Lightweight Description Logics and Branching Time: a Troublesome Marriage

Víctor Gutiérrez-Basulto
Universität Bremen, Germany
University of Liverpool, UK
victor@informatik.uni-bremen.de

Jean Christoph Jung and **Thomas Schneider**
Universität Bremen, Germany
{jeanjung, tschneider}@informatik.uni-bremen.de

Abstract

We study branching-time temporal description logics (BTDLs) based on the temporal logic CTL in the presence of rigid (*time-invariant*) roles and general TBoxes. There is evidence that, if full CTL is combined with the classical $\mathcal{ALC}$ in this way, reasoning becomes undecidable. In this paper, we begin by substantiating this claim, establishing undecidability for a fragment of this combination. In view of this negative result, we then investigate BTDLs that emerge from combining fragments of CTL with lightweight DLs from the $\mathcal{EL}$ and *DL-Lite* families. We show that even rather inexpressive BTDLs based on $\mathcal{EL}$ exhibit very high complexity. Most notably, we identify two *convex* fragments which are undecidable and hard for non-elementary time, respectively. For BTDLs based on *DL-Lite*$^{\mathcal{N}}_{bool}$, we obtain tight complexity bounds that range from PSPACE to EXPTIME.

1 Introduction

Classical description logics (DLs) such as those that underlie the W3C recommendation OWL, are decidable fragments of first order logic and aim at the representation of and reasoning about *static* knowledge. With the objective of enhancing DLs with means to capture temporal aspects of knowledge, over the last 20 years much effort has been spent on the study of temporal description logics (TDLs) as discussed in detail in the surveys (Artale and Franconi 2000; Lutz, Wolter, and Zakharyaschev 2008) and the references therein. TDLs are useful for applications requiring to describe dynamic aspects of the application domain. For example, in the medical domain the description of a term might refer to temporal patterns; hence a medical ontology like SNOMED CT (Baader, Ghilardi, and Lutz 2008) should model the knowledge that, for example, the disease *diabetes* may potentially lead to several ocular disorders in the future.

A prominent approach to TDLs, following Schild's original proposal (Schild 1993), is to combine classical DLs with the standard temporal logics LTL, CTL and CTL* based on a two-dimensional product-like semantics in the style of many-dimensional DLs (Gabbay et al. 2003). In the construction of temporal DLs, besides the logics used in the temporal and DL component, there are a number of further design choices. For example, we can choose whether we apply temporal operators to concepts, roles or TBoxes. We can moreover define some DL roles or concepts as rigid, meaning that they will not change their interpretation over time: for example, since every human has the same genetic disorders during their life, the relation "has genetic disorder" between humans and diseases should be modelled by a rigid role hasGeneticDisease. Consequently, it has been argued that applications of TDLs require rigid roles and general TBoxes; e.g., in temporal data modeling (Artale et al. 2012) or in medical ontologies such as SNOMED CT.

Remarkably, the landscape of the expressivity and computational complexity of linear-time TDLs based on LTL (LTDLs) is well-understood today (Artale et al. 2007; Baader, Ghilardi, and Lutz 2008; Franconi and Toman 2011; Artale et al. 2012). In particular, various design choices have been explored, and both lightweight and expressive DLs have been considered. Notably, it has been shown that LTDLs based on the basic DL $\mathcal{ALC}$ become undecidable in the presence of a general TBox as soon as temporal operators are applied not only to concepts but also to roles, or rigid roles are allowed.

It is interesting to note that, more generally, combinations of DLs and modal logics allowing for rigid roles and general TBoxes tend to have very high computational complexity. Indeed, the standard reasoning problem for DLs is concept satisfiability with respect to a TBox. Via the well-known correspondence of DL combinations with product modal logics (PMLs) (Gabbay et al. 2003, Chapters 3, 14), this reasoning problem relates to the global consequence problem in a product modal logic. For many PMLs, global consequence is known to be undecidable (Gabbay et al. 2003).

In the context of LTDLs, with the aim to attain decidability in the presence of rigid roles and general TBoxes, recent work investigates lightweight DLs from the $\mathcal{EL}$ and *DL-Lite* families as the DL component (Artale et al. 2007; 2012). This choice is supported by the fact that lightweight DLs typically allow for tractable reasoning, and motivated by their importance in applications such as ontology-based data access or the representation of huge bio- and medical ontologies (Baader, Brandt, and Lutz 2005; Artale et al. 2009). The results of this work on LTDLs are two-fold: the studied LTDLs based on *DL-Lite* are decidable while those based on $\mathcal{EL}$ are undecidable. Undecidability in the $\mathcal{EL}$ case is explained by an interaction of two phenomena: the first is

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non-convexity – roughly speaking, the ability to simulate disjunctions, here using properties of linear-time, as $\Diamond \mathsf{X} \sqcap \Diamond \mathsf{Y}$ implies that one of $\Diamond(\mathsf{X} \sqcap \Diamond \mathsf{Y})$ and $\Diamond(\mathsf{Y} \sqcap \Diamond \mathsf{X})$ is true, where $\Diamond \mathsf{X}$ reads as *eventually* in the future X holds. The second is the known fact that $\mathcal{EL}$ with disjunctions can encode $\mathcal{ALC}$; hence TDLs based on $\mathcal{EL}$ are as hard as the corresponding $\mathcal{ALC}$ variant, which is undecidable in the case of LTDLs.

From the viewpoint of some DL applications, it has been argued that representing the existence of different *possible futures*, inherent to a branching-time structure, is necessary for a more appropriate modeling (Gutiérrez-Basulto, Jung, and Lutz 2012); for example, in medical ontologies where a symptom or disease might evolve in different ways in the future. For instance consider the statement *'each patient having the autoimmune disorder diabetes will* possibly *develop glaucoma in the future'*. In LTDLs this can be modeled as

$$\mathsf{Patient} \sqcap \exists \mathsf{hasAutoDis.Diabetes} \sqsubseteq \Diamond \exists \mathsf{develops.Glaucoma},$$

meaning that each diabetes patient will *eventually* develop glaucoma, thus excluding the possibility that a glaucoma will never be developed if diabetes is controlled. In branching-time TDLs we can use the existential path quantifier $\mathbf{E}$ together with $\Diamond$ to more carefully state that there exists a *possible future* where the patient develops glaucoma, leaving open the existence of other possible futures.

Interestingly, opposite to LTDLs, the study of TDLs based on branching-time temporal logics (BTDLs) has been rather limited; mainly focusing on decidability boundaries obtained in the context of first-order branching temporal logic (Hodkinson, Wolter, and Zakharyaschev 2002; Bauer et al. 2004). Only recently tight elementary bounds were presented for combinations with the DLs $\mathcal{ALC}$ and $\mathcal{EL}$ in the case where only *local* roles are allowed (Gutiérrez-Basulto, Jung, and Lutz 2012). Hence, the aim of this paper is to deepen the study of BTDLs by considering rigid roles, that is, we study the case where temporal operators are applied to concepts, and rigid roles and general TBoxes are allowed. In particular, we will analyze the decidability and complexity of the standard reasoning problems. In the light of the known results from the literature, we will focus on BTDLs based on lightweight DLs from the $\mathcal{EL}$ and *DL-Lite* families, but we will lay the groundwork by reconsidering $\mathcal{ALC}$.

Our study starts with the combination of CTL and $\mathcal{ALC}$. We show that its fragment with only the $\mathbf{E}\Diamond$, $\mathbf{A}\Box$ operators is undecidable, combining results for products of 'transitive' modal logics with a well-known DL-technique for reasoning about transitive roles (Tobies 2001). Notably, we obtain a more general result, covering a whole family of PMLs.

We continue by investigating several BTDLs based on $\mathcal{EL}$. We identify non-convex BTDLs based on $\mathcal{EL}$ and thus show undecidability of their subsumption problem similarly to the results described above. As a next step, we focus on identifying convex BTDLs based on $\mathcal{EL}$. Studying them is motivated by the fact that convex logics have a good chance to be computationally well-behaved: examples of convex combinations of DLs with modal logics are the tractable combination of $\mathcal{EL}$ with the CTL operator $\mathbf{E}\Diamond$ in the case where only *local* roles are allowed (Gutiérrez-Basulto, Jung, and Lutz 2012), and various combinations of the modal logic $\mathsf{S5}$ with $\mathcal{EL}$, which have been shown to be easier than the $\mathcal{ALC}$ variant (Lutz and Schröder 2010; Gutiérrez-Basulto et al. 2011). For BTDLs, we indeed identify three convex fragments, based on the temporal operator sets $\{\mathbf{E}\bigcirc\}$, $\{\mathbf{E}\Diamond\}$, and $\{\mathbf{E}\Diamond, \mathbf{A}\Box\}$. Surprisingly, we then show that these fragments exhibit very high complexity: undecidable and hard for non-elementary for the $\mathbf{E}\bigcirc$ and $\mathbf{E}\Diamond$ cases, respectively. The proofs of these results are challenging and technically involved because the absence of disjunction makes it difficult to use standard techniques for establishing lower complexity bounds, such as tilings.

Finally, we investigate BTDLs based on $\textit{DL-Lite}^{\mathcal{N}}_{bool}$, showing that the technique developed by Artale et al. (2012) to prove decidability of LTDLs based on *DL-Lite* is robust in the sense that it can be carried over to *most* BTDLs based on CTL. We obtain tight complexity bounds, namely PSPACE- and EXPTIME-completeness.

Throughout the paper, we will omit most technical proof details which, together with additional technical notation, can be found in the long version available online: http://tinyurl.com/kr14tdl

2 Preliminaries

We introduce $\text{CTL}_{\mathcal{ALC}}$, a temporal DL based on the classical DL $\mathcal{ALC}$. The other temporal DLs studied in this paper are fragments or variants of $\text{CTL}_{\mathcal{ALC}}$, introduced later as needed. Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be countably infinite sets of *concept names* and *role names*, respectively. We assume that $\mathsf{N_R}$ is partitioned into two countably infinite sets $\mathsf{N_{rig}}$ and $\mathsf{N_{loc}}$ of *rigid role names* and *local role names*, respectively. $\text{CTL}_{\mathcal{ALC}}$*-concepts* C are defined by the following grammar

$$C := \top \mid A \mid \neg C \mid C \sqcap D \mid \exists r.C \mid \mathbf{E}\bigcirc C \mid \mathbf{E}\Box C \mid \mathbf{E}(C \mathcal{U} D)$$

where A ranges over $\mathsf{N_C}$, r over $\mathsf{N_R}$. As usual, we use $\bot$ to abbreviate the concept $\neg\top$, $C \sqcup D$ for $\neg(\neg C \sqcap \neg D)$, and $\forall r.C$ for $\neg\exists r.\neg C$. We additionally have the following temporal abbreviations (Clarke, Grumberg, and Peled 1999): $\mathbf{E}\Diamond C{=}\mathbf{E}(\top \mathcal{U} C)$, $\mathbf{A}\Box C{=}\neg\mathbf{E}\Diamond\neg C$, $\mathbf{A}\Diamond C{=}\neg\mathbf{E}\Box\neg C$, $\mathbf{A}(C \,\mathcal{U}\, D) = \neg\mathbf{E}(\neg D \,\mathcal{U}\, (\neg C \wedge \neg D)) \wedge \neg\mathbf{E}\Box\neg D$. We moreover consider the *strict* $(\cdot^{<})$ versions of the temporal operators: $\mathbf{E}\Box^{<} C = \mathbf{E}\bigcirc\mathbf{E}\Box C$, $\mathbf{E}(C \mathcal{U}^{<} D) = \mathbf{E}\bigcirc\mathbf{E}(C \mathcal{U} D)$.

The semantics of classical, non-temporal DLs, such as $\mathcal{ALC}$ is given in terms of *interpretations* of the form $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$, where Δ is a non-empty set called the *domain* and $\cdot^{\mathcal{I}}$ is an *interpretation function* that maps each $A \in \mathsf{N_C}$ to a subset $A^{\mathcal{I}} \subseteq \Delta$ and each $r \in \mathsf{N_R}$ to a binary relation $r^{\mathcal{I}} \subseteq \Delta \times \Delta$. The semantics of $\text{CTL}_{\mathcal{ALC}}$ is given in terms of temporal interpretations, which are infinite trees in which every node is associated with a classical interpretation. For the purposes of this paper, a *tree* is a directed graph $T = (W, E)$ where $W \subseteq (\mathbb{N} \setminus \{0\})^*$ is a prefix-closed non-empty set of *nodes* and $E = \{(w, wc) \mid wc \in W, w \in \mathbb{N}^*, c \in \mathbb{N}\}$ a set of *edges*; we generally assume that $wc \in W$ and $c' < c$ implies $wc' \in W$ and that E is a total relation. The node $\varepsilon \in W$ is the *root* of T. For brevity and since E can be reconstructed from W, we will usually identify T with W. In the context of temporal DLs we refer to nodes of T as *time points or worlds*.

A *temporal interpretation* is a structure $\mathfrak{I} = (\Delta, T, \{\mathcal{I}_w\}_{w\in W})$ where $T = (W, E)$ is a tree, and for each $w \in W$, $\mathcal{I}_w$ is an interpretation with domain Δ, such that $r^{\mathcal{I}_w} = r^{\mathcal{I}_{w'}}$ for all $r \in \mathsf{N}_{\mathsf{rig}}$ and $w, w' \in W$. We usually write $A^{\mathfrak{I},w}$ instead of $A^{\mathcal{I}_w}$, and intuitively $d \in A^{\mathfrak{I},w}$ means that in the interpretation $\mathfrak{I}$, the object d is an instance of the concept name A *at time point* w. Moreover, note that we make the *constant domain assumption*, that is, all time points share the same domain Δ. Intuitively, this means that objects are not created or destroyed over time.

We now define the semantics of $\text{CTL}_{\mathcal{ALC}}$-concepts. A *path* in a tree $T = (W, E)$ starting at a node w is a minimal set $\pi \subseteq W$ such that $w \in \pi$ and for each $w' \in \pi$, there is a $c \in \mathbb{N}$ with $w'c \in \pi$. We use $\mathsf{Paths}(w)$ to denote the set of all paths starting at the node w. For a path $\pi = w_0 w_1 w_1 \cdots$ and $i \geq 0$, we use $\pi[i]$ to denote w_i. The mapping $\cdot^{\mathfrak{I},w}$ is extended from concept names to $\text{CTL}_{\mathcal{ALC}}$-concepts as follows:

$$
\begin{aligned}
\top^{\mathfrak{I},w} = \Delta; \qquad (C \sqcap D)^{\mathfrak{I},w} &= C^{\mathfrak{I},w} \cap D^{\mathfrak{I},w};\\
(\exists r.C)^{\mathfrak{I},w} &= \{d \in \Delta \mid \exists e.(d,e) \in r^{\mathfrak{I},w} \wedge e \in C^{\mathfrak{I},w}\};\\
(\mathbf{E}\bigcirc C)^{\mathfrak{I},w} &= \{d \in \Delta \mid d \in C^{\mathfrak{I},\pi[1]} \text{ for some } \pi \in \mathsf{Paths}(w)\};\\
(\mathbf{E}\square C)^{\mathfrak{I},w} &= \{d \in \Delta \mid \forall j \geq 0. d \in C^{\mathfrak{I},\pi[j]} \text{ for some } \pi \in \mathsf{Paths}(w)\};\\
(\mathbf{E}(C\mathcal{U}D))^{\mathfrak{I},w} &= \{d \in \Delta \mid \exists j \geq 0.(d \in D^{\mathfrak{I},\pi[j]} \wedge (\forall 0 \leq k < j.\, d \in C^{\mathfrak{I},\pi[k]})) \text{ for some } \pi \in \mathsf{Paths}(w)\}.
\end{aligned}
$$

A *general $\text{CTL}_{\mathcal{ALC}}$-TBox $\mathcal{T}$* is a finite set of *concept inclusions (CIs)* $C \sqsubseteq D$ with C, D $\text{CTL}_{\mathcal{ALC}}$ concepts.

A temporal interpretation $\mathfrak{I}$ is a *model* of a concept C if $C^{\mathfrak{I},\varepsilon} \neq \emptyset$; it is a *model* of $C \sqsubseteq D$, written $\mathfrak{I} \models C \sqsubseteq D$, if $C^{\mathfrak{I},w} \subseteq D^{\mathfrak{I},w}$ for all $w \in W$; it is a model of a TBox $\mathcal{T}$ if it satisfies all CIs in $\mathcal{T}$. Thus, a TBox $\mathcal{T}$ is interpreted globally in the sense that it has to be satisfied at *every* time point.

We say that a concept C is *satisfiable with respect to a $\text{CTL}_{\mathcal{ALC}}$-TBox $\mathcal{T}$* if there is a common model of C and $\mathcal{T}$. We then consider the following *concept satisfiability problem*: given a concept C and a TBox $\mathcal{T}$, decide whether there is a common model of C and $\mathcal{T}$.

We will denote fragments of $\text{CTL}_{\mathcal{ALC}}$ by putting the available temporal operators as a superscript; for example, $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\square}_{\mathcal{ALC}}$ denotes the fragment having as the only operators available $\mathbf{E}\Diamond$ and $\mathbf{A}\square$.

3 Undecidability of CTL combined with $\mathcal{ALC}$

We begin our study by looking at $\text{CTL}_{\mathcal{ALC}}$. To our knowledge, there are no decidability or complexity results specifically for this combination in the literature. However, there are two results about logics that are close to $\text{CTL}_{\mathcal{ALC}}$: On the one hand, it follows from results by (Hodkinson, Wolter, and Zakharyaschev 2002) that a combination of the two-variable guarded fragment of first-order logic (which subsumes $\mathcal{ALC}$) with CTL is undecidable, but that logic is more expressive than our $\text{CTL}_{\mathcal{ALC}}$: for example, it allows for CIs in the scope of temporal operators. On the other hand, there is a close correspondence between $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\square}_{\mathcal{ALC}}$ and the PML $\mathsf{S4} \times \mathsf{K}$, whose classical satisfiability problem is decidable (Gabbay et al. 2003), though hard for non-elementary time (Göller, Jung, and Lohrey 2012). This correspondence is based on the fact that $\mathsf{S4} \times \mathsf{K}$ has the tree model property and thus $\mathsf{S4} \times \mathsf{K}$ is the logic of all bi-modal Kripke frames (relational structures) that are the product of the reflexive and transitive closure of a tree ($\mathsf{S4}$) and a tree (K). Therefore, in $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\square}_{\mathcal{ALC}}$ with one rigid role and no further roles, the CTL branching time structures correspond to the $\mathsf{S4}$-dimension and $\mathcal{ALC}$ corresponds to K.

The correspondence with PMLs, however, is only of restricted use for our study of the concept satisfiability problem *relative to global TBoxes:* this problem corresponds to the global consequence problem in modal logic which, to the best of our knowledge, has not been studied for $\mathsf{S4} \times \mathsf{K}$. Therefore, the above result for classical satisfiability cannot easily be transferred to $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\square}_{\mathcal{ALC}}$. We will in fact show that concept satisfiability relative to global TBoxes in $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\square}_{\mathcal{ALC}}$ is undecidable, exploiting undecidability of satisfiability for a family of transitive PMLs which includes $\mathsf{S4} \times \mathsf{K4}$ (Gabelaia et al. 2005). We prove that the global consequence problem of $\mathsf{S4} \times \mathsf{K}$ can encode the satisfiability problem of $\mathsf{S4} \times \mathsf{K4}$, using a technique by Tobies (2001).

We first rephrase a consequence of Gabelaia et al.'s result in BTDL notation and will then sketch how to employ Tobies's technique to establish undecidability of $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\square}_{\mathcal{ALC}}$.

Theorem 1 (Gabelaia et al. 2005) *The following problem is undecidable for $\mathcal{L} \in \{\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\square}_{\mathcal{ALC}}, \text{CTL}^{\mathbf{E}\Diamond^<,\mathbf{A}\square^<}_{\mathcal{ALC}}\}$. Given an $\mathcal{L}$-concept C that uses at most one rigid role r and no other roles, is C satisfiable in a temporal interpretation $\mathfrak{I} = (\Delta, T, \{\mathcal{I}_w\}_{w\in W})$ where $r^{\mathfrak{I},w}$ is transitive for all w ?*

We sketch how to employ Tobies's technique to transfer this result to our standard concept satisfiability problem for $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\square}_{\mathcal{ALC}}$ and $\text{CTL}^{\mathbf{E}\Diamond^<,\mathbf{A}\square^<}_{\mathcal{ALC}}$.

Theorem 2 *Concept satisfiability w.r.t. general TBoxes for $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\square}_{\mathcal{ALC}}$ and $\text{CTL}^{\mathbf{E}\Diamond^<,\mathbf{A}\square^<}_{\mathcal{ALC}}$ with rigid roles is undecidable.*

Proof. We only sketch the proof for $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\square}_{\mathcal{ALC}}$; the strict variant can be treated using the same arguments. Consider a $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\square}_{\mathcal{ALC}}$-concept C that uses at most one rigid role r and no other roles, and assume that C is in negation normal form (NNF), i.e., negation occurs only in front of atomic concepts in C. We use $\mathsf{cl}(C)$ to denote the set of NNFs of all subconcepts of C and of their negations, and $\sim D$ to denote the NNF of $\neg D$ for any $D \in \mathsf{cl}(C)$. For every $\forall r.D \in \mathsf{cl}(C)$, reserve a fresh concept name X_D. We define a translation $\cdot^{\mathsf{tr}}$ from $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\square}_{\mathcal{ALC}}$-concepts to $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\square}_{\mathcal{ALC}}$-concepts.

$$
\begin{aligned}
A^{\mathsf{tr}} &= A^{\mathsf{tr}} & (\forall r.D)^{\mathsf{tr}} &= X_D\\
(\neg A)^{\mathsf{tr}} &= \neg A^{\mathsf{tr}} & (\exists r.D)^{\mathsf{tr}} &= \neg X_{\sim D}\\
(D_1 \sqcap D_2)^{\mathsf{tr}} &= D_1^{\mathsf{tr}} \sqcap D_2^{\mathsf{tr}} & (\mathbf{E}\Diamond D)^{\mathsf{tr}} &= \mathbf{E}\Diamond D\\
(D_1 \sqcup D_2)^{\mathsf{tr}} &= D_1^{\mathsf{tr}} \sqcup D_2^{\mathsf{tr}} & (\mathbf{A}\square D)^{\mathsf{tr}} &= \mathbf{A}\square D
\end{aligned}
$$

Let $\mathcal{T}_C$ be the following TBox.

$$\mathcal{T}_C = \{X_D \equiv \forall r.D^{\mathsf{tr}},\ X_D \sqsubseteq \forall r.X_D \mid \forall r.D \in \mathsf{cl}(C)\}$$

It suffices to show that C is satisfiable in an interpretation that makes r transitive everywhere iff C^{tr} is satisfiable w.r.t. $\mathcal{T}_C$ in $\mathrm{CTL}_{\mathcal{ALC}}^{\mathbf{E}\Diamond,\mathbf{A}\Box}$. For the "⇒" direction, take a model $\mathfrak{I}$ of C with r being transitive, and additionally interpret X_D as $\forall r.D$. Then a standard induction shows that the modified $\mathfrak{I}$ satisfies C^{tr}, and transitivity of r ensures that it is a model of $\mathcal{T}_C$. For the "⇐" direction, take a model $\mathfrak{J}$ of C^{tr} and $\mathcal{T}_C$ and extend the interpretation of r in every world to its transitive closure. Then $\mathfrak{J}$ being a model of $\mathcal{T}_C$ can be used to show that the modified $\mathfrak{J}$ still satisfies C. ❑

In fact, we are able to prove a more general result of independent interest in the field of PMLs. All notation used in the following theorem is standard for PMLs (Gabbay et al. 2003), except for $\mathfrak{C}^+$, which refers to the class of all transitive frames in the frame class $\mathfrak{C}$.

Theorem 3 *For any classes $\mathfrak{C}_1, \mathfrak{C}_2$ of frames, where $\mathfrak{C}_2$ contains only transitive frames and both $\mathfrak{C}_1^+, \mathfrak{C}_2$ contain frames of arbitrarily large finite or infinite depth, the global satisfiability problem for $\mathsf{Log}(\mathfrak{C}_1 \times \mathfrak{C}_2)$ is undecidable.*

4 CTL combined with $\mathcal{EL}$

In this section, we consider fragments of $\mathrm{CTL}_{\mathcal{EL}}$, the fragment of $\mathrm{CTL}_{\mathcal{ALC}}$ that disallows the constructor $\neg$ (and thus also the abbreviations $C \sqcup D$, $\forall r.C$, $\mathbf{A}\Box, \ldots$). As an example, consider the following $\mathrm{CTL}_{\mathcal{EL}}$-TBox:

$$\exists \mathsf{hasAutoDis}.\mathsf{Diabetes} \sqsubseteq \mathbf{E}\Diamond(\exists \mathsf{develops}.\mathsf{Glaucoma} \sqcap \mathbf{E}\Diamond\exists \mathsf{hasTreat}.\mathsf{EyeDrops})$$

$$\mathbf{E}\Diamond\exists \mathsf{hasTreat}.\mathsf{EyeDrops} \sqsubseteq \mathbf{E}\Diamond\exists \mathsf{hasEffect}.\mathsf{ChangeEyeColor}$$

It says that everyone with diabetes may eventually develop a glaucoma which may eventually be treated using eye drops, and that such a treatment may lead to a change in eye color.

Because of the absence of negation, satisfiability in $\mathrm{CTL}_{\mathcal{EL}}$ is trivial; as in non-temporal $\mathcal{EL}$, we therefore consider subsumption as the central reasoning problem. Formally, a concept D *subsumes* a concept C w.r.t. a $\mathrm{CTL}_{\mathcal{EL}}$ TBox $\mathcal{T}$, written $\mathcal{T} \models C \sqsubseteq D$, if $C^{\mathfrak{I}} \subseteq D^{\mathfrak{I}}$ for all temporal interpretations $\mathfrak{I}$ that are a model of $\mathcal{T}$. For example, the TBox above implies that $\exists \mathsf{hasAutoDis}.\mathsf{Diabetes} \sqsubseteq \mathbf{E}\Diamond\exists \mathsf{hasEffect}.\mathsf{ChangeEyeColor}$. We then consider the following *concept subsumption problem:* given concepts C, D and a $\mathrm{CTL}_{\mathcal{EL}}$ TBox $\mathcal{T}$, decide whether $\mathcal{T} \models C \sqsubseteq D$.

In the rest of the section we focus on the study of several fragments of $\mathrm{CTL}_{\mathcal{EL}}$. In this context, we view each of the operators $\mathbf{E}\bigcirc$, $\mathbf{E}\Diamond$, $\mathbf{E}\Box$, $\mathbf{A}\Diamond$, $\mathbf{A}\Box$, $\mathbf{E}\mathcal{U}$ and their strict versions as primitive instead of as an abbreviation.

4.1 Non-Convex Fragments

We next show undecidability of certain fragments of $\mathrm{CTL}_{\mathcal{EL}}$ using techniques developed in the context of *linear-time* temporal extensions of $\mathcal{EL}$ (Artale et al. 2007). In particular, the non-convexity of these extensions, and therefore their capability to 'express' disjunction, is exploited to show that they are as complex as the corresponding $\mathcal{ALC}$ variant.

Before we present the undecidability results we recall the standard notion of convexity.

