ for all $i \leqslant k$. Since h is a homomorphism, we have

- $w_0 = w_i$ and $(x_0, x_i) \in r_i^{\mathfrak{I},w_0}$ for every $i \leqslant m$ and
- $x_0 = x_i$ and $(w_0, w_0) \in R$ for every $i = m+1, \ldots, \ell$.

Furthermore, by applying the induction hypothesis to all these $h(x_{D_i})$, we obtain $\langle w_i, x_i\rangle \in D_i^{\mathfrak{I}}$. Hence, we get (i)–(ii).

❑

The *canonical model of C* is the temporal interpretation $\mathfrak{I}_C = (\Delta_C, T_C, (\mathcal{I}_{C,w})_{w\in W_C})$ with $T_C = (W_C, R_C)$, whose components are constructed in two steps.

Step 1. We construct a finite fragment of T_C.

$$W_C^- = \{x_C\} \cup \{v \in V_C \mid v \text{ has an incoming } \bigcirc\text{-edge in } E_C\}$$

$$R_C^- = \{(w, w') \mid \exists x : w \xrightarrow{\mathsf{N_R}}{}^{*}_{t_C} x \xrightarrow{\bigcirc}_{t_C} w'\}$$

Step 2. Since temporal interpretations are based on infinite trees, we need to "artificially" continue every path in (W_C^-, R_C^-) ad infinitum. For this purpose, let

$$\mathsf{leaves}_C = \{v \in W_C^- \mid v \text{ has no outgoing } \bigcirc\text{-edges in } E_C\}.$$

For every $v \in \mathsf{leaves}_C$, we set $v_0 = v$ and introduce a sequence $(v_i)_{i\geqslant 1}$ of fresh worlds. Now we can finish the construction of $\mathfrak{I}_C$.

$$\Delta_C = \{x_C\} \cup \{v \in V_C \mid v \text{ has incoming } r\text{-edge}, r \in \mathsf{N_R}\}$$
$$W_C = W_C^- \cup \{v_i \mid v \in \mathsf{leaves}_C,\ i \geqslant 1\}$$
$$R_C = R_C^- \cup \{\{(v_i, v_{i+1}) \mid v \in \mathsf{leaves}_C,\ i \geqslant 0\}$$
$$A^{\mathfrak{I}_C,w} = \{x \in \Delta_C \mid \quad w \xrightarrow{\mathsf{N_R}}{}^*_{t_C} x \ \&\ A \in L_C(x) \text{ or } x \xrightarrow{\bigcirc}{}^*_{t_C} w \ \&\ A \in L_C(w)\}$$
$$r^{\mathfrak{I}_C,w} = \{(x, x') \mid \exists v : x \xrightarrow{\bigcirc}{}^*_{t_C} v \xrightarrow{r}_{t_C} x'\} \quad (2)$$

In particular, the construction ensures that every r is rigid (because v in the definition of $r^{\mathfrak{I}_C,w}$ is independent of w), and $A^{\mathfrak{I}_C,v_i} = \emptyset$ for any v_i with $i \geqslant 1$ added in Step 2.

Lemma 15 *For every* $\mathrm{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$*-concept* C*, temporal interpretation* $\mathfrak{I}$*,* $w_0 \in W$ *and* $x_0 \in \Delta$*, we have*

1. $\langle w_0, x_0\rangle \in C^{\mathfrak{I}}$ *if and only if there is a homomorphism* h *from* $G_{\mathfrak{I}_C}$ *to* $G_{\mathfrak{I}}$ *with* $h(\langle x_C, x_C\rangle) = \langle w_0, x_0\rangle$*;*
2. $\langle x_C, x_C\rangle \in C^{\mathfrak{I}_C}$.

Proof. To prove Points 1 and 2, we will make use of a homomorphism h_C from t_C to $G_{\mathfrak{I}_C}$ with $h_C(x_C) = \langle x_C, x_C\rangle$, which we define as follows. For any $x \in V_C$, set

$$h_C(x) = \langle y, z\rangle,$$

where y is the unique element from W_C with $y \xrightarrow{\mathsf{N_R}}{}^*_{t_C} x$ and z is the unique element from Δ_C with $z \xrightarrow{\bigcirc}{}^*_{t_C} x$. By construction of G_C, we have either $y = x$ (if x has an incoming $\bigcirc$-edge) or $z = x$ (if x has an incoming r-edge), or both if $x = x_C$.

Claim 1 *h_C is a homomorphism from t_C to $G_{\mathfrak{I}_C}$ with $h(x_C) = \langle x_C, x_C\rangle$.*

Proof of Claim. Since $h(x_C) = \langle x_C, x_C\rangle$ is immediate, it remains to show Properties 1–2 of a homomorphism; Property 3 is not required because C does not contain any $\mathbf{E}\Diamond$ operator.

Property 1. Let $A \in L_{t_C}(x)$ and $h_C(x) = \langle y, z\rangle$ as above. To show that $A \in L_{G_{\mathfrak{I}_C}}(y, z)$, we distinguish the following two cases (see above).

If $y = x$, then by construction of $\mathcal{I}_{C,x}$, we have $z \in A^{\mathfrak{I}_C,x}$. By definition of $G_{\mathfrak{I}_C}$, this implies $A \in L_{G_{\mathfrak{I}_C}}(x, z)$.

If $z = x$, then by construction of $\mathcal{I}_{C,y}$, we have $x \in A^{\mathfrak{I}_C,y}$. By definition of $G_{\mathfrak{I}_C}$, this implies $A \in L_{G_{\mathfrak{I}_C}}(y, x)$.

Property 2. Let $x_1 \xrightarrow{r}_{t_C} x_2$ and $h_C(x_i) = (y_i, z_i)$. Since x_2 has an incoming r-edge in t_C, we have $x_2 \in \Delta_C$. Hence $z_2 = x_2$ and $y_1 = y_2 =: y$. By construction of $\mathcal{I}_{C,y}$, we get $(z_1, x_2) \in r^{\mathfrak{I}_C,y}$. Due to the construction of $E_{\mathfrak{I}_C}$, we get $(\langle y, z_1\rangle, r, \langle y, x_2\rangle) \in E_{\mathfrak{I}_C}$, which means that $h_C(x_1) \xrightarrow{r}_{G_{\mathfrak{I}_C}} h_C(x_2)$.

The case $x_1 \xrightarrow{\bigcirc}_{t_C} x_2$ is analogous.

Property 3 is void because C is from $\mathrm{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$.

We are now ready to prove Points 1 and 2.

Point 1. "$\Leftarrow$". Let h be a homomorphism from $G_{\mathfrak{I}_C}$ to $G_{\mathfrak{I}}$ with $h(\langle x_C, x_C\rangle) = \langle w_0, x_0\rangle$. Then the composition h' of h_C with h is a homomorphism from t_C to $G_{\mathfrak{I}}$ with $h'(x_C) = \langle w_0, x_0\rangle$. By Lemma 14 "$\Leftarrow$", we have $\langle w_0, x_0\rangle \in C^{\mathfrak{I}}$.

"$\Rightarrow$". Let $\langle w_0, x_0\rangle \in C^{\mathfrak{I}}$. By Lemma 14 "$\Rightarrow$", there is a homomorphism from t_C to $G_{\mathfrak{I}}$ with $h(x_C) = \langle w_0, x_0\rangle$. We construct a homomorphism h' from $G_{\mathfrak{I}_C}$ to $G_{\mathfrak{I}}$ with $h'(\langle x_C, x_C\rangle) = \langle w_0, x_0\rangle$ according to the following intuition: h' can be directly obtained from h for those pairs $\langle w, x\rangle$ which directly correspond to a node v in t_C – i.e., either $w \xrightarrow{\mathsf{N_R}}{}^*_{t_C} x = v$, or $x \xrightarrow{\bigcirc}{}^*_{t_C} w = v$. All remaining pairs $\langle w, x\rangle$ have been added to $G_{\mathfrak{I}_C}$ due to rigidity of some r; consequently we find corresponding images for h' in $G_{\mathfrak{I}}$, where r is interpreted as rigid too.

In detail, h' is constructed from h in four steps.

1. For all $\langle w, x\rangle$ with $w \xrightarrow{\mathsf{N_R}}{}^*_{t_C} x$, set $h'(\langle w, x\rangle) = h(x)$.
2. For all $\langle w, x\rangle$ with $x \xrightarrow{\bigcirc}{}^*_{t_C} w$, set $h'(\langle w, x\rangle) = h(w)$.
3. For all other $\langle w, x\rangle$, if $h(w) = \langle w_1, x_1\rangle$ and $h(x) = \langle w_2, x_2\rangle$, then set $h'(\langle w, x\rangle) = \langle w_1, x_2\rangle$.
4. For all $v \in \mathsf{leaves}_C$, let $\langle w_0, x'_C\rangle = h'(\langle v_0, x_C\rangle)$. Fix some path $w_0 R w_1 R w_2 R \ldots$ of worlds in $\mathfrak{I}$. For every $x \in \Delta_C$, let $\langle w_0, x'\rangle = h'(\langle v_0, x\rangle)$; and for all $i \geqslant 1$ set $h'(\langle v_i, x\rangle) = \langle w_i, x'\rangle$).

The construction ensures $h'(\langle x_C, x_C\rangle) = h(x_C) = \langle w_0, x_0\rangle$. It remains to show that h' is a homomorphism. As above, we can omit Property 3 because C does not contain any $\mathbf{E}\Diamond$ operator.

Property 1. Since h is a homomorphism from t_C to $G_{\mathfrak{I}}$, we have that (i) $L_C(v) \subseteq L_{G_{\mathfrak{I}}}(h(v))$ for all $v \in V_C$. To show that (ii) $L_{G_{\mathfrak{I}_C}}(\langle w, x\rangle) \subseteq L_{G_{\mathfrak{I}}} h'(\langle w, x\rangle))$ for all $\langle w, x\rangle \in V_{G_{\mathfrak{I}_C}}$, we distinguish three cases.

If $\langle w, x\rangle$ satisfies the condition in Step 1, then $h'(\langle w, x\rangle) = h(x)$ by construction of h', and $L_{G_{\mathfrak{I}_C}}(\langle w, x\rangle) = L_C(x)$ by construction of $\mathfrak{I}_C$. Hence we get (ii) via (i).

If $\langle w, x\rangle$ satisfies the condition in Step 2, then we can argue analogously.

If $\langle w, x\rangle$ satisfies the condition in one of Steps 3 or 4, then Inclusion (ii) holds trivially because $L_{G_{\mathfrak{I}_C}}(\langle w, x\rangle) = \emptyset$ by construction of $\mathfrak{I}_C$.

Property 2, case $\langle w, x\rangle \xrightarrow{r}_{G_{\mathfrak{I}_C}} \langle w', x'\rangle$. Let $\langle w, x\rangle \xrightarrow{r}_{G_{\mathfrak{I}_C}} \langle w', x'\rangle$, which implies $w = w'$. In order to show

(i) $h'(\langle w, x\rangle) \xrightarrow{r}_{G_{\mathfrak{I}}} h'(\langle w, x'\rangle)$,

we analyze the possible relations between x, w, x' in t_C. First, from the assumption we conclude $(x, x') \in r^{\mathfrak{I}_C,w}$, due to the construction of $G_{\mathfrak{I}_C}$. By construction of $r^{\mathfrak{I}_C,w}$, there is some $v \in W_C$ with

(ii) $x \xrightarrow{\bigcirc}{}^*_{t_C} v$ and

(iii) $v \xrightarrow{r}_{t_C} x'$.

We now distinguish the following cases according to the construction of $h'(\langle w, x\rangle)$ and $h'(\langle w, x'\rangle)$; the notation "*(s, t)*" stands for "$\langle w, x\rangle$ and $\langle w, x'\rangle$ satisfy the conditions of Step s and t, respectively".

Case 1: (1, 1). Then, by construction of h', we have $h'(\langle w,x\rangle) = h(x)$ and $h'(\langle w,x'\rangle) = h(x')$. Since h is a homomorphism, we obtain (i).

Case 2: (2, 2). This means $x \xrightarrow{\bigcirc}^*_{t_C} w$ and $x' \xrightarrow{\bigcirc}^*_{t_C} w$, which contradicts (ii) and (iii), given that t_C is a tree.

Case 3: (1, 2). This means $w \xrightarrow{\mathsf{N_R}}^*_{t_C} x$ and $x' \xrightarrow{\bigcirc}^*_{t_C} w$, which contradicts (ii) and (iii), given that t_C is a tree.

Case 4: (2, 1). This means $x \xrightarrow{\bigcirc}^*_{t_C} w$ and $w \xrightarrow{\mathsf{N_R}}^*_{t_C} x'$. Because of (ii) and (iii) and since t_C is a tree, we even have $w = v$, that is,

(iv) $w \xrightarrow{r}_{t_C} x'$.

By construction of h' in (b) and (a), we have that $h'(\langle w,x\rangle) = h(w)$ and $h'(\langle w,x'\rangle) = h(x')$. Since h is a homomorphism, we get via (iv) that $h(w) \xrightarrow{r}_{G_\mathfrak{I}} h(x')$, hence (i).

Case 5: (1, 3). This means that

(v) $w \xrightarrow{\mathsf{N_R}}^*_{t_C} x$

and neither $w \xrightarrow{\mathsf{N_R}}^*_{t_C} x'$ nor $x' \xrightarrow{\bigcirc}^*_{t_C} w$. Let $h(x) =: \langle w_1, x_1\rangle$. By construction of h', we get

(vi) $h'(\langle w,x\rangle) = \langle w_1, x_1\rangle$.

Furthermore, let $h(w) = \langle w_2, x_2\rangle$ and $h(x') = \langle w_3, x_3\rangle$. Then we have by construction step 3 that

(vii) $h'(\langle w,x'\rangle) = \langle w_2, x_3\rangle$.

