Extension of the Schoenberg theorem to integrally conditionally positive definite functions

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Abstract

The celebrated Schoenberg theorem establishes a relation between positive definite and conditionally positive definite functions. In this paper, we consider the classes of real-valued functions $\mathrm{P}(J)$ and $\mathrm{CP}(J)$, which are positive definite and respectively, conditionally positive definite, with respect to a given class of test functions J. For suitably chosen J, the classes $\mathrm{P}(J)$ and $\mathrm{CP}(J)$ contain classically positive definite (respectively, conditionally positive definite) functions, as well as functions which are singular at the origin. The main result of the paper is a generalization of Schoenberg's theorem to such function classes.

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1. Introduction

A function $f : \mathbb{R}^d \to \mathbb{R}$ is *positive definite* if f is even, i.e. $f(\mathbf{x}) = f(-\mathbf{x}) \;\forall\, \mathbf{x} \in \mathbb{R}^d$, and

$$\sum_{i,j=1}^{n} f(\mathbf{x}_i - \mathbf{x}_j)\, \xi_i\, \xi_j \geq 0 \tag{1}$$

for all $n \in \mathbb{N}$, all $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^d$ and all $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{R}$. A function $f : \mathbb{R}^d \to \mathbb{R}$ is *conditionally positive definite* if f is even and the inequality (1) holds for all $n \in \mathbb{N}$, all $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^d$ and all $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{R}$ satisfying $\sum_{i=1}^n \xi_i = 0$. Conditionally *negative* definite functions can be defined similarly, by reversing the inequality in (1). By [13, Prop. 4.4], these functions coincide with the *negative definite* functions defined in [13, Def. 4.3], provided $f(\mathbf{0}) \geq 0$.

The notions of positive definite and conditionally positive definite functions can be generalized to complex-valued functions, two-variable kernels, as well as functions and kernels defined on groups, see e.g. [1, 19, 3]. We restrict our attention to the case when f is a real-valued function on $\mathbb{R}^d$, however the functions defined above are automatically classically positive definite (respectively, conditionally positive definite) in the complex sense. The celebrated Schoenberg theorem (which is also valid for complex-valued kernels defined on more general sets) establishes a relation between the two types of functions we have introduced.

Theorem 1.1. (Schoenberg [14, Th. 2], [1, Th. C.3.2], [13, Prop. 4.4], [3, p.74])
A function $f : \mathbb{R}^d \to \mathbb{R}$ is conditionally positive definite if and only if for all $t > 0$, the functions e^{tf} are positive definite.

Note that both positive definite and conditionally positive definite functions have finite values at zero. The main result of this paper is a generalization of Theorem 1.1 to the case where functions can be singular at the origin, see Theorem 2.1.

The following definition extends the definition given in [5, p. 54], where functions defined on $\mathbb{R}$ were considered.

Definition 1.1. *Let J be a set of real-valued measurable functions defined on $\mathbb{R}^d$. We define a function $f : \mathbb{R}^d \to \mathbb{R}$ to be positive definite for J if f is even a.e. (that is, $f(\mathbf{x}) = f(-\mathbf{x})$ for almost all $\mathbf{x} \in \mathbb{R}^d$), and for every $h \in J$, the integral*

$$\Phi_f(h) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{x} - \mathbf{y}) h(\mathbf{x}) h(\mathbf{y})\, d\mathbf{x} d\mathbf{y} \tag{2}$$

exists as a Lebesgue integral and is non-negative.

Similarly, we extend the definition of conditionally positive definite functions.

Definition 1.2. *A function $f : \mathbb{R}^d \to \mathbb{R}$ is conditionally positive definite for a set J of test functions if f is even a.e. and the integral in* (2) *exists as a Lebesgue integral and is non-negative, for every $h \in \{g \in J : \int_{\mathbb{R}^d} g(\mathbf{x})\, d\mathbf{x} = 0\}$, assuming this set is not empty.*

In association with [7], the functions defined in Definitions 1.1 and 1.2 may also be called integrally (conditionally) positive definite functions for J.

Another notable extension of the set of positive definite functions is the set of functions with k negative squares; for these functions, the matrix $(f(\mathbf{x}_i - \mathbf{x}_j))_{i,j=1}^n$ can have up to k negative eigenvalues. Functions with 0 negative squares are positive definite functions and the class of conditionally positive definite functions is a subclass of functions with 1 negative square; see e.g. [12, Lemma 3.12.3]. In [18], some key results of [5] were extended to this class of functions.

The classes of functions which are positive definite and conditionally positive definite for J will be denoted by $\mathrm{P}(J)$ and $\mathrm{CP}(J)$, respectively. By $\mathrm{P_C}$ and $\mathrm{CP_C}$, we denote the sets of classical positive definite and conditionally positive definite functions on $\mathbb{R}^d$.

We shall also use the following notation. Let $p \in [1, \infty]$. By $\mathrm{L}_0^p(\mathbb{R}^d)$ we denote the set of real-valued functions in $\mathrm{L}^p(\mathbb{R}^d)$ with compact essential support. By $\mathrm{L}_{\mathrm{loc}}^p(\mathbb{R}^d)$ we denote the class of functions

$$\mathrm{L}_{\mathrm{loc}}^p(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \ \mid \ f \in \mathrm{L}^p(K) \ \text{ for all compact sets } \mathrm{K} \subset \mathbb{R}^d \right\} .$$

Furthermore, by $\mathrm{C}_0^r(\mathbb{R}^d)$ we denote the space of r times continuously differentiable functions on $\mathbb{R}^d$ with compact support, and by $\mathrm{C}^\infty(\mathbb{R}^d)$ we denote the space of infinitely differentiable functions on $\mathbb{R}^d$.

Let $p \in [1, \infty)$. By $f_n \xrightarrow{\mathrm{L}_{\mathrm{loc}}^p} f$ as $n \to \infty$, we mean that f_n converges to f in the $\mathrm{L}_{\mathrm{loc}}^p$ sense as $n \to \infty$; that is,

$$\lim_{n\to\infty} \|f_n - f\|_{p,K} = \lim_{n\to\infty} \left(\int_K |f_n(\mathbf{x}) - f(\mathbf{x})|^p \, d\mathbf{x} \right)^{\frac{1}{p}} = 0$$

for any compact set $K \subset \mathbb{R}^d$.

A non-constant function $f : (0, \infty) \to [0, \infty)$ is *completely monotone* if f is infinitely differentiable and $(-1)^n f^{(n)} > 0$, for all non-negative integers n. The family of all non-constant, completely monotone functions will be denoted CM.

The concept of positive definite functions was extended to positive definite distributions by L. Schwartz [15, Chapter VII, §9]. For any $\phi \in \mathcal{D}(\mathbb{R}^d)$, that is, for any ϕ in the set $\mathrm{C}_0^\infty(\mathbb{R}^d)$ with the topology usual for the theory of distributions, we have for the distribution T_f associated with any real-valued locally integrable function f, $\mathrm{T}_f(\phi) = \int_{\mathbb{R}^d} f(\mathbf{x})\phi(\mathbf{x})\, d\mathbf{x}$, $\mathrm{T}_f(\phi * \phi(-\cdot)) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{x}-\mathbf{y})\phi(\mathbf{x})\phi(\mathbf{y})\, d\mathbf{x}d\mathbf{y}$, see e.g. [19, Sec. 7]. Hence, for a real-valued function $f \in \mathrm{L}_{\mathrm{loc}}^1(\mathbb{R}^d)$, T_f is a positive definite distribution if and only if $f \in \mathrm{P}(\mathrm{C}_0^\infty(\mathbb{R}^d))$.

The rest of the paper is organized as follows. In Section 2 we establish some properties of the functional classes $\mathrm{P}(J)$ and $\mathrm{CP}(J)$, and formulate the main result of the paper, Theorem 2.1. In Section 3 we provide several algorithmic schemes for constructing functions from $\mathrm{P}(J)$ and $\mathrm{CP}(J)$. We also discuss an important practical implication of Theorem 2.1, which states that $f \in \mathrm{CP}(J)$ if and only if the functional (2) is convex on a suitable class of functions. Section 4 contains the proof of Theorem 2.1, split into a series of lemmas, and in Section 5 we develop further the theory of the class of functions $\mathrm{CP}(J)$. Finally, the appendix contains the proof of two auxiliary technical lemmas.

2. Extension of the Schoenberg theorem and properties of $\mathrm{P}(J)$ and $\mathrm{CP}(J)$

2.1. Properties of the functional classes $\mathrm{P}(J)$ *and* $\mathrm{CP}(J)$

The following two simple properties proceed immediately from the definitions of $\mathrm{P}(J)$ and $\mathrm{CP}(J)$. If $J_1 \subseteq J_2$, then $\mathrm{P}(J_2) \subseteq \mathrm{P}(J_1)$ and $\mathrm{CP}(J_2) \subseteq \mathrm{CP}(J_1)$. Also, if $c_i \geq 0$ for any $i = 1, \dots, n$, and $f_1, f_2, \dots, f_n \in \mathrm{P}(J)$ (or $\mathrm{CP}(J)$), then $\sum_{i=1}^n c_i f_i \in \mathrm{P}(J)$ (respectively, $\sum_{i=1}^n c_i f_i \in \mathrm{CP}(J)$).

Some non-trivial properties of the functional classes $\mathrm{P}(J)$ have been established in the pioneering paper [5], where complex-valued functions on $\mathbb{R}$ are considered. For most classes of functions J, these properties also hold true for the real-valued positive definite functions defined in Definition 1.1; see Appendix for details.

Properties of the set $\mathrm{P}(J)$ are highly dependent on the choice of the set J of test functions. For continuous functions f and J being the set of continuous functions with compact support, $f \in \mathrm{P}(J)$ and $f \in \mathrm{P_C}$ are equivalent; see Proposition 6.2. We prove an analogous result for $\mathrm{CP}(J)$ in Proposition 5.1.

In view of the fundamental results of [5], $\mathrm{P}(J)$ is most interesting when $J = \mathrm{L}_0^q(\mathbb{R})$, for $q \geq 1$. We now list six properties of the functional classes $\mathrm{P}(\mathrm{L}_0^q(\mathbb{R}))$ established in [5].