Definition 1 *A logic $\mathcal{L}$ is called* convex *if, for all $\mathcal{L}$-TBoxes $\mathcal{T}$ and all $\mathcal{L}$-concepts $C, D_1, \ldots, D_n$, $n \geq 2$, whenever $\mathcal{T} \models C \sqsubseteq D_1 \sqcup \cdots \sqcup D_n$, then $\mathcal{T} \models C \sqsubseteq D_i$ for some $i \in \{1, \ldots, n\}$.*

Theorem 4 *Concept subsumption w.r.t. general TBoxes is undecidable for*

(a) $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond,\mathbf{A}\Diamond}$	(b) $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond,\mathbf{E}\bigcirc}$	(c) $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\mathcal{U}}$
(d) $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\mathcal{U}^<}$	(e) $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond^<,\mathbf{E}\Box^<}$	(f) $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond,\mathbf{E}\Box}$

Proof. The proof is in two steps. First, we show the non-convexity (in opposition to convexity) of these fragments of $\mathrm{CTL}_{\mathcal{EL}}$. Second, we reduce from satisfiability w.r.t. TBoxes in the corresponding $\mathrm{CTL}_{\mathcal{ALC}}$ fragment.

Lemma 5 *The fragments* (a)–(f) *of $\mathrm{CTL}_{\mathcal{EL}}$ are non-convex.*

Proof. The non-convexity of (a)–(d) was previously established in (Gutiérrez-Basulto, Jung, and Lutz 2012), for the witnesses see Figure 1. In the same place, the fragments (e) and (f) were stated to be convex, too; however, we show here that this is not the case. Interestingly, this shows that the convexity of $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$, which we show later (Theorem 6), is rather fragile since $\mathbf{E}\Diamond$ and $\mathbf{E}\Box$ allow only for existential quantification and there is no possibility of talking about 'next' time points, which in principle might make us think of $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond,\mathbf{E}\Box}$ as a temporal analog of $\mathcal{EL}$. Alas, the presence of both operators makes it non-convex. We begin by presenting the witnesses used to prove non-convexity and their intuition.

To show non-convexity of $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond,\mathbf{E}\Box}$ we use the following witness.

$$\mathcal{T} = \{C \sqsubseteq \mathbf{E}\Diamond(D \sqcap A),\ D \sqsubseteq \mathbf{E}\Diamond(C \sqcap A)\} \text{ and }$$
$$D_1 = \mathbf{E}\Diamond(C \sqcap D),\ D_2 = \mathbf{E}\Box\mathbf{E}\Diamond A.$$

Intuitively, $\mathcal{T}$ stipulates that every instance d of C has the possibility of eventually being an instance of D and A, and thereafter has the possibility of eventually being an instance of C and A, but there is a choice on whether d becomes an instance of C and D at the same time point, or whether d infinitely alternates between C and D.

Although the disjunction expressed in $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond,\mathbf{E}\Box}$ relies on the non-strict interpretation of $\mathbf{E}\Diamond$ and $\mathbf{E}\Box$, the following shows that the strict version is also non-convex.

$$\mathcal{T} = \{C \sqsubseteq \mathbf{E}\Diamond^< B,\ B \sqsubseteq D \sqcap \mathbf{E}\Box^< D\} \text{ and }$$
$$D_1 = \mathbf{E}\Diamond^<\mathbf{E}\Diamond^< B,\ D_2 = \mathbf{E}\Box^< D$$

Intuitively, every instance d of C has two choices on becoming an instance of B for the first time, namely, either it does it at the immediate successor, or at some point after it.

Formally, in both cases we show non-convexity by proving that $\mathcal{T} \models C \sqsubseteq \bigsqcup D_i$, but $\mathcal{T} \not\models C \sqsubseteq D_i$. ❑

Having established the non-convexity of the above logics we can devise a reduction from the satisfiability problem w.r.t. TBoxes in the corresponding $\mathcal{ALC}$ variant. First, recall that $\mathrm{CTL}_{\mathcal{ALC}}^{\mathbf{E}\Diamond,\mathbf{A}\Box}$ and $\mathrm{CTL}_{\mathcal{ALC}}^{\mathbf{E}\Diamond^<,\mathbf{A}\Box^<}$ are undecidable (Theorem 2). Note moreover that all fragments above contain at least the $\mathbf{E}\Diamond$ or $\mathbf{E}\Diamond^<$ operator. Hence all the $\mathcal{ALC}$ variants of these fragments are undecidable.

$\mathbf{E}\Diamond\mathbf{A}\Diamond$	$\mathcal{T}=\emptyset,\ C=\mathbf{A}\Diamond A\sqcap\mathbf{A}\Diamond B$ $D_1=\mathbf{E}\Diamond(A\sqcap\mathbf{E}\Diamond B),\ D_2=\mathbf{E}\Diamond(B\sqcap\mathbf{E}\Diamond A)$
$\mathbf{E}\Diamond\mathbf{E}\bigcirc$	$\mathcal{T}=\emptyset,\ C=\mathbf{E}\Diamond A,\ D_1=A,\ D_2=\mathbf{E}\bigcirc\mathbf{E}\Diamond A$
$\mathbf{E}\mathcal{U}$	$\mathcal{T}=\emptyset,\ C=\mathbf{E}(A\mathcal{U}B),\ D_1=B,\ D_2=A$
$\mathbf{E}\mathcal{U}^<$	$\mathcal{T}=\emptyset,\ C=\mathbf{E}(A\mathcal{U}^<B),\ D_1=\mathbf{E}(A\mathcal{U}^<A),$ $D_2=\mathbf{E}(B\mathcal{U}^<B)$

Figure 1: Non-convexity witnesses

The crucial point of the reduction is the elimination of disjunction, that is, encoding CIs $C\sqsubseteq A\sqcup B$ present in the $\mathcal{ALC}$ variants. To this aim, we use the non-convexity witnesses (*cf.* witnesses presented above and Fig. 1) which, as discussed, allow to describe 'choices'. ❏

4.2 Convex Fragments

In this section, we will show that the logics $\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$, $\text{CTL}^{\mathbf{E}\Diamond}_{\mathcal{EL}}$, and $\text{CTL}^{\mathbf{E}\Diamond\mathbf{A}\Box}_{\mathcal{EL}}$ are convex, which suggests that they have limited expressive power. Indeed, convex logics are typically computationally well-behaved (Oppen 1980; Baader, Brandt, and Lutz 2005; Lutz and Wolter 2012). Our convexity result is contrasted by two observations: first, the linear-time TDL based on $\Diamond$ and $\mathcal{EL}$ with rigid roles is non-convex (Artale et al. 2007) which, intuitively, is due to the linear structure of time. Second, we will show later that $\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$ is undecidable because its expressive power is sufficient to encode the halting problem for two-counter machines, which is equivalent to that of Turing machines.

Theorem 6 *$\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$, $\text{CTL}^{\mathbf{E}\Diamond}_{\mathcal{EL}}$, and $\text{CTL}^{\mathbf{E}\Diamond,\mathbf{A}\Box}_{\mathcal{EL}}$ are convex.*

Proof. (sketch) We start with $\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$ and first observe that $\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$, if restricted to a single role that is rigid, is a notational variant of a fragment of the product modal logic $\mathsf{KD}\times\mathsf{K}$, where K is the basic (uni-)modal logic and KD is the modal logic of the class of frames where every state has a successor. We can therefore use standard results and machinery from product modal logics – namely unravelling tolerance, the standard translation to first-order logic, and first-order axiomatizations of product frames – to achieve two things: we will translate $\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$-TBoxes and -concepts into a fragment of first-order logic that generalizes the Horn fragment, and we will axiomatize the rigidity of roles and the properties of the temporal successor relation in the same fragment. This will allow us to encode logical consequence in $\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$, even with multiple non-rigid roles, as first-order formulas in this fragment. Using the fact that such formulas are preserved under a standard operation on structures called *direct product*, it will then be easy to establish convexity.

In the following, we sketch some details of this argument. Consider a $\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$-TBox $\mathcal{T}$ and $\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$-concepts $C,D_1,\ldots,D_n$, $n\geq 2$. Let $r_1,\ldots,r_k$ be the rigid roles occurring in $\mathcal{T},C,D_1,\ldots,D_n$, and let $r_{k+1},\ldots,r_\ell$ be the non-rigid roles. We denote by $\mathsf{ST}(\mathcal{T})$ the standard translation of $\mathcal{T}$ into first-order logic. Assuming $\mathcal{T}$ to be in a normal form that extends the usual one for $\mathcal{EL}$ (cf. Baader, Brandt, and Lutz 2005), it is easy to see that $\mathsf{ST}(\mathcal{T})$ is equivalent to a conjunction of embedded implicational dependencies (EIDs) (Fagin 1982), i.e., to a conjunction of FO-sentences

$$\forall x_1\ldots x_m.((E_1\wedge\cdots\wedge E_n)\rightarrow\exists x_{m+1}\ldots x_k.(F_1\wedge\cdots\wedge F_\ell))$$

where each E_i is a relational formula $Px_{j_1}\ldots x_{j_d}$ and each F_i is either a relational formula or an equality $x_j=x_{j'}$.

Furthermore, we need to express rigidity of the roles $r_1,\ldots,r_k$ as well as the property that the temporal "direct successor" relation is a total tree. For rigidity, we borrow from the theory of product modal logics (Gabbay et al. 2003): we have to say that each pair (R_i,S) of binary predicates representing the rigid role r_i and the temporal successor relation is left-commutative, right-commutative and satisfies the Church-Rosser property. For example, left-commutativity is given by the following EID.

$$\mathsf{LC}=\textstyle\bigwedge_{i=1}^{k}\forall xyz(R_ixy\wedge Syz\rightarrow\exists u(Sxu\wedge R_iuz))$$

The other two properties can be expressed via analogous EIDs RC and CRP. For enforcing the total tree, we only need to say that S is total because standard translations of $\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$-TBoxes are preserved under unravelling and under taking point-generated substructures. Totality of S can be expressed by the EID $\mathsf{D}=\forall x\exists y(Sxy)$.

We now consider the conjunction

$$\varphi(\mathcal{T})=\mathsf{LC}\wedge\mathsf{RC}\wedge\mathsf{CRP}\wedge\mathsf{D}\wedge\mathsf{ST}(\mathcal{T})$$

which, by classical results (Chang and Keisler 1990; Fagin 1982) is preserved under *direct products*: If n structures $\mathcal{M}_1,\ldots,\mathcal{M}_n$ satisfy an EID, then so does their direct product, which has as its domain the cross-product of the domains of $\mathcal{M}_1,\ldots,\mathcal{M}_n$ and interprets every m-ary predicate P as the set of m-tuples whose projections to the domain of each $\mathcal{M}_i$ are in the interpretation of P.

In order to show convexity of $\text{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$ via the contrapositive, we observe that the statement

$$\mathcal{T}\not\models C\sqsubseteq D_i \text{ for all } i \quad\text{implies}\quad \mathcal{T}\not\models C\sqsubseteq D_1\sqcup\cdots\sqcup D_n$$

is equivalent to

$$\varphi(\mathcal{T})\not\models\forall x.(Cx\rightarrow D_ix)\text{ for all } i\quad\text{implies}$$
$$\varphi(\mathcal{T})\not\models\forall x.(Cx\rightarrow D_1x\vee\cdots\vee D_nx).$$

The claim of this property can be established easily, consulting the direct product of the n structures witnessing the non-entailments in the hypotheses.

To carry this reasoning over to $\text{CTL}^{\mathbf{E}\Diamond}_{\mathcal{EL}}$ and $\text{CTL}^{\mathbf{E}\Diamond\mathbf{A}\Box}_{\mathcal{EL}}$, we replace totality in the above argument with reflexivity and transitivity. Consequently, we have to replace the conjunct D in $\varphi(\mathcal{T})$ with the conjunction of the EIDs $\mathsf{T}=\forall x(xR_hx)$ and $\mathsf{4}=\forall xyz(xR_hy\wedge yR_hz\rightarrow xR_hz)$. Unfortunately, $\mathsf{ST}(\mathcal{T})$ now contains translations of axioms containing $\mathbf{A}\Box$, which are not generally equivalent to EIDs. Using additional arguments, however, we can show that $\mathsf{ST}(\mathcal{T})$ for these two logics is then preserved under direct products *of structures where the relation S representing the temporal successor relation is interpreted as a total relation*. Therefore, the above $\varphi(\mathcal{T})$ remains preserved under direct products of such structures. Since $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ and $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond\mathbf{A}\Box}$ are restricted to such structures, the above argument goes through.

It is worth recalling that convexity of both $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ and $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ does not imply that $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc,\mathbf{E}\Diamond}$ is convex: in order to translate a TBox using both the $\mathbf{E}\bigcirc$ and the $\mathbf{E}\Diamond$ operator into FO, we would have to use two binary relations $S_{\mathbf{E}\bigcirc}$ and $S_{\mathbf{E}\Diamond}$ and state that $S_{\mathbf{E}\Diamond}$ is the reflexive transitive closure of $S_{\mathbf{E}\bigcirc}$, which is not expressible in FO. ❑

We continue our study by showing that $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ lacks the *finite model property (FMP)*: consider the TBox $\mathcal{T}$ consisting of the following concept inclusions

$$A \sqsubseteq \exists r.A, \quad A \sqsubseteq \mathbf{E}\bigcirc B, \quad \exists r.B \sqsubseteq B',$$
$$\exists r.B' \sqsubseteq B', \qquad \mathbf{E}\bigcirc(B \sqcap B') \sqsubseteq C, \qquad \exists r.C \sqsubseteq C,$$

with r a rigid role. It is not hard to see that $\mathcal{T} \not\models A \sqsubseteq C$ but for every finite model $\mathfrak{I}$ of $\mathcal{T}$, $\mathfrak{I} \models A \sqsubseteq C$. In fact, we show that this also holds if the two occurrences of $\mathbf{E}\bigcirc$ in $\mathcal{T}$ are replaced by $\mathbf{E}\Diamond$. Therefore, the following holds.

Theorem 7 *$\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ and $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ lack the FMP.*

Interestingly, in the same way the TBox presented above enforces a temporal tree with infinite branching degree, relying on rigid roles.

We next study the complexity of subsumption relative to general TBoxes in $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ and $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$. In Theorem 2, we already showed that the $\mathcal{ALC}$ variant of the latter is undecidable. For the former, observe that the corresponding $\mathcal{ALC}$-version is a notational variant of the product logic $\mathsf{K} \times \mathsf{K}$, for which the global consequence problem (and thus reasoning relative to global TBoxes) is known to be undecidable.

We will prove undecidability for $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ and a nonelementary lower bound for $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$, thus showing that even the convex fragments have a high complexity. These high complexity results are surprising because convex logics often exhibit better computational properties. Specifically, the lack of expressiveness prohibits reductions that were previously used to show undecidability particularly for product modal logics, for instance reductions from tiling problems: without disjunction we cannot express that in some point of the grid one of the tile types has to be true. However, we show how to exploit rigidity of the roles for proving these results. Let us remark that the proven lower bounds remain valid when we adopt the *expanding domain assumption* (Gabbay et al. 2003); in particular, the same proofs work.

4.3 Undecidability with the Next Operator

We proceed to show undecidability of the combination of $\mathcal{EL}$ with $\mathbf{E}\bigcirc$.

Theorem 8 *Concept subsumption w.r.t. general $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$-TBoxes is undecidable.*

We present a reduction from the reachability problem for *two-counter machines (TCM)*. A TCM is a pair (Q, P) with states $Q = \{0, \ldots, n\}$, where 0 is the *initial state* and n is the *halting state*, and corresponding instructions $P = \{I_0, \ldots, I_{n-1}\}$. Additionally, there are two *registers* 1 and 2. For all $i < n$, we have that

- either $I_i = \mathsf{inc}(c, j)$ is an *incrementation instruction* with $c \in \{1, 2\}$ a register and j the subsequent state;
- or $I_i = \mathsf{dec}(c, j, k)$ is a *decrementation instruction* with $c \in \{1, 2\}$ a register and j the subsequent state if register c contains value 0 and k the subsequent state otherwise.

A *configuration* is a triple (i, n, m) where i is the current state and n and m are the contents of registers 1 and 2, respectively. We write $(i, n_1, n_2) \Rightarrow_M (j, m_1, m_2)$ if one of the following holds:

- $I_i = \mathsf{inc}(c, j)$, $m_c = n_c + 1$ and $m_{\overline{c}} = n_{\overline{c}}$ (where $\overline{c} = 3 - c$);
- $I_i = \mathsf{dec}(c, j, k)$, $m_c = n_c = 0$ and $m_{\overline{c}} = n_{\overline{c}}$;
- $I_i = \mathsf{dec}(c, k, j)$, $n_c > 0$, $m_c = n_c - 1$, and $m_{\overline{c}} = n_{\overline{c}}$.

The *computation of M* is the unique longest configuration sequence $(p_0, n_0, m_0) \Rightarrow_M (p_1, n_1, m_1) \Rightarrow_M \ldots$ such that $p_0 = n_0 = m_0 = 0$. The *halting problem for TCM*, i.e., to decide whether a given TCM M reaches state n, is undecidable (Minsky 1967).

For our purposes, it is convenient to define a variant of the halting problem. Fix some TCM $M = (Q, I)$ with $Q = \{0, \ldots, n\}$. An *extended state (e-state) x for M* is either $x = q$ for some $q \in Q$ with I_q an increment instruction or a pair $x = (q, f)$ with $f \in \{0, +\}$ and I_q a decrement instruction. Denote with $\overline{x}$ the state q if $x = q \in Q$ or $x = (q, f)$. A finite sequence of e-states $x_0, \ldots, x_k$ is called *halting* if $x_k = n$ and there are numbers $n_0, m_0, \ldots, n_k, m_k$ such that $x_0 = n_0 = m_0 = 0$ and $(\overline{x_i}, n_i, m_i) \Rightarrow_M (\overline{x_{i+1}}, n_{i+1}, m_{i+1})$ for all i, and additionally, if $x_i = (q, 0)$ and $I_q = \mathsf{dec}(c, \cdot, \cdot)$ then either $c = 1$ and $n_i = 0$ or $c = 2$ and $m_i = 0$, and if $x_i = (q, +)$ and $I_q = \mathsf{dec}(c, \cdot, \cdot)$, then either $c = 1$ and $n_i > 0$ or $c = 2$ and $m_i > 0$. It is easy to verify that M halts iff there is a halting sequence of e-states.

For the reduction let M be as above and assume without loss of generality that I_0 is an inc-instruction. Moreover, define a set of rigid roles, consisting of one role for each e-state:

$$\mathsf{ROL} = \{r_i \mid I_i = \mathsf{inc}(\cdot, \cdot)\} \cup \{r_{i0}, r_{i+} \mid I_i = \mathsf{dec}(\cdot, \cdot, \cdot)\}.$$

Finally, introduce concept names $Q_0, \ldots, Q_n$ representing the states of M, for each $c \in \{1, 2\}$ concepts C_c marking elements that contribute to the value in register c and concepts Tail_c and Head_c that mark the bounds of register c.

We construct a $\text{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$-TBox $\mathcal{T}_M$ and identify two concept names A, B such that:

Lemma 9 *$\mathcal{T}_M \models A \sqsubseteq B$ iff there is a halting sequence of e-states.*

The idea is (i) to generate all sequences of e-states for M using the roles in ROL and (ii) to check if one of them is halting. Point (i) is done using the concept inclusions

$$A \sqsubseteq \exists r_0.(S \sqcap A), \qquad A \sqsubseteq \exists r.A \text{ for all } r \in \mathsf{ROL} \setminus \{r_0\}$$

where the first inclusion additionally ensures that after creating an r_0 successor, some special concept name S is satisfied. Intuitively, domain elements satisfying A in some world stipulate an infinite tree where the path to each node corresponds to *the reverse of a sequence of e-states*. For example, the branch $r_{3+}r_{20}r_{2+}r_1r_0$ of such a tree corresponds to the sequence $0, 1, (2,+), (2, 0), (3,+)$ of e-states for M.

For achieving Point (ii), we simulate the behavior of M *backwards* over the tree mentioned above. A configuration (i, n_1, n_2) of M will be represented in the temporal dimension by a time point that has a (time) successor for each register: one successor labeled with Head_1 and another with Head_2. Both successors have some (not necessarily direct) successor that is labeled with Tail_c and all worlds between Tail_c and Head_c are labeled with C_c. Further, the distance between Tail_c and Head_c is precisely the value n_c of the register c. Moreover, all these worlds are labeled with the current state Q_i.

Note that a valid computation of M always starts in state 0 and that by the above CI all corresponding nodes are labeled with S. Thus, we can use

$$S \sqsubseteq \mathbf{E}\bigcirc\textstyle\bigsqcap_{c\in\{1,2\}}\mathbf{E}\bigcirc(\mathsf{Head}_c \sqcap \mathsf{Tail}_c \sqcap Q_0)$$

in order to enforce (the encoding of) an initial configuration of M starting in every node labeled with S. The begin of the counter never changes along a computation:

$$\exists r.\mathsf{Head}_c \sqsubseteq \mathsf{Head}_c \text{ for all } r \in \mathsf{ROL}, c \in \{1, 2\}.$$

It remains to describe how the ends Tail_c, the states Q_i, and the intermediate labels C_c are updated according to the instructions I. For instructions $I_i = \mathsf{inc}(c, j)$, we add the inclusions

$$\begin{aligned}
\exists r_i.(Q_i \sqcap \mathsf{Tail}_c) &\sqsubseteq Q_j \sqcap C_c \sqcap \mathbf{E}\bigcirc(Q_j \sqcap \mathsf{Tail}_c)\\
\exists r_i.(Q_i \sqcap C_c) &\sqsubseteq Q_j \sqcap C_c\\
\exists r_i.(Q_i \sqcap \mathsf{Tail}_{\overline{c}}) &\sqsubseteq Q_j \sqcap \mathsf{Tail}_{\overline{c}}\\
\exists r_i.(Q_i \sqcap C_{\overline{c}}) &\sqsubseteq Q_j \sqcap C_{\overline{c}}
\end{aligned}$$

For instructions $I_i = \mathsf{dec}(c, j, k)$, add the inclusions

$$\begin{aligned}
\exists r_{i0}.(Q_i \sqcap \mathsf{Head}_c \sqcap \mathsf{Tail}_c) &\sqsubseteq Q_j \sqcap \mathsf{Tail}_c\\
\exists r_{i0}.(Q_i \sqcap C_{\overline{c}}) &\sqsubseteq Q_j \sqcap C_{\overline{c}}\\
\exists r_{i0}.(Q_i \sqcap \mathsf{Tail}_{\overline{c}}) &\sqsubseteq Q_j \sqcap \mathsf{Tail}_{\overline{c}}\\
\exists r_{i+}.(Q_i \sqcap C_c \sqcap \mathbf{E}\bigcirc\mathsf{Tail}_c) &\sqsubseteq Q_k \sqcap \mathsf{Tail}_c\\
\exists r_{i+}.(Q_i \sqcap C_c \sqcap \mathbf{E}\bigcirc C_c) &\sqsubseteq Q_k \sqcap C_c\\
\exists r_{i+}.(Q_i \sqcap C_{\overline{c}}) &\sqsubseteq Q_k \sqcap C_{\overline{c}}\\
\exists r_{i+}.(Q_i \sqcap \mathsf{Tail}_{\overline{c}}) &\sqsubseteq Q_k \sqcap \mathsf{Tail}_{\overline{c}}
\end{aligned}$$

Note that in order to correctly update a configuration along some role r_i or r_{if}, $f \in \{0,+\}$, it is required that (1) the state of the configuration is in fact i, i.e., labeled with Q_i, and (2) for decrementation instructions $\mathsf{dec}(c, \cdot, \cdot)$ the mentioned conditions are satisfied. For example, if $f = 0$, then the value currently encoded in register c has to be zero. If this is not the case, none of the above CIs for register c can be applied; thus, no information is propagated for register c and the sequence under consideration does not correspond to a valid computation.

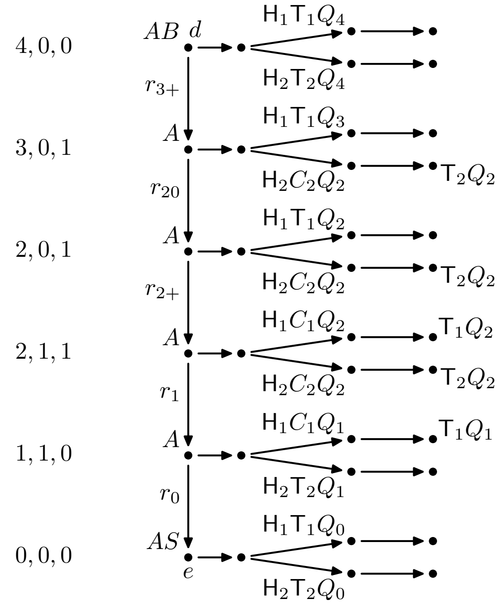


Figure 2: Example computation for $\mathcal{T}_M$.

Finally, a halting configuration can be detected by checking whether the encoding of some configuration has label Q_n in both registers. This is expressed by the CI

$$\mathbf{E}\bigcirc(\mathbf{E}\bigcirc(\mathsf{Head}_1 \sqcap Q_n) \sqcap \mathbf{E}\bigcirc(\mathsf{Head}_2 \sqcap Q_n)) \sqsubseteq B$$

in order to label accepting configurations with B.

Example 1 *Let $M = (Q, I)$ be a TCM with $Q = \{0, \ldots, 4\}$ and $I_0 = \mathsf{inc}(1, 1)$, $I_1 = \mathsf{inc}(2, 2)$, and $I_2 = \mathsf{dec}(1, 3, 2)$, $I_3 = \mathsf{dec}(2, 2, 4)$. It is easy to check that M halts and that the witnessing sequence of e-states is $0, 1, (2,+), (2, 0), (3,+), 4$. The construction of $\mathcal{T}_M$ implies that for any instance d of A, there is another domain element that is reachable via the sequence of roles $r_{3+}, r_{20}, r_{2+}, r_1, r_0$. This sequence is depicted in Figure 2. Note that by $\mathcal{T}_M$, e is labeled with S and starting from e we find the encoding of the initial configuration of M (the temporal relation is drawn horizontally and* H, T *abbreviate* $\mathsf{Head}, \mathsf{Tail}$*). $\mathcal{T}_M$ induces the configurations from bottom to top. The encoding of the configurations is found in the right side of the picture; the corresponding configurations are depicted on the left side. It can be seen that the top-most configuration is actually halting and enforces finally label B in point d. Note that only the consequences of* one *sequence of e-states are depicted. In particular, in a full model, all instances of A are also instances of B.*

4.4 Hardness for Non-Elementary Time with the Transitive Future Operator

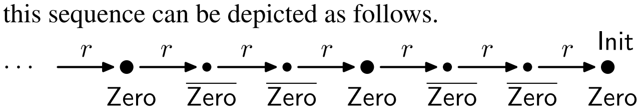
We begin by noting that the reduction presented in the previous section does not easily transfer to the case when we consider the transitive future operator $\mathbf{E}\Diamond$ instead of the next operator $\mathbf{E}\bigcirc$. The main problem is that we cannot update the concept Tail_c for a decrement instruction because the fact *'there is a future with* Tail_c *but no* C_c *until then'* is not expressible. This does not come as a surprise since it is known that the impossibility of talking about next time points brings considerable technical difficulties. For example, for proving lower bounds for products of transitive logics quite intricate techniques are required (Gabelaia et al. 2005).

In this section, we show that also the combination of $\mathbf{E}\Diamond$ with $\mathcal{EL}$ exhibits high computational complexity.

Theorem 10 *Concept subsumption w.r.t. general* $\mathit{CTL}^{\mathbf{E}\Diamond}_{\mathcal{EL}}$*-TBoxes is inherently nonelementary.*

In order to prove the non-elementary lower bound, we use a technique in the style of (Stockmeyer 1974), which has already been used in the context of many-dimensional modal logics; for instance, for concept satisfiability for $\mathrm{LTL}_{\mathcal{ALC}}$ *without* TBoxes (Gabbay et al. 2003). In a nutshell, this technique consists of two steps: the first step is to find some encoding for numbers and show that arbitrarily big elementary numbers can be 'enforced'. In the second step, a standard reduction is used to show, e.g., k-ExpTime-hardness of the problem for every $k \geq 1$.