Because of (v) and h being a homomorphism, we have that $h(w) \xrightarrow{\mathsf{N_R}}^*_{G_\mathfrak{I}} h(x)$. By construction of $G_\mathfrak{I}$, this means that $w_2 = w_1$; hence

(vii′) $h'(\langle w,x'\rangle) = \langle w_1, x_3\rangle$.

Furthermore, because of (ii) and h being a homomorphism, we have that $h(x) \xrightarrow{\bigcirc}^*_{G_\mathfrak{I}} h(v)$. By construction of $G_\mathfrak{I}$, this means that $x_1 = x_4$, where $h(v) = \langle w_4, x_4\rangle$. Hence

(vi′) $h'(\langle w,x\rangle) = \langle w_1, x_4\rangle$.

Finally, because of (iii) and h being a homomorphism, we have that $h(v) \xrightarrow{r}_{G_\mathfrak{I}} h(x')$, By construction of $G_\mathfrak{I}$, this means that $w_4 = w_3$ and $(x_4, x_3) \in r^{\mathfrak{I},w_3}$. Since r is rigid, we also get $(x_4, x_3) \in r^{\mathfrak{I},w_1}$ and, by construction of $G_\mathfrak{I}$ together with (vi′) and (vii′), $h'(\langle w,x\rangle) \xrightarrow{r}_{G_\mathfrak{I}} h'(\langle w,x'\rangle)$, which is (i) as required.

Case 6: (2, 3). This means that

(viii) $x \xrightarrow{\bigcirc}^*_{t_C} w$

and neither $w \xrightarrow{\mathsf{N_R}}^*_{t_C} x'$ nor $x' \xrightarrow{\bigcirc}^*_{t_C} w$. Let $h(w) =: \langle w_1, x_1\rangle$. By construction of h', we get

(ix) $h'(\langle w,x\rangle) = \langle w_1, x_1\rangle$.

Furthermore, let $h(x) = \langle w_2, x_2\rangle$ and $h(x') = \langle w_3, x_3\rangle$. Then we have by construction step 3 that

(x) $h'(\langle w,x'\rangle) = \langle w_1, x_3\rangle$.

Because of (viii), (ii) and (iii), we have

(xi) $w \xrightarrow{\bigcirc}^*_{t_C} v$.

From this and because h being a homomorphism, we get that $h(w) \xrightarrow{\bigcirc}^*_{G_\mathfrak{I}} h(v)$. By construction of $G_\mathfrak{I}$, this means that $x_1 = x_4$, where $h(v) = \langle w_4, x_4\rangle$. Hence

(ix′) $h'(\langle w,x\rangle) = \langle w_1, x_4\rangle$.

Now, because of (iii) and h being a homomorphism, we have that $h(v) \xrightarrow{r}_{G_\mathfrak{I}} h(x')$, By construction of $G_\mathfrak{I}$, this means that $w_4 = w_3$ and $(x_4, x_3) \in r^{\mathfrak{I},w_3}$. Since r is rigid, we also get $(x_4, x_3) \in r^{\mathfrak{I},w_1}$ and, by construction of $G_\mathfrak{I}$ together with (ix′) and (x), $h'(\langle w,x\rangle) \xrightarrow{r}_{G_\mathfrak{I}} h'(\langle w,x'\rangle)$, which is (i) as required.

Case 7: (3, 1). This means that neither $w \xrightarrow{\mathsf{N_R}}^*_{t_C} x$ nor $x \xrightarrow{\bigcirc}^*_{t_C} w$, but $w \xrightarrow{\mathsf{N_R}}^*_{t_C} x'$. These contradict (ii) and (iii), given that t_C is a tree.

Case 8: (3, 2). This means that neither $w \xrightarrow{\mathsf{N_R}}^*_{t_C} x$ nor $x \xrightarrow{\bigcirc}^*_{t_C} w$, but $w \xrightarrow{\bigcirc}^*_{t_C} x'$. The argument is analogous to Case 5, swapping the temporal and DL dimensions.

Case 9: (3, 3). Let $h(w) =: \langle w_1, x_1\rangle$, $h(x) = \langle w_2, x_2\rangle$ and $h(x') = \langle w_3, x_3\rangle$. Then we have by construction step 3 that

(xii) $h'(\langle w,x\rangle) = \langle w_1, x_2\rangle$ and

(xiii) $h'(\langle w,x'\rangle) = \langle w_1, x_3\rangle$.

Because of (ii) and h being a homomorphism, we have that $h(x) \xrightarrow{\bigcirc}^*_{G_\mathfrak{I}} h(v)$. By construction of $G_\mathfrak{I}$, this means that $x_2 = x_4$, where $h(v) = \langle w_4, x_4\rangle$. Hence

(xii′) $h'(\langle w,x\rangle) = \langle w_1, x_4\rangle$.

Finally, because of (iii) and h being a homomorphism, we have that $h(v) \xrightarrow{r}_{G_\mathfrak{I}} h(x')$, By construction of $G_\mathfrak{I}$, this means that $w_4 = w_3$ and $(x_4, x_3) \in r^{\mathfrak{I},w_3}$. Since r is rigid, we also get $(x_4, x_3) \in r^{\mathfrak{I},w_1}$ and, by construction of $G_\mathfrak{I}$ together with (vi′) and (vii′), $h'(\langle w,x\rangle) \xrightarrow{r}_{G_\mathfrak{I}} h'(\langle w,x'\rangle)$, which is (i) as required.

Case 10: At least one of $\langle w,x\rangle$ and $\langle w,x'\rangle$ satisfies the conditions of Step 4 Then *both* pairs satisfy the conditions of Step 4, and we have $w = v_i$ for some $v \in \mathsf{leaves}_C$ and $i \geqslant 1$. Since $\langle v_i, x\rangle \xrightarrow{r}_{G_{\mathfrak{I}_C}} \langle v_i, x'\rangle$ by assumption, we have $\langle v_0, x\rangle \xrightarrow{r}_{G_{\mathfrak{I}_C}} \langle v_0, x'\rangle$ by construction of $\mathfrak{I}_C$ (r is rigid). From the previous cases, we can now conclude $h'(\langle v_0, x\rangle) \xrightarrow{r}_{G_\mathfrak{I}} h'(\langle v_0, x'\rangle)$. Now consider those two homomorphic images: let $\langle w_0, \overline{x}\rangle = h'(\langle v_0, x\rangle)$ and $\langle w_0, \overline{x}'\rangle = h'(\langle v_0, x'\rangle)$. Then, by construction of h' in Step 4, we get $h'(\langle v_i, x\rangle) = \langle w_i, \overline{x}\rangle$ and $h'(\langle v_i, x'\rangle) = \langle w_i, \overline{x}'\rangle$ for some w_i with $w_0 R^i w_i$. Since r is rigid in $G_\mathfrak{I}$, we get $\langle w_i, \overline{x}\rangle \xrightarrow{r}_{G_\mathfrak{I}} \langle w_i, \overline{x}'\rangle$, i.e., $h'(\langle v_i, x\rangle) \xrightarrow{r}_{G_\mathfrak{I}} h'(\langle v_i, x'\rangle)$, which is (i) as required.

Property 2, case $\langle w,x\rangle \xrightarrow{\bigcirc}_{G_{\mathfrak{I}_C}} \langle w',x'\rangle$. We can proceed as in the proof for Property 2, swapping the dimensions (worlds and the temporal relation $\bigcirc$ versus individuals and the DL role r). The assumption here is that $\langle w,x\rangle \xrightarrow{\bigcirc}_{G_{\mathfrak{I}_C}} \langle w',x\rangle$. Cases 1–9 are analogous, and we only need to deal with *Case 10: At least one of $\langle w,x\rangle$ and $\langle w',x\rangle$ satisfies the conditions of Step 4.* Then $w = v_i$ and $w' = v_{i+1}$ for some $v \in \mathsf{leaves}_C$ and

$i \geqslant 0$. Due to the construction of h', we have $h'(\langle v_i, x\rangle) = \langle w_i, \overline{x}\rangle$ and $h'(\langle v_{i+1}, x\rangle) = \langle w_{i+1}, \overline{x}\rangle$ for some w_i, w_{i+1} with $w_0 R^i w_i R w_{i+1}$ in $\mathfrak{I}$ and w_0 being such that $h'(\langle v_0, x\rangle) = \langle w_0, y\rangle$. By definition of $G_\mathfrak{I}$, we get $\langle w_i, \overline{x}\rangle \xrightarrow{\bigcirc}_{\mathfrak{I}_C} \langle w_{i+1}, \overline{x}\rangle$, which is the required $h'(\langle w, x\rangle) \xrightarrow{\bigcirc}_{G_\mathfrak{I}} h(\langle w', x\rangle)$.

Point 2. Follows from "b ⇒ a" via h_C. ❏

Lemma 3. *For all* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$*-concepts* C *and all* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$*-concepts* D*, we have*

$$\models C \sqsubseteq D \qquad \textit{if and only if} \qquad \langle x_C, x_C\rangle \in D^{\mathfrak{I}_C}.$$

Proof. First assume $\models C \sqsubseteq D$. Then, for all temporal interpretations $\mathfrak{I}$ and all $\langle w, x\rangle$, we have that $\langle w, x\rangle \in C^\mathfrak{I}$ implies $\langle w, x\rangle \in D^\mathfrak{I}$. Since $\langle x_C, x_C\rangle \in C^{\mathfrak{I}_C}$ by Lemma 15 (2), we obtain $\langle x_C, x_C\rangle \in D^{\mathfrak{I}_C}$.

For the reverse direction, assume $\not\models C \sqsubseteq D$. Then there is some temporal interpretation $\mathfrak{I}$ and some $\langle w, x\rangle$ with $\langle w, x\rangle \in C^\mathfrak{I} \setminus D^\mathfrak{I}$. By Lemma 15, there is a homomorphism h from $G_{\mathfrak{I}_C}$ to $G_\mathfrak{I}$ with $h(\langle x_C, x_C\rangle) = \langle w, x\rangle$. Now Lemma 13 implies that $\langle x_C, x_C\rangle \notin D^{\mathfrak{I}_C}$. ❏

We are now ready to prove the second part of Theorem 2.

Theorem 16 *Concept subsumption relative to the empty TBox for* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ *and* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ *can be decided in polynomial time.*

Proof. We start with $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$. Let C, D be $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$-concepts. By Lemma 3, it suffices to check whether $\langle x_C, x_C\rangle \in D^{\mathfrak{I}_C}$. Since the canonical model $\mathfrak{I}_C$ is based on an infinite tree $T_C = (W_C, R_C)$, we cannot construct it fully. Instead, we use a finite representation $\mathfrak{I}_C^{\mathsf{pre}}$ whose unraveling along the temporal direction starting from the world x_C yields $\mathfrak{I}_C$. We call $\mathfrak{I}_C^{\mathsf{pre}}$ the *canonical pre-model for* C.

More precisely, we construct $\mathfrak{I}_C^{\mathsf{pre}}$ as follows. In Step 1, we build W_C^- and R_C^- as above. In Step 2, we introduce a single fresh world v_1 for every $v \in \mathsf{leaves}_C$ and set

$$W_C^{\mathsf{pre}} = W_C^- \cup \{v_1 \mid v \in \mathsf{leaves}_C\}$$
$$R_C^{\mathsf{pre}} = R_C^- \cup \{\{(v_0, v_1), (v_1, v_1) \mid v \in \mathsf{leaves}_C\}$$

In addition, Δ_C^{pre} and $\cdot^{\mathfrak{I}_{C,w}^{\mathsf{pre}}}$ are defined as for $\mathfrak{I}_C$. Since $(W_C^{\mathsf{pre}}, R_C^{\mathsf{pre}})$ is not a total tree, $\mathcal{I}_{C,w}^{\mathsf{pre}}$ is not a temporal interpretation. However, we can define the extension of $\cdot^{\mathcal{I}_{C,w}^{\mathsf{pre}}}$ to arbitrary $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ concepts as usual. It is then an easy consequence of the construction that, if we unravel $\mathfrak{I}_C^{\mathsf{pre}}$ along the temporal dimension starting from the root x_C, we obtain a temporal interpretation isomorphic to $\mathfrak{I}_C$. Since $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ is tolerant of unraveling along the temporal dimension, we have that $\langle x_C, x_C\rangle \in D^{\mathfrak{I}_C}$ is equivalent to $\langle x_C, x_C\rangle \in D^{\mathfrak{I}_C^{\mathsf{pre}}}$. Therefore the following procedure decides $\models C \sqsubseteq D$.

1. Construct the description tree t_C.
2. Transform t_C into the canonical pre-model $\mathfrak{I}_C^{\mathsf{pre}}$ for C.
3. Check whether $\langle x_C, x_C\rangle \in D^{\mathfrak{I}_C^{\mathsf{pre}}}$. Output "yes" if the answer is positive, and "no" otherwise.

The first two steps can be performed in time polynomial in the size of C and yield polynomially-sized structures, which is an easy consequence of the constructions above. The third step is standard model checking of a bimodal formula, which can be done in polynomial time too, via a straightforward bottom-up labeling procedure [Clarke *et al.*, 1999].