(i) If $f \in \mathrm{P}(\mathrm{L}_0^1(\mathbb{R}))$, then f is essentially bounded and almost everywhere equal to a continuous, classically positive definite function on $\mathbb{R}$.
(ii) If $f \in \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$, then $f \in \mathrm{L}_{\mathrm{loc}}^1(\mathbb{R})$.
(iii) If f is continuous and classically positive definite on $\mathbb{R}$, then $f \in \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$.
(iv) If $f \in \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$ and g is continuous and classically positive definite on $\mathbb{R}$, then $fg \in \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$.
(v) $\mathrm{P}(\mathrm{L}_0^q(\mathbb{R})) \subseteq \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$ for all $q \in [1, 2]$.
(vi) $\mathrm{P}(\mathrm{L}_0^2(\mathbb{R})) = \mathrm{P}(\mathrm{L}_0^q(\mathbb{R})) = \mathrm{P}(\mathrm{C}_0^r(\mathbb{R}))$ for all $q \in [2, \infty]$, $r \in [0, \infty]$.

The essential boundedness of the function in property (i) follows from [5, Th. 5], whilst the fact that it is almost everywhere equal to a continuous, classically positive definite function follows from [6, Sec. 6]. For a proof of (ii), see [5, Lemma 1]. In Lemma 4.3 we prove that functions in $\mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}))$ are also locally integrable. Property (iii) follows directly from Bochner's theorem [4, Chapter IV.20] and Definition 1.1. For a proof of (iv), see [5, Th. 1]. Property (v) follows since $\mathrm{L}_0^2(\mathbb{R}) \subseteq \mathrm{L}_0^q(\mathbb{R})$ for all $1 \le q \le 2$, and property (vi) can be proved using [5, Lemma 1] and the fact that for any $0 \le r \le \infty$, $\mathrm{C}_0^r(\mathbb{R})$ is dense in $\mathrm{L}_0^2(\mathbb{R})$.

The properties (i)–(vi) above generalize to arbitrary $d > 1$, as well as to the case when $\mathbb{R}$ is replaced with any locally compact Abelian group G; see e.g. [8, 9, 17]. We note that for such a group G, property (i) follows from two applications of the uniform boundedness theorem, see [17, Th. 2.4].

It is clear that property (v) holds for conditionally positive definite functions. In Propositions 5.1 and 5.2 respectively, we prove that properties (iii) and (vi) are also valid for such functions.

Properties (v) and (vi) demonstrate that as q increases from 1 to 2, $\mathrm{P}(\mathrm{L}_0^q(\mathbb{R}^d))$ and $\mathrm{CP}(\mathrm{L}_0^q(\mathbb{R}^d))$ increase from smaller classes of functions to larger such classes, but for all $q \ge 2$, $\mathrm{P}(\mathrm{L}_0^q(\mathbb{R}^d))$ and $\mathrm{CP}(\mathrm{L}_0^q(\mathbb{R}^d))$ remain unchanged. Functions in $\mathrm{P}(\mathrm{L}_0^2(\mathbb{R}^d))$ and $\mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$ need not be bounded or continuous, and hence can tend to $+\infty$ at zero.

2.2. The main result: an extended Schoenberg theorem

As follows from the discussion above, the widest and hence most interesting classes of positive definite and conditionally positive definite functions are $\mathrm{P}(\mathrm{L}_0^2(\mathbb{R}^d))$ and $\mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$. Using this as motivation, we consider these two classes of functions in the rest of the paper. Our main result is the following theorem.

Theorem 2.1. *Let $f : \mathbb{R}^d \to \mathbb{R}$. If there exist $t_0 > 0$ and $p > 1$ such that $e^{t|f|} \in \mathrm{L}_{\mathrm{loc}}^p(\mathbb{R}^d)$ for all $0 < t \le t_0$, then*

$$f \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d)) \iff e^{tf} \in \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}^d)) \ \ (0 < t \le t_0). \tag{3}$$

This theorem will be proved in Section 4.

3. Some examples and possible applications

3.1. Examples of functions in $\mathrm{P}(\mathrm{L}_0^2(\mathbb{R})) \setminus \mathrm{P}_C$

Continuous, classically positive definite functions belong to $\mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$, see Proposition 6.2. We will now demonstrate how to find functions in $\mathrm{P}(\mathrm{L}_0^2(\mathbb{R})) \setminus \mathrm{P}_C$; that is, functions in $\mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$, which are unbounded at the origin and therefore not positive definite in the classical sense. To do this, we use the following result of [11].

Proposition 3.1. [11] *Let $g \in \mathrm{CM}$. If $f = g(|\cdot|) \in \mathrm{L}_{\mathrm{loc}}^1(\mathbb{R})$, then $f \in \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$.*

Proposition 3.1 suggests the following strategy for finding examples of functions $f \in \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$, which are not classically positive definite. Take any completely monotone function $g \in \mathrm{CM}$; check whether g is unbounded at zero; for $f = g(|\cdot|)$, check whether $f \in \mathrm{L}_{\mathrm{loc}}^1(\mathbb{R})$; if both of these conditions are satisfied, then $f \in \mathrm{P}(\mathrm{L}_0^2(\mathbb{R})) \setminus \mathrm{P}_C$.

There are several ways of constructing completely monotone functions. One of the simplest is based on the following statement, see [13, Th. 3.6].

Proposition 3.2. *If $u \in \mathrm{CM}$ and h is a Bernstein function, then $g := u \circ h \in \mathrm{CM}$.*

We provide four basic functions $u \in \mathrm{CM}$, which are especially useful in this context: $u_1(x) = x^{-\alpha}$ with $\alpha > 0$, $u_2(x) = e^{-\beta x}$ with $\beta > 0$, $u_3(x) = \log(1 + 1/x)$, and $u_4(x) = e^{1/x}$ $(x > 0)$.

Another way of constructing completely monotone functions is based on the fact that the derivative of a Bernstein function is always completely monotone, see [13, p.18]. A long list of Bernstein functions can be found in [13, Chapter 15]. Going through the first fifty Bernstein functions in this list, we find that in the following cases, the corresponding functions f, belong to the set $\mathrm{P}(\mathrm{L}^2_0(\mathbb{R})) \setminus \mathrm{P}_C$: 1, 7, 8, 9, 11, 12, 13, 16, 17, 19, 23, 25, 27, 31, 33, 34, 36, 38, 40, 41, 42, 43, 44, 45.

In particular, the following functions belong to $\mathrm{P}(\mathrm{L}^2_0(\mathbb{R})) \setminus \mathrm{P}_C$:

$$
\begin{aligned}
f_1(x) &= |x|^{-\alpha},\ 0 < \alpha < 1;\\
f_8(x) &= |x|^{\alpha-1}/(1+|x|)^{\alpha+1},\ 0 < \alpha < 1;\\
f_{11}(x) &= \left(\alpha|x|^{\alpha-1}(1-|x|^{\beta}) - \beta|x|^{\beta-1}(1-|x|^{\alpha})\right)/(1-|x|^{\alpha})^2,\ 0 < \alpha < \beta < 1\,;\\
f_{16}(x) &= \left(\alpha_1|x|^{-\alpha_1-1} + \ldots + \alpha_n|x|^{-\alpha_n-1}\right)/\left(|x|^{-\alpha_1} + \ldots + |x|^{-\alpha_n}\right)^2,\ 0 \le \alpha_1, \ldots, \alpha_n \le 1;\\
f_{19}(x) &= \left(1 - (\lambda\sqrt{|x|} - 1)e^{-\lambda\sqrt{|x|}}\right)/\sqrt{|x|},\ \lambda > 0;\\
f_{23}(x) &= |x|\,(1+1/|x|)^{1+|x|}\log(1+1/|x|) \qquad (x \in \mathbb{R}\setminus\{0\}).
\end{aligned}
$$

3.2. Extension to functions on $\mathbb{R}^d$

The main result of [11], extended to functions on $\mathbb{R}^d$, is the following.

Proposition 3.3. *Let $f \in \mathrm{L}^1(\mathbb{R}^d)$. Then*

$$f \in \mathrm{P}(\mathrm{L}^2_0(\mathbb{R}^d)) \text{ if and only if } \hat{f} \ge 0,$$

where $\hat{f}$ denotes the Fourier transform of f.

If $g \in \mathrm{CM}$ is bounded, then $f = g(\|\cdot\|^2) : \mathbb{R}^d \to [0,\infty)$ is continuous and classically positive definite for any $d \in \mathbb{N}$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$; see [2, Th. 1.6]. Proposition 3.3 can be used to prove the next result, which generalizes this observation to potentially unbounded completely monotone functions.

Proposition 3.4. *Let $g \in \mathrm{CM}$ and $f = g(\|\cdot\|^2) : \mathbb{R}^d \to [0,\infty)$. If $f \in \mathrm{L}^1(\mathbb{R}^d)$, then $f \in \mathrm{P}(\mathrm{L}^2_0(\mathbb{R}^d))$.*

Proof. By [13, Th.1.4], g is the Laplace transform of a non-negative measure μ on $[0,\infty)$. By the Fubini theorem, for any $\mathbf{u} \in \mathbb{R}^d$,

$$
\begin{aligned}
\hat{f}(\mathbf{u}) &= (2\pi)^{-\frac{d}{2}}\int_{[0,\infty)}\int_{\mathbb{R}^d} e^{-\|\mathbf{x}\|^2 t}e^{-i\mathbf{x}\cdot\mathbf{u}}\,d\mathbf{x}\,\mu(dt)\\
&= \int_{[0,\infty)}\prod_{i=1}^{d}\left((2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} e^{-x_i^2 t}e^{-ix_iu_i}\,dx_i\right)\mu(dt)\\
&= \int_{[0,\infty)}\prod_{i=1}^{d}\left((2t)^{-\frac{1}{2}}e^{-u_i^2/4t}\right)\mu(dt) = \int_{[0,\infty)}(2t)^{-\frac{d}{2}}e^{-\|\mathbf{u}\|^2/4t}\mu(dt) \ge 0.
\end{aligned}
$$

Thus, $f \in \mathrm{P}(\mathrm{L}^2_0(\mathbb{R}^d))$ by Proposition 3.3. □

The following result is a direct consequence of Proposition 3.4, and provides a basis for finding examples of functions in $\mathrm{P}(\mathrm{L}^2_0(\mathbb{R}^d))$ with $d \ge 2$, which are unbounded at zero.