We will proceed along these lines. For step 1, the encoding of a number n, we will provide a TBox $\mathcal{T}$ such that in any model of $\mathcal{T}$ containing a sequence of domain elements connected via a rigid role r the following holds. If some element of the sequence, say d, is an instance of the concept name Init, then every element that can reach d via $k \cdot n$ steps along r is an instance of a concept name Zero, and all other elements are instances of a concept name $\overline{\mathsf{Zero}}$. For $n = 3$, this sequence can be depicted as follows.



Formally, we say that a TBox $\mathcal{T}$ *counts modulo* n if there exist concept names Init, Zero, $\overline{\mathsf{Zero}}$, Fail, and a rigid role r such that for all $i \geq 0$ there are models $\mathfrak{I}$ of $\mathcal{T}$ and $\exists r^i.\mathsf{Init}$[1] with $\mathsf{Fail}^{\mathfrak{I},w} = \emptyset$ for all worlds w, and in all such models $\mathfrak{I}$ it holds for all $i \geq 0$:

(i) $\mathfrak{I} \models \exists r^i.\mathsf{Init} \sqsubseteq \mathsf{Zero}$ iff $i \equiv 0 \bmod n$;
(ii) $\mathfrak{I} \models \exists r^i.\mathsf{Init} \sqsubseteq \overline{\mathsf{Zero}}$ iff $i \not\equiv 0 \bmod n$.

Intuitively, the concept Fail is necessary as negation is not available; it is used to ensure that complementary concept names behave in a complementary way. The use of Fail is straightforward, so in the proof sketches below we will concentrate on points (i) and (ii).

We next show that there are 'small TBoxes that count modulo large numbers'. For making this precise, let us introduce the function $\mathsf{exp} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$:

$$\mathsf{exp}(1, n) := 2^n; \quad \mathsf{exp}(\ell + 1, n) := \mathsf{exp}(\ell, n) \cdot 2^{\mathsf{exp}(\ell,n)}.$$

Lemma 11 *For each* $k, n \geq 1$*, there is a polynomially sized (in* k, n*)* $\mathit{CTL}^{\mathbf{E}\Diamond}_{\mathcal{EL}}$*-TBox* $\mathcal{T}_{k,n}$ *that counts modulo* $\mathsf{exp}(k, n)$*.*

Proof. (sketch) The proof is by induction on k. For $k = 1$, we describe an $\mathcal{EL}$-TBox that counts modulo 2^n. To this aim, we introduce concept names $X_1, \overline{X_1}, \ldots, X_n, \overline{X_n}$, $C_1, \overline{C_1}, \ldots, C_n, \overline{C_n}$ modeling n bits and carry bits (and their negations), respectively, of a counter and take the CIs

$$\exists r.X_1 \sqsubseteq \overline{X_1} \sqcap C_1 \qquad \exists r.\overline{X_1} \sqsubseteq X_1 \sqcap \overline{C_1}$$
$$C_i \sqcap \exists r.X_{i+1} \sqsubseteq \overline{X_{i+1}} \sqcap C_{i+1}$$
$$C_i \sqcap \exists r.\overline{X_{i+1}} \sqsubseteq X_{i+1} \sqcap \overline{C_{i+1}}$$
$$\overline{C_i} \sqcap \exists r.X_{i+1} \sqsubseteq X_{i+1} \sqcap \overline{C_{i+1}}$$
$$\overline{C_i} \sqcap \exists r.\overline{X_{i+1}} \sqsubseteq \overline{X_{i+1}} \sqcap \overline{C_{i+1}}$$

For initializing the counter and for propagating Fail we include the following CIs for $1 \leq i \leq n$:

$$\mathsf{Init}_1 \sqsubseteq \mathsf{Zero}_1, \qquad \mathsf{Zero}_1 \equiv \overline{X_1} \sqcap \ldots \sqcap \overline{X_n},$$
$$X_i \sqsubseteq \overline{\mathsf{Zero}_1}, \qquad X_i \sqcap \overline{X_i} \sqsubseteq \mathsf{Fail}, \qquad C_i \sqcap \overline{C_i} \sqsubseteq \mathsf{Fail}$$

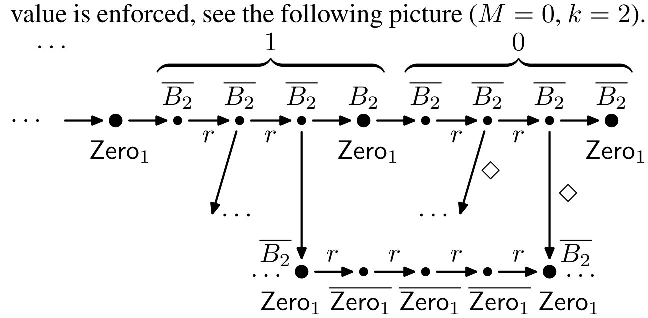
It should be clear that the TBox $\mathcal{T}_{1,n}$ consisting of all these CIs counts modulo 2^n.

For the induction step $k{>}1$, a number is now represented in binary in $\mathsf{exp}(k-1, n)$ consecutive domain elements along the r-chain; we use additional concepts B_k ($\overline{B_k}$) expressing that a bit is set (not set). We proceed 'inductively':

(1) If $\mathsf{Init_k}$ is satisfied in some domain element (in some world), then enforce that the next $\mathsf{exp}(k-1, n)$ domain elements satisfy $\overline{B_k}$; this encodes the number 0.
(2) If a sequence of $\mathsf{exp}(k-1, n)$ domain elements encodes some number, say M, then the preceding sequence of $\mathsf{exp}(k-1, n)$ domain elements is enforced to encode the number $M + 1$.

Clearly, for both steps (1) and (2) it is required to determine $\mathsf{exp}(k-1, n)$ consecutive elements, which is possible due to the induction hypothesis. In particular, for step (2) we mark the begin of the encoding of some number using $\mathcal{T}_{k-1,n}$: a new number begins whenever the concept $\mathsf{Zero_{k-1}}$ is satisfied. Also by induction hypothesis, we can mark all domain elements that are *not* the begin of some number.

For the actual incrementation necessary in Point (2), it suffices to communicate between domain elements that have distance (again: along r) precisely $\mathsf{exp}(k-1, n)$ as these domain elements represent the same bit positions in two successive numbers. This is realized by enforcing in each element of the chain a possible future where the bit of this element is stored. In the domain element having precisely distance $\mathsf{exp}(k-1, n)$, this bit is accessed and the correct value is enforced, see the following picture ($M = 0$, $k = 2$).



It remains to mark the elements of the sequence with Zero_k and $\overline{\mathsf{Zero_k}}$, respectively. First, note that $\mathsf{Zero_k}$ can hold only

[1] $\exists r^i$ abbreviates sequences of $\exists r$ with length i.

at the begin of the encoding of some number. So in all 'non-begin' elements we imply $\overline{\mathsf{Zero_k}}$. For distinguishing the begin elements it suffices to detect the encoding of a number where all $\mathsf{exp}(k-1, n)$ bits are set to 1. Then, starting from the next domain element we will find the encoding of a 0, and we mark its begin with $\mathsf{Zero_k}$. ❏

In the second step we employ the TBoxes $\mathcal{T}_{k,n}$ to reduce the word-problem for deterministic Turing machines with k-fold exponential space restriction to subsumption in $\text{CTL}^{\mathbf{E}\Diamond}_{\mathcal{EL}}$. Theorem 10 is then a direct corollary.

Theorem 12 *Concept subsumption w.r.t. general* $\text{CTL}^{\mathbf{E}\Diamond}_{\mathcal{EL}}$*-TBoxes is* k-EXPSPACE*-hard for every* $k \geq 1$.

Proof. (sketch) Fix any $k \geq 1$ and let M be a deterministic Turing machine which runs within $\mathsf{exp}(k, p(n))$ space for inputs of length n for some polynomial $p(n)$. Let w be some input of length n and set $N := p(n)$. By Lemma 11, there is a TBox $\mathcal{T}_{k,N}$ that counts modulo $\mathsf{exp}(k, N)$ and whose size is polynomial in k, N and hence also polynomial in k, n.

For the reduction, we first include the CI $A \sqsubseteq \exists r.A$ that generates an infinite and rigid r-chain. For every domain element satisfying A, we enforce a possible future that simulates M (again backwards) along the r-chain: $A \sqsubseteq \mathbf{E}\Diamond\mathsf{Init}_M$. A configuration of M is represented by a sequence of $\mathsf{exp}(k, N)$ domain elements connected via r using concept names for every symbol from the tape alphabet, for every state, and for the head. The borders between two configurations are marked using $\mathcal{T}_{k,N}$. We proceed inductively along the lines of the proof of Lemma 11:

(1') enforce the initial configuration of M (including w);

(2') if a sequence of $\mathsf{exp}(k, N)$ domain elements between two borders encodes a configuration of M, the preceding sequence of $\mathsf{exp}(k, N)$ domain elements is enforced to encode the successor configuration.

Obviously, step (1') can be done using a polynomial TBox starting from the concept Init_M. For enforcing tape symbols in step (2') it suffices to communicate between domain elements that have an r-distance of precisely $\mathsf{exp}(k, N)$. Finally, for updating the head position, we might have to communicate between elements having distance $\mathsf{exp}(k, N) \pm 1$. The previous ideas can be easily adapted to this case.

It remains to add the CI $\mathbf{E}\Diamond Q_{acc} \sqsubseteq B$ in order to detect an accepting configuration. It is routine to show that M accepts w iff the constructed TBox implies $A \sqsubseteq B$. ❏

5 Decidability of CTL with *DL-Lite*$^{\mathcal{N}}_{bool}$

In this section, we show that the technique developed by (Artale et al. 2012) to prove decidability of TDLs based on *linear-time* temporal logic and *DL-Lite* is robust in the sense that it can be carried over to *some* TDLs based on CTL and *DL-Lite*$^{\mathcal{N}}_{bool}$ (Artale et al. 2009).

CTL-*DL-Lite*$^{\mathcal{N}}_{bool}$ *concepts* C, D are defined by the following grammar:

$$\bot \mid A \mid \geq q\, r \mid \neg C \mid C \sqcap D \mid \mathbf{E}\bigcirc C \mid \mathbf{E}\Box C \mid \mathbf{E}(C\mathcal{U}D),$$

where q is a positive integer given in binary, A ranges over concept names and r ranges over $\{r, r^- \mid r \in \mathsf{N_R}\}$.

The semantics for Booleans and temporal operators is defined as in Section 2. We interpret *inverse roles* and *number restrictions* in a temporal interpretation $\mathfrak{I}$ as follows:

$$\begin{aligned} (r^-)^{\mathfrak{I},w} &= \{(d', d) \mid (d, d') \in r^{\mathfrak{I},w}\}; \\ (\geq q\, r)^{\mathfrak{I},w} &= \{d \mid \sharp\{d' \mid (d, d') \in r^{\mathfrak{I},w}\} \geq q\}. \end{aligned}$$

Our following result shows that three fragments of CTL-*DL-Lite*$^{\mathcal{N}}_{bool}$ behave in a similar way as temporal extensions of *DL-Lite*$^{\mathcal{N}}_{bool}$ based on linear time: reasoning in these fragments is not harder than in the component logics.

Theorem 13 *Concept satisfiability w.r.t. general TBoxes is* EXPTIME*-complete*
1. for CTL-*DL-Lite*$^{\mathcal{N}}_{bool}$ *without local roles,*
2. for $\text{CTL}^{\mathbf{E}\mathcal{U},\mathbf{E}\Box}$-*DL-Lite*$^{\mathcal{N}}_{bool}$,
and PSPACE*-complete*
3. for $\text{CTL}^{\mathbf{E}\Diamond}$-*DL-Lite*$^{\mathcal{N}}_{bool}$.

The fragments of points 2,3 do not allow the operator $\mathbf{E}\bigcirc$.

Proof. (sketch) For the upper bound, the proof follows the two-step technique proposed by (Artale et al. 2012). First, we reduce from CTL-*DL-Lite*$^{\mathcal{N}}_{bool}$ to the one-variable fragment $\mathcal{QCTL}^1$ of *first-order branching temporal logic* $\mathcal{QCTL}$ (Hodkinson, Wolter, and Zakharyaschev 2002). Then, the result is further reduced to a $\mathcal{QCTL}^1$-formula *without occurrences of existential quantifiers*, which is essentially a CTL-formula. This way we get the same upper bounds as for the propositional CTL fragments. The lower bounds are inherited from the corresponding fragments (Meier et al. 2009).

The elimination of the quantifiers in the second step does not work for full CTL-*DL-Lite*$^{\mathcal{N}}_{bool}$ because the original shifting technique (Artale et al. 2012, Lemma 4.2) relies on the past being unbounded, which is not the case for the standard CTL semantics. However, the problem can be circumvented by (1) disallowing local roles – which makes shifting superfluous – or (2) restricting the temporal operators to $\mathbf{E}\mathcal{U}$ and $\mathbf{E}\Box$ which are tolerant to a certain variant of unravelling into the temporal direction. Unravelling tolerance is related to stutter invariance (Lamport 1983) and does not apply to $\mathbf{E}\bigcirc$ and therefore to the strict variants $\mathbf{E}\Box^<$ and $\mathbf{E}\mathcal{U}^<$.

We note that the only temporal operators used to encode the TBox and rigidity of roles in the first step are $\mathbf{E}\Diamond$ and $\mathbf{A}\Box$, which are available in $\text{CTL}^{\mathbf{E}\Diamond}$-*DL-Lite*$^{\mathcal{N}}_{bool}$. Hence our translation works for $\text{CTL}^{\mathbf{E}\Diamond}$-*DL-Lite*$^{\mathcal{N}}_{bool}$, too. ❏

6 Conclusions and Future Work

In this paper, we have made progress towards understanding the landscape of the computational complexity and expressivity of BTDLs allowing for rigid roles and general TBoxes. In particular, we showed for some BTDLs based on $\mathcal{EL}$ that we cannot easily conclude undecidability via non-convexity, as in the case of similar LTDLs, since they can be shown to be convex. However, we have been able to squelch the hope that these convex logics are practical because we established a non-elementary lower bound and undecidability, respectively.

The problems left open include determining the precise complexity of $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ and of unrestricted CTL-*DL-Lite*$_{bool}^{\mathcal{N}}$. For the latter, we plan to pursue an automata-based approach in the style of (Gutiérrez-Basulto, Jung, and Lutz 2012).

Our investigation provides several research directions that are worth exploring in the near future – for example, one could consider reasoning in the presence of *acyclic TBoxes* instead of general ones, as usually the computational properties of DLs over acyclic TBoxes are better. Furthermore, complexity might also drop when assuming *expanding domains* instead of constant domains.

For BTDLs based on *DL-Lite*, the investigation can be broadened in two ways: by considering restrictions such as *DL-Lite*$_{core}$ or by studying the effect of adding role inclusions. In fact, LTDLs based on *DL-Lite*$_{core}$ are less complex than their *DL-Lite*$_{bool}$ counterpart (Artale et al. 2012). Another possibility is the study of BTDLs with the more expressive *temporal roles*; e.g., where $\mathbf{E}\Diamond$ and $\mathbf{A}\Box$ are applied to roles. LTDLs with temporal roles are undecidable (Artale et al. 2012).

Finally, we believe that our results, together with the suggested restriction to acyclic and empty TBoxes, can have an impact on the study of PMLs, given the known correspondence between PMLs and combinations of DLs and modal-like logics with rigid roles and empty TBoxes: it seems that PMLs with good computational properties may be found among their *positive fragments*, some of which are notational variants of $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$, $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$and $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond,\mathbf{A}\Box}$. For instance, the satisfiability problems of $\mathsf{K} \times \mathsf{K}$ and $\mathsf{S4} \times \mathsf{K}$ are non-elementary, and there is hope that the complexity of their positive fragments is only elementary.

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Appendix

A Proofs for Section 3

We will first employ Tobies's technique to transfer Gabelaia et al.'s general result into a general undecidability result for the global consequence problem of a family of PMLs. Undecidability of $\text{CTL}^{\mathbf{E}\diamond,\mathbf{A}\Box}_{\mathcal{ALC}}$ with rigid roles will then be a straightforward consequence.

We need the following notation for bimodal product logics, see (Gabbay et al. 2003) for details. Let PROP be a countable set of propositional variables, which we will usually denote by $p_1, p_2, \ldots$. Bimodal formulas are built according to the following grammar, where $p \in \mathsf{PROP}$.

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \boxminus\varphi \mid \boxbar\varphi$$

We use the common abbreviations $\bot = \neg\top$, $\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi)$, $\diamondminus\varphi = \neg\boxminus\neg\varphi$, and $\diamondbar\varphi = \neg\boxbar\neg\varphi$.

The semantics is given in terms of product frames and models.

- A *Kripke frame* is a pair $\mathcal{F} = (W, R)$, where W is a nonempty set and $R \subseteq W \times W$ a binary relation.
- The *product* of two Kripke frames $\mathcal{F} = (W, R)$ and $\mathcal{G} = (X, S)$, written $\mathcal{F} \times \mathcal{G}$, is a triple $(W \times X, R_h, R_v)$ where $(x, y)R_h(x', y')$ iff xRx' and $y = y'$, and $(x, y)R_v(x', y')$ iff $x = x'$ and ySy'. We call $\mathcal{F} \times \mathcal{G}$ a *product frame*.
- An *product model* is a pair $\mathcal{M} = (\mathcal{F}, V)$ where $\mathcal{F} = (W, R_h, R_v)$ is a product frame and V is a *valuation*, i.e., a function $V : \mathsf{PROP} \to 2^W$. We say that $\mathcal{M}$ is *based on* $\mathcal{F}$ and call the pair $(\mathcal{M}, w)$ with $w \in W$ a *pointed model*.
- The satisfaction relation between (pointed) models and formulas is defined as follows.

$$\begin{aligned}
\mathcal{M}, w &\models p \text{ iff } w \in V(p), \quad p \in \mathsf{PROP}\\
\mathcal{M}, w &\models \top\\
\mathcal{M}, w &\models \neg\varphi \text{ iff } \mathcal{M}, w \not\models \varphi\\
\mathcal{M}, w &\models \varphi \wedge \psi \text{ iff } \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi\\
\mathcal{M}, w &\models \boxminus\varphi \text{ iff for all } v \text{ with } wR_hv : \mathcal{M}, v \models \varphi\\
\mathcal{M}, w &\models \boxbar\varphi \text{ iff for all } v \text{ with } wR_vv : \mathcal{M}, v \models \varphi\\
\mathcal{M} &\models \varphi \text{ iff for all } w \in W : \mathcal{M}, w \models \varphi
\end{aligned}$$

- We say that a bimodal formula φ is *satisfiable* (with respect to a class $\mathfrak{C}$ of frames) if there is a pointed model $(\mathcal{M}, w)$ (with $\mathcal{M}$ based on a frame from $\mathfrak{C}$) such that $\mathcal{M}, w \models \varphi$. We further say that φ is *valid* in a frame $\mathcal{F} = (W, R_h, R_v)$ if $\mathcal{M}, w \models \varphi$ for all models $\mathcal{M}$ based on $\mathcal{F}$ and all worlds $w \in W$.

Various PMLs can be defined as the theories of certain classes of product frames: Let $\mathfrak{C}_1, \mathfrak{C}_2$ be classes of Kripke frames. Then $\mathsf{Log}(\mathfrak{C}_1 \times \mathfrak{C}_2)$ is the set of all formulas that are valid in all frames $\mathcal{F}_1 \times \mathcal{F}_2$ with $\mathcal{F}_i \in \mathfrak{C}_i$, $i = 1, 2$. The *satisfiability problem* for $\mathsf{Log}(\mathfrak{C}_1 \times \mathfrak{C}_2)$ is the question whether a given formula φ is satisfiable with respect to $\mathfrak{C}_1 \times \mathfrak{C}_2$ – this is equivalent to asking whether $\neg\varphi \in \mathsf{Log}(\mathfrak{C}_1 \times \mathfrak{C}_2)$. The *global consequence problem* for $\mathsf{Log}(\mathfrak{C}_1 \times \mathfrak{C}_2)$ is the question whether, for given formulas φ, ψ, if $\mathcal{M} \models \varphi$, then $\mathcal{M} \models \psi$, for any model $\mathcal{M}$ based on a frame from $\mathfrak{C}_1 \times \mathfrak{C}_2$.

The following general undecidability result is known for PMLs over transitive product frames. A transitive frame $\mathcal{F} = (W, R)$ has *finite depth* $k \geqslant 0$ if there is a longest path $w_0Rw_1Rw_2R\ldots$ with $w_i \in W$ and *not* $w_{i+1}Rw_i$ which has length k, and *infinite depth* otherwise.

Theorem 14 (Gabelaia et al. 2005) *For any classes $\mathfrak{C}_1, \mathfrak{C}_2$ of transitive frames both containing frames of arbitrarily large finite or infinite depth, the satisfiability problem for $\mathsf{Log}(\mathfrak{C}_1 \times \mathfrak{C}_2)$ is undecidable.*

We want to transfer this result to the global consequence problem, sacrificing transitivity of $\mathfrak{C}_1$. This is achieved by the following lemma, which uses the notation $\mathfrak{C}^+ = \{\mathcal{F} \in \mathfrak{C} \mid \mathcal{F} \text{ is transitive}\}$, for an arbitrary frame class $\mathfrak{C}$.

Lemma 15 *Let $\mathfrak{C}_1, \mathfrak{C}_2$ be classes of frames. Then the satisfiability problem for $\mathsf{Log}(\mathfrak{C}_1 \times \mathfrak{C}_2^+)$ is polytime-reducible to the global consequence problem for $\mathsf{Log}(\mathfrak{C}_1 \times \mathfrak{C}_2)$.*

Proof. Let φ be a bimodal formula, and let $\mathsf{cl}(\varphi)$ be the set of all subformulas of φ. We first define a translation $\cdot^{\mathsf{tr}}$ from bimodal formulas to bimodal formulas as follows, using a fresh atomic proposition x_ψ for every $\boxminus\psi \in \mathsf{cl}(\varphi)$.

$$\begin{aligned}
p^{\mathsf{tr}} &= p \quad (p \in \mathsf{PROP}) & (\psi_1 \wedge \psi_2)^{\mathsf{tr}} &= \psi_1^{\mathsf{tr}} \wedge \psi_2^{\mathsf{tr}}\\
\top^{\mathsf{tr}} &= \top & (\boxminus\psi)^{\mathsf{tr}} &= \boxminus\psi^{\mathsf{tr}}\\
(\neg\psi)^{\mathsf{tr}} &= \neg\psi^{\mathsf{tr}} & (\boxbar\psi)^{\mathsf{tr}} &= x_\psi
\end{aligned}$$

We further set

$$f(\varphi) = \bigwedge_{\boxbar\psi \in \mathsf{cl}(\varphi)} (x_\psi \leftrightarrow \boxbar\psi^{\mathsf{tr}} \;\wedge\; x_\psi \to \boxbar x_\psi).$$

We want to show that φ is satisfiable with respect to $\mathfrak{C}_1 \times \mathfrak{C}_2^+$ if and only if $\neg\varphi^{\mathsf{tr}}$ is *not* a global consequence of $f(\varphi)$ with respect to $\mathfrak{C}_1 \times \mathfrak{C}_2$. This establishes the claim of the lemma.

For the "⇒" direction, let φ be satisfiable in a product model $\mathcal{M} = (\mathcal{F}_h \times \mathcal{F}_v, V)$ with $\mathcal{F}_h = (W_h, R_h) \in \mathfrak{C}_1$ and $\mathcal{F}_v = (W_v, R_v) \in \mathfrak{C}_2^+$ (i.e., R_v is transitive). Hence there is a world $w_0 \in W_h \times W_v$ with $\mathcal{M}, w_0 \models \varphi$. We construct a new product model $\mathcal{M}' = (\mathcal{F}_h \times \mathcal{F}_v, V')$ that globally satisfies $f(\varphi)$ but not $\neg\varphi^{\mathsf{tr}}$. We set

$$\begin{aligned}
V'(x_\psi) &= \{w \in W_h \times W_v \mid \mathcal{M}, w \models \boxbar\psi\} && \text{for all } x_\psi,\\
V'(p) &= V(p) && \text{for all other } p \in \mathsf{PROP}.
\end{aligned}$$

The following claim can be proven by a straightforward induction on the structure of ψ.

Claim 1 *For all $\psi \in \mathsf{cl}(\varphi)$ and all $w \in W_h \times W_v$, we have that $\mathcal{M}, w \models \psi$ iff $\mathcal{M}', w \models \psi^{\mathsf{tr}}$.*

Indeed, the Boolean and atomic cases, as well as the case $\psi = \boxminus\vartheta$, are obvious. For the case $\psi = \boxbar\vartheta$, we argue that $\mathcal{M}, w \models \boxbar\vartheta$ is equivalent to $\mathcal{M}', w \models x_\vartheta$, (due to the construction of V'), and $x_\vartheta = (\boxbar\vartheta)^{\mathsf{tr}}$.

It remains to show $\mathcal{M} \models f(\varphi)$ and $\mathcal{M} \not\models \neg\varphi^{\mathsf{tr}}$. For $\mathcal{M} \models f(\varphi)$, we consider the two conjuncts for each $\boxbar\psi \in \mathsf{cl}(\varphi)$ separately.

- $\mathcal{M}' \models x_\psi \leftrightarrow \boxplus\psi^{\mathsf{tr}}$:

$$\begin{aligned}\mathcal{M}', w \models x_\psi &\Leftrightarrow \mathcal{M}, w \models \boxplus\psi\\ &\Leftrightarrow \forall v(wR_v v \Rightarrow \mathcal{M}, v \models \psi)\\ &\Leftrightarrow \forall v(wR_v v \Rightarrow \mathcal{M}', v \models \psi^{\mathsf{tr}})\\ &\Leftrightarrow \mathcal{M}', w \models \boxplus\psi^{\mathsf{tr}}\end{aligned}$$

The first equivalence is due to the definition of $V'(x_\psi)$, and the third is via the previous claim.

- $\mathcal{M}' \models x_\psi \to \boxplus x_\psi$:

$$\begin{aligned}\mathcal{M}', w \models x_\psi &\Leftrightarrow \mathcal{M}, w \models \boxplus\psi\\ &\Rightarrow \mathcal{M}, w \models \boxplus\boxplus\psi\\ &\Leftrightarrow \forall v(wR_v v \Rightarrow \mathcal{M}, v \models \boxplus\psi)\\ &\Leftrightarrow \forall v(wR_v v \Rightarrow \mathcal{M}', v \models x_\psi)\\ &\Leftrightarrow \mathcal{M}', w \models \boxplus x_\psi\end{aligned}$$

The first equivalence is due to the definition of $V'(x_\psi)$, the implication holds because R_v is transitive, and the second-last equivalence is due to the definition of $\cdot^{\mathsf{tr}}$ and the previous claim.

The remaining claim $\mathcal{M} \not\models \neg\varphi^{\mathsf{tr}}$ follows from $\mathcal{M}, w_0 \models \varphi$ which, by the previous claim, implies $\mathcal{M}', w_0 \models \varphi^{\mathsf{tr}}$.