The $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ part can be obtained by modifying the construction of the canonical model $\mathfrak{I}_C$, replacing all $\xrightarrow{\bigcirc}_{t_C}$ with $\xrightarrow{\Diamond}_{t_C}$. This way, every $\Diamond$-edge in t_C will induce a new world and a single R-edge in $\mathfrak{I}_C$. It should be noted that it is not necessary for the canonical model to capture transitivity of the $\Diamond$-edges in t_C – as before, it suffices to show that $\mathfrak{I}_C$ s still a model of C (Lemma 14) and homomorphically embeds into every model of C (Lemma 15). This requires two simple modifications to the proofs of Lemmas 14 and 15: (1) Replace all occurrences of R and $\xrightarrow{\bigcirc}_{t_C}$ with R^+ and $\xrightarrow{\Diamond}_{t_C}$, respectively. (2) In the proof of Lemma 15, the extensive argument of h' satisfying Property 2 of a homomorphism now proves Property 3, and Property 2 is void instead. Finally, Lemma 3 continues to follow directly from Lemma 13 (which is untouched by this modification) and the modified Lemma 15. ❏

C.5 Adding back local roles

In order to incorporate local roles in the definitions and proofs above, one change and some explanations are necessary. The change has the purpose to make the construction of the canonical model $\mathfrak{I}_C$ from the description tree t_C sensitive to the distinction between local and rigid roles. This requires to divide the definition of $r^{\mathfrak{I},w}$ in Equation (2) into two cases: for rigid roles r, we continue using (2); for local roles, we need

$$r^{\mathfrak{I}_C,w} = \{(x, x') \mid w \xrightarrow{\mathsf{N_R}}{}^*_{r_C} x \xrightarrow{r}_{t_C} x'\}.$$

Surprisingly, this modification has no impacts on the proof of Lemma 15. One would expect that, in "⇒" of Point 1, the construction of the homomorphism h' from h needs to be adapted to the modified shape of $\mathfrak{I}_C$. However, closer inspection of this part of the proof reveals that the arguments go through unchanged: First, it is still true that h' can be directly obtained from h for all pairs $\langle w, x\rangle$ which directly correspond to a node v in t_C – and these are still the pairs satisfying either $w \xrightarrow{\mathsf{N_R}}{}^*_{t_C} x = v$ or $x \xrightarrow{\bigcirc}{}^*_{t_C} w = v$. Second, it is still true that all remaining pairs $\langle w, x\rangle$ have been added to $G_{\mathfrak{I}_C}$ due to rigidity of some r; we call these pairs *rigidity-imposed*. Consequently the four steps of the construction of h' can be kept. Third, it remains to justify all references to rigid roles in Cases 5–10 of the subsequent case distinction. Indeed, Cases 5–9 are determined by Step 3 of the construction of h', which is only necessary for rigidity-imposed pairs $\langle w, x\rangle$; hence the references to r being rigid are justified. Case 10 is restricted to rigid roles altogether because the original assumption $\langle w, x\rangle \xrightarrow{r}_{G_{\mathfrak{I}_C}} \langle w, x'\rangle$ is never satisfied if r is local and $w = v_i$ for some $v \in \mathsf{leaves}_C$ and $i \geqslant 1$, due to the modified construction of $r^{\mathfrak{I}_C,w}$.

C.6 The logic $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$

To prove the technical lemmas necessary for the first part of Theorem 2, we denote by $\mathbb{U}_C$ the set of all expansion vectors for C.

Lemma 17 *For all* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$*-concepts* C, D*, we have that* $\models C \sqsubseteq D$ *if and only if* $\models C[U] \sqsubseteq D$ *for all* $U \in \mathbb{U}_C$*.*

Proof. We prove the contrapositives of both directions.

"$\Rightarrow$." Assume $\not\models C[U] \sqsubseteq D$ for some $U = (u_1, \ldots, u_n) \in \mathbb{U}_C$. Then there is some temporal interpretation $\mathfrak{I} = (\Delta, T, (\mathcal{I}_w)_{w\in W})$ based on $T = (W, R)$ and some $w \in W, x \in \Delta$ such that $\langle w, x\rangle \in C[U]^{\mathfrak{I}} \setminus D^{\mathfrak{I}}$. By Lemma 14, there is a homomorphism h from $t_{C[U]}$ to $G_{\mathfrak{I}}$ with $h(x_{C[U]}) = \langle w, x\rangle$. According to the definition of a description tree, t_C can be obtained from $t_{C[U]}$ by replacing, for each $i = 1, \ldots, n$, the corresponding chain of u_i $\bigcirc$-edges with a single $\Diamond$-edge and deleting the intermediate nodes. Hence, we can take h' as the restriction of h to the nodes in t_C, which is obviously a homomorphism from t_C to $G_{\mathfrak{I}}$, and which satisfies $h(x_C) = \langle w, x\rangle$. By Lemma 14, we get $\langle w, x\rangle \in C^{\mathfrak{I}}$. Hence $\not\models C \sqsubseteq D$.

"$\Leftarrow$." Assume $\not\models C \sqsubseteq D$. Then there is some temporal interpretation $\mathfrak{I} = (\Delta, T, (\mathcal{I}_w)_{w\in W})$ based on $T = (W, R)$ and some $w \in W, x \in \Delta$ such that $\langle w, x\rangle \in C^{\mathfrak{I}} \setminus D^{\mathfrak{I}}$. By Lemma 14, there is a homomorphism h from t_C to $G_{\mathfrak{I}}$ with $h(x_C) = \langle w, x\rangle$.

We first construct $U = (u_1, \ldots, u_n)$ as follows. Let $v_i \xrightarrow{\Diamond}_{t_C} v_i'$ be the edge representing the i-th occurrence of $\mathbf{E}\Diamond$ in C. Since h is a homomorphism, there are $w_i, w_i' \in W$ and $y_i \in \Delta$ with $h(v_i) = \langle w_i, y_i\rangle$ and $h(v_i') = \langle w_i', y_i\rangle$, as well as $(w_i, w_i') \in R^*$. The latter means that there are $w_i = w_{i,0}, w_{i,1}, \ldots, w_{i,m(i)} = w_i'$ with $(w_{i,j}, w_{i,j+1}) \in R$ for all $i \leq m(i)$. Set $u_i = m(i)$.

We can now construct a homomorphism h' from $t_{C[U]}$ to $G_{\mathfrak{I}}$ as an extension of h: according to the definition of a description tree, $t_{C[U]}$ can be obtained from t_C by replacing, for each $i = 1, \ldots, n$, the above edge $v_i \xrightarrow{\Diamond}_{t_C} v_i'$ with the corresponding chain $v_{i,0} = v_0 \xrightarrow{\bigcirc}_{t_{C[U]}} v_{i,1} \xrightarrow{\bigcirc}_{t_{C[U]}} \ldots \xrightarrow{\bigcirc}_{t_{C[U]}} v_{i,u_i} = v_i'$, introducing new nodes $v_{i,1}, \ldots, v_{i,u_i-1}$. By setting $h'(v_{i,j}) = \langle w_{i,j}, y_i\rangle$ for these new $v_{i,j}$ and $h'(v) = h(v)$ for all remaining nodes v, we obviously obtain a homomorphism. Furthermore, we get $h'(x_{C[U]}) = h(x_C) = \langle w, x\rangle$ which, by Lemma 14, leads to $\langle w, x\rangle \in C[U]^{\mathfrak{I}}$. Hence $\not\models C[U] \sqsubseteq D$. ❑

Lemma 4. *For all* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$*-concepts* C, D*, we have*
$\models C \sqsubseteq D$ *if and only if* $\models C[\overline{U}] \sqsubseteq D$ *for all* $\overline{U} \in \mathbb{U}_C^{\mathsf{td}(D)+1}$*.*
Proof. The "$\Rightarrow$" direction follows from Lemma 17.

To prove the contrapositive of the "$\Leftarrow$" direction, assume $\not\models C \sqsubseteq D$, and let $m = \mathsf{td}(D) + 1$. By Lemma 17, there is some expansion vector $U = (u_1, \ldots, u_n) \in \mathbb{U}_C$ with $\not\models C[U] \sqsubseteq D$. By Lemma 3, we have

(i) $\langle x_{C[U]}, x_{C[U]}\rangle \notin D^{\mathfrak{I}_{C[U]}}$.

From U, we construct $\overline{U} = (\overline{u}_1, \ldots, \overline{u}_n) \in \mathbb{U}_C^m$ by setting $\overline{u}_i = \min(u_i, m)$ for all $i \leqslant k$. Again by Lemma 3, it suffices to show

(ii) $\langle x_{C[\overline{U}]}, x_{C[\overline{U}]}\rangle \notin D^{\mathfrak{I}(t_{C[\overline{U}]})}$.

It thus remains to show the contrapositive "not (ii) $\Rightarrow$ not (i)".

Assume "not (ii)". Then, by Lemma 14, there is a homomorphism h from t_D to $G_{\mathfrak{I}_{C[\overline{U}]}}$ with $h(x_D) = \langle x_{C[\overline{U}]}, x_{C[\overline{U}]}\rangle$. To establish "not (i)", we thus need to construct a homomorphism h' from t_D to $G_{\mathfrak{I}_{C[U]}}$ with $h'(x_D) = \langle x_{C[U]}, x_{C[U]}\rangle$.

To construct h', let $e_1, \ldots, e_k \in E_C$ be the $\Diamond$-edges $v \xrightarrow{\Diamond}_{t_c} v'$ in t_C, where e_i corresponds to the i-th occurrence of $\mathbf{E}\Diamond$ in C. Let $E' \subseteq \{e_1, \ldots, e_k\}$ be the set of *affected* e_i: those edges whose expansion differs from $t_{C[\overline{U}]}$ to $t_{C[U]}$ (and thus from $G_{\mathfrak{I}_{C[\overline{U}]}}$ to $G_{\mathfrak{I}_{C[U]}}$). Let the expansion of an affected edge $e_i = v \xrightarrow{\Diamond}_{t_c} v'$ in $t_{C[\overline{U}]}$ and $t_{C[U]}$ be

$$v = v_1 \xrightarrow{\bigcirc}_{t_{C[\overline{U}]}} \ldots \xrightarrow{\bigcirc}_{t_{C[\overline{U}]}} v_{m+1} = v' \quad \text{and} \tag{3}$$
$$v = v_1 \xrightarrow{\bigcirc}_{t_{C[\overline{U}]}} \ldots \xrightarrow{\bigcirc}_{t_{C[\overline{U}]}} v_{m+1} \ldots \xrightarrow{\bigcirc}_{t_{C[\overline{U}]}} v_{u_i+1} = v',$$

respectively. Since the expansions differ, we have $u_i > \overline{u}_i = m$.

For each affected edge $e_i \in E'$, we set

- $P(e_i)$ to be the set of those pairs of domain elements and individuals in $\mathfrak{I}_{C[\overline{U}]}$ which correspond to all intermediate nodes in (3):
$$P(e_i) = \{\langle v_i, x\rangle \mid i \in \{2, \ldots, m\},\ x \in \Delta\},$$
where Δ is the domain of $\mathfrak{I}_{C[\overline{U}]}$
- $W(e_i)$ to be the set of all paths in t_D of maximal length which are affected by the expansion of e_i in $t_{C[\overline{U}]}$:

$$\begin{aligned} W(e_i) = \{ & y_1 \xrightarrow{\bullet_1}_{t_D} y_2 \xrightarrow{\bullet_2}_{t_D} \ldots \xrightarrow{\bullet_k}_{t_D} y_{k+1} \mid \\ & \forall \ell \leq k+1 : y_\ell \in V_D \\ & \text{and } \forall \ell \leq k : \bullet_\ell \in \mathsf{N_R} \cup \{\Diamond, \bigcirc\} \\ & \text{and } \forall \ell \leq k+1 : h(y_\ell) \in P(e_i) \\ & \text{and } \forall y' \in V_D\ \forall \bullet \in \mathsf{N_R} \cup \{\Diamond, \bigcirc\} : \\ & \quad \text{if } y' \xrightarrow{\bullet}_{t_D} y_1, \text{ then } h(y') \notin P(e_i) \quad \text{and} \\ & \quad \text{if } y_{k+1} \xrightarrow{\bullet}_{t_D} y', \text{ then } h(y') \notin P(e_i)\} \end{aligned}$$

We can now define h' node-wise, distinguishing between nodes that are affected by the expansion and those which are not. We start with affected nodes y_ℓ that occur on some path $y_1 \xrightarrow{\bullet_1}_{t_D} y_2 \xrightarrow{\bullet_2}_{t_D} \ldots \xrightarrow{\bullet_k}_{t_D} y_{k+1}$ in $W(e_i)$ for some affected edge $e_i \in E'$. Since the expansion of e_i in $t_{C[\overline{U}]}$ has length m, we have that $h(y_\ell) = \langle v_q, x\rangle$ for some $q \leq m+1$ and some x in the domain of $\mathfrak{I}_{C[\overline{U}]}$. Since $t_{C[\overline{U}]}$ is a tree and p is of maximal length with the above properties, we get by construction of $P(e_i)$ that y_1 has no incoming r-edge.

We distinguish the following three cases.

(a) If y_1 has an incoming $\Diamond$-edge, then set $h'(y_\ell) = \langle v_{q+u_i-m}, x\rangle$.

(b) If y_1 does not have an incoming $\Diamond$-edge and there is some $\ell' < \ell$ with $\bullet_\ell = \Diamond$, then set $h'(y_\ell) = \langle v_{q+u_i-m}, x\rangle$.

(c) Otherwise set $h'(y_\ell) = h(y_\ell)$ and $h'(z_\ell) = h(z_\ell)$.

For all other (unaffected) nodes, set $h'(y) = h(y)$.

The intuition behind this construction is to guarantee that, for every affected path p in t_D, if p is long enough to reach via h the end of the expansion of a $\Diamond$-edge e_i in $G_{\mathfrak{I}_{C[\overline{U}]}}$, then p also reaches exactly the end of the expansion of e_i in $G_{\mathfrak{I}_{C[U]}}$. The above construction ensures this: since the expansion of e_i is longer than $m = \mathsf{td}(D)$, there is at least one $\Diamond$-edge on p in t_D, and Cases (a)–(c) in the construction of h' cause the first $\Diamond$-edge in p to be mapped via h' to a chain of $\bigcirc$-edges in $G_{\mathfrak{I}_{C[U]}}$ that bridges the difference in the lengths of the expansions of e_i in $t_{C[\overline{U}]}$ and $t_{C[U]}$.