Proposition 3.5. *Let $g \in \mathrm{CM} \cap \mathrm{L}^1_{\mathrm{loc}}((0,\infty))$. For any $s > 0$, define*

$$f(\mathbf{x}) = g(\|\mathbf{x}\|^2)e^{-s\|\mathbf{x}\|^2}, \quad \mathbf{x} \in \mathbb{R}^d. \tag{4}$$

Then, $f \in \mathrm{P}(\mathrm{L}^2_0(\mathbb{R}^d))$, for any $d \ge 2$.

Proof. Let $s > 0$ and $d \ge 2$. Since the product of completely monotone functions is also completely monotone, see e.g. [13, Corollary 1.6], it follows that $g e^{-s}|_{(0,\infty)} \in \mathrm{CM}$, where $e^{-s}|_{(0,\infty)}$ denotes the restriction of e^{-s} to the domain $(0,\infty)$. Moreover, $f \in \mathrm{L}^1(\mathbb{R}^d)$, for a change of variables to polar coordinates gives,

$$\int_{\mathbb{R}^d} f(\mathbf{x})\,d\mathbf{x} = \omega_{d-1}\int_0^1 g(r^2)e^{-sr^2}r^{d-1}\,dr + \omega_{d-1}\int_1^\infty g(r^2)e^{-sr^2}r^{d-1}\,dr$$
$$\leq \omega_{d-1}\int_0^1 g(x)\,x^{\frac{d-2}{2}}\,dx + \omega_{d-1}\,g(1)\int_1^\infty e^{-sr^2}r^{d-1}\,dr < \infty,$$

where ω_{d-1} denotes the volume of unit $(d-1)$-dimensional ball and r is the radius. Thus, $f \in \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}^d))$ by Proposition 3.4. □

The derivatives of the functions from [13, Chapter 15] listed in Section 3.1 are completely monotone, locally integrable on $(0,\infty)$, and singular at the origin. The corresponding functions in $\mathrm{P}(\mathrm{L}_0^2(\mathbb{R}^d))$ can be constructed using (4).

3.3. Examples of functions in $\mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d)) \setminus \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}^d))$

Firstly, we assume $d=1$. It is clear that if $f \in \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$, then $f \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}))$. We will now show how to construct examples of functions $f \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}))$, such that f has a singularity at zero (so that $f \notin \mathrm{CP}_C$) and $f \notin \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$.

In view of Proposition 3.2 with $u(x) = x^{-t}$ $(t>0)$, if h is a Bernstein function, then $g_t = h^{-t} \in \mathrm{CM}$ for all $t>0$. Define $v_t := h^{-t}(|\cdot|)$ and let $t_0 > 0$. By Proposition 3.1, if $v_t \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ for all $0 < t \leq t_0$, then $v_t \in \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$ for all $0 < t \leq t_0$. Define $f := -\log(h(|\cdot|))$, so that $e^{tf} = v_t$. By Theorem 2.1, if $e^{t|f|} \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R})$ for any $0 < t \leq t_0$ and some $p > 1$, then $f \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}))$. It only remains to check the last condition, $f \notin \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$.

Consider the following simple examples. In all three cases we choose $p = 2$ and $t_0 = 1/4$, so that $v_t \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ and $e^{t|f|} \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R})$ for all $0 < t \leq t_0$.

1. Take $h(x) = x$, then $g_t(x) = x^{-t}$, $v_t(x) = |x|^{-t}$ and $f(x) = -\log|x|$. All conditions are satisfied, hence $f \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}))$. It is easy to see that $f \notin \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$. Indeed, take the test function $\phi(x) = 1$ for $0 < x < 8$, $\phi(x) = 0$ otherwise. Then

$$\Phi_f(\phi) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)\phi(x)\phi(y)\,dxdy = -\int_0^8\int_0^8 \log|x-y|\,dxdy = 96(1-2\log 2) < 0\,.$$

2. Take $h(x) = x+\sqrt{x}$, then $g_t(x) = (x+\sqrt{x})^{-t}$, $v_t(x) = (|x|+\sqrt{|x|})^{-t}$ and $f = -\log(|\cdot|+\sqrt{|\cdot|}) \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}))$. To prove $f \notin \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$, we take the same test function ϕ as in the example above, giving

$$\Phi_f(\phi) = -\int_0^8\int_0^8 \log\left(|x-y| + \sqrt{|x-y|}\right)\,dxdy \simeq -75.20631216 < 0\,.$$

3. Take $h(x) = \Gamma(x+\frac{1}{2})/\Gamma(x)$, where Γ is the Gamma function. We have $f = -\log\Gamma(|\cdot|+\frac{1}{2}) + \log\Gamma(|\cdot|)$. To prove $f \notin \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}))$, again we take the same test function ϕ as above, finding

$$\Phi_f(\phi) = -\int_0^8\int_0^8 \left(\log\Gamma(|x-y|+1/2) - \log\Gamma(|x-y|)\right)\,dxdy \simeq -10.83 < 0\,.$$

All three examples are built on the same principle: if h is a Bernstein function and some regularity conditions are satisfied, then $f = -\log h(|\cdot|) \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}))$.

We now indicate how to construct functions in $\mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d)) \setminus \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}^d))$. Similarly to above, define $v_t(\mathbf{x}) := h^{-t}(\|\mathbf{x}\|^2)e^{-t\|\mathbf{x}\|^2}$, $\mathbf{x} \in \mathbb{R}^d$, and let $t_0 > 0$. By Proposition 3.5, if $h^{-t} \in \mathrm{L}^1_{\mathrm{loc}}((0,\infty))$ for all $0 < t \leq t_0$, then $v_t \in \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}^d))$ for all $0 < t \leq t_0$. Define

$$f(\mathbf{x}) := -\log\left(h(\|\mathbf{x}\|^2)\exp(\|\mathbf{x}\|^2)\right) = -\log\left(h(\|\mathbf{x}\|^2)\right) - \|\mathbf{x}\|^2 \quad (\mathbf{x} \in \mathbb{R}^d),$$

so that $e^{tf} = v_t$. By Theorem 2.1, if $e^{t|f|} \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$ for any $0 < t \leq t_0$ and some $p > 1$, then $f \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$. It only remains to check the last condition, $f \notin \mathrm{P}(\mathrm{L}_0^2(\mathbb{R}^d))$.

3.4. Convexity of the functional (2)

For any $M \in \mathbb{R}$, define $\mathrm{L}^2_{0,M}(\mathbb{R}^d)$ to be the set of functions $h \in \mathrm{L}_0^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} h(\mathbf{x})\,d\mathbf{x} = M$.

A functional $\Phi : \mathrm{L}^2_{0,M}(\mathbb{R}^d) \to \mathbb{R}$ is *convex* if

$$\Phi\left((1-\alpha)\,h_1 + \alpha\,h_2\right) \leq (1-\alpha)\,\Phi\left(h_1\right) + \alpha\,\Phi\left(h_2\right) \tag{5}$$

for all $0 \leq \alpha \leq 1$ and all $h_1, h_2 \in \mathrm{L}^2_{0,M}(\mathbb{R}^d)$.

Proposition 3.6. *For any* $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ *and* $M \in \mathbb{R}$, *the functional* $\Phi_f : \mathrm{L}^2_{0,M}(\mathbb{R}^d) \to \mathbb{R}$ *defined in* (2) *is convex if, and only if,* $f \in \mathrm{CP}(\mathrm{L}^2_0(\mathbb{R}^d))$.

Proof. For $\Phi = \Phi_f$, we can rewrite the l.h.s. in (5) as follows,

$$\Phi_f\left((1-\alpha)\, h_1 + \alpha\, h_2\right) = (1-\alpha)\,\Phi_f\,(h_1) + \alpha\,\Phi_f\,(h_2) - \alpha\,(1-\alpha)\,A_f(h_1,h_2),$$

where

$$A_f(h_1,h_2) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{x}-\mathbf{y})(h_1-h_2)(\mathbf{x})(h_1-h_2)(\mathbf{y})\,d\mathbf{x}d\mathbf{y}.$$

Hence, Φ_f is convex if and only if $A_f(h_1,h_2) \geq 0$, for any $h_1,\ h_2 \in \mathrm{L}^2_{0,M}(\mathbb{R}^d)$.

One direction is clear, since if $h_1, h_2 \in \mathrm{L}^2_{0,M}(\mathbb{R}^d)$, then $\int_{\mathbb{R}^d}(h_1-h_2)(\mathbf{x})\,d\mathbf{x} = 0$. Thus, if $f \in \mathrm{CP}(\mathrm{L}^2_0(\mathbb{R}^d))$, then Φ_f is convex on $\mathrm{L}^2_{0,M}(\mathbb{R}^d)$. For the reverse implication, let $h \in \mathrm{L}^2_0(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} h(\mathbf{x})\,d\mathbf{x} = 0$, and let K denote the compact support of h. Take $h_1(\mathbf{x}) = h(\mathbf{x}) + M$ for $\mathbf{x} \in K$, $h_1(\mathbf{x}) = 0$ otherwise, and $h_2(\mathbf{x}) = M$ for $\mathbf{x} \in K$, $h_2(\mathbf{x}) = 0$ otherwise. Then, $h_1,\ h_2 \in \mathrm{L}^2_{0,M}(\mathbb{R}^d)$ and $h = h_1 - h_2$. □

If we choose $M = 1$ and consider a subset of $\mathrm{L}^2_{0,1}(\mathbb{R}^d)$ containing only functions which are non-negative almost everywhere, then we can interpret these functions as probability density functions. Since any probability measure can be approximated by a sequence of probability measures which have densities (with respect to the Lebesgue measure), we can extend the functional $\Phi_f : \mathrm{L}^2_{0,1}(\mathbb{R}^d) \to \mathbb{R}$ to the functional Φ_f, defined on the set of probability measures μ on $\mathbb{R}^d$, or a given Borel set $X \subset \mathbb{R}^d$, by

$$\Phi_f(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\,d\mu(x)\,d\mu(y). \tag{6}$$

Proposition 3.6 then implies that for $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$, the functional Φ_f, defined in (6), is convex on the set of probability measures if and only if $f \in \mathrm{CP}(\mathrm{L}^2_0(\mathbb{R}^d))$.