For the "$\Leftarrow$" direction, assume that $\neg\varphi^{\mathsf{tr}}$ is not *not* a global consequence of $f(\varphi)$ with respect to $\mathfrak{C}_1 \times \mathfrak{C}_2$. That is, there is a product model $\mathcal{M} = (\mathcal{F}_h \times \mathcal{F}_v, V)$ with $\mathcal{F}_h = (W_h, R_h) \in \mathfrak{C}_1$ and $\mathcal{F}_v = (W_v, R_v) \in \mathfrak{C}_2$ such that $\mathcal{M} \models f(\varphi)$ and, for some world $w_0 \in W_h \times W_v$, $\mathcal{M}, w_0 \models \varphi^{\mathsf{tr}}$. We construct a new product model $\mathcal{M}' = (\mathcal{F}_h \times \mathcal{F}_v^+, V')$ that satisfies φ. We set

$$\begin{aligned}&\mathcal{F}_v^+ = (W_v, R_v^+) &&(R_v^+ \text{ is the transitive closure of } R_v),\\ &V'(p) = V(p) &&\text{for all } p \in \mathsf{PROP}.\end{aligned}$$

Satisfiability of φ in $\mathcal{M}'$ now follows from $\mathcal{M}, w_0 \models \varphi^{\mathsf{tr}}$ and the following claim.

Claim 2 *For all $\psi \in \mathsf{cl}(\varphi)$ and all $w \in W_h \times W_v$, we have that $\mathcal{M}, w \models \psi^{\mathsf{tr}}$ iff $\mathcal{M}', w \models \psi$.*

To prove the claim, we proceed by induction on ψ. The Boolean and atomic cases, as well as the case $\psi = \boxminus\vartheta$, are obvious. For the case $\psi = \boxplus\vartheta$, we treat both directions separately.

For "$\Rightarrow$", we argue:

$$\begin{aligned}\mathcal{M}, w \models (\boxplus\vartheta)^{\mathsf{tr}} &\Rightarrow \forall v(wR_v^+ v \Rightarrow \mathcal{M}', v \models \vartheta)\\ &\Leftrightarrow \mathcal{M}, w \models x_\vartheta\\ &\Leftrightarrow \mathcal{M}', w \models \boxplus\vartheta\end{aligned}$$

The first equivalence is due to the definition of $\cdot^{\mathsf{tr}}$. The implication in the center is justified as follows. Take an arbitrary v with wR_v^+v. Then there are $u_1, \dots, u_n \in W_v \times W_h$ with $u_1 = w$, $u_n = v$, and $u_i R_v u_{i+1}$ for all $i = 1, \dots, n-1$. Now $\mathcal{M}, w \models x_\vartheta$ means that $\mathcal{M}, u_1 \models x_\vartheta$. Since $\mathcal{M}$ globally satisfies the second conjunct of $f(\varphi)$, this implies $\mathcal{M}, u_2 \models x_\vartheta$. We can iterate this argument $n-2$ more times and obtain $\mathcal{M}, u_{n-1} \models x_\vartheta$. Since $\mathcal{M}$ globally satisfies the first conjunct of $f(\varphi)$, we conclude $\mathcal{M}, u_n \models \vartheta^{\mathsf{tr}}$. Hence, due to the induction hypothesis and because $u_n = v$, we get $\mathcal{M}', v \models \vartheta$.

For "$\Leftarrow$", we argue:

$$\begin{aligned}\mathcal{M}', w \models \boxplus\vartheta &\Leftrightarrow \forall v(wR_v^+ v \Rightarrow \mathcal{M}', v \models \vartheta)\\ &\Rightarrow \forall v(wR_v v \Rightarrow \mathcal{M}', v \models \vartheta)\\ &\Leftrightarrow \forall v(wR_v v \Rightarrow \mathcal{M}, v \models \vartheta^{\mathsf{tr}})\\ &\Leftrightarrow \mathcal{M}, w \models \boxplus\vartheta^{\mathsf{tr}}\\ &\Leftrightarrow \mathcal{M}, w \models x_\vartheta\\ &\Leftrightarrow \mathcal{M}, w \models (\boxplus\vartheta)^{\mathsf{tr}}\end{aligned}$$

The implication in line 2 holds because $R_v \subseteq R_v^+$, the subsequent equivalence is due to the induction hypothesis, the second-last equivalence follows from the first conjunct of $f(\varphi)$, and the last equivalence is due to the definition of $\cdot^{\mathsf{tr}}$.

This finishes the proof. ❏

As a consequence of Theorem 14 and Lemma 15, we obtain:

Theorem 3 *For any classes $\mathfrak{C}_1, \mathfrak{C}_2$ of frames, where $\mathfrak{C}_2$ contains only transitive frames and both $\mathfrak{C}_1, \mathfrak{C}_2^+$ contain frames of arbitrarily large finite or infinite depth, the global satisfiability problem for $\mathsf{Log}(\mathfrak{C}_1 \times \mathfrak{C}_2)$ is undecidable.*

The transfer to $\text{CTL}_{\mathcal{ALC}}^{\mathbf{E}\diamond,\mathbf{A}\square}$ with rigid roles and its strict version is now straightforward.

Theorem 2 *Concept satisfiability w.r.t. general TBoxes for $\text{CTL}_{\mathcal{ALC}}^{\mathbf{E}\diamond,\mathbf{A}\square}$ and $\text{CTL}_{\mathcal{ALC}}^{\mathbf{E}\diamond^<,\mathbf{A}\square^<}$ with rigid roles is undecidable.*

Proof. For the non-strict case, we recall that $\text{CTL}_{\mathcal{ALC}}^{\mathbf{E}\diamond,\mathbf{A}\square}$ with a single, rigid, role is essentially a notational variant of $\mathsf{S4} \times \mathsf{K} = \mathsf{Log}(\mathfrak{C}_1 \times \mathfrak{C}_2)$, where $\mathfrak{C}_1$ is the class of all reflexive and transitive frames and $\mathfrak{C}_2$ is the class of all frames. We can therefore reduce the complement of the global consequence problem for $\mathsf{S4} \times \mathsf{K}$ to satisfiability w.r.t. general TBoxes for $\text{CTL}_{\mathcal{ALC}}^{\mathbf{E}\diamond,\mathbf{A}\square}$ with rigid roles if we take care of the subtle difference that CTL-trees are always total, whereas S4-frames may be finite.

We therefore use the following reduction. Let φ, ψ be bimodal formulas. We construct $\text{CTL}_{\mathcal{ALC}}^{\mathbf{E}\diamond,\mathbf{A}\square}$-concepts C_φ, C_ψ from φ, ψ inductively in the usual way, using a rigid role r and atomic concepts C_p for every $p \in \mathsf{PROP}$:

$$\begin{aligned}&C_\top = \top\\ &C_{\neg\vartheta} = \neg C_\vartheta && C_{\boxminus\vartheta} = \mathbf{A}\square C_\vartheta\\ &C_{\vartheta_1 \wedge \vartheta_2} = C_{\vartheta_1} \sqcap C_{\vartheta_2} && C_{\boxplus\vartheta} = \forall r.C_\vartheta\end{aligned}$$

It is now easy to prove that ψ is *not* a global consequence of φ iff $\neg C_\psi$ is satisfiable w.r.t. $\mathcal{T} = \{\top \sqsubseteq C_\varphi\}$: for the "$\Rightarrow$" direction, take an $\mathsf{S4} \times \mathsf{K}$-model $\mathcal{M}$ globally satisfying φ but not ψ. Then the $\mathsf{S4} \times \mathsf{K}$-model obtained from $\mathcal{M}$ by unravelling into the S4-direction and re-inserting the reflexive edges globally satisfies φ but not ψ. This model, viewed as a $\text{CTL}_{\mathcal{ALC}}^{\mathbf{E}\diamond,\mathbf{A}\square}$-interpretation, is a model of $\mathcal{T}$ and $\neg C_\psi$.

For the "$\Leftarrow$" direction, take a $\text{CTL}_{\mathcal{ALC}}^{\mathbf{E}\diamond,\mathbf{A}\square}$-model $\mathcal{I}$ of $\mathcal{T}$ and $\neg C_\psi$. Viewed as a Kripke model, $\mathcal{I}$ globally satisfies φ but not ψ.

For the strict case, we use the same reduction with the only difference that we do not re-insert any reflexive edges in the model construction for the "⇒" direction. ❑

It remains to note that the chain of reductions used in the proofs of Lemma 15 and Theorem 2 produces a general TBox of the form

$$\top \sqsubseteq \bigsqcap_{\psi} (C_{x_\psi} \leftrightarrow \forall r.C_{\psi^{\mathsf{tr}}} \ \wedge\ C_{x_\psi} \rightarrow \forall r.C_{x_\psi}),$$

which does not seem to be equivalent to a terminology. Therefore, we are not able to strengthen the result of Theorem 2 to apply to *cyclic terminologies*.

B Proofs for Section 4.1

The fragment $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond,\mathbf{E}\Box}$ of $\mathrm{CTL}_{\mathcal{EL}}$ is non-convex. Consider the following setting:

$$\mathcal{T} = \{C \sqsubseteq \mathbf{E}\Diamond(D \sqcap A),\ D \sqsubseteq \mathbf{E}\Diamond(C \sqcap A)\} \text{ and}$$

$$D_1 = \mathbf{E}\Diamond(C \sqcap D),\ D_2 = \mathbf{E}\Box\mathbf{E}\Diamond A$$

Lemma 16 *$\mathcal{T} \models C \sqsubseteq \bigsqcup D_i$ but $\mathcal{T} \not\models C \sqsubseteq D_i$ for $1 \leq i \leq 2$.*

Proof. For the former, let $\mathfrak{I}$ be a model of $\mathcal{T}$ and $d \in C^{\mathfrak{I},w}$ for some $w \in W$. Since

$$d \in (\mathbf{E}\Diamond(D \sqcap A))^{\mathfrak{I},w} \text{ and } D \sqsubseteq \mathbf{E}\Diamond(C \sqcap A)$$

there exists a $j \geq 0$ such that $d \in (D \sqcap A \sqcap \mathbf{E}\Diamond(C \sqcap A))^{\mathfrak{I},\pi[j]}$ for some $\pi \in \mathsf{Paths}(w)$. Then, by semantics, there exists a $k \geq j$ such that $d \in (C \sqcap A)^{\mathfrak{I},\pi'[k]}$ for some $\pi' \in \mathsf{Paths}(\pi[j])$. We can distinguish two cases (1) $k = j$ or (2) $k > j$:

- if the first case holds, then $d \in (C \sqcap D \sqcap A)^{\mathfrak{I},\pi[j]}$. Therefore, $d \in \mathbf{E}\Diamond(C \sqcap D)^{\mathfrak{I},\pi[\varepsilon]}$;
- if the second case holds, then $d \in (C \sqcap \mathbf{E}\Diamond(D \sqcap A))^{\mathfrak{I},\pi'[k]}$.

Clearly, if we are in the second case, then the same two cases can be distinguish again. Hence, if always the second case holds, then $d \in (\mathbf{E}\Box\mathbf{E}\Diamond A)^{\mathfrak{I},w}$. Therefore

$$\mathcal{T} \models C \sqsubseteq \mathbf{E}\Diamond(C \sqcap D) \sqcup \mathbf{E}\Box\mathbf{E}\Diamond A.$$

Now, we proceed to prove that $\mathcal{T} \not\models C \sqsubseteq D_i$, for $1 \leq i \leq 2$. We begin by constructing a temporal model $\mathfrak{I} = (\Delta, T, \{\mathcal{I}_w\}_{w \in W})$ of $\mathcal{T}$ with $\Delta = \{d\}$ and T a 1-ary tree with $w_1 = \varepsilon \cdot 1$ such that $d \in C^{\mathfrak{I},\varepsilon}$ and $d \notin D_2^{\mathfrak{I},\varepsilon}$ by setting

$$\begin{array}{ll} C^{\mathfrak{I},\{\varepsilon,w_1\}} := \{d\}; & C^{\mathfrak{I},w} := \emptyset, \text{ for } w \notin \{\varepsilon, w_1\}; \\ A^{\mathfrak{I},\{\varepsilon,w_1\}} := \{d\}; & A^{\mathfrak{I},w} := \emptyset; \text{ for } w \notin \{\varepsilon, w_1\}; \\ D^{\mathfrak{I},w_1} := \{d\}; & D^{\mathfrak{I},w} := \emptyset, \text{ for } w \neq w_1. \end{array}$$

Clearly, $\mathfrak{I}$ is a model of $\mathcal{T}$. Finally, note that $d \notin A^{\mathfrak{I},w}$ for all $w \neq \{\varepsilon, w_i\}$, therefore $d \notin (\mathbf{E}\Box\mathbf{E}\Diamond A)^{\mathfrak{I},\varepsilon}$.

Next, we analogously construct a model $\mathfrak{I} = (\Delta, T, \{\mathcal{I}_w\}_{w \in W})$ of $\mathcal{T}$ with $\Delta = \{d\}$ and T a 1-ary tree such that $d \in C^{\mathfrak{I},\varepsilon}$ and $d \notin D_1^{\mathfrak{I},\varepsilon}$ by setting

$$\begin{array}{ll} C^{\mathfrak{I},w_{\mathsf{even}}} := \{d\}; & C^{\mathfrak{I},w_{\mathsf{odd}}} := \emptyset; \\ D^{\mathfrak{I},w_{\mathsf{odd}}} := \{d\}; & D^{\mathfrak{I},w_{\mathsf{even}}} := \emptyset; \\ A^{\mathfrak{I},\{w_{\mathsf{even}},w_{\mathsf{odd}}\}} := \{d\}. & \end{array}$$

where w_{even} and w_{odd} denote the worlds whose distance to the root is an even (including 0) or odd number, respectively.

Clearly, $\mathfrak{I}$ is a model of $\mathcal{T}$. Finally, note that $d \in C^{\mathfrak{I},w_{\mathsf{even}}}$ but $d \notin D^{\mathfrak{I},w_{\mathsf{even}}}$, and $d \in D^{\mathfrak{I},w_{\mathsf{odd}}}$ but $d \notin C^{\mathfrak{I},w_{\mathsf{odd}}}$. Therefore, $d \notin \mathbf{E}\Diamond(C \sqcap D)^{\mathfrak{I},\varepsilon}$. ❑

The fragment $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond^<,\mathbf{E}\Box^<}$ of of $\mathrm{CTL}_{\mathcal{EL}}$ is non-convex. Consider the following setting:

$$\mathcal{T} = \{C \sqsubseteq \mathbf{E}\Diamond^< B,\ B \sqsubseteq D \sqcap \mathbf{E}\Box^< D\} \text{ and}$$

$$D_1 = \mathbf{E}\Diamond^<\mathbf{E}\Diamond^< B,\ D_2 = \mathbf{E}\Box^< D$$

Lemma 17 *$\mathcal{T} \models C \sqsubseteq \bigsqcup D_i$ but $\mathcal{T} \not\models C \sqsubseteq D_i$ for $1 \leq i \leq 2$.*

Proof. For the former, let $\mathfrak{I}$ be a model of $\mathcal{T}$, and $d \in C^{\mathfrak{I},w}$ for some $w \in W$. Hence, $d \in (\mathbf{E}\Diamond^< B)^{\mathfrak{I},w}$, that is, there exists a $j > 0$ such that $d \in B^{\mathfrak{I},\pi[j]}$ for some $\pi \in \mathsf{Paths}(w)$. Then, we can distinguish two cases (1) $j = 1$ or (2) $j > 1$:

- if the first case holds, then $d \in (D \sqcap \mathbf{E}\Box^< D)^{\mathfrak{I},\pi[1]}$. Hence, by semantics, $d \in (\mathbf{E}\Box^< D)^{\mathfrak{I},w}$.
- if the second case holds, then $d \in (\mathbf{E}\Diamond^< B)^{\mathfrak{I},\pi[1]}$. Therefore, by semantics $d \in (\mathbf{E}\Diamond^<\mathbf{E}\Diamond^< B)^{\mathfrak{I},w}$.

From these cases, we can conclude that

$$\mathcal{T} \models C \sqsubseteq \mathbf{E}\Box^< D \sqcup \mathbf{E}\Diamond^<\mathbf{E}\Diamond^< B.$$

Now, we proceed to prove that $\mathcal{T} \not\models C \sqsubseteq D_i$, for $1 \leq i \leq 2$. We begin by constructing a model $\mathfrak{I} = (\Delta, T, \{\mathcal{I}_w\}_{w \in W})$ of $\mathcal{T}$ with $\Delta = \{d\}$ and T a 1-ary tree with $w_1 = \varepsilon \cdot 1$ and $w_2 = w_1 \cdot 1$, such that $d \in C^{\mathfrak{I},\varepsilon}$ and $d \notin D_2^{\mathfrak{I},\varepsilon}$ by setting

$$\begin{array}{ll} C^{\mathfrak{I},\varepsilon} := \{d\}; & C^{\mathfrak{I},w} := \emptyset, \text{ for } w \neq \varepsilon; \\ B^{\mathfrak{I},w_2} := \{d\}; & B^{\mathfrak{I},w} := \emptyset; \text{ for } w \neq w_2; \\ D^{\mathfrak{I},\{w_2,w\}} := \{d\}, & \text{for all } w \text{ of the form } w_2 \cdot u; \\ D^{\mathfrak{I},\{\varepsilon,w_1\}} = \emptyset. & \end{array}$$

Clearly, $\mathfrak{I}$ is a model of $\mathcal{T}$. Finally, note that $d \notin D^{\mathfrak{I},w_1}$. Therefore, $d \notin (\mathbf{E}\Box^< B)^{\mathfrak{I},\varepsilon}$.

Next, we analogously construct a model $\mathfrak{I} = (\Delta, T, \{\mathcal{I}_w\}_{w \in W})$ of $\mathcal{T}$ with $\Delta = \{d\}$ and T a 1-ary tree with $w_1 = \varepsilon \cdot 1$ such that $d \in C^{\mathfrak{I},\varepsilon}$ and $d \notin D_1^{\mathfrak{I},\varepsilon}$ by setting

$$\begin{array}{ll} C^{\mathfrak{I},\varepsilon} := \{d\}; & C^{\mathfrak{I},w} := \emptyset, \text{ for } w \neq \varepsilon; \\ B^{\mathfrak{I},w_1} := \{d\}; & B^{\mathfrak{I},w} := \emptyset, \text{ for } w \neq w_1; \\ D^{\mathfrak{I},w} := \{d\}, \text{ for all } w. & \end{array}$$

Clearly, $\mathfrak{I}$ is a model of $\mathcal{T}$. Finally, note that $d \notin D^{\mathfrak{I},w_1 \cdot 1}$. Therefore, $d \notin (\mathbf{E}\Diamond^<\mathbf{E}\Diamond^< B)^{\mathfrak{I},\varepsilon}$.

❑

Now that we have established the non-convexity of the above logics we can devise a reduction from the satisfiability problem w.r.t. TBoxes in the corresponding $\mathcal{ALC}$ variant. We provide a reduction for $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond,\mathbf{A}\Diamond}$, and discuss how to do it for the remaining fragments.

Lemma 18 *Concept subsumption w.r.t. TBoxes for* $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond,\mathbf{A}\Diamond}$ *is undecidable.*

Proof. The proof is by reduction of the satisfiability problem w.r.t. TBoxes for $\mathrm{CTL}_{\mathcal{ALC}}^{\mathbf{E}\Diamond,\mathbf{A}\Diamond}$. Suppose that an $\mathrm{CTL}_{\mathcal{ALC}}^{\mathbf{E}\Diamond,\mathbf{A}\Diamond}$ TBox $\mathcal{T}$ and a concept name A_0 are given for which satisfiability is to be decided. First, we manipulate the TBox $\mathcal{T}$ with some satisfiability preserving operations:

* Ensure that negation $\neg$ occurs *only* in front of concept names: for every subconcept $\neg C$ in $\mathcal{T}$ with C complex, introduce a fresh concept name A, replace $\neg C$ with $\neg A$, and add $A \sqsubseteq C$ and $C \sqsubseteq A$ to $\mathcal{T}$.

* Eliminate negation: for every subconcept $\neg A$, introduce a fresh concept name $\overline{A}$, replace every occurrence of $\neg A$ with $\overline{A}$, and add $\top \sqsubseteq A \sqcup \overline{A}$ and $A \sqcap \overline{A} \sqsubseteq \bot$ to $\mathcal{T}$.

* Eliminate disjunction: modulo introduction of new concept names, we may assume that $\sqcup$ occurs in $\mathcal{T}$ only in the form

  (i) $A \sqcup B \sqsubseteq C$ and (ii) $C \sqsubseteq A \sqcup B$,

  where A and B are concept names and C is disjunction free.

  The former kind of inclusion is replaced with $A \sqsubseteq C$ and $B \sqsubseteq C$. The latter one is replaced with

$$\begin{aligned} C &\sqsubseteq \exists r.(M \sqcap \mathbf{A}\Diamond X \sqcap \mathbf{A}\Diamond Y) \\ \exists r.(M \sqcap \mathbf{E}\Diamond(X \sqcap \mathbf{E}\Diamond Y)) &\sqsubseteq A \qquad (1) \\ \exists r.(M \sqcap C \sqcap \mathbf{E}\Diamond(Y \sqcap \mathbf{E}\Diamond X)) &\sqsubseteq B \end{aligned}$$

  where r is a fresh rigid role name and M, X, Y are fresh concept names. We denote by $\mathcal{T}'$ the TBox obtained by the the previous manipulations.

  Claim 3 *A_0 is satisfiable w.r.t. $\mathcal{T}$ iff A_0 is satisfiable w.r.t. $\mathcal{T}'$.*

  Proof of the claim. Clearly, if A_0 is satisfiable w.r.t. $\mathcal{T}$', then A_0 is satisfiable w.r.t. $\mathcal{T}$.

  For the other direction, we assume that A_0 is satisfiable w.r.t. $\mathcal{T}$. Let $\mathfrak{I} = (\Delta, T, \{\mathcal{I}_w\}_{w\in W})$ be a model of A_0 and $\mathcal{T}$. We construct a temporal interpretation $\mathfrak{J} = (\Delta, T, \{\mathcal{J}_w\}_{w\in W})$ as follows. As proposed by (Artale et al. 2007) w.l.o.g. we assume Δ to be an infinite domain. Consider a CI $C \sqsubseteq A \sqcup B$, set

  (I) for all $w \in W$,

$$A^{\mathfrak{J},w} = A^{\mathfrak{I},w} \text{ for all } A \in \mathsf{N_C}\backslash\{M, X, Y\};$$
$$s^{\mathfrak{J},w} = s^{\mathfrak{I},w} \text{ for all } s \in \mathsf{N_R}\backslash\{r\}.$$

  (II) Interpret M as follows: for all $w \in W$, if $d \in C^{\mathfrak{I},w}$, then choose exactly one d' such that $(d, d') \in r^{\mathfrak{I},w}$ and set $M^{\mathfrak{J},w} = \{d'\}$. We choose d' such that $M^{\mathfrak{J},w} \cap M^{\mathfrak{J},w'}$ for all $w \neq w'$.

  (III) Interpret r as follows: $r^{\mathfrak{J},w}$ is a forest of infinite outdegree, that is, for each $d \in \Delta$ there exists infinitely many $d' \in \Delta$ such that $(d, d') \in r^{\mathfrak{J},w}$ and for each d' there is a unique d such that $(d, d') \in r^{\mathfrak{J},w}$.

  (IV) Interpret X and Y as follows: assume $(d, d') \in r^{\mathfrak{J},w}$, $d \in C^{\mathfrak{J},w}$ and $d' \in M^{\mathfrak{J},w}$. We then have that $d \in (A \sqcup B)^{\mathfrak{I},w}$, we distinguish the following cases:

  1. if $d \in B^{\mathfrak{I},w}$, include $d' \in Y^{\mathfrak{J},w}$ and $d' \in X^{\mathfrak{J},w'}$ for all $w' \in W$ of the form wc ;

  2. if $d \in A^{\mathfrak{I},w}\backslash B^{\mathfrak{I},w}$, include $d \in X^{\mathfrak{J},w}$ and $d \in Y^{\mathfrak{J},w'}$ for all w' of the form wc.

  Now it is standard to show that $\mathfrak{J}$ satisfies A_0 and $\mathcal{T}'$.

  This finishes the proof of the claim.

* The TBox $\mathcal{T}'$ contains only the operators $\sqcap$, $\exists$, $\top$, $\bot$, and $\mathbf{E}\Diamond$, $\mathbf{A}\Diamond$. We now reduce satisfiability of A_0 w.r.t. $\mathcal{T}'$ to (non-)subsumption in $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond,\mathbf{A}\Diamond}$. To this aim we use the reduction proposed by (Baader, Brandt, and Lutz 2005) for the extension of $\mathcal{EL}$ with $\bot$: Introduce a fresh concept name L and replace every occurrence of $\bot$ with L and extend $\mathcal{T}'$ with $\exists r.\mathsf{L} \sqsubseteq \mathsf{L}$, for every role r from $\mathcal{T}'$ $\mathbf{E}\Diamond\mathsf{L} \sqsubseteq \mathsf{L}$. It is not hard to see that A_0 is satisfiable w.r.t. $\mathcal{T}'$ iff $\mathcal{T}'' \not\models A_0 \sqsubseteq \mathsf{L}$.

❑

For the rest of the listed fragments from Theorem 4 in Section 4.1 we can show it by using their respective non-convexity witnesses (*cf.* Figure 1, Section 4.1) as in Equation (1) of Lemma 18.

C Proofs for Section 4.2

Before we can prove Theorem 6, we need to recall the relevant notions and results from description logics, product modal logics, and the theory of first-order Horn sentences.

Let us fix a DL vocabulary consisting of concept names $A_1, A_2, \dots$ and role names $r_1, r_2, \dots$ A $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$-TBox $\mathcal{T}$ is in *normal form* if all axioms have one of the following forms:

$$\begin{array}{lll} \top \sqsubseteq A_i & & \\ A_i \sqsubseteq A_j & A_i \sqsubseteq \exists r_j.A_k & A_i \sqsubseteq \mathbf{E}\bigcirc A_j \\ A_i \sqcap A_j \sqsubseteq A_k & \exists r_i.A_j \sqsubseteq A_k & \mathbf{E}\bigcirc A_i \sqsubseteq A_j. \end{array}$$

It is clear that every arbitrary $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$-TBox can be transformed into normal form using operations analogous to those described for an extension of $\mathcal{EL}$ in (Brandt 2004). The normal form for $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ is defined in the same way, replacing $\mathbf{E}\bigcirc$ with $\mathbf{E}\Diamond$. For defining the normal form for $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond\mathbf{A}\Box}$, we additionally allow axioms of the form

$$A_i \sqsubseteq \mathbf{A}\Box A_j \qquad \mathbf{A}\Box A_i \sqsubseteq A_j. \qquad (2)$$

Let us furthermore fix a first-order (FO) vocabulary containing the unary predicates $A_1, A_2, \ldots$ and the binary predicates $S, R_1, R_2, \ldots$ The *standard translation* from $\mathrm{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$ into first-order logic (FO) consists of two functions ST_x and ST_y mapping $\mathrm{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$-concepts C to FO formulas with one free variable. These two functions are defined simultaneously via induction on C as follows.

$$\begin{aligned}
\mathsf{ST}_x(A_i) &= A_i x\\
\mathsf{ST}_x(C \sqcap D) &= \mathsf{ST}_x(C) \wedge \mathsf{ST}_x(D)\\
\mathsf{ST}_x(\exists r_i.C) &= \exists y.(R_i xy \wedge \mathsf{ST}_y(C))\\
\mathsf{ST}_x(\mathbf{E}\bigcirc C) &= \exists y.(Sxy \wedge \mathsf{ST}_y(C))
\end{aligned}$$

ST_y is defined as ST_x with x and y interchanged. The *standard translation* of a $\mathrm{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$-TBox $\mathcal{T}$ is given by

$$\mathsf{ST}(\mathcal{T}) = \{\forall x.(\mathsf{ST}_x(C) \to \mathsf{ST}_x(D)) \mid C \sqsubseteq D \in \mathcal{T}\}.$$

The standard translation for $\mathrm{CTL}^{\mathbf{E}\Diamond}_{\mathcal{EL}}$ is defined in the same way, using the same translation for $\mathbf{E}\Diamond C$ as for $\mathbf{E}\bigcirc C$. The different semantics of $\mathbf{E}\Diamond$ and $\mathbf{E}\bigcirc$ will later be considered by axiomatizing transitivity and reflexivity. For the standard translation for $\mathrm{CTL}^{\mathbf{A}\Box}_{\mathcal{EL}}$, we add the following cases and its analogon for ST_y.

$$\mathsf{ST}_x(\mathbf{A}\Box C) = \forall y.(Sxy \to \mathsf{ST}_y(C))$$

An *embedded implicational dependency (EID)* (Fagin 1982) is an FO-sentence of the form

$$\forall x_1 \ldots x_m.((E_1 \wedge \cdots \wedge E_n) \to \exists x_{m+1} \ldots x_k.(F_1 \wedge \cdots \wedge F_\ell))$$

with $n \geq 1$, $k \geq m$, $\ell \geq 1$, and where each E_i is a relational formula $Px_{j_1} \ldots x_{j_d}$, each x_j appears in some E_i, and each F_i is either a relational formula or an equality $x_j = x_{j'}$. It is easy to see that, for any $\mathrm{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$-TBox or $\mathrm{CTL}^{\mathbf{E}\Diamond}_{\mathcal{EL}}$-TBox $\mathcal{T}$, $\mathsf{ST}(\mathcal{T})$ is equivalent to a conjunction of embedded implicational dependencies (EIDs):

Lemma 19 *If $\mathcal{T}$ is a $\mathrm{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$-TBox or a $\mathrm{CTL}^{\mathbf{E}\Diamond}_{\mathcal{EL}}$-TBox in normal form, then $\mathsf{ST}(\mathcal{T})$ is equivalent to a conjunction of embedded implicational dependencies (EIDs).*

Proof. It suffices to show the statement for $\mathrm{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$-TBoxes because the standard translation of $\mathrm{CTL}^{\mathbf{E}\Diamond}_{\mathcal{EL}}$-TBoxes treats $\mathbf{E}\Diamond$ exactly as $\mathbf{E}\bigcirc$. We only need to show that the standard translation of every single axiom α in normal form is (equivalent to) an EID – we proceed by cases.