It remains to prove that h' is a homomorphism from t_D to $G_{\mathfrak{I}_{C[U]}}$ with $h'(x_D) = \langle x_{C[U]}, x_{C[U]}\rangle$; the latter property follows directly from the construction.

For any $y \in V_D$, let t_D^y be the subtree of t_D with root y, and let h'_y be the restriction of h' to the nodes in t_D^y. It remains to prove the following claim.

Claim 2 *For all $y \in V_D$, h'_y is a homomorphism from t_D^y to $G_{\mathfrak{I}_{C[U]}}$.*

Proof of Claim. We proceed by induction on the depth $\mathsf{d}_D(y)$ of y in t_D.

For the base case, let $\mathsf{d}_D(y) = 0$. If $h'(y) = h(y)$, then the claim holds trivially. Otherwise, y occurs on some maximal path affected by the expansion of some affected edge e_i, and $h(y) \in P(e_i)$; furthermore, $h'(y)$ has been constructed from $h(y)$ in Case (a) or (b) above. Then we have $L_{C[\overline{U}]}(h(y)) = L_{C[U]}(h'(y)) = \emptyset$, and Property 1 of being a homomorphism holds for h' because it does for h. Properties 2–4 are satisfied trivially: there are no edges in t_D^y.

For the induction step, the induction hypothesis yields Property 1 of being a homomorphism for all nodes except y and Properties 2–4 for all edges not originating in y. For the remaining nodes and edges, we consider Properties 1–4 in turn.

Property 1. If $h'_y(y) = h(y)$, then Property 1 holds for h' because it does for h. Otherwise, the construction of h' ensures that $L_{C[\overline{U}]}(h(y)) = L_{C[U]}(h'_y(y)) = \emptyset$; consequently, Property 1 holds for h' because it does for h.

Property 2. Let $y \xrightarrow{r}_{t_D} y'$. Since h is a homomorphism, we have $h(y) \xrightarrow{r}_{G_{\mathfrak{I}_{C[\overline{U}]}}} h(y')$. If $h'_y(y) = h(y)$, then the construction of h' ensures that $h'_y(y') = h(y')$, and Property 2 is immediate. Otherwise, *both* of $h(y)$ and $h(y')$ are in $P(e_i)$ for some affected edge e_i; then Cases (a) and (b) in the construction of h' ensure that $h'_y(y) \xrightarrow{r}_{G_{\mathfrak{I}_{C[U]}}} h'(y')$.

Property 3. Let $y \xrightarrow{\bigcirc}_{t_D} y'$. Then neither y nor y' are among the affected y_ℓ in the sense of the construction of h'. Hence the construction ensures $h'_y(y) = h(y)$ and $h'_y(y') = h(y')$, and Property 3 for h' follows from Property 3 for h.

Property 4. Let $y \xrightarrow{\Diamond}_{t_D} y'$. Since h is a homomorphism, we have $h(y) \xrightarrow{\Diamond}{}^{k}_{G_{\mathfrak{I}_{C[\overline{U}]}}} h(y')$ for some $k \geqslant 0$. Let $h(y) = \langle w, x\rangle$ for some world w and some domain element x from $\mathfrak{I}_{C[\overline{U}]}$. Then $h(y') = \langle w', x\rangle$ for some world w' with $wR^k w'$.

We first consider the case $h'_y(y) = h(y)$. If y' is not among the affected y_ℓ, then $h'_y(y') = h(y')$, and we get Property 4 for h' from Property 4 for h. Otherwise, $h'(y')$ must have been constructed in Case (b) or (c) for some edge e_i. In Case (c) we repeat the previous argument. In Case (b) we conclude that y is some affected y_ℓ as well; hence $w = v_q$ and $w' = v_{q+m-u_i}$, and by construction of h' we get $h'_y(y) \xrightarrow{\Diamond}{}^{k+m-u_i}_{G_{\mathfrak{I}_{C[\overline{U}]}}} h'_y(y')$.

Finally, if $h'_y(y) \neq h(y)$, then y is some affected y_ℓ for some e_i, and $h'(y)$ has been constructed in Step (a) or (b). In addition, y' is affected for the same e_i, and $h'(y')$ has been constructed in Step (b). The construction ensures that $h'_y(y) \xrightarrow{\Diamond}{}^{k}_{G_{\mathfrak{I}_{C[\overline{U}]}}} h'_y(y')$. ❑

The following is an easy consequence of Lemmas 3 and 4.

Lemma 18 *For all $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$-concepts C, D with $m = \mathsf{td}(D) + 1$, we have that $\models C \sqsubseteq D$ if and only if $\langle x_{C[U]}, x_{C[U]}\rangle \in D^{\mathfrak{I}_{C[U]}}$ for all $U \in \mathbb{U}_C^m$.*

Now we can prove the first part of Theorem 2.

Theorem 19 *Concept subsumption w.r.t. the empty TBox for $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$ is* coNP*-complete.*

Proof. For the upper bound, we proceed analogously to the proof of Theorem 16, using the criterion in Lemma 18 to decide whether $\models C \sqsubseteq D$ for two given $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$-concepts C, D. As above, we use the finite canonical pre-models in place of the infinite canonical models, which is correct because $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$ too is tolerant of unraveling along the temporal dimension. The following procedure decides $\models C \sqsubseteq D$.

1. Nondeterministically guess an expansion vector $U \in \mathbb{U}_C^m$, $m = \mathsf{td}(D) + 1$.
2. Construct the description tree $t_{C[U]}$.
3. Transform $t_{C[U]}$ into the canonical pre-model $\mathfrak{I}_{C[U]}^{\mathsf{pre}}$ for $C[U]$.
4. Check whether $\langle x_{C[U]}, x_{C[U]}\rangle \in D^{\mathfrak{I}_{C[U]}^{\mathsf{pre}}}$.

We have already argued that Steps 2–4 can be performed in polynomial time. In addition, the new nondeterministic Step 1 requires the guessing of $n \cdot \log m$ bits, where n is the number of occurrences of $\mathbf{E}\Diamond$ in C.

For the lower bound, we use a straightforward reduction from $\mathcal{EL}$ with transitive roles, for short $\mathcal{EL}^+$, to $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$. The syntax of $\mathcal{EL}^+$ is obtained by extending the definition of $\mathcal{EL}$-concepts with existential restrictions $\exists r^+.C$, where r is a role name and C is an arbitrary concept. The semantics of such restrictions is as follows: given an interpretation $\mathcal{I}$, we have

$$x \in (\exists r^+.C)^{\mathcal{I}} \quad \text{if} \quad \#\{y \mid (x,y) \in (r^{\mathcal{I}})^+ \;\&\; y \in C^{\mathcal{I}}\} \geq 1,$$

where $(r^{\mathcal{I}})^+$ is the transitive closure of $r^{\mathcal{I}}$. From the proof of [Haase, 2007, Lemma 16], it follows that already the fragment of $\mathcal{EL}^+$ with a single role is coNP-complete.

We use the following translation $t(\cdot)$ of $\mathcal{EL}^+$-concepts with a single role r into $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\diamond}$-concepts.

$$\begin{aligned} t(A) &= A \\ t(C \sqcap D) &= t(C) \sqcap t(D) \\ t(\exists r.C) &= \mathbf{E}\bigcirc\, t(C) \\ t(\exists r^+.C) &= \mathbf{E}\bigcirc\mathbf{E}\diamond\, t(C) \end{aligned}$$

It suffices to prove the following claim.

Claim 3 *For all $\mathcal{EL}^+$-concepts C, D, we have $\models C \sqsubseteq D$ if and only if $\models t(C) \sqsubseteq t(D)$.*

We prove both directions via contraposition. For the "if" direction, let $\not\models C \sqsubseteq D$. Then $d_0 \in C^\mathcal{I} \setminus D^\mathcal{I}$ for some interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ and some $d_0 \in \Delta$. We construct $T = W, R$ and a temporal interpretation $\mathfrak{J} = (\Delta', T, (\mathcal{J}_w)_{w\in W})$ as follows.

- $T = (W, R)$ is the unraveling of $(\Delta^\mathcal{I}, r^\mathcal{I})$.
- $\Delta' = \{x\}$ for a fresh domain element x.
- For every concept name A and path $\pi d \in W$, we set $A^{\mathcal{J},\pi d} = \begin{cases} \{x\} & \text{if } d \in A^\mathcal{I} \\ \emptyset & \text{otherwise} \end{cases}$
- The interpretation of roles is irrelevant since $t(\cdot)$ does not use any roles.

It is now straightforward to show inductively for all $\mathcal{EL}^+$-concepts E and $d \in (\Delta')^\mathcal{I}$ that $d \in E^\mathcal{I}$ if and only if $x \in t(E)^{\mathfrak{J},d}$. This implies $x \in t(C)^{\mathfrak{J},d_0} \setminus t(D)^{\mathfrak{J},d_0}$. Hence we have $\not\models t(C) \sqsubseteq t(D)$ as desired.

For the "only if" direction, let $\not\models t(C) \sqsubseteq t(D)$. Then $x_0 \in t(C)^{\mathfrak{I},w_0} \setminus t(D)^{\mathfrak{I},w_0}$ for some temporal interpretation $\mathfrak{I} = (\Delta, T, (\mathcal{I}_w)_{w\in W})$ based on $T = (W, R)$ and some $w_0 \in W$ and $x_0 \in \Delta$. We construct an interpretation $\mathcal{J} = (\Delta^\mathcal{J}, \cdot^\mathcal{J})$ as follows.

- $\Delta^\mathcal{J} = W$
- $r^\mathcal{J} = R$
- $A^\mathcal{J} = \{w \in W \mid x_0 \in A^{\mathfrak{I},w}\}$

It is now straightforward to show inductively for all $\mathcal{EL}^+$-concepts E and $w \in W$ that $x_0 \in t(E)^{\mathfrak{I},w}$ if and only if $w \in E^\mathcal{J}$. This implies $w_0 \in C^\mathcal{J} \setminus D^\mathcal{J}$. Hence we have $\not\models C \sqsubseteq D$.

❏

D Proofs for Section 5

Lemma 7 *Let $\mathcal{T}$ be an acyclic $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\diamond}$-TBox in normal form. Then for all $A, B \in \mathsf{CN}$, we have $\mathcal{T} \models A \sqsubseteq B$ iff, after exhaustive rule application, $B \in Q(A, AA)$.*

Proof. For the "if"-direction, the strategy is to show that a certain interpretation induced by the data structure used in our algorithm embeds to any model of A. For doing so, we need to adapt classical techniques from the $\mathcal{EL}$ family of DLs to capture rigid roles.

A *temporal pre-interpretation* is a tuple $\mathfrak{P} = (\Delta, W, (\mathcal{P}_w)_{w\in W}, (E_d)_{d\in\Delta})$ where Δ is the domain, W is a set of worlds, $\mathcal{P}_w$ is a standard DL interpretation for all $w \in W$ and E_d is a binary relation on W for all $d \in D$. Pre-interpretations can be thought of as variants of temporal interpretations where we have a temporal relation E_d for every domain element $d \in \Delta$. Moreover, roles are not necessarily interpreted rigid. More precisely, we have

$$\begin{array}{lll} d \in (\exists r.A)^{\mathfrak{P},w} & \text{iff} & \text{exists } e \in A^{\mathfrak{P},w} \text{ with } (d,e) \in r^{\mathfrak{P},w}; \\ d \in (\mathbf{E}\diamond A)^{\mathfrak{P},w} & \text{iff} & \text{exists } v \in W \text{ with } (w,v) \in E_d \\ & & \text{and } d \in A^{\mathfrak{P},v}. \end{array}$$

Next, we define the notion of *embeddings*, which can, intuitively, be thought of as variants of homomorphisms capturing both temporal and DL-dimension. Given a pre-interpretation $\mathfrak{P} = (\Delta, W, (\mathcal{P}_w)_{w\in W}, (E_d)_{d\in\Delta})$ and a temporal interpretation $\mathfrak{I} = (\Delta', W', (\mathcal{I}_w)_{w\in W'})$, and elements $d \in \Delta$, $d' \in \Delta'$, $w \in W$, and $w' \in W$, we say that $(\mathfrak{P}, d, w)$ *embeds into* $(\mathfrak{I}, d', w')$ if there is a partial function $h : (\Delta \times W) \to (\Delta' \times W')$ such that $h(d, w) = (d', w')$ and for all (e, v) such that $h(e, v)$ is defined:

- $e \in A^{\mathfrak{P},v}$ implies $e' \in A^{\mathfrak{I},v'}$ for $h(e, v) = (e', v')$;
- $(e, f) \in r^{\mathfrak{P},v}$ implies that there are e', f', v' with $h(e, v) = (e', v')$, $h(f, v) = (f', v')$ and $(e', f') \in r$;
- $(v, u) \in E_e$ implies that there are v', u', e' such that $h(e, v) = h(e', v')$, $h(e, u) = (e', u')$, and $(v', u') \in E'$.