4. Proof of Theorem 2.1

The proof of Theorem 2.1 will proceed in several steps, formulated as lemmas. Lemmas 4.1 and 4.2 are simple technical results which demonstrate that $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathrm{P}(\mathrm{L}^2_0(\mathbb{R}^d))$ and $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathrm{CP}(\mathrm{L}^2_0(\mathbb{R}^d))$ are closed subsets of $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$. These results will be used in the proofs of Lemmas 4.7 and 4.8, respectively.

Lemma 4.1. *Let* $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ *and* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of functions in* $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathrm{P}(\mathrm{L}^2_0(\mathbb{R}^d))$. *If* $f_n \xrightarrow{\mathrm{L}^1_{\mathrm{loc}}} f$ *as* $n \to \infty$, *then* $f \in \mathrm{P}(\mathrm{L}^2_0(\mathbb{R}^d))$.

Proof. Let $h \in \mathrm{L}^2_0(\mathbb{R}^d)$ and $h^*(\mathbf{z}) = h(-\mathbf{z})$ ($\mathbf{z} \in \mathbb{R}^d$). Let K be the compact support of $h * h^*$. Since $f_n \xrightarrow{\mathrm{L}^1_{\mathrm{loc}}} f$ as $n \to \infty$,

$$\begin{aligned}\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f-f_n)(\mathbf{x}-\mathbf{y})h(\mathbf{x})h(\mathbf{y})\,d\mathbf{x}d\mathbf{y}\right| &= \left|\int_{\mathbb{R}^d}(f-f_n)(\mathbf{z})\int_{\mathbb{R}^d}h(\mathbf{x})h(\mathbf{x}-\mathbf{z})\,d\mathbf{x}d\mathbf{z}\right| \\ &= \left|\int_K (f-f_n)(\mathbf{z})(h*h^*)(\mathbf{z})\,d\mathbf{z}\right| \leq \|f-f_n\|_{1,K}\|h*h^*\|_\infty,\end{aligned} \tag{7}$$

using Young's inequality in the final step. □

The following statement can be proved in an analogous manner.

Lemma 4.2. *Let* $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ *and* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of functions in* $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathrm{CP}(\mathrm{L}^2_0(\mathbb{R}^d))$. *If* $f_n \xrightarrow{\mathrm{L}^1_{\mathrm{loc}}} f$ *as* $n \to \infty$, *then* $f \in \mathrm{CP}(\mathrm{L}^2_0(\mathbb{R}^d))$.

Functions in $\mathrm{CP}(\mathrm{L}^2_0(\mathbb{R}^d))$ need not be bounded at the origin, they need only be locally integrable. This fact is proved in the following lemma and is later used to ensure the existence of the integral in (2), for particular spaces J of test functions.

Lemma 4.3. *If* $f \in \mathrm{CP}(\mathrm{L}^2_0(\mathbb{R}^d))$, *then* $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Proof. Let $\mathrm{K} \subset \mathbb{R}^d$ be any compact set and $\mathrm{I}=[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d] \subset \mathbb{R}^d$ be such that $\mathrm{K} \subset \mathrm{I}$. Let $c = \max\{|a_1|, |b_1|, |a_2|, \ldots, |b_d|\} > 0$.

Let $\psi \in \mathrm{L}_0^2(\mathbb{R})$ be such that ψ is positive and continuous on $[-2c, 2c]$, and $\int_{\mathbb{R}} \psi(x)\,dx = 0$. For any $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, define $\Psi(\mathbf{x}) := \psi(x_1)\psi(x_2)\ldots\psi(x_d)$. Then $\Psi \in \mathrm{L}_0^2(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \Psi(\mathbf{x})\,d\mathbf{x} = 0$ and

$$g(\mathbf{z}) = \int_{[-c,\,c]^d} \Psi(\mathbf{z}+\mathbf{y})\Psi(\mathbf{y})\,d\mathbf{y} = \prod_{i=1}^{d} \int_{-c}^{c} \psi(z_i + y_i)\psi(y_i)\,dy_i \qquad (\mathbf{z} \in \mathbb{R}^d)$$

is positive and continuous on $[-c,\,c]^d$. Thus,

$$\begin{aligned} \inf_{\mathbf{z}\in[-c,\,c]^d} g(\mathbf{z}) \int_{[-c,\,c]^d} |f(\mathbf{z})|\,d\mathbf{z} \le \int_{[-c,\,c]^d} |f(\mathbf{z})g(\mathbf{z})|\,d\mathbf{z} &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(\mathbf{z})\Psi(\mathbf{z}+\mathbf{y})\Psi(\mathbf{y})|\,d\mathbf{y}d\mathbf{z} \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(\mathbf{x}-\mathbf{y})\Psi(\mathbf{x})\Psi(\mathbf{y})|\,d\mathbf{x}d\mathbf{y} \qquad (8) \end{aligned}$$

where (8) follows from the Fubini theorem. Since $f \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$, the integral in (8) exists. Hence

$$\int_{\mathrm{K}} |f(\mathbf{z})|\,d\mathbf{z} \le \int_{\mathrm{I}} |f(\mathbf{z})|\,d\mathbf{z} \le \int_{[-c,\,c]^d} |f(\mathbf{z})|\,d\mathbf{z} < \infty.$$

□

Note that we can replace $\mathrm{L}_0^2(\mathbb{R}^d)$ in Lemma 4.3 with a more general set J of functions defined on $\mathbb{R}^d$, provided that for any $c > 0$, J contains a function h, which is positive almost everywhere on $[-c,\,c]^d$ and $\int_{\mathbb{R}^d} h(\mathbf{x})\,d\mathbf{x} = 0$.

Lemma 4.4. *Let $\phi \in \mathrm{L}_0^2(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} \phi(\mathbf{x})\,d\mathbf{x} < \infty$. If $f \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$, then there exists a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_f)$ and a mapping $k : \mathbb{R}^d \to \mathcal{H}$, $\mathbf{z} \mapsto k_{\mathbf{z}}$ such that*

$$\|k_{\mathbf{x}} - k_{\mathbf{y}}\|_f^2 = \mathrm{C} - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\mathbf{x}-\mathbf{u})\phi(\mathbf{y}-\mathbf{v})\,d\mathbf{u}d\mathbf{v} \qquad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^d) \tag{9}$$

where

$$\mathrm{C} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{s}-\mathbf{t})\phi(\mathbf{s})\phi(\mathbf{t})\,d\mathbf{s}d\mathbf{t} \tag{10}$$

is independent of $\mathbf{x}$ *and* $\mathbf{y}$.

Proof. Suppose $f \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$. Let V be the subset of $\mathrm{L}_0^2(\mathbb{R}^d)$ defined by

$$\mathrm{V} = \left\{ h \in \mathrm{L}_0^2(\mathbb{R}^d) \;\middle|\; h(\mathbf{u}) = \sum_{i=1}^{m} a_i\,\phi(\mathbf{x}_i - \mathbf{u}) \ (\mathbf{u} \in \mathbb{R}^d) \text{ for some } m \in \mathbb{N}, \right.$$
$$\left. \mathbf{x}_i \in \mathbb{R}^d,\ a_i \in \mathbb{R};\ \text{s.t.} \int_{\mathbb{R}^d} h(\mathbf{u})\,d\mathbf{u} = 0 \right\}.$$

V is a vector space. For $\Phi, \Psi \in \mathrm{V}$, define

$$\langle \Phi, \Psi \rangle_f := \frac{1}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{x}-\mathbf{y})\Phi(\mathbf{x})\Psi(\mathbf{y})\,d\mathbf{x}d\mathbf{y}.$$

Since $f \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$, the mapping

$$(\Phi, \Psi) \mapsto \langle \Phi, \Psi \rangle_f$$

is a bilinear, symmetric and non-negative form on V. Set

$$\mathrm{V}' = \{h' \in \mathrm{V} \mid \langle h', h' \rangle_f = 0\}.$$

V' is a subspace of V since for any $g', h' \in \mathrm{V}'$,

$$\langle g' + h', g' + h' \rangle_f = 2\,\langle g', h' \rangle_f \le 2\,\langle g', g' \rangle_f^{\frac{1}{2}}\,\langle h', h' \rangle_f^{\frac{1}{2}} = 0$$

by the Cauchy-Schwarz inequality. On the quotient space V/V', define

$$\langle [g], [h] \rangle_f := \langle g, h \rangle_f, \tag{11}$$

where $[g],[h]$ denote the equivalence classes in V/V'. The inner product in (11) is well-defined since

$$\langle g+g', h+h'\rangle_f = \langle g,h\rangle_f \qquad (g,h\in\mathrm{V},\, g',h'\in\mathrm{V}').$$

To see this, note that

$$\langle g,h'\rangle_f \le \langle g,g\rangle_f^{\frac12}\,\langle h',h'\rangle_f^{\frac12} = 0$$

and

$$-\langle g,h'\rangle_f = \langle g,-h'\rangle_f \le \langle g,g\rangle_f^{\frac12}\,\langle -h',-h'\rangle_f^{\frac12} = 0,$$

by the Cauchy-Schwarz inequality. Thus, $\langle g,h'\rangle_f = 0$ and similarly, $\langle g',h\rangle_f = 0$. Let $(\mathcal{H},\langle\cdot,\cdot\rangle_f)$ be the Hilbert space completion of V/V'; then, in particular, V/V' is dense in $\mathcal{H}$.