- $\alpha = (\top \sqsubseteq A_i)$. Then α contributes to $\mathsf{ST}(\mathcal{T})$ the conjunct $\forall x.A_i x$, which is an EID (the case $m = k$ in the definition of EIDs means that there is no existential quantifier).
- $\alpha = (A_i \sqsubseteq A_j)$. Then α contributes to $\mathsf{ST}(\mathcal{T})$ the conjunct $\forall x.(A_i x \to A_j x)$, which is an EID.
- $\alpha = (A_i \sqcap A_j \sqsubseteq A_k)$. Then, analogously, α contributes to $\mathsf{ST}(\mathcal{T})$ the conjunct $\forall x.(A_i x \wedge A_j x \to A_k x)$, which is an EID.
- $\alpha = (A_i \sqsubseteq \exists r_j.A_k)$. Then α contributes to $\mathsf{ST}(\mathcal{T})$ the conjunct $\forall x.(A_i x \to \exists y.(R_j xy \wedge A_j y))$, which is an EID.
- $\alpha = (\exists r_i.A_j \sqsubseteq A_k)$. Then α contributes to $\mathsf{ST}(\mathcal{T})$ the conjunct $\forall x.(\exists y.(R_i xy \wedge A_j x) \to A_j y)$, which is equivalent to the EID $\forall xy.((R_i xy \wedge A_j x) \to A_j y)$.
- $\alpha = (A_i \sqsubseteq \mathbf{E}\bigcirc A_j)$. Analogous to $\alpha = (A_i \sqsubseteq \exists r_j.A_k)$, with R_j replaced by S.
- $\alpha = (\mathbf{E}\Diamond.A_j \sqsubseteq A_k)$. Analogous to $\alpha = (\exists r_i.A_j \sqsubseteq A_k)$, with R_i replaced by S.

❏

We are emphasizing EIDs because of their preservation under direct products. This property will play a central rôle in the proof of Theorem 6. Given n first-order structures $\mathcal{M}_i = (\Delta^{\mathcal{M}_i}, \cdot^{\mathcal{M}_i})$, $i = 1, \ldots, n$, the *direct product* $\prod_{i=1,\ldots,n} \mathcal{M}_i$ is the structure $\mathcal{M} = (\Delta^{\mathcal{M}}, \cdot^{\mathcal{M}})$ with

$$\begin{aligned}
&\Delta^{\mathcal{M}} = \Delta^{\mathcal{M}_1} \times \cdots \times \Delta^{\mathcal{M}_n} \qquad \text{and}\\
&P^{\mathcal{M}} = \{(\langle d_{1,1}, \ldots, d_{1,n}\rangle, \ldots, \langle d_{m,1}, \ldots, d_{m,n}\rangle) \mid\\
&\qquad (d_{1,i}, \ldots, d_{m,i}) \in P^{\mathcal{M}_i} \text{ for all } i = 1, \ldots, n,\\
&\qquad \text{for any } m\text{-ary predicate symbol } P.
\end{aligned}$$

We say that an FO sentence φ is *preserved under direct products* if, for all structures $\mathcal{M}_1, \ldots, \mathcal{M}_n$: if $\mathcal{M}_i \models \varphi$ for $i = 1, \ldots, n$, then $\prod_{i=1,\ldots,n} \mathcal{M}_i \models \varphi$. In fact, the standard definition of direct products and preservation allows for an arbitrary cardinality of structures to be multiplied (Chang and Keisler 1990), but the special case given here suffices for our purposes. One of the central results in (Fagin 1982) is the following.

Theorem 20 (Fagin 1982) *Every conjunction of EIDs is preserved under direct products.*

The following observation is obvious.

Fact 21 *If two sentences φ, ψ are each preserved under direct products, then so is their conjunction $\varphi \wedge \psi$.*

We can now state the preservation properties of the standard translation of TBoxes.

Lemma 22 *1. For each logic $\mathcal{L} \in \{\mathrm{CTL}^{\mathbf{E}\bigcirc}_{\mathcal{EL}}, \mathrm{CTL}^{\mathbf{E}\Diamond}_{\mathcal{EL}}\}$, the standard translation $\mathsf{ST}(\mathcal{T})$ of any $\mathcal{L}$-TBox $\mathcal{T}$ is preserved under direct products.*

2. The standard translation $\mathsf{ST}(\mathcal{T})$ of any $\mathrm{CTL}^{\mathbf{E}\Diamond\mathbf{A}\Box}_{\mathcal{EL}}$-TBox $\mathcal{T}$ is preserved under direct products of structures where the relation S representing the temporal successor relation is interpreted as a total relation.

Proof.

1. follows from Lemma 19 and Theorem 20.
2. For $\mathrm{CTL}^{\mathbf{E}\Diamond\mathbf{A}\Box}_{\mathcal{EL}}$, we need to take additional care of the translation of the axioms involving $\mathbf{A}\Box$, given in (2).
   - Every axiom $\alpha = A_i \sqsubseteq \mathbf{A}\Box A_j$ in $\mathcal{T}$ contributes to $\mathsf{ST}(\mathcal{T})$ the conjunct $\forall x.(A_i x \to \forall y.(Sxy \to A_j y))$, which is equivalent to the EID $\forall xy.((A_i x \wedge Sxy) \to \exists z.A_j y)$.
   - Every axiom $\alpha = (\mathbf{A}\Box A_i \sqsubseteq A_j)$ in $\mathcal{T}$ contributes to $\mathsf{ST}(\mathcal{T})$ the conjunct
   $$\forall x.(\forall y.(Sxy \to A_i y) \to A_j x), \qquad (3)$$
   and we want to show that (3) is preserved under taking direct products of structures where the relation S

representing the temporal successor relation is interpreted as a total relation. Let $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be structures with $\mathcal{M}_k \models (3)$ for $k = 1, \ldots, n$, and let $\mathcal{M} = \prod_{k=1,\ldots,n} \mathcal{M}_k$. To show that $\mathcal{M} \models (3)$, we take an arbitrary element $\langle d_1, \ldots, d_n \rangle \in \Delta^{\mathcal{M}}$ and show that $\mathcal{M} \models \forall y.(Sxy \to A_i y)[\langle d_1, \ldots, d_n \rangle)]$ implies $\mathcal{M} \models A_j x[\langle d_1, \ldots, d_n \rangle]$.
Assume that $\mathcal{M} \models \forall y.(Sxy \to A_i y)[\langle d_1, \ldots, d_n \rangle]$, i.e., for every $\langle d'_1, \ldots, d'_n \rangle \in \Delta^{\mathcal{M}}$, $S^{\mathcal{M}} \langle d_1, \ldots, d_n \rangle \langle d'_1, \ldots, d'_n \rangle$ implies $A_i^{\mathcal{M}} \langle d_1, \ldots, d_n \rangle$. We first claim that $\mathcal{M}_k \models \forall y.(Sxy \to A_i y)[d_k]$, for each $k = 1, \ldots, n$: Take some d'_k with $S^{\mathcal{M}_1} d_k d'_k$. Since $S^{\mathcal{M}_i}$, is total for all i, there are $d'_1, \ldots, d'_{k-1}, d'_{k+1}, \ldots, d'_n$ with $S^{\mathcal{M}_\ell} d_\ell d'_\ell$ for all $\ell \neq k$. Hence, $S^{\mathcal{M}} \langle d_1, \ldots, d_n \rangle \langle d'_1, \ldots, d'_n \rangle$ which, by assumption, implies $A_i^{\mathcal{M}} \langle d_1, \ldots, d_n \rangle$. Then, due to the product construction, we have that $A_i^{\mathcal{M}_k} d_k$. This finishes the proof of the intermediate claim $\mathcal{M}_k \models \forall y.(Sxy \to A_i y)[d_k]$.
We now use the previous claim and the assumption $\mathcal{M}_k \models (3)$ to conclude that $\mathcal{M}_k \models A_j x[d_k]$. The product construction ensures $\mathcal{M} \models A_j x[\langle d_1, \ldots, d_n \rangle]$. Since we started with the assumption $\mathcal{M} \models \forall y.(Sxy \to A_i y)[\langle d_1, \ldots, d_n \rangle]$, we have thus proven $\mathcal{M} \models (3)$.

❏

We are now ready to prove Theorem 6.

Theorem 6 *The logics $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$, $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$, and $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond\mathbf{A}\Box}$ are convex.*

Proof. We start with $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$. Consider a $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$-TBox $\mathcal{T}$ and $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$-concepts $C, D_1, \ldots, D_n$, $n \geq 2$. Let $r_1, \ldots, r_k$ be the rigid roles occurring in $\mathcal{T}, C, D_1, \ldots, D_n$, and let $r_{k+1}, \ldots, r_\ell$ be the non-rigid roles. Furthermore, we assume w.l.o.g. that $\mathcal{T}$ is in normal form and that C and all D_i are atomic concepts. The latter assumption does not restrict generality because non-atomic C or D_i can always be "defined away" in $\mathcal{T}$ before normalizing $\mathcal{T}$.

In order to express entailment statements like $\mathcal{T} \models C' \sqsubseteq D'$ as entailment statements in first-order logic, we need to express rigidity of the roles $r_1, \ldots, r_k$ as well as the property that the temporal "direct successor" relation is a total tree. For rigidity, we borrow from the theory of product modal logics (Gabbay et al. 2003): we have to say that all $R_1, \ldots, R_k$ are left-commutative, right-commutative and satisfy the Church-Rosser property:

$$\bigwedge_{i=1}^{k} \forall xyz(R_i xy \wedge Syz \to \exists u(Sxu \wedge R_i uz)) \tag{4}$$

$$\bigwedge_{i=1}^{k} \forall xyz(Sxy \wedge R_i yz \to \exists u(R_i xu \wedge Suz)) \tag{5}$$

$$\bigwedge_{i=1}^{k} \forall xyz(R_i xy \wedge Sxz \to \exists u(Syu \wedge R_i zu)) \tag{6}$$

These three formulas are conjunctions of EIDs.

Furthermore, we need to say that S, which represents the temporal "direct successor" relation, is a total tree. Since standard translations of $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$-TBoxes are preserved under unravelling and under taking point-generated substructures, it suffices to say that S is total, which can be achieved by the following EID.

$$\mathsf{D} \qquad \forall x \exists y(Sxy) \tag{7}$$

We can now conclude, using Fact 21, that the conjunction

$$\varphi(\mathcal{T}) = (4) \wedge (5) \wedge (6) \wedge (7) \wedge \mathsf{ST}(\mathcal{T})$$

is preserved under direct products. In order to show convexity of $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$, we observe that

$$\mathcal{T} \models C \sqsubseteq D_1 \sqcup \cdots \sqcup D_n \quad \text{implies} \quad \mathcal{T} \models C \sqsubseteq D_i \text{ for some } i$$

is equivalent to

$$\mathcal{T} \not\models C \sqsubseteq D_i \text{ for all } i \quad \text{implies} \quad \mathcal{T} \not\models C \sqsubseteq D_1 \sqcup \cdots \sqcup D_n,$$

which is equivalent to

$$\begin{aligned} &\varphi(\mathcal{T}) \not\models \forall x.(Cx \to D_i x) \text{ for all } i \quad \text{implies} \\ &\varphi(\mathcal{T}) \not\models \forall x.(Cx \to D_1 x \vee \cdots \vee D_n x). \end{aligned} \tag{8}$$

To establish (8) for $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$, assume that $\varphi(\mathcal{T}) \not\models \forall x.(Cx \to D_i x)$ for all $i \in \{1, \ldots, n\}$. Then there are structures $\mathcal{M}_i$, $i = 1, \ldots, n$, with $\mathcal{M}_i \models \varphi(\mathcal{T})$ and $\mathcal{M}_i \models \exists x.(Cx \wedge \neg D_i x)$. Therefore, there are domain elements $d_i \in \Delta^{\mathcal{M}_i}$ with $\mathcal{M}_i \models (Cx \wedge \neg D_i x)[d_i]$. Hence, the product construction ensures that $\mathcal{M} \models (Cx \wedge \neg D_1 x \wedge \cdots \wedge \neg D_n x)[\langle d_1, \ldots, d_n \rangle]$, which means that $\mathcal{M} \models \exists x.(Cx \wedge \neg D_1 x \wedge \neg D_n x)$. Since, additionally $\varphi(\mathcal{T})$ is preserved under direct products, $\mathcal{M}$ witnesses $\varphi(\mathcal{T}) \not\models \forall x.(Cx \to D_1 x \vee \cdots \vee D_n x)$.

To carry this reasoning over to $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ and $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond\mathbf{A}\Box}$, we replace KD in the above argument with S4, the modal logic of reflexive transitive frames. Consequently, we have to replace the axiom D in $\varphi(\mathcal{T})$ with the following two EIDs.

$$\mathsf{T} \qquad \forall x(xR_h x) \tag{9}$$

$$\mathsf{4} \qquad \forall xyz(xR_h y \wedge yR_h z \to xR_h z) \tag{10}$$

Furthermore, $\mathsf{ST}(\mathcal{T})$ changes only in one respect: axioms containing $\mathbf{A}\Box$ have to be translated, and we have shown in Lemma 22 (2) that $\mathsf{ST}(\mathcal{T})$ is then preserved under direct products of structures where the relation S representing the temporal successor relation is interpreted as a total relation. Therefore, the above $\varphi(\mathcal{T})$ remains preserved under direct products of such structures. Since $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ and $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond\mathbf{A}\Box}$ are restricted to such structures, the above argument goes through.

It is worth recalling that convexity of both $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc}$ and $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\Diamond}$ does not imply that $\mathrm{CTL}_{\mathcal{EL}}^{\mathbf{E}\bigcirc\mathbf{E}\Diamond}$ is convex: in order to translate a TBox using both the $\mathbf{E}\bigcirc$ and the $\mathbf{E}\Diamond$ operator into FO, we would have to use two binary relations $S_{\mathbf{E}\bigcirc}$ and $S_{\mathbf{E}\Diamond}$ and state that $S_{\mathbf{E}\Diamond}$ is the reflexive transitive closure of $S_{\mathbf{E}\bigcirc}$, which is not expressible in FO. ❏

Theorem 7 *$CTL^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$ and $CTL^{\mathbf{E}\Diamond}_{\mathcal{EL}}$ lack the FMP.*

Proof. For seeing this consider the $\mathcal{T}$ consisting of the following concept inclusions:

$$A \sqsubseteq \exists r.A \qquad A \sqsubseteq \mathbf{E}\bigcirc B$$
$$\exists r.B \sqsubseteq B' \qquad \exists r.B' \sqsubseteq B'$$
$$\mathbf{E}\bigcirc(B \sqcap B') \sqsubseteq C \qquad \exists r.C \sqsubseteq C$$

We can now show that $\mathcal{T}$ satisfies the following:

Claim:

1. $\mathcal{T} \not\models A \sqsubseteq C$ and
2. for every finite model $\mathfrak{I}$ of $\mathcal{T}$ we have that $\mathfrak{I} \models A \sqsubseteq C$.

Proof of the claim: For Point 1, consider the temporal interpretation $\mathfrak{I} = (\Delta, W, \{\mathcal{I}_w\}_{w\in W})$ with $\Delta = \mathbb{N}$ with an infinite A-chain and construct the minimal model from it. That is, set:

$W = \{\varepsilon\} \cup \mathbb{N}$ (the tree consisting of a root and ω many children at depth 1)

$$A^{\mathfrak{I},\varepsilon} = \Delta; \qquad A^{\mathfrak{I},n} = \emptyset,\ n \in \mathbb{N};$$
$$B^{\mathfrak{I},\varepsilon} = \emptyset; \qquad B^{\mathfrak{I},n} = \{n\},\ n \in \mathbb{N};$$
$$B'^{\mathfrak{I},\varepsilon} = \emptyset; \qquad B'^{\mathfrak{I},n} = \{m \mid m > n\},\ n \in \mathbb{N};$$
$$C^{\mathfrak{I},w} = \emptyset,\ w \in W.$$

It is easy to see that $\mathfrak{I} \models \mathcal{T}$ and $\mathfrak{I} \not\models A \sqsubseteq C$.

For Point 2, assume that there is some finite model $\mathfrak{I}$ of $\mathcal{T}$ and $d_0 \in \Delta$, $w \in W$ with $d_0 \in A^{\mathfrak{I},w}$. By the first CI, there has to be a sequence of elements $d_1, d_2, \dots$ such that $(d_i, d_{i+1}) \in r^{\mathfrak{I},w}$ and $d_i \in A^{\mathfrak{I},w}$ for all $i \geq 0$. Since Δ is finite by assumption, there are $i < j$ with $d_i = d_j$. Hence there is a cycle $d_i, d_i+1, \dots, d_{j-1}, d_j = d_i$. We can now use this cycle to show that $d_i \in C^{\mathfrak{I},w}$, which, by the last CI, implies that $d_0 \in C^{\mathfrak{I},w}$.

Since $d_i, \dots, d_j \in A^{\mathfrak{I},w}$ and due to the second CI, there are successor worlds $v_i, \dots, v_j$ of w such that $d_k \in B^{\mathfrak{I},v_k}$ for all $k = i, \dots, j$. Because of the third and fourth CI and due to the r-cycle, we also get $d_k \in B'^{\mathcal{I},v_k}$ for all $k = i, \dots, j$. The fifth CI then yields $d_k \in C^{\mathfrak{I},w}$ for all $k = i, \dots, j$.

This finishes the proof of the claim.

It is not hard to see that the proof also works if $\mathbf{E}\bigcirc$ is replaced with $\mathbf{E}\Diamond$ in $\mathcal{T}$. ❑

D Proofs for Section 4.3

Theorem 8 *Concept subsumption w.r.t. general TBoxes for $CTL^{\mathbf{E}\bigcirc}_{\mathcal{EL}}$ is undecidable.*

Proof. We begin by recalling $\mathcal{T}_M$.

$$A \sqsubseteq \exists r_0.(S \sqcap A) \tag{11}$$
$$A \sqsubseteq \exists r.A \qquad \forall r \in \mathsf{ROL} \setminus \{r_0\} \tag{12}$$
$$\exists r.\mathsf{Head}_c \sqsubseteq \mathsf{Head}_c \qquad \forall r \in \mathsf{ROL}, c \in \{1,2\} \tag{13}$$
$$S \sqsubseteq \mathbf{E}\bigcirc(\mathbf{E}\bigcirc(\mathsf{Head}_1 \sqcap \mathsf{Tail}_1 \sqcap Q_0) \sqcap \mathbf{E}\bigcirc(\mathsf{Head}_2 \sqcap \mathsf{Tail}_2 \sqcap Q_0)) \tag{14}$$
$$\mathbf{E}\bigcirc(\mathbf{E}\bigcirc(\mathsf{Head}_1 \sqcap Q_n) \sqcap \mathbf{E}\bigcirc(\mathsf{Head}_2 \sqcap Q_n)) \sqsubseteq B \tag{15}$$

For all $i \in Q$ with $I_i = \mathsf{inc}(c, j)$ $\mathcal{T}_M$ contains

$$\exists r_i.(Q_i \sqcap \mathsf{Tail}_c) \sqsubseteq Q_j \sqcap C_c \sqcap \mathbf{E}\bigcirc(Q_j \sqcap \mathsf{Tail}_c) \tag{16}$$
$$\exists r_i.(Q_i \sqcap C_c) \sqsubseteq Q_j \sqcap C_c \tag{17}$$
$$\exists r_i.(Q_i \sqcap \mathsf{Tail}_{\overline{c}}) \sqsubseteq Q_j \sqcap \mathsf{Tail}_{\overline{c}} \tag{18}$$
$$\exists r_i.(Q_i \sqcap C_{\overline{c}}) \sqsubseteq Q_j \sqcap C_{\overline{c}} \tag{19}$$

For all $i \in Q$ with $I_i = \mathsf{dec}(c, j, k)$ we have the following concept inclusions in $\mathcal{T}_M$:

$$\exists r_{i0}.(Q_i \sqcap \mathsf{Head}_c \sqcap \mathsf{Tail}_c) \sqsubseteq Q_j \sqcap \mathsf{Tail}_c \tag{20}$$
$$\exists r_{i0}.(Q_i \sqcap C_{\overline{c}}) \sqsubseteq Q_j \sqcap C_{\overline{c}} \tag{21}$$
$$\exists r_{i0}.(Q_i \sqcap \mathsf{Tail}_{\overline{c}}) \sqsubseteq Q_j \sqcap \mathsf{Tail}_{\overline{c}} \tag{22}$$
$$\exists r_{i+}.(Q_i \sqcap C_c \sqcap \mathbf{E}\bigcirc\mathsf{Tail}_c) \sqsubseteq Q_k \sqcap \mathsf{Tail}_c \tag{23}$$
$$\exists r_{i+}.(Q_i \sqcap C_c \sqcap \mathbf{E}\bigcirc C_c) \sqsubseteq Q_k \sqcap C_c \tag{24}$$
$$\exists r_{i+}.(Q_i \sqcap C_{\overline{c}}) \sqsubseteq Q_k \sqcap C_{\overline{c}} \tag{25}$$
$$\exists r_{i+}.(Q_i \sqcap \mathsf{Tail}_{\overline{c}}) \sqsubseteq Q_k \sqcap \mathsf{Tail}_{\overline{c}} \tag{26}$$

We proceed to prove the correctness of the reduction.

Lemma 9 *$\mathcal{T}_M \models A \sqsubseteq B$ iff M halts on $(0, 0)$.*

Proof. "if": Assume M halts on $(0, 0)$. This means that there is a finite sequence $(p_0, n_0, m_0) \Rightarrow_M \dots \Rightarrow_M (p_\ell, n_\ell, m_\ell)$ such that $p_0 = 0$, $p_\ell = n$, $n_0 = m_0 = 0$. Define a sequence $a_0, \dots, a_{\ell-1} \in \mathsf{ROL}^\ell$ as

$$a_i = \begin{cases} r_{p_i} & \text{if } I_{p_i} = \mathsf{inc}(\cdot,\cdot); \\ r_{p_i 0} & \text{if } I_{p_i} = \mathsf{dec}(c,\cdot,\cdot), (c = 1 \text{ and } n_i = 0) \text{ or} \\ & \qquad (c = 2 \text{ and } m_i = 0); \\ r_{p_i +} & \text{if } I_{p_i} = \mathsf{dec}(c,\cdot,\cdot), (c = 1 \text{ and } n_i > 0) \text{ or} \\ & \qquad (c = 2 \text{ and } m_i > 0). \end{cases}$$

Now, let $\mathfrak{I} = (\Delta, W, \{\mathcal{I}_w\}_{w\in W})$ be any model of $\mathcal{T}_M$ and $d \in \Delta$, $w \in W$ such that $d \in A^{\mathfrak{I},w}$.

Claim 1. There are domain elements $d_0, \dots, d_\ell$ such that

1. $d_0 = d$;
2. $d_i \in A^{\mathfrak{I},w}$ for all $0 \leq i \leq \ell$;
3. $(d_i, d_{i+1}) \in a^{\mathfrak{I},w}_{\ell-i-1}$ for all $0 \leq i < \ell$;
4. $d_\ell \in S^{\mathfrak{I},w}$.

Proof of Claim 1. The proof is by induction on ℓ. For $\ell = 0$ it is trivial. So let $\ell > 0$. By induction hypothesis, we have $d_{\ell-1} \in A^{\mathfrak{I},w}$. As $\mathfrak{I} \models \mathcal{T}_M$, in particular $\mathfrak{I} \models A \sqsubseteq \exists r_0.(S \sqcap A)$ and $\mathfrak{I} \models A \sqsubseteq \exists r.A$ for all $r \in \mathsf{ROL} \setminus \{r_0\}$ by CIs (11) and (12). In any case, this yields an element d_ℓ satisfying points 2 and 3. Note moreover that $a_0 = r_0$ and thus the CI (11) implies that we can choose d_ℓ such that it satisfies Point 4. This finishes the proof of the claim.

Let now $d_0, \ldots, d_\ell$ be the sequence that exists due to Claim 1. For the sake of readibility, we will write $w < w'$ in place of $E(w, w')$ in what follows.

Claim 2. For all $0 \leq i \leq \ell$ there exists a world w' with $w < w'$ and sequences of worlds $w_0 < \ldots < w_N$ and $v_0 < \ldots < v_M$ with $w' < w_0$, $w' < v_0$, $N = n_{\ell-i}$, and $M = m_{\ell-i}$ such that:

1. $d_i \in \mathsf{Head}_1^{\mathfrak{I},w_0}$ and $d_i \in \mathsf{Head}_2^{\mathfrak{I},v_0}$;
2. $d_i \in \mathsf{Tail}_1^{\mathfrak{I},w_N}$ and $d_i \in \mathsf{Tail}_2^{\mathfrak{I},v_M}$;
3. $d_i \in C_1^{\mathfrak{I},w_j}$ for all $0 \leq j < N$ and $d_i \in C_2^{\mathfrak{I},v_j}$ for all $0 \leq j < M$;
4. $d_i \in Q_{p_{\ell-i}}^{\mathfrak{I},w_j}$ for all $0 \leq j \leq N$ and $d_i \in Q_{p_{\ell-i}}^{\mathfrak{I},v_j}$ for all $0 \leq j \leq M$.

Proof of Claim 2. The proof is again by induction. For the induction base assume $i = \ell$. Notice that by assumption, $a_0 = r_0$. Thus, Point 4 of Claim 1 yields that $d_\ell \in S^{\mathfrak{I},w}$. By CI (14), we immediately get that the induction base is satisfied.