We show now how the data structure of our algorithm gives rise to temporal pre-interpretations. For a fixed concept name $A \in \mathsf{CN}$, define $\mathfrak{P}_A = (\Delta_A, W_A, (\mathcal{P}_v)_{v\in W_A}, (E_d)_{d\in\Delta_A})$ as follows:

- the domain Δ_A is defined as the set of all sequences $A_0 w_0 r_0 A_1 w_1 r_1 \cdots r_{n-1} A_n$ such that $A_0 = A$ and for all $0 \leq i < n$ we have $(A_i, w_i) \xrightarrow{r_i} (A_{i+1}, A_{i+1}A_{i+1})$; we define $\mathsf{tail}(A_0 \cdots A_n) = A_n$;
- the set W_A is the set of all sequences $w_0 \cdots w_n$ such that $w_0 = AA$, $w_i \in W$ for all i, and for $0 \leq i < n$ either $(w_i, w_{i+1}) \in E$ or there are $B, C \in \mathsf{CN}$ with $(B, w_i) \xrightarrow{r} (C, CC)$ and $(CC, w_{i+1}) \in E$; we set $\mathsf{tail}(w_0 \cdots w_n) = w_n$.
- the auxiliary $\overline{E}$ is defined based on E as $\{(w, w') \in W_A \times W_A \mid (\mathsf{tail}(w), \mathsf{tail}(w')) \in E\}$;
- E_d is defined based on $\overline{E}$, $d = d' \cdot vrX$, and $v = w_0 \cdots w_n$ as follows:
$$\begin{aligned} E_d = {} & \{(w_0 \cdots w_i, w_0 \cdots w_{i+1}) \mid 0 \leq i < n\} \cup \\ & \{(w, w') \in \overline{E} \mid (v, w) \in \overline{E}^*\}; \end{aligned}$$
- $d \in B^{\mathfrak{P},w}$ for $B \in \mathsf{CN}$ iff $d = d' \cdot w_0 r D$ and one of the following is the case:
  - $w_0 = \mathsf{tail}(w)$ and $B \in Q(D, DD)$;
  - $w_0 \neq \mathsf{tail}(w)$, $(\mathsf{tail}(w), w_0) \in E^*$, and $B \in Q(D, \mathbf{E}\diamond D)$;
  - $w_0 \neq \mathsf{tail}(w)$, $(\mathsf{tail}(w), w_0) \notin E^*$, and $B \in Q(D, \mathsf{tail}(w))$;
- $(d, d') \in r^{\mathfrak{P},w}$ iff $d' = d \cdot w_0 r D$ and one of the following is the case:

– $\mathsf{tail}(w) = w_0$ and $(\mathsf{tail}(d), w_0) \xrightarrow{r} (D, DD)$;
– $\mathsf{tail}(w) \neq w_0$, $(\mathsf{tail}(w), w_0) \in E^*$, and $(\mathsf{tail}(d), \mathsf{tail}(w)) \xrightarrow{r} (D, \mathbf{E}\Diamond D)$;
– $\mathsf{tail}(w) \neq w_0$, $(\mathsf{tail}(w), w_0) \notin E^*$, and $(\mathsf{tail}(d), \mathsf{tail}(w)) \xrightarrow{r} (D, \mathsf{tail}(w))$.

Claim 1. *Let $A \in \mathsf{CN}$ be any concept name and $\mathfrak{I}$ be a temporal interpretation and $d \in \Delta, w \in W$ such that $\mathfrak{I} \models \mathcal{T}$ and $d \in A^{\mathfrak{I},w}$. Then $(\mathfrak{P}_A, A, AA)$ embeds into $(\mathfrak{I}, d, w)$ during the* FORWARD*-phase of the algorithm.*

Proof of Claim 1. We prove the Claim by induction over the number of rule applications of the algorithm. For the induction base, observe that $(\mathfrak{P}_A, A, AA)$ clearly embeds into any model of A. Note that due to acyclicity of the TBox it is w.l.o.g. to assume that **F1** is prioritized such that it is applied only to worlds AX that are already reachable via E from AA.

For rule **F1**, assume that $C \in Q(B, BX)$ and $C \equiv \mathbf{E}\Diamond C' \in \mathcal{T}$ and fix an embedding h of $(\mathfrak{P}_A, A, AA)$ into $(\mathfrak{I}, d, w)$. For every $\sigma \in \Delta_A, \omega \in W_A$ such that $\sigma \in C^{\mathfrak{P}_A,\omega}$ because $C \in Q(B, BX)$ we modify h as follows. Suppose $h(\sigma, \omega) = (e, v)$. By induction hypothesis, we have that $e \in C^{\mathfrak{I},v}$. Since $\mathfrak{I} \models \mathcal{T}$, there is some world v_0 with $(v, v_0) \in E$ and $e \in C'^{\mathfrak{I},v_0}$. Moreover, by definition, there is an infinite sequence of worlds $v_1, v_2, \ldots$ such that $(v_i, v_{i+1}) \in E$ for all $i \geq 0$. We put $h(\sigma, \omega \cdot BC' \cdot (B\top)^i) = (e, v_i)$ for all $i \geq 0$. It is routine to verify that h after this modification is an embedding of $(\mathfrak{P}'_A, A, AA)$ into $(\mathfrak{I}, d, w)$ where $\mathfrak{P}'_A$ is the pre-interpretation associated with the data structure after the rule application. Note that here we need the assumption about the priorization of **F1**.

For rule **F2**, assume that $C \in Q(B, w)$ and $C \equiv \exists r.D \in \mathcal{T}$ and fix an embedding h_A of $(\mathfrak{P}_A, A, AA)$ into $(\mathfrak{I}, d, w)$. For every $\sigma \in \Delta_A, \omega \in W_A$ such that $\sigma \in C^{\mathfrak{P}_A,\omega}$ because $C \in Q(B, w)$, we modify h_A as follows. Suppose $h_A(\sigma, \omega) = (e, v)$. By induction hypothesis, we have that $e \in C^{\mathfrak{I},v}$. Since $\mathfrak{I} \models \mathcal{T}$, there is some $e' \in D^{\mathfrak{I},v}$ with $(e, e') \in r^{\mathfrak{I},v}$. By induction hypothesis, there is an embedding h_D from $(\mathfrak{P}_D, D, DD)$ into $(\mathfrak{I}, e', v)$. Put $h_A(\sigma \cdot wrD \cdot \sigma', \omega \cdot \omega') := h_D(D \cdot \sigma', \omega')$ for all $D \cdot \sigma' \in \Delta_D$ and $\omega' \in W_D$. Moreover, suppose that $\omega = w_0 \cdots w_n$ and denote with $v_0, \ldots, v_n$ the n predecessor worlds of v, i.e., $(v_i, v_{i+1}) \in E$ for all $0 \leq i < n$ and $v_n = v$. Then put for all $\sigma' \in \Delta_D$ and $0 \leq i < n$: $h_A(\sigma \cdot \sigma', w_0 \cdots w_i) = (f, v_i)$ where $h_D(\sigma', v_0 \cdots v_i) = (f, \cdot)$.

It is not hard to verify that h_A obtained after such modification is an embedding of $(\mathfrak{P}'_A, A, AA)$ into $(\mathfrak{I}, d, w)$ where $\mathfrak{P}'_A$ is again the updated pre-interpretation.

For rule **F3**, the statement is an immediate consequence of the induction hypothesis and the definition of embeddings. This finishes the proof of Claim 1.

Claim 2. Let h be an embedding of $(\mathfrak{P}_A, A, AA)$ into some $(\mathfrak{I}, e, v)$ constructed in the proof of the previous Claim. Then, the following points are true:

(A) $h(d, w_1) = (d_1, v_1)$ and $h(d, w_2) = (d_2, v_2)$ implies $d_1 = d_2$;

(B) $h(d_1, w) = h(d_1, v_1)$ and $h(d_2, w) = (d_2, v_2)$ implies $v_1 = v_2$;

(C) for every $d \in \Delta_A$, there is some $w \in W_A$ with $h(d, w)$ defined;

(D) for every $w \in W_A$, there is some $d \in \Delta_A$ with $h(d, w)$ defined.

Proof of Claim 2. The statement is a direct consequence of the construction in Claim 1.

Claim 3. If $(\mathfrak{P}_A, A, AA)$ embeds into $(\mathfrak{I}, e, v)$, then $(\mathfrak{P}'_A, A, AA)$ embeds into $(\mathfrak{I}, e, v)$ where $\mathfrak{P}'_A$ is the updated pre-interpretation after applying the completion rules **C1** and **C2** of the intermediate phase of the algorithm.

Proof of Claim 3. Let h be the embeddding of $(\mathfrak{P}_A, A, AA)$ into $(\mathfrak{I}, e, v)$ constructed in the proof of the previous Claim. By Points (A) and (C) from Claim 2, there is a unique element $e \in \Delta$ such that $h(d, w) = (e, v)$ for some w; we denote this element with $[d]_h$. Likewise, by Points (B) and (D), there is a unique world $v \in W$ such that $h(d, w) = (e, v)$ for some e. Thus, the mapping h' defined by taking

$$h'(d, w) = ([d]_h, [w]_h)$$

is well-defined. We show that h' is the required embedding. We have to verify the three conditions for embeddings:

- Assume $d \in B^{\mathfrak{P}'_A,w}$, then by definition, we have $d \in B^{\mathfrak{P}_A,w}$. Thus $h(d, w)$ is defined and we have $h'(d, w) = h(d, w) = (e, v)$ for some e, v. As h is an embedding, we have $e \in B^{\mathfrak{I},v}$.
- Assume $(e, f) \in r^{\mathfrak{P}'_A,w}$, $h'(e, w) = (\overline{e}, w_1)$, and $h'(f, w) = (\overline{f}, w_2)$. The definition of h' and Point (B) of Claim 2 imply that $\overline{w} := w_1 = w_2$. Since $(e, f) \in r^{\mathfrak{P}'_A,w}$, we have $f = e \cdot w_0 rD$ and one of the three possibilities in the definition of the interpretation of role names is the case. We distinguish cases.
  – If $w_0 = \mathsf{tail}(w)$ and $(\mathsf{tail}(e), w_0) \xrightarrow{r} (D, DD)$, then already $(e, f) \in r^{\mathfrak{P}_A,w}$, i.e., after the exhaustive application of **F1-F3**. Since h' is an extension of h, and h was an embedding by assumption, we also have that $(\overline{e}, \overline{f}) \in r^{\mathfrak{I},\overline{w}}$.
  – If $w_0 \neq \mathsf{tail}(w)$, $(\mathsf{tail}(w), w_0) \in E^*$, and $(\mathsf{tail}(e), \mathsf{tail}(w)) \xrightarrow{r} (D, \mathbf{E}\Diamond D)$, then we also have $(\mathsf{tail}(e), w_0) \xrightarrow{r} (D, DD)$. Thus, there is some v such that $(e, f) \in r^{\mathfrak{P}_A,v}$, i.e., before application of rule **C2**. Since h is an embedding, there are e', f', v' such that $h(e, v) = (e', v')$, $h(f, v) = (f', v')$, and $(e', f') \in r^{\mathfrak{I},v'}$. The latter and rigidity of roles imply that $(e', f') \in r^{\mathfrak{I},\overline{w}}$. Moreover, by definition of h', we have $\overline{e} = [e]_h$ and $\overline{f} = [f]_h$. Since additionally $h(e, v) = (e', v')$ and $h(f, v) = (f', v')$, we get $\overline{e} = e'$ and $\overline{f} = f'$, thus $(\overline{e}, \overline{f}) \in r^{\mathfrak{I},\overline{w}}$.
  – If $w_0 \neq \mathsf{tail}(w)$, $(\mathsf{tail}(w), w_0) \notin E^*$, and $(\mathsf{tail}(e), \mathsf{tail}(w)) \xrightarrow{r} (D, \mathsf{tail}(w))$, then we also have $(\mathsf{tail}(e), w_0) \xrightarrow{r} (D, DD)$. Thus, there is some v such that $(e, f) \in r^{\mathfrak{P}_A,v}$, i.e., before application of rule **C2**. We can proceed as in the previous case.

- Assume $(v, w) \in E'_d$, $h'(d, v) = (d_1, \overline{v})$, and $h'(d, w) = (d_2, \overline{w})$. The definition of h' and Point (A) of Claim 2 imply that $\overline{d} := d_1 = d_2$. Assume $d = d' \cdot \widehat{w} r D$ and $\widehat{w} = w_0 \dots w_n$. Then by definition of E'_d, we have to distinguish the following two cases:
  - If there is $i \in \{0, \dots, n-1\}$ such that $v = w_i$, $w = w_{i+1}$, then already $(v, w) \in E_d$, i.e., before any application of rule **C1**. Since h is an embedding, there are e', v', w' such that $h(e, v) = (e', v')$, $h(e, w) = (e', w'$ and $(v', w') \in E$. Since, by definition of h', $\overline{v} = [v]_h$ and $h(e, v) = (e', v')$, we have $\overline{v} = v'$; analogously, we get $\overline{w} = w'$. Thus $(\overline{v}, \overline{w}) \in E_{\mathfrak{I}}$.
  - If $(\overline{w}, v) \in \overline{E}'^*$ and $(v, w) \in \overline{E}'$, then either already $(v, w) \in E_d$ (i.e., before any application of **C1**) or $(v, w) \in E'_d$ by an application of rule **C1**, i.e., **C1** put $(\mathsf{tail}(v), \mathsf{tail}(w)) \in E$. Thus, there is X such that $(X, \mathsf{tail}(v)) \xrightarrow{s} (B, BB)$ and $(BB, \mathsf{tail}(w)) \in E$. By rules **F2** and **C2** and the construction of $\mathfrak{P}_A$, we know that $X = \mathsf{tail}(d) = D$. Hence, there is a domain element $e = d \cdot vsB$. By definition of $\mathfrak{P}_A$, we get $(v, w) \in E_e$. Since h is an embedding, there are e, v', w' such that $h(e, v) = (e', v')$, $h(e, w) = (e', w')$, and $(v', w') \in E_{\mathfrak{I}}$. Since, by definition of h', $\overline{v} = [v]_h$ and $h(e, v) = (e', v')$, we have $\overline{v} = v'$; analogously, we get $\overline{w} = w'$. Thus $(\overline{v}, \overline{w}) \in E_{\mathfrak{I}}$.