Let $\tilde{\mathbf{x}}\in\mathbb{R}^d$. For any $\mathbf{z}\in\mathbb{R}^d$, set $k_{\mathbf{z}} := [\phi(\mathbf{z}-\cdot)-\phi(\tilde{\mathbf{x}}-\cdot)]\in\mathcal{H}$. Then, for any $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$,

$$\begin{aligned}\|k_{\mathbf{x}}-k_{\mathbf{y}}\|_f^2 &= \langle k_{\mathbf{x}}-k_{\mathbf{y}}, k_{\mathbf{x}}-k_{\mathbf{y}}\rangle_f\\ &= \frac12\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\mathbf{x}-\mathbf{u})\phi(\mathbf{x}-\mathbf{v})\,d\mathbf{u}d\mathbf{v}\\ &\quad+\frac12\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\mathbf{y}-\mathbf{u})\phi(\mathbf{y}-\mathbf{v})\,d\mathbf{u}d\mathbf{v}\\ &\quad-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\mathbf{x}-\mathbf{u})\phi(\mathbf{y}-\mathbf{v})\,d\mathbf{u}d\mathbf{v},\end{aligned}\tag{12}$$

since

$$\begin{aligned}\langle k_{\mathbf{x}},k_{\mathbf{x}}\rangle_f &= \frac12\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\mathbf{x}-\mathbf{u})\phi(\mathbf{x}-\mathbf{v})\,d\mathbf{u}d\mathbf{v}\\ &\quad-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\mathbf{x}-\mathbf{u})\phi(\tilde{\mathbf{x}}-\mathbf{v})\,d\mathbf{u}d\mathbf{v}\\ &\quad+\frac12\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\tilde{\mathbf{x}}-\mathbf{u})\phi(\tilde{\mathbf{x}}-\mathbf{v})\,d\mathbf{u}d\mathbf{v},\end{aligned}$$

$$\begin{aligned}2\langle k_{\mathbf{x}},k_{\mathbf{y}}\rangle_f &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\mathbf{x}-\mathbf{u})\phi(\mathbf{y}-\mathbf{v})\,d\mathbf{u}d\mathbf{v}\\ &\quad-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\tilde{\mathbf{x}}-\mathbf{u})\phi(\mathbf{y}-\mathbf{v})\,d\mathbf{u}d\mathbf{v}\\ &\quad-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\mathbf{x}-\mathbf{u})\phi(\tilde{\mathbf{x}}-\mathbf{v})\,d\mathbf{u}d\mathbf{v}\\ &\quad+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\tilde{\mathbf{x}}-\mathbf{u})\phi(\tilde{\mathbf{x}}-\mathbf{v})\,d\mathbf{u}d\mathbf{v}\end{aligned}$$

and

$$\begin{aligned}\langle k_{\mathbf{y}},k_{\mathbf{y}}\rangle_f &= \frac12\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\mathbf{y}-\mathbf{u})\phi(\mathbf{y}-\mathbf{v})\,d\mathbf{u}d\mathbf{v}\\ &\quad-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\tilde{\mathbf{x}}-\mathbf{u})\phi(\mathbf{y}-\mathbf{v})\,d\mathbf{u}d\mathbf{v}\\ &\quad+\frac12\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\phi(\tilde{\mathbf{x}}-\mathbf{u})\phi(\tilde{\mathbf{x}}-\mathbf{v})\,d\mathbf{u}d\mathbf{v}.\end{aligned}$$

By a simple change of variables, each of the first two integrals in (12) is equal to $\mathrm{C}/2$, where C is defined by (10). Thus, formula (9) follows. □

We will refer to $\mathcal{H}$ as the Hilbert space, and to k as the mapping, associated with f and ϕ.

Lemma 4.4 can be considered as a generalised version of the *GNS construction*, which is a widely celebrated technique in the literature, see e.g. [1, Th. C.2.3]. In fact, (9) is a direct extension of [1, Th. C.2.3 (i)]. Note that in [1], it is assumed that conditionally positive definite functions vanish at the origin. The following result is analogous to [1, Lemma C.3.1] and [3, Chapter 3, Lemma 2.1].

Lemma 4.5. *Let $f\in\mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$ and $\phi\in\mathrm{L}_0^2(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d}\phi(\mathbf{x})\,d\mathbf{x}<\infty$. Let $\mathcal{H}$ and k denote the associating Hilbert space and mapping respectively. Fix $\mathbf{x}_0\in\mathbb{R}^d$. The kernel*

$$g(\mathbf{x},\mathbf{y}) = \|k_{\mathbf{x}}-k_{\mathbf{x}_0}\|_f^2 + \|k_{\mathbf{y}}-k_{\mathbf{x}_0}\|_f^2 - \|k_{\mathbf{x}}-k_{\mathbf{y}}\|_f^2 \qquad (\mathbf{x},\mathbf{y}\in\mathbb{R}^d)\tag{13}$$

is a classical positive definite kernel, see e.g. [19, Eq. 8.1].

Proof. As in [1, p. 373], a straightforward calculation gives

$$g(\mathbf{x},\mathbf{y}) = 2\langle k_{\mathbf{x}} - k_{\mathbf{x}_0}, k_{\mathbf{y}} - k_{\mathbf{x}_0}\rangle_f \qquad (\mathbf{x},\mathbf{y} \in \mathbb{R}^d).$$

Therefore, for any $n \in \mathbb{N}$, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^d$ and $v_1, v_2, \ldots, v_n \in \mathbb{R}$,

$$\sum_{i,j=1}^{n} g(\mathbf{x}_i,\mathbf{x}_j)\, v_i\, v_j = 2\left\| \sum_{i=1}^{n} v_i(k_{\mathbf{x}_i} - k_{\mathbf{x}_0})\right\|_f^2 \geq 0.$$

□

For $f \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$ and $\phi \in \mathrm{L}_0^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \phi(\mathbf{x})\, d\mathbf{x} < \infty$, with associating Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_f)$ and mapping k, define

$$\mathrm{K}(\mathbf{x},\mathbf{y}) := \|k_{\mathbf{x}} - k_{\mathbf{y}}\|_f^2 \qquad (\mathbf{x},\mathbf{y} \in \mathbb{R}^d).$$

It follows from (9) that $\mathrm{K}(\mathbf{x}+\mathbf{a},\mathbf{y}+\mathbf{a}) = \mathrm{K}(\mathbf{x},\mathbf{y})$ for any $\mathbf{a} \in \mathbb{R}^d$, and hence $\mathrm{K}(\mathbf{x},\mathbf{y}) = \mathrm{K}(\mathbf{x}-\mathbf{y},\mathbf{0})$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$. Define $\tilde{f}(\mathbf{z}) := \mathrm{K}(\mathbf{z},\mathbf{0})$ for all $\mathbf{z} \in \mathbb{R}^d$. Then, $\mathrm{K}(\mathbf{x},\mathbf{y}) = \tilde{f}(\mathbf{x}-\mathbf{y})$ for any $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$.

The kernel $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, defined in Lemma 4.5, is positive definite. Hence, so is $t^n g^n$ for any $t > 0$, $n \in \mathbb{N}$, since the product of positive definite kernels is also positive definite, see e.g. [1, Prop. C.1.6 (iv)]. Consequently, e^{tg} is a classically positive definite kernel.

Let $t > 0$. It follows from (13) that

$$e^{-t\mathrm{K}(\mathbf{x},\mathbf{y})} = e^{tg(\mathbf{x},\mathbf{y})} \times \left(e^{-t\mathrm{K}(\mathbf{x},\mathbf{x}_0)}\, e^{-t\mathrm{K}(\mathbf{y},\mathbf{x}_0)}\right) \qquad (\mathbf{x},\mathbf{y} \in \mathbb{R}^d).$$

The kernel $e^{-t\mathrm{K}(\,\cdot\,,\mathbf{x}_0)}\, e^{-t\mathrm{K}(\,\cdot\,,\mathbf{x}_0)} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is positive definite, since

$$\sum_{i,j=1}^{n} \left(e^{-t\mathrm{K}(\mathbf{x}_i,\mathbf{x}_0)}\, e^{-t\mathrm{K}(\mathbf{x}_j,\mathbf{x}_0)}\right) v_i\, v_j = \left(\sum_{i=1}^{n} v_i\, e^{-t\mathrm{K}(\mathbf{x}_i,\mathbf{x}_0)}\right)^2 \geq 0$$

for any $n \in \mathbb{N}$, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^d$ and $v_1, v_2, \ldots, v_n \in \mathbb{R}$; as in [1, p. 374]. Hence, $e^{-t\mathrm{K}}$ is a positive definite kernel, and therefore, $e^{-t\tilde{f}}$ is a classically positive definite function, as in (1). By Theorem 1.1, it follows that $-\tilde{f}$ is conditionally positive definite. Thus, $-\tilde{f} + \alpha$ is conditionally positive definite for any $\alpha \in \mathbb{R}$.

The next lemma highlights a connection between classically conditionally positive definite functions and functions which are conditionally positive definite with respect to $\mathrm{L}_0^2(\mathbb{R}^d)$. In particular, we observe that $\mathrm{L}_{\mathrm{loc}}^p(\mathbb{R}^d) \cap \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$ is the closure of $\mathrm{C}^\infty(\mathbb{R}^d) \cap \mathrm{CP}_\mathrm{C}$. We use this result in the proof of Lemma 4.7.

Lemma 4.6. *Let $p \in [1,\infty)$ and $f \in \mathrm{L}_{\mathrm{loc}}^p(\mathbb{R}^d) \cap \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$. Then, there is a sequence $(f_n)_{n\in\mathbb{N}}$ of infinitely differentiable, classically conditionally positive definite functions, such that $f_n \xrightarrow{\mathrm{L}_{\mathrm{loc}}^p} f$ as $n \to \infty$.*

Proof. Let $\psi : \mathbb{R} \to \mathbb{R}$ denote the bump function

$$\psi(x) = \begin{cases} c_0 \exp\left(\frac{1}{|x|^2-1}\right), & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$$

where $c_0 > 0$ is the constant chosen such that $\int_{\mathbb{R}} \psi(x)\, dx = 1$. For any $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, define $\Psi(\mathbf{x}) := \psi(x_1)\psi(x_2)\ldots\psi(x_d)$ and

$$\Psi_n(\mathbf{x}) := n^d\, \Psi(n\,\mathbf{x}) \qquad (n \in \mathbb{N}). \tag{14}$$

Then, for any $n \in \mathbb{N}$, $\Psi_n \in \mathrm{C}_0^\infty(\mathbb{R}^d)$ is even and has compact support $[-\frac{1}{n}, \frac{1}{n}]^d$. Moreover, $\int_{\mathbb{R}^d} \Psi_n(\mathbf{x})\, d\mathbf{x} = 1$ for all $n \in \mathbb{N}$.