For the induction step assume $i < \ell$. By Point 3 of Claim 1, we have $(d_i, d_{i+1}) \in a_{\ell-i-1}^{\mathfrak{I},w}$ and as $a_{\ell-i-1}$ is rigid, $(d_i, d_{i+1}) \in a_{\ell-i-1}^{\mathfrak{I},\overline{w}}$ for all $\overline{w} \in \{w_0, \ldots, w_N, v_0, \ldots, v_M\}$. First observe that by (13) and Point 1 from the induction hypothesis, we get

$$d_i \in \mathsf{Head}_1^{\mathfrak{I},w_0} \text{ and } d_i \in \mathsf{Head}_2^{\mathfrak{I},v_0},$$

hence d_i satisfies Point 1. For proving points 2-4, we make a case distinction on $a_{\ell-i-1}$. Let us start with the simplest case $a_{\ell-i-1} = r_x$ for some $x = p_{\ell-i-1}$ with $I_x = \mathsf{inc}(c, k)$ and assume $c = 1$ (the case $c = 2$ is symmetric). The semantics of TCMs yields that $n_{\ell-i} = n_{\ell-(i+1)} + 1$, $m_{\ell-i} = m_{\ell-(i+1)}$, and $p_{\ell-i} = k$.

By CIs (18) and (19) together with Points 2,3,4 from the hypothesis, we get $d_i \in \mathsf{Tail}_2^{\mathfrak{I},v_M}$, $d_i \in C_2^{\mathfrak{I},v_j}$ for all $0 \leq j < M$, and $d_i \in Q_k^{\mathfrak{I},v_j}$ for all $0 \leq j \leq M$. Thus, the second parts (involving worlds v_j) of Points 2,3,4 are satisfied for d_i.

On the other hand, CIs (16) together with Points 2 and 4 from the induction hypothesis imply the existence of a world $w_{N+1} > w_N$ such that $d_i \in \mathsf{Tail}_1^{\mathfrak{I},w_{N+1}}$ and $d_i \in Q_k^{\mathfrak{I},w_{N+1}}$. CI (17) together with Points 3 and 4 imply $d_i \in C_1^{\mathfrak{I},w_j}$ for all $0 \leq j \leq N$, and $d_i \in Q_k^{\mathfrak{I},w_j}$ for all $0 \leq j \leq N$. Hence, also the first parts of Points 2,3,4 (those involving w_j) are fulfilled.

Let now be $a_{\ell-i-1} = r_{x0}$ with $x = p_{\ell-i-1}$ and $I_x = \mathsf{dec}(c, k, k')$ and assume $c = 1$ (the case $c = 2$ is again symmetric) and thus $n_{\ell-(i+1)} = 0$. The semantics of TCMs yields that $n_{\ell-i} = n_{\ell-(i+1)} = 0$, $m_{\ell-i} = m_{\ell-(i+1)}$, and $p_{\ell-i} = k$.

By CI (20) together with Points 1,2,4 from the hypothesis, we get $d_i \in \mathsf{Tail}_1^{\mathfrak{I},w_0}$, and $d_i \in Q_k^{\mathfrak{I},w_0}$. Thus, the first parts (involving worlds w_j) of Points 2,3,4 are satisfied for d_i.

On the other hand, CIs (21) and (22) together with Points 2, 3 and 4 from the induction hypothesis imply that $d_i \in \mathsf{Tail}_2^{\mathfrak{I},v_M}$, $d_i \in Q_k^{\mathfrak{I},v_j}$ for all $0 \leq j \leq M$, and $d_i \in C_2^{\mathfrak{I},v_j}$ for all $0 \leq j \leq M$. Hence, also the second parts of Points 2,3,4 are fulfilled.

For the final case assume $a_{\ell-i-1} = r_{x+}$ with $x = p_{\ell-i-1}$ and $I_x = \mathsf{dec}(c, k, k')$ and assume $c = 1$ (the case $c = 2$ is again symmetric) and thus (by definition of $a_{\ell-i-1}$) $n_{\ell-(i+1)} > 0$. The semantics of TCMs yields that $n_{\ell-i} = n_{\ell-(i+1)} - 1$, $m_{\ell-i} = m_{\ell-(i+1)}$, and $p_{\ell-i} = k'$.

Note that $N > 0$ by induction hypothesis. In particular, we have

$$d_{i+1} \in (C_1 \sqcap Q_{p_{\ell-(i+1)}} \sqcap \mathbf{E}\bigcirc\mathsf{Tail}_1)^{\mathfrak{I},w_{N-1}}.$$

By CI (23), we have $d_i \in (Q_k \sqcap \mathsf{Tail}_1)^{\mathfrak{I},w_{N-1}}$. Moreover, CIs (24) together with the induction hypothesis implies $d_i \in (Q_k \sqcap C_1)^{\mathfrak{I},w_j}$ for all $0 \leq j < N-1$. Thus, the first parts of Points 2,3,4 are satisfied.

On the other hand, CIs (25) and (26) together with Points 2, 3 and 4 from the induction hypothesis imply that $d_i \in \mathsf{Tail}_2^{\mathfrak{I},v_M}$, $d_i \in Q_k^{\mathfrak{I},v_j}$ for all $0 \leq j \leq M$, and $d_i \in C_2^{\mathfrak{I},v_j}$ for all $0 \leq j \leq M$. Hence, also the second parts of Points 2,3,4 are fulfilled.

This finishes the proof of Claim 2.

Now, Claim 2 applied to d_0 implies the existence of worlds w', w_0, v_0 such that $w < w'$, $w' < w_0$, and $w' < v_0$ with

$$d_0 \in (\mathsf{Head}_1 \sqcap Q_n)^{\mathfrak{I},w_0} \quad \text{and} \quad d_0 \in (\mathsf{Head}_2 \sqcap Q_n)^{\mathfrak{I},v_0}.$$

By CI (15), we must have $d_0 \in B^{\mathfrak{I},w}$. By Point 1 of Claim 1 we have $d_0 = d$.

This finishes the proof of the "if"-direction.

"only if": Assume M does not halt on $(0, 0)$. We define a model $\mathfrak{I} = (\Delta, W, \{\mathcal{I}_w\}_{w \in W})$ and identify $\widehat{d} \in \Delta$ and $\widehat{w} \in W$ such that $\widehat{d} \in A^{\mathfrak{I},\widehat{w}}$ and $\widehat{d} \notin B^{\mathfrak{I},\widehat{w}}$.

The idea is to define a *minimal model* of A relative to $\mathcal{T}_M$, i.e., in this model we satisfy only the concepts that are implied by $\mathcal{T}_M$. Intuitively, the minimal model is the tree generated by CIs (11) and (12) where the simulation of the computation of M is realized in disjoint worlds for every node of the tree.

- $\Delta = \mathsf{ROL}^*$; $\Delta_0 = \mathsf{ROL}^* \cdot \{r_0\}$;
- $W = \{w_0\} \cup \{w_d \mid d \in \Delta_0\} \cup \{(d, c, i) \mid d \in \Delta_0, c \in \{1, 2\}, i \geq 0\}$ such that
  - $w_0 < w_d$ for all $d \in \Delta_0$;
  - $w_d < (d, c, 0)$ for all $c \in \{1, 2\}$, $d \in \Delta_0$;
  - $(d, c, i) < (d, c, i+1)$ for all $c \in \{1, 2\}$, $d \in \Delta_0$, and $i \geq 0$;
- $r^{\mathfrak{I},w} = \{(s, s \cdot r) \mid s \in \mathsf{ROL}^*\}$ for all $r \in \mathsf{ROL}$ and $w \in W$;
- $A^{\mathfrak{I},w} = \begin{cases} \Delta & w = w_0 \\ \emptyset & \text{otherwise;} \end{cases}$
- $S^{\mathfrak{I},w} = \begin{cases} \Delta_0 & w = w_0 \\ \emptyset & \text{otherwise;} \end{cases}$

- $B^{\mathfrak{I},w} = \begin{cases} \Delta & \text{if } w \neq w_0 \\ \Delta \setminus \{\varepsilon\} & \text{if } w = w_0. \end{cases}$;

For the definition of concept names we need some more machinery. For $w = (d, c, i) \in W$ put $w+1 := (d, c, i+1)$. Now, define a function $\pi : \Delta_0 \times W \to 2^{\mathsf{sig}(\mathcal{T}_M)}$ by taking:

$$\pi(d, w) = \begin{cases} \{\mathsf{Head}_c, \mathsf{Tail}_c, Q_0\} & \text{if } w = (d, c, 0), \\ & \quad c \in \{1, 2\} \text{ and} \\ & \qquad d \in \Delta_0; \\ \emptyset & \text{otherwise.} \end{cases}$$

Now, let be π^* the function obtained from π by closing off under the following rules:

Bot: if $\mathsf{Head}_c \in \pi_i(d, w)$ and d' is a prefix of d then put $\mathsf{Head}_c \in \pi_i(d', w)$;

- Let $d = d' \cdot r_i$ and $I_i = \mathsf{inc}(c, j)$.
  - Inc-1: if $\{Q_i, \mathsf{Tail}_c\} \subseteq \pi(d, w)$, then put $\{Q_j, C_c\} \subseteq \pi(d', w)$ and $\{Q_j, \mathsf{Tail}_c\} \subseteq \pi(d', w+1)$;
  - Inc-2: if $\{Q_i, C_c\} \subseteq \pi(d, w)$, then put $\{Q_j, C_c\} \subseteq \pi_i(d', w)$;
  - Inc-3: if $\{Q_i, \mathsf{Tail}_{\overline{c}}\} \subseteq \pi(d, w)$ then put $\{Q_j, \mathsf{Tail}_{\overline{c}}\} \subseteq \pi_i(d', w)$;
  - Inc-4: if $\{Q_i, C_{\overline{c}}\} \subseteq \pi(d, w)$ then put $\{Q_m, C_c\} \subseteq \pi_i(d', w)$.
- Let $d = d' \cdot r_{i0}$ and $I_i = \mathsf{dec}(c, j, k)$.
  - Dec-1: if $\{Q_i, \mathsf{Tail}_c, \mathsf{Head}_c\} \subseteq \pi(d, w)$, then put $\{Q_j, \mathsf{Tail}_c\} \subseteq \pi(d', w)$;
  - Dec-2: if $\{Q_i, C_{\overline{c}}\} \subseteq \pi(d, w)$, then put $\{Q_j, C_{\overline{c}}\} \subseteq \pi_i(d', w)$;
  - Dec-3: if $\{Q_i, \mathsf{Tail}_{\overline{c}}\} \subseteq \pi(d, w)$ then put $\{Q_j, \mathsf{Tail}_{\overline{c}}\} \subseteq \pi_i(d', w)$;
- Let $d = d' \cdot r_{i+}$ and $I_i = \mathsf{dec}(c, j, k)$.
  - Dec-4: if $\{Q_i, C_c\} \subseteq \pi(d, w)$ and $\mathsf{Tail}_c \in \pi(d, w+1)$, then put $\{Q_k, \mathsf{Tail}_c\} \subseteq \pi(d', w)$;
  - Dec-5: if $\{Q_i, C_c\} \subseteq \pi(d, w)$ and $C_c \in \pi(d, w+1)$, then put $\{Q_k, C_c\} \subseteq \pi(d', w)$;
  - Dec-6: if $\{Q_i, C_{\overline{c}}\} \subseteq \pi(d, w)$, then put $\{Q_k, C_{\overline{c}}\} \subseteq \pi_i(d', w)$;
  - Dec-7: if $\{Q_i, \mathsf{Tail}_{\overline{c}}\} \subseteq \pi(d, w)$ then put $\{Q_k, \mathsf{Tail}_{\overline{c}}\} \subseteq \pi_i(d', w)$;

Using π^*, we set for all concept names $X \in \mathsf{sig}(\mathcal{T}_M) \setminus \{S, A, B\}$:

$$d \in X^{\mathfrak{I},w} \qquad \Longleftrightarrow \qquad X \in \pi^*(d, w) \qquad (\dagger)$$

It should be clear that $\varepsilon \in A^{\mathfrak{I},w_0}$ and $\varepsilon \notin B^{\mathfrak{I},w_0}$. Thus, $\mathfrak{I} \not\models A \sqsubseteq B$ and it remains to show that $\mathfrak{I}$ is a model of $\mathcal{T}_M$. By the definition of the interpretation of A, S, and the role names it is easy to see that the concept inclusions (11) and (12) are satisfied.

For CI (14) let $d \in S^{\mathfrak{I},w}$. Hence $w = w_0$ and $d \in \Delta_0$. By definition of π, we have $\{\mathsf{Head}_c, \mathsf{Tail}_c, Q_0\} \subseteq \pi(d, (d, c, 0))$ for both $c \in \{1, 2\}$. By (†), $d \in (\mathsf{Head}_c \sqcap \mathsf{Tail}_c \sqcap Q_0)^{\mathfrak{I},(d,c,0)}$ for $c \in \{1, 2\}$. Finally, by definition of W, we have $w < w_d$, $w_d < (d, 1, 0)$, and $w_d < (d, 2, 0)$. Thus, d satisfies the concept in the right-hand side of (14).

For CI (13) assume that $e \in \mathsf{Head}_c^{\mathfrak{I},w}$ and $(d, e) \in r^{\mathfrak{I},w}$. By the former and (†), we get $\mathsf{Head}_c \in \pi^*(e, w)$; by the latter, we get that $e = d \cdot r$. Thus, by Rule Bot, $\mathsf{Head}_c \in \pi^*(d, w)$. Finally, (†) yields $d \in \mathsf{Head}_c^{\mathfrak{I},w}$.

The concept inclusions (16)-(26) are satisfied since π^* is closed under the rules. Take as examples the inclusions (17), (20) and (23); the others can be treated analogously.

- For (16) assume $e \in (Q_i \sqcap C_c)^{\mathfrak{I},w}$ and $(d, e) \in r_i^{\mathfrak{I},w}$. By the former and (†), we get $\{Q_i, C_c\} \subseteq \pi^*(e, w)$ and by the latter, we get $e = \cdot r_i$. As π^* is closed under Rule Inc-2, we have $\{Q_j, C_c\} \subseteq \pi^*(d, w)$. Applying (†) again yields $d \in (Q_j \sqcap C_c)^{\mathfrak{I},w}$.
- For (20) assume $e \in (Q_i \sqcap \mathsf{Head}_c \sqcap \mathsf{Tail}_c)^{\mathfrak{I},w}$ and $(d, e) \in r_{i0}^{\mathfrak{I},w}$. By the former and (†), we get $\{Q_i, \mathsf{Head}_c, \mathsf{Tail}_c\} \subseteq \pi^*(d, w)$ and by the latter, we get $e = d \cdot r_{i0}$. As π^* is closed under Rule Dec-1, we have $\{Q_j, \mathsf{Tail}_c\} \subseteq \pi^*(d, w)$. Applying (†) again yields $d \in (Q_j \sqcap \mathsf{Tail}_c)^{\mathfrak{I},w}$.
- For (23) assume $e \in (Q_i \sqcap C_c \sqcap \mathbf{E}\bigcirc\mathsf{Tail}_c)^{\mathfrak{I},w}$ and $(d, e) \in r_{i+}^{\mathfrak{I},w}$. The latter implies $e = d \cdot r_{i+}$ and by the former, we get $e \in (Q_i \sqcap C_c)^{\mathfrak{I},w}$ and $e \in (\mathbf{E}\bigcirc\mathsf{Tail}_c)^{\mathfrak{I},w}$. By the former, we have $\{Q_i, C_c\} \subseteq \pi^*(e, w)$. It is easy to see that $\pi^*(e, w)$ can be non-empty only in case w is of the form (e', c', j). As there is only one successor world of w, namely $w + 1$, $e \in (\mathbf{E}\bigcirc\mathsf{Tail}_c)^{\mathfrak{I},w}$ implies $e \in \mathsf{Tail}_c^{\mathfrak{I},w+1}$ and thus, by (†), $\mathsf{Tail}_c \in \pi^*(e, w+1)$. As π^* is closed under Rule Dec-4, we have $\{Q_k, \mathsf{Tail}_c\} \subseteq \pi(d, w)$. Applying (†) again yields $d \in (Q_k \sqcap \mathsf{Tail}_c)^{\mathfrak{I},w}$.

For the last CI (15) assume that $d \in \mathbf{E}\bigcirc(\mathbf{E}\bigcirc(\mathsf{Head}_1 \sqcap Q_n) \sqcap \mathbf{E}\bigcirc(\mathsf{Head}_2 \sqcap Q_n))^{\mathfrak{I},w}$ but $d \notin B^{\mathfrak{I},w}$. By definition of the interpretation of B, we get $d = \varepsilon$ and $w = w_0$. Thus, there are some worlds w', w_1, w_2 such that $w < w'$, $w' < w_1$, and $w' < w_2$ such that $d \in (\mathsf{Head}_1 \sqcap Q_n)^{\mathfrak{I},w_1}$ and $d \in (\mathsf{Head}_2 \sqcap Q_n)^{\mathfrak{I},w_2}$. Clearly, $w' = w_e$ for some $e \in \Delta_0$ and $w_1 = (e, c_1, 0)$ and $w_2 = (e, c_2, 0)$. By (†), we get $\{\mathsf{Head}_1, Q_n\} \subseteq \pi^*(d, w_1)$ and $\{\mathsf{Head}_2, Q_n\} \subseteq \pi^*(d, w_2)$. It is easy to see that $\mathsf{Head}_c \in \pi^*(d, (e, c', 0))$ implies $c' = c$.[2] Hence $c_1 = 1$ and $c_2 = 2$.

In the remainder of the proof, we need the following two properties of π^*. Intuitively, the first property restricts the worlds where actually some concept is implied, namely only in worlds that witness a computation. The second property expresses that if no state concept Q_j is implied in a given point, then also in the future of the computation no state concept will be implied.

Claim 3. For all d, e, c we have

- $\pi^*(d, (e, c, i)) \neq \emptyset$ only for prefixes d of e.
- If there is no $j \geq 0$ and $0 \leq i \leq n$ such that $Q_i \in \pi^*(d, (e, c, j))$, then this holds for all prefixes d' of d.

[2] This can easily be shown by an induction over the rule applications to close π.

Proof of the Claim. The proof is by induction on the number of rule applications. Obviously, both points are true for π. For the induction step note that a rule can add Q_i to $\pi(d,(e,c,j))$ if there is some $d' = d \cdot r$ such that $\pi(d',(e,c,j)) \neq \emptyset$. By induction hypothesis, d' is a prefix of e, thus also d is a prefix of e. For the second point, observe that if there is no $k \geq 0$ and $0 \leq i \leq n$ such that $Q_i \in \pi^*(d',(e,c,k))$ then no rule is applicable to d.

This finishes the proof of Claim 3.

Let $e = e_1 \cdots e_\ell$ and for each $0 \leq i \leq \ell$ $E_i = e_1 \cdots e_{\ell-i}$, i.e., in particular, $E_\ell = \varepsilon = d$.

Claim 4. For every $0 \leq i \leq \ell$, there is a unique $q_i \in \{0,\ldots,\ell\}$ such that for every $c \in \{1,2\}$ there is a unique $n_i^c \geq 0$ with:

(a) $\mathsf{Tail}_c \in \pi^*(E_i,(e,c,j_c))$;

(b) $Q_{q_i} \in \pi^*(E_i,(e,c,j))$ for every $j \leq n_i^c$;

(c) $e_{\ell-i} = r_{q_i x}$, for some $x \in \{0,1,\varepsilon\}$

(d) $C_c \in \pi^*(E_i,(e,c,j))$ for every $j < n_i^c$;

(e) $(q_0,n_0^1,n_0^2) \Rightarrow_M \ldots \Rightarrow_M (q_i,n_i^1,n_i^2)$ is (the inital part of) the computation of M on input $(0,0)$.

Proof of Claim 4. The proof is by induction on i. For $i = 0$ observe that $E_i = e$ and thus $\pi(e,(e,c,0)) = \{Q_0, \mathsf{Head}_c, \mathsf{Tail}_c\}$ and $\pi(e,(e,c,j)) = \emptyset$ for $j > 0$ and $\pi(d,(e,c,j))$ is only nonempty for prefixes d of e. By an induction on the number of rule applications, it is easily seen that this last property is maintained and thus no rule can add anything new to $\pi^*(e,(e,c,j))$. Hence, $\pi^*(e,(e,c,j)) = \pi(e,(e,c,j))$ for all $j \geq 0$. We put $q_\ell = 0$ then $e_\ell = r_0$ and conditions (a) to (d) hold. Further, we can uniquely set $n_0^1 = 0$ and $n_0^2 = 0$ it remains to note that $(0,0,0)$ is the initial part of the computation of M on input $(0,0)$.

For the induction step assume that $E_{i-1} = E_i \cdot r$ for some $r \in \mathsf{ROL}$. By induction, there is precisely one state q_{i-1} and integers n_{i-1}^1, n_{i-1}^2 such that $\mathsf{Tail}_c \in \pi^*(E_{i-1},(e,c,n_{i-1}^c))$ for $c \in \{1,2\}$. We first prove (c). Assume r is not of the form $r_{q_{i-1}x}$ for some $x \in \{0,1,\varepsilon\}$. Then, no rule can be applied and $\pi^*(E_i,(e,c,j))$ does not contain any state concept name Q_x for all $j \geq 0$. By Claim 3, we know in particular that $Q_n \notin \pi^*(\varepsilon,(e,c,0))$. This contradicts the original assumption and thus proves (c).

To prove the other points we make a case distinction on r.

- $r = r_{q_i}$ and $I_{q_i} = \mathsf{inc}(c,k)$ for some $c \in \{1,2\}$. Rules Inc-1 to Inc-4 together with the hypothesis imply:
  - $\mathsf{Tail}_c \in \pi^*(E_i,(e,c,j)$ iff $j = n_{i-1}^c + 1$;
  - $\mathsf{Tail}_{\overline{c}} \in \pi^*(E_i,(e,\overline{c},j))$ iff $j = n_{i-1}^{\overline{c}}$;
  - $Q_m \in \pi^*(E_i,(e,c,j))$ iff $m = k$ and $j \leq n_{i-1}^c + 1$;
  - $Q_m \in \pi^*(E_i,(e,\overline{c},j))$ iff $m = k$ and $j \leq n_{i-1}^{\overline{c}}$.
  - $C_c \in \pi^*(E_i,(e,c,j))$ iff $j < n_{i-1}^c + 1$;
  - $C_{\overline{c}} \in \pi^*(E_i,(e,\overline{c},j))$ iff $j < n_{i-1}^{\overline{c}}$.

  Obviously, q_i, n_i^1, and n_i^2 are uniquely defined by the above by setting $q_i = k$, $n_i^c = n_{i-1}^c + 1$, and $n_i^{\overline{c}} = n_{i-1}^c$. It is now also easy to check that $(q_{i-1},n_{i-1}^1,n_{i-1}^2) \Rightarrow_M (q_i,n_i^1,n_i^2)$. Hence all conditions (a)-(e) are satisfied.

- $r = r_{q_i 0}$ and $I_{q_i} = \mathsf{dec}(c,k,k')$ for some $c \in \{1,2\}$. Rules Dec-1 to Dec-3 together with the hypothesis imply:
  - $\mathsf{Tail}_c \in \pi^*(E_i,(e,c,j)$ iff $j = n_{i-1}^c = 0$;
  - $\mathsf{Tail}_{\overline{c}} \in \pi^*(E_i,(e,\overline{c},j))$ iff $j = n_{i-1}^{\overline{c}}$;
  - $Q_m \in \pi^*(E_i,(e,c,j))$ iff $m = k$ and $j = n_{i-1}^c = 0$;
  - $Q_m \in \pi^*(E_i,(e,\overline{c},j))$ iff $m = k$ and $j \leq n_{i-1}^{\overline{c}}$.
  - $C_c \notin \pi^*(E_i,(e,c,j))$ for all $j \geq 0$;
  - $C_{\overline{c}} \in \pi^*(E_i,(e,\overline{c},j))$ iff $j < n_{i-1}^{\overline{c}}$.

  Observe that the third point in this list is a consequence of Dec-1 and implies $n_{i-1}^c = 0$: Assume the contrary, then $\pi^*(E_i,(e,c,j))$ does not contain any state concept Q_x for any x and $j \geq 0$. Claim 3 particularly yields $Q_n \notin \pi^*(\varepsilon,(e,c,0))$, contradiction.

  Further, q_i, n_i^1, and n_i^2 are uniquely defined by the above by setting $q_i = k$, $n_i^c = 0$ and $n_i^{\overline{c}} = n_{i-1}^{\overline{c}}$. It is easy to verify that $(q_{i-1},n_{i-1}^1,n_{i-1}^2) \Rightarrow_M (q_i,n_i^1,n_i^2)$. Hence all conditions (a)-(e) are satisfied.

- $r = r_{q_i +}$ and $I_{q_i} = \mathsf{dec}(c,k,k')$ for some $c \in \{1,2\}$. Rules Dec-4 to Dec-7 together with the hypothesis imply:
  - $\mathsf{Tail}_c \in \pi^*(E_i,(e,c,j)$ iff $j = n_{i-1}^c - 1$ and $n_{i-1}^c > 0$;
  - $\mathsf{Tail}_{\overline{c}} \in \pi^*(E_i,(e,\overline{c},j))$ iff $j = n_{i-1}^{\overline{c}}$;
  - $Q_m \in \pi^*(E_i,(e,c,j))$ iff $m = k'$, $j \leq n_{i-1}^c - 1$, and $n_{i-1}^c > 0$;
  - $Q_m \in \pi^*(E_i,(e,\overline{c},j))$ iff $m = k'$ and $j < n_{i-1}^{\overline{c}}$.
  - $C_c \in \pi^*(E_i,(e,c,j))$ iff $j < n_{i-1}^c - 1$;
  - $C_{\overline{c}} \in \pi^*(E_i,(e,\overline{c},j))$ iff $j < n_{i-1}^{\overline{c}}$.

  Observe that the third point in this list is a consequence of Dec-4 and implies $n_{i-1}^c > 0$: Assume otherwise, then $\pi^*(E_i,(e,c,j))$ does not contain any state concept Q_x for any x and $j \geq 0$. Again, Claim 3 yields $Q_n \notin \pi^*(\varepsilon,(e,c,0))$, contradiction.

  Obviously, q_i, n_i^1, and n_i^2 are uniquely defined by the above by setting $q_i = k'$, $n_i^c = n_{i-1}^c - 1$, and $n_i^{\overline{c}} = n_{i-1}^{\overline{c}}$. It is now also easy to check that $(q_{i-1},n_{i-1}^1,n_{i-1^2}) \Rightarrow_M (q_i,n_i^1,n_i^2)$. Hence all conditions (a)-(e) are satisfied.

This finishes the proof of Claim 4.

By Point (e) of Claim 4 together with the assumption that $Q_n \in \pi^*(\varepsilon,(e,c,0))$ for all $c \in \{1,2\}$ we get $q_\ell = n$. Hence, there is an accepting computation of M on input $(0,0)$. This is a contradiction to the assumption and, thus, finally shows that CI (15) is actually satisfied. Thus, $\mathfrak{I} \models \mathcal{T}$ but $\mathfrak{I} \not\models A \sqsubseteq B$.

This finishes the proof of the "only-if" direction, and of Lemma 9. ❑

This finishes the proof of Theorem 8. ❑

E Proofs for Section 4.4

Lemma 11 *For each $k,n \geq 1$, there is a polynomially sized (in k,n) CTL$^{\mathbf{E}\Diamond}_{\mathcal{EL}}$-TBox $\mathcal{T}_{k,n}$ that counts modulo* $\mathsf{exp}(k,n)$.