This finishes the proof of Claim 3.

Finally, it should be clear that the following claim is an immediate consequence of the definition of embeddings.

Claim 4. h' constructed so far remains an embedding after exhaustive application of rules **B1**-**B3**.

To finish the proof of the "if"-direction, assume that $B \in Q(A, AA)$ and an arbitrary model $\mathfrak{I} = (\Delta, W, (\mathcal{I}_w)_{w \in W})$ of $\mathcal{T}$ and $d \in \Delta, w \in W$ with $d \in A^{\mathfrak{I}, w}$. Combining Claims 1-4, we have shown that there is an embedding of $(\mathfrak{P}_A, A, AA)$ into $(\mathfrak{I}, d, w)$ such that $h(A, AA) = (d, w)$. By definition of $\mathfrak{P}_A$, we know that $A \in B^{\mathfrak{P}_A, AA}$. By definition of embedding, we know that $d \in B^{\mathfrak{I}, w}$.

For showing the "only if" direction, assume that $B \notin Q(A, AA)$. We provide a temporal model $\mathfrak{I} = (\Delta, T, (\mathcal{I}_v)_{v \in W^*})$ of $\mathcal{T}$ such that there is a domain element $d \in \Delta$ and a world $w \in W$ with $d \in A^{\mathfrak{I}, w}$ and $d \notin B^{\mathfrak{I}, w}$.

We first introduce some auxiliary notions. A *world path* is a sequence $w_0 \cdots w_n$ such that $w_i \in W$ for all i and $(w_i, w_{i+1}) \in E$ for $0 \leq i < n$. We denote with W^* the set of all world paths with $w_0 = AA$ and with $\overline{E}$ the extension of E to W^*, i.e., $\overline{E} = \{(w, w') \in W^* \times W^* \mid (\mathsf{tail}(w), \mathsf{tail}(w')) \in E\}$. Set now

$$T = (W^*, \overline{E})$$

We define sequences $\Delta_0, \Delta_1, \dots$ and partial mappings $\pi_0, \pi_1, \dots$ with $\pi_i : \Delta_i \times W^* \to 2^{\mathsf{CN}}$, and $\theta_0, \theta_1, \dots$ with $\theta_i : \Delta_i \to \mathsf{CN}$. Our desired set Δ is obtained in the limit.

To start the construction of $\mathfrak{I}$, set

- $\Delta_0 = \{d_0\}$;
- $\pi_0(d_0, w) = Q(A, \mathsf{tail}(w))$ for all $w \in W^*$;
- $\theta_0(d) = A$,

where A is the concept name from the left-hand side from the left-hand side of the subsumption.

For the *induction step*, we start with setting $\Delta_i = \Delta_{i-1}$, $\pi_i = \pi_{i-1}$ and then inductively proceed according to the rules in Figure 6. Finally, set $\Delta = \bigcup_{i \geq 0} \Delta_i$. The temporal interpretation $\mathfrak{I} = (\Delta, T, (\mathcal{I}_w)_{w \in W^*})$ is given by

$$A^{\mathfrak{I}, w} = \{d \in \Delta \mid A \in \pi(d, w)\};$$

$$r^{\mathfrak{I}, w} = \{(d, d') \mid d' \in \Delta \text{ because } \mathbf{M} \text{ was applied to } \pi(d, v) \text{ and } \exists r.\theta(d') \text{ for some } v\}.$$

M If $A' \in \pi_i(d, w)$ and $A' \equiv \exists r.A \in \mathcal{T}$, then

1. add d' to Δ_i
set $\pi_i(d', w) = Q(A, AA)$; $\theta_i(d') = A$

2. For all $w' \neq w$ such that $(w', w) \in \overline{E}$,
set $\pi_i(d', w') = Q(A, \mathbf{E}\Diamond A)$

3. For all w' with $Q(d', w')$ undefined after **1** and **2**,
set $\pi_i(d', w') = Q(A, \mathsf{tail}(w'))$

Figure 6: Induction step rules

It is clear that $\mathfrak{I} \models A \not\sqsubseteq B$, so it remains to show that $\mathfrak{I}$ is a model of the TBox. We make a case distinction according to the possible definitions in $\mathcal{T}$.

- $A \equiv A_1 \sqcap A_2$: By definition of $\mathfrak{I}$, $d \in A^{\mathfrak{I}, w}$ iff $A \in \pi(d, w)$. By construction, $\pi(d, w)$ is some set $Q(\cdot, \cdot)$ from our algorithm. By rule **F3**, $\pi(d, w)$ contains also A_1, A_2, which yields $d \in A_i^{\mathfrak{I}, w}$ for $i = 1, 2$. The other direction is analagous using **B3**.
- $A \equiv \exists r.B$:
  Assume first that $d \in A^{\mathfrak{I}, w}$ and thus $A \in \pi(d, w)$. By construction, $\pi(d, w)$ is some set $Q(X, u)$ from our algorithm. By rule **F2**, $(X, u) \xrightarrow{r} (B, BB)$. Additionally, by **M-1**, we have that $\pi(d', w) = Q(B, BB)$ and by definition of the interpretation of role names, $(d, d') \in r^{\mathfrak{I}, w}$. By initialization, $B \in Q(B, BB)$ thus $B \in \pi(d', w)$ and $d' \in B^{\mathfrak{I}, w}$. This implies $d \in (\exists r.B)^{\mathfrak{I}, w}$.
  For the other direction, assume $d \in (\exists r.B)^{\mathfrak{I}, w}$, i.e., there is some d' with $(d, d') \in r^{\mathfrak{I}, w}$ and $d' \in B^{\mathfrak{I}, w}$. The former implies that there is some v such that d' was created due to application of **M** to $X \in \pi(d, v)$, $X \equiv \exists r.\theta(d') \in \mathcal{T}$. Put $Y = \theta(d)$ and $Y' = \theta(d')$. By construction, we know that $\pi(d, w)$ and $\pi(d', w)$ are of the form $Q(Y, y)$ and $Q(Y', y')$, respectively. By rule **F2**, we have $(Y, y) \xrightarrow{r} (Y', Y'Y')$. We distinguish cases:
  - If $v = w$, then, by **M-1**, $\pi(d', w) = Q(Y', Y'Y')$, $\pi(d, w) = Q(Y, y)$, $A' \in Q(Y, y)$, and $A' \equiv$

$\exists r.Y \in \mathcal{T}$. Since $d' \in B^{\mathcal{I},w}$, $B \in Q(Y', Y'Y')$. By rule **B2**, we have that $A \in Q(Y, y)$, thus $d \in A^{\mathfrak{I},w}$.

  - If $(w, v) \in \overline{E}$, then $\pi(d', w) = Q(Y', \mathbf{E}\Diamond Y')$ by **M-2**, thus $B \in Q(Y', \mathbf{E}\Diamond Y')$. By the completion rule **C2 a)** from the middle phase of the algorithm, we have that $(Y, \mathsf{tail}(w)) \xrightarrow{r} (Y', \mathbf{E}\Diamond Y')$. By rule **B2**, $A \in Q(Y, \mathsf{tail}(w))$. By construction of $\mathfrak{I}$ and the fact that $(w, v) \in \overline{E}$, we have that $\pi(d, w) = Q(Y, \mathsf{tail}(w))$, thus $A \in \pi(d, w)$ and $d \in A^{\mathfrak{I},w}$.
  - If $(v, w) \in \overline{E}$, then $\pi(d', w) = Q(Y', W)$ where $W = \mathsf{tail}(w)$. By the completion rule **C2 b)** from the middle phase of our algorithm, we have that $(Y, W) \xrightarrow{r} (Y', W)$. By rule **B2**, $A \in Q(Y, W)$. Since $(v, w) \in \overline{E}$, $\pi(d, w)$ is defined using **M-3** and we have that $\pi(d, w) = Q(Y, W)$. Hence $A \in \pi(d, w)$ and $d \in A^{\mathfrak{I},w}$.
  - Otherwise, $\pi(d', w) = Q(Y', W)$ where $W = \mathsf{tail}(w)$. By **M-3**, $\pi(d, w)$ is defined as $Q(Y, W)$. By the completion rule **C2 c)** from the middle phase of our algorithm, we have that $(Y, W) \xrightarrow{r} (Y', W)$. By rule **B2**, $A \in Q(Y, W)$, thus $d \in A^{\mathfrak{I},w}$.

- $A \equiv \mathbf{E}\Diamond B$:

  Assume first that $d \in A^{\mathfrak{I},w}$. By construction, $\pi(d, w)$ is some set $Q(X, u)$ from our algorithm such that $\mathsf{tail}(w) = u$. By rule **F1**, we have $(u, XB) \in E$. By definition of $\mathfrak{I}$, $w' = w \cdot XB \in W^*$ and $(w, w') \in \overline{E}$. By initialization of the algorithm, we have that $B \in Q(X, XB)$. Moreover, by definition of $\mathfrak{I}$, we have $\pi(d, w') = Q(X, XB)$ and hence $d \in B^{\mathfrak{I},w'}$. Thus, $d \in (\mathbf{E}\Diamond B)^{\mathfrak{I},w}$.

  For the other direction assume that $d \in (\mathbf{E}\Diamond B)^{\mathfrak{I},w}$, i.e., there are $w_0, \ldots, w_n$ with $w_0 = w$, $(w_i, w_{i+1}) \in \overline{E}$ for all $0 \leq i < n$ and $d \in B^{\mathfrak{I},w_n}$. By definition of W^* and $\overline{E}$, there are $v_0, \ldots, v_n \in W$ such that $(v_i, v_{i+1}) \in E$ for all $0 \leq i < n$. We distinguish cases:

  - If $d = d_0$, then $\pi(d, w_i) = Q(Y, v_i)$ and $v_i = \mathsf{tail}(w_i)$ for all $0 \leq i \leq n$. By rule **B1**, we have that $A \in Q(Y, v_0)$ and hence $A \in \pi(d, w_0)$. Thus, $d \in A^{\mathfrak{I},w}$.
  - If $d \neq d_0$ and for each $0 \leq i \leq n$, $\pi(d, w_i)$ is defined using **M-3** or **M-1**, we have $v_i = \mathsf{tail}(w_i)$ and $\pi(d, w_i) = Q(Y, v_i)$ for all $0 \leq i \leq n$. Thus, we can proceed as in the previous rule.
  - If $\pi(d, w_j)$ is defined using **M-2** for some j, then so is $\pi(d, w_0)$ by **M-2**. Thus, $v_0 = \mathbf{E}\Diamond X$ and $\pi(d, w_0) = Q(Y, \mathbf{E}\Diamond X)$ for some X. By rule **B1**, we have $A \in Q(Y, \mathbf{E}\Diamond X)$ and hence $d \in A^{\mathfrak{I},w}$.

❏

D.1 Local Roles

Figure 7 shows the rules needed, extending those in Section 5, in order to deal with local roles. In particular, a variant (**B2'**) of **B2** is included, and **F2** is slightly modified to specify that it is only applied to rigid roles. We use $\mathsf{RIG}(\mathcal{T})$to denote the set of rigid roles occurring in $\mathcal{T}$.

F1 If $B \in Q(A, AX)$ & $B \equiv \mathbf{E}\Diamond B' \in \mathcal{T}$, add (AX, AB') to E

F2 If $B \in Q(A, w)$, $B \equiv \exists r.B' \in \mathcal{T}$ and $r \in \mathsf{RIG}(\mathcal{T})$, set $(A, w) \xrightarrow{r} (B', B'B')$

F3 If $B \in Q(A, w)$ & $B \equiv A_1 \sqcap A_2 \in \mathcal{T}$, add A_1, A_2 to $Q(A, w)$

B1 If $B \in Q(A, w)$, $(w', w) \in E^*$, and $X \equiv \mathbf{E}\Diamond B \in \mathcal{T}$, add X to $Q(A, w')$

B2 If $A \in Q(B, w)$, $(A', w') \xrightarrow{r} (B, w)$, and $X \equiv \exists r.A \in \mathcal{T}$ add X to $Q(A', w')$

B2' If $A \in Q(B, w)$, $A \equiv \exists r.A'$, $B' \in Q(A', A'A')$ and $B'' \equiv \exists r.B' \in \mathcal{T}$, add B'' to $Q(B, w)$

B3 If $A_1, A_2 \in Q(B, w)$ & $A \equiv A_1 \sqcap A_2 \in \mathcal{T}$ add A to $Q(B, w)$

Figure 7: FORWARD- and BACKWARD-completion rules

E $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\circ}$ and $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ with Increasing Domains

In the design of TDLs one can make various assumptions on the domain, e.g., *increasing domains*, that is, each world w in W comes equipped with a domain Δ^w such that $\Delta^w \subseteq \Delta^{w'}$ for all successor worlds w' of w.

Formally, a *temporal interpretation with increasing domains* based on an infinite tree $T = (W, E)$ is a structure $\mathfrak{I} = (T, (\mathcal{I}_w)_{w \in W})$ such that, for each $w \in W$, $\mathcal{I}_w$ is a DL interpretation with domain Δ^w; for all $w, w' \in W$, $(w, w') \in E$ implies $\Delta^w \subseteq \Delta^{w'}$; and $r^{\mathcal{I}_w} = r^{\mathcal{I}_{w'}}$ for all $r \in \mathsf{N}_{\mathsf{rig}}$ and $w, w' \in W$.