Applying Lemma 4.4 to the functions Ψ_n and f, we find

$$\|k_{n,\mathbf{x}} - k_{n,\mathbf{y}}\|_f^2 = \mathrm{C}_n - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\Psi_n(\mathbf{x}-\mathbf{u})\Psi_n(\mathbf{y}-\mathbf{v})\, d\mathbf{u}d\mathbf{v} \quad (\mathbf{x},\mathbf{y} \in \mathbb{R}^d,\ n \in \mathbb{N}),$$

where

$$\mathrm{C_n} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{s}-\mathbf{t})\Psi_n(\mathbf{s})\Psi_n(\mathbf{t})\, d\mathbf{s}d\mathbf{t} \in \mathbb{R}$$

and $k_{n,\mathbf{z}}$ is the equivalence class $[\Psi_n(\mathbf{z}-\cdot)-\Psi_n(\tilde{\mathbf{x}}-\cdot)]$ in $\mathcal{H}$ ($\mathbf{z}\in\mathbb{R}^d$). Let

$$\tilde{f}_n(\mathbf{z}) = \|k_{n,\mathbf{z}} - k_{n,\mathbf{0}}\|_f^2 \qquad (\mathbf{z}\in\mathbb{R}^d).$$

From Lemma 4.5 and the ensuing remarks, it follows that $-\tilde{f}_n+\alpha$ is conditionally positive definite for any $\alpha\in\mathbb{R}$, $n\in\mathbb{N}$. In particular,

$$\begin{aligned} f_n(\mathbf{z}) := -\tilde{f}_n(\mathbf{z}) + \mathrm{C_n} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{u}-\mathbf{v})\Psi_n(\mathbf{z}-(\mathbf{u}-\mathbf{y}))\Psi_n(\mathbf{z}-(\mathbf{x}-\mathbf{v}))\,d\mathbf{u}d\mathbf{v} \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{z}+\mathbf{t}-\mathbf{s})\Psi_n(\mathbf{z}-\mathbf{s})\Psi_n(\mathbf{z}-\mathbf{t})\,d\mathbf{s}d\mathbf{t} \quad (\mathbf{z}\in\mathbb{R}^d) \end{aligned} \tag{15}$$

defines a classically conditionally positive definite function. Note that we have used the evenness of f and Ψ_n in order to arrive at the above equation. On rewriting (15), again by using the fact that Ψ_n is even, we obtain

$$f_n(\mathbf{z}) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{t}-(\mathbf{s}-\mathbf{z}))\Psi_n(\mathbf{s}-\mathbf{z})\Psi_n(\mathbf{z}-\mathbf{t})\,d\mathbf{s}d\mathbf{t} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f*\Psi_n(\mathbf{t}))\,\Psi_n(\mathbf{z}-\mathbf{t})\,d\mathbf{t}.$$

Hence, for any $n\in\mathbb{N}$,

$$f_n = f*\eta_n, \tag{16}$$

where $\eta_n=\Psi_n*\Psi_n=n^d(\Psi*\Psi)(n\,\cdot)$. Using the properties of Ψ_n, it follows that for any $n\in\mathbb{N}$, $\eta_n\in\mathrm{C}_0^\infty(\mathbb{R}^d)$ has compact support $[-\frac{2}{n},\frac{2}{n}]^d$, and $\int_{\mathbb{R}^d}\eta_n(\mathbf{x})\,d\mathbf{x}=1$. Hence, $f_n\in\mathrm{C}^\infty(\mathbb{R}^d)$ for all $n\in\mathbb{N}$, and $f_n \xrightarrow{\mathrm{L}^p_{\mathrm{loc}}} f$ as $n\to\infty$ by [16, Th. 1.18]. □

Proposition 5.3 demonstrates that the converse to Lemma 4.6 is also true. In the following result, we establish one direction of the equivalence (3) in Theorem 2.1.

Lemma 4.7. *Let $f\in\mathrm{CP}(\mathrm{L}^2_0(\mathbb{R}^d))$. If there exist $t_0>0$ and $p>1$ such that $e^{t|f|}\in\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$ for all $0<t\le t_0$, then $e^{tf}\in\mathrm{P}(\mathrm{L}^2_0(\mathbb{R}^d))$ for all $0<t\le t_0$.*

Proof. Suppose $t_0>0$ and $p>1$ are such that $e^{t|f|}\in\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$ for all $0<t\le t_0$. Then, it follows that $e^{tf}\in\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$ for all $0<t\le t_0$, and $f\in\mathrm{L}^q_{\mathrm{loc}}(\mathbb{R}^d)$ for any $1\le q<\infty$.

The functions f_n, as defined in the proof of Lemma 4.6, are conditionally positive definite in the classical sense. Thus, by Theorem 1.1, e^{tf_n} is positive definite for any $t>0$, $n\in\mathbb{N}$. Moreover, e^{tf_n} is continuous for all $t>0$, $n\in\mathbb{N}$, since f_n is continuous for any $n\in\mathbb{N}$. By property (iii) of Section 2, it follows that $e^{tf_n}\in\mathrm{P}(\mathrm{L}^2_0(\mathbb{R}^d))$ for any $t>0$, $n\in\mathbb{N}$.

By Lemma 4.1, we need only show that $e^{tf_n}\xrightarrow{\mathrm{L}^1_{\mathrm{loc}}} e^{tf}$ as $n\to\infty$. Let $\epsilon>0$, $0<t\le t_0$ and $K\subset\mathbb{R}^d$ be a compact set. Let $n_0\in\mathbb{N}$ and define $\hat{K}:=K+\left[-\frac{2}{n_0},\frac{2}{n_0}\right]^d$. By Lemma 4.6, there exists $n_*\in\mathbb{N}$ such that for all $n\ge n_*$,

$$\|f_n-f\|_{q,K} = \left(\int_K |f_n(\mathbf{x})-f(\mathbf{x})|^q\,d\mathbf{x}\right)^{\frac{1}{q}} < \frac{\epsilon}{2t\,\|e^{t|f|}\|_{p,\hat{K}}} \tag{17}$$

where $\frac{1}{p}+\frac{1}{q}=1$. W.l.o.g., we assume $n_*\ge n_0$.

Let $n>n_*$. We partition $K=K_1\cup K_2$, where $K_1=\{\mathbf{x}\in K\,|\,f(\mathbf{x})\le f_n(\mathbf{x})\}$ and $K_2=\{\mathbf{x}\in K\,|\,f(\mathbf{x})>f_n(\mathbf{x})\}$. By the Mean Value Theorem,

$$\left|\left(e^{tf_n}-e^{tf}\right)(\mathbf{x})\right| = t\,e^{t\,\xi_n(\mathbf{x})}\,|f_n(\mathbf{x})-f(\mathbf{x})| \qquad \text{f. a. a. } \mathbf{x}\in\mathbb{R}^d,$$

where $\xi_n(\mathbf{x})$ lies between $f_n(\mathbf{x})$ and $f(\mathbf{x})$. Therefore,

$$\begin{aligned} \int_K \left|\left(e^{tf_n}-e^{tf}\right)(\mathbf{x})\right|dx &= \int_{K_1} t\,e^{t\xi_n(\mathbf{x})}|(f_n-f)(\mathbf{x})|\,d\mathbf{x} + \int_{K_2} t\,e^{t\xi_n(\mathbf{x})}|(f_n-f)(\mathbf{x})|\,d\mathbf{x} \\ &\le \int_{K_1} t\,e^{tf_n(\mathbf{x})}|(f_n-f)(\mathbf{x})|\,d\mathbf{x} + \int_{K_2} t\,e^{tf(\mathbf{x})}|(f_n-f)(\mathbf{x})|\,d\mathbf{x} \\ &\le \int_{K} t\,e^{tf_n(\mathbf{x})}|(f_n-f)(\mathbf{x})|\,d\mathbf{x} + \int_{K} t\,e^{tf(\mathbf{x})}|(f_n-f)(\mathbf{x})|\,d\mathbf{x} \\ &\le t\left(\|e^{tf_n}\|_{p,K} + \|e^{tf}\|_{p,K}\right)\|f_n-f\|_{q,K}, \end{aligned} \tag{18}$$

using Hölder's inequality in the last step. Next, for almost all $x \in \mathbb{R}^d$,

$$0 \le e^{tf(\mathbf{x})} = \sum_{j=0}^{\infty} \frac{t^j\, f(\mathbf{x})^j}{j!} \le e^{t\,|f(\mathbf{x})|},$$

and thus, since $K \subset \hat{K}$,

$$\|e^{tf}\|_{p,K} \le \|e^{t\,|f|}\|_{p,K} \le \|e^{t\,|f|}\|_{p,\hat{K}}. \tag{19}$$

By Jensen's inequality, see e.g. [10, Th. 1.8.1], and (16), it follows that for any $\mathbf{x} \in \mathbb{R}^d$,

$$e^{tf_n(\mathbf{x})} = \exp\left(t\int_{\mathbb{R}^d} f(\mathbf{x}-\mathbf{z})\eta_n(\mathbf{z})d\mathbf{z}\right) \le \int_{\mathbb{R}^d} e^{tf(\mathbf{x}-\mathbf{z})}\eta_n(\mathbf{z})d\mathbf{z} = e^{tf} * \eta_n(\mathbf{x}),$$

and hence,

$$\|e^{tf_n}\|_{p,K} \le \|e^{tf} * \eta_n\|_{p,K}.$$

Moreover,

$$\big(e^{tf} * \eta_n\big)(\mathbf{x})\,\chi_K(\mathbf{x}) \le \big((e^{tf}\chi_{\hat{K}}) * \eta_n\big)(\mathbf{x}) \qquad (\mathbf{x} \in \mathbb{R}^d).$$

This follows since for $\mathbf{x} \in K$ and $n \ge n_0$,

$$\begin{aligned}\big((e^{tf}\chi_{\hat{K}}) * \eta_n\big)(\mathbf{x}) &= \int_{[-\frac{2}{n},\frac{2}{n}]^d} \Big(e^{tf(\mathbf{x}-\mathbf{y})}\chi_{\hat{K}}(\mathbf{x}-\mathbf{y})\Big)\,\eta_n(\mathbf{y})\,d\mathbf{y}\\ &= \int_{[-\frac{2}{n},\frac{2}{n}]^d} e^{tf(\mathbf{x}-\mathbf{y})}\eta_n(\mathbf{y})\,d\mathbf{y} = \big(e^{tf} * \eta_n\big)(\mathbf{x}).\end{aligned}$$

Thus, using Young's inequality, it proceeds that

$$\|e^{tf} * \eta_n\|_{p,K} \le \|e^{tf}\|_{p,\hat{K}}\,\|\eta_n\|_1 = \|e^{tf}\|_{p,\hat{K}} \le \|e^{t\,|f|}\|_{p,\hat{K}},$$

and thus,

$$\|e^{tf_n}\|_{p,K} \le \|e^{t\,|f|}\|_{p,\hat{K}}. \tag{20}$$

From (17), (18), (19) and (20), we conclude that

$$\int_K \big|\big(e^{tf_n} - e^{tf}\big)(\mathbf{x})\big|\,d\mathbf{x} < \epsilon.$$

□

The following lemma concludes the proof of Theorem 2.1.