Proof. We begin with giving the TBox in an inductive way. In the induction base we provide a TBox that counts modulo 2^n. For the induction step we exploit the ability of counting modulo $\mathsf{exp}(k-1,n)$ in order to count modulo $\mathsf{exp}(k,n)$. As the definition of "counting" indicates, the counting is done along the DL dimension. We use the temporal dimension for the induction step. We need the following concept names:

- $\mathsf{Init}_i, 1 \leq i \leq k$: to initialize a counter of level i,
- $\mathsf{Zero}_i, 1 \leq i \leq k$: to mark the positions where the counter is 0,
- $\mathsf{Prop}_i, 2 \leq i \leq k$: to enforce an initial value of 0
- $\mathsf{Lvl}_i, 1 \leq i \leq k$: indicates the level of the counter,
- $\mathsf{Border}_i, 2 \leq i \leq k$: indicates the borders of the encoding of a counter value in level i,
- $B_i, \overline{B_i}, 2 \leq i \leq k$: bit variable for counter of level i
- $X_1, \overline{X_1}, \ldots, X_n, \overline{X_n}$ and $C_1, \overline{C_1}, \ldots, C_n, \overline{C_n}$ for encoding the bits and carry bits, respectively, of the level-1 counter,
- $Y_i, \overline{Y_i}, 2 \leq i \leq k$: an auxiliary variable to communicate between two consecutive numbers,
- $\mathsf{Ones}_i, 2 \leq i \leq k$: markers to detect long sequences of B_i,
- $\mathsf{Flip}_i, 2 \leq i \leq k$: the carry bit for increasing the level-i counter by one, and finally
- $\mathsf{Count}, \mathsf{Check}, \mathsf{Act}$: markers to distinguish different uses of the counters.

Let us first show how the level-1 counter is realized. For initializing it, we add the CIs:

$$\mathsf{Init}_1 \sqsubseteq \mathsf{Zero}_1, \qquad \mathsf{Zero}_1 \equiv \overline{X_1} \sqcap \ldots \sqcap \overline{X_n},$$
$$X_i \sqsubseteq \overline{\mathsf{Zero}_1}, \qquad X_i \sqcap \overline{X_i} \sqsubseteq \mathsf{Fail}, \qquad C_i \sqcap \overline{C_i} \sqsubseteq \mathsf{Fail}$$

The counting is now realized in the standard way using the following concept inclusions:

$$\exists r.X_1 \sqsubseteq \overline{X_1} \sqcap C_1 \tag{27}$$
$$\exists r.\overline{X_1} \sqsubseteq X_1 \sqcap \overline{C_1} \tag{28}$$
$$C_i \sqcap \exists r.X_{i+1} \sqsubseteq \overline{X_{i+1}} \sqcap C_{i+1} \tag{29}$$
$$C_i \sqcap \exists r.\overline{X_{i+1}} \sqsubseteq X_{i+1} \sqcap \overline{C_{i+1}} \tag{30}$$
$$\overline{C_i} \sqcap \exists r.X_{i+1} \sqsubseteq X_{i+1} \sqcap \overline{C_{i+1}} \tag{31}$$
$$\overline{C_i} \sqcap \exists r.\overline{X_{i+1}} \sqsubseteq \overline{X_{i+1}} \sqcap \overline{C_{i+1}} \tag{32}$$

For the induction step note that for every $k > 1$ we have $\mathcal{T}_{k-1,n} \subseteq \mathcal{T}_{k,n}$. In particular, $\mathcal{T}_{1,n}$ is always included. We start with including the following CI for every $1 < i \leq k$:

$$\mathsf{Init}_i \sqsubseteq \mathsf{Lvl}_i \sqcap \mathsf{Zero}_i \sqcap \mathsf{Prop}_i \sqcap \mathbf{E}\Diamond(\mathsf{Count} \sqcap \mathsf{Init}_{i-1}) \tag{33}$$

The concept names Lvl_i and Count are propagated along r-chains:

$$\exists r.\mathsf{Count} \sqsubseteq \mathsf{Count} \qquad \exists r.\mathsf{Lvl}_i \sqsubseteq \mathsf{Lvl}_i \qquad \text{for } 1 \leq i \leq k \tag{34}$$

Intuitively, an level-i counter for $i > 1$ is realized (again inductively) by first enforcing a sequence of $\mathsf{exp}(i-1,n)$ zeros along the DL dimension, and then enforcing that, if there is a sequence of length $\mathsf{exp}(i-1,n)$ encoding some number, say M, then the preceding sequence of the same length encodes $M+1$. Since the numbers are next to each other along the DL dimension, we need to define the *borders*:

$$\mathsf{Border}_i \equiv \mathsf{Lvl}_i \sqcap \mathbf{E}\Diamond(\mathsf{Count} \sqcap \mathsf{Zero}_{i-1}) \tag{35}$$
$$\overline{\mathsf{Border}_i} \equiv \mathsf{Lvl}_i \sqcap \mathbf{E}\Diamond(\mathsf{Count} \sqcap \overline{\mathsf{Zero}_{i-1}}) \tag{36}$$

More precisely, we use the following CIs for initializing the counter to 0 (using the concept names $B_i, \overline{B_i}$):

$$\overline{\mathsf{Border}_i} \sqcap \exists r.\mathsf{Prop}_i \sqsubseteq \mathsf{Prop}_i \tag{37}$$
$$\mathsf{Prop}_i \sqsubseteq \overline{B_i} \tag{38}$$

In order to correctly increment the level-i counter, we have to communicate between domain elements that have r-distance of $\mathsf{exp}(i-1,n)$. For this purpose, we introduce the following set of CIs for every $1 < i \leq k$:

$$B_i \sqsubseteq \mathbf{E}\Diamond(\mathsf{Init}_{i-1} \sqcap \mathsf{Act} \sqcap Y_i) \qquad \exists r.Y_i \sqsubseteq Y_i \tag{39}$$
$$\overline{B_i} \sqsubseteq \mathbf{E}\Diamond(\mathsf{Init}_{i-1} \sqcap \mathsf{Act} \sqcap \overline{Y_i}) \qquad \exists r.\overline{Y_i} \sqsubseteq \overline{Y_i} \tag{40}$$

These CIs allow to create a new world and store the value of the current bit there. This bit value is then transferred *precisely* $\mathsf{exp}(i-1,n)$ steps back along the r-chain. The following CIs mark the relevant position with Check:

$$\mathsf{Lvl}_{i-1} \sqcap \overline{\mathsf{Zero}_{i-1}} \sqcap \exists r.\mathsf{Act} \sqsubseteq \mathsf{Act} \qquad 1 < i \leq k \tag{41}$$
$$\mathsf{Lvl}_{i-1} \sqcap \mathsf{Zero}_{i-1} \sqcap \exists r.\mathsf{Act} \sqsubseteq \mathsf{Check} \qquad 1 < i \leq k \tag{42}$$

Finally, the next group of CIs realizes increment of a level-i counter:

$$\mathsf{Border}_i \sqsubseteq \mathsf{Flip}_i \tag{43}$$
$$\mathsf{Flip}_i \sqcap \mathbf{E}\Diamond(\mathsf{Check} \sqcap Y_i) \sqsubseteq \overline{B_i} \tag{44}$$
$$\overline{\mathsf{Flip}_i} \sqcap \mathbf{E}\Diamond(\mathsf{Check} \sqcap Y_i) \sqsubseteq B_i \tag{45}$$
$$\mathsf{Flip}_i \sqcap \mathbf{E}\Diamond(\mathsf{Check} \sqcap \overline{Y_i}) \sqsubseteq B_i \tag{46}$$
$$\overline{\mathsf{Flip}_i} \sqcap \mathbf{E}\Diamond(\mathsf{Check} \sqcap \overline{Y_i}) \sqsubseteq \overline{B_i} \tag{47}$$
$$\overline{\mathsf{Border}_i} \sqcap \exists r.(\mathsf{Flip}_i \sqcap \overline{B_i}) \sqsubseteq \mathsf{Flip}_i \tag{48}$$
$$\overline{\mathsf{Border}_i} \sqcap \exists r.(\mathsf{Flip}_i \sqcap B_i) \sqsubseteq \overline{\mathsf{Flip}_i} \tag{49}$$
$$\overline{\mathsf{Border}_i} \sqcap \exists r.\overline{\mathsf{Flip}_i} \sqsubseteq \overline{\mathsf{Flip}_i} \tag{50}$$

The concept Zero_i needs to be enforced at a border element if all the successor elements on the way to the next border element satisfy B_i. For this purpose we use the auxiliary variable Ones_i. Starting at a border, it marks sequences of elements all satisfying B_i:

$$\mathsf{Border}_i \sqcap B_i \sqsubseteq \mathsf{Ones}_i \tag{51}$$
$$\overline{B_i} \sqsubseteq \overline{\mathsf{Ones}_i} \tag{52}$$
$$\overline{\mathsf{Border}_i} \sqcap \exists r.\mathsf{Ones}_i \sqcap B_i \sqsubseteq \mathsf{Ones}_i \tag{53}$$
$$\overline{\mathsf{Border}_i} \sqcap \exists r.\overline{\mathsf{Ones}_i} \sqsubseteq \overline{\mathsf{Ones}_i} \tag{54}$$

We enforce the correct evaluation of Zero_i and $\overline{\mathsf{Zero}_i}$ with the following set of CIs.

$$\overline{\mathsf{Border}_i} \sqsubseteq \overline{\mathsf{Zero}_i} \tag{55}$$
$$\mathsf{Border}_i \sqcap \exists r.\mathsf{Ones}_i \sqsubseteq \mathsf{Zero}_i \tag{56}$$
$$\mathsf{Border}_i \sqcap \exists r.\overline{\mathsf{Ones}_i} \sqsubseteq \overline{\mathsf{Zero}_i} \tag{57}$$

It remains to specify the usage of the concept name Fail. In general, Fail is used to enforce that complementary concept names behave in a complementary way. Thus, we include the following CI for every pair of concept names $(D, \overline{D})$:

$$D \sqcap \overline{D} \sqsubseteq \mathsf{Fail} \tag{58}$$

Finally, we propagate Fail using the following CIs:

$$\mathbf{E}\Diamond\mathsf{Fail} \sqsubseteq \mathsf{Fail} \quad \text{and} \quad \exists r.\mathsf{Fail} \sqsubseteq \mathsf{Fail} \tag{59}$$

Clearly, the size of $\mathcal{T}_{k,n}$ is polynomial in k, n. So it remains to prove that (a) for all $i \geq 0$ the constructed $\mathcal{T}_{k,n}$ and $\exists r^i.\mathsf{Init}$ is satisfiable in models where Fail is interpreted empty in every world and (b) that in such models $\mathfrak{I}$ it holds for all $i \geq 0$:

(i) $\mathfrak{I} \models \exists r^i.\mathsf{Init} \sqsubseteq \mathsf{Zero}$ iff $i \equiv 0 \bmod n$;

(ii) $\mathfrak{I} \models \exists r^i.\mathsf{Init} \sqsubseteq \overline{\mathsf{Zero}}$ iff $i \not\equiv 0 \bmod n$.

Part (a) is done via constructing a *minimal model* of $\mathcal{T}_{k,n}$ and $\exists r^i.\mathsf{Init}$ and show that Fail is never satisfied. This is straightforward and we will concentrate on part (b).

The proof is by induction on k. The induction base $k = 1$ follows directly from Equations (27)-(32).

For the induction step, fix some $k > 1$ and let $\mathfrak{I} = (\Delta, W, \{\mathcal{I}_w\}_{w \in W})$ be a model of $\mathcal{T}_{k,n}$ and $\exists r^m.\mathsf{Init}$. Thus, there are a world $w \in W$ and domain elements $d_0, \ldots, d_m \in \Delta$ such that $(d_i, d_{i-1}) \in r^{\mathfrak{I},w}$, $d_0 \in \mathsf{Init}_k^{\mathfrak{I},w}$, and $d_i \notin \mathsf{Fail}^{\mathfrak{I},w}$ for all i. By CI (33), there is some $w' > w$ such that both $d_0 \in \mathsf{Init}_{k-1}^{\mathfrak{I},w'}$ and $d_0 \in \mathsf{Count}^{\mathfrak{I},w'}$. By the first CI in (34), $d_i \in \mathsf{Count}^{\mathfrak{I},w'}$ for all $0 \leq i \leq m$. Observe that $d_i \notin \mathsf{Fail}^{\mathfrak{I},w'}$ for all i since otherwise $d_i \in \mathsf{Fail}^{\mathfrak{I},w}$ by CI (59). By applying induction hypothesis to $d_0, \ldots, d_m$, w', and $k - 1$, we obtain that

(i) $d_i \in \mathsf{Zero}_{k-1}^{\mathfrak{I},w'}$ iff $i \equiv 0 \bmod M$ and

(ii) $d_i \in \overline{\mathsf{Zero}_{k-1}}^{\mathfrak{I},w'}$ iff $i \not\equiv 0 \bmod M$,

where we define $M := \mathsf{exp}(k-1, n)$. By CIs (35) and (36), we get

(i') $d_i \in \mathsf{Border}_k^{\mathfrak{I},w}$ iff $i \equiv 0 \bmod M$ and

(ii') $d_i \in \overline{\mathsf{Border}_k}^{\mathfrak{I},w}$ iff $i \not\equiv 0 \bmod M$.

We divide the sequence $d_0, \ldots, d_m$ of domain elements into subsequences $s_0, \ldots, s_u$ each of length M by taking $s_0 = d_0, \ldots, d_{M-1}$, $s_1 = d_M, \ldots, d_{2M-1}$, and so on. Moreover, denote with s^j the j-th element of such a sequence s (starting with 0). Finally, define the functions $\mathsf{bit}_k : \Delta \to \{0, 1, \mathsf{undef}\}$ and $\mathsf{val}_k : \{s_0, \ldots, s_u\} \to \mathbb{N} \cup \{\mathsf{undef}\}$ as follows:

$$\mathsf{bit}_k(d) = \begin{cases} 0 & d \in \overline{B_k}^{\mathfrak{I},w} \setminus B_k^{\mathfrak{I},w} \\ 1 & d \in B_k^{\mathfrak{I},w} \setminus \overline{B_k}^{\mathfrak{I},w} \\ \mathsf{undef} & \text{otherwise.} \end{cases}$$

$$\mathsf{val}_k(s) = \begin{cases} \mathsf{undef} & \exists i.\mathsf{bit}_k(s^i) = \mathsf{undef} \\ \sum_{i=0}^{M-1} \mathsf{bit}_k(s^i) \cdot 2^i & \text{otherwise.} \end{cases}$$

It should be clear that, if defined, $\mathsf{val}_k(s) \in \{0, \ldots, 2^M - 1\}$ for all sequences s defined above. Let us verify the following claim.

Claim 1. For $i \in \{0, \ldots, u\}$ we have

(a) $\mathsf{bit}_k(s_i^j) \neq \mathsf{undef}$ for all $0 \leq j < M$, moreover

(b) $\mathsf{val}_k(s_i) = i \bmod 2^M$.

Proof of Claim 1. The proof is by induction on i. For the induction base fix $i = 0$ and abbreviate $s := s_0$. We show that $\mathsf{val}_k(s) = 0$, i.e., $\mathsf{bit}_k(s^j) = 0$ for every $0 \leq j < M$. As $s = d_0, \ldots, d_{M-1}$, we have that $s^0 \in \mathsf{Init}_k^{\mathfrak{I},w}$ and thus, by CI (33), $s^0 \in \mathsf{Prop}_k^{\mathfrak{I},w}$. By CI (37) and item (ii') above, we have that $s^j \in \mathsf{Prop}_k^{\mathfrak{I},w}$ for all $0 \leq j < M$. Now, CI (38) implies $s^j \in \overline{B_k}^{\mathfrak{I},w}$ for all $0 \leq j < M$. By the assumption that $s^j \notin \mathsf{Fail}^{\mathfrak{I},w}$ and CI (58), we obtain that $s^j \notin B_k^{\mathfrak{I},w}$, and thus, $\mathsf{bit}_k(s^j) = 0$ for all $0 \leq j < M$. Thus, $\mathsf{val}_k(s_0) = 0$.

For the induction step, assume $i > 0$. For this purpose abbreviate $s := s_i$ and $t := s_{i-1}$, i.e., $s = d_{iM}, \ldots, d_{(i+1)M-1}$ and $t = d_{(i-1)M}, \ldots, d_{iM-1}$. We show the following points via an inductive argument for all $0 \leq j < M$:

P1 If for all $x < j$ it is $t^x \in B_k^{\mathfrak{I},w}$ then $s^j \in \mathsf{Flip}_k^{\mathfrak{I},w}$; otherwise $s^j \in \overline{\mathsf{Flip}_k}^{\mathfrak{I},w}$.

P2 If either $t^j \in B_k^{\mathfrak{I},w}$ and $s^j \in \overline{\mathsf{Flip}_k}^{\mathfrak{I},w}$ or $t^j \in \overline{B_k}^{\mathfrak{I},w}$ and $s^j \in \mathsf{Flip}_k^{\mathfrak{I},w}$ then $s^j \in B_k^{\mathfrak{I},w}$; otherwise $s^j \in \overline{B_k}^{\mathfrak{I},w}$.

It should be clear that **P2** implies part (a) of the Claim, because of the first CI in (58) and the assumption $s^x \notin \mathsf{Fail}^{\mathfrak{I},w}$ for all $0 \leq x < M$. Moreover, properties **P1** and **P2** also imply part (b) of the Claim as they specify precisely the process of incrementing the M-bit number represented by $t (= s_{i-1})$ with Flip_k acting as the carry bit. Hence, $\mathsf{val}_k(s_i) = (\mathsf{val}_k(s_{i-1}) + 1) \bmod 2^M$ and we are done.

We start with j=0. For showing **P1**, observe that $s^0 \in \mathsf{Border}_k^{\mathfrak{I},w}$ by item (i') above. By CI (43), we have that $s^0 \in \mathsf{Flip}_k^{\mathfrak{I},w}$. For verifying **P2**, observe first that by induction hypothesis of part (a) of the Claim we have either $t^0 \in B_k^{\mathfrak{I},w}$ or $t^0 \in \overline{B_k}^{\mathfrak{I},w}$. W.l.o.g., let us assume that $t^0 \in B_k^{\mathfrak{I},w}$ (the other case works analogously). By the first CI in (39), there is some world $v > w$ with (I) $t^0 \in \mathsf{Init}_{k-1}^{\mathfrak{I},v}$, (II) $t^0 \in \mathsf{Act}^{\mathfrak{I},v}$ and (III) $t^0 \in Y_k^{\mathfrak{I},v}$. By (I) and (33) we have $t^0 \in \mathsf{Lvl}_{k-1}^{\mathfrak{I},v}$, thus by CI (34) also $s^0 \in \mathsf{Lvl}_{k-1}^{\mathfrak{I},v}$. Also by (I), we can apply the (outermost) induction hypothesis to $t^0, t^1, \ldots, t^{M-1}, s^0, v$ and $k - 1$ yielding $t^x \in \overline{\mathsf{Zero}_{k-1}}^{\mathfrak{I},v}$ for all $0 < x < M$ and $s^0 \in \mathsf{Zero}_{k-1}^{\mathfrak{I},v}$. Together with (II) and CI (41) we obtain $t^x \in \mathsf{Act}^{\mathfrak{I},v}$ for all $0 \leq x < M$. By CI (42), we get $s^0 \in \mathsf{Check}^{\mathfrak{I},v}$. Finally, (III) together with the second CI in (39) implies $s^0 \in Y_k^{\mathfrak{I},v}$. Overall, we have $s^0 \in (\mathsf{Flip}_k \sqcap \mathbf{E}\Diamond(\mathsf{Check} \sqcap Y_k))^{\mathfrak{I},w}$. The CI in (44) yields $s^0 \in \overline{B_k}^{\mathfrak{I},w}$.

Let now be $0 < j < M$ and let us again start with **P1**. Observe that by item (ii') above, $s^j \in \overline{\mathsf{Border}_k}^{\mathfrak{I},w}$. Assume first

$s^{j-1} \in \overline{\mathsf{Flip}_k}^{\mathfrak{I},w}$. Then, by induction hypothesis (on **P1**), there is some $x < j-1$ such that $t^x \notin B_k^{\mathfrak{I},w}$. Hence, in order to verify **P1** we have to show that $s^j \in \overline{\mathsf{Flip}_k}^{\mathfrak{I},w}$. However, this follows directly from CI (50). Assume on the other hand $s^{j-1} \in \mathsf{Flip}_k^{\mathfrak{I},w}$. According to the induction hypothesis of part (a) of the Claim we distinguish two cases:

- $s^{j-1} \in B_k^{\mathfrak{I},w}$: According to the induction hypothesis of **P2** we have that $t^{j-1} \in \overline{B_k}^{\mathfrak{I},w}$. For proving **P1**, we have to show that $s^j \in \overline{\mathsf{Flip}_k}^{\mathfrak{I},w}$. However, this follows immediately from CI (49).
- $s^{j-1} \in \overline{B_k}^{\mathfrak{I},w}$: Analogously.

Verifying **P2** for $j > 0$ works analogously to the case $j = 0$ using the CIs (44)-(47), respectively.

This finishes the proof of the claim.

We show another property of the sequences $s_0, \ldots, s_u$.

Claim 2. For all $s \in \{s_0, \ldots, s_u\}$ and all $0 \leq j < M$ we have: if $s^x \in B_k^{\mathfrak{I},w}$ for all $0 \leq x \leq j$, then $s^j \in \mathsf{Ones}_k^{\mathfrak{I},w}$; otherwise $s^j \in \overline{\mathsf{Ones}_k}^{\mathfrak{I},w}$.

Proof of Claim 2. The proof is by a straightforward induction on j. For $j = 0$ observe that $s^0 \in \mathsf{Border}_k^{\mathfrak{I},w}$ by item (i') above. If additionally $s^0 \in B_k^{\mathfrak{I},w}$ then CI (51) implies $s^0 \in \mathsf{Ones}_k^{\mathfrak{I},w}$. Otherwise CI (52) implies $s^0 \in \overline{\mathsf{Ones}_k}^{\mathfrak{I},w}$. Let now be $j > 0$. We distinguish two cases (according to part (a) of Claim 1):

- If $s^x \in B_k^{\mathfrak{I},w}$ for all $x \leq j$ then by induction hypothesis, $s^{j-1} \in \mathsf{Ones}_k^{\mathfrak{I},w}$. Together with $s^j \in B_k^{\mathfrak{I},w}$, the concept inclusion (53) implies $s^j \in \mathsf{Ones}_k^{\mathfrak{I},w}$.
- If there is some $x \leq j$ with $s^x \in \overline{B_k}^{\mathfrak{I},w}$, then fix such an x. If $x = j$, CI (52) implies $s^j \in \overline{\mathsf{Ones}_k}^{\mathfrak{I},w}$. If $x < j$, the induction hypothesis implies $s^{j-1} \in \overline{\mathsf{Ones}_k}^{\mathfrak{I},w}$ and CI (54) implies $s^j \in \overline{\mathsf{Ones}_k}^{\mathfrak{I},w}$.

This finishes the proof of the claim.

We are now ready to establish the statement. For this it suffices to prove the following two items:

- $d_i \in \mathsf{Zero}_k^{\mathfrak{I},w}$ iff $i \equiv 0 \bmod \mathsf{exp}(k,n)$ and
- $d_i \in \overline{\mathsf{Zero}_k}^{\mathfrak{I},w}$ iff $i \not\equiv 0 \bmod \mathsf{exp}(k,n)$.

We show here only the "if"-directions of both points; the "only if" then follows because Fail is assumed to be empty and CI (58).

Observe first that $i \not\equiv 0 \bmod \mathsf{exp}(k-1,n)$ implies $d_i \in \overline{\mathsf{Border}_k}^{\mathfrak{I},w}$ by item (ii') above. Hence, $d_i \in \overline{\mathsf{Zero}_k}^{\mathfrak{I},w}$ by CI (55). Hence, it remains to consider those i such that $i \equiv 0 \bmod \mathsf{exp}(k-1,n)$. By item (i') above, we have that $d_i \in \mathsf{Border}_k^{\mathfrak{I},w}$. Moreover, by definition we have $i = xM$ for some $x \geq 0$ and d_i is the start of the sequence s_x. If $x = 0$, the statement follows from the fact that $d_0 \in \mathsf{Init}_k^{\mathfrak{I},w}$ and thus, by CI (33), $d_0 \in \mathsf{Zero}_k^{\mathfrak{I},w}$. For $x > 0$, we distinguish cases on $\mathsf{val}_k(s_x)$; specifically, we will show that these cases are equivalent to the first and second point above, respectively, given that $i \equiv 0 \bmod \mathsf{exp}(k-1,n)$.

- If $\mathsf{val}_k(s_x) = 0$, then $\mathsf{val}_k(s_{x-1}) = \mathsf{exp}(k,n) - 1$ by Claim 1(b). By definition of val_k, we have that $\mathsf{bit}_k(s_{x-1}^z) = 1$ for all $0 \leq z < M$, hence $s_{x-1}^z \in B_k^{\mathfrak{I},w}$ for all z. By Claim 2, we have that $s_{x-1}^{M-1} = d_{i-1} \in \mathsf{Ones}_k^{\mathfrak{I},w}$. By CI (56), we obtain $d_i \in \mathsf{Zero}_k^{\mathfrak{I},w}$.
- If $\mathsf{val}_k(s_x) > 0$, then $\mathsf{val}_k(s_{x-1}) < \mathsf{exp}(k,n) - 1$. By definition of val_k, there is some $0 \leq z < M$ with $\mathsf{bit}_k(s_{x-1}^z) = 0$, hence $s_{x-1}^z \in \overline{B_k}^{\mathfrak{I},w}$. By Claim 2, we have that $s_{x-1}^{M-1} = d_{i-1} \in \overline{\mathsf{Ones}_k}^{\mathfrak{I},w}$. By CI (57), we obtain $d_i \in \overline{\mathsf{Zero}_k}^{\mathfrak{I},w}$.

The mentioned equivalence is seen as follows. By Claim 1(b), $\mathsf{val}_k(s_x) = 0$ iff $x \equiv \bmod 2^{\mathsf{exp}(k-1,n)}$, i.e., x is a multiple of $2^{\mathsf{exp}(k-1,n)}$. Obviously , this is equivalent to $i = x \cdot M$ is a multiple of $M \cdot 2^{\mathsf{exp}(k-1,n)} = \mathsf{exp}(k,n)$.

This finishes the proof of Lemma 11. ❑

Theorem 12 *Reasoning relative to* $CTL_{\mathcal{EL}}^{\mathbf{E}\Diamond}$*-TBoxes is* k-EXPSPACE*-hard for every* $k \geq 1$.

Proof. Let $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ be a deterministic TM with space bounded by $\mathsf{exp}(k, p(n))$ on inputs of length n for some polynomial $p(n)$. Set $N := p(n)$. By Lemma 11, there is a TBox $\mathcal{T}_{k,N}$ that counts modulo $\mathsf{exp}(k,N)$ and whose size is polynomial in k, N and hence also polynomial in k, n.

Our aim is to construct in polynomial time a TBox $\mathcal{T}_{M,w,k}$ and concepts A, B such that $\mathcal{T}_{M,w,k} \models A \sqsubseteq B$ iff M accepts w.

- all symbols from $\mathsf{sig}(\mathcal{T}_{k,N})$
- concept names $A_a, \overline{A_a}$ for every $a \in \Gamma$ (in particular $A_b, \overline{A_b}$ for the *blank* symbol),
- concept names H_q for every state $q \in Q$ indicating head position and current state, and $\overline{H}$ for no state,
- concept names Y_a and Y_q for synchronizing two consecutive configurations,
- a concept name Init and concept names $W_1, \ldots, W_n$ for encoding the input word and a symbol B for the rest of the tape, and
- an auxiliary symbol Tape which is used to indicate the encoding of the tape of M.

As every configuration of M has at most size $\mathsf{exp}(k,N)$, we include $\mathcal{T}_{k,N}$; moreover, let Init_k, Zero_k, $\overline{\mathsf{Zero}_k}$, r the concept and role names witnessing that $\mathcal{T}_{k,N}$ counts modulo $\mathsf{exp}(k,N)$. A configuration is encoded along a sequence of $\mathsf{exp}(k,N)$ domain elements, but in reverse direction, i.e., the left-most symbol of the tape is encoded in the right-most domain element.