It is known that, for most Boolean-complete TDLs, reasoning with constant domains is not easier than with increasing domains [Gabbay *et al.*, 2003, Proposition 3.32]. We show that, unexpectedly, subsumption relative to acyclic $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\circ}$- and $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ TBoxes with increasing domains is harder than with constant domains. Intuitively, this can be explained by the fact that increasing domains allow rigid roles to mimic the behaviour of the $\mathbf{A}\Box$-operator. Indeed, we will show that subsumption relative to acyclic $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond,\mathbf{A}\Box}$ TBoxes is PSPACE-hard.

Theorem 8 *Concept subsumption relative to acyclic* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\circ}$ *and* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ *TBoxes with rigid roles and increasing domains is* PSPACE*-hard.*

Proof. The proof is by reduction of the validity problem for quantified Boolean formulas. A *quantified Boolean formula (QBF)* φ is of the form $Q_1 x_1 \ldots Q_n x_k.\psi$, where $Q_i \in \{\exists, \forall\}$ and ψ is a Boolean with only variables $x_1, \ldots, x_k$. From now on, w.l.o.g. we assume ψ to be in conjunctive normal form, that is, $\psi = c_1 \wedge \ldots \wedge c_n$. We aim at constructing in polynomial time an acyclic TBox $\mathcal{T}_\varphi$ such that for certain concept names L_0, E_0, we have that $\mathcal{T}_\varphi \models L_0 \sqsubseteq E_0$ iff φ is valid.

In a nutshell, a model of $\mathcal{T}_\varphi$ and L_0 is an evaluation tree for φ, that is, a binary tree of depth k such that at each level $i \leq k$ one of the nodes sets the variable x_i to true and the other to false. In other words, we aim at representing with each node at level i a truth assignment to the variables $x_1, \ldots, x_i$. More precisely, we use the following signature.

- concept names $L_0, \ldots, L_k$ to distinguish the levels of a binary tree of depth n.
- rigid roles r_{ij},, for $1 \leq i \leq n$, to represent the truth of a clause c_i at level j of the evaluation tree.
- concept names $E_0, \ldots, E_k$ to evaluate in a bottom-up fashion ψ.

For $1 \leq j \leq k$, we use abbreviations P_j and N_j

- we use a concept name P_j to denote the following conjunction

$$\bigsqcap_{\substack{i \leq n \\ x_j \text{ occurs } \textit{positively} \text{ in } c_i}} \exists r_i.\top$$

- we use a concept name N_j to denote the following conjunction

$$\bigsqcap_{\substack{i \leq n \\ x_j \text{ occurs } \textit{negatively} \text{ in } c_i}} \exists r_i.\top$$

We are now ready to define the axioms of $\mathcal{T}_\varphi$ providing the core of the reduction. We start by defining the evaluation tree (4). The propagation of the truth of a clause at jth-level to the deeper levels is achieved by the rigid roles in the definitions of P_j and N_j. Finally, we use concepts E_i to evaluate φ such that quantifiers are respected (5)-(7). For $0 \leq \ell < k$,

$$L_\ell \equiv \mathbf{E}\Diamond(L_{\ell+1} \sqcap \mathsf{P}_{\ell+1}) \sqcap \mathbf{E}\Diamond(L_{\ell+1} \sqcap \mathsf{N}_{\ell+1}) \tag{4}$$

$$E_k \equiv L_k \sqcap \exists r_1.\top \sqcap \ldots \sqcap \exists r_n.\top \tag{5}$$

For $0 \leq i < k$,

$$E_i \equiv L_i \sqcap \mathbf{E}\Diamond(L_{i+1} \sqcap E_{i+1}), \text{ if } Q_{i+1} = \exists \tag{6}$$

For $0 \leq i < k$,

$$\begin{aligned} E_i \equiv L_i \sqcap \mathbf{E}\Diamond(L_{i+1} \sqcap \mathsf{P}_{i+1} \sqcap E_{i+1}) \sqcap \\ \mathbf{E}\Diamond(L_{i+1} \sqcap \mathsf{N}_{i+1} \sqcap E_{i+1}), \text{ if } Q_{i+1} = \forall \end{aligned} \tag{7}$$

Following the intuitions provided above, it is not hard to see that $\mathcal{T}_\varphi \models L_0 \sqsubseteq E_0$ iff φ is valid.

For $\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$, the reduction clearly goes through by simply replacing $\mathbf{E}\Diamond$ with $\mathbf{E}\bigcirc$.

❏

F Proofs for Section 6

Theorem 9 *Concept subsumption relative to acyclic* $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\Box}_{\mathcal{EL}}$*-TBoxes with rigid roles is* PSPACE*-complete.*

Proof. The proof of the lower bound is by reduction of the validity problem for quantified Boolean formulasA *quantified Boolean formula (QBF)* φ is of the form $Q_1 x_1 \ldots Q_n x_k.\psi$, where $Q_i \in \{\exists, \forall\}$ and ψ is a Boolean formula with only variables $x_1, \ldots, x_k$. From now on, w.l.o.g. we assume ψ to be in conjunctive normal form, that is, $\psi = c_1 \wedge \ldots \wedge c_n$. We aim at constructing in polynomial time an acyclic TBox $\mathcal{T}_\varphi$ such that for certain concept names L_0, E_0, we have that $\mathcal{T}_\varphi \models L_0 \sqsubseteq E_0$ iff φ is valid.

In a nutshell, a model of $\mathcal{T}_\varphi$ and L_0 is an evaluation tree for φ, that is, a binary tree of depth k such that at each level $i \leq k$ one of the nodes sets the variable x_i to true and the other to false. In other words, we aim at representing with each node at level i a truth assignment to the variables $x_1, \ldots, x_i$. More precisely, we use the following signature. In particular, we do not make use of roles.

- concept names $L_0, \ldots, L_k$ to distinguish the levels of a binary tree of depth n.
- concept names $C_{i,j}$, for $i \leq n$, $j \leq k$, to represent the 'truth' of clause c_i at level j of the evaluation tree.
- concept names $E_0, \ldots, E_k$ to evaluate in a bottom-up fashion ψ.

Moreover, we use the following abbreviations.

- For, $j \leq k$, we use a concept name P_j to denote the following conjunction

$$\bigsqcap_{\substack{i \leq n \\ x_j \text{ occurs } \textit{positively} \text{ in } c_i}} C_{i,j}$$

- For, $j \leq k$, we use a concept name N_j to denote the following conjunction

$$\bigsqcap_{\substack{i \leq n \\ x_j \text{ occurs } \textit{negatively} \text{ in } c_i}} C_{i,j}$$

We are now ready to define the axioms of $\mathcal{T}_\varphi$ providing the core of the reduction. We start by defining the evaluation tree (8). We then propagate the truth of a clause at jth-level to the deeper levels (9). Note that the use of abbreviations P_ℓ and N_ℓ and definitions $C_{i,j}$ allow to establish the truth value of c_i up to a 'partial' assignment of length ℓ. Finally, we use concepts E_i to evaluate φ such that quantifiers are respected (10)-(12).

For $0 \leq \ell < k$,

$$L_\ell \equiv \mathbf{E}\Diamond(L_{\ell+1} \sqcap \mathsf{P}_{\ell+1}) \sqcap \mathbf{E}\Diamond(L_{\ell+1} \sqcap \mathsf{N}_{\ell+1}) \tag{8}$$

For $1 \leq i \leq n$, $1 \leq j < k$,

$$C_{i,j} \equiv \mathbf{A}\Box\, C_{i,j+1}, \tag{9}$$

$$E_k \equiv L_k \sqcap C_{1,k} \sqcap \ldots \sqcap C_{n,k} \tag{10}$$

For $0 \leq i < k$,

$$E_i \equiv L_i \sqcap \mathbf{E}\Diamond(L_{i+1} \sqcap E_{i+1}), \text{ if } Q_{i+1} = \exists \tag{11}$$

For $0 \leq i < k$,

$$\begin{aligned} E_i \equiv L_i \sqcap \mathbf{E}\Diamond(L_{i+1} \sqcap \mathsf{P}_{i+1} \sqcap E_{i+1}) \sqcap \\ \mathbf{E}\Diamond(L_{i+1} \sqcap \mathsf{N}_{i+1} \sqcap E_{i+1}), \text{ if } Q_{i+1} = \forall \end{aligned} \tag{12}$$

Following the intuitions provided above, it is not hard to see that $\mathcal{T}_\varphi \models L_0 \sqsubseteq E_0$ iff φ is valid. This concludes the proof.

The proof straightforwardly works for $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{A}\Box}$ by replacing above $\mathbf{E}\Diamond$ with $\mathbf{E}\bigcirc$.

The upper bound is an immediate consequence of Lemmas 20 and 21 proved below and the observation that `complete` can be implemented using only polynomial space. ❑

We divide the proof of Lemma 10 into two lemmas, Lemma 20 for termination (and argument for running in PSPACE) and Lemma 21 for correctness.

Lemma 20 *On input $\mathcal{T}, A, B$, Algorithm 1 always terminates and at any time maintains only polynomially sized structures. More precisely, there is a polynomial p, such that the size of every trace that is constructed in `expand` of Algorithm 1 is bounded by $p(n)$, where $n = |\mathcal{T}|$.*

Proof. First observe that, due to acyclicity of $\mathcal{T}$ the expansion in Line 13 ($\exists r.B$) can only be applied n times along a trace. Thus, it is sufficient to show that traces become periodic after a polynomial application of the expansion in Line 17 ($\mathbf{E}\Diamond B$). For this purpose, let us take a trace with suffix $(d, w_1) \cdots (d, w_k)$ and let $A_1, \ldots, A_k$ be the concept names such that $\mathbf{E}\Diamond A_i$ lead to creation of w_i.

Claim. If $A = A_i = A_j$ for $i < j$, then $Q(d, w_i) \subseteq Q(d, w_j)$ after application of `complete`.

Proof of the Claim. Note that $Q(d, w_i)$ and $Q(d, w_j)$ are both initialized with $\{A, \top\}$. It is routine to verify (via induction on the number of rule applications of **R1**-**R12**) that every rule applied to $Q(d, w_i)$ can also be applied to $Q(d, w_j)$. This finishes the proof of the Claim.

Thus, for $k > n(n + 1)$, there is a concept name A and a sequence $i_0 < \ldots < i_n$ such that $A_{i_0} = \ldots = A_{i_n}$. By the Claim, we have

$$Q(d, w_{i_0}) \subseteq Q(d, w_{i_2}) \subseteq \ldots \subseteq Q(d, w_{i_n}).$$

Since $Q(d, w_{i_n}) \subseteq \mathsf{CN}$ and $|\mathsf{CN}| \leq n$, we have that there are different $j, j' \in \{i_0, \ldots, i_n\}$ such that $Q(d, w_j) = Q(d, w_{j'})$. Consequently, the trace is periodic. Since the expansion in Line 13 is applied at most n times along each trace, we get the lemma for $p(n) = n^2(n + 1)$. ❑

Lemma 21 *On every input $\mathcal{T}, A_0, B_0$, Algorithm 1 returns `true` iff $\mathcal{T} \models A_0 \sqsubseteq B_0$.*

Proof. For the "⇒"-direction, we show that after every call to `complete` the trace together with Q embeds into every model of $\mathcal{T}$ and A_0. As traces can become cyclic (due to Line 8), we have to consider *unravellings* of traces. It is important to notice that because of the structure of the algorithm, there is at most one cycle in E. Let (σ, E, R) be a trace with $\sigma = (d_0, w_0) \cdots (d_n, w_n)$ and Q a mapping. The unraveling $\mathfrak{I}^u = (\Delta^u, W^u, (\mathcal{I}_w^u)_{w \in W})$ of a trace is defined as follows:

- $\Delta^u = \{d_0, \ldots, d_n\}$;
- W^u is the set of sequences of worlds $v_0 \cdots v_k$ such that $v_0 = w_0$ and $(v_i, v_{i+1}) \in E$ for all $0 \leq i < n$;
- $r^{\mathfrak{I}^u, w} = R(r)$;
- $E^u = \{(u, v) \in W^u \times W^u \mid (\mathsf{tail}(u), \mathsf{tail}(v)) \in E\}$;
- $d \in A^{\mathfrak{I}^u, w}$ iff $A \in Q(d, \mathsf{tail}(w))$.

Let now be $\mathfrak{I} = (\Delta, W, (\mathcal{I}_w)_{w \in W})$ a temporal interpretation and $d \in \Delta, w \in W$. We say that the unraveling $\mathfrak{I}^u$ of a trace *embeds into* $(\mathfrak{I}, d, w)$ if there are functions h_Δ, h_W such that:

- $h_\Delta(d_0) = d$ and $h_W(w_0) = w$;
- $(d_i, d_j) \in r^{\mathfrak{I}^u, w}$ implies $(h_\Delta(d_i), h_\Delta(d_j) \in r^{\mathfrak{I}, w'}$ for all $w' \in W$;
- $(w_i, w_j) \in E^u$ implies $(h_W(w_i), h_W(w_j)) \in E^*$;
- $d \in A^{\mathfrak{I}^u, w}$ implies $h_\Delta(d) \in A^{\mathfrak{I}, h_W(w)}$.

Assume now that $\mathfrak{I} \models \mathcal{T}$ and choose d, w such that $d \in A_0^{\mathfrak{I}, w}$. We show by induction on the number of calls to `complete` that the unraveling of (σ, E, R) together with Q embeds into $(\mathfrak{I}, d, w)$ where $(\sigma, E, R), Q$ is the state of the algorithm after calling `complete`.

The induction base is immediate: we set $h_\Delta(d_0) = d$ and $h_W(w_0) = w$. Since $Q(d_0, w_0) = \{A_0, \top\}$, it remains to note that $d \in A_0^{\mathfrak{I}, w}$ by assumption. The first call to `complete` applies only rules **R1** and **R2** for closing under conjunction. Clearly, h_Δ, h_W remains an embedding since $\mathfrak{I} \models \mathcal{T}$.