Lemma 4.8. *Let $f : \mathbb{R}^d \to \mathbb{R}$. If there exists $t_0 > 0$ such that $e^{t|f|} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ for any $0 < t \le t_0$, then*

$$e^{tf} \in \mathrm{P}(\mathrm{L}^2_0(\mathbb{R}^d))\ \ (0 < t \le t_0) \implies f \in \mathrm{CP}(\mathrm{L}^2_0(\mathbb{R}^d)).$$

Proof. Suppose $e^{tf} \in \mathrm{P}(\mathrm{L}^2_0(\mathbb{R}^d))$ for all $0 < t \le t_0$. Then, for any $\phi \in \mathrm{L}^2_0(\mathbb{R}^d)$ and $0 < t \le t_0$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{tf(\mathbf{x}-\mathbf{y})}\phi(\mathbf{x})\phi(\mathbf{y})\,d\mathbf{x}d\mathbf{y} \ge 0$$

and hence,

$$\frac{1}{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{tf(\mathbf{x}-\mathbf{y})}\phi(\mathbf{x})\phi(\mathbf{y})\,d\mathbf{x}d\mathbf{y} \ge 0.$$

Let $\psi \in \mathrm{L}^2_0(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d}\psi(\mathbf{x})\,d\mathbf{x} = 0$. Then, for any $0 < t \le t_0$,

$$\begin{aligned}\frac{1}{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{tf(\mathbf{x}-\mathbf{y})}\psi(\mathbf{x})\psi(\mathbf{y})d\mathbf{x}d\mathbf{y} &= \frac{1}{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{tf(\mathbf{x}-\mathbf{y})}\psi(\mathbf{x})\psi(\mathbf{y})d\mathbf{x}d\mathbf{y} - \frac{1}{t}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(\mathbf{x})\psi(\mathbf{y})d\mathbf{x}d\mathbf{y}\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{e^{tf(\mathbf{x}-\mathbf{y})}-1}{t}\right)\psi(\mathbf{x})\psi(\mathbf{y})d\mathbf{x}d\mathbf{y}\end{aligned}$$

Define $f_t := (e^{tf}-1)/t$, $(0<t\le t_0)$. It follows that $f_t \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$ for all $0<t\le t_0$. To prove $f \in \mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$, we need only show that $f_t \xrightarrow{\mathrm{L}^1_{\mathrm{loc}}} f$ as $t\to 0$, by Lemma 4.2. Let $K\subset\mathbb{R}^d$ be a compact set. Then,

$$\begin{aligned}\int_K\left|\left(\frac{e^{tf(\mathbf{x})}-1}{t}\right)-f(\mathbf{x})\right|d\mathbf{x} &= \int_K\left|t\sum_{j=2}^{\infty}\frac{t^{j-2}\,f(\mathbf{x})^j}{j!}\right|d\mathbf{x} \le \int_K t\sum_{j=2}^{\infty}\frac{t_0^{j-2}\,|f(\mathbf{x})|^j}{j!}\,d\mathbf{x}\\ &=\frac{t}{t_0^2}\int_K\sum_{j=2}^{\infty}\frac{t_0^j\,|f(\mathbf{x})|^j}{j!}d\mathbf{x}=\frac{t}{t_0^2}\int_K\Big(e^{t_0|f(\mathbf{x})|}-1-t_0|f(\mathbf{x})|\Big)d\mathbf{x}\\ &\le\frac{t}{t_0^2}\,\|e^{t_0|f|}\|_{1,K}\to 0 \qquad (t\to 0).\end{aligned}$$

□

5. Further results

In this section we prove two corollaries of Theorem 2.1 and establish an important result, analogous to property (vi) of Section 2, concerning the class $\mathrm{CP}(J)$. Firstly, we show that functions which are continuous and conditionally positive definite for $\mathrm{L}_0^2(\mathbb{R}^d)$ are conditionally positive definite in the classical sense.

Proposition 5.1. *Let $f:\mathbb{R}^d\to\mathbb{R}$ be continuous. Then, $f\in\mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$ if and only if f is classically conditionally positive definite.*

Proof. Suppose f is continuous and classically conditionally positive definite. By Theorem 1.1, e^{tf} is classically positive definite for all $t>0$. Moreover, for any $t>0$, e^{tf} is continuous since f is continuous. Using property (iii) of Section 2, it follows that $e^{tf}\in\mathrm{P}(\mathrm{L}_0^2(\mathbb{R}^d))$ for all $t>0$. By Theorem 2.1, we conclude that $f\in\mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))$.

For the reverse implication, consider the following. For $n\in\mathbb{N}$, let Ψ_n denote the functions defined in (14). For any $N\in\mathbb{N}$ and any $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N\in\mathbb{R}^d$, define

$$\Phi_n(\mathbf{x}) := \sum_{i=1}^{N}\xi_i\,\Psi_n(\mathbf{x}-\mathbf{x}_i)\quad(\mathbf{x}\in\mathbb{R}^d, n\in\mathbb{N}),$$

where $\xi_1,\xi_2,\ldots,\xi_N\in\mathbb{R}$ are such that $\sum_{i=1}^N\xi_i=0$. Then, $\Phi_n\in\mathrm{L}_0^2(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d}\Phi_n(\mathbf{x})\,d\mathbf{x}=0$ for all $n\in\mathbb{N}$. Furthermore, since f is continuous,

$$\begin{aligned}\sum_{i,j=1}^{N}f(\mathbf{x}_i-\mathbf{x}_j)\,\xi_i\,\xi_j &= \lim_{n\to\infty}\sum_{i,j=1}^{N}\int_{\mathbf{x}_j+[-\frac{1}{n},\frac{1}{n}]^d}\int_{\mathbf{x}_i+[-\frac{1}{n},\frac{1}{n}]^d}f(\mathbf{x}-\mathbf{y})\Phi_n(\mathbf{x})\Phi_n(\mathbf{y})\,d\mathbf{x}d\mathbf{y}\\ &=\lim_{n\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}f(\mathbf{x}-\mathbf{y})\Phi_n(\mathbf{x})\Phi_n(\mathbf{y})\,d\mathbf{x}d\mathbf{y}\ge 0.\end{aligned}$$

□

The next result demonstrates that for $p\ge 2$, $\mathrm{CP}(\mathrm{L}_0^p(\mathbb{R}^d))$ remains the same.

Proposition 5.2. $\mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))=\mathrm{CP}(\mathrm{L}_0^p(\mathbb{R}^d))=\mathrm{CP}(\mathrm{C}_0^r(\mathbb{R}^d))$ $(p\in(2,\infty],\ r\in[0,\infty])$.

Proof. Let $p\in(2,\infty]$, $r\in[0,\infty]$. Since $\mathrm{C}_0^r(\mathbb{R}^d)\subset\mathrm{L}_0^p(\mathbb{R}^d)\subset\mathrm{L}_0^2(\mathbb{R}^d)$, it follows directly that $\mathrm{CP}(\mathrm{L}_0^2(\mathbb{R}^d))\subset\mathrm{CP}(\mathrm{L}_0^p(\mathbb{R}^d))\subset\mathrm{CP}(\mathrm{C}_0^r(\mathbb{R}^d))$. For the reverse implication, consider the following. Let $\phi\in\mathrm{L}_0^2(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d}\phi(\mathbf{x})\,d\mathbf{x}=0$. For $n\in\mathbb{N}$, let Ψ_n denote the functions defined in (14), and set

$$\psi_n := \phi * \Psi_n \quad (n\in\mathbb{N}).$$

Then, $\psi_n\in\mathrm{C}_0^r(\mathbb{R}^d)\subset\mathrm{L}_0^p(\mathbb{R}^d)$ for all $r\in[0,\infty]$, $p\in(2,\infty]$, $n\in\mathbb{N}$, and by the Fubini theorem,

$$\begin{aligned}\int_{\mathbb{R}^d}\psi_n(\mathbf{x})\,d\mathbf{x}=\int_{\mathbb{R}^d}(\phi*\Psi_n)(\mathbf{x})\,d\mathbf{x}&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi(\mathbf{x}-\mathbf{y})\,\Psi_n(\mathbf{y})\,d\mathbf{y}d\mathbf{x}\\ &=\int_{\mathbb{R}^d}\phi(\mathbf{z})\,d\mathbf{z}\int_{\mathbb{R}^d}\Psi_n(\mathbf{y})\,d\mathbf{y}=0.\end{aligned}$$

Moreover, $\psi_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} \phi$ as $n \to \infty$, by [16, Th. 1.18].