The basic idea is that each model $\mathfrak{I}$ of $\mathcal{T}_{M,w,k}$ will take the form of an infinite r-chain, enforced by the CI

$$A \sqsubseteq \exists r.A.$$

In every point of the root world, one computation of the Turing machine is initiated:

$$A \sqsubseteq \mathbf{E}\Diamond\mathsf{Init}$$

together with

$$\mathsf{Init} \sqsubseteq \mathsf{Tape} \sqcap \mathsf{Init}_k \sqcap W_1 \sqcap H_{q_0} \qquad \exists r.\mathsf{Tape} \sqsubseteq \mathsf{Tape}.$$

As above, the computation is performed backwards along the r-chain. The initial configuration for the input w is enforced by the following CIs:

$$\exists r.W_i \sqsubseteq W_{i+1} \qquad W_i \sqsubseteq A_{a_i}$$

$$\exists r.W_n \sqsubseteq B \qquad \overline{\mathsf{Zero}_k} \sqcap \exists r.B \sqsubseteq B \qquad B \sqsubseteq A_b$$

It remains to enforce that two consecutive configurations satisfy the transition relation δ of M, in particular:

- tape symbols in non-head positions remain unchanged,
- symbols in the head position are changed according to δ, and
- head position and state are updated according to δ.

For all of them, the idea is to compare the cell content with the state that is $\mathsf{exp}(k,N)$ r-steps away. As in the proof of Lemma 11 one can do this by creating a temporal successor, storing the information of the current position there, and accessing the information $\mathsf{exp}(k,N)$ steps later (backwards along the r-chain). More precisely, for achieving the first item above we include for each $a \in \Gamma$ the following group of CIs:

$$A_a \sqcap \overline{H} \sqsubseteq \mathbf{E}\Diamond(Y_a \sqcap \mathsf{Init}_k \sqcap \mathsf{Act}) \qquad \exists r.Y_a \sqsubseteq Y_a$$

$$\mathsf{Zero}_k \sqcap \exists r.\mathsf{Act} \sqsubseteq \mathsf{Check} \qquad \overline{\mathsf{Zero}_k} \sqcap \exists r.\mathsf{Act} \sqsubseteq \mathsf{Act}$$

$$\mathsf{Tape} \sqcap \mathbf{E}\Diamond(\mathsf{Check} \sqcap Y_a) \sqsubseteq A_a \qquad \text{for all } a \neq b$$

For the second and third item assume that $\delta(a,q) = (b,p,d)$ with $d \in \{L,R\}$. We introduce the CI

$$A_a \sqcap H_q \sqsubseteq \mathbf{E}\Diamond(Y_b \sqcap \mathsf{Init}_k \sqcap \mathsf{Act}).$$

For moving the head to the left ($d = L$) we use the CIs

$$A_a \sqcap H_q \sqsubseteq \mathbf{E}\Diamond(Y_p \sqcap \exists r.\mathsf{Init}_k \sqcap \mathsf{Act})$$

together with

$$\exists r.Y_q \sqsubseteq Y_q \quad \mathsf{Tape} \sqcap \mathbf{E}\Diamond(\mathsf{Check} \sqcap Y_q) \sqsubseteq H_q \quad \text{for all } q \in Q$$

For moving the head to the right ($d = R$) we use the CIs

$$A_a \sqcap H_q \sqsubseteq \mathbf{E}\Diamond(\mathsf{Act}_R \sqcap Y_p) \qquad \exists r.\mathsf{Act}_R \sqsubseteq \mathsf{Init}_k \sqcap \mathsf{Act}.$$

Additionally, we need to ensure that between two state symbols H_q, $H_{q'}$ along the r-chain, everywhere $\overline{H}$ is satisfied. This is done by additionally introducing for each $q \in Q$ the CI

$$\mathsf{Tape} \sqcap \mathbf{E}\Diamond(\overline{\mathsf{Zero}} \sqcap \mathsf{Act} \sqcap Y_q) \sqsubseteq \overline{H}$$

It remains to explain the failure-concept Fail. As in the proof of Lemma 11, it is used to enforce that complementary concept names behave as expected. For example, we have to recognize when Check is satisfied more than once along some r-chain:

$$\exists r.\mathsf{Check} \sqsubseteq \overline{\mathsf{Check}} \qquad \exists r.\overline{\mathsf{Check}} \sqsubseteq \overline{\mathsf{Check}}$$

$$\mathsf{Check} \sqcap \overline{\mathsf{Check}} \sqsubseteq \mathsf{Fail} \qquad \mathsf{Check} \sqcap \mathsf{Act} \sqsubseteq \mathsf{Fail}$$

$$\mathsf{Act} \sqcap \mathsf{Act}' \sqsubseteq \mathsf{Fail} \qquad \mathsf{Check} \sqcap \mathsf{Act}' \sqsubseteq \mathsf{Fail}$$

Further, there are some standard conditions which are covered by the CI

$$D \sqcap \overline{D} \sqsubseteq \mathsf{Fail}$$

for each complementary pair $(D, \overline{D})$ of concept names (including for instance $H_q, \overline{H}$).

The failure concept Fail is propagated along the r-chain using

$$\exists r.\mathsf{Fail} \sqsubseteq \mathsf{Fail}.$$

Finally, we include the following two CIs:

$$A \sqcap \mathbf{E}\Diamond\mathsf{Fail} \sqsubseteq B \qquad A \sqcap \mathbf{E}\Diamond H_{q_{acc}} \sqsubseteq B.$$

It is straightforward to show that M accepts w iff $\mathcal{T}_{M,w} \models A \sqsubseteq B$ using the techniques developed and demonstrated in the proof of Lemma 11. ❑

F Proofs for Section 5

Theorem 13. *Concept satisfiability w.r.t. TBoxes is* ExpTime*-complete*
1. for CTL-*DL-Lite*$^{\mathcal{N}}_{bool}$ *without local roles,*
2. for CTL$^{\mathbf{E}\mathcal{U},\mathbf{E}\Box}$-*DL-Lite*$^{\mathcal{N}}_{bool}$*,*
and PSpace*-complete*
3. for CTL$^{\mathbf{E}\Diamond}$-*DL-Lite*$^{\mathcal{N}}_{bool}$*.*
To prove Theorem 13, we proceed in two steps.

From CTL-*DL-Lite*$^{\mathcal{N}}_{bool}$ to $\mathcal{QCTL}^1$

We give a reduction from CTL-*DL-Lite*$^{\mathcal{N}}_{bool}$ to the one-variable fragment $\mathcal{QCTL}^1$ of the *first-order branching temporal logic* $\mathcal{QCTL}$ (Hodkinson, Wolter, and Zakharyaschev 2002).

Let $\mathcal{T}$ and C be the input TBox and concept formulated in CTL-*DL-Lite*$^{\mathcal{N}}_{bool}$, and $\mathsf{ROL}(\mathcal{T},C)$ the set of roles (rigid, local) occurring in $\mathcal{T}$ and C and their inverses. We use $\mathsf{inv}(r)$ to denote the inverse of a role r, that is, $\mathsf{inv}(r) = r^-$ and $\mathsf{inv}(r^-) = r$. Furthermore, let $Q_{(C,\mathcal{T})}$ be the set containing 1 and all q such that $\geq q\ r$ occurs in $\mathcal{T}$ or C.

Following the technique by Artale et al. (2012), we map concept names A to unary predicates $A(x)$ and number restrictions $\geq q\ r$ to unary predicates $E^r_q(x)$. Note that the domain and range of a role r can be respectively encoded by $E^r_1(x)$ and $E^{r^-}_1(x)$.

We define the translation from CTL-*DL-Lite*$^{\mathcal{N}}_{bool}$ concepts C to $\mathcal{QCTL}^1$ formulas $C^\dagger(x)$.

- $A^\dagger = A(x)$
- $\bot^\dagger = \bot$
- $(\neg C)^\dagger = \neg C^\dagger$
- $(C \sqcap D)^\dagger = C^\dagger \wedge D^\dagger$
- $(\geq qr)^\dagger = E^r_q(x)$
- $(\mathbf{E}\bigcirc C)^\dagger = \mathbf{E}\bigcirc C^\dagger$
- $(\mathbf{E}\Box C)^\dagger = \mathbf{E}\Box C^\dagger$
- $(\mathbf{E}C\mathcal{U}D)^\dagger = \mathbf{E}(C^\dagger \mathcal{U} D^\dagger)$

We encode the TBox $\mathcal{T}$ using the following $\mathcal{QCTL}^1$ sentence φ.

$$\varphi = \bigwedge_{C \sqsubseteq D \in \mathcal{T}} \mathbf{A}\Box\, \forall x\, (C^\dagger(x) \rightarrow D^\dagger(x)) \qquad (60)$$

We next ensure that unary predicates $E^r_q(x)$ properly capture the semantics of number restrictions across the whole temporal model:

- Each element with at least q' r-successors has at least q r-successors, for each $q < q'$:

$$\bigwedge_{r \in \mathsf{ROL}(C,\mathcal{T})} \bigwedge_{\substack{q,q' \in Q_{(C,\mathcal{T})} \\ q' > q}} \mathbf{A}\Box\, \forall x\, (E^r_{q'}(x) \rightarrow E^r_q(x)) \quad (61)$$

- If r is a rigid role, then every element with at least q r-successors at some time point has at least q r-successors at all time points:

$$\bigwedge_{\substack{r \in \mathsf{ROL}(C,\mathcal{T}) \\ r \text{ is rigid}}} \bigwedge_{q \in Q_{(C,\mathcal{T})}} \mathbf{A}\Box\, \forall x\, (\mathbf{E}\Diamond E^r_q(x) \rightarrow \mathbf{A}\Box E^r_q(x)) \quad (62)$$

- If the domain of a role is not empty, then its range neither:

$$\bigwedge_{r \in \mathsf{ROL}(C,\mathcal{T})} \mathbf{A}\Box(\exists x (\exists r)^\dagger(x) \rightarrow \exists x (\exists \mathsf{inv}(r))^\dagger(x)) \quad (63)$$

We denote by ψ the conjunction of formulas (61)-(63). Finally, we define the $\mathcal{QCTL}^1$ translation of $\mathcal{T}$ as follows.

$$\varphi_\mathcal{T} = \varphi \wedge \psi$$

The translation of C is $C^\dagger$. It can be easily checked that the size of the translation is polynomial on the size of $\mathcal{T}$ and C.

Theorem 23 *A CTL-DL-Lite$^{\mathcal{N}}_{bool}$ concept C is satisfiable w.r.t. a CTL-DL-Lite$^{\mathcal{N}}_{bool}$ TBox $\mathcal{T}$ iff $\varphi_\mathcal{T} \wedge C^\dagger$ is satisfiable.*

Proof. "$\Leftarrow$:" Let $\mathfrak{M} = (T, D, I)$ be a $\mathcal{QCTL}$model with a countable domain D such that $\mathfrak{M}, \varepsilon \models \varphi_\mathcal{T} \wedge C^\dagger$. We provide a temporal model $\mathfrak{I} = (\Delta, T, \{\mathcal{I}_w\}_{w \in W})$ of $\mathcal{T}$ such that $C^{\mathfrak{I},\varepsilon} \neq \emptyset$.

We define sequences $\Delta_0, \Delta_1, \ldots$ partial mappings π_1, π_2 with $\pi_i : \Delta_i \rightarrow D$, and relations $\mathsf{R}^r_0, \mathsf{R}^r_1, \ldots$ with $r \in \mathsf{ROL}(C,\mathcal{T})$ and $\mathsf{R}^r_i \subseteq \Delta_i \times \Delta_i \times W$. We obtain our desired sets Δ and R^r in the limit.

- *Initial step.* Set
  - $\Delta_0 = \{d\}$;
  - $\pi_0(d) = a$ such that $\mathfrak{M}, \varepsilon \models C^\dagger[a]$;
  - $\mathsf{R}^r_0 = \emptyset$ for all $r \in \mathsf{ROL}(C,\mathcal{T})$.
- Completion step. We observe that, for all $w \in W$ and $a \in D$,
  - by Equation (61), $\mathfrak{M}, w \models E^r_q[a]$ implies $\mathfrak{M}, w \models E^r_1[a]$, and
  - by Equation (63), there exists a $a' \in D$ such that $\mathfrak{M}, w \models E^{r^-}_1[a']$.

  Set $\Delta_i = \Delta_{i-1}$ and $\pi_i = \pi_{i-1}$, and proceed according to the following rules:

  1. If $\mathfrak{M}, w \models E^r_q[\pi_i(d)]$ for some $d \in \Delta_i$ and $w \in W$ such that there is no $E^r_{q'}$ with $q' > q$ and $\mathfrak{M}, w \models E^r_{q'}[\pi_i(d)]$, then
     - add $e_1, \ldots, e_q$ to Δ_i and set $\pi(e_i) = a$ for some a with $\mathfrak{M}, w \models E^{r^-}_1[a]$;
     - if r is a local role, then add (d, e_i, w) to R^r_i; if r is rigid, then for all $w' \in W$, add (d, e_i, w') to R^r_i.
  2. If $\mathfrak{M}, w \models E^{r^-}_q[\pi_i(d)]$ for some $d \in \Delta_i$ and $w \in W$ such that there is no $E^{r^-}_{q'}$ with $q' > q$ and $\mathfrak{M}, w \models E^{r^-}_{q'}[\pi_i(d)]$, then
     - add $e_1, \ldots, e_q$ to Δ_i and set $\pi(e_i) = a$ for some a with $\mathfrak{M}, w \models E^{r^-}_1[a]$;
     - if r is a local role, then add (e_i, d, w) to $\mathsf{R}^{r^-}_i$; if r is rigid, then for all $w' \in W$, add (e_i, d, w') to $\mathsf{R}^{r^-}_i$.

Finally, set $\Delta = \bigcup_{i \geq 0} \Delta_i$, and $\mathsf{R}^r = \bigcup_{i \geq 0} \mathsf{R}^r_i$, for all $r \in \mathsf{ROL}(C,\mathcal{T})$. The temporal interpretation $\mathfrak{I} = (\Delta, T, \{\mathcal{I}_w\}_{w \in W})$ is then given by

$$A^{\mathfrak{I},w} = \{d \in \Delta \mid \mathfrak{M}, w \models A[\pi(d)]\};$$
$$r^{\mathfrak{I},w} = \{(d, d') \in \Delta \times \Delta \mid (d, d', w) \in \mathsf{R}^r\}.$$

Now, it is straightforward to see by structural induction that the following claim holds.

Claim 4 *For all $d \in \Delta$, $w \in W$ and concepts D in $\mathcal{T}$, we have that*

$$\mathfrak{M}, w \models D^\dagger(\pi(d)) \textit{ iff } d \in D^{\mathfrak{I},w}.$$

We can now conclude from the previous claim and Equation (60) that $\mathfrak{I}$ is a model of the TBox, and also of C.

The other direction is direct. ❏

From $\mathcal{QCTL}^1$ to propositional CTL

We begin by recalling the well known relation between *KB satisfiability* and concept satisfiability relative to TBoxes: a concept C is satisfiable w.r.t. a TBox $\mathcal{T}$ if the KB $\mathcal{K}_{C,\mathcal{T}} = (\mathcal{T} \cup \{A \sqsubseteq C\}, \{A(a)\})$ is satisfiable, where A is a fresh concept name and a is a fresh individual name. From now on, we focus on the corresponding KB satisfiability problem. It should be clear that we can straightforwardly modify the translation above, obtaining $(\mathcal{K}_{C,\mathcal{T}})^\dagger$, and state Theorem 23 in terms of KB satisfiability.

We next show that we can construct a CTL formula that is equi-satisfiable with $(\mathcal{K}_{C,\mathcal{T}})^\dagger$ above. From now on, we slightly abuse notation by considering $\mathcal{T}$ in $\varphi_\mathcal{T}$ as the TBox component of $\mathcal{K}_{C,\mathcal{T}}$, that is, $\mathcal{T} = \mathcal{T} \cup \{A \sqsubseteq C\}$. We first observe that $\varphi_\mathcal{T}$ is of the form

$$\begin{aligned}\varphi_\mathcal{T} \quad &= \mathbf{A}\Box\, \forall x\, \varphi(x) \wedge \\ &\bigwedge_{r \in \mathsf{ROL}(C,\mathcal{T})} \mathbf{A}\Box(\exists x(\exists r)^\dagger(x) \to \exists x(\exists \mathsf{inv}(r))^\dagger(x))\end{aligned}$$

where $\varphi(x)$ is a quantifier-free $\mathcal{QCTL}^1$ formula with x being the only variable occurring in it. Moreover, $(A(a))^\dagger$ is a ground formula. We will now attempt to replace the second conjunct of the formula above with a formula without existential quantifiers. In the case of CTL, where time is assumed to have an initial point, Artale et al.'s shifting technique will not work off the shelf. We have therefore identified a fragment where shifting can be avoided and another where shifting can be made to work with additional efforts.

We call a $\mathcal{QCTL}$ model $\mathfrak{M}$ *nice* if, for every role name $r \in \mathcal{K}_{C,\mathcal{T}}$, either both $(\exists r)^\dagger$ and $(\exists r^-)^\dagger$ are empty at all worlds w or both $(\exists r)^\dagger$ and $(\exists r^-)^\dagger$ are non-empty at all worlds w. The existence of nice models for satisfiable $\mathcal{K}_{C,\mathcal{T}}$ is crucial for eliminating the existential quantifiers.

Lemma 24 *For every C and $\mathcal{T}$ in* CTL-*DL-Lite*$^\mathcal{N}_{bool}$*, if $\mathcal{K}_{C,\mathcal{T}}$ is satisfiable and*

1. *C and $\mathcal{T}$ use only rigid roles, or*
2. *C and $\mathcal{T}$ use only* $\mathbf{E}\Box$ *and* $\mathbf{E}\mathcal{U}$ *as temporal operators, or*
3. *C and $\mathcal{T}$ use only* $\mathbf{E}\Diamond$ *as temporal operator,*

then there is a nice model $\mathfrak{M}$ satisfying $(\mathcal{K}_{C,\mathcal{T}})^\dagger$.

Proof. We know, by Theorem 23, that there is a model $\mathfrak{M}'$ such that $\mathfrak{M}', \varepsilon \models (\mathcal{K}_{C,\mathcal{T}})^\dagger$. In case $\mathfrak{M}', w \not\models E_1^r[a]$ for every $w \in T$ and $a \in \Delta^{\mathfrak{M}'}$, we also have $\mathfrak{M}', w \not\models E_1^{r^-}[a]$ for every $w \in T$ and $a \in \Delta^{\mathfrak{M}'}$ due to Equation 63; hence $\mathfrak{M}'$ is nice. Otherwise, pick some $a \in \Delta^{\mathfrak{M}'}$ and $w_0 \in T$ with $\mathfrak{M}', w_0 \models E_1^r[a]$. Again due to Equation 63, there is some $a' \in \Delta^{\mathfrak{M}'}$ with $\mathfrak{M}', w_0 \models E_1^{r^-}[a']$.

In Case (1), where C and $\mathcal{T}$ use only rigid roles, Equation 62 ensures that $\mathfrak{M}', w \models E_1^r[a]$ and $\mathfrak{M}', w \models E_1^{r^-}[a']$ for all $w \in T$; hence $\mathfrak{M}'$ is nice.

In Case (2), where C and $\mathcal{T}$ use only $\mathbf{E}\Box$ and $\mathbf{E}\mathcal{U}$ as temporal operators, we cannot argue as in Case (1) because local roles may not satisfy Equation 62; hence, there may be worlds w with $\mathfrak{M}', w \not\models E_1^r[b]$ or $\mathfrak{M}', w \not\models E_1^{r^-}[b]$ for any $b \in \Delta^{\mathfrak{M}'}$, and thus $\mathfrak{M}'$ may not be nice. We will therefore adapt the solution in Artale et al. (2012) of turning $\mathfrak{M}'$ into a nice model $\mathfrak{M}$ to our purposes: extend the domain with additional copies of a and a' per world w. In the case of linear time, which is modelled by the integers, this can be easily achieved by "shifting" the whole linear history (future and past) of a and a' by every positive and negative integer to the future and past, respectively, introducing a fresh copy of a and a' for every shifting distance (Artale et al. 2012).

We cannot do this easily because, in CTL-*DL-Lite*$^\mathcal{N}_{bool}$, the past is bounded: if we shift the history of a to a world w in the future of w_0, we do not find enough "past" of a in $\mathfrak{M}'$ to determine how to interpret the longer past of the new copy of a. To circumvent this problem, we introduce an intermediate step where we "unravel" $\mathfrak{M}'$ into the temporal direction, which means that we create paths where a copy of w_0 occurs in every depth d greater than the depth d_0 of w_0. As a result, in every depth $d \le d_0$ in the temporal tree, there will be a copy w_0^d of w_0 such that $\mathfrak{M}'', w_0^d \models E_1^r[a]$ and $\mathfrak{M}'', w_0^d \models E_1^{r^-}[a']$. We can then create copies of a for every world w in $\mathfrak{M}''$, using the history of a in w_0^d such that d is the depth of w. This approach only works for temporal operators that are tolerant to this form of unravelling, namely $\mathbf{E}\Box$ and $\mathbf{E}\mathcal{U}$ which are the ones present in CTL$^{\mathbf{E}\Box,\mathbf{E}\mathcal{U}}$-*DL-Lite*$^\mathcal{N}_{bool}$.[3] More precisely, our construction will look as follows.

Step 1. Start with $\mathfrak{M}'$ based on the temporal tree T with $w_0 \in T$ as above. Assume w.l.o.g. that every world in T has infinite, countable outdegree (if a world w has finite outdegree, just add infinitely many copies of some path starting from w, which preserves all $\mathcal{QCTL}$ formulas, and countability can be assumed due to the Löwenheim-Skolem theorem). This means that T is isomorphic to the tree (V, E) with $V = \mathbb{N}^*$ and $E = \{(w, w \cdot n) \mid w \in V,\ n \in \mathbb{N}\}$. Let V_0 consist of w_0 and all its descendants. Construct the tree $T'' = (V'', E'')$ with $V'' \subseteq V \times \mathbb{N}$ and

$$\begin{aligned} V'' &= V \times \{0\} \,\cup\, V_0 \times \{1, 2, \dots\} \\ E'' &= \{(\langle w, 0\rangle, \langle w \cdot n, 0\rangle) \mid w \in V,\ n \in \mathbb{N}\} \\ &\quad \cup \{(\langle w_0, i\rangle, \langle w_0, i+1\rangle) \mid i \in \mathbb{N}\} \\ &\quad \cup \{(\langle w, i\rangle, \langle w \cdot n, i\rangle) \mid w \in V_0,\ n, i \in \mathbb{N}\} \end{aligned}$$

Intuitively speaking, T'' is obtained by creating a path starting at w_0 where w_0 repeats infinitely often, and attaching a copy of the subtree of w_0 to each of the new copies. We construct $\mathfrak{M}''$ by interpreting every world $\langle w, i\rangle$ the same way as w was interpreted in $\mathfrak{M}'$. Then $\mathfrak{M}', \varepsilon \models (\mathcal{K}_{C,\mathcal{T}})^\dagger$ implies $\mathfrak{M}'', \varepsilon \models (\mathcal{K}_{C,\mathcal{T}})^\dagger$ due to a standard unravelling argument. Furthermore, every world in the temporal tree of $\mathfrak{M}''$ has infinite, countable outdegree as well.

Step 2. The temporal tree T'' of $\mathfrak{M}''$ is again isomorphic to the tree (V, E) with $V = \mathbb{N}^*$ and $E = \{(w, w \cdot n) \mid w \in V,\ n \in n\}$. Due to the copying of w_0, we find a world w_0^d with $\mathfrak{M}'', w_0^d \models E_1^r[a]$ and $\mathfrak{M}'', w_0^d \models E_1^{r^-}[a']$ at every depth $d \ge d_0$. Extend the domain by a fresh copy of a and a' for every point in the tree T''. The history of every (a, v) is interpreted by $\mathfrak{M}''$ as the history of a in the world w_0^d is interpreted by $\mathfrak{M}'$, where d is the depth of v if that is at least d_0, otherwise $d = d_0$.

In Case (3), we follow the same arguments of Case (2). In fact recall that $\mathbf{E}\Diamond C$ is an abbreviation of $\mathbf{E}(\top \mathcal{U} C)$. ❑

[3] Since our form of unravelling will create paths with arbitrary repetitions of a world, tolerance to it is related to stutter invariance (Lamport 1983) of linear temporal logic, which says that, if a linear temporal model is modified by repeating the same world (and its interpretation) a finite amount of times, the resulting model satisfies the same formulas as the original one. This property fails for temporal operators that can distinguish the direct next time point from any other point in the future, namely $\mathbf{E}\bigcirc$ and therefore the strict variants of $\mathbf{E}\mathcal{U}$ and $\mathbf{E}\Box$.

We are now in the position of introducing a variant $(\mathcal{K}_{C,\mathcal{T}})^\ddagger$ of $(\mathcal{K}_{C,\mathcal{T}})^\dagger$ without existential quantifiers. For every role name r, we introduce two fresh constants d_r and d_{r^-}; moreover, we introduce two fresh propositional variables p_r and p_{r^-}. We define $\varphi'_{\mathcal{T}}$ as follows:

$$\mathbf{A}\Box\, \forall x\, \varphi(x) \;\wedge\; \bigwedge_{r \in \mathsf{ROL}(C,\mathcal{T})} \Big(p_r \to (\exists \mathsf{inv}(r))^\dagger(d_{\mathsf{inv}(r)}) \;\wedge\; \mathbf{A}\Box\, \forall x\, ((\exists r)^\dagger(x) \to \mathbf{A}\Box\, p_r)\Big).$$

We have then that $(\mathcal{K}_{C,\mathcal{T}})^\ddagger = \varphi'_{\mathcal{T}} \wedge A(a)$. The main idea is to ensure via the 'marker' p_r that r is not empty at the root using the constant d_r. Now, with the help of Lemma 24, we next show that it is enough to consider $(\mathcal{K}_{C,\mathcal{T}})^\ddagger$.

Proposition 25 *A CTL-DL-Lite$^{\mathcal{N}}_{bool}$ KB $\mathcal{K}_{C,\mathcal{T}}$ is satisfiable iff the $\mathcal{QCTL}^1$-sentence $(\mathcal{K}_{C,\mathcal{T}})^\ddagger$ is satisfiable.*

Proof. By Lemma 24 above, if a KB $\mathcal{K}_{C,\mathcal{T}}$ is satisfiable then there is a model $\mathfrak{M}$ where each role r occurring in $\mathcal{K}_{C,\mathcal{T}}$ is either empty or not empty at all time points. We construct a model for $(\mathcal{K}_{C,\mathcal{T}})^\ddagger$ by distinguishing the two cases above: (1) for a non empty-role r set $p_r^{\mathfrak{M},w}$ and $p_{r'}^{\mathfrak{M},w}$ to true for all $w \in W$, and take respectively some elements in $E_r^{\mathfrak{M},\varepsilon}$ $E_{r^-}^{\mathfrak{M},\varepsilon}$ as d_r and d_{r^-}. (2) If r is an empty-role then set $p_r^{\mathfrak{M},w}$ and $p_{r'}^{\mathfrak{M},w}$ to false, and take some arbitrary elements of the domain as d_r and d_{r^-}.

For the other direction, note that if $(\mathcal{K}_{C,\mathcal{T}})^\ddagger$ then $(\mathcal{K}_{C,\mathcal{T}})^\dagger$, and thus, by Theorem 23, $\mathcal{K}$ is satisfiable. ❑

The previous Lemma helps us to reach our goal: since $(\mathcal{K}_{C,\mathcal{T}})^\ddagger$ has no existential quantifiers, we can see it as a CTL formula by instantiating all universally quantified formulas by all the constants in the formula.