For the induction step, we make a case distinction on which part of the algorithm has been applied before calling `complete`.

Case 1: Expansion of the trace because of $A \in Q(\hat{d}, \hat{w})$ such that $A \equiv \exists r.B \in \mathcal{T}$. This causes a new element $(d', \hat{w})$ and consequently a new domain element $d' \in \Delta^u$ such that $(\hat{d}, d') \in r^{\mathfrak{I}^u, v}$ for all v and $d' \in B^{\mathfrak{I}^u, \hat{w}}$ (note that, due to the structure of the algorithm – expansion of $\exists r$ before $\mathbf{E}\Diamond$ – E is acyclic in this case). By induction, there is an embedding h_Δ, h_W from $(\mathfrak{I}, d_0, w_0)$ into $(\mathfrak{I}, d, w)$; in particular, suppose $h_\Delta(\hat{d}) = e$ and $h_W(\hat{w}) = v$. By definition of embedding, we know that $e \in A^{\mathfrak{I}, v}$. Since $\mathfrak{I} \models \mathcal{T}$, there is an $e' \in B^{\mathfrak{I}, v}$ such that $(e, e') \in r^{\mathfrak{I}, v}$. It should be clear that h'_Δ defined as the extension of h_Δ with $h'_\Delta(d') = e'$ together with h_W is an embedding (before calling `complete`). It is then routine to verify that rules **R1**-**R12** maintain the properties of an embedding.

Case 2: Expansion of the trace because of $A \in Q(\hat{d}, \hat{w})$ such that $A \equiv \mathbf{E}\Diamond B \in \mathcal{T}$. Analogous to the previous case.

Case 3: Detection of periodicity; suppose σ, Q is periodic at (i, j) and $\sigma = (d_0, w_0) \cdots (d_n, w_n)$, i.e., $i < j$, $d_i = d_j = d_n$, and $Q(d_i, w_i) = Q(d_j, w_j)$. Then the edge (w_{j-1}, w_i) is added to E. By induction, there is an embedding h_Δ, h_W of $(\mathfrak{I}^u, d_0, w_0)$ into $(\mathfrak{I}, d, w)$. Let $d := d_i = d_j$, $\hat{d} := h_\Delta(d)$ and $v_k = h_W(w_k)$ for all $i \leq m \leq j$. By construction of $\mathfrak{I}^u$ and the definition of embedding, we have that

$$A \in Q(d, w_i) \text{ implies } \hat{d} \in A^{\mathfrak{I}, v_i}. \qquad (*)$$

By construction and $(*)$, we have that $\hat{d} \in (\mathbf{A}\Box Q_{\mathsf{cert}}(d) \sqcap Q(d, w_{j-1}))^{\mathfrak{I}, v_{j-1}}$. We need the following auxiliary claim, the proof of which is straightforward by induction on the number of rule applications.

Claim 1. For a trace with infix $(d, v_1) \cdots (d, v_m)$, we have

$$\mathcal{T} \models \mathbf{A}\Box Q_{\mathsf{cert}}(d) \sqcap Q(d, v_i) \sqsubseteq \mathbf{E}\Diamond Q(d, v_j)$$

for all $1 \leq i < j \leq m$.

We apply Claim 1 to σ and the worlds w_{j-1} and w_j from the precondition and obtain that there is a world u_0 such that $\hat{d} \in Q(d, w_j)^{\mathfrak{I}, u_0}$. We put $h_W(w_0 \cdots w_{j-1} w_i) := v'_j$. We repeat the above argument now for w_i, w_{i+1}. By construction of $\mathfrak{I}^u$ and $(*)$ and $Q(d, w_i) = Q(d, w_j)$, we know that $\hat{d} \in Q(d, w_i)^{\mathfrak{I}, u_0}$. By Claim 1 and $\mathfrak{I} \models \mathcal{T}$, we know that there is a world u_1 such that $(u_0, u_1) \in E^*$ and $\hat{d} \in Q(d, w_{i+1})^{\mathfrak{I}, u_1}$. We can repeatedly apply the same arguments and obtain an embedding of $(\mathfrak{I}^u, d_0, w_0)$ into $(\mathfrak{I}, d, w)$ in the limit. Again, rules **R1**-**R12** applied in `complete` preserve embeddings. This finishes the proof of Case 3 and thus of the induction step.

To finish the "⇒"-direction, assume that Algorithm 1 returns `true` on input $\mathcal{T}, A_0, B_0$, i.e., $B_0 \in Q(d_0, w_0)$ after termination. Moreover, suppose that $\mathfrak{I} = (\Delta, W, (\mathcal{I}_w)_{w \in W})$ is a model of $\mathcal{T}$ and let $d \in \Delta$, $w \in W$ such that $d \in A_0^{\mathfrak{I}, w}$. By what was said above, there is an embedding of the final trace before terminating into $(\mathfrak{I}, d, w)$. By definition of embedding and $B_0 \in Q(d_0, w_0)$, we have that $d \in B_0^{\mathfrak{I}, w}$. This proves that $\mathcal{T} \models A \sqsubseteq B$.

For the "⇐"-direction, assume that Algorithm 1 returns `false`, i.e., $B_0 \notin Q(d_0, w_0)$. We construct a temporal interpretation $\mathfrak{I} = (\Delta, W, (\mathcal{I}_w)_{w \in W})$ and $d \in \Delta$, $w \in W$ such that $d \in A_0^{\mathfrak{I}, w}$ but not $d \in B_0^{\mathfrak{I}, w}$.

Note that the $Q(d, w)$ might change during the run of Algorithm 1; however, in what follows, we denote with $\overline{Q}(d, w)$ the maximal set $Q(d, w)$ that appears (well-defined since $Q(d, w)$ only grows). Analogously, we use $\overline{Q}_{\mathbf{A}\Box}(d, w)$ and $\overline{Q}_{\mathsf{cert}}(d)$. Moreover, we denote with $\overline{W}$ the set of all worlds created during the run of the algorithm, which are not truncated at some point due to periodicity (cf. Line 9); similarly, we define $\overline{E}$ and $\overline{R}(r)$ for each $r \in \mathsf{ROL}$. Finally, observe that, if a trace element (d, w) is created due to $\exists r.B$, we have that $\overline{Q}(d, w') = \overline{Q}(d, w'')$ for all w', w'' with $(w', w) \in E^+$, $(w'', w) \in E^+$. This holds as well for different d, d'. This justifies to write $\overline{Q}_{\mathbf{E}\Diamond}(B)$ for $\overline{Q}(d, w')$.

Intuitively, we want to use the "unraveling" of the structures $\overline{W}, \overline{E}, \overline{R}(r)$ to define the desired interpretation $\mathfrak{I}$. However, doing so naively does not suffice since some points (d, w) would satisfy concepts of the form $\mathbf{A}\Box B$ which are not enforced. Thus, we will define a sequence of interpretation $\mathfrak{I}_0, \mathfrak{I}_1, \ldots$ and the desired $\mathfrak{I}$ is defined in the limit.

In our construction, domain elements $\sigma \in \Delta_i$ take the form $\sigma = (r_1, d_1, w_1) \cdots (r_n, d_n, w_n)$ where r_i are rigid role names, the d_i are domain elements created during the run of the algorithm, and the w_i are worlds created during the construction of $\mathfrak{I}$. Worlds are sequences $\omega = w_0 \ldots w_k$ of worlds w_i created in the algorithm or special worlds of the form $\langle \sigma, \omega \rangle$ whose purpose is to "break" unintended $\mathbf{A}\Box B$. We abbreviate $\sigma^{\downarrow} = d_n$ and $\omega^{\downarrow} = w_k$.

We start with an interpretation $\mathfrak{I}_0 = (\Delta_0, (W_0, E_0), (\mathcal{I}_{0,w})_{w \in W})$ along with a function $\pi_0 : \Delta_0 \times W_0 \to 2^{\mathsf{CN}}$ defined as follows:

- $\Delta_0 = \{(r, d_0, w_0)\}$ where r is any role (not important);
- $W_0 = \{w_0\}$; $E_0 = \emptyset$;
- $\pi_0((d, d_0, w_0), w_0) = \overline{Q}(d_0, w_0)$.

Now, $\mathfrak{I}_{i+1}, \pi_{i+1}$ are obtained from $\mathfrak{I}_i, \pi_i$ by setting $\Delta_{i+1} = \Delta_i$, $W_{i+1} = W_i$, $E_{i+1} = E_i$, $\pi_{i+1} = \pi_i$, and applying one of the rules in Figure 8. Note that rules **C1** and **C2** are just a form of unraveling. However, **C3** adds additional worlds to break unwanted concepts of the form $\mathbf{A}\Box B$. In rule **C2**, we need the following property of our algorithm:

Claim 2. For all $A \in \overline{Q}(d, w)$, $A \equiv \mathbf{E}\Diamond B \in \mathcal{T}$, there is a w' such that $(w, w') \in \overline{E}^*$ and $B \in \overline{Q}(d, w')$.

Thus, there is a well-defined selection function $\mathsf{wit}(d, w, B)$ that for all $A \in \overline{Q}(d, w)$, $A \equiv \mathbf{E}\Diamond B \in \mathcal{T}$ returns such a w' from Claim 2.

C1	If $A \in \pi_i(\sigma, \omega)$ and $A \equiv \exists r.B \in \mathcal{T}$, then add $\sigma' = \sigma \cdot (r, d', \omega)$ where d' is the element that was added in the expansion step applied to $A \in \overline{Q}(\sigma^{\downarrow}, \omega^{\downarrow})$ and set $\pi_{i+1}(\sigma', \omega) = \overline{Q}(d', \omega^{\downarrow})$;
C2	If $\sigma \in A^{\mathfrak{I}_i, \omega}$ and $A \equiv \mathbf{E}\Diamond B \in \mathcal{T}$, then add $\omega' = \omega \cdot v$ to W_i, where $v = \mathsf{wit}(\sigma^{\downarrow}, \omega^{\downarrow}, B)$; set $\pi_{i+1}(\sigma', \omega') = \overline{Q}(\sigma'^{\downarrow}, \omega'^{\downarrow})$ for all $\sigma' \in \Delta_i$ such that $\overline{Q}(\sigma'^{\downarrow}, \omega'^{\downarrow})$ defined;
C3	For some $\sigma \in \Delta_i, \omega \in W_i$, add $\langle \sigma, \omega \rangle$ to W_{i+1} and put $(\omega, \langle \sigma, \omega \rangle) \in E_{i+1}$.

Figure 8: Induction step rules

Note that the application of rules **C1**-**C3** leaves π_i undefined for some $(\sigma, \omega) \in \Delta_i \times W_i$. For defining π_i also for those pairs, we proceed as follows. We say that σ *is introduced in* ω if σ is of the form $\sigma' \cdot (\_, \_, \omega)$. Note that for every σ, the ω where it is introduced is uniquely determined. For some $\sigma \in \Delta_i, \omega \in W_i$ with $\pi_i(\sigma, \omega)$ undefined, $d = \sigma^{\downarrow}$, $w = \omega^{\downarrow}$, and ω' the world where σ was introduced in, we put:

- if $w \in \overline{W}$ and $(\omega, \omega') \in E_i^+$, then $\pi_i(\sigma, \omega) = \overline{Q}_{\mathbf{E}\Diamond}(B)$ where B is the concept name due to which d was created;
- if $w \in \overline{W}$ and $(\omega', \omega) \in E_i^+$, then $\pi_i(\sigma, \omega) = \overline{Q}_{\mathbf{A}\Box}(d, \omega'^{\downarrow})$;
- if $w \in \overline{W}$ and $(\omega', \omega) \notin E_i^*$ and $(\omega, \omega') \notin E_i^*$, then $\pi_i(\sigma, \omega) = \overline{Q}_{\mathsf{cert}}(d)$;
- if $w = \langle \sigma', \omega' \rangle$ and $(\omega, \omega') \in E_i^*$ then $\pi_i(\sigma, \omega) = \overline{Q}_{\mathbf{E}\Diamond}(B)$, where B is the concept name due to which d was created;
- if $w = \langle \sigma', \omega' \rangle$ and $(\omega', \omega) \in E_i^+$, then $\pi_i(\sigma, \omega) = \overline{Q}_{\mathbf{A}\Box}(d, \omega'^{\downarrow})$;
- if $w = \langle \sigma', \omega' \rangle$ and $(\omega', \omega) \notin E_i^*$ and $(\omega, \omega') \notin E_i^*$, then $\pi_i(\sigma, \omega) = \overline{Q}_{\mathsf{cert}}(d)$.

The desired temporal interpretation $\mathfrak{I} = (\Delta, W, (\mathcal{I}_w)_{w \in W})$ is obtained in the limit. We first put

$$W = \bigcup_{i \geq 0} W_i, E = \bigcup_{i \geq 0} E_i, \Delta = \bigcup_{i \geq 0} \Delta_i, \pi = \bigcup_{i \geq 0} \pi_i.$$

Then, the interpretation of role and concept names is as follows:

$$r^{\mathfrak{I},\omega} = \{(\sigma, \sigma \cdot (r, d, w)) \mid \sigma, \sigma \cdot (r, d, w) \in \Delta\};$$
$$A^{\mathfrak{I},\omega} = \{\sigma \mid A \in \pi(\sigma, \omega)\};$$

It is not hard to verify the following:

Claim 3. The constructed $\mathfrak{I}$ is a model of $\mathcal{T}$.

As $(r, d_0, w_0) \in A_0^{\mathfrak{I},w_0}$ but $(r, d_0, w_0) \notin B_0^{\mathfrak{I},w_0}$, this finishes the proof of the Theorem. ❑