Let $r \in [0, \infty]$ and suppose $f \in \mathrm{CP}(\mathrm{C}^r_0(\mathbb{R}^d))$. Since $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ by Lemma 4.3, the integral

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{x} - \mathbf{y}) \phi(\mathbf{x}) \phi(\mathbf{y}) \, d\mathbf{x} d\mathbf{y} = \int_{\mathbb{R}^d} (f * \phi) \, \phi$$

exists as a Lebesgue integral. By a change of variables and the Fubini theorem

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{x} - \mathbf{y}) \phi(\mathbf{x}) \phi(\mathbf{y}) \, d\mathbf{x} d\mathbf{y} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{z}) \phi(\mathbf{z} + \mathbf{y}) \phi(\mathbf{y}) \, d\mathbf{y} d\mathbf{z}.$$

Then,

$$\begin{aligned}
&\sup_{\mathbf{z} \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} (\phi(\mathbf{z} + \mathbf{y}) \phi(\mathbf{y}) - \psi_n(\mathbf{z} + \mathbf{y}) \psi_n(\mathbf{y})) d\mathbf{y} \right| \\
&\quad \leq \sup_{\mathbf{z} \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\phi(\mathbf{z} + \mathbf{y})| |(\phi - \psi_n)(\mathbf{y})| d\mathbf{y} + \sup_{\mathbf{z} \in \mathbb{R}^d} \int_{\mathbb{R}^d} |(\phi - \psi_n)(\mathbf{z} + \mathbf{y})| |\psi_n(\mathbf{y})| d\mathbf{y} \\
&\quad \leq \|\phi\|_2 \, \|\phi - \psi_n\|_2 + \|\phi - \psi_n\|_2 \, \|\psi_n\|_2 \ \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence, $\left| \int_K f(\mathbf{z}) g_n(\mathbf{z}) \, d\mathbf{z} \right| \to 0$ as $n \to \infty$, where g_n denotes the function $g_n(\mathbf{z}) = \int_{\mathbb{R}^d} (\phi(\mathbf{z} + \mathbf{y}) \phi(\mathbf{y}) - \psi_n(\mathbf{z} + \mathbf{y}) \psi_n(\mathbf{y})) \, d\mathbf{y}$ and K denotes its compact support. Thus,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{x} - \mathbf{y}) \phi(\mathbf{x}) \phi(\mathbf{y}) \, d\mathbf{x} d\mathbf{y} = \lim_{n \to \infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{x} - \mathbf{y}) \psi_n(\mathbf{x}) \psi_n(\mathbf{y}) \, d\mathbf{x} d\mathbf{y} \geq 0,$$

and the result follows, for $\mathrm{CP}(\mathrm{C}^r_0(\mathbb{R}^d)) \subset \mathrm{CP}(\mathrm{L}^2_0(\mathbb{R}^d))$. □

The following proposition shows that the converse to Lemma 4.6 is true. Indeed, if there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of classically conditionally positive definite functions such that $f_n \xrightarrow{\mathrm{L}^p_{\mathrm{loc}}} f$ as $n \to \infty$, then f is conditionally positive definite for $\mathrm{L}^2_0(\mathbb{R}^d)$.

Proposition 5.3. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be such that $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d)$ for some $p \in [1, \infty)$. Then, $f \in \mathrm{CP}(\mathrm{L}^2_0(\mathbb{R}^d))$ if and only if there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of infinitely differentiable, classically conditionally positive definite functions, such that $f_n \xrightarrow{\mathrm{L}^p_{\mathrm{loc}}} f$ as $n \to \infty$.*

Proof. One direction is proved in Lemma 4.6. For the reverse implication, we note that for any $n \in \mathbb{N}$, $f_n \in \mathrm{CP}(\mathrm{L}^2_0(\mathbb{R}^d))$ by Proposition 5.1. By Lemma 4.2, we need only show that $f_n \xrightarrow{\mathrm{L}^1_{\mathrm{loc}}} f$ as $n \to \infty$. This follows directly, since for any compact set $K \subset \mathbb{R}^d$,

$$\int_K |f_n(\mathbf{x}) - f(\mathbf{x})| \, d\mathbf{x} \leq |K|^{\frac{1}{q}} \left(\int_K |f_n(\mathbf{x}) - f(\mathbf{x})|^p \, d\mathbf{x} \right)^{\frac{1}{p}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $|K|$ denotes the Lebesgue measure of K. □

6. Appendix

The following proposition highlights the connection between real-valued functions which are positive definite with respect to a set of complex-valued test functions, and which are positive definite with respect to the subset of real-valued test functions. Roughly speaking, as in the classical case, real-valued positive definite functions are automatically positive definite in a complex sense.

Proposition 6.1. *Let $f : \mathbb{R}^d \to \mathbb{R}$ and $\hat{J}$ denote a vector space of complex-valued functions on $\mathbb{R}^d$, such that if $\phi \in \hat{J}$, then $\overline{\phi} \in \hat{J}$ and $|\phi| \in \hat{J}$. Let $\hat{J}_{\mathbb{R}} := \{\phi \in \hat{J} \mid \phi \text{ is real-valued}\}$. Then, $f \in \mathrm{P}(\hat{J}_{\mathbb{R}})$ if and only if $f \in \mathrm{P}(\hat{J})$.*

Proof. One direction is clear since $\hat{J}_{\mathbb{R}} \subseteq \hat{J}$. For the reverse implication, consider the following. Let $\psi \in \hat{J}$ and suppose $f \in \mathrm{P}(\hat{J}_{\mathbb{R}})$. First we prove the existence of the integral

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\mathbf{x} - \mathbf{y}) \psi(\mathbf{x}) \overline{\psi(\mathbf{y})} \, d\mathbf{x} d\mathbf{y}. \tag{21}$$

Indeed,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(\mathbf{x}-\mathbf{y})\psi(\mathbf{x})\overline{\psi(\mathbf{y})}|\, d\mathbf{x}d\mathbf{y} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(\mathbf{x}-\mathbf{y})|\,|\psi(\mathbf{x})|\,|\psi(\mathbf{y})|\, d\mathbf{x}d\mathbf{y}$$
$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(\mathbf{x}-\mathbf{y})\,\tilde{\psi}(\mathbf{x})\,\tilde{\psi}(\mathbf{y})|\, d\mathbf{x}d\mathbf{y} \tag{22}$$

where $\tilde{\psi} = |\psi| \in \hat{J}_{\mathbb{R}}$. The integral in (22) exists since $f \in \mathrm{P}(\hat{J}_{\mathbb{R}})$. Hence, it follows that the integral in (21) exists in the Lebesgue sense. Next we show the non-negativity of (21), which in turn, proves that $f \in \mathrm{P}(\hat{J})$. ψ can be re-written as

$$\psi = \mathrm{Re}\,(\psi) + i\,\mathrm{Im}\,(\psi)$$

where $\mathrm{Re}\,(\psi) : \mathbb{R}^d \to \mathbb{R}$ and $\mathrm{Im}\,(\psi) : \mathbb{R}^d \to \mathbb{R}$. Moreover,

$$\mathrm{Re}\,(\psi) = \frac{\psi + \overline{\psi}}{2} \in \hat{J} \quad \text{and} \quad \mathrm{Im}\,(\psi) = \frac{\psi - \overline{\psi}}{2i} \in \hat{J}.$$

Thus, $\mathrm{Re}\,(\psi), \mathrm{Im}\,(\psi) \in \hat{J}_{\mathbb{R}}$. Let $a := \mathrm{Re}\,(\psi)$, $b := \mathrm{Im}\,(\psi)$ and

$$t\,[u, v] := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{x}-\mathbf{y})\, u(\mathbf{x})\,\overline{v(\mathbf{y})}\, d\mathbf{x}d\mathbf{y} \qquad (u, v \in \hat{J}).$$

Then

$$t\,[a, b] + t\,[b, a] = t\,[a + b, a + b] - t\,[a, a] - t\,[b, b]$$

and

$$-i\,(t\,[a, b] - t\,[b, a]) = t\,[\psi, \psi] - t\,[a, a] - t\,[b, b] \tag{23}$$

are finite, since $f \in \mathrm{P}(\hat{J}_{\mathbb{R}})$. Hence, $t\,[a, b]$ and $t\,[b, a]$ individually exist since both the sum $t\,[a, b] + t\,[b, a]$, and the difference $t\,[a, b] - t\,[b, a]$, exist. This allows us to use the Fubini theorem, which in conjunction with the evenness of f gives

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{x}-\mathbf{y})\, a(\mathbf{x})\, b(\mathbf{y})\, d\mathbf{x}d\mathbf{y} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(\mathbf{x}-\mathbf{y})\, b(\mathbf{x})\, a(\mathbf{y})\, d\mathbf{x}d\mathbf{y}.$$

Hence, $t\,[a, b] = t\,[b, a]$, and it follows from (23) that

$$t\,[\psi, \psi] = t\,[a, a] + t\,[b, b] \geq 0.$$

□

Note that we can replace P with CP throughout, provided the integral in (21) exists for all $\psi \in \hat{J}$ such that $\int_{\mathbb{R}^d} \psi(\mathbf{x})\, d\mathbf{x} = 0$. In the above proof, the existence of the integral follows under the assumption that $f \in \mathrm{P}(\hat{J}_{\mathbb{R}})$, and since $\hat{J}$ is closed under the operation $|\cdot|$, see (22). A similar approach will not work in the case of proving conditional positive definiteness, since for $\psi \in \hat{J}$ such that $\int_{\mathbb{R}^d} \psi(\mathbf{x})\, d\mathbf{x} = 0$, although we have $\tilde{\psi} = |\psi| \in \hat{J}_{\mathbb{R}}$, as in (22), it doesn't necessarily follow that $\int_{\mathbb{R}^d} \tilde{\psi}(\mathbf{x})\, d\mathbf{x} = 0$. Hence, assuming $f \in \mathrm{CP}(\hat{J}_{\mathbb{R}})$ does not guarantee the existence of the integral in (21). However, for the function spaces we are mainly interested in, namely $\mathrm{L}^2_0(\mathbb{R}^d)$, the integral in (21) automatically exists for all real and complex-valued functions $\psi \in \mathrm{L}^2_0(\mathbb{R}^d)$, since $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ by Lemma 4.3.

For completeness, we demonstrate that under certain conditions on our function, Definition 1.1 coincides with the original definition of positive definite functions, see (1). In particular, a continuous function is classically positive definite if and only if it is positive definite w.r.t. $\mathrm{C}_0(\mathbb{R}^d)$.

Proposition 6.2. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be continuous. Then, $f \in \mathrm{P}(\mathrm{C}_0(\mathbb{R}^d))$ if and only if f is classically positive definite.*

Proof. Let $f \in \mathrm{P}_{\mathrm{C}}$. By properties (iii) and (vi) in Section 2, for functions defined on $\mathbb{R}^d$, it follows directly that $f \in \mathrm{P}(\mathrm{C}_0(\mathbb{R}^d))$. For the reverse implication, consider the same argument as in the proof of Proposition 5.1, however, we no longer require the condition that $\sum_{i=1}^{N} \xi_i = 0$. □